LESSON 10 - 19 MAY 2021

HIGHER ORDER SOBOLEV SPACES

We can of course generalize the definition of Sobolev function to higher order derivatives:

DEFINITION Let $k \geq 2$ be an integer, $1 \leq p \leq +\infty$. Let $I \subseteq \mathbb{R}$ be an open set. We define

$$W^{k,p}(I) := \{ u \in W^{k-1,p}(I) \mid u' \in W^{k-1,p}(I) \}.$$

For $p=2$ we set

$$H^k(I) := W^{k,2}(I).$$

REMARK $u \in W^{k,p}(I)$ iff $\exists\ g_1, \dots, g_k \in L^p(I)$ s.t.

$$\int_I u \varphi^{(j)} dx = (-1)^j \int_I g_j \varphi \, dx \quad , \quad \forall \varphi \in C_c^\infty(I) \ , \ \forall j = 1, \dots, k \ ,$$

i.e. u admits weak derivatives up to order k. (easy check)

NOTATION In view of the above remark, and due to the uniqueness of weak derivatives, if $u \in W^{k,p}(I)$ we denote by

$$u^{(j)} := g_j \quad , \quad j = 1, \dots, k$$

the j-th weak derivative.

PROPOSITION 7.33 Let $I \subseteq \mathbb{R}$ be open, $k \geq 2$ be an integer, $1 \leq p \leq +\infty$. Then, the space $W^{k,p}(I)$ is Banach with the norm

$$\boxed{\|u\|_{W^{k,p}} := \|u\|_{L^p} + \sum_{j=1}^{k} \|u^{(j)}\|_{L^p}}$$

Moreover $H^k(I)$ is Hilbert with the inner product

$$\boxed{\langle u, v \rangle_{H^k} := \langle u, v \rangle_{L^2} + \sum_{j=1}^{k} \langle u^{(j)}, v^{(j)} \rangle_{L^2}}$$

(The proof is obtained following the lines of the proof of PROPOSITION 7.16)

REMARK $I \subseteq \mathbb{R}$ open, $k \geq 2$, $1 \leq p \leq +\infty$. Then $W^{k,p}(I) \subseteq C^{k-1}(\bar{I})$.

(Proof is consequence of THEOREM 7.19. For example, for $k=2$ we have that if $u \in W^{2,p}(I)$, then by definition $u' \in W^{1,p}(I)$.
As $W^{1,p}(I) \subset C(\bar{I})$ by THM 7.19, we get that $u' \in C(\bar{I})$. Therefore

$$u \in W^{1,p}(I), \quad u' \in C(I)$$

and thus, by PROPOSITION 7.22 we get $u \in C^1(\bar{I})$, concluding that

$$W^{2,p}(I) \subset C^1(\bar{I}).$$

Similarly, one can conclude the other cases.).

THE SPACE $W_0^{1,p}$

When dealing with Dirichlet type boundary conditions, it is useful to introduce the space $W_0^{1,p}$, which will be the space of functions $u \in W^{1,p}$ s.t. $u = 0$ on ∂I.

DEFINITION Let $I \subseteq \mathbb{R}$ be open, $1 \leq p < +\infty$. The space $W_0^{1,p}(I)$ is defined as the CLOSURE of $C_c^1(I)$ in $W^{1,p}(I)$. We denote

$$H_0^1(I) := W_0^{1,2}(I).$$

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, while $H_0^1(I)$ is equipped with the inner product of $H^1(I)$.

REMARK
- $W_0^{1,p}$ is a SEPARABLE BANACH space
- $W_0^{1,p}$ is REFLEXIVE for $1 < p < +\infty$
- H_0^1 is a SEPARABLE HILBERT space

(These follow from PROPOSITION 7.16 and the fact that $W_0^{1,p}$ is closed by definition)

REMARK By THEOREM 7.24 we know that $C_c^1(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$. Therefore

$$W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R}).$$

THEOREM 7.34 Let $I \subseteq \mathbb{R}$ be open, $1 \leq p < +\infty$. They are equivalent:

(a) $u \in W_0^{1,p}(I)$

(b) $u = 0$ on ∂I

We only prove the easy implication of THEOREM 7.34, that is, (a) $\Rightarrow$ (b).

Proof

(a) $\Rightarrow$ (b): By definition, if $u \in W_0^{1,p}(I)$ there $\exists \{u_n\} \subseteq C_c^1(I)$ s.t. $u_n \to u$ strongly in $W^{1,p}(I)$. By the embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$ (THEOREMS 7.19 and 7.27) we get that $u_n \to u$ uniformly in $\bar{I}$. As $u_n = 0$ on ∂I we then conclude $u = 0$ on ∂I.

(b) $\Rightarrow$ (a): See THEOREM 8.12 in BREZIS – "Functional Analysis, Sobolev Spaces and PDE", SPRINGER 2011. $\square$

POINCARÉ INEQUALITIES

THEOREM 7.35 (POINCARÉ INEQUALITY)

Let $I=(a,b)$ be bounded, $1\le p<+\infty$. There $\exists$ $C>0$ (depending only on $|I|$) s.t.

$$\|u\|_{W^{1,p}(I)} \le C\,\|u'\|_{L^p(I)}, \quad \forall u\in W^{1,p}_0(I).$$

In particular $\|u\|_{W^{1,p}(I)}$ and $\|u'\|_{L^p(I)}$ are equivalent norms on $W^{1,p}_0(I)$.

We give two proofs: the first one is more direct, while the second one is more abstract, but useful for proving generalizations.

WARNING The Poincaré Inequality does not hold in $W^{1,p}(a,b)$ (think of constants)

Proof 1 Let $u\in W^{1,p}_0(a,b)$. As $u(a)=0$ by THEOREM 7.34, we get

$$|u(x)| = |u(x)-u(a)|$$

(Here use THEOREM 7.19) $\longrightarrow$ $$= \left|\int_a^x u'(x)\,dx\right| \le \|u'\|_{L^1(a,b)}$$

Therefore $\|u\|_{L^\infty(a,b)} \le \|u'\|_{L^1(a,b)}$. Then

$$(*)\quad \|u\|^p_{L^p(a,b)} = \int_a^b |u|^p\,dx \le (b-a)\,\|u\|^p_{L^\infty(a,b)} \le (b-a)\,\|u'\|^p_{L^1(a,b)}$$

By Hölder's inequality we get

$$(**)\quad \|u'\|_{L^1(a,b)} \le \left(\int_a^b |u'|^p\,dx\right)^{1/p}\left(\int_a^b 1^{p'}\,dx\right)^{1/p'}$$

$$= \|u'\|_{L^p(a,b)}\,(b-a)^{1/p'}$$

Thus, by (*),

$$\|u\|_{L^p(a,b)} \overset{(*)}{\leq} (b-a)^{1/p} \, \|u'\|_{L^1(a,b)}$$

Since $\frac{1}{p}+\frac{1}{p'}=1$

$$\overset{(**)}{\leq} (b-a)^{1/p+1/p'} \, \|u'\|_{L^p(a,b)} = (b-a) \, \|u'\|_{L^p(a,b)}$$

Now

$$\|u\|_{W^{1,p}(a,b)} = \|u\|_{L^p(a,b)} + \|u'\|_{L^p(a,b)} \leq (b-a+1) \, \|u'\|_{L^p(a,b)}$$

therefore we conclude setting $C := b-a+1 = |I|+1$. $\square$

Proof 2 Assume by contradiction that the inequality does not hold. Then we can find a sequence $\{u_n\} \subseteq W_0^{1,p}(a,b)$ s.t.

$$(*) \qquad \|u_n\|_{L^p} \geq n \, \|u_n'\|_{L^p} \, , \qquad \forall n \in \mathbb{N}.$$

As the norm is homogeneous, up to rescaling u_n by $\|u_n\|_{L^p}$, we can assume that $\|u_n\|_{L^p} = 1$, $\forall n \in \mathbb{N}$. Then, from (*), we get

$$(**) \qquad \|u_n\|_{L^p} = 1 \, , \qquad \|u_n'\|_{L^p} \leq \frac{1}{n} \, , \qquad \forall n \in \mathbb{N}.$$

In particular $\{u_n\}$ is bounded in $W^{1,p}(a,b)$. By the SOBOLEV EMBEDDING THM 7.27 (point (c)) we know that $W^{1,p}(a,b) \hookrightarrow L^p(a,b)$ compactly. Thus $\overline{\{u_n\}}$ is compact in $L^p(a,b)$. In particular $\{u_n\}$ admits a subsequence s.t.

$$u_{n_k} \to u \text{ strongly in } L^p(a,b).$$

Moreover, from (**) we know that $\|u_{n_k}'\|_{L^p} \leq \frac{1}{n_k}$, $\forall k \in \mathbb{N}$. Therefore

$$u_{n_k}' \to 0 \text{ strongly in } L^p(a,b).$$

Thus, from REMARK 7.17 we conclude that $u_{n_k} \to u$ strongly in $W^{1,p}(a,b)$, with $u' = 0$ in the weak sense.

Therefore, by definition of weak derivative, we get

$$\int_a^b u \varphi' \, dx = -\int_a^b u' \varphi \, dx \overset{u'=0}{=} 0 \quad , \quad \forall \varphi \in C_c^1(a,b)$$

and the DBR LEMMA 7.13 implies that $u = c$ a.e. on (a,b), for some $c \in \mathbb{R}$.

Now recall that $W_0^{1,p}(a,b)$ is closed by definition, therefore, as $u_{n_k} \to u$ in $W^{1,p}(a,b)$, and $\{u_n\} \subseteq W_0^{1,p}(a,b)$, we get that $u \in W_0^{1,p}(a,b)$.

By THEOREM 7.34 we then have $u(a) = u(b) = 0$. Since $u = c$, this implies $c = 0$ and

$$u \equiv 0 .$$

However, taking the limit as $k \to +\infty$ in the first condition in (**) gives

$$\|u\|_{L^p} = 1 ,$$

which is a contradiction, as $u \equiv 0$. □

When dealing with BC which are more general than homogeneous Dirichlet BC, the above Poincaré inequality is useless.

Therefore we look for a more general version. In order to do that, notice that the Poincaré Inequality

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^p(I)} \quad , \quad \forall u \in W_0^{1,p}(I)$$

holds because non-zero constant functions do not belong to $W_0^{1,p}$.

This simple observation motivates the following generalization of THEOREM 7.35.

THEOREM 7.36 (GENERALIZED POINCARÉ INEQUALITY)

Let $I=(a,b)$ be bounded, $1 \leq p < +\infty$. Let $V \subseteq W^{1,p}(I)$ be a SUBSPACE s.t.

(i) V is closed in $W^{1,p}(I)$

(ii) If $u \in V$ is constant, then $u \equiv 0$.

Then there $\exists$ $C > 0$

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^p(I)}, \quad \forall u \in V.$$

In particular $\|u\|_{W^{1,p}(I)}$ and $\|u'\|_{L^p(I)}$ are equivalent norms on V.

(The proof of THEOREM 7.36 can be obtained following the lines of PROOF 2 of THEOREM 7.35. It is left for exercise in the Exercises Course).

EXAMPLE 7.37 We give some examples of subspaces $V \subseteq W^{1,p}$ satisfying the assumptions of THEOREM 7.36:

- $V = \{ u \in W^{1,p}(a,b) \mid u(p) = 0 \}$ for $p \in [a,b]$ fixed
  (V is closed by the embedding $W^{1,p}(a,b) \hookrightarrow C[a,b]$)

- $V = \{ u \in W^{1,p}(a,b) \mid \int_a^b u \, dx = 0 \}$

- $V = \{ u \in W^{1,p}(a,b) \mid \int_E u \, dx = 0 \}$, for $E \subseteq [a,b]$ with $|E| > 0$

8. EULER-LAGRANGE EQUATION, SOBOLEV CASE

We now analyze variational problems in Sobolev space. First we generalize the following theorems we proved in the C^1 setting: consider the spaces

$$X = \{ u \in C^1[a,b] \mid u(a) = \alpha,\ u(b) = \beta \}, \qquad V = \{ u \in C^1[a,b] \mid u(a) = u(b) = 0 \},$$

the Lagrangian $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, \xi)$, and the functional

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx, \qquad u \in X.$$

① **THEOREM 4.5:** L continuous and C^1 WRT s, ξ.

1) If u_0 minimizes F over X then u_0 solves

(INTEGRAL ELE) $$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0, \quad \forall v \in V}$$

2) If $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ minimizes F over X, then u_0 solves

(ELE) $$\boxed{\begin{cases} \dfrac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0) \right] = L_s(x, u_0, \dot{u}_0), & \forall x \in [a,b] \\ u_0(a) = \alpha,\ u_0(b) = \beta \end{cases}}$$

② **THEOREM 5.4:** $L \in C^1$, $u_0 \in X$ solution to (INTEGRAL ELE).

1) If L is CONVEX in s, ξ then u_0 is minimizer of F.

2) If L is STRICTLY CONVEX in s, ξ, then u_0 is the UNIQUE minimizer of F.

We start by relaxing the assumptions on L, by just requiring measurability. Precisely, we will require L to be a Carathéodory function:

DEFINITION 8.1 $\Omega \subseteq \mathbb{R}^d$ open, $L : \Omega \times \mathbb{R}^n \to \mathbb{R}$. We say that L is a CARATHÉODORY FUNCTION if

1) $y \mapsto L(x,y)$ is continuous for a.e. $x \in \Omega$,

2) $x \mapsto L(x,y)$ is Lebesgue measurable for all $y \in \mathbb{R}^n$.

NOTATION Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$. Whenever we say that the Lagrangian L is Carathéodory we mean that

$$\Omega = (a,b), \quad d = 1, \; n = 2 \quad \text{and} \quad y = (s,\xi)$$

in DEFINITION 8.1.

EXAMPLE $L : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $L(x,s,\xi) := \alpha(x) + g(s,\xi)$ is Carathéodory if $\alpha : (0,1) \to \mathbb{R}$ is measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

PROPOSITION 8.2 Let $\Omega \subseteq \mathbb{R}^d$ be open, $L : \Omega \times \mathbb{R}^n \to \mathbb{R}$ Carathéodory, $u : \Omega \to \mathbb{R}^n$ measurable. Then $g : \Omega \to \mathbb{R}$ defined by

$$g(x) := L(x, u(x))$$

is measurable.

(Proof is omitted. It is obvious by approximation by step functions – see PROPOSITION 3.7 in the book by Dacorogna).

WEAK EULER-LAGRANGE EQUATION

Let $p \geq 1$, $a<b$, and define the space

$$X := \{ u \in W^{1,p}(a,b) \mid u(a) = \alpha, \ u(b) = \beta \}$$

Note
- X is well-defined, since $W^{1,p}$ functions are continuous by THEOREM 7.19 (so $u(a)$ and $u(b)$ make sense)
- X is an AFFINE space with reference vector space $W_0^{1,p}(a,b)$ (since functions in $W_0^{1,p}(a,b)$ vanish on a, b, by THEOREM 7.34).

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

WEAK DERIVATIVE

The Sobolev version of THEOREM 4.5 is as follows:

ASSUMPTION 8.3 Assume L, L_s, L_ξ are Carathéodory functions.
Suppose that either of the following holds:

(H1) $\forall R > 0$, $\exists$ $\alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $p' := \frac{p}{p-1}$, $\beta = \beta(R)$ such that $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

$$|L(x,s,\xi)|, \ |L_s(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$$

$$|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1}$$

(H2) $\forall R > 0$, $\exists$ $\alpha_1 \in L^1(a,b)$, $\beta = \beta(R)$ such that $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

$$|L(x,s,\xi)|, \ |L_s(x,s,\xi)|, \ |L_\xi(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$$

THEOREM 8.4 Suppose the above ASSUMPTION 8.3 holds.
Let $u_0 \in X$ be a minimizer for F over X.

1) If (H1) holds then u_0 satisfies the weak form of ELE

$$\text{(W-ELE)} \qquad \boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx = 0, \qquad \forall\, v \in W_0^{1,p}(a,b)}$$

2) If (H2) holds then u_0 satisfies the weaker form of ELE

$$\text{(W'-ELE)} \qquad \boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx = 0, \qquad \forall\, v \in C_c^\infty(a,b)}$$

3) If in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ then u_0 satisfies the classical ELE

$$\text{(ELE)} \qquad \boxed{\frac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0)\right] = L_s(x, u_0, \dot{u}_0), \qquad \forall\, x \in [a,b]}$$

Proof **Step 1. F is well-defined:** let $u \in W^{1,p}(a,b)$. Then both u and $\dot{u}$ are measurable. Since L is Carathéodory, by PROPOSITION 8.2 we get that $g : (a,b) \to \mathbb{R}$ defined by

$$g(x) := L(x, u(x), \dot{u}(x))$$

is measurable. Thus g can be integrated, with the integral possibly being unbounded.

Next we need to show that F is bounded.

Since $W^{1,p}(a,b) \hookrightarrow L^\infty(a,b)$ (THEOREM 7.27), we get $u \in L^\infty(a,b)$. Therefore

$$|u(x)| \leq \|u\|_\infty \quad \text{a.e. in } (a,b).$$

Choose $R = \|u\|_\infty$ in (H1) or (H2), so that there exist $\alpha_1 \in L^1(a,b)$, $\beta = \beta(R)$ s.t.

$$(*) \qquad |L(x,s,\xi)| \leq \alpha_1(x) + \beta|\xi|^p, \quad \forall x \in (a,b),\ |s| \leq \|u\|_\infty,\ \xi \in \mathbb{R}.$$

Thus $(x, u(x), \dot{u}(x)) \in (a,b) \times [-\|u\|_\infty, \|u\|_\infty] \times \mathbb{R}$, and

$$|F(u)| \leq \int_a^b |L(x,u(x),\dot{u}(x))|\, dx$$

$$\overset{(*)}{\leq} \int_a^b \alpha_1(x)\, dx + \beta \int_a^b |\dot{u}|^p\, dx < +\infty \quad \to \alpha_1 \in L^1,\ \dot{u} \in L^p$$

showing that F is well-defined.

Step 2. Gâteaux derivative of F:

CASE OF (H1): Assume (H1). We show that for every $u \in W^{1,p}$ the functional F is Gâteaux differentiable in every direction $v \in W^{1,p}$, by proving that

$$(**) \qquad \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v}\, dx$$

Since we are assuming that L_s, L_ξ are Carathéodory, this means that the maps

$$(s,\xi) \mapsto L_s(x,s,\xi) \quad , \quad (s,\xi) \mapsto L_\xi(x,s,\xi)$$

are continuous for a.e $x \in (a,b)$ fixed. Therefore we can apply the standard chain rule to conclude that the map

$$t \longmapsto L(x, u + t\varepsilon v, \dot{u} + \varepsilon t \dot{v})$$

is differentiable, with

$$(***) \qquad \frac{d}{dt}\left\{ L(x, u+t\varepsilon v, \dot{u} + \varepsilon t \dot{v}) \right\} = \varepsilon L_s(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, v + \varepsilon L_\xi(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, \dot{v}$$

Now set

$$g(x,\varepsilon) := \int_0^1 L_s(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, v + L_\xi(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, \dot{v} \, dt$$

Then

def of F

$$\frac{1}{\varepsilon}\left\{ F(u + tv) - F(u) \right\} = \frac{1}{\varepsilon} \int_a^b \left\{ L(x, u + tv, \dot{u} + t\dot{v}) - L(x,u,\dot{u}) \right\} dx$$

(Fundamental Thm of Calculus) $$= \frac{1}{\varepsilon} \int_a^b \left[\int_0^1 \frac{d}{dt}\left\{ L(x, u + t\varepsilon v, \dot{u} + \varepsilon t \dot{v}) \right\} dt \right] dx$$

(by $(***)$ and def of g) $$= \int_a^b g(x,\varepsilon)\, dx$$

In order to prove (**) it is then sufficient to show that

$$\text{(C)} \qquad \lim_{\varepsilon \to 0} \int_a^b g(x,\varepsilon)\, dx = \int_a^b \underbrace{L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v}}_{= g(x,0) \text{ by definition of } g}\, dx$$

IDEA To show (C) we use DOMINATED CONVERGENCE: i.e., we need to show

$$\text{(A)} \quad \lim_{\varepsilon \to 0} g(x,\varepsilon) = g(x,0) \quad \text{for a.e. } x \in (a,b)$$

$$\text{(B)} \quad \sup_{0<\varepsilon<1} |g(x,\varepsilon)| \le |\Lambda(x)| \quad \text{for a.e. } x \in (a,b), \text{ for some } \Lambda \in L^1(a,b)$$

To do that, first notice that by the embedding $W^{1,p}(a,b) \hookrightarrow L^\infty(a,b)$ we get $u + \varepsilon t v \in L^\infty(a,b)$ for all $\varepsilon > 0$, $t \in [0,1]$.
In particular, for $0 < \varepsilon < 1$, $t \in [0,1]$ we get

$$\text{(B)} \quad |u(x) + \varepsilon t v(x)| \le \|u\|_\infty + \|v\|_\infty \quad \text{a.e. on } (a,b).$$

Thus set $R := \|u\|_\infty + \|v\|_\infty$ in (H1), to obtain the existence of $\alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $\beta = \beta(R)$ s.t.

$$\text{(1)} \quad |L(x,s,\xi)| \le \alpha_1(x) + \beta |\xi|^p$$

$$\text{(2)} \quad |L_s(x,s,\xi)| \le \alpha_1(x) + \beta |\xi|^p \qquad , \ \forall x \in (a,b),\ |s| \le R,\ \xi \in \mathbb{R}$$

$$\text{(3)} \quad |L_\xi(x,s,\xi)| \le \alpha_2(x) + \beta |\xi|^{p-1}$$

We now show A : need DOMINATED CONVERGENCE, as $g(x,\varepsilon)$ is itself an integral.
For a.e. $x \in (a,b)$ we know that the maps

$$(s,\xi) \mapsto L_s(x,s,\xi), \qquad (s,\xi) \mapsto L_\xi(x,s,\xi)$$

are continuous (as L_s, L_ξ Carathéodory). Therefore

$$\lim_{\varepsilon \to 0} \left\{ L_s(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, v + L_\xi(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, \dot{v} \right\} =$$

$$= L_s(x, u, \dot{u})\, v + L_\xi(x, u, \dot{u})\, \dot{v}$$

for all $t \in [0,1]$ and a.e. $x \in (0,1)$.
Moreover, as $u + t\varepsilon v$ satisfies (B), we can invoke (2) to get

$$|L_s(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, v| \overset{(2)}{\leq} \left[\alpha_1(x) + \beta\, |\dot{u} + t\varepsilon \dot{v}|^p\right] |v|$$

(as $\varepsilon, t \in (0,1)$, and using $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $p \geq 1$)
$$\leq \left[\alpha_1(x) + \beta\, 2^{p-1}\left(|\dot{u}|^p + |\dot{v}|^p\right)\right] |v|, \quad \forall t \in [0,1]$$

and the RHS belongs to $L^1(0,1)$ since x is fixed.
Similarly, using (3), one also shows that

$$|L_\xi(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, \dot{v}| \leq C(x), \qquad \forall t \in [0,1]$$

so that $C(x) \in L^1(0,1)$, being a constant (x is fixed). Then by DOMINATED CONVERGENCE

$$\lim_{\varepsilon \to 0} g(x,\varepsilon) = \lim_{\varepsilon \to 0} \int_0^1 L_s(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, v + L_\xi(x, u + t\varepsilon v, \dot{u} + t\varepsilon \dot{v})\, \dot{v}\, dt$$

$$= \int_0^1 L_s(x, u, \dot{u})\, v + L_\xi(x, u, \dot{u})\, \dot{v}\, dt = g(x,0)$$

showing A.

We now prove (B): we need to estimate $g(x,\varepsilon)$:

$$|g(x,\varepsilon)| \leq \int_0^1 |L_s(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, v|\, dt + \int_0^1 |L_\xi(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, \dot v|\, dt$$

For the first integral we use (2):

$$\int_0^1 |L_s(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, v|\, dt \overset{(2)}{\leq} \int_0^1 [\alpha_1(x) + \beta |\dot u + t\varepsilon \dot v|^p]\, |v(x)|\, dt$$

(as $\varepsilon, t \in (0,1)$ and using $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $p \geq 1$) $\leq \int_0^1 [\alpha_1(x) + \beta 2^{p-1}(|\dot u|^p + |\dot v|^p)]\, |v(x)|\, dt$

(as nothing depends on t anymore) $= [\alpha_1(x) + \beta 2^{p-1}(|\dot u|^p + |\dot v|^p)]\, |v(x)|$

(since $v \in W^{1,p} \hookrightarrow L^\infty$) $\leq \underbrace{[\alpha_1(x) + \beta 2^{p-1}(|\dot u|^p + |\dot v|^p)]}_{\in L^1(a,b) \text{ since } \alpha_1 \in L^1(a,b),\ \dot u, \dot v \in L^p(a,b)} \|v\|_\infty$

For the second integral we use (3):

$$\int_0^1 |L_\xi(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, \dot v|\, dt \overset{(3)}{\leq} \int_0^1 [\alpha_2(x) + \beta |\dot u + \varepsilon t \dot v|^{p-1}]\, |\dot v|\, dt$$

(the first term does not depend on t) $\rightarrow = \underbrace{\alpha_2(x)\, |\dot v(x)|}_{\in L^1(a,b) \text{ by Hölder as } \alpha_2 \in L^{p'},\ \dot v \in L^p} + \underbrace{\beta |\dot v(x)| \int_0^1 |\dot u + \varepsilon t \dot v|^{p-1}\, dt}_{\text{This one is estimated separately below}}$

$$\beta |\dot v(x)| \int_0^1 |\dot u + \varepsilon t \dot v|^{p-1}\, dt \leq \sup_{t \in [0,1]} \underbrace{\beta |\dot v(x)|\, |\dot u(x) + \varepsilon t \dot v(x)|^{p-1}}_{\in L^1(a,b)}$$

$\in L^1(a,b)$ by Hölder, as $|\dot u + \varepsilon t \dot v(x)|^{p-1} \in L^{p'}(a,b)$ since $\dot u + \varepsilon t \dot v \in L^p$

Thus, $\exists\, \Lambda \in L^1(a,b)$ s.t. $|g(x,\varepsilon)| \leq \Lambda(x)$ for a.e. $x \in (a,b)$, $\forall\, 0 < \varepsilon < 1$, showing (B).

Using the same argument of PROPOSITION 2.3 it is immediate to check that the above implies

$$F_g'(u_0)(v) = 0 .$$

Therefore we conclude that

$$\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx = 0 \qquad , \qquad \forall v \in W_0^{1,p}(a,b)$$

proving that u_0 solves (W-ELE).

- Assume (H2). For what already proved, F is gâteaux differentiable at u_0 $\forall$ directions in $C^\infty(a,b)$, with

$$F_g'(u_0)(v) = \int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx \quad , \quad \forall v \in C^\infty(a,b)$$

Let $v \in C_c^\infty(a,b)$ be arbitrary. Thus $u_0 + \varepsilon v \in X$, $\forall \varepsilon \in \mathbb{R}$ (as $v(a) = v(b) = 0$)
Since u_0 is a minimizer, as above we can show $F_g'(u_0)(v) = 0$, i.e.

$$\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx = 0 \qquad , \qquad \forall v \in C_c^\infty(a,b)$$

proving that u_0 solves (W'-ELE).

Then (C) follows by DOMINATED CONVERGENCE, showing that F is gâteaux diff. at each $u \in W^{1,p}(a,b)$ with

$$F_g'(u)(v) = \int_a^b L_s(x,u,\dot{u})\,v + L_\xi(x,u,\dot{u})\,\dot{v}\,dx \quad , \quad \forall v \in W^{1,p}(a,b)$$

<u>CASE OF (H2)</u> : Assume (H2). By similar arguments we can show that F is Gâteaux differentiable for every $u \in W^{1,p}$, in every direction $v \in C^\infty(a,b)$, with

$$F_g'(u)(v) = \int_a^b L_s(x,u,\dot{u})\,v + L_\xi(x,u,\dot{u})\,\dot{v}\,dx$$

The difference WRT the case of (H1) is that now the bound on L_ξ is different, but since $v \in C^\infty(a,b)$ (not $v \in W^{1,p}$ as in the previous case) all the estimates work.

<u>Step 3. Show ELE</u> : Suppose now that $u_0 \in X$ minimizes F over X.

- Assume (H1). For what already proved, F is gâteaux differentiable at u_0 $\forall$ directions in $W^{1,p}(a,b)$, with

$$F_g'(u_0)(v) = \int_a^b L_s(x,u_0,\dot{u}_0)\,v + L_\xi(x,u_0,\dot{u}_0)\,\dot{v}\,dx \quad , \quad \forall v \in W^{1,p}(a,b)$$

Let $v \in W_0^{1,p}(a,b)$ be arbitrary. Thus $u_0 + \varepsilon v \in X$, $\forall \varepsilon \in \mathbb{R}$ (as $v(a) = v(b) = 0$)
Since u_0 is a minimizer, we get

$$F(u_0) \le F(u_0 + \varepsilon v)$$

- Assume that in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$. Since L satisfies at least one between (H1) and (H2) by assumption, we deduce that u_0 solves either (W-ELE) or (W'-ELE). In both cases, we have

$$\int_a^b L_s(x,u_0,\dot{u}_0)\, v + L_\xi(x,u_0,\dot{u}_0)\, \dot{v} \, dx = 0 \quad , \quad \forall v \in C_c^\infty(a,b)$$

As u_0 and L are C^2, we can integrate by parts the above, and use that $v(a) = v(b) = 0$ to get

$$\int_a^b \left\{ L_s(x,u_0,\dot{u}_0) - \left[L_\xi(x,u_0,\dot{u}_0) \right]' \right\} v \, dx = 0 \quad , \quad \forall v \in C_c^\infty(a,b)$$

By the standard FLCV LEMMA 3.4 we deduce (ELE) $\square$

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OTHER BOUNDARY CONDITIONS

So far we only dealt with Dirichlet Boundary Conditions. What about other BC? For example one could set the problem in

$$\{ u \in W^{1,p}(a,b) \mid u(a) = \alpha \}$$

This space is well-defined, since Sobolev functions are continuous (THEOREM 7.19).

REMARK Inspecting the proof of THEOREM 8.4 we notice:

- Step 1 - F is well-defined: here we only used the growth assumptions on L to prove that $|F(u)| < +\infty \quad \forall u \in W^{1,p}(a,b)$

- Step 2 - Gâteaux derivative: Here we used (H1) and (H2) separately:

  - Assuming (H1) we proved that F is Gâteaux differentiable at each $u \in W^{1,p}(a,b)$, in every direction $v \in W^{1,p}(a,b)$

  - Assuming (H2) we proved that F is Gâteaux differentiable at each $u \in W^{1,p}(a,b)$, in every direction $v \in C^{\infty}(a,b)$

- Step 3 - Showing ELE: Here we used the BC

  - Assuming (H1), we chose the variations $v \in W^{1,p}_0(a,b)$. This allowed to deduce (W-ELE), since F was diff. in every direction in $W^{1,p}$

  - Assuming (H2), we chose the variations $v \in C^{\infty}_c(a,b)$. This allowed to deduce (W'-ELE), since F was diff. in every direction in C^{∞}

Therefore, we deduce the following general result.

THEOREM 8.5 Let $p \geq 1$, $a<b$. Let $X \subseteq W^{1,p}(a,b)$ be an AFFINE SPACE with reference vector space $V \subseteq W^{1,p}(a,b)$.

Suppose $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies ASSUMPTION 8.3.
Define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx .$$

Let $u_0 \in X$ be a minimizer for F over X. Then:

1) If (H1) holds then u_0 satisfies the weak form of ELE

(W-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx = 0 , \quad \forall\, v \in V}$$

2) If (H2) holds then u_0 satisfies the weaker form of ELE

(W'-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx = 0 , \quad \forall\, v \in C^\infty(a,b) \cap V}$$

3) If in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ then u_0 satisfies the classical ELE

(ELE)
$$\boxed{\begin{cases} \dfrac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0) \right] = L_s(x, u_0, \dot{u}_0) , & \forall\, x \in [a,b] \\ \text{BC from the integration by parts of (W'-ELE)} \end{cases}}$$

SUFFICIENT CONDITIONS FOR MINIMALITY

We now address the generalization of THEOREM 5.4. Let us recall the setting:

Let $p \geq 1$, $a<b$, and let $X \subseteq W^{1,p}(a,b)$ be an affine space over $V \subseteq W^{1,p}(a,b)$.

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx$$

The Sobolev version of THEOREM 5.4 (with general BC) is as follows:

THEOREM 8.6 Suppose L satisfies Assumption 8.3.

Assume $u_0 \in X$ solves (W-ELE) or (ELE) in THEOREM 8.5.

1) If

$(s,\xi) \longmapsto L(x,s,\xi)$ is CONVEX for a.e. $x \in (a,b)$

then u_0 is a minimizer for F on X.

2) If

$(s,\xi) \longmapsto L(x,s,\xi)$ is STRICTLY CONVEX for a.e. $x \in (a,b)$

then u_0 is the UNIQUE minimizer for F on X.

(The proof carries out exactly like the one of THEOREM 5.4, with straightforward changes. THEOREM 5.2 can be used because L_s, L_ξ are Carathéodory. Hence L is C^1 WRT (s,ξ), for a.e. $x \in (a,b)$ fixed).

9. DIRECT METHOD

GOAL The Direct Method is used to prove existence of minimizers

We state a very general version of the direct method, for functionals $F: X \to \mathbb{R}$ with X space with a notion of convergence.

DEFINITION 9.1 Let X be a set, and $Seq(X)$ the set of all sequences in X:

$$Seq(X) := \{ f: \mathbb{N} \to X \}.$$

A NOTION of CONVERGENCE on X is a subset N of

$$\underbrace{Seq(X)}_{\text{contains sequences}} \times \underbrace{X}_{\text{contains limits}}$$

NOTATION Thus a notion of convergence is a list of sequences with corresponding limit. Therefore, whenever we say that $\{x_n\} \subseteq X$ converges to $x_0 \in X$, in symbols $x_n \to x_0$, we mean that $(\{x_n\}, x_0) \in N$ notion of convergence.

EXAMPLES

- X = topological space. A notion of convergence is for example the list of all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x_0$ WRT to τ, for some $x_0 \in X$
- The above example contains all the well-known cases: Metric spaces, Normed spaces with weak or strong convergence, Hilbert spaces, $\mathbb{R}^d$.

DEFINITION 9.2 X space with notion of convergence. We say that $K \subseteq X$ is (sequentially) compact if every sequence $\{x_n\} \subseteq K$ admits a subsequence such that $x_{n_k} \to x_0$ with $x_0 \in K$.

DEFINITION 9.3 X space with notion of convergence. A function $f: X \to \mathbb{R}$ is

- **CONTINUOUS** if for all sequences $x_n \to x_0$ we have

$$f(x_n) \to f(x_0).$$

- **LOWER SEMICONTINUOUS (LSC)** if for all $x_n \to x_0$ we have

$$f(x_0) \leq \liminf_{n \to +\infty} f(x_n).$$

THEOREM 9.4 **(DIRECT METHOD)**

X space with notion of convergence, $f: X \to \mathbb{R}$. Assume that

(i) X is compact

(ii) f is LSC

WRT the SAME notion of convergence

Then the problem

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution: i.e., $\exists\, \hat{x} \in X$ s.t. $f(\hat{x}) = I$.

Proof Exactly like the Weierstraß Theorem of Analysis 1.

By the properties of the infimum $\exists$ infimizing sequence $\{y_n\} \subseteq f(X)$ s.t.

$$y_n \to I.$$

Note that, a priori, $I \in [-\infty, +\infty]$.

By def. of image $\exists \{x_n\} \subseteq X$ s.t. $y_n = f(x_n)$, $\forall n \in \mathbb{N}$. Thus

$$f(x_n) \to I.$$

As X is compact, there $\exists$ a subsequence s.t. $x_{n_k} \to \hat{x}$ for some $\hat{x} \in X$. Then

$$\underset{I \text{ is inf}}{I \leq} f(\hat{x}) \underset{\substack{f \text{ is LSC} \\ \text{and } x_{n_k} \to \hat{x}}}{\leq} \liminf_{k \to +\infty} f(x_{n_k}) \underset{f(x_n) \text{ is convergent}}{=} \lim_{n \to +\infty} f(x_n) = I$$

Thus $f(\hat{x}) = I$, concluding that $\hat{x}$ is a minimizer and that I is finite. $\square$

REMARK The direct method is deceivingly simple. The highly non-trivial task is finding a notion of convergence on X s.t. (i) - (ii) hold. Note that:

- If we have many convergent sequences then (i) is easy and (ii) hard
- If we have few convergent sequences then (i) is hard and (ii) easy

Therefore (i) and (ii) are in competition, and and finding a notion of convergence s.t. both hold is delicate.

Let us now see some variants of the direct method.

DEFINITION 9.5 X space with notion of convergence. A function $f: X \to \mathbb{R}$ is **COERCIVE** if $\exists K \subseteq X$ compact s.t.

$$\inf \{ f(x) \mid x \in K \} = \inf \{ f(x) \mid x \in X \}$$

EXAMPLE $X = \mathbb{R}$, $f(x) = x^2$. Then f is coercive (i.e. $K = [-1, 1]$)

THEOREM 9.6 X space with notion of convergence, $f: X \to \mathbb{R}$ s.t.

(i) f is COERCIVE

(ii) f is LSC

Then

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution.

Proof As f is coercive then by definition $\exists K \subseteq X$ compact s.t.

$$(*) \quad I = \inf \{ f(x) \mid x \in K \}$$

As K is compact and f is LSC on K (as f is LSC on X), we can apply THEOREM 9.4 to obtain $\hat{x} \in K$ s.t. $f(\hat{x}) = \inf \{ f(x) \mid x \in K \}$. Thus, by $(*)$, $f(\hat{x}) = I$ and we conclude. $\square$

THEOREM 9.7 X space with notion of convergence, $f: X \to \mathbb{R}$ s.t.

(i) $\exists M \geq 0$, $\exists K \subseteq X$ compact s.t. $\{x \in X \mid f(x) \leq M\} \neq \emptyset$ and

$$\{x \in X \mid f(x) \leq M\} \subseteq K$$

(ii) f is LSC

Then

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution $\hat{x} \in X$ s.t. $f(\hat{x}) \leq M$.

Proof We want to show that $\tilde{K} := \{ x \in X \mid f(x) \le M \}$ is compact. So let $\{x_n\} \subseteq \tilde{K}$. As $\tilde{K} \subseteq K$ and K is compact, there $\exists$ a subsequence and $x_0 \in K$ s.t.

$$x_{n_k} \to x_0 .$$

Since $\{x_{n_k}\} \subseteq \tilde{K}$ and f is LSC, we get

$$f(x_0) \le \liminf_{n \to +\infty} f(x_{n_k}) \le M$$

f LSC and $x_{n_k} \to x_0$ As $\{x_{n_k}\} \subseteq \tilde{K}$

proving that $x_0 \in \tilde{K}$ and so that $\tilde{K}$ is compact. We now have two cases:

- $I = M$: then by def. of $\tilde{K}$ and of infimum

  $$\tilde{K} = \{ x \in X \mid f(x) \le I \} = \{ x \in X \mid f(x) = I \}.$$

  Thus $\tilde{K}$ is exactly set of minimizers. Since $\tilde{K} \neq \emptyset$ by assumption, we conclude.

- $I < M$: Let $\{x_n\} \subseteq X$ be an infimizing sequence, i.e., such that

  $$f(x_n) \to I .$$

  Since $I < M$, we conclude that $\{x_n\} \subseteq \tilde{K}$. (upon discarding a finite number of indices). As $\tilde{K}$ is compact, $\exists$ a subsequence and $\hat{x} \in \tilde{K}$ s.t. $x_{n_k} \to \hat{x}$. Now

  $$I \le f(\hat{x}) \le \liminf_{k \to +\infty} f(x_{n_k}) = \lim_{n \to +\infty} f(x_n) = I ,$$

  def of inf $x_{n_k} \to \hat{x}$ and f is LSC as $f(x_n)$ is convergent

  Thus $f(\hat{x}) = I$ and we conclude that $\hat{x}$ is a minimizer. $\square$

DIRECT METHOD – ACTION PLAN

Given $F : X \to \mathbb{R}$, the minimization problem

$$(P) \qquad \inf \{ F(x) \mid x \in X \}$$

can be studied in the following way:

① **WEAK FORMULATION**: Extend F to a functional $\hat{F} : \hat{X} \to \mathbb{R}$ with $X \subseteq \hat{X}$, i.e., to a larger space
(Typically $\hat{X}$ will by a SOBOLEV SPACE, rather than the usual C^1, C^∞ or C^1_{pw})

② **COMPACTNESS**: Prove that the sublevel sets of $\hat{F}$ are compact WRT some appropriate notion of convergence on $\hat{X}$

③ **LOWER SEMICONTINUITY**: Prove that $\hat{F}$ is LSC WRT the same notion of convergence of point ②.

④ **REGULARITY**: At this point one can apply THEOREM 9.7 and conclude the $\exists$ of a solution $\bar{x} \in \hat{X}$ to

$$\inf \{ \hat{F}(x) \mid x \in \hat{X} \}$$

The last step consists in showing that

$$\bar{x} \text{ is more regular, i.e., } \bar{x} \in X$$

Note that, as $\hat{F} = F$ on X, this immediately implies that $\bar{x}$ solves the original minimization problem (P)

EXAMPLE 9.8 Set $X = \{ u \in C^1[0,1] \mid u(0)=0, \ u(1)=1 \}$ and

$$F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx \,, \qquad u \in X.$$

Note that the Lagrangian appearing in F is non-linear. Thus the associated ELE is hard (maybe impossible) to solve explicitly).

We then resort to our ACTION PLAN for the DIRECT METHOD:

(1) **WEAK FORMULATION**: We extend F to the larger space

$$\hat{X} := \{ u \in H^1(0,1) \mid u(0)=0, \quad u(1)=1 \}.$$

Note that $\hat{X}$ is well-defined, since H^1 functions are continuous by THEOREM 7.19. Therefore the Dirichlet Boundary conditions appearing in $\hat{X}$ make sense.

The extension of F to $\hat{X}$ is trivially defined by

$$\hat{F}(u) := \int_0^1 \underset{\substack{\uparrow \\ \text{WEAK DERIVATIVE}}}{\dot{u}^2} + \sin(u^5)\, dx \,, \qquad \forall u \in H^1(0,1),$$

Note that $\hat{F}$ is well-defined, since

- $\dot{u} \in L^2(0,1)$ as $u \in H^1(0,1)$
- $\sin(u^5) \in L^1(0,1)$ as $H^1(0,1) \hookrightarrow L^\infty(0,1)$ by the SOBOLEV EMBEDDING THEOREM 7.27 (or, more simply, because $|\sin x| \leq 1$)

Moreover $\hat{F} = F$ on X, since if $u \in C^1[0,1]$, then its weak derivative $\dot{u}$ coincides a.e. with the classical derivative.

② COMPACTNESS : We need to show that there $\exists\ M>0$ s.t. the sublevel

$$K := \{ u \in \hat{X} \mid \hat{F}(u) \leq M \}$$

is non-empty, and compact WRT some notion of convergence on $H^1(0,1)$. Clearly we can choose $M := F(u)$ with $u(x) := x$, so that $K \neq \emptyset$.

As notion of convergence we take the weak convergence on H^1. We have to show that K is compact. Hence assume that $\{u_n\} \subseteq K$, that is,

$$\{u_n\} \subseteq \hat{X} \quad \text{and} \quad \hat{F}(u_n) \leq M, \quad \forall\, n \in \mathbb{N}.$$

As $|\sin x| \leq 1$, we get

$$\int_0^1 \dot{u}_n^2 \, dx - 1 \leq \int_0^1 \dot{u}_n^2 + \sin(u^5)\, dx = \hat{F}(u_n) \leq M \quad \Rightarrow \quad \boxed{\|\dot{u}_n\|_{L^2} \leq \sqrt{M+1}}$$

Thus $\{\dot{u}_n\}$ is uniformly bounded in $L^2(0,1)$. Since $L^2(0,1)$ is Hilbert separable, by Banach-Alaoglu Theorem we conclude that there $\exists$ a subseq. and $\hat{v} \in L^2(0,1)$ s.t.

$$\dot{u}_n \rightharpoonup \hat{v} \quad \text{weakly in } L^2(0,1).$$

Moreover by the Hölder estimate of THEOREM 7.23 (with $p=2$) we get

$$|u_n(x) - u_n(y)| \leq \|u_n'\|_{L^2} \, |x-y|^{1/2}$$

(*)

$$\leq \sqrt{M+1}\, |x-y|^{1/2}, \qquad \forall\, x,y \in [0,1],$$

showing that $\{u_n\}$ is EQUI-CONTINUOUS.

Using the boundary condition $u_n(0)=0$ we also get

$$|u_n(x)| = |u_n(x) - u_n(0)| \overset{*}{\leq} \sqrt{M+1} \quad , \quad \forall x \in [0,1],$$

showing that $\{u_n\}$ is UNIFORMLY BOUNDED in $C[0,1]$. Therefore we can apply ASCOLI-ARZELA' THEOREM 7.28 to conclude that $\overline{\{u_n\}}$ is COMPACT in $C[0,1]$. Then $\exists$ a subsequence and $\hat{u} \in C[0,1]$ s.t.

$$u_{n_k} \to \hat{u} \quad \text{uniformly in } [0,1].$$

In particular $u_{n_k} \to \hat{u}$ strongly in $L^2(0,1)$ and so

$$u_{n_k} \rightharpoonup \hat{u} \quad \text{weakly in } L^2(0,1).$$

Recalling that $\dot{u}_{n_k} \rightharpoonup \hat{v}$ weakly in $L^2(0,1)$, by REMARK 7.18 we get that

$$u_{n_k} \rightharpoonup \hat{u} \text{ weakly in } H^1(0,1), \quad \text{with } \hat{u}' = \hat{v} \text{ in the weak sense.}$$

In particular $\hat{u} \in H^1(0,1)$, and $\hat{u}(0)=0$, $\hat{u}(1)=1$ by the uniform convergence. Thus $\hat{u} \in \hat{X}$. As norms are weakly lower semicontinuous, we get that

$$\int_0^1 (\hat{u}')^2 dx \leq \liminf_{k \to +\infty} \int_0^1 u_{n_k}' dx$$

Also, since $u_{n_k} \to \hat{u}$ uniformly,

$$\lim_{k \to +\infty} \int_0^1 \sin(u_n^5) dx = \int_0^1 \sin(\hat{u}^5) dx$$

Therefore

$$\hat{F}(\hat{u}) = \int_0^1 (\hat{u}')^2 + \sin(\hat{u}^5) dx \leq \liminf_{k \to +\infty} \int_0^1 \dot{u}_n^2 + \sin(u_n^5) dx = \liminf_{k \to +\infty} \hat{F}(u_n) \leq M$$

showing that $\hat{F}(\hat{u}) \leq M$. Thus $\hat{u} \in K$, proving that K is weakly compact.

③ **LOWER SEMICONTINUITY:**

(Here we could conclude with the same arguments of point ②. But it is instructive to make a separate argument.)

We need to prove that $\hat{F}$ is lower semicontinuous WRT the weak convergence of H^1, that is,

$$(S) \quad u_n \to u \text{ weakly in } H^1(0,1) \quad \Rightarrow \quad \hat{F}(u) \le \liminf_{n \to +\infty} \hat{F}(u_n)$$

From the SOBOLEV EMBEDDING THEOREM 7.27 we have $H^1(0,1) \hookrightarrow C[0,1]$ compactly. Now recall that COMPACT OPERATORS transform weakly convergent sequences into strongly convergent sequences (PROPOSITION 7.31). Therefore

$$u_n \to u \text{ weakly in } H^1(0,1) \quad \Rightarrow \quad u_n \to u \text{ uniformly in } [0,1]$$

From the weak lower semicontinuity of the norm, we obtain

$$\int_0^1 \dot{u}^2 \, dx \le \liminf_{n \to +\infty} \int_0^1 \dot{u}_n^2 \, dx \qquad (\text{since } \dot{u}_n \to \dot{u} \text{ weakly in } L^2(0,1))$$

Moreover, as $u_n \to u$ uniformly, we also have

$$\int_0^1 \sin(u^5) \, dx = \lim_{n \to +\infty} \int_0^1 \sin(u_n^5) \, dx.$$

Thus $\hat{F}(u) \le \liminf_{n \to +\infty} \hat{F}(u_n)$, and (S) is proven.

Therefore, by THEOREM 9.7 we conclude the existence of $\bar{u} \in \hat{X}$ s.t.

$$\boxed{\hat{F}(\bar{u}) = \inf \{ \hat{F}(u) \mid u \in \hat{X} \}}$$

④ **REGULARITY**: We wish to show that $\bar{u} \in \hat{X}$ actually belongs to X, so that it automatically solves the original problem

$$F(\bar{u}) = \inf\{ F(u) \mid u \in X \}$$

CLAIM All minimizers of $\hat{F}$ in $\hat{X}$ belong to $C^\infty(0,1)$.

HOW TO PROVE IT
- 4.1 : WRITE THE WEAK ELE FOR $\hat{F}$
- 4.2 : SHOW THAT u_0 IS CONTINUOUS (where u_0 is minimizer)
- 4.3 : BOOTSTRAP ARGUMENT

Proof of Claim Let $u_0 \in \hat{X}$ be a minimizer for $\hat{F}$. We want to apply THEOREM 8.4 (with $p=2$) to derive the ELE.

Since u_0 is not regular for now, we can only hope that either the WEAK ELE, or worse the VERY WEAK ELE, hold. So let us check ASSUMPTION 8.3.
In our case the Lagrangian is

$$L(x,s,\xi) = \xi^2 + \sin(s^5)$$

- L is C^∞, therefore L, L_s, L_ξ are Carathéodory functions

- We check (H1): we need to show that $\forall R>0$, $\exists\, \alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $\beta = \beta(R)$ s.t.

(1) $|L(x,s,\xi)| \leq \alpha_1(x) + \beta|\xi|^p$

(2) $|L_s(x,s,\xi)| \leq \alpha_1(x) + \beta|\xi|^p$, $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

(3) $|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta|\xi|^{p-1}$

Notice that in our case $a=0$, $b=1$ and $p=2$, so that $p'=2$.
Let us check (1):

$$|L(x,s,\xi)| = |\xi^2 + \sin(s^5)| \leq 1 + \xi^2, \quad \forall x \in (0,1),\ s,\xi \in \mathbb{R}$$

Therefore it looks like we can choose $\alpha_1 \equiv 1$ independent on x (since the RHS does not depend on x) and, $\beta \equiv 1$ independent on R (since the estimate holds for all $s \in \mathbb{R}$).

Let us see if this choice of α_1 and β works for (2):

$$|L_s(x,s,\xi)| = |5 s^4 \cos(s^5)| \leq 5\,|s|^4$$

This estimate can resemble (2) only if we assume $|s| \leq R$, in which case we get

$$|L_s(x,s,\xi)| \leq 5R^4, \quad \forall x \in (0,1),\ |s| \leq R,\ \xi \in \mathbb{R}.$$

This is saying that we should have $\alpha_1 \equiv 5R^4$ and $\beta \equiv 0$.

Let us look into (3):

$$|L_\xi(x,s,\xi)| = 2|\xi|, \quad \forall x \in (0,1),\ s \in \mathbb{R},\ \xi \in \mathbb{R}$$

Therefore (3) is satisfied for $\alpha_2 \equiv 0$ and $\beta \equiv 2$.

Then, it is immediate to check that L satisfies (1), (2), (3) for

$$\alpha_1(x) \equiv \max\{1, 5R^4\}, \quad \alpha_2(x) \equiv 0, \quad \beta = 2$$

Since $\alpha_1, \alpha_2 \in L^1(0,1)$, we get that (H1) holds.

Therefore L satisfies ASSUMPTION 8.3, and since $u_0 \in X$ is a minimizer of $\hat{F}$ over $\hat{X}$, by THEOREM 8.4 we get that u_0 satisfies the WEAK ELE

$$\int_0^1 L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \, dx = 0, \quad \forall v \in W_0^{1,2}(0,1),$$

which in our case reads

$$\int_0^1 5u_0^4 \cos(u_0^5)\, v + 2\dot{u}_0 \dot{v} \, dx = 0, \qquad \forall v \in W_0^{1,2}(0,1).$$

Rearranging we get

$$\text{(W-ELE)} \qquad \int_0^1 \underbrace{2\dot{u}_0}_{f}\, \underbrace{\dot{v}}_{\dot{\varphi}} \, dx = -\int_0^1 \underbrace{5u_0^4 \cos(u_0^5)}_{g}\, \underbrace{v}_{\varphi} \, dx, \qquad \forall v \in W_0^{1,2}(0,1).$$

Recalling that $C_c^1(0,1) \subseteq W_0^{1,2}(0,1)$, (W-ELE) is saying that $f := 2\dot{u}_0$ is weakly differentiable, with weak derivative given by $g := 5u_0^4 \cos(u_0^5)$, that is

$$(*) \qquad (2\dot{u}_0)' = 5u_0^4 \cos(u_0^5) \qquad \text{weakly}$$

We use $(*)$ to prove regularity of u_0. Note that

$$\int_0^1 |g|^2 dx \underset{|\cos x| \leq 1}{\leq} 25 \int_0^1 |u_0|^8 \, dx$$

and the RHS is finite, since $u_0 \in W^{1,2}(0,1)$ and $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1)$ continuously by THEOREM 7.27.

This shows $g \in L^2(0,1)$. But then

$$\dot{f} = g \text{ weakly}, \; g \in L^2 \;\Rightarrow\; f \in W^{1,2}(0,1) \;\Rightarrow\; \dot{u}_0 \in W^{1,2}(0,1)$$

$f = 2\dot{u}_0$

But

THM 7.19

$$\dot{u}_0 \in W^{1,2}(0,1) \;\Rightarrow\; \dot{u}_0 \in C[0,1]$$

(Note: here $\dot{u}_0$ is still a weak derivative)

Now, by PROPOSITION 7.22 we have that, as $u_0 \in W^{1,2}$ and the weak derivative $\dot{u}_0$ is continuous, then

$$\boxed{u_0 \in C^1[0,1]}$$

Then

$$u_0 \text{ is } C^1 \;\Rightarrow\; g = 5u_0^4 \cos(u_0^5) \text{ is } C^1 \;\Rightarrow\; g \in C^0$$

(As $(2\dot{u}_0)' = g$ weakly) $\Rightarrow$ $2\dot{u}_0$ has continuous weak derivative

$$(\text{PROP 7.22}) \;\Rightarrow\; \dot{u}_0 \in C^1 \;\Rightarrow\; u_0 \in C^2$$

this is true because $\dot{u}_0$ is a classical derivative

Now that we proved $u_0 \in C^2$, we can employ the BOOTSTRAP argument.

BOOTSTRAP: Since now we know $u_0 \in C^2$, the relationship

$$(2\dot{u}_0)' = 5u_0^4 \cos(u_0^5) \qquad \text{weakly}$$

holds in the classical sense (the weak derivative of a diff. function is just the classical derivative), i.e.,

$$(**) \qquad 2\ddot{u}_0 = 5u_0^4 \cos(u_0^5) \quad , \quad \forall x \in [0,1].$$

Then, as $u_0 \in C^2$, the RHS of (**) belongs to C^2, and so

$$\ddot{u}_0 \in C^2 \Rightarrow u_0 \in C^4$$

Again, as $u_0 \in C^4$, the RHS of (**) belongs to C^4, and so

$$\ddot{u}_0 \in C^4 \Longrightarrow u_0 \in C^6$$

Proceeding with the botstrap argument we conclude that

$$\ddot{u}_0 \in C^k \Longrightarrow u_0 \in C^{k+2}$$

and therefore $u_0 \in C^\infty(0,1)$. $\square$

To summarize, in EXAMPLE 9.8 we proved the following:

PROPOSITION Let $X := \{ u \in C^1[0,1] \mid u(0)=0,\ u(1)=1 \}$ and

$$F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx .$$

Then F admits a minimizer $\bar{u} \in X \cap C^\infty(0,1)$.

NOTE: The remarkable feature of this ACTION PLAN for the DIRECT METHOD is that we never tried to solve the ELE equation, but just use abstract arguments to prove $\exists$ of a minimizer, and then the structure of ELE to recover Regularity.

LESSON 12 - 2 JUNE 2021

GENERAL EXISTENCE RESULT IN SOBOLEV

Let $p > 1$, $a < b$, and consider the space

$$X := \{ u \in W^{1,p}(a,b) \mid u(a) = \alpha, \ u(b) = \beta \}$$

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F : W^{1,p}(a,b) \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

THEOREM 9.9 Let $p > 1$. Assume L is a Carathéodory function. Suppose that the following conditions hold:

(M1) $\xi \mapsto L(x,s,\xi)$ is CONVEX for a.e. $x \in (a,b)$ and $\forall s \in \mathbb{R}$.

(M2) $\exists \ q \in [1,p)$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ s.t.

$$L(x,s,\xi) \geq \alpha_1 |\xi|^p + \alpha_2 |s|^q + \alpha_3 , \quad \begin{array}{l} \text{for a.e. } x \in (a,b) \\ \forall \ (s,\xi) \in \mathbb{R} \times \mathbb{R} \end{array}$$

Set

$$m := \inf \{ F(u) \mid u \in X \} .$$

1. If $m < +\infty$, then $\exists \ u_0 \in X$ which minimizes F over X.

2. If in addition $(s,\xi) \mapsto L(x,s,\xi)$ is STRICTLY CONVEX for a.e. $x \in (a,b)$, then the minimizer is UNIQUE.

REMARK Assumptions (M1)-(M2) in THEOREM 9.9 cannot be weakened. I will leave some exercises for the Exercise Course to show this claim.

Proof of THEOREM 9.9

Step 1. F is well-defined: Let $u \in W^{1,p}(a,b)$. The map

$$x \mapsto L(x, u(x), \dot{u}(x))$$

is measurable by PROPOSITION 8.2, since L is Carathéodory and $u, \dot{u}$ are measurable. Therefore $x \mapsto L(x, u(x), \dot{u}(x))$ can be integrated and $F(u)$ is well-defined, possibly being infinite.

Step 2. F is weakly LSC:

The proof of weak LSC is very difficult under the assumptions given: see THEOREMS 3.30, 4.1 in B. DACOROGNA - "DIRECT METHODS IN THE CALCULUS OF VARIATIONS", SPRINGER, 2008.

Instead, we prove LSC under much stronger assumptions, just to give an idea of what lies behind it.

Just for this step, assume then that

- $L \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$
- $(s, \xi) \mapsto L(x, s, \xi)$ is convex for every $x \in [a,b]$. (Note that this implies (M1))
- $\exists \, \beta \geq 0$ s.t.

$$|L_s(x,s,\xi)|, \ |L_\xi(x,s,\xi)| \leq \beta \left(1 + |s|^{p-1} + |\xi|^{p-1}\right), \quad \forall x \in [a,b], \ s, \xi \in \mathbb{R}.$$

We now show that F is weakly LSC, that is,

$$u_n \rightharpoonup u_0 \text{ weakly in } W^{1,p}(a,b) \implies F(u_0) \leq \liminf_{n \to +\infty} F(u_n)$$

Indeed, since L is C^1 and convex WRT (s, ξ), by THEOREM 5.2 we get

$$(*) \qquad L(x, u_n(x), \dot{u}_n(x)) \geq L(x, u_0(x), \dot{u}_0(x)) + L_s(x, u_0, \dot{u}_0)(u_n - u_0) + L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)$$

Notice that

$$(**) \qquad L_s(x, u_0, \dot{u}_0), \ L_\xi(x, u_0, \dot{u}_0) \in L^{p'}(a,b)$$

since

$$\int_a^b |L_s(x, u_0, \dot{u}_0)|^{p'} dx \overset{\text{ASSUMPTION}}{\leq} \beta^{p'} \int_a^b (1 + |u_0|^{p-1} + |\dot{u}_0|^{p-1})^{p'} dx$$

$$\left(\begin{array}{l} p' = \frac{p}{p-1} \text{ and} \\ (a+b)^{p'} \leq 2^{p'-1}(a^{p'} + b^{p'}) \end{array} \right) \leq \beta^{p'} C \int_a^b |u_0|^p + |\dot{u}_0|^p \, dx$$

$$= \beta^{p'} C \, \|u_0\|^p_{W^{1,p}} < +\infty$$

The same calculation shows that also $L_\xi(x, u_0, \dot{u}_0) \in L^{p'}(a,b)$.
Then, since $u_n, u_0 \in W^{1,p}(a,b)$, from (**) and Hölder's inequality we get

$$L_s(x, u_0, \dot{u}_0)(u_n - u_0), \ L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0) \in L^1(a,b).$$

Therefore we can integrate $\circledast$ to get

$$F(u_n) \geq F(u_0) + \int_a^b L_s(x, u_0, \dot{u}_0)(u_n - u_0)\, dx$$

$\circledast\!\circledast\!\circledast$

$$+ \int_a^b L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)\, dx$$

Now, $u_n \rightharpoonup u_0$ weakly in $W^{1,p}(a,b)$. In particular $u_n \rightharpoonup u_0$, $\dot{u}_n \rightharpoonup \dot{u}_0$ weakly in $L^p(a,b)$. Since $\circledast\circledast$ holds, by definition of weak convergence we get

$$\int_a^b L_s(x, u_0, \dot{u}_0)(u_n - u_0)\, dx, \quad \int_a^b L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)\, dx \to 0$$

as $n \to +\infty$. Taking the liminf in $\circledast\circledast\circledast$ yields weak LSC for F.

Step 3. F has COMPACT sublevels :

We are going to prove this part with the original assumptions. So fix $M \in \mathbb{R}$ and let

$$K := \{ u \in X \mid F(u) \leq M \}$$

From (M2) we deduce that $\exists\, M$ such that $K \neq \emptyset$.

By (M2) we have

$$F(u) \overset{(M2)}{\geq} \alpha_1 \|\dot{u}\|^p_{L^p} + \alpha_2 \|u\|^q_{L^q} + \alpha_3 (b-a)$$

$\circledast$

$$\geq \alpha_1 \|\dot{u}\|^p_{L^p} - |\alpha_2| \|u\|^q_{L^q} - |\alpha_3|(b-a)$$

By Hölder inequality we get

Hölder with exponents $p/q > 1$, $(p/q)' = \frac{p}{p-q}$

$$\|u\|_{L^q}^q = \int_a^b |u|^q \, dx \leq \left(\int_a^b |u|^p \, dx \right)^{q/p} \left(\int_a^b 1^{\frac{p}{p-q}} \, dx \right)^{\frac{p-q}{p}}$$

$$= \|u\|_{L^p}^q \, (b-a)^{\frac{p-q}{p}}$$

Then from (*)

$$F(u) \overset{(*)}{\geq} \alpha_1 \|\dot{u}\|_{L^p}^p - |\alpha_2| \, \|u\|_{L^q}^q - |\alpha_3| (b-a)$$

(**)

$$\geq \alpha_1 \|\dot{u}\|_{L^p}^p - C_1 \|u\|_{L^p}^q - C_2$$

for some $C_1, C_2 \in \mathbb{R}$. Moreover, if $u \in X$, we have

$$|u(x)| = |u(a) - u(a) + u(x)|$$

(as $u(a) = \alpha$) $\quad \leq \alpha + |u(x) - u(a)|$

(THEOREM 7.23, as $p > 1$) $\quad \leq \alpha + \|\dot{u}\|_{L^p} |x-a|^{1-1/p}$

$$\leq \alpha + \|\dot{u}\|_{L^p} |b-a|^{1-1/p}$$

and so, integrating the above,

$$\|u\|_{L^p}^q \leq c \left\{ 1 + \|\dot{u}\|_{L^p}^q \right\}, \qquad \forall x \in X.$$

Using (**) and the above, we then get some $C_1, C_2 \in \mathbb{R}$ s.t.

$$F(u) \geq \alpha_1 \|\dot{u}\|_{L^p}^p - C_1 \|\dot{u}\|_{L^p}^q - C_2$$

Now let $\{u_n\} \subseteq K$. Then

$$\underbrace{\alpha_1 \|\dot{u}_n\|^p_{L^p} - C_1 \|\dot{u}_n\|^q_{L^p} - C_2}_{\text{Polynomial in } \|\dot{u}_n\|_{L^p}} \overset{\text{Estimate above}}{\leq} F(u_n) \underset{\text{Since } \{u_n\} \subseteq K}{\leq} M$$

As $p > q \geq 1$, we deduce that $\|\dot{u}_n\|_{L^p}$ must be bounded uniformly, i.e.

$$(**) \quad \|\dot{u}_n\|_{L^p} \leq \tilde{C}, \quad \forall n \in \mathbb{N}.$$

Since we already proved that

$$\|u\|^q_{L^p} \leq c \{1 + \|\dot{u}\|^q_{L^p}\}, \quad \forall u \in X,$$

from $(**)$ we get

$$\|u_n\|_{W^{1,p}} \leq \tilde{C}, \quad \forall n \in \mathbb{N}.$$

Recalling that $W^{1,p}$ is a REFLEXIVE BANACH space for $1 < p < +\infty$ (PROPOSITION 7.16) from BANACH - ALAOGLU we conclude the existence of $u_0 \in W^{1,p}(a,b)$ s.t.

$$u_{n_k} \rightharpoonup u_0 \quad \text{weakly in } W^{1,p}(a,b),$$

along some subsequence. By weak LSC of F we also get

$$F(u_0) \leq \liminf_{k \to +\infty} F(u_{n_k}) \underset{\text{As } \{u_n\} \subseteq K}{\leq} M$$

Finally, from the COMPACT embedding $W^{1,p}(a,b) \hookrightarrow C[a,b]$ for $p > 1$ (THEOREM 7.27) we get, by PROPOSITION 7.31,

$$u_{n_k} \to u_0 \quad \text{uniformly in } [a,b].$$

Since $\{u_n\} \subseteq X$, and so $u_n(a) = \alpha$, $u_n(b) = \beta$ $\forall n \in \mathbb{N}$, we conclude

$$u_0(a) = \alpha, \quad u_0(b) = \beta$$

showing that $u_0 \in X$. In total $u_0 \in K$, proving that K is compact.

Step 4. Existence of a minimizer :

So far we have shown that :

- F is weakly LSC in $W^{1,p}(a,b)$
- $\exists\, M \in \mathbb{R}$ s.t.

$$K := \{ u \in X \mid F(u) \leq M \}$$

is non-empty and weakly compact in X.

Thus by the DIRECT METHOD (THEOREM 9.7) we conclude the existence of $\hat{u} \in X$ s.t.

$$F(\hat{u}) = \inf \{ F(u) \mid u \in X \}.$$

Step 5. Uniqueness : Usual stuff : follows as in the proof of THEOREM 5.4, with straightforward adaptations. $\square$

10. RELAXATION

LESSON 13 - 9 JUNE 2021

LSC ENVELOPE

NOTATION In the following we denote the extended real numbers by

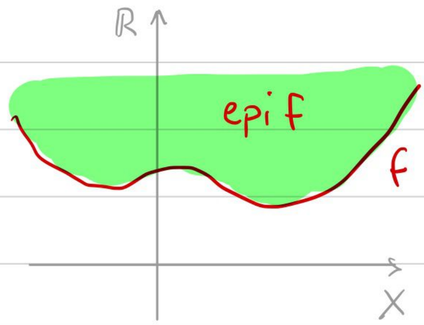
$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

DEFINITION 10.1 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. We say that f is LOWER SEMICONTINUOUS (LSC) at $x_0 \in X$ if

$$x_n \to x_0 \text{ in } (X, d) \Rightarrow f(x_0) \leq \liminf_{n \to +\infty} f(x_n)$$

PROPOSITION 10.2 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. They are equivalent:

1. f is LSC



2. For all $x \in X$ it holds

$$f(x) \leq \liminf_{y \to x} f(y)$$

3. The epigraph of f

$$\text{epi } f := \{ (x, y) \in X \times \mathbb{R} \mid y \geq f(x) \}$$

is closed in $X \times \mathbb{R}$.

4. For all $M \in \mathbb{R}$ the sublevel

$$\{ x \in X \mid f(x) \leq M \}$$

is closed in X.

(Proof is easy, but omitted)

PROPOSITION 10.3 (Sup of LSC is LSC)

(X, d) metric space, I arbitrary set of indices, $f_i : X \to \overline{\mathbb{R}}$ LSC for all $i \in I$. Then $f : X \to \overline{\mathbb{R}}$ defined by

$$f(x) := \sup \{ f_i(x) \mid i \in I \}$$

is LSC.

<u>Proof</u> Let $x_n \to x_0$ in X. Then

$$\liminf_{n \to +\infty} f(x_n) \geq \liminf_{n \to +\infty} f_i(x_n) \geq f_i(x_0)$$

As f is defined as the supremum

As f_i is LSC

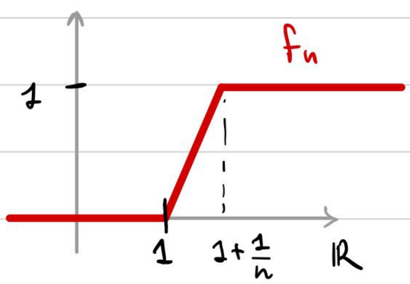
Taking the supremum for $i \in I$ allows to conclude. □

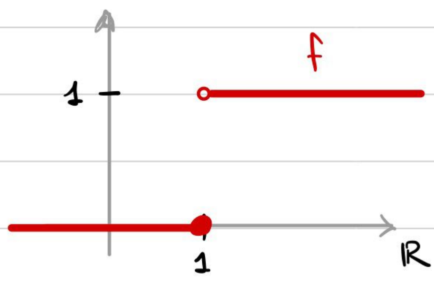
REMARK Let $f_i : X \to \mathbb{R}$ be a family of CONTINUOUS functions for $i \in I$. Then

$$(*) \quad f(x) := \sup \{ f_i(x) \mid x \in X \}$$

is in general only LSC.

For example consider f_n as in the picture. Clearly f defined by $(*)$ is <u>not</u> continuous.





$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

DEFINITION 10.4 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ a function.
The **LSC ENVELOPE** of f is the function $\hat{f}: X \to \overline{\mathbb{R}}$ defined by:

$$\hat{f}(x) := \sup \{ g(x) \mid g: X \to \overline{\mathbb{R}} \text{ is LSC}, \ g \le f \text{ on } X \}$$

REMARK
1. The LSC ENVELOPE is well-defined, since we can always consider $g \equiv -\infty$. Thus the class in which we take the sup is non-empty.
2. The LSC envelope $\hat{f}$ is LSC (by PROPOSITION 10.3)

NOTE The LSC envelope is not straightforward to compute. For this reason we introduce a more practical notion of envelope (called RELAXATION). Eventually we will prove that the two notions coincide.

RELAXATION

DEFINITION 10.5 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ a function.
The **RELAXATION** of f is the function $\overline{f}: X \to \overline{\mathbb{R}}$ defined by

$$(*) \qquad \overline{f}(x) := \inf \{ \liminf_{n \to +\infty} f(x_n) \mid \{x_n\} \subseteq X, \ x_n \to x \}$$

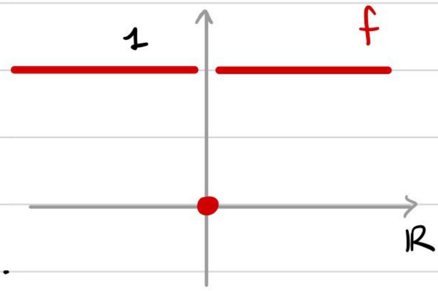
WARNING The relaxation in $(*)$ is **NOT** equivalent to

$$\overline{f}(x) \neq \liminf_{y \to x} f(y)$$

This is because the above limit does not allow to take $y \equiv x$, whereas in $(*)$ we can take $x_n \equiv x$.

For example, consider $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



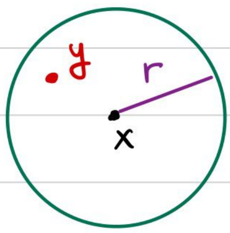
Then the relaxation is $\overline{f}(x) \equiv f(x)$.
However

$$\lim_{y \to 0} f(y) = 1$$

GOAL We aim to prove that LSC ENVELOPE and RELAXATION coincide.

LEMMA 10.6 (X, d) metric space, $f : X \to \mathbb{R}$ a function. Then

$$\boxed{\forall x \in X,\ \forall r > 0,\ \forall \varepsilon > 0,\ \exists y \in X \text{ s.t. } d(x,y) \le r \text{ and } f(y) \le \overline{f}(x) + \varepsilon}$$



Proof Fix $x \in X$, $r > 0$ and $\varepsilon > 0$. By definition of Relaxation and of infimum, $\exists \{x_n\} \subseteq X$ s.t.

$$(*) \qquad x_n \to x \quad \text{and} \quad \liminf_{n \to +\infty} f(x_n) \le \overline{f}(x) + \frac{\varepsilon}{2}$$

By the properties of liminf $\exists$ a subsequence $\{x_{n_k}\}$ s.t.

$$\liminf_{n \to +\infty} f(x_n) = \lim_{k \to +\infty} f(x_{n_k}).$$

From (*) we get

$$x_{n_k} \to x \qquad \text{and} \qquad \lim_{k \to +\infty} f(x_{n_k}) \leq \overline{f}(x) + \varepsilon/2 .$$

Therefore, $\exists\, N \in \mathbb{N}$ sufficiently large such that

$$d(x_N, x) < r, \qquad f(x_N) \leq \overline{f}(x) + \varepsilon .$$

Setting $y := x_N$ yields the thesis. □

DEFINITION 10.7 (RECOVERY SEQUENCE)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$. We say that $\{x_n\} \subseteq X$ is a **RECOVERY SEQUENCE** for f at $x \in X$ if

$$x_n \to x \qquad \text{and} \qquad \overline{f}(x) = \lim_{n \to +\infty} f(x_n)$$

LEMMA 10.8 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. For all $x \in X$ there exists a Recovery Sequence $\{x_n\} \subseteq X$.

Proof Use LEMMA 10.6 with $\varepsilon = 1/n$, $r = 1/n$ to find $y_n \in X$ s.t.

$$d(x, y_n) < \frac{1}{n}, \qquad f(y_n) \leq \overline{f}(x) + \frac{1}{n}, \qquad \forall n \in \mathbb{N}.$$

Therefore $y_n \to x$ and

$$(*) \qquad \limsup_{n \to +\infty} f(y_n) \leq \limsup_{n \to +\infty} \overline{f}(x) + \frac{1}{n} = \overline{f}(x).$$

On the other hand

$$\bar{f}(x) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n\to+\infty} f(x_n) \;\middle|\; \{x_n\} \subseteq X,\; x_n \to x \right\}.$$

$$\text{(Since } y_n \to x\text{)} \to \; \leq \liminf_{n\to+\infty} f(y_n) \leq \limsup_{n\to+\infty} f(y_n) \overset{(*)}{\leq} \bar{f}(x)$$

showing that $\bar{f}(x) = \lim_{n\to+\infty} f(y_n)$. Thus $\{y_n\}$ is Recovery Sequence for f at x. $\square$

PROPOSITION 10.9 (Equivalence of LSC ENVELOPE and RELAXATION)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$ function. We have

1. $\hat{f}$ is LSC and $\hat{f}(x) \leq f(x) \quad \forall x \in X$,
2. $\bar{f}$ is LSC and $\bar{f}(x) \leq f(x) \quad \forall x \in X$,
3. $\bar{f}(x) = \hat{f}(x), \quad \forall x \in X$.

Proof 1 $\hat{f}$ is the supremum of LSC functions, hence it is LSC by PROP 10.3. The inequality is obvious by definition of $\hat{f}$.

2 We first show the inequality: Consider the sequence $\bar{x}_n \equiv x$. Then

$$\bar{f}(x) \overset{\text{def}}{:=} \inf \left\{ \liminf_{n\to+\infty} f(x_n) \;\middle|\; \{x_n\} \subseteq X,\; x_n \to x \right\}$$

$$\leq \liminf_{n\to+\infty} f(\bar{x}_n) \underset{\bar{x}_n \equiv x}{=} f(x).$$

We show that $\bar{f}$ is LSC. So let $x_n \to x_0$ be arbitrary. We want to prove

$$f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

$\forall n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = 1/n$, $\varepsilon = 1/n$ to find $y_n \in X$ s.t.

$$(*) \qquad d(x_n, y_n) < \frac{1}{n}, \qquad f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}, \qquad \forall n \in \mathbb{N}$$

Since $x_n \to x_0$, the first condition implies $y_n \to x_0$. Therefore

$$\bar{f}(x_0) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n\to+\infty} f(z_n) \mid \{z_n\} \subseteq X,\ z_n \to x_0 \right\}$$

$$(\text{As } y_n \to x_0) \to \quad \leq \liminf_{n\to+\infty} f(y_n) \overset{(*)}{\leq} \liminf_{n\to+\infty} \left[\bar{f}(x_n) + \frac{1}{n} \right]$$

$$(\text{Property of } \liminf) \quad \leq \liminf_{n\to+\infty} \bar{f}(x_n) + \liminf_{n\to+\infty} \frac{1}{n} = \liminf_{n\to+\infty} \bar{f}(x_n),$$

showing that $\bar{f}$ is LSC.

3 • $\bar{f}(x) \geq \hat{f}(x)$: Let $x_n \to x$ be arbitrary. Then by 1

$$\liminf_{n\to+\infty} f(x_n) \underset{f \geq \hat{f}}{\geq} \liminf_{n\to+\infty} \hat{f}(x_n) \underset{\hat{f} \text{ is LSC}}{\geq} \hat{f}(x)$$

Taking the infimum for all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x$, we obtain the thesis.

- $\hat{f}(x) \geq \bar{f}(x)$: $\bar{f}$ is LSC and $\bar{f} \leq f$ by ②, thus

def of $\hat{f}$ As $\bar{f}$ is competitor

$$\hat{f}(x) = \sup \{ g(x) \mid g : X \to \bar{\mathbb{R}},\ g \text{ LSC},\ g \leq f \text{ on } X \} \geq \bar{f}(x)$$

□

NOTE In the following $\bar{f}$ and $\hat{f}$ will be used interchangeably, depending on which is the most convenient.

RELATIONSHIP BETWEEN inf / min f AND inf / min $\bar{f}$

The next proposition shows why RELAXATION and LSC ENVELOPE are useful.

PROPOSITION 10.10 (X, d) metric space, $f : X \to \bar{\mathbb{R}}$ function. Then

$$\boxed{\inf_{x \in X} f(x) = \inf_{x \in X} \bar{f}(x) = \inf_{x \in X} \hat{f}(x)}$$

Proof Since $\bar{f} = \hat{f}$ by PROPOSITION 10.9, we only need to show the first equality.

$\boxed{\geq}$ This is clear, since $f \geq \bar{f}$ by PROPOSITION 10.9

$\boxed{\leq}$ Let $\{x_n\}$ be an infimizing sequence for $\bar{f}$, i.e.,

$$\overline{f}(x_n) \to \inf_{x \in X} \overline{f}(x) .$$

For all $n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = 1$, $\varepsilon = 1/n$, so that $\exists$ $\{y_n\} \subseteq X$ s.t.

$$\circledast \qquad d(x_n, y_n) < 1 \quad \text{and} \quad f(y_n) \leq \overline{f}(x_n) + \frac{1}{n} , \qquad \forall\, n \in \mathbb{N}.$$

Then

$\{x_n\}$ is infimizing $\qquad$ $\{\overline{f}(x_n)\}$ is convergent

$$\inf_{x \in X} \overline{f}(x) = \lim_{n \to +\infty} \overline{f}(x_n) = \liminf_{n \to +\infty} \overline{f}(x_n)$$

(As $\frac{1}{n} \to 0$) $\to$
$$= \liminf_{n \to +\infty} \left[\overline{f}(x_n) + \frac{1}{n} \right]$$

$$\overset{\circledast}{\geq} \liminf_{n \to +\infty} f(y_n) \geq \inf_{x \in X} f(x)$$

def of inf

$\square$

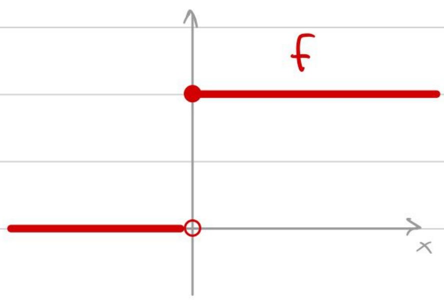
WARNING The statement of PROPOSITION 10.10 only holds on the whole X. In general one has

$$\inf_{x \in A} f(x) > \inf_{x \in A} \overline{f}(x)$$

for $A \subset X$.

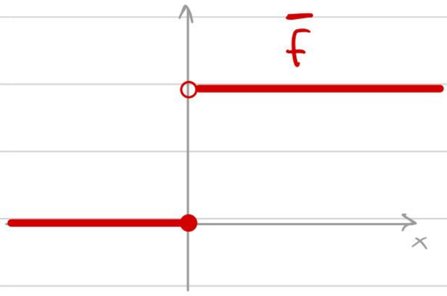
For example consider $X = \mathbb{R}$ and

$$f(x) := \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$



Then

$$\bar{f}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



For $A = [0, +\infty)$ we have

$$\inf_{x \in A} f(x) = 1 \quad , \quad \inf_{x \in A} \bar{f}(x) = 0$$

However the thesis of PROPOSITION 10.10 holds when A is open:

PROPOSITION 10.11 (X, d) metric space, $f : X \to \bar{\mathbb{R}}$ function, $A \subset X$ open. Then

$$\boxed{\inf_{x \in A} f(x) = \inf_{x \in A} \bar{f}(x) = \inf_{x \in A} \hat{f}(x)}$$

Proof Since $\bar{f} = \hat{f}$ by PROPOSITION 10.9, we only need to show the first equality.

$\boxed{\geq}$ This is clear, since $f \geq \bar{f}$ by PROPOSITION 10.9

$\boxed{\leq}$ Let $\{x_n\}$ be an infimizing sequence for $\bar{f}$ over A, i.e., $\{x_n\} \subseteq A$ and

$$\bar{f}(x_n) \to \inf_{x \in A} \bar{f}(x) .$$

Since A is open, $\forall n \in \mathbb{N}$, $\exists\, r_n > 0$ s.t. $B_{r_n}(x_n) \subset A$.

For all $n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = r_n$, $\varepsilon = 1/n$, so that $\exists$ $\{y_n\} \subseteq X$ s.t.

$$(*) \qquad d(x_n, y_n) < r_n \quad \text{and} \quad f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}.$$

The first condition tells us that $y_n \in B_{r_n}(x_n)$, so that $\{y_n\} \subset A$. Then

$$\inf_{x \in A} \bar{f}(x) \overset{\{x_n\} \text{ is infimizing}}{=} \lim_{n \to +\infty} \bar{f}(x_n) \overset{\{\bar{f}(x_n)\} \text{ is convergent}}{=} \liminf_{n \to +\infty} \bar{f}(x_n)$$

(As $\frac{1}{n} \to 0$) $\rightarrow$
$$= \liminf_{n \to +\infty} \left[\bar{f}(x_n) + \frac{1}{n} \right]$$

$$\overset{(*)}{\geq} \liminf_{n \to +\infty} f(y_n) \geq \inf_{x \in A} f(x)$$

def of inf, since $\{y_n\} \subset A$

$\square$

Now recall the definition of COERCIVE function (DEFINITION 9.5)

DEFINITION X space with notion of convergence. A map $f : X \to \overline{\mathbb{R}}$ is COERCIVE if $\exists$ $K \subset X$ compact s.t.

$$\inf_{x \in X} f(x) = \inf_{x \in K} f(x)$$

For coercive functions on metric space, the following holds:

PROPOSITION 10.12 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$ COERCIVE. Then $\overline{f}$ admits minimum over X and

$$\boxed{\inf_{x \in X} f(x) = \min_{x \in X} \overline{f}(x)}$$

WARNING PROP 10.12 is saying that if f is COERCIVE then the minimum of $\overline{f}$ exists and is equal to the infimum of f.
It is NOT saying that f admits minimum. This is false in general.

Proof As f coercive, $\exists\, K \subset X$ compact s.t.

$$\inf_{x \in X} f(x) = \inf_{x \in K} f(x).$$

By PROPOSITION 10.9 we have that $\overline{f}$ is LSC. As K is compact, from THEOREM 9.4 (DIRECT METHOD) we have that $\overline{f}$ admits minimum on K, i.e.,

$$(*) \qquad \inf_{x \in K} \overline{f}(x) = \min_{x \in K} \overline{f}(x)$$

We CLAIM that $\overline{f}$ admits minimum over X, with

$$(**) \qquad \min_{x \in X} \overline{f}(x) = \min_{x \in K} \overline{f}(x)$$

Let $y \in X$ be arbitrary, and let $\{y_n\} \subset X$ be a RECOVERY SEQUENCE for f at y (which $\exists$ by LEMMA 10.8), i.e.,

$$\overline{f}(y) = \lim_{n \to +\infty} f(y_n)$$

Then

$$\bar{f}(y) \overset{\text{Recovery}}{=} \lim_{n\to+\infty} f(y_n) \overset{\text{Def of inf}}{\geq} \inf_{x\in X} f(x) \overset{\text{Coercivity of } f}{=} \inf_{x\in K} f(x)$$

$$\overset{\left(\substack{f \geq \bar{f} \text{ by} \\ \text{PROP } 10.9}\right)}{\geq} \inf_{x\in K} \bar{f}(x) \overset{(*)}{=} \min_{x\in K} \bar{f}(x)$$

Since y was arbitrary, we get

$$\inf_{x\in X} \bar{f}(x) \geq \min_{x\in K} \bar{f}(x)$$

The reverse inequality is obvious, as $K\subset X$. We conclude (**). Therefore

$$\inf_{x\in X} f(x) \underset{\text{PROP } 10.10}{=} \inf_{x\in X} \bar{f}(x) \underset{(**)}{=} \min_{x\in X} \bar{f}(x)$$

□

PROPOSITION 10.13 (Behavior of infimizing sequences)

(X,d) metric space, $f: X \to \bar{\mathbb{R}}$. Suppose that $\{x_n\} \subseteq X$ is s.t.

$$x_n \to x_0 \quad \text{and} \quad f(x_n) \to \inf_{x\in X} f(x)$$

(i.e. $\{x_n\}$ infimizing for f)

Then x_0 is a minimizer for $\bar{f}$, i.e.,

$$\bar{f}(x_0) = \inf_{x\in X} \bar{f}(x)$$

Proof $\displaystyle \inf_{x\in X} \bar{f}(x) \le \bar{f}(x_0) \le \liminf_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} f(x_n)$

def of inf   def of LSC envelope, as $x_n \to x_0$ as $f(x_n)$ convergent

$$= \inf_{x\in X} f(x) = \inf_{x\in X} \bar{f}(x) \Rightarrow x_0 \in \operatorname*{argmin}_{x\in X} \bar{f}(x)$$

assumption   PROP 10.10

$\square$

COROLLARY 10.14 (X,d) metric space, $f: X \to \bar{\mathbb{R}}$. Assume that $\exists\, M>0$ and $K \subseteq X$ compact s.t.

$$\tilde{K} := \{x\in X \mid f(x) < M\} \neq \emptyset \quad \text{and} \quad \tilde{K} \subseteq K.$$

If $\{x_n\} \subseteq X$ is infimizing for f, i.e.,

$$f(x_n) \to \inf_{x\in X} f(x)$$

then $\exists$ subsequence and $x_0 \in X$ s.t.

$$x_{n_k} \to x_0 \quad \text{and} \quad x_0 \in \operatorname*{argmin}_{x\in X} \bar{f}(x).$$

Proof Since $\tilde{K} \neq \emptyset$, it means that $I < M$, where $I := \inf\{f(x) \mid x\in X\}$. As $f(x_n) \to I$, we then conclude that $\exists\, N\in\mathbb{N}$ s.t.

$$x_n \in \tilde{K}, \quad \forall\, n \geq N.$$

As $\tilde{K} \subseteq K$ and K is compact, then $\exists\, x_0 \in K$ and a subsequence s.t. $x_{n_k} \to x_0$. We then conclude from PROP 10.13, since $\{x_{n_k}\}$ is an infimizing sequence for f. $\square$

COMPUTING THE RELAXATION

We will see 2 strategies to compute the relaxation.

PROPOSITION 10.15 (STRATEGY 1)

(X, d) metric space, $F : X \to \overline{\mathbb{R}}$. Suppose that $g : X \to \overline{\mathbb{R}}$ is s.t.

1. (liminf inequality) For all $x \in X$ and $x_n \to x$ it holds

$$g(x) \leq \liminf_{n \to +\infty} F(x_n)$$

2. (limsup inequality) For all $x \in X$, $\exists\, x_n \to x$ s.t.

$$\limsup_{n \to +\infty} F(x_n) \leq g(x)$$

Then $g = \overline{F}$.

NOTE If 1 and 2 hold, then the limsup in 2 is actually a limit.

Proof $\boxed{g \leq \overline{F}}$ Let $x_n \to x$ be arbitrary. By 1 we have

$$g(x) \leq \liminf_{n \to +\infty} F(x_n)$$

Since $\{x_n\}$ is arbitrary, taking the infimum over all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x$, we get $g \leq \overline{F}$.

$\boxed{\bar{f} \leq g}$ Conversely, let $\{x_n\}$ be the sequence existing by ②. Then

$$\bar{f}(x) \underset{\substack{\uparrow \\ \text{def of } \bar{f} \\ \text{since } x_n \to x}}{\leq} \liminf_{n\to+\infty} f(x_n) \underset{\substack{\uparrow \\ \text{properties of} \\ \liminf / \limsup}}{\leq} \limsup_{n\to+\infty} f(x_n) \underset{\substack{\uparrow \\ \text{by } ②}}{\leq} g(x)$$

showing that $\bar{f} \leq g$ and concluding. □

We now look at a second strategy to compute the relaxation.

DEFINITION 10.16 (ENERGY DENSE SUBSETS)

(X,d) metric space, $f: X \to \bar{\mathbb{R}}$. A subset $D \subseteq X$ is ENERGY DENSE WRT f if

$$\forall x \in X, \ \exists \{x_n\} \subseteq D \ \text{ s.t. } \ x_n \to x \ \text{ and } \ f(x_n) \to f(x).$$

REMARK ① Suppose $f: X \to \mathbb{R}$ is continuous. Then $D \subseteq X$ is Energy Dense WRT to f iff it is Dense.

② $D \subseteq X$ is Energy Dense WRT f iff

$$\{(x, f(x)), \ x \in D\} \subseteq X \times \bar{\mathbb{R}}$$

is dense in $X \times \bar{\mathbb{R}}$.

LEMMA 10.17 (X,d) metric space, $\varphi, \psi : X \to \overline{\mathbb{R}}$. Let $D \subseteq X$. Suppose that

(i) $\varphi(x) \leq \psi(x) \quad \forall x \in D$

(ii) D is Energy Dense WRT ψ

(iii) φ is LSC

Then

$$\varphi(x) \leq \psi(x), \quad \forall x \in X.$$

Proof Let $x \in X$. By (ii) there $\exists \{x_n\} \subseteq D$ s.t. $x_n \to x$ and $\psi(x_n) \to \psi(x)$.

Then

$$\underbrace{\varphi(x) \leq \liminf_{n\to+\infty} \varphi(x_n)}_{\varphi \text{ is LSC and } x_n \to x} \underbrace{\leq}_{\text{By (i), since } \{x_n\} \subseteq D} \liminf_{n\to+\infty} \psi(x_n) \underbrace{=}_{\text{As } \psi(x_n) \to \psi(x)} \psi(x)$$

$\square$

PROPOSITION 10.18 (STRATEGY 2)

(X,d) metric space, $f : X \to \overline{\mathbb{R}}$. Suppose that $g : X \to \overline{\mathbb{R}}$ satisfies

1. g is LSC
2. $g(x) \leq f(x), \quad \forall x \in X$
3. $\exists D \subseteq X$ Energy Dense WRT g, s.t.

$$\forall x \in D, \ \exists \{x_n\} \subseteq X \text{ s.t. } x_n \to x \text{ and } \limsup_{n\to+\infty} f(x_n) \leq g(x)$$

Then $\overline{f} = g$.

Proof $\boxed{g \leq \overline{f}}$ Let $x_n \to x$ be arbitrary. Then

$$\liminf_{n \to +\infty} f(x_n) \overset{②}{\geq} \liminf_{n \to +\infty} g(x_n) \overset{①}{\geq} g(x)$$

Taking the infimum for all $x_n \to x$, we conclude $\overline{f} \geq g$.

$\boxed{\overline{f} \leq g}$ Set $\varphi := \overline{f}$, $\psi := g$. Let us verify the assumptions of LEMMA 10.17:

(i) $\varphi(x) \leq \psi(x) \quad \forall x \in D$ (TRUE because of ③ and definition of $\overline{f}$)

(ii) D is Energy Dense WRT ψ (TRUE: it is assumed in ③)

(iii) φ is LSC (TRUE because $\varphi = \overline{f}$ and $\overline{f}$ is LSC by PROP 10.9)

Therefore by LEMMA 10.17 we have that $\varphi \leq \psi$ on X, i.e. $\overline{f} \leq g$ on X. $\square$

EXTENSION BY RELAXATION : CONVEX CASE

LESSON 14
16 JUNE 2021

Setting : $(\hat{X}, d)$ metric space, $X \subseteq \hat{X}$ and $f : X \to \overline{\mathbb{R}}$.

QUESTION Find $\hat{f} : \hat{X} \to \overline{\mathbb{R}}$ which extends f in a meaningful way.

IDEA Extend f on $\hat{X}$ by setting

$$(*) \qquad \hat{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in X \\ +\infty & \text{if} \quad x \in \hat{X} \setminus X \end{cases}$$

Then consider $\hat{f} := \overline{f}$. In the following f is always extended according to $(*)$

EXAMPLE $\hat{X} = L^2(a,b)$, $X = C^1[a,b]$, $F : X \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b \dot{u}^2 \, dx \quad , \quad \forall \, u \in X .$$

Extend F to $+\infty$ on $\hat{X} \setminus X$. Then set $\hat{F} := \overline{F}$. (relax in L^2)

CLAIM We have that

weak derivative

$$\hat{F}(u) = G(u) := \begin{cases} \int_a^b \dot{u}^2 \, dx & \text{if} \quad u \in H^1(a,b) \\ +\infty & \text{if} \quad u \in L^2 \setminus H^1 \end{cases}$$

(proof left as exercise. One can employ STRATEGY 2 in this case)

IN GENERAL We want to compute relaxation for $F: C^1[a,b] \to \overline{\mathbb{R}}$

$$F(u) := \int_a^b \psi(\dot{u})\, dx \,, \qquad \psi: \mathbb{R} \to \mathbb{R} \,, \quad \psi = \psi(\xi)$$

Under some assumptions the relaxation of F in $L^p(a,b)$ is given by

$$\hat{F}: L^p(a,b) \to \overline{\mathbb{R}} \,, \qquad \hat{F}(u) := \begin{cases} \int_a^b \psi(\dot{u})\, dx \,, & \text{if } u \in W^{1,p}(a,b) \\ +\infty & \text{otherwise in } L^p(a,b) \end{cases}$$

THEOREM 10.19 Consider F, $\hat{F}$ as above. Assume $\psi: \mathbb{R} \to \mathbb{R}$ is

1. Convex

2. $\exists\, A > 0$, $B \in \mathbb{R}$, $p \in (1, +\infty)$ s.t.

$$\psi(\xi) \geq A\,|\xi|^p - B \,, \qquad \forall\, \xi \in \mathbb{R}.$$

Then $\overline{F} = \hat{F}$ in $L^p(a,b)$.

Proof We use STRATEGY 2 (PROP 10.18). We need to show that:

1. $\hat{F}$ is LSC in $L^p(a,b)$

2. $\hat{F}(u) \leq F(u)$, $\forall\, u \in L^p(a,b)$ (Here $F(u) := +\infty$ if $u \notin C^1[a,b]$)

3. $\exists\, D \subseteq L^p(a,b)$ Energy Dense WRT $\hat{F}$, s.t.

$$\forall u \in D \,, \ \exists \{u_n\} \subseteq C^1[a,b] \text{ s.t. } u_n \to u \text{ and } \limsup_{n \to +\infty} F(u_n) \leq \hat{F}(u)$$

Strongly in $L^p(a,b)$

Checking 1 : Need to show that if $u_n \to u$ in $L^p(a,b)$ then

$$\hat{F}(u) \le \liminf_{n\to+\infty} \hat{F}(u_n). \quad (*)$$

If RHS is $+\infty$ then $(*)$ is trivial. Then WLOG we can assume that

$$\hat{F}(u_n) \le M, \quad \forall n \in \mathbb{N}.$$

From the growth assumption on ψ we get

$$\int_a^b A|\dot{u}_n|^p - B \, dx \le \hat{F}(u_n) \le M$$

so that

$$\int_a^b |\dot{u}_n|^p \, dx \le \frac{M + (b-a)B}{A}$$

proving that $\{\dot{u}_n\}$ is bounded in $L^p(a,b)$. Thus, up to subsequences

$$\dot{u}_n \rightharpoonup v \quad \text{weakly in } L^p(a,b)$$

As $u_n \to u$ strongly in $L^p(a,b)$, in particular we get

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(a,b)$$

Thus, from REMARK 7.18 (trivially adaptable to $W^{1,p}$ case) we get

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(a,b)$$

Now $(*)$ can be shown as in THEOREM 9.9 (if we assume ψ is C^1 and growth of ψ from above). In general, see THM 3.6 in BUTTAZZO, GIAQUINTA, HILDEBRANDT

Checking 2 : This is obvious by definition of F, $\hat{F}$, and by the fact that weak derivatives coincide with classical ones for maps in $C^1[a,b]$.

Checking 3 : Set

$$D := \{ u : [a,b] \to \mathbb{R} \mid u \text{ continuous and piecewise linear} \}$$

CLAIM D is Energy Dense WRT $\hat{F}$

[Given $\hat{u} \in L^p(a,b)$ we need to find $\{u_n\} \subseteq D$ s.t.

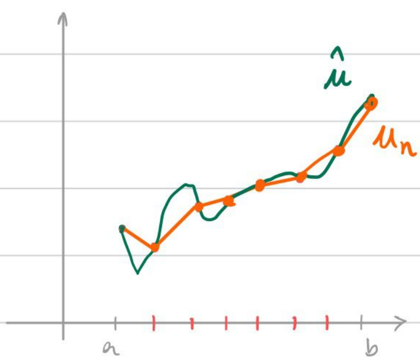
$$(*) \qquad u_n \to \hat{u} \text{ in } L^p \quad \text{and} \quad \hat{F}(u_n) \to \hat{F}(\hat{u})$$

- If $\hat{u} \notin W^{1,p}(a,b)$ then $\hat{F}(\hat{u}) = +\infty$. Now it is easy to approximate u in L^p with a sequence in D and obtain $(*)$ by LSC of $\hat{F}$:

$$+\infty = \hat{F}(\hat{u}) \leq \liminf_{n \to +\infty} \hat{F}(u_n) \quad \Rightarrow \quad \lim_{n \to +\infty} \hat{F}(u_n) = +\infty .$$

- If $\hat{u} \in W^{1,p}(a,b)$ and $\hat{F}(\hat{u}) = +\infty$, then proceed as above.

- If $\hat{u} \in W^{1,p}(a,b)$ and $\hat{F}(\hat{u}) < +\infty$: by THEOREM 7.19 we know that $\hat{u} \in C[a,b]$. Then construct u_n as in picture



Divide $[a,b]$ in sub-intervals I_i of amplitude $1/n$.
Define u_n by linear interpolation of values of $\hat{u}$ on the grid.

As the mesh-size goes to zero as $n \to +\infty$ and $\hat{u}$ is uniformly continuous in $[a,b]$ we get

$$u_n \to \hat{u} \quad \text{uniformly in } [a,b] \qquad \text{(easy check)}$$

Then in particular

$$u_n \to \hat{u} \quad \text{strongly in } L^p(a,b)$$

Moreover, it holds that

$$(**) \qquad \hat{F}(u_n) \le \hat{F}(\hat{u}) \quad , \quad \forall n \in \mathbb{N} .$$

Indeed

$$\hat{F}(u_n) \overset{\text{def of } u_n}{=} \sum_{k=1}^{N} \int_{I_k} \psi(\dot{u}_n)\, dx$$

Now, consider the problem:

$$(P) \qquad \min \left\{ \int_{I_k} \psi(\dot{u})\, dx \;\middle|\; u \in W^{1,p}(I_k) \;,\; u_{|\partial I_k} = \hat{u}_{|\partial I_k} \right\}$$

Since $\psi = \psi(\xi)$, and ψ is convex, one immediately sees that the straight line solves (P) (by Jensen's Inequality THEOREM 6.8). Thus

$$(***) \quad \int_{I_k} \psi(\dot{u}_n)\, dx \le \int_{I_k} \psi(\dot{u})\, dx \quad , \quad \forall u \in W^{1,p}(I_k) \text{ s.t. } u_{|\partial I_k} = \hat{u}_{|\partial I_k}$$

Since $\hat{u}$ satisfies the Dirichlet BC, we get

$$\hat F(u_n) \overset{\text{def of } u_n}{=} \sum_{k=1}^{N} \int_{I_k} \Psi(\dot u_n)\, dx \overset{\circledast\circledast}{\leq} \sum_{k=1}^{N} \int_{I_k} \Psi(\hat u')\, dx$$

$$= \int_a^b \Psi(\hat u')\, dx = \hat F(\hat u)$$

so that (**) holds. Taking the limsup in (**) yields

(LS) $\displaystyle \limsup_{n \to +\infty} \hat F(u_n) \leq \hat F(\hat u).$

By (1) we know that $\hat F$ is LSC in $L^p(a,b)$, so that

(LI) $\displaystyle \hat F(\hat u) \leq \liminf_{n \to +\infty} \hat F(u_n)$

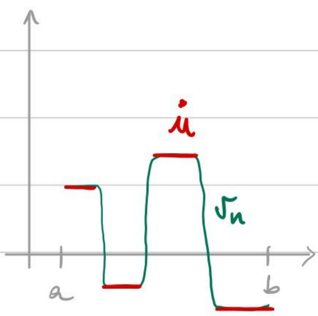
since $u_n \to u$ in $L^p(a,b)$ by construction

From (LS)-(LI) we conclude $\hat F(u_n) \to \hat F(\hat u)$, and (*) follows.]

CLAIM $\forall u \in D$, $\exists \{u_n\} \subseteq C^1[a,b]$ s.t.

$u_n \to u$ in $L^p(a,b)$ and $\displaystyle \limsup_{n \to +\infty} F(u_n) \leq \hat F(u)$

[Usual approximation argument: for $u \in D$, we approximate $\dot u$ with some smooth v_n and then define u_n as the primitive of v_n.]

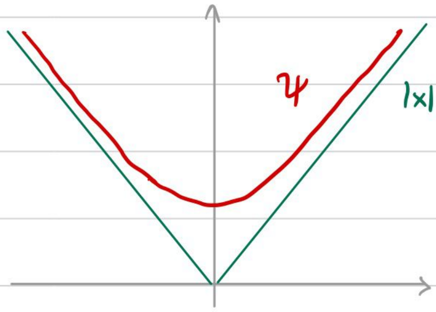


Therefore (1), (2), (3) hold and so PROP 10.18 implies $\bar F = \hat F$ in $L^p(a,b)$.

$\square$

WARNING The thesis of THEOREM 10.19 is FALSE for $p=1$.

[For example consider



$$\psi(\xi) := \sqrt{1+\xi^2}$$

which is CONVEX and such that

$$\psi(\xi) \geq |\xi| \quad , \qquad \forall \xi \in \mathbb{R}.$$

Consider the functional $F : C^1[-1,1] \to \mathbb{R}$

$$F(u) := \int_{-1}^{1} \sqrt{1+\dot{u}^2}\, dx$$

and $\hat{F} : W^{1,1}(-1,1) \to \mathbb{R}$

$$\hat{F}(u) := \begin{cases} \int_{-1}^{1} \sqrt{1+\dot{u}^2}\, dx & \text{if } u \in W^{1,1}(-1,1) \\ +\infty & \text{otherwise} \end{cases}$$

Then

$$\boxed{\overline{F} \neq \hat{F}}$$

In fact, let

$$\hat{u}(x) := \begin{cases} 0 & \text{if } x \in [-1,0) \\ J & \text{if } x \in [0,1] \end{cases} \quad , \qquad J \in \mathbb{R}$$

Then $\hat{F}(\hat{u}) = +\infty$, since $\hat{u} \notin W^{1,1}(-1,1)$. But $\overline{F}(\hat{u}) = |J|$.
In this case $\overline{F}$ is finite on the space $BV(-1,1)$.]

EXAMPLE Consider the functional of EXAMPLE 9.8 :

$$F : C^1[a,b] \to \mathbb{R} \,, \qquad F(u) := \int_a^b \dot{u}^2 + \sin(u^5) \, dx$$

We can write $F = G + H$ with

$$G(u) := \int_a^b \psi(\dot{u}) \, dx \,, \qquad \psi(\xi) := \xi^2 \,, \qquad H(u) := \int_a^b \sin(u^5) \, dx$$

ψ is convex and satisfies the growth assumption of THEOREM 10.19 with $p = 2$, $A = 1$, $B = 0$. Therefore the relaxation of G in $L^2(a,b)$ is

$$\bar{G}(u) = \begin{cases} \int_a^b \dot{u}^2 \, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

Also notice that H is continuous in $L^2(a,b)$. Thus $\bar{H} = H$ (exercise)
Then one can prove (exercise)

$$\bar{F} = \bar{G} + H$$

showing that

$$\bar{F}(u) = \begin{cases} \int_a^b \dot{u}^2 + \sin(u^5) \, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise in } L^2(a,b) \end{cases}$$

is the relaxation of F in $L^2(a,b)$, as anticipated in EXAMPLE 9.8.

EXTENSION BY RELAXATION : NON - CONVEX CASE

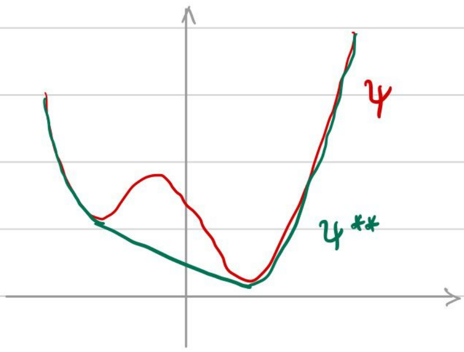
We want to generalize THEOREM 10.19 to NON-CONVEX Lagrangians.

DEFINITION 10.20 (CONVEX ENVELOPE)

Let $\psi : \mathbb{R} \to \mathbb{R}$. The CONVEX ENVELOPE of ψ is the map $\psi^{**} : \mathbb{R} \to \overline{\mathbb{R}}$

$$\boxed{\psi^{**}(x) := \sup \left\{ g(x) \mid g : \mathbb{R} \to \overline{\mathbb{R}} \text{ is convex}, \; g \leq \psi \text{ in } \mathbb{R} \right\}}$$

PROPOSITION 10.21 (PROPERTIES OF ψ^{**})



Let $\psi : \mathbb{R} \to \mathbb{R}$. Then

1. ψ^{**} is convex
2. $\psi^{**} \leq \psi$ on $\mathbb{R}$
3. The supremum in the definition of ψ^{**} is a maximum
4. We have

$$\psi^{**}(x) = \sup \left\{ mx + q \mid my + q \leq \psi(y), \; \forall y \in \mathbb{R} \right\}$$

i.e. ψ^{**} is the supremum of all lines below the graph of ψ.

(Proof is omitted)

THEOREM 10.22 Let $\psi : \mathbb{R} \to \mathbb{R}$ and define $F : C^1[a,b] \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b \psi(\dot{u})\, dx$$

Suppose $\exists\, p \in (1, +\infty)$ and $A > 0$, $B \in \mathbb{R}$ s.t.

$$(*) \qquad \psi(\xi) \geq A |\xi|^p - B, \qquad \forall\, \xi \in \mathbb{R}.$$

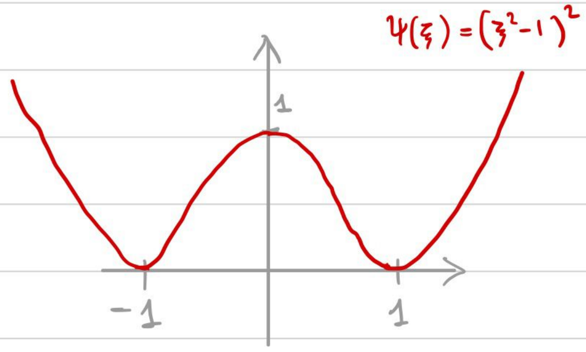
Then $\overline{F} = \hat{F}$ in $L^p(a,b)$, where

$$\hat{F}(u) := \begin{cases} \int_a^b \psi^{**}(\dot{u})\, dx, & \text{if } u \in W^{1,p}(a,b) \\ +\infty, & \text{otherwise in } L^p(a,b) \end{cases}$$

(Proof is omitted. It is similar to the proof of THEOREM 10.19)

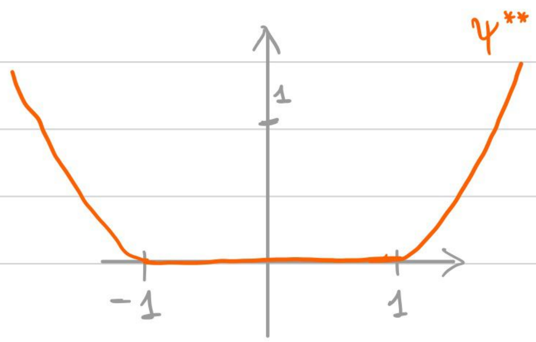
EXAMPLE Consider the functional of EXAMPLE 5.6 : $F : C^1[a,b] \to \mathbb{R}$,

$$F(u) := \int_a^b (\dot{u}^2 - 1)^2\, dx$$



DOUBLE WELL

The Lagrangian is $\psi(\xi) := (\xi^2 - 1)^2$ which satisfies $(*)$ with $p = 4$ (for some $A > 0$, $B \in \mathbb{R}$). Thus THM 10.22 implies that the relaxation of F in $L^4(a,b)$ is given by



$$\overline{F}(u) = \begin{cases} \int_a^b \psi^{**}(\dot{u})\, dx & \text{if } u \in W^{1,4}(a,b) \\ +\infty & \text{otherwise in } L^4(a,b) \end{cases}$$

$$\psi^{**}(\xi) = \begin{cases} \psi(\xi) & \text{if } |\xi| > 1 \\ 0 & \text{if } |\xi| \leq 1 \end{cases}$$

11. GAMMA - CONVERGENCE

DEFINITION 11.1 (Γ- CONVERGENCE)

(X,d) metric space, $f_n, f : X \to \overline{\mathbb{R}}$. We say that $f_n \xrightarrow{\Gamma} f$, Γ-converges, if

① (Γ-liminf inequality) $\forall x \in X$, $\forall$ $x_n \to x$ it holds

$$f(x) \leq \liminf_{n \to +\infty} f_n(x_n)$$

② (Γ-limsup inequality) $\forall x \in X$, $\exists$ $x_n \to x$ such that

$$\limsup_{n \to +\infty} f_n(x_n) \leq f(x).$$

REMARK If ① and ② hold, then the limsup in ② is actually a limit.

NOTATION The sequences $\{x_n\}$ satisfying ② are called RECOVERY SEQUENCES

RELATIONSHIP WITH POINTWISE CONVERGENCE

PROPOSITION 11.2 (X,d) metric space, $f : X \to \overline{\mathbb{R}}$. Define $f_n := f$, $\forall n \in \mathbb{N}$. Then

$$f_n \xrightarrow{\Gamma} \overline{f}$$

Proof ① Γ-liminf inequality: Let $x_n \to x$. Then

def of $\bar{f}$

$$\bar{f}(x) = \inf \{ \liminf_{n \to +\infty} f(z_n) \mid z_n \to x \}$$

$$\leq \liminf_{n\to+\infty} f(x_n) \underset{f_n \equiv f}{=} \liminf_{n\to+\infty} f_n(x_n)$$

② Γ-limsup inequality: Let $x \in X$. By LEMMA 10.8 $\exists\, x_n \to x$ s.t.

$$\bar{f}(x) = \lim_{n\to+\infty} f(x_n)$$

As $f_n \equiv f$, we conclude. □

REMARK PROPOSITION 11.2 implies that Γ-convergence is not related to pointwise convergence. Indeed if $f_n \equiv f$, $f \neq \bar{f}$ then

- $f_n \to f$ pointwise but $f_n \overset{\Gamma}{\not\to} f$ (because Γ-limit is unique)
- $f_n \xrightarrow{\Gamma} \bar{f}$ but $f_n \not\to \bar{f}$ pointwise (because pointwise limit is unique)

However, under additional assumptions, uniform conv. implies Γ-conv.

PROPOSITION 11.3 (X,d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$. Suppose:

(i) $f_n \to f$ uniformly on compact sets of X

(ii) f is LSC.

Then $f_n \xrightarrow{\Gamma} f$.

(Proof will be left as an exercise)

STABILITY PROPERTIES

We now investigate stability properties of Γ- conv. WRT continuous perturbations.

PROPOSITION 11.4 (Stability)

(X, d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$ s.t. $f_n \xrightarrow{\Gamma} f$. Assume $g : X \to \mathbb{R}$ is continuous. Then

$$f_n + g \xrightarrow{\Gamma} f + g$$

(Proof is consequence of PROPOSITION 11.5 below)

A simple generalization of the above is the following.

PROPOSITION 11.5 (Stability)

(X, d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$ s.t. $f_n \xrightarrow{\Gamma} f$. Assume $g_n, g : X \to \mathbb{R}$ are such that:

(i) $g_n \to g$ uniformly on compact sets of X,

(ii) g is continuous.

Then

$$f_n + g_n \xrightarrow{\Gamma} f + g$$

(Proof will be left as an exercise)

Γ-liminf and Γ-limsup

As usual with limits, they don't always exist. For this reason one introduces notions of Γ-liminf and Γ-limsup.

DEFINITION 11.6 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. We define

$$\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) := \inf\left\{ \liminf_{n\to+\infty} f_n(x_n) \;\middle|\; \{x_n\} \subseteq X, \; x_n \to x \right\}$$

$$\Gamma\text{-}\limsup_{n\to+\infty} f_n(x) := \inf\left\{ \limsup_{n\to+\infty} f_n(x_n) \;\middle|\; \{x_n\} \subseteq X, \; x_n \to x \right\}$$

PROPOSITION 11.7 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. Then Γ-liminf f_n and Γ-limsup f_n always exist and satisfy

$$\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) \le \Gamma\text{-}\limsup_{n\to+\infty} f_n(x), \qquad \forall x \in X.$$

Moreover $f_n \xrightarrow{\Gamma} f$ for some $f : X \to \overline{\mathbb{R}}$ if and only if

$$(*) \qquad \boxed{\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) = \Gamma\text{-}\limsup_{n\to+\infty} f_n(x), \qquad \forall x \in X.}$$

Proof The first part of the statement is trivial. Suppose now that $f_n \xrightarrow{\Gamma} f$. Let $x_n \to x$. Then by the Γ-liminf inequality

$$f(x) \le \liminf_{n\to+\infty} f_n(x_n)$$

and so, taking the inf for all $x_n \to x$ yields

$$(1) \qquad f(x) \le \Gamma\text{-}\liminf_{n\to+\infty} f_n(x), \qquad \forall x \in X$$

On the other hand the Γ-limsup inequality says there $\exists\, x_n \to x$ s.t.

$$\limsup_{n \to +\infty} f_n(x_n) \leq f(x)$$

By def. of Γ-limsup we get

$$② \qquad \Gamma\text{-}\limsup_{n \to +\infty} f_n(x) \leq f(x) \qquad , \qquad \forall\, x \in X$$

Therefore from ①-② we infer

$$\Gamma\text{-}\limsup_{n \to +\infty} f_n(x) \leq \Gamma\text{-}\liminf_{n \to +\infty} f_n(x) \qquad , \qquad \forall\, x \in X$$

As the other inequality was already proven in the first part of the statement, we conclude $*$.

Conversely, assume $*$ and set

$$f(x) := \Gamma\text{-}\liminf_{n \to +\infty} f_n(x) \overset{*}{=} \Gamma\text{-}\limsup_{n \to +\infty} f_n(x)$$

We want to prove that $f_n \xrightarrow{\Gamma} f$. So we need to check Γ-liminf and Γ-limsup inequalities:

- Γ-liminf ineq: Let $x_n \to x$. Then

$$\liminf_{n \to +\infty} f_n(x_n) \underset{\text{def of } \Gamma\text{-liminf } f_n}{\geq} \Gamma\text{-}\liminf_{n \to +\infty} f_n(x) \underset{\text{def of } f}{=} f(x)$$

• Γ-limsup ineq: Let $x \in X$. Since (*) holds, one can show that there there $\exists \{x_n\}$ s.t.

$$(3) \qquad x_n \to x \quad \text{and} \quad \liminf_{n\to+\infty} f_n(x_n) = \limsup_{n\to+\infty} f_n(x_n) = f(x)$$

concluding. $\square$

LESSON 15 - 23 JUNE 2021

FUNDAMENTAL THEOREM OF Γ-CONVERGENCE

We now want to show that the Γ-limit captures the asymptotic behavior of minimizers for a sequence $f_n : X \to \overline{\mathbb{R}}$.

LEMMA 11.8 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$, $f_n \xrightarrow{\Gamma} f$. Then f is LSC.

Proof Assume $x_n \to x$. We need to show

$$f(x) \leq \liminf_{n\to+\infty} f(x_n).$$

Since $f_k \xrightarrow{\Gamma} f$ we know that for each x_n there $\exists$ a recovery sequence $\{y_k\}$ s.t.

$$\lim_{k\to+\infty} y_k = x_n \quad , \quad f(x_n) = \lim_{k\to+\infty} f_k(y_k)$$

Therefore by a diagonal argument we can find $\{\tilde{y}_n\}$ s.t.

$$(*) \qquad d(\tilde{y}_n, x_n) < \frac{1}{n} \quad , \quad |f_n(\tilde{y}_n) - f(x_n)| < \frac{1}{n} \quad , \quad \forall n \in \mathbb{N}.$$

Since $x_n \to x$, the first condition implies $\tilde{y}_n \to x$. Therefore,

Γ- liminf inequality as $\tilde{y}_n \to x$ — Second condition in ⊛ — As $1/n \to 0$

$$f(x) \le \liminf_{n\to+\infty} f_n(\tilde{y}_n) \le \liminf_{n\to+\infty} \left[f(x_n) + 1/n \right] = \liminf_{n\to+\infty} f(x_n)$$

concluding that f is LSC. □

PROPOSITION 11.9

(X, d) metric space, $f_n : X \to \overline{\mathbb{R}}$, $f_n \xrightarrow{\Gamma} f$.

① Let $A \subseteq X$ be open. Then

$$\limsup_{n\to+\infty} \left\{ \inf_{x\in A} f_n(x) \right\} \le \inf_{x\in A} f(x)$$

② Let $K \subseteq X$ be compact. Then

$$\liminf_{n\to+\infty} \left\{ \inf_{x\in K} f_n(x) \right\} \ge \inf_{x\in K} f(x)$$

Proof ① Fix $\varepsilon > 0$. By definition of inf there $\exists$ $\hat{x} \in A$ s.t.

$$\circledast \qquad f(\hat{x}) \le \inf_{x\in A} f(x) + \varepsilon$$

Let x_n be a recovery sequence for $\hat{x}$, i.e.

$$x_n \to \hat{x} \qquad \text{and} \qquad f(\hat{x}) \ge \limsup_{n\to+\infty} f_n(x_n)$$

Since A is open, $\hat{x} \in A$, and $x_n \to \hat{x}$, then $x_n \in A$ for $n >> 0$. Then

$$\inf_{x\in A} f(x) + \varepsilon \overset{(*)}{\geq} f(\hat{x}) \overset{x_n \text{ rec. seq. for } \hat{x}}{\geq} \limsup_{n\to+\infty} f_n(x_n) \overset{\text{As } x_n\in A \text{ for } n>>0}{\geq} \limsup_{n\to+\infty} \inf_{x\in A} f_n(x)$$

As ε is arbitrary, we conclude.

(2) Let $\{x_n\} \subseteq K$ be a sequence of quasi-minimizers, i.e.

$$(**) \qquad f_n(x_n) \leq \inf_{x\in K} f_n(x) + \frac{1}{n}, \qquad \forall n\in\mathbb{N}.$$

Since K is compact, up to extracting a subsequence, we can suppose

$$x_n \to \hat{x} \quad \text{for some} \quad \hat{x}\in K.$$

Then

$$\inf_{x\in K} f(x) \overset{\text{As } \hat{x}\in K}{\leq} f(\hat{x}) \overset{\Gamma\text{-liminf ineq., as } x_n\to\hat{x}}{\leq} \liminf_{n\to+\infty} f_n(x_n)$$

$$\overset{(**)}{\leq} \liminf_{n\to+\infty} \left[\inf_{x\in K} f_n(x) + \frac{1}{n}\right] \overset{\text{as } 1/n\to 0}{=} \liminf_{n\to+\infty} \left[\inf_{x\in K} f_n(x)\right]$$

$\square$

DEFINITION 11.10 (EQUICOERCIVITY)

(X,d) metric space, $f_n : X \to \mathbb{R}$. We say that $\{f_n\}$ is EQUICOERCIVE if $\exists\, K\subseteq X$ non empty and compact s.t.

$$\inf\{f_n(x) : x\in X\} = \inf\{f_n(x) : x\in K\}, \qquad \forall n\in\mathbb{N}.$$

(K is independent on n)

REMARK 11.11 (X,d) metric space, $f_n : X \to \mathbb{R}$. Suppose that there $\exists$ $M \in \mathbb{R}$ s.t. the set

$$\{ x \in X \mid f_n(x) \le M , \ \forall n \in \mathbb{N} \}$$

is non empty and pre-compact. Then $\{f_n\}$ is EQUICOERCIVE.

Proof Set $K := \{ x \in X \mid f_n(x) \le M , \ \forall n \in \mathbb{N} \}$. This is non empty, compact, and satisfies the condition $\inf_{x \in X} f_n = \inf_{x \in K} f_n$ for all $n \in \mathbb{N}$. ∎

We are finally able to prove the main result of this section.

THEOREM 11.12 (CONVERGENCE OF MINIMUMS AND MINIMIZERS)

(X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. Suppose that:

(i) $\{f_n\}$ is equicoercive WRT the compact set K

(ii) $f_n \xrightarrow{\Gamma} f$ for some $f : X \to \overline{\mathbb{R}}$

Then: 1 f admits minimum on X

2 As $n \to +\infty$ we have $\inf_{x \in X} f_n(x) \to \min_{x \in X} f(x)$

3 Assume $\{x_n\}$ is a sequence of almost-minimizers, i.e.,

$$\lim_{n \to +\infty} \left\{ f_n(x_n) - \inf_{x \in X} f_n(x) \right\} = 0 .$$

Suppose that $x_{n_k} \to \hat{x}$. Then $\hat{x}$ is minimum for f over X.

Proof ① By LEMMA 11.8 we know that the Γ-limit f is LSC.
Since K is compact, by the DIRECT METHOD (THM 9.4)
there $\exists\ \hat{x} \in K$ s.t.

$$(K) \qquad f(\hat{x}) = \min_{x \in K} f(x) \qquad \text{(} f \text{ admits minimum on } K\text{)}$$

We claim that

$$(*) \qquad f(\hat{x}) = \min_{x \in X} f(x) \qquad (\hat{x} \text{ minimizes } f \text{ on } X)$$

Indeed let $y \in X$ be arbitrary. Then there $\exists$ a recovery sequence $\{y_n\}$ s.t.

$$y_n \to y \quad \text{and} \quad f(y) = \lim_{n \to +\infty} f_n(y_n)$$

Then

$$f(y) = \lim_{n\to+\infty} f_n(y_n) \overset{\{f_n(y_n)\}\text{ is convergent}}{=} \liminf_{n\to+\infty} f_n(y_n) \overset{\text{def of inf}}{\geq} \liminf_{n\to+\infty} \inf_{x\in X} f_n(x)$$

$$\overset{\text{Equicoercivity}}{=} \liminf_{n\to+\infty}\left\{\inf_{x\in K} f_n(x)\right\} \overset{\text{PROP 11.9, point ② as } K \text{ is compact}}{\geq} \inf_{x\in K} f(x) \overset{\text{by } (K)}{=} f(\hat{x})$$

and so $(*)$ holds.

② We have:

$$\inf_{x\in X} f(x) \overset{\text{By PROPOSITION 11.9 point ①, since } X \text{ is open}}{\geq} \liminf_{n\to+\infty}\left\{\inf_{x\in X} f_n(x)\right\} \overset{\text{Equicoercivity}}{=} \liminf_{n\to+\infty}\left\{\inf_{x\in K} f_n(x)\right\}$$

$$\overset{\text{By PROPOSITION 11.9 point ②, as } K \text{ is compact}}{\geq} \inf_{x\in K} f(x) \overset{\text{by } (K) \text{ and } (*)}{=} \min_{x\in X} f(x)$$

proving that

$$(**) \qquad \liminf_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} = \min_{x\in X} f(x)$$

Now let $\hat{x}$ be minimizer for f on X, which exists by point (1).
Let $\{x_n\}$ be a recovery sequence, i.e.

$$x_n \to \hat{x} \qquad \text{and} \qquad f(\hat{x}) = \lim_{n\to+\infty} f_n(x_n)$$

Clearly

$$\inf_{x\in X} f_n(x) \leq f_n(x_n), \qquad \forall\, n \in \mathbb{N}$$

Taking the limsup in the above yields

$$(***) \qquad \limsup_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} \leq \limsup_{n\to+\infty} f_n(x_n) \underset{\{x_n\} \text{ is recovery sequence}}{=} f(\hat{x}) \underset{\hat{x} \text{ is minimizer}}{=} \min_{x\in X} f(x)$$

Therefore

$$\min_{x\in X} f(x) \overset{(**)}{=} \liminf_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} \overset{\text{property of liminf / limsup}}{\leq} \limsup_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} \overset{(***)}{\leq} \min_{x\in X} f(x)$$

concluding.

③ Let $\{x_n\}$ be a sequence of quasi-minimizers s.t. $x_{n_k} \to \hat{x}$. Then

$$f(\hat{x}) \overset{\Gamma\text{-liminf inequality}}{\leq} \liminf_{k\to+\infty} f_{n_k}(x_{n_k}) = \liminf_{k\to+\infty} \Big\{ \underbrace{f_{n_k}(x_{n_k}) - \inf_{x\in X} f_{n_k}(x)}_{\to 0 \text{ by assumption}} + \inf_{x\in X} f_{n_k}(x) \Big\}$$

$$= \liminf_{k\to+\infty} \Big\{ \inf_{x\in X} f_{n_k}(x) \Big\} \underset{\text{point ② of this Theorem}}{=} \min_{x\in X} f(x)$$

showing that $\hat{x}$ minimizes f over X. □

EXAMPLE 11.13 Consider the functionals $F_n : C^1[0,1] \to \mathbb{R}$ defined by

$$F_n(u) := \int_0^1 n\dot{u}^2 + (u - \arctan x)^2 \, dx$$

QUESTION What is the limit of $M_n := \inf\{F_n(u) : u \in C^1[0,1]\}$.

Extend F_n to $L^2(0,1)$ by setting $F_n := +\infty$ in $L^2 \setminus C^1$. Thus

$$M_n = \inf\{F_n(u) \mid u \in L^2(0,1)\}.$$

CLAIM $F_n \xrightarrow{\Gamma} F$ in $L^2(0,1)$, with

$$F(u) := \begin{cases} \int_0^1 (u - \arctan x)^2 \, dx, & \text{if } u \text{ is constant} \\ +\infty, & \text{otherwise in } L^2(0,1) \end{cases}$$

<u>Proof of CLAIM</u> Note that $F_n = G_n + H$ with

$$G_n(u) := \begin{cases} \int_0^1 n\dot{u}^2\,dx & \text{if } u \in C^1[0,1] \\ +\infty & \text{otherwise} \end{cases}, \qquad H(u) := \int_0^1 (u - \arctan x)^2\,dx$$

Clearly H is continuous in $L^2(0,1)$. Therefore, by PROPOSITION 11.4, it is sufficient to compute the Γ-limit of G_n. We have that

$$G_n \xrightarrow{\Gamma} G, \qquad \text{with} \qquad G(u) := \begin{cases} 0, & \text{if } u \text{ is constant} \\ +\infty, & \text{otherwise in } L^2(0,1) \end{cases}$$

- Γ-liminf inequality: suppose $u_n \to u$ in $L^2(0,1)$. We need to show

$$\circledast \qquad G(u) \leq \liminf_{n\to+\infty} G_n(u_n).$$

WLOG we can assume the RHS to be finite, so there $\exists$ a subsequence s.t.

$$G_{n_k}(u_{n_k}) \leq M, \qquad \forall\, k \in \mathbb{N}.$$

This means

$$\int_0^1 \dot{u}_{n_k}^2\,dx \leq \frac{M}{n_k},$$

which implies

$$\dot{u}_{n_k} \to 0 \quad \text{strongly in } L^2(0,1).$$

Since we are assuming $u_n \to u$ strongly in $L^2(0,1)$, REMARK 7.17 implies

$$u_{n_k} \to u \quad \text{strongly in } W^{1,2}(0,1), \quad \text{with } \dot{u} = 0 \quad \text{weakly}.$$

Thus $u \in W^{1,2}(0,1)$ with $\dot{u} \in C[0,1]$. Therefore $u \in C^1[0,1]$ by PROPOSITION 7.22. Hence the relationship $\dot{u} = 0$ also holds in the classical sense (as the weak derivative of a differentiable function coincides with the classical one)
Since $[0,1]$ is connected then

$$u \in C^1[0,1], \quad \dot{u} = 0 \quad \Rightarrow \quad u = \text{constant}$$

Thus $G(u) = 0$ by definition and $(*)$ holds (being $G_n \geq 0$).

- Γ-limsup inequality: Let $u \in L^2(0,1)$. We need to construct a recovery sequence.

  - If u is not constant, then $G(u) = +\infty$. Thus setting $u_n := u$, $\forall n \in \mathbb{N}$ we get $u_n \to u$ and, trivially,

$$\limsup_{n \to +\infty} G_n(u_n) \leq +\infty = G(u).$$

  - If u is constant, then $G(u) = 0$. Again set $u_n := u$, $\forall n \in \mathbb{N}$. Then $u_n \to u$. Moreover, as u is constant, then $u \in C^\infty[0,1]$ and $\dot{u} = 0$. Therefore

$$G_n(u_n) = G_n(u) = \int_0^1 n \, \dot{u}^2 \, dx = 0, \qquad \forall n \in \mathbb{N}$$

and the Γ-limsup inequality trivially holds.

Then $G_n \xrightarrow{\Gamma} G$ and so $F_n = G_n + H \xrightarrow{\Gamma} G + H = F$, by PROPOSITION 11.4. $\square$

In order to apply THEOREM 11.12, we also need to show that the sequence of functionals F_n is EQUICOERCIVE in $L^2(0,1)$.

CLAIM $\{F_n\}$ is EQUICOERCIVE in $L^2(0,1)$.

Proof of Claim By REMARK 11.11 it is sufficient to show $\exists$ of $M \in \mathbb{R}$ s.t.

$$K := \{ u \in L^2(0,1) \mid F_n(u) \leq M \}$$

is non-empty and pre-compact. First of all, note that

$$F_n(0) = \int_0^1 (\arctan x)^2 \, dx \leq \left(\frac{\pi}{2}\right)^2 < 10 \quad , \quad \forall n \in \mathbb{N}.$$

We then choose $M := 10$, so that $K \neq \emptyset$. We are left to show that K is pre-compact in $L^2(0,1)$. Indeed,

$$F_n(u) \leq 10 \overset{\text{def of } F_n}{\Rightarrow} \begin{cases} \displaystyle\int_0^1 \dot{u}^2 \, dx \leq \frac{10}{n} \\ \displaystyle\int_0^1 (u - \arctan x)^2 \, dx \leq 10 \end{cases} \Rightarrow \|u\|_{W^{1,2}} \leq C$$

for some $C > 0$ not depending on n and on u. Thus

$$K = \{ u \in L^2(0,1) \mid F_n(u) \leq 10 \} \subseteq \tilde{K} := \{ u \in W^{1,2}(0,1) \mid \|u\|_{W^{1,2}} \leq C \}$$

Note that $\tilde{K}$ is compact in $L^2(0,1)$, thanks to the compact embedding $W^{1,2}(0,1) \hookrightarrow L^2(0,1)$ of THEOREM 7.27.

Therefore K is pre-compact, since $\overline{K}$ is closed and contained in the compact $\tilde{K}$. $\square$

Thus we have shown

(i) $\{F_n\}$ is EQUICOERCIVE in $L^2(0,1)$

(ii) $F_n \xrightarrow{\Gamma} F$ in $L^2(0,1)$

From THEOREM 11.12 we then get

$$\inf_{u \in L^2(0,1)} F_n(u) \longrightarrow \min_{u \in L^2(0,1)} F(u),$$

that is,

$$M_n \to M := \min_{u \in L^2(0,1)} F(u)$$

Since $F(u) < +\infty$ if and only if u is constant, then

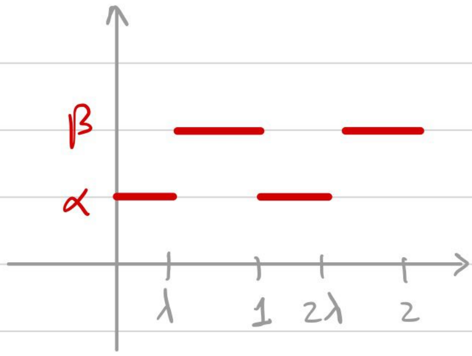
$$M = \min \left\{ \int_0^1 (\lambda - \arctan x)^2 \, dx \;\middle|\; \lambda \in \mathbb{R} \right\}$$

which can be computed explicitly.

APPLICATION : HOMOGENIZATION PROBLEMS

DEFINITION 11.14 For $\alpha, \beta \in \mathbb{R}$, $\lambda \in (0,1)$ define

$$A(x) := \begin{cases} \alpha & \text{if} \quad x \in [0, \lambda) \\ \beta & \text{if} \quad x \in [\lambda, 1) \end{cases}$$



Extend A to $\mathbb{R}$ by periodicity. For $n \in \mathbb{N}$ set

$$A_n(x) := A(nx)$$

FACT $A_n(x) \rightharpoonup \underbrace{\lambda\alpha + (1-\lambda)\beta}_{\text{Average of } A \text{ in } [0,1]}$ weakly in $L^p(a,b)$, $\forall\, 1 \leq p < +\infty$

Define $F_n : C^1[a,b] \to \mathbb{R}$ by

$$F_n(u) := \int_a^b A_n(x)\, \dot{u}(x)\, dx$$

Extend F_n to $+\infty$ on $L^2(a,b) \setminus C^1[a,b]$. Define $F : L^2(a,b) \to \overline{\mathbb{R}}$ by

$$F(u) := \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

One might expect $F_n \xrightarrow{\Gamma} cF$ with $c = \lambda\alpha + (1-\lambda)\beta$. However this is FALSE

THEOREM 11.15 Suppose $\alpha, \beta > 0$. Consider F_n, F as above. Then

$$F_n \xrightarrow{\Gamma} cF, \qquad c := \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}}$$

(Harmonic mean of A in $[0,1]$: $\int_0^1 \frac{1}{A(x)}\, dx$)

In order to prove the above, consider the following:

CELL - PROBLEM For $\ell > 0$, consider the problem:

$$\min \left\{ \int_0^1 A(x)\, \dot{u}^2 \, dx \;\middle|\; u(0)=0 ,\ u(1) = \ell \right\}$$

This is called cell-problem because A has only one oscillation in $[0,1]$

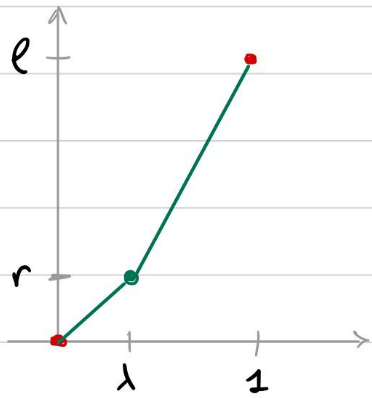
LEMMA 11.16 Let $\alpha, \beta > 0$. The cell-problem has solution

$$\min \left\{ \int_0^1 A(x)\, \dot{u}^2 \, dx \;\middle|\; u(0)=0 ,\ u(1) = \ell \right\} = c\ell^2 , \qquad c := \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}}$$

Proof Since $A = \alpha$ in $[0, \lambda)$ and $A = \beta$ in $[\lambda, 1)$, the separate problems in $[0,\lambda]$ and $[\lambda, 1]$ become

$$\min \left\{ \alpha \int_0^\lambda \dot{u}^2 \, dx \;\middle|\; u(0)=0 ,\ u(\lambda) = r \right\}$$

$$\min \left\{ \beta \int_\lambda^1 \dot{u}^2 \, dx \;\middle|\; u(\lambda)=r ,\ u(1) = \ell \right\}$$



We already know that the above problems are solved by straight lines u_1, u_2 respectively. In particular

$$\dot{u}_1 = \frac{r}{\lambda} , \qquad \dot{u}_2 = \frac{\ell - r}{1-\lambda}$$

Define

$$u_r(x) := \begin{cases} u_1 & \text{if } x \in [0, \lambda) \\ u_2 & \text{if } x \in [\lambda, 1) \end{cases}$$

It is easy to show that

$$\circledast \quad \min\left\{ \int_0^1 A(x)\,\dot{u}^2\,dx \;\middle|\; u(0)=0,\ u(1)=\ell \right\} = \min\left\{ \int_0^1 A(x)\,\dot{u}_r^2\,dx \;:\; r\in\mathbb{R} \right\}$$

Now

$$\int_0^1 A(x)\,\dot{u}_r^2\,dx = \int_0^\lambda A(x)\,\dot{u}_1^2\,dx + \int_\lambda^1 A(x)\,\dot{u}_2^2\,dx = \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda}$$

so that

$$\min_{r\in\mathbb{R}} \int_0^1 A(x)\,\dot{u}_r^2\,dx = \min_{r\in\mathbb{R}} \left\{ \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} \right\}$$

Now

$$\alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} = Ar^2 + Br + C, \qquad \begin{cases} A = \dfrac{\alpha}{\lambda} + \dfrac{\beta}{1-\lambda} \\[2ex] B = -\dfrac{2\ell\beta}{1-\lambda} \\[2ex] C = \dfrac{\beta}{1-\lambda}\,\ell^2 \end{cases}$$

which is minimized at $r = -\dfrac{B}{2A}$. Substituting into Ar^2+Br+C, we obtain the minimum $-\dfrac{B^2}{4A}+C$, i.e.,

$$\min_{r\in\mathbb{R}} \left\{ \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} \right\} = -\frac{B^2}{4A} + C = \frac{\alpha\beta}{\lambda\beta + (1-\lambda)\alpha}\,\ell^2 = c\,\ell^2$$

Recalling $\circledast$, we conclude. □

REMARK Solving the cell-problem is equivalent to solving

$$\min\left\{ c\int_0^1 \dot{u}^2\,dx \;\middle|\; u(0)=0,\; u(1)=e \right\}, \qquad c := \frac{1}{\frac{\lambda}{\alpha}+\frac{1-\lambda}{\beta}}$$

(Indeed, the solution to the above is given by the straight line $u(x) = ex$, so that $c\int_0^1 \dot{u}^2 dx = ce^2$)

LEMMA 11.17 (RESCALED CELL – PROBLEM)

The rescaled cell-problem satisfies

Harmonic average of A

$$\min\left\{ \int_{k/n}^{\frac{k+1}{n}} \underbrace{A(nx)}\,\dot{u}^2\,dx \;\middle|\; u\left(\frac{k}{n}\right)=A,\; u\left(\frac{k+1}{n}\right)=B \right\} = cn\,(B-A)^2$$

$A(nx)$ values α in $\left[\frac{k}{n}, \frac{k}{n}+\frac{\lambda}{n}\right)$ and β in $\left[\frac{k}{n}+\frac{\lambda}{n}, \frac{k+1}{n}\right)$

Thus $A(nx)$ has only one oscillation in $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, and this is still a cell-problem

Proof Same as LEMMA 11.16. □

REMARK Solving the rescaled cell-problem is equivalent to solving

$$\min\left\{ c\int_{k/n}^{\frac{k+1}{n}} \dot{u}^2\,dx \;\middle|\; u\left(\frac{k}{n}\right)=A,\; u\left(\frac{k+1}{n}\right)=B \right\}, \qquad c := \frac{1}{\frac{\lambda}{\alpha}+\frac{1-\lambda}{\beta}}$$

(Indeed, the solution to the above is given by the straight line u with

$$\dot{u} = \frac{B-A}{\frac{k+1}{n}-\frac{k}{n}} = n(B-A)$$

so that

$$c\int_{k/n}^{\frac{k+1}{n}} \dot{u}^2\,dx = c\,n^2(B-A)^2\cdot\frac{1}{n} = cn\,(B-A)^2$$)

Γ-LIMINF INEQUALITY FOR THEOREM 11.15

We now sketch the proof of the Γ-liminf inequality in THEOREM 11.15.
Let $u_n \to u$ strongly in $L^2(a,b)$. We need to prove that

$$(*) \qquad c\,F(u) \leq \liminf_{n\to+\infty} F_n(u_n)$$

WLOG assume RHS finite, i.e., $\exists$ a subsequence s.t.

$$F_{n_k}(u_{n_k}) \leq M \,, \qquad \forall\, k \in \mathbb{N}.$$

In particular $\{u_{n_k}\} \subseteq C^1[a,b]$ and

$$\int_a^b A_{n_k}(x)\, \dot{u}_{n_k}^2 \, dx \leq M \,, \qquad \forall\, k \in \mathbb{N}.$$

Now $A_{n_k} \geq \min\{\alpha, \beta\} > 0$, from which we deduce that $\{\dot{u}_{n_k}\}$ is bounded in $L^2(a,b)$. Thus, $\exists\, v \in L^2(a,b)$ s.t.

$$\dot{u}_{n_{k_j}} \rightharpoonup v \quad \text{weakly in } L^2(a,b)$$

As $u_n \to u$ in $L^2(a,b)$, from REMARK 7.18 we conclude that

$$u_{n_{k_j}} \rightharpoonup u \quad \text{weakly in } W^{1,2}(a,b)$$

Since the above limit does not depend on the subsequence, we get convergence along the whole sequence, i.e.,

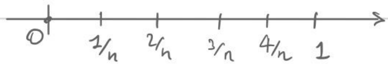
$$(W) \qquad u_n \rightharpoonup u \quad \text{weakly in } W^{1,2}(a,b)$$

The compact embedding $W^{1,2}(a,b) \hookrightarrow C[a,b]$ (see THEOREM 7.27) implies

$$(U) \qquad u_n \to u \quad \text{uniformly in } [a,b] \,, \qquad \{u_n\}, u \text{ continuous}$$

• **Step 1:** Assume $u_n \to u$ uniformly in $[0,1]$.
We want to prove ⊛ localized to $[0,1]$ (i.e. $a=0$, $b=1$)

Divide $[0,1]$ in subintervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. Then



$$\int_0^1 A(nx)\, \dot{u}_n^2 \, dx = \sum_{k=0}^{n-1} \underbrace{\int_{\frac{k}{n}}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx}$$

u_n is competitor for RESCALED CELL-PROBLEM
WITH $A = u_n\left(\frac{k}{n}\right)$, $B = u_n\left(\frac{k+1}{n}\right)$

$$(\text{LEMMA } 11.17) \geq \sum_{k=0}^{n-1} c\, n \left[u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right) \right]^2$$

$$\left(n \sum \alpha_k^2 \geq \left(\sum \alpha_k\right)^2 \right) \geq c \left[\sum_{k=0}^{n-1} u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right) \right]^2$$

$$= c \left[u_n(1) - u_n(0) \right]^2$$

As $u_n \to u$ uniformly, we get

$$\liminf_{n \to +\infty} \int_0^1 A(nx)\, \dot{u}_n^2 \, dx \geq \liminf_{n \to +\infty} c \left[u_n(1) - u_n(0) \right]^2$$

$$(u_n \to u \text{ uniformly}) = c \left[u(1) - u(0) \right]^2$$

NOTE The above would be ⊛ if $u(x) = mx + q$ line, since

$$c \int_0^1 \dot{u}^2 \, dx = c\, m^2 = c \left[u(1) - u(0) \right]^2$$

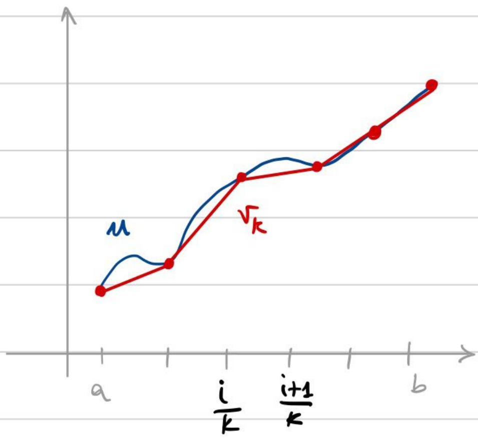
• **Step 2:** Assume $u \in L^2(\tilde a, \tilde b)$ with $\tilde a < \tilde b$ arbitrary.
Suppose $u_n \to u$ uniformly in $[\tilde a, \tilde b]$. By a rescaling argument, and proceeding as in STEP 1, one can show

$$\text{(L)} \qquad \liminf_{n \to +\infty} \int_{\tilde a}^{\tilde b} A_n(x)\, \dot u_n^2 \, dx \geq c\, (\tilde b - \tilde a) \left[\frac{u(\tilde b) - u(\tilde a)}{\tilde b - \tilde a} \right]^2$$

NOTE Again, (L) is $\circledast$ if $u(x) = mx + q$

• **Step 3:** General case. Assume $u_n \to u$ in $L^2(a,b)$. Then WLOG (W)-(U) hold and u_n, u are continuous.

Divide (a,b) in k equal parts, $k \in \mathbb{N}$ (suppose $\frac{b-a}{k} \in \mathbb{N}$)
Let v_k be the linear interpolation of u on the grid (recall u continuous)



By applying (L) to u_n on $\left[\frac{i}{k}, \frac{i+1}{k}\right]$ we get (this is possible as $u_n \to u$ uniformly)

$$\liminf_{n \to +\infty} \int_{\frac{i}{k}}^{\frac{i+1}{k}} A(nx)\, \dot u_n^2 \, dx \overset{(L)}{\geq} c\, \frac{1}{k} \left(\frac{u\left(\frac{i+1}{k}\right) - u\left(\frac{i}{k}\right)}{\frac{1}{k}} \right)^2$$

$$= c \int_{\frac{i}{k}}^{\frac{i+1}{k}} \dot v_k^2 \, dx \qquad \left(\text{as } \dot v_k = \frac{u\left(\frac{i+1}{k}\right) - u\left(\frac{i}{k}\right)}{1/k} \text{ on } \left[\frac{i}{k}, \frac{i+1}{k}\right] \right)$$

Summing over i we find

$$(**) \qquad \liminf_{n \to +\infty} F_n(u_n) \geq c\, F(v_k), \qquad \forall\, k \in \mathbb{N}$$

Now, one can check that as the grid width goes to zero, we have

$$v_k \to u \quad \text{uniformly}, \quad \text{and} \quad F(v_k) \nearrow F(u)$$

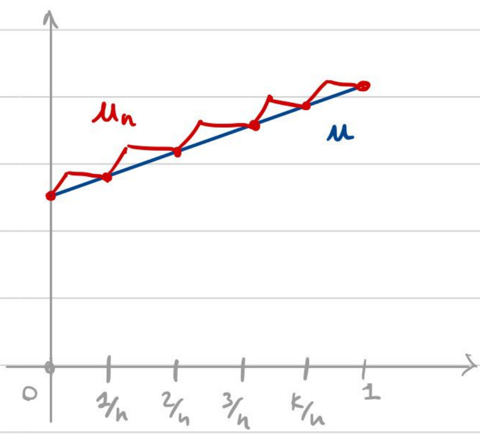
Thus, taking the sup for $k \in \mathbb{N}$ in (**) yields (*), concluding the Γ-liminf inequality.

Γ-LIMSUP INEQUALITY FOR THEOREM 11.15

We now sketch the proof of the Γ-limsup inequality in THEOREM 11.15

- **Step 1** : $u = \ell x + q$ in $[0,1]$. Then we can chose the recovery sequence u_n in the following way:

Divide $[0,1]$ in sub-intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. In each sub-int. u_n is the solution to the rescaled cell-problem with data $A = u\left(\frac{k}{n}\right)$, $B = u\left(\frac{k+1}{n}\right)$



In this way the energy in each $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ is

$$\int_{k/n}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx \underset{\substack{\uparrow \\ u_n \text{ solution to rescaled} \\ \text{cell-problem (LEMMA 11.17)}}}{=} c\, n \left[u\left(\frac{k+1}{n}\right) - u\left(\frac{k}{n}\right) \right]^2 \underset{\substack{\uparrow \\ u \text{ straight line}}}{=} c \int_{k/n}^{\frac{k+1}{n}} \dot{u}^2 \, dx$$

and the total energy becomes :

$$F_n(u_n) = \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx$$

$$= c \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} \dot{u}^2 \, dx \quad = c\, F(u)$$

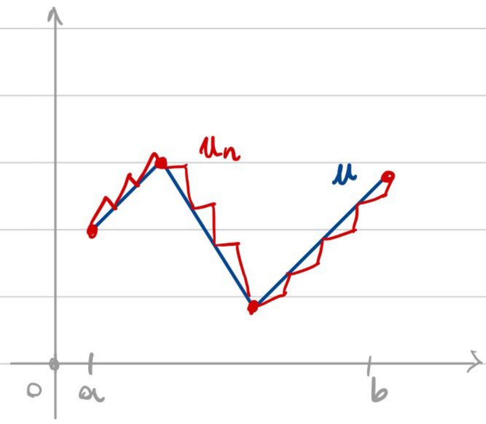
Thus $F_n(u_n) \to c\, F(u)$ trivially. One can also show that, as the grid refines,

$$u_n \to u \quad \text{uniformly in } [0,1],$$

concluding the Γ-limsup inequality.

- **Step 2:** u piecewise affine in $[a,b]$.

To obtain u_n just divide $[a,b]$ into the sub-intervals in which u is affine and in each of those define u_n as in STEP 1.



- **Step 3:** Let $u \in L^2(a,b)$ be arbitrary.

REMARK 11.18 In general, it is sufficient to show the Γ-limsup inequality for elements in D, where $D \subseteq X$ is an energy dense set wRT the Γ-limit.

(This is easily proven with a diagonal argument)

Choose D as the set of PIECEWISE AFFINE FUNCTIONS. Then D is energy dense WRT cF (easy check). The Γ-limsup inequality holds in D by STEP 2. Therefore we conclude the Γ-limsup inequality in $L^2(a,b)$ by REMARK 11.18.

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