5. SUFFICIENT CONDITIONS FOR MINIMALITY

LESSON 6
21 APRIL 2021

So far we have shown that solutions to a minimization problem for integral functionals also solve the associated EULER-LAGRANGE EQUATION.

QUESTION: Are solutions to (ELE) minimizers? If YES, how do we prove it?

To answer the above, we will analyze 4 methods:

WARNING: This is in general not true: we can have $\bar{u}$ sol. to (ELE) but not minimizer

(1) CONVEXITY
(2) TRIVIAL LEMMA
} NOW

(3) CALIBRATIONS
(4) WEIERSTRASS FIELDS
} LATER in the course (if we have time!)

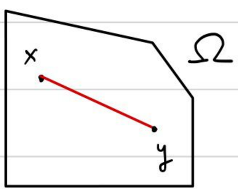
(1) CONVEXITY

If the Lagrangian $L = L(x,s,p)$ is convex in s,p, we will prove that solutions to (ELE) are minimizers.

DEFINITION 5.1

Let $\Omega \subseteq \mathbb{R}^d$. We say that Ω is CONVEX if

$$\lambda x + (1-\lambda) y \in \Omega, \quad \forall\, x,y \in \Omega, \ \lambda \in [0,1].$$

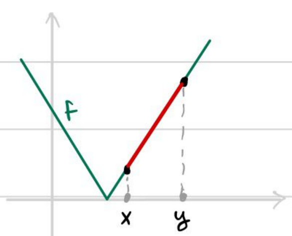


CONVEX SET Ω

Let $f: \Omega \to \mathbb{R}$, with Ω convex. We say that f IS CONVEX if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x,y \in \Omega$, $\lambda \in [0,1]$.

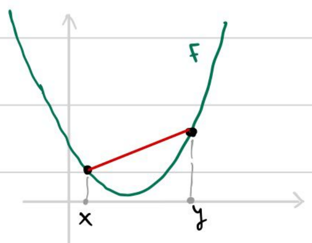


f CONVEX BUT NOT STRICTLY

We say that f is STRICTLY CONVEX if

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y),$$

for all $x,y \in \Omega$, s.t. $x \neq y$ and $\lambda \in (0,1)$.



f STRICTLY CONVEX

For regular convex functions the following result holds:

THEOREM 5.2 Let $\Omega \subseteq \mathbb{R}^d$ be open convex, $f : \Omega \to \mathbb{R}$, $f \in C^1(\Omega)$. Then

1) f is convex iff

(f above tangent planes) $\rightsquigarrow$ $$f(y) \geq f(x) + \nabla f(x) \cdot (y-x), \quad \forall x, y \in \Omega$$

2) f is strictly convex iff

$$f(y) > f(x) + \nabla f(x) \cdot (y-x), \quad \forall x, y \in \Omega, \ x \neq y$$

Assume in addition that $f \in C^2(\Omega)$

3) f is convex iff the HESSIAN $\nabla^2 f$ is POSITIVE SEMI-DEFINITE, i.e.,

$$y^T \nabla^2 f(x) \, y \geq 0, \quad \forall x \in \Omega, \ y \in \mathbb{R}^d$$

4) Assume $\nabla^2 f$ is POSITIVE DEFINITE, i.e.,

$$y^T \nabla^2 f(x) \, y > 0, \quad \forall x \in \Omega, \ y \in \mathbb{R}^d \setminus \{0\}$$

Then f is strictly convex.

(The proof is standard, from analysis courses. See B. DACOROGNA - "INTRODUCTION TO THE CALCULUS OF VARIATIONS", IMPERIAL COLLEGE PRESS, 2004 - THEOREM 1.5)

WARNING: The converse of (4) does not hold.

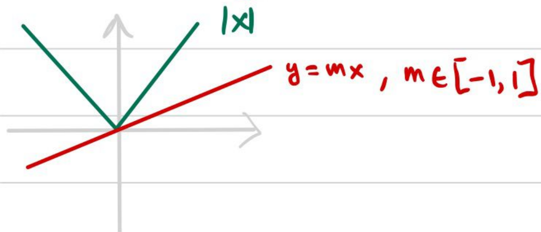
If instead we have no regularity, then we get:

THEOREM 5.3 Let $\Omega \subseteq \mathbb{R}^d$ be convex, and $f : \Omega \to \mathbb{R}$ convex. Let $\bar{x} \in \Omega$. Then $\exists\, m \in \mathbb{R}^d$ s.t.

$$f(y) \geq f(\bar{x}) + m \cdot (y - \bar{x}), \quad \forall y \in \Omega$$

(Proof is omitted. This result is saying that if f convex then $\partial f(\bar{x}) \neq \phi$, i.e., the SUBDIFFERENTIAL of f at $\bar{x}$ is non-empty. For a proof see R.T. ROCKAFELLAR – "CONVEX ANALYSIS", PRINCETON UNIVERSITY PRESS, 1970 – THEOREM 23.4)

NOTE: For $f : [a,b] \to \mathbb{R}$ one can take $m \in [f'_-(\bar{x}), f'_+(\bar{x})]$ left and right derivatives.



APPLICATION TO COV

Let $X := \{u \in C^1[a,b] \mid u(a) = \alpha, u(b) = \beta\}$, $V := \{v \in C^1[a,b] \mid v(a) = v(b) = 0\}$.
Let $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,p)$ and define $F : X \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx.$$

THEOREM 5.4 Suppose $L \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ and let $\bar{u} \in X$ be a solution to ELE in INTEGRAL FORM, i.e.,

$$(*) \quad \int_a^b L_s(x, \bar{u}, \bar{u}')\, v + L_p(x, \bar{u}, \bar{u}')\, v' \, dx = 0, \quad \forall v \in V.$$

(1) If $(s,p) \mapsto L(x,s,p)$ is CONVEX for all $x \in [a,b]$ fixed, then $\bar{u}$ is a minimizer for F in X.

(2) If $(s,p) \mapsto L(x,s,p)$ is STRICTLY CONVEX for all $x \in [a,b]$, then $\bar{u}$ is the unique minimizer of F in X.

NOTE: If u solves ELE in DIFFERENTIAL FORM then it solves ELE in INTEGRAL FORM

<u>Proof</u> (1) Let $w \in X$ be arbitrary and set $v := w - \bar{u}$.
Then $v \in V$ i.e. $v(a) = v(b) = 0$. We have

$$F(w) = F(\bar{u} + v) = \int_a^b L(x, \bar{u} + v, \bar{u}' + v')\, dx$$

As L is C^1 and is convex in s, p, we can apply Theorem 5.2 and obtain

$$L(x, s+\tilde{s}, p+\tilde{p}) \geq L(x, s, p) + L_s(x,s,p)\tilde{s} + L_p(x,s,p)\tilde{p}, \quad \begin{array}{l} \forall s, \tilde{s}, p, \tilde{p} \in \mathbb{R} \\ \forall x \in [a,b] \end{array}$$

Apply the above with $s = \bar{u}$, $\tilde{s} = v$, $p = \bar{u}'$, $\tilde{p} = v'$,

$$\begin{aligned} F(w) &= \int_a^b L(x, \bar{u} + v, \bar{u}' + v')\, dx \geq \\ &\geq \int_a^b L(x, \bar{u}, \bar{u}')\, dx + \underbrace{\int_a^b L_s(x, \bar{u}, \bar{u}')\, v + L_p(x, \bar{u}, \bar{u}')\, v'\, dx}_{=0 \text{ by } (*), \text{ since } v \in V} \\ &= F(\bar{u}) \end{aligned}$$

showing that $\bar{u}$ minimizes F over X.

(2) Assume $\bar{u}$ and $\hat{u}$ both minimize F over X. Set $m := \min\{F(u) \mid u \in X\}$.
Therefore $F(\hat{u}) = F(\bar{u}) = m$, and also $F(u) \geq m$, $\forall u \in X$.

Define $w := \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}$, so $w \in X$. By convexity of L (just using the definition)

$$\begin{aligned} L(x, w, w') &= L\left(x, \tfrac{1}{2}\bar{u} + \tfrac{1}{2}\hat{u}, \tfrac{1}{2}\bar{u}' + \tfrac{1}{2}\hat{u}'\right) \\ &\leq \tfrac{1}{2} L(x, \bar{u}, \bar{u}') + \tfrac{1}{2} L(x, \hat{u}, \hat{u}') \end{aligned}$$

Integrating the above inequality we obtain

$$m \leq F(w) = \int_a^b L(x, w, w')\,dx \leq$$

Since $w \in X$

$$\leq \frac{1}{2}\int_a^b L(x, \bar{u}, \bar{u}')\,dx + \frac{1}{2}\int_a^b L(x, \hat{u}, \hat{u}')\,dx =$$

$$= \frac{1}{2}F(\bar{u}) + \frac{1}{2}F(\hat{u})$$

$$= \frac{1}{2}m + \frac{1}{2}m = m$$

Thus, all the inequalities in the above chain are actually equalities, and we get

$$(*) \quad \int_a^b \left\{ \frac{1}{2}L(x, \bar{u}, \bar{u}') + \frac{1}{2}L(x, \hat{u}, \hat{u}') - L\left(x, \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}, \frac{1}{2}\bar{u}' + \frac{1}{2}\hat{u}'\right) \right\} dx = 0$$

Now, by convexity of L, the INTEGRAND in $(*)$ is always ≥ 0. Hence, by continuity, we conclude that

$$L\left(x, \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}, \frac{1}{2}\bar{u}' + \frac{1}{2}\hat{u}'\right) = \frac{1}{2}L(x, \bar{u}, \bar{u}') + \frac{1}{2}L(x, \hat{u}, \hat{u}'), \quad \forall x \in [a,b]$$

Since L is STRICTLY CONVEX, the above is possible iff $\bar{u}(x) = \hat{u}(x)$ and $\bar{u}'(x) = \hat{u}'(x)$, $\forall x \in [a,b]$.

Thus $\bar{u} \equiv \hat{u}$ and the minimizer is unique. $\square$

EXAMPLE Let $L: \mathbb{R} \to \mathbb{R}$, $L = L(p)$. Assume $L \in C^2(\mathbb{R})$. Define

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha, u(b) = \beta \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = 0, v(b) = 0 \}$$

Consider $F: X \to \mathbb{R}$ defined by

$$F(u) := \int_a^b L(\dot{u})\, dx$$

We can then write ELE in DIFFERENTIAL FORM:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x)) \Big] = L_s(x, u(x), \dot{u}(x)) \quad , & \forall x \in (a,b) \\ u(a) = \alpha \ , \quad u(b) = \beta \end{cases}$$

which in this case reads

$$\text{(ELE)} \quad \begin{cases} \dfrac{d}{dx}\Big[L'(\dot{u}(x)) \Big] = 0 \quad , & \forall x \in (a,b) \\ u(a) = \alpha \ , \quad u(b) = \beta . \end{cases}$$

Now, the above ODE implies that

$$\boxed{L'(\dot{u}) = \text{CONSTANT}}$$

Therefore the straight line

$$\bar{u}(x) := \frac{\beta - \alpha}{b - a}(x - a) + \alpha$$

is ALWAYS a solution to (ELE).

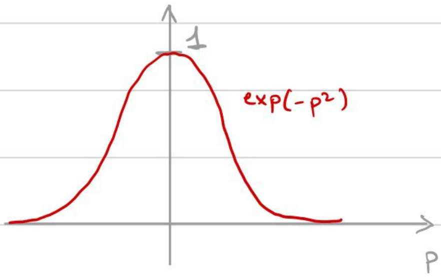
QUESTION When does $\bar{u}$ also solve

$$\text{(P)} \qquad F(\bar{u}) = \min \{ F(u) \mid u \in X \} \qquad ?$$

CASE 1 Assume L CONVEX. As $\bar{u}$ solves (ELE), in particular it solves ELE in INTEGRAL FORM. Then by THEOREM 5.4 we have that $\bar{u}$ solves (P).

CASE 2 If we do not assume convexity, then in general $\bar{u}$ DOES NOT solve (P). For example let

$$L(p) := \exp(-p^2)$$



Let us consider the case with zero Dirichlet conditions, i.e.,

$$X = V = \{ u \in C^1[0,1] \mid u(0) = u(1) = 0 \}.$$

Note that in this setting our straight line is $\bar{u} \equiv 0$. Then $\bar{u}$ solves (ELE), but is it solution to (P) ?

Clearly L is not convex, so THEOREM 5.4 cannot be applied.

FACT The minimization problem (P) has NO SOLUTION and

$$m := \inf \{ F(u) \mid u \in X \} = 0$$

(This will be left as an exercise)

Therefore $\bar{u} \equiv 0$ solves (ELE) but DOES NOT solve (P).

② TRIVIAL LEMMA

Given $\bar{u}$ solution to ELE, we want to know if $\bar{u}$ is also a minimizer. A possible way to answer this question is given by the following Lemma.

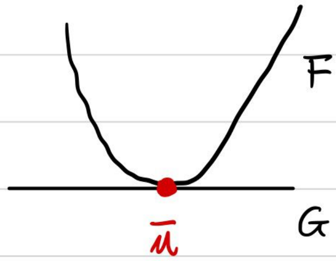
LEMMA 5.5 (LEMMA TRIVIAL)

Let X be a set and $F, G : X \to \mathbb{R}$ functionals. Assume that

(i) $F(u) \geq G(u)$, $\forall u \in X$

(ii) $\bar{u} \in X$ is a minimizer for G on X

(iii) $F(\bar{u}) = G(\bar{u})$.



Then $\bar{u}$ is a minimizer for F. If in addition $\bar{u}$ is the unique minimizer of G, then $\bar{u}$ is the unique minimizer of F.

Proof Let $u \in X$ be arbitrary. Then

$$F(u) \overset{(i)}{\geq} G(u) \overset{(ii)}{\geq} G(\bar{u}) \overset{(iii)}{=} F(\bar{u}),$$

showing that $\bar{u}$ minimizes F.

Assume now that $\bar{u}$ is the unique minimizer of G. Then, for the first part of the statement, we know that $\bar{u}$ also minimizes F. Suppose that $\bar{w} \in X$ is another minimizer for F. Then

$$G(\bar{w}) \underset{(i)}{\leq} F(\bar{w}) \underset{\substack{\text{minimality of } \bar{u} \\ \text{and } \bar{w} \text{ for } F}}{=} F(\bar{u}) \overset{(iii)}{=} G(\bar{u})$$

Thus $G(\bar{w}) = G(\bar{u})$, being $\bar{u}$ minimizer for G. $\Rightarrow \bar{u} = \bar{w}$ as the minimizer of G is unique. $\square$

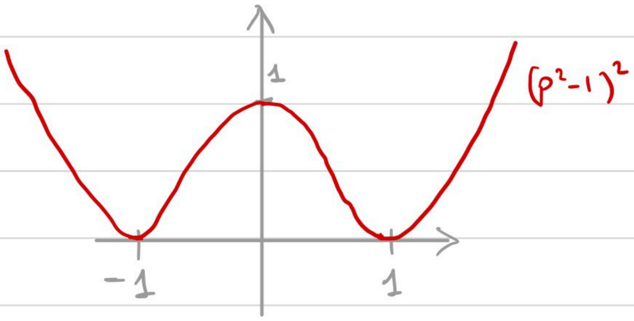
COMMENT: The above lemma requires to find a functional G satisfying (i),(ii),(iii). This is not always obvious. However in the future we will see a systematic way to construct G from F.

EXAMPLE 5.6 $X = \{ u \in C^1[0,1] \mid u(0) = 1, \; u(1) = 3 \}$

$F : X \to \mathbb{R}$ defined by

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 \, dx$$

Note that the Lagrangian is $L = L(p) = (p^2-1)^2$, which is not convex. Such Lagrangian is very typical and is named DOUBLE WELL.



DOUBLE WELL

The ELE for the minimum problem associated to F is

$$\begin{cases} \dfrac{d}{dx} L_p(\dot{u}) = 0 \,, & \forall x \in (0,1) \\ u(0) = 1, \; u(1) = 3 \end{cases}$$

From $L_p(\dot{u})' = 0$ we deduce $L_p(\dot{u}) = \text{CONSTANT}$. Therefore the line

$$\bar{u}(x) := 2x + 1$$

satisfies the BOUNDARY CONDITIONS and ELE.

NOTE If L was CONVEX we could have concluded that $\bar{u}$ minimizes F, by THEOREM 5.4. However L is not convex, so we need to proceed differently.

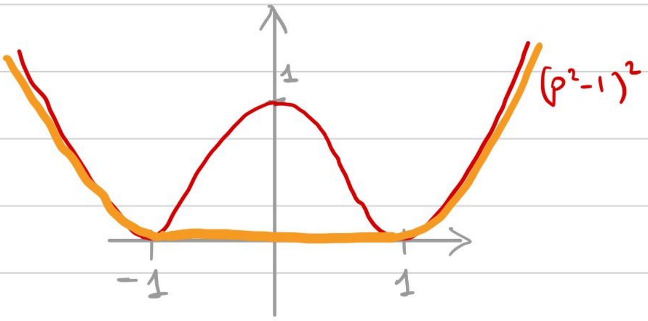
CLAIM $\bar{u}$ is the unique minimizer of F in X.

Proof We make use of the TRIVIAL LEMMA. We need to find G satisfying the assumptions.

IDEA: Find a Lagrangian $\hat{L}$ such that $\hat{L} \leq L$ and that the functional

$$G(u) := \int_0^1 \hat{L}(\dot{u})\,dx \quad , \quad u \in X$$

is likely to admit unique minizer. Ideally we also want $\hat{L}$ to be CONVEX, so we can apply THEOREM 5.4 to G.



A good idea is then to convexify L, by setting

$$\hat{L}(p) := \begin{cases} L(p) & \text{if } |p| \geq 1 \\ 0 & \text{if } |p| \leq 1 \end{cases}$$

Notice that $\hat{L}$ is convex. We now verify (i), (ii), (iii) from LEMMA 5.5:

(i) $F(u) \leq G(u)$, $\forall u \in X$: True because $\hat{L} \leq L$ pointwise.

(ii) $\bar{u}$ minimizes G: True because $\hat{L}$ depends only on p. Therefore the line $\bar{u}$ is solution of ELE for G:

$$\begin{cases} \dfrac{d}{dx} \hat{L}_p(\bar{u}') = 0 \\ \bar{u}(0) = 1, \quad \bar{u}(1) = 2 \end{cases}$$

Therefore $\bar{u}$ minimizes G by THEOREM 5.4, as $\hat{L}$ is CONVEX.

(iii) $F(\bar{u}) = G(\bar{u})$: True because $\bar{u}' \equiv 2$, and $\hat{L}(2) = L(2)$ by definition.

Therefore $\bar{u}$ minimizes F by LEMMA TRIVIAL 5.5.

Also note that $\hat{L}$ is STRICTLY CONVEX in a neighborhood of $p = 2$ (that is, in a neighborhood of $\bar{u}' \equiv 2$). Thus (by a slightly more general version of THEOREM 5.4 we conclude that $\bar{u}$ is the unique minimizer of G.
By Lemma 5.5 we then have that $\bar{u}$ is the unique minimizer of F. □

EXAMPLE 5.7 (VARIATION ON EXAMPLE 5.6)

Let us consider the same Lagrangian $L(p) = (p^2 - 1)^2$ as in EXAMPLE 5.6

However this time we look for a minimum of F over the set

$$X = \{ u \in C^1[a,b] \mid u(0) = 0 \ , \ u(1) = 0 \}$$

Note : The only difference is we have changed the DIRICHLET BC

Let's try to show that the line passing through $(0,0)$ and $(1,0)$, i.e.,

$$\bar{u}(x) \equiv 0$$

(which solves ELE associated to F) is a minimizer for F.
We immediately see that the above strategy fails, because by definition

$$\hat{L}(0) = 0 \ , \quad \text{while} \quad L(0) = 1$$

Thus (iii) does not hold and we cannot apply LEMMA 5.5 to F, G and $\bar{u}$.

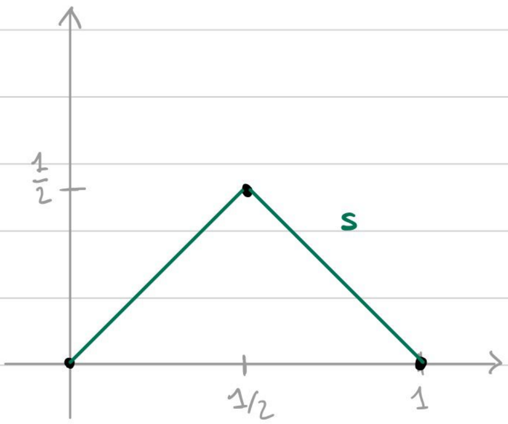
What is going on ?

Since the constraints are $u(0)=0$, $u(1)=0$, then $\bar{u}$ is NOT a minimizer for F. More in general:

CLAIM F admits no minimizer in X. Moreover

$$m := \inf \{ F(u) \mid u \in X \} = 0$$

Proof The idea is that, since $L(1) = L(-1) = 0$, we can construct a function $\tilde{s} \in X$ s.t. $|\tilde{s}'| \approx 1$ and so $F(\tilde{s}) \approx 0$. This is possible because the points $(0,0)$, $(1,0)$ are sufficiently close. To construct $\tilde{s}$, define $s : [0,1] \to \mathbb{R}$ by



$$s(x) := \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ -x+1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Notice that $s(0) = 0$, $s(1) = 0$ and $s' \in \{-1, 1\}$. Thus $F(s) = 0$.

The only problem is that s is not C^1. However, we can "ROUND the CORNER" at $x = 1/2$ by paying a small amount of energy (see WORKSHEET 3)
Thus we can define $\tilde{s} : [0,1] \to \mathbb{R}$ s.t.

$$\tilde{s} \in C^1[0,1], \quad \tilde{s}(0) = \tilde{s}(1) = 0, \quad \tilde{s}'(x) = \pm 1 \text{ for } x \in [0,1] \setminus I, \quad F(\tilde{s}) = \varepsilon$$

with $I = (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ for some $\delta > 0$ and $\varepsilon > 0$ arbitrary. Then $\tilde{s} \in X$, so that $m \le F(\tilde{s}) = \varepsilon$. As ε is arbitrary, we conclude $m = 0$ (as $F \ge 0$).

Finally, to see that the infimum is not attained, if it existed $\bar{u} \in X$ s.t. $F(\bar{u}) = 0$, then in particular

$$F(\bar{u}) = \int_0^1 ((\bar{u}')^2 - 1)^2 \, dx = 0 \Rightarrow \bar{u}' \in \{-1, 1\} \text{ for all } x \in [0,1].$$

However, as $\bar{u}'$ is continuous, we can only have $\bar{u}' \equiv 1$, or $\bar{u}' \equiv -1$, which are not possible since we must have $\bar{u}(0) = \bar{u}(1) = 0$ by the DIRICHLET BC. Thus F admits no minimizer. $\square$

NOTE In general if we define $X := \{ u \in C^1[0,1] \mid u(0) = \alpha, \, u(1) = \beta \}$ and

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 \, dx, \qquad u \in X.$$

then

- If $|\beta - \alpha| > 1$, then the unique minimizer of F is the straight line

$$\bar{u}(x) = (\beta - \alpha)x + \alpha$$

which can be shown as in EXAMPLE 5.6.

- If $|\beta - \alpha| \leq 1$ then F admits no minimizers and the infimum is 0. This can be shown by adapting the arguments of EXAMPLE 5.7.

SUMMARY OF INDIRECT METHOD

Given a minimization problem, the strategy is as follows:

1. Finding necessary conditions for minimality: ELE + BC
2. Solve ELE + BC (This is possible in very few cases: linear differential equations and not much more)

3. Prove that STATIONARY POINTS found in 2 are minimizers:

   → Using CONVEXITY

   → Using TRIVIAL LEMMA

6. L^p SPACES REVISION

REFERENCE W. RUDIN – "REAL AND COMPLEX ANALYSIS"
Mc GRAW – HILL, 2001

MEASURE THEORY

σ- Algebra Let Ω be a SET. Denote by $P(\Omega)$ the set of all subsets of Ω. A collection $\mathcal{A} \subseteq P(\Omega)$ is called a σ- ALGEBRA if

(1) $\phi \in \mathcal{A}$

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, where $A^c := \Omega \setminus A$

(3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n=1}^{+\infty} A_n \in \mathcal{A}$

The sets in $\mathcal{A}$ are called MEASURABLE.
The pair $(\Omega, \mathcal{A})$ is called MEASURE SPACE

NOTATION If $\mathcal{G} \subseteq P(\Omega)$ is a collection of sets, we denote by $\sigma(\mathcal{G})$ the smallest σ- algebra on Ω containing $\mathcal{G}$, that is,

$$\sigma(\mathcal{G}) := \bigcap \{\mathcal{A} \subseteq P(\Omega) \mid \mathcal{A} \text{ is } \sigma\text{-algebra}, \ \mathcal{G} \subseteq \mathcal{A}\}$$

BOREL SETS If τ is a topology over Ω, we call $\sigma(\tau)$ the BOREL σ-algebra. The elements of $\sigma(\tau)$ are called BOREL SETS

MEASURES A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a MEASURE if

(1) $\mu(\phi) = 0$

COUNTABLY ADDITIVE → (2) $\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} \mu(A_n)$ whenever $\{A_n\} \subseteq \mathcal{A}$ and they are pairwise disjoint, i.e., $A_i \cap A_j = \phi$ if $i \neq j$

The triple $(\Omega, \mathcal{A}, \mu)$ is called MEASURABLE SPACE

TERMINOLOGY

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space.

- μ is called COMPLETE if for all $E \in \mathcal{A}$ s.t. $\mu(E)=0$, then every $F \subseteq E$ satisfies $F \in \mathcal{A}$.

- μ is σ-FINITE if $\exists \{\Omega_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ s.t.

$$\Omega = \bigcup_{n=1}^{+\infty} \Omega_n \quad \text{and} \quad \mu(\Omega_n) < +\infty, \ \forall n \in \mathbb{N}.$$

- μ is FINITE if $\mu(\Omega) < +\infty$

- The sets $E \in \mathcal{A}$ s.t. $\mu(E)=0$ are called NULL SETS

- We say that a property holds μ-ALMOST EVERYWHERE in Ω (abbreviated in μ-a.e.) if $\exists$ $E \in \mathcal{A}$ s.t. $\mu(E)=0$ and the property holds for all $x \in \Omega \setminus E$.

OUTER MEASURES

Ω set. A set map $\mu^*: P(\Omega) \to [0,+\infty]$ is called OUTER MEASURE if

(a) $\mu^*(\emptyset)=0$

Monotonic → (b) $\mu^*(E) \leq \mu^*(F)$ for all $E \subseteq F \subseteq \Omega$

Sub-additive → (c) $\mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(E_n)$, for all $\{E_n\}_{n\in\mathbb{N}} \subseteq \Omega$

To construct an outer measure we usually start with a family $\mathcal{G} \subseteq P(\Omega)$ of elementary sets (e.g. cubes in $\mathbb{R}^d$), for which we have a desired notion of measure $\rho: \mathcal{G} \to [0,+\infty]$.

PROPOSITION 6.1 Let $\Omega \neq \emptyset$, $\mathcal{G} \subseteq \mathcal{P}(\Omega)$, $\rho : \mathcal{G} \to [0, +\infty]$. Assume that

- $\emptyset \in \mathcal{G}$ and $\rho(\emptyset) = 0$,
- $\exists \, \{\Omega_n\}_{n \in \mathbb{N}} \subseteq \mathcal{G}$ s.t. $\Omega = \bigcup_{n=1}^{+\infty} \Omega_n$

Define $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ by

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{+\infty} \rho(E_n) \;\middle|\; \{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{G}, \; E \subseteq \bigcup_{n=1}^{+\infty} E_n \right\}$$

Then μ^* is an OUTER MEASURE.

The problem with outer measures is that they are not additive on disjoint sets. To solve this problem, we restrict μ^* on a smaller collection of sets $\mathcal{A}^* \subseteq \mathcal{P}(\Omega)$:

μ^*-MEASURABLE SETS Given $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ outer measure, we say that $E \subseteq \Omega$ is **μ^*-MEASURABLE** if

$$\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c), \quad \forall F \subseteq \Omega$$

THEOREM 6.2 **(CARATHÉODORY)**

Let $\Omega \neq \emptyset$ and let $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ be an outer measure. Define

$$\mathcal{A}^* := \{ E \subseteq \Omega \mid E \text{ is } \mu^*\text{-measurable} \}.$$

Then $\mathcal{A}^*$ is a σ-algebra and $\mu^* : \mathcal{A}^* \to [0, +\infty]$ is a COMPLETE MEASURE.

THE LEBESGUE MEASURE

On $\mathbb{R}^d$ we can construct a particular measure called the LEBESGUE MEASURE.

For $x \in \mathbb{R}^d$, $r>0$, define $Q(x,r) := x + \left(-\frac{r}{2}, \frac{r}{2}\right)^d$, the CUBE of side length r centered at x. Introduce the collection of cubes $\mathcal{G} \subseteq \mathcal{P}(\mathbb{R}^d)$ as

$$\mathcal{G} := \{ Q(x,r) \mid x \in \mathbb{R}^d, r>0 \} \cup \{\emptyset\}$$

and $\rho : \mathcal{G} \to [0,+\infty)$ s.t. $\rho(\emptyset) := 0$ and $\rho(Q(x,r)) := r^d$. We can then define $\mathcal{L}_0^d : \mathcal{P}(\mathbb{R}^d) \to [0,+\infty]$ as

$$\mathcal{L}_0^d(E) := \inf \left\{ \sum_{i=1}^{+\infty} r_i^d \;\middle|\; E \subseteq \bigcup_{i=1}^{+\infty} Q(x_i, r_i) \right\}$$

$$= \inf \left\{ \sum_{i=1}^{+\infty} \rho(E_i) \;\middle|\; \{E_i\} \subseteq \mathcal{G},\ E \subseteq \bigcup_{i=1}^{+\infty} E_i \right\}$$

i.e., cover E with cubes and sum up the volumes (counting overlapping). Then take the smallest outcome.

By PROPOSITION 6.1 we have that $\mathcal{L}_0^d$ is an outer measure, called the LEBESGUE OUTER MEASURE. It can be shown that

- $\mathcal{L}_0^d(Q(x,r)) = r^d$
- $\mathcal{L}_0^d$ is TRANSLATION INVARIANT:

$$\mathcal{L}_0^d(x+E) = \mathcal{L}_0^d(E), \quad \forall x \in \mathbb{R}^d, E \subseteq \mathbb{R}^d$$

Define

$$\mathcal{L}^* := \{ E \subseteq \mathbb{R}^d \mid E \text{ is } \mathcal{L}_0^d\text{-measurable} \}$$

Then by THEOREM 6.2 we have that:

① $\mathcal{L}^*$ is a σ-algebra, called the σ-ALGEBRA of LEBESGUE MEASURABLE SETS

② $\mathcal{L}_0^d$ restricted to $\mathcal{L}^*$ is a COMPLETE MEASURE. We denote it by $\mathcal{L}^d$ and call it the d-DIMENSIONAL LEBESGUE MEASURE

Notice that $\mathcal{L}^d$ is <u>not</u> FINITE $(\mathcal{L}(\mathbb{R}^d) = +\infty)$ but it is σ-FINITE, since

$$\mathbb{R}^d = \bigcup_{n=1}^{+\infty} Q(0,n) \quad \text{and} \quad \mathcal{L}^d(Q(0,n)) = n^d < +\infty .$$

Moreover, if we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra of $\mathbb{R}^d$ WRT the Euclidean topology, we have

$$\boxed{\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{L}^*}$$

i.e., all Borel sets of $\mathbb{R}^d$ are Lebesgue measurable.

<u>WARNING</u> The inclusion $\mathcal{B}(\mathbb{R}^d) \subsetneq \mathcal{L}^*$ is STRICT : $\exists$ sets in $\mathcal{L}^*$ which is not Borel measurable. Thus $\mathcal{L}^d$ restricted to $\mathcal{B}(\mathbb{R}^d)$ is <u>not</u> COMPLETE.

<u>WARNING</u> There exist sets $E \subseteq \mathbb{R}^d$ which are <u>NOT</u> Lebesgue measurable.

INTEGRABILITY

On a measurable space $(\Omega, \mathcal{A}, \mu)$ we can define the notion of integrability.

MEASURABLE FUNCTIONS

Let X, Y be non-empty sets, $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras on X and Y respectively. A function $u : X \to Y$ is **MEASURABLE** if

$$u^{-1}(E) \in \mathcal{A} \quad \text{for all} \quad E \in \mathcal{B}.$$

If X, Y are topological spaces and $\mathcal{A}, \mathcal{B}$ are Borel σ-algebras then measurable functions are called **BOREL FUNCTIONS**.

REMARK 6.3

1. If $(X, \mathcal{A})$ is a measurable space and $u : X \to \mathbb{R}$ with $\mathbb{R}$ equipped with the Borel σ-algebra, then

$$u \text{ is measurable} \iff u^{-1}((a, +\infty)) \in \mathcal{A} \text{ for every } a \in \mathbb{R}.$$

If instead $u : X \to [-\infty, +\infty]$ (always with Borel σ-algebra) then

$$u \text{ is measurable} \iff u^{-1}((a, +\infty]) \in \mathcal{A} \text{ for every } a \in \mathbb{R}.$$

2. If X, Y are topological spaces equipped with Borel σ-algebras then

$$u : X \to Y \text{ CONTINUOUS} \Rightarrow u \text{ Borel}$$

3. The composition of measurable functions is measurable. In particular if $(X, \mathcal{A})$ is a measure space and $u : X \to \mathbb{R}$ is measurable, then u^p, $|u|$, cu and

$$u^+ := \begin{cases} u & \text{if } u(x) \geq 0 \\ 0 & \text{if } u(x) < 0 \end{cases}, \quad u^- := \begin{cases} -u & \text{if } u(x) \leq 0 \\ 0 & \text{if } u(x) > 0 \end{cases}$$

are all measurable, for $p \geq 1$, $c \in \mathbb{R}$.

④ Moreover if $v : X \to \mathbb{R}$ is measurable then $u+v$, uv, $\min\{u,v\}$, $\max\{u,v\}$ are measurable.

⑤ Let $(X, \mathcal{A})$ be a measurable space and $u_n : X \to [-\infty, +\infty]$ be measurable. Then the functions

$$\sup_{n \in \mathbb{N}} u_n \quad , \quad \inf_{n \in \mathbb{N}} u_n \quad , \quad \liminf_{n \to +\infty} u_n \quad , \quad \limsup_{n \to +\infty} u_n$$

are measurable.

⑥ Let $(X, \mathcal{A}, \mu)$ be a measurable space. Assume that μ is COMPLETE. If $u_n : X \to [-\infty, +\infty]$ are measurable and

$$u(x) := \lim_{n \to +\infty} u_n(x) \quad \text{exists for } \mu\text{-a.e. } x \in X$$

then u is measurable.

⑦ Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure. Suppose $u : X \to Y$ is measurable. If $v : X \to Y$ is s.t.

$$u(x) = v(x) \quad \text{for } \mu\text{-a.e. } x \in X$$

then v is also measurable.

We are now ready to introduce integrals. For a measurable space $(X, \mathcal{A})$ and $E \subseteq X$ we define the CHARACTERISTIC FUNCTION of E as

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that χ_E is measurable if $E \in \mathcal{A}$.

SIMPLE FUNCTIONS $(X, \mathcal{A})$ measurable space. A SIMPLE FUNCTION is a measurable map $s: X \to \mathbb{R}$ such that $s(X)$ is finite, i.e., there exist disjoint sets $E_1, .., E_N \in \mathcal{A}$, $N \in \mathbb{N}$ and $c_1, .., c_N \in \mathbb{R}$ distinct, s.t.

$$(*) \qquad s(x) = \sum_{i=1}^{N} c_i \chi_{E_i}(x) \quad , \quad \forall x \in X.$$

THEOREM 6.4 $(X, \mathcal{A})$ measure space, $u: X \to [0, +\infty]$ measurable. Then there exists a sequence $\{s_n\}$ of SIMPLE FUNCTIONS s.t. $0 \leq s_1 \leq s_2 \leq \dots$ and $s_n(x) \to u(x)$ for all $x \in X$.

LEBESGUE INTEGRAL Let $(X, \mathcal{A}, \mu)$ be a measurable space. The LEBESGUE INTEGRAL is defined in 3 steps:

1. Let $s \geq 0$ a step function of the form $(*)$. We define the LEBESGUE INTEGRAL of s on a set $E \in \mathcal{A}$ by

$$\boxed{\int_E s(x)\, d\mu(x) := \sum_{i=1}^{N} c_i\, \mu(E \cap E_i)}$$

where if $c_i = 0$ and $\mu(E \cap E_i) = +\infty$ we adopt the standard convention

$$c_i\, \mu(E \cap E_i) := 0.$$

2. Let $u: X \to [0, +\infty]$ be a measurable function (note that $u \geq 0$). The LEBESGUE INTEGRAL of u over a set $E \in \mathcal{A}$ is defined as

$$\boxed{\int_E u(x)\, d\mu(x) := \sup \left\{ \int_E s\, d\mu \;\middle|\; s \text{ simple},\ 0 \leq s \leq u \right\}}$$

(This is well posed thanks to THEOREM 6.4)

③ Let $u : X \to [-\infty, +\infty]$ be measurable. Note that $u = u^+ - u^-$ with $u^+, u^- \geq 0$. The LEBESGUE INTEGRAL of u over a set $E \in \mathcal{A}$ is defined

$$\boxed{\int_E u(x)\, d\mu := \int_E u^+ \, d\mu - \int_E u^- \, d\mu}$$

If $\int_E u^+ d\mu$ and $\int_E u^- d\mu$ are FINITE then u is said to be LEBESGUE INTEGRABLE WRT μ.

REMARK Let $(X, \mathcal{A}, \mu)$ be a measurable space. Let $u, v : X \to [-\infty, +\infty]$ be measurable.

① If $0 \leq u \leq v$ then $\int_E u \, d\mu \leq \int_E v \, d\mu$, $\forall E \in \mathcal{A}$

② If $c \in [0, +\infty]$, then $\int_E cu \, d\mu = c \int_E u \, d\mu$, $\forall E \in \mathcal{A}$ $\quad (0 \cdot (\pm\infty) := 0)$

③ Let $E \in \mathcal{A}$ and $u \geq 0$. Then $\int_E u \, d\mu = 0$ iff $u(x) = 0$ for μ-a.e. $x \in E$.

④ If $E \in \mathcal{A}$ and $\mu(E) = 0$ then $\int_E u \, d\mu = 0$

⑤ If $E \in \mathcal{A}$ then $\int_E u \, d\mu = \int_X \chi_E u \, d\mu$

⑥ u is LEBESGUE INTEGRABLE iff $\int_E |u| \, d\mu < +\infty$ for all $E \in \mathcal{A}$.

⑦ If u is LEBESGUE INTEGRABLE then

$$\mu(\{x \in X : |u(x)| = +\infty\}) = 0.$$

(8) If u, v are integrable and $\alpha, \beta \in \mathbb{R}$ then $\alpha u + \beta v$ is integrable, and

$$\int_X (\alpha u + \beta v)\, d\mu = \alpha \int_X u\, d\mu + \beta \int_X v\, d\mu .$$

(9) If u, v are integrable and $u = v$ μ-a.e. in X, then

$$\int_X u\, d\mu = \int_X v\, d\mu$$

(10) If u is integrable then

$$\left| \int_X u\, d\mu \right| \leq \int_X |u|\, d\mu$$

UNFORGETTABLE THEOREMS

We recall a few theorems concerning the Lebesgue integral:

THEOREM 6.5 (MONOTONE CONVERGENCE)

Let $(X, \mathcal{A}, \mu)$ be a measurable space, and $u_n : X \to [0, +\infty]$ s.t.

1. u_n is measurable $\forall n \in \mathbb{N}$
2. $0 \leq u_1(x) \leq u_2(x) \leq \dots$ for all $x \in X$
3. $u_n(x) \to u(x)$ as $n \to +\infty$ for all $x \in X$

Then

$$\lim_{n \to +\infty} \int_X u_n\, d\mu = \int_X u\, d\mu$$

THEOREM 6.6 (FATOU'S LEMMA)

Let $(X, \mathcal{A}, \mu)$ be a measurable space. If $u_n : X \to [0, +\infty]$ is a sequence of measurable functions, then

$$\int_X \liminf_{n \to +\infty} u_n(x)\, d\mu(x) \leq \liminf_{n \to +\infty} \int_X u_n(x)\, d\mu(x)$$

THEOREM 6.7 (DOMINATED CONVERGENCE)

Let $(X, \mathcal{A}, \mu)$ be a measurable space and $u_n : X \to [-\infty, +\infty]$ a sequence of measurable functions. Suppose that:

1. $u_n(x) \to u(x)$ as $n \to +\infty$, for μ-a.e. $x \in X$
2. $\exists\, v$ Lebesgue integrable such that

$$|u_n(x)| \leq v(x), \quad \forall n \in \mathbb{N} \text{ and } \mu\text{-a.e. } x \in X.$$

Then u is Lebesgue integrable and

$$\lim_{n \to +\infty} \int_X |u_n - u|\, d\mu = 0$$

THEOREM 6.8 (JENSEN'S INEQUALITY)

Let $(X, \mathcal{A}, \mu)$ be measurable space, with $\mu(X) = 1$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. For all $u : X \to \mathbb{R}$ integrable we have

$$\varphi\left(\int_X u\, d\mu \right) \leq \int_X \varphi \circ u\, d\mu .$$

Finally we recall FUBINI'S and TONELLI'S THEOREMS. We first need:

PRODUCT MEASURE Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces. On the cartesian product $X_1 \times X_2$ define the PRODUCT σ-ALGEBRA

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \bigcap \left\{ \mathcal{A} \subseteq P(X_1 \times X_2) \mid \mathcal{A} \text{ is a } \sigma\text{-algebra}, (E_1 \times E_2) \in \mathcal{A}, \ \forall E_i \in \mathcal{A}_i \right\}$$

Thus $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest σ-algebra on $X_1 \times X_2$ containing all the sets of the form $E_1 \times E_2$ with $E_i \in \mathcal{A}_i$. Whenever μ_1, μ_2 are σ-FINITE, there exists a unique measure $\mu_1 \otimes \mu_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0, +\infty]$ such that

$$(\mu_1 \otimes \mu_2)(E_1 \times E_2) = \mu_1(E_1)\, \mu_2(E_2), \qquad \forall E_i \in \mathcal{A}_i$$

(it can be constructed via PROP 6.1 and THM 6.2). The measure $\mu_1 \otimes \mu_2$ is called PRODUCT MEASURE between μ_1 and μ_2.

NOTE For the Lebesgue measure it holds that $\mathcal{L}^{d_1} \otimes \mathcal{L}^{d_2} = \mathcal{L}^{d_1+d_2}$.

THEOREM 6.9 (TONELLI)

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces, with μ_1, μ_2 σ-finite. Let $u : X_1 \times X_2 \to \mathbb{R}$ be measurable WRT $\mathcal{A}_1 \otimes \mathcal{A}_2$, and s.t.

(a) For μ_1-a.e. $x \in X_1$ the map $y \in X_2 \mapsto u(x,y)$ is measurable and it holds

$$\int_{X_2} |u(x,y)| \, d\mu_2(y) < +\infty \qquad \text{for } \mu_1\text{-a.e. } x \in X_1$$

(b) $\displaystyle \int_{X_1} \left(\int_{X_2} |u(x,y)| \, d\mu_2(y) \right) d\mu_1(x) < +\infty$

Then u is INTEGRABLE WRT the product measure $\mu_1 \otimes \mu_2$.

THEOREM 6.10 (FUBINI)

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces, with μ_1, μ_2 σ-finite. Let $u: X_1 \times X_2 \to [-\infty, +\infty]$ be measurable WRT $\mathcal{A}_1 \otimes \mathcal{A}_2$ and integrable WRT $\mu_1 \otimes \mu_2$. Then

① For μ_1-a.e. $x \in X_1$ the map $y \in X_2 \mapsto u(x,y)$ is measurable and

$$\int_{X_2} |u(x,y)| \, d\mu_2(y) < +\infty, \qquad \int_{X_1} \left\{ \int_{X_2} |u(x,y)| \, d\mu_2(y) \right\} d\mu_1(x) < +\infty$$

② For μ_2-a.e. $y \in X_2$ the map $x \in X_1 \mapsto u(x,y)$ is measurable and

$$\int_{X_1} |u(x,y)| \, d\mu_1(x) < +\infty, \qquad \int_{X_2} \left\{ \int_{X_1} |u(x,y)| \, d\mu_1(x) \right\} d\mu_2(y) < +\infty$$

③ The so-called FUBINI'S FORMULA holds:

$$\int_{X_1 \times X_2} |u(x,y)| \, d(\mu_1 \otimes \mu_2)(x,y) = \int_{X_1} \left\{ \int_{X_2} |u(x,y)| \, d\mu_2(y) \right\} d\mu_1(x)$$

$$= \int_{X_2} \left\{ \int_{X_1} |u(x,y)| \, d\mu_1(x) \right\} d\mu_2(y)$$

L^p SPACES

Let $(X, \mathcal{A}, \mu)$ be a measurable space. For $p \geq 1$ we set

$$L^p(X, \mu) := \left\{ u : X \to \mathbb{R} \;\middle|\; u \text{ measurable}, \int_X |u|^p \, d\mu < +\infty \right\}$$

In other words, $u \in L^p(X, \mu)$ iff u is μ-INTEGRABLE.

For the case $p = +\infty$ we have an ad-hoc definition

$$L^\infty(X,\mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \ \exists\, C > 0 \text{ s.t. } |u(x)| \le C \ \ \mu\text{-a.e. in } X \}$$

The condition $|u(x)| \le C$ for μ-a.e. $x \in X$ is called ESSENTIAL BOUNDEDNESS.

WARNING The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are NOT functions, but classes of equivalence of functions, where the equivalence is

$$u \sim v \iff u(x) = v(x) \text{ for } \mu\text{-a.e. } x \in X$$

This is not an issue, since in this case $\int_X u \, d\mu = \int_X v \, d\mu$. Therefore $L^p(X,\mu)$ and $L^\infty(X,\mu)$ have to be understood as QUOTIENT SPACES WRT $\sim$

THEOREM 6.11 Let $1 \le p \le +\infty$ and define the CONJUGATE EXPONENT $p' := \frac{p}{p-1}$. If $u \in L^p(X,\mu)$, $v \in L^{p'}(X,\mu)$ then

$$uv \in L^1(X,\mu) \quad \text{and} \quad \underbrace{\|uv\|_1 \le \|u\|_p \, \|v\|_{p'}}_{\text{HÖLDER'S INEQUALITY}}$$

THEOREM 6.12 $L^p(X,\mu)$, $L^\infty(X,\mu)$ are Banach spaces with the norms

$$\|u\|_p := \left(\int_X |u|^p \, d\mu \right)^{1/p}, \quad u \in L^p(X,\mu)$$

$$\|u\|_\infty := \inf \{ C : |u(x)| \le C \ \ \mu\text{-a.e. in } X \}, \quad u \in L^\infty(X,\mu)$$

Moreover $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_X uv \, d\mu, \quad u, v \in L^2(X,\mu)$$

A standard corollary of the proof of THEOREM 6.12 is the following.

PROPOSITION 6.13 Let $\{u_n\} \subseteq L^p(X,\mu)$ and suppose $u_n \to u$ strongly. Then there exist a subsequence u_{n_k} and $h \in L^p(X,\mu)$ s.t.

(a) $u_{n_k}(x) \to u(x)$ as $k \to +\infty$ for μ-a.e. $x \in X$

(b) $\sup_k |u_{n_k}(x)| \leq h(x)$ for μ-a.e. $x \in X$

THEOREM 6.14 (DUALITY)

Let $1 < p < +\infty$. Then $L^p(X,\mu)^* \cong L^{p'}(X,\mu)$, with isometry

$$L^{p'}(X,\mu) \longrightarrow L^p(X,\mu)^*$$
$$u \longmapsto \left(v \mapsto \int_X u v \, d\mu \right)$$

In particular, as $(p')' = p$, we have that $L^p(X,\mu)$ is REFLEXIVE.
Also $L^1(X,\mu)^* \cong L^\infty(X,\mu)$.

WARNING It is NOT TRUE that $L^\infty(X,\mu)^* \cong L^1(X,\mu)$.

We now recall a result about SEPARABILITY of L^p spaces. We need first the following definition

SEPARABLE MEASURE SPACE

Let $(X,\mathcal{A})$ be a SEPARABLE measure space, i.e., $\exists \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ s.t. $\sigma(\{E_n\}) = \mathcal{A}$, where

$$\sigma(\{E_n\}) := \bigcap \{ \mathcal{M} \mid \mathcal{M} \text{ is } \sigma\text{-algebra on } X,\ \{E_n\} \subseteq \mathcal{M} \},$$

i.e., $\sigma(\{E_n\})$ is the smallest σ-algebra on X which contains $\{E_n\}$.

EXAMPLE

- $\mathbb{R}^d$ is separable with the Borel σ-algebra.
- $(\mathbb{R}^d, \mathcal{L}^*)$ is separable, where $\mathcal{L}^*$ is the σ-algebra of Lebesgue measurable sets
- (X, d) separable metric space, τ_d topology induced by d. Then $(X, \sigma(\tau_d))$ is a separable measure space.

THEOREM 6.15 (SEPARABILITY) Let $(X, \mathcal{A}, \mu)$ be a SEPARABLE measure space. Then $L^p(X, \mu)$ equipped with the standard norm is SEPARABLE, for all $1 \leq p < +\infty$.
The space $L^\infty(X, \mu)$ is in general NOT separable.

We summarize the above results in a table

	REFLEXIVE	SEPARABLE	DUAL SPACE
L^p with $1<p<+\infty$	YES	YES	$L^{p'}$
L^1	NO	YES	L^∞
L^∞	NO	NO	Strictly bigger than L^1

Finally we conclude with a useful density result:

THEOREM 6.16 Consider $(\mathbb{R}^d, \mathcal{L}^*, \mathcal{L}^d)$, where $\mathcal{L}^*$ is the LEBESGUE σ-algebra and $\mathcal{L}^d$ is the d-dimensional LEBESGUE MEASURE. Let $1 \leq p < +\infty$. Then $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, i.e.,

$$\forall u \in L^p(\mathbb{R}^d),\ \forall \varepsilon > 0,\ \exists v \in C_c(\mathbb{R}^d) \text{ s.t. } \|u - v\|_p \leq \varepsilon.$$

STRONG COMPACTNESS IN L^p

We conclude with a STRONG COMPACTNESS criterion for L^p spaces. To this end, given $f: \mathbb{R}^d \to \mathbb{R}$, $h \in \mathbb{R}^d$, we define the SHIFT of f by h as the function $\tau_h f: \mathbb{R}^d \to \mathbb{R}$

$$(\tau_h f)(x) := f(x+\tau), \qquad \forall x \in \mathbb{R}^d.$$

THEOREM 6.17 (FRÉCHET - KOLMOGOROV)

Let $1 \le p < +\infty$ and $A \subseteq L^p(\mathbb{R}^d)$. For a measurable set $\Omega \subseteq \mathbb{R}^d$ with finite measure, we denote by $A_{|\Omega}$ the restrictions to Ω of the functions in A, i.e.,

$$A_{|\Omega} = \{ v: \Omega \to \mathbb{R} \mid \exists u \in A \text{ s.t. } v = u_{|\Omega} \}.$$

Assume that

1. A is **BOUNDED**: i.e., $\exists M > 0$ s.t. $\|u\|_{L^p(\mathbb{R}^d)} \le M, \quad \forall u \in A$

2. A is **EQUI-INTEGRABLE**: i.e.,

$$\lim_{|h| \to 0} \left\{ \sup_{u \in A} \| \tau_h u - u \|_{L^p(\mathbb{R}^d)} \right\} = 0.$$

Then the closure of $A_{|\Omega}$ in $L^p(\Omega)$ is COMPACT, i.e., if $\{u_n\} \subseteq \overline{A_{|\Omega}}$, $\exists\, \bar{u} \in \overline{A_{|\Omega}}$ and a subsequence n_k such that

$$u_{n_k} \to \bar{u} \quad \text{as } k \to +\infty \quad \text{strongly in } L^p(\Omega)$$

OTHER MEASURE THEORETIC RESULTS

THEOREM 6.18 (ABSOLUTE CONTINUITY OF LEBESGUE INTEGRAL)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in L^1(X;\mu)$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\mu(E) < \delta \quad \Rightarrow \quad \left| \int_E u \, d\mu \right| < \varepsilon .$$

THEOREM 6.19 (EGOROFF)

Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) < +\infty$.
Suppose $f_n : X \to \mathbb{R}$ is a sequence s.t.

$$f_n \to f \quad \text{a.e. in } X$$

Then $\forall \varepsilon > 0$, $\exists$ $E_\varepsilon \in \mathcal{A}$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E_\varepsilon$, i.e.

$$\lim_{n \to +\infty} \sup_{x \in X \setminus E_\varepsilon} |f_n(x) - f(x)| = 0 .$$

THEOREM 6.20 (LUSIN)

Let $\Omega \subseteq \mathbb{R}^d$ with $|\Omega| < +\infty$. Let $u : \Omega \to \mathbb{R}$ be Lebesgue measurable.
Then $\forall \varepsilon > 0$, $\exists K \subseteq \Omega$ compact such that

$$|\Omega \setminus K| < \varepsilon \quad \text{and} \quad u_{|K} \text{ is CONTINUOUS.}$$

7. SOBOLEV SPACES

REFERENCE:
GIOVANNI LEONI – "A FIRST COURSE IN SOBOLEV SPACES"
AMERICAN MATHEMATICAL SOCIETY, 2017

So far our minimization problems were mostly set in C^1. Often a solution did not exist in C^1, however in many examples we saw that we could find an infimizing sequence converging to something piecewise C^1, i.e.,

$$C^1_{pw}[a,b] := \{ u \in C[a,b] \mid \exists \{x_0, \dots, x_N\} \subseteq [a,b] \text{ with } a = x_0 < x_1 < \dots < x_N = b, \; u \in C^1[x_i, x_{i+1}], \; i = 0, \dots, N-1 \}$$

However these functional spaces are not very convenient to work with, due to their lack of completeness WRT weaker norms (e.g. the L^p convergence). The default functional spaces for setting variational problems are (nowadays and in the past 60-70 years) SOBOLEV SPACES.

In order to define Sobolev spaces, we rely on previous knowledge about L^p spaces (LEBESGUE SPACES). A self-contained summary of definitions and properties can be found in SECTION 6 of these notes. (L^p SPACES REVISION)

Here we just recall the definition of L^p spaces, to establish some notation.

L^p SPACES Let $(X, \mathcal{A}, \mu)$ be a measurable space, where X set, $\mathcal{A}$ is σ-algebra over X and $\mu : \mathcal{A} \to [0, +\infty]$ is a measure.
For $1 \leq p < +\infty$ and $p = +\infty$ we set, respectively:

$$L^p(X, \mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \int_X |u|^p \, d\mu < +\infty \}$$

$$L^\infty(X, \mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \exists C > 0 \text{ s.t. } |u(x)| \leq C \;\; \mu\text{-a.e. in } X \}$$

When we say μ-a.e. we mean that a certain property holds in $X \setminus E$, where $\mu(E) = 0$.

WARNING The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are <u>NOT</u> functions, but classes of equivalence of functions, where the equivalence is

$$u \sim v \iff u(x) = v(x) \quad \text{for } \mu\text{- a.e. } x \in X$$

This is not an issue, since in this case $\int_X u \, d\mu = \int_X v \, d\mu$. Therefore $L^p(X,\mu)$ and $L^\infty(X,\mu)$ have to be understood as QUOTIENT SPACES WRT $\sim$

RECALL $L^p(X,\mu)$, $L^\infty(X,\mu)$ are Banach spaces with the norms

$$\|u\|_p := \left(\int_X |u|^p \, d\mu \right)^{1/p} \quad , \quad u \in L^p(X,\mu) \quad , \quad 1 \leq p < +\infty ,$$

$$\|u\|_\infty := \inf \{ C : |u(x)| \leq C \ \ \mu\text{-a.e. in } X \} \quad , \quad u \in L^\infty(X,\mu)$$

Moreover $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_X u v \, d\mu \quad , \quad u, v \in L^2(X,\mu)$$

NOTE In the following the definition of L^p will be employed in this setting:

- X will always be an OPEN SET OF $\mathbb{R}^d$
- $\mathcal{A}$ is the d-dimensional LEBESGUE σ- Algebra
- $\mu = dx = \mathcal{L}^d$ the d - dimensional LEBESGUE MEASURE

Thus we will always write $L^p(X)$ in place of $L^p(X,\mu)$, as there is no ambiguity.

We need to introduce versions of the FLCV and DBR Lemma for L^p functions.
For that we need tools to smoothen functions, i.e., convolutions.

CONVOLUTIONS

DEFINITION 7.1 Let $u, v : \mathbb{R} \to \mathbb{R}$. The **CONVOLUTION** between u and v, is defined as

$$\boxed{(u * v)(x) := \int_{\mathbb{R}} u(x-y)\, v(y)\, dy}$$

for all $x \in \mathbb{R}$ s.t. the RHS is FINITE.

REMARK It is immediate to check that, whenever the convolution is finite,

$$u * v = v * u \qquad \text{and} \qquad u * (v * w) = (u * v) * w$$

for $u, v, w : \mathbb{R} \to \mathbb{R}$.

The following Theorem gives a sufficient condition for $u * v$ to be well-defined.

THEOREM 7.2 (YOUNG)

Let $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Then for a.e. $x \in \mathbb{R}$ the map $y \mapsto u(x-y)\, v(y)$ is integrable, so that $u * v$ is finite. Moreover $u * v \in L^p(\mathbb{R})$, with

$$(*) \qquad \| u * v \|_p \leq \| u \|_1 \, \| v \|_p .$$

Proof
- $p = +\infty$: this is immediate, since for a.e. $x \in \mathbb{R}$

$$|(u * v)(x)| \leq \int_{\mathbb{R}} |u(x-y)|\, |v(y)|\, dy \leq \|v\|_\infty \int_{\mathbb{R}} |u(x-y)|\, dy = \|v\|_\infty \, \|u\|_1 .$$

Taking the essential supremum in the above inequality we obtain $(*)$

• $p=1$: Set $\psi(x,y) := u(x-y)v(y)$. For a.e. $y \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} |\psi(x,y)|\, dx = |v(y)| \int_{\mathbb{R}} |u(x-y)|\, dx = |v(y)|\, \|u\|_1 < +\infty$$

Integrating WRT x we get

$$(**) \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dx \right\} dy = \|v\|_1 \|u\|_1 < +\infty$$

Then ψ satisfies the assumptions of TONELLI'S THEOREM (THEOREM 6.9) and we infer $\psi \in L^1(\mathbb{R}\times\mathbb{R})$ (where $\mathbb{R}\times\mathbb{R}$ is equipped with the 2-dimensional Lebesgue measure). We can then apply FUBINI'S THEOREM (THEOREM 6.10) to get that

$$\int_{\mathbb{R}} |\psi(x,y)|\, dy < +\infty \quad \text{for a.e. } x \in \mathbb{R}.$$

and also

$$(***) \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dy \right\} dx \overset{\text{FUBINI}}{=} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dx \right\} dy \overset{(**)}{=} \|v\|_1 \|u\|_1$$

Therefore

$$\int_{\mathbb{R}} |(u*v)(x)|\, dx \overset{\text{def. of convolution}}{=} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} u(x-y)v(y)\, dy \right| dx \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |u(x-y)v(y)|\, dy \right\} dx$$

$$\overset{\text{def of } \psi}{=} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dy \right\} dx \overset{(***)}{=} \|u\|_1 \|v\|_1 ,$$

which is exactly $(*)$.

- $1 < p < +\infty$: The functions $|u|, |v|^p \in L^1(\mathbb{R})$. Thus, from the case $p=1$, we know that $y \mapsto |u(x-y)|\,|v(y)|^p$ belongs to $L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$. In particular

$$|u(x-\cdot)|^{1/p}\,|v(\cdot)| \in L^p(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}.$$

Moreover, as $u \in L^1(\mathbb{R})$, we also have

$$|u(x-\cdot)|^{1/p'} \in L^{p'}(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}$$

where we chose p' as the HÖLDER CONJUGATE, i.e.

$$p' := \frac{p}{p-1}, \quad \text{so that} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

From HÖLDER INEQUALITY (THEOREM 6.11) we get, for a.e. $x \in \mathbb{R}$,

$$|(u * v)(x)| \leq \int_{\mathbb{R}} |u(x-y)|\,|v(y)|\,dy$$

$$= \int_{\mathbb{R}} \underbrace{|u(x-y)|^{1/p'}}_{\in L^{p'}} \underbrace{|u(x-y)|^{1/p}\,|v(y)|}_{\in L^p}\,dy$$

$$\text{(HÖLDER)} \leq \left(\int_{\mathbb{R}} |u(x-y)|\,dy \right)^{1/p'} \left(\int_{\mathbb{R}} |u(x-y)|\,|v(y)|^p\,dy \right)^{1/p}$$

$$= \|u\|_1^{1/p'} \cdot \left[(|u| * |v|^p)(x) \right]^{1/p}$$

Taking the p-power of the above we get

$$(**) \quad |(u*v)(x)|^p \leq \|u\|_1^{p/p'} \, (|u| * |v|^p)(x) \qquad \text{for a.e. } x \in \mathbb{R}$$

Now, as $|u|, |v|^p \in L^1(\mathbb{R})$, we can apply $(*)$ for the case $p=1$ to get:

$$(***) \quad \| \, |u| * |v|^p \, \|_1 \leq \|u\|_1 \, \| \, |v|^p \, \|_1 = \|u\|_1 \, \|v\|_p^p$$

By integrating $(**)$:

$$\int_{\mathbb{R}} |(u*v)(x)|^p \, dx \overset{(**)}{\leq} \|u\|_1^{p/p'} \int_{\mathbb{R}} |(|u| * |v|^p)(x)| \, dx$$

and so

$$= \|u\|_1^{p/p'} \, \| \, |u| * |v|^p \, \|_1$$

$$\overset{(***)}{\leq} \|u\|_1^{p/p'} \, \|u\|_1 \, \|v\|_p^p$$

As $\frac{p}{p'} + 1 = p$ $\longrightarrow$ $= \|u\|_1^p \, \|v\|_p^p$.

Taking the $\frac{1}{p}$-power of the above inequality yields $(*)$. ◻

We now need the notion of SUPPORT for L^p functions. Indeed, as elements of L^p are actually equivalence classes, and thus defined a.e., the definition of support we used for continuous functions makes no sense:

EXAMPLE $u := \chi_{\mathbb{Q}}$. As the Lebesgue measure of $\mathbb{Q}$ is zero, u belongs to the same equivalence class of $v \equiv 0$. Using the classical definition of support we get

$$\operatorname{supp} u = \overline{\{x \in \mathbb{R} \mid u(x) \neq 0\}} = \overline{\mathbb{Q}} = \mathbb{R} \quad , \text{ while } \quad \operatorname{supp} v = \emptyset .$$

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DEFINITION 7.3 (SUPPORT)

Let $u : \mathbb{R}^d \to \mathbb{R}$. Let $\{w_i\}_{i\in I}$ be the family of all open sets in $\mathbb{R}^d$ s.t.

$$u = 0 \text{ a.e. on } w_i, \quad \forall i \in I.$$

We define the support of u as

$$\operatorname{Supp} u := \mathbb{R}^d \setminus \bigcup_{i\in I} w_i .$$

REMARK The above definition makes sense, since it is possible to show that:

1. $u = 0$ a.e. on $\bigcup_{i\in I} w_i$
2. If $u_1 = u_2$ a.e. on $\mathbb{R}^d$, then $\operatorname{supp} u_1 = \operatorname{supp} u_2$.
3. If u is continuous, then DEFINITION 7.3 coincides with the classical one, i.e.,

$$\operatorname{Supp} u = \mathbb{R}^d \setminus \bigcup_{i\in I} w_i = \overline{\{x \in \mathbb{R} \mid u(x) \neq 0\}}$$

EXAMPLE Again consider $u := \chi_{\mathbb{Q}}$. As $\mathcal{L}(\mathbb{Q}) = 0$, we know that $u = 0$ a.e. on $\mathbb{R}$. Therefore $u = 0$ a.e. on w, for all $w \subseteq \mathbb{R}$ open. It follows that $\operatorname{Supp} u = \mathbb{R} \setminus \bigcup_{i\in I} w_i = \mathbb{R} \setminus \mathbb{R} = \emptyset$. This coincides with $\operatorname{supp} v$, where $v \equiv 0$.

The support of a convolution can be estimated in the following way:

PROPOSITION 7.4 Let $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Then

$$(*) \quad \operatorname{Supp}(u * v) \subset \overline{\operatorname{Supp} u + \operatorname{supp} v}$$

The sum in $(*)$ is defined as $E + F := \{x + y,\ x \in E, y \in F\}$, where $E, F \subseteq \mathbb{R}$ subsets.

(The proof of PROPOSITION 7.4 will be left as an Exercise in the EX. COURSE)

The main point of introducing convolutions is that they have a smoothing effect. To make this statement rigorous we need the definition of LOCAL INTEGRABILITY.

DEFINITION 7.5 Let $\Omega \subset \mathbb{R}^d$ be open. Let $1 \le p \le +\infty$. We say that $u : \Omega \to \mathbb{R}$ is LOCALLY INTEGRABLE on Ω, if

$$u \chi_K \in L^p(\Omega) \quad \text{for all} \quad K \subset \Omega,\ K \text{ compact}.$$

The space of locally integrable functions on Ω is denoted by $L^p_{loc}(\Omega)$

REMARK We have that $L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$, for all $\Omega \subseteq \mathbb{R}^d$ open

(This is not true for $L^p(\Omega)$ with Ω unbounded)

THEOREM 7.6 (Smoothing via convolutions)

(a) Let $u \in C_c(\mathbb{R})$, $v \in L^1_{loc}(\mathbb{R})$. Then $(u * v)(x)$ is well-defined $\forall x \in \mathbb{R}$ and

$$u * v \in C(\mathbb{R})$$

(b) Let $k \geq 1$, $u \in C^k_c(\mathbb{R})$, $v \in L^1_{loc}(\mathbb{R})$. Then $(u * v) \in C^k(\mathbb{R})$ and

$$\frac{d^k}{dx^k}[u * v] = u^{(k)} * v$$

In particular, if $u \in C^\infty_c(\mathbb{R})$ and $v \in L^1_{loc}(\mathbb{R})$, then $(u * v) \in C^\infty(\mathbb{R})$.

(The proof of this Theorem will be left as an ex. for the Exercises Course)

DEFINITION 7.7 (MOLLIFIERS) A sequence of MOLLIFIERS is any sequence $\rho_n : \mathbb{R} \to \mathbb{R}$ s.t.

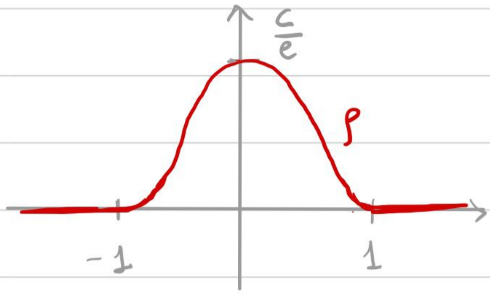
$$\rho_n \in C^\infty_c(\mathbb{R}), \quad \rho_n \geq 0, \quad \operatorname{supp} \rho_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right], \quad \int_{\mathbb{R}} \rho_n(x)\,dx = 1, \quad \forall n \in \mathbb{N}.$$

The most commonly used sequence of mollifiers is defined as follows:

EXAMPLE **(STANDARD MOLLIFIERS)**

Let $\rho : \mathbb{R} \to \mathbb{R}$ be as in REMARK 3.2, i.e.,

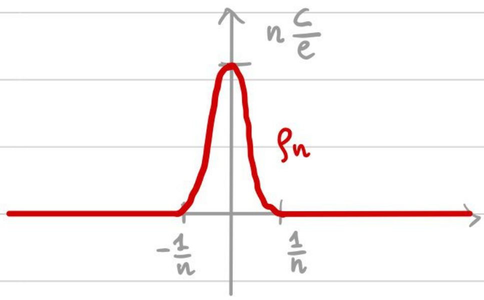
$$\rho(x) := \begin{cases} C \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



where $C \in \mathbb{R}$ is $C := \left(\int_{\mathbb{R}} \rho(x)\,dx \right)^{-1}$. Then $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$, $\operatorname{supp} \rho \subset [-1,1]$, $\int_{\mathbb{R}} \rho = 1$.

In particular $\rho_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\boxed{\rho_n(x) := n\,\rho(nx)}$$



is a sequence of mollifiers.

PROPOSITION 7.8 Let $u \in C(\mathbb{R})$ and $\{\rho_n\}$ be a sequence of mollifiers. Then $\rho_n * u \to u$ uniformly on compact sets, i.e. for each $K \subset \mathbb{R}$ compact we have

$$\lim_{n \to +\infty} \max_{x \in K} |(\rho_n * u)(x) - u(x)| = 0.$$

(Also the proof of this is left as exercise for the EX. COURSE)

THEOREM 7.9 Let $1 \leq p < +\infty$, $u \in L^p(\mathbb{R})$, $\{\rho_n\}$ a sequence of mollifiers. Then

$$\rho_n * u \to u \quad \text{STRONGLY in } L^p(\mathbb{R}).$$

<u>Proof</u> Let $\varepsilon > 0$. By THEOREM 6.16 $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$. Thus $\exists\, \tilde{u} \in C_c(\mathbb{R})$ such that

$$(*) \qquad \| u - \tilde{u} \|_p < \varepsilon .$$

By PROPOSITION 7.8 we have that $\rho_n * \tilde{u} \to \tilde{u}$ uniformly on compact sets. Moreover PROPOSITION 7.4 says that

$$\operatorname{supp}(\rho_n * \tilde{u}) \subset \overline{\operatorname{supp}\rho_n + \operatorname{supp}\tilde{u}} \overset{\text{by def of } \rho_n}{\subseteq} \left[-\tfrac{1}{n}, \tfrac{1}{n}\right] + \operatorname{supp}\tilde{u} \subseteq [-1,1] + \operatorname{supp}\tilde{u}$$

In particular

$$\begin{aligned}
\operatorname{supp}(\rho_n * \tilde{u} - \tilde{u}) &\subset \left(\operatorname{supp}(\rho_n * \tilde{u}) \cup \operatorname{supp}\tilde{u}\right) \\
(**) \qquad &\subset \left(\{[-1,1] + \operatorname{supp}\tilde{u}\} \cup \operatorname{supp}\tilde{u} \right) \\
&= [-1,1] + \operatorname{supp}\tilde{u} =: K
\end{aligned}$$

and K is compact, as $\tilde{u}$ is compactly supported. Then

$$\begin{aligned}
\int_{\mathbb{R}} |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \, dx &\overset{(**)}{=} \int_K |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \, dx \\
&\leq \max_{x \in K} |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \cdot |K|
\end{aligned}$$

and the RHS goes to zero as $n \to +\infty$, since $\rho_n * \tilde{u} \to \tilde{u}$ uniformly on compact sets, and $|K| < +\infty$ being K bounded (as it is compact). Thus

$$(***) \qquad \rho_n * \tilde{u} \to \tilde{u} \quad \text{STRONGLY in } L^p(\mathbb{R})$$

Now notice that $\rho_n \in L^1(\mathbb{R})$ with $\|\rho_n\|_1 = 1$ by definition. Moreover $(u - \tilde{u}) \in L^p(\mathbb{R})$. Therefore by YOUNG INEQUALITY (THEOREM 7.2)

$$\text{(Y)} \qquad \|\rho_n * (u - \tilde{u})\|_p \leq \|\rho_n\|_1 \, \|u - \tilde{u}\|_p = \|u - \tilde{u}\|_p$$

Finally, by adding and subtracting we can estimate

$$\|\rho_n * u - u\|_p \leq \|\rho_n * (u - \tilde{u})\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p + \|u - \tilde{u}\|_p$$

$$\overset{\text{(Y)}}{\leq} \|u - \tilde{u}\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p + \|u - \tilde{u}\|_p$$

$$= 2\|u - \tilde{u}\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p$$

Recalling that $\|u - \tilde{u}\|_p < \varepsilon$ and $\|\rho_n * \tilde{u} - \tilde{u}\|_p \to 0$ by (***), we get

$$0 \leq \limsup_{n \to +\infty} \|\rho_n * u - u\|_p \leq 2\varepsilon$$

As $\varepsilon > 0$ was arbitrary, we conclude $\|\rho_n * u - u\|_p \to 0$. $\square$

COROLLARY 7.10 Let $I \subset \mathbb{R}$ be open, $1 \leq p < +\infty$. Then $C_c^\infty(I)$ is dense in $L^p(I)$.

FLCV and DBR in L^p

We are now ready to prove L^p versions of the FLCV and DBR Lemma.

LEMMA 7.11 (FLCV in L^p)

Let $I \subset \mathbb{R}$ be open. Suppose $u \in L^1_{loc}(I)$ is s.t.

$$\int_I u v \, dx = 0, \qquad \forall v \in C_c^\infty(I).$$

Then $u = 0$ a.e. on I.

Proof Let $\psi \in L^\infty(\mathbb{R})$ be s.t. $\operatorname{supp} \psi$ is compact and contained in I. Let $\psi_n := \rho_n * \psi$, with ρ_n mollifier. Then

$$\operatorname{supp} \psi_n \subset \overline{\operatorname{supp} \rho_n + \operatorname{supp} \psi} \overset{\text{by def of } \rho_n}{=} \overline{\left[-\frac{1}{n}, \frac{1}{n}\right] + \operatorname{supp} \psi}$$

by PROPOSITION 7.4. Then there $\exists\, N \in \mathbb{N}$ s.t. $\operatorname{supp} \psi_n \subset I$ for all $n \geq N$. Moreover $\psi_n \in C^\infty(\mathbb{R})$ by THEOREM 7.6. Thus $\psi_n \in C_c^\infty(I)$ for all $n \geq N$. By assumption we get

$$(*) \qquad \int_I u \psi_n \, dx = 0, \qquad \forall n \geq N.$$

Notice that $\psi \in L^1(\mathbb{R})$, being compactly supported and in $L^\infty(\mathbb{R})$. Then by THEOREM 7.9 we have $\psi_n \to \psi$ strongly in $L^1(\mathbb{R})$. Therefore, up to subsequencies (not relabelled), we have $\psi_n \to \psi$ pointwise a.e. in $\mathbb{R}$ (see PROPOSITION 6.13). Also, by YOUNG'S INEQUALITY, we get $\|\psi_n\|_{L^\infty(\mathbb{R})} \leq \|\rho_n\|_{L^1(\mathbb{R})} \|\psi\|_{L^\infty(\mathbb{R})} = \|\psi\|_{L^\infty(\mathbb{R})}$, as $\|\rho_n\|_{L^1(\mathbb{R})} = 1$ by definition. Therefore

$$u \psi_n \to u \psi \quad \text{a.e. in } I, \qquad |u \psi_n| \leq \|\psi\|_{L^\infty(\mathbb{R})} |u| \in L^1(I)$$

Therefore we can invoke DOMINATED CONVERGENCE (THEOREM 6.7) to conclude

$$\int_I u \psi_n \, dx \to \int_I u \psi \, dx \qquad \text{as } n \to +\infty.$$

However $\int_I u \psi_n \, dx = 0$ for n sufficiently large, by (*), and so

$$\int_I u \psi \, dx = 0 .$$

Therefore we have proven that

(**) $\int_I u \psi \, dx = 0$, $\forall \psi \in L^\infty(I)$ s.t. $\operatorname{supp}\psi$ is compact, $\operatorname{supp}\psi \subset I$

To obtain our thesis we now choose a function ψ in (**) in a clever way:
Let $K \subset I$ be compact and define

$$\tilde{\psi}(x) := \begin{cases} \operatorname{sign} u(x) & \text{if } x \in K \\ 0 & \text{if } x \in \mathbb{R} \setminus K \end{cases} \qquad \left(\operatorname{sign} a = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases} \right)$$

Therefore $\tilde{\psi} \in L^\infty(\mathbb{R})$ and $\operatorname{supp}\tilde{\psi} \subset K \subset I$. Then, from (**),

$$0 \overset{(**)}{=} \int_I u \tilde{\psi} \, dx = \int_K u \operatorname{sign} u \, dx = \int_K |u| \, dx .$$

Then $u = 0$ a.e. on K by the properties of the Lebesgue integral (see the REMARK after THEOREM 6.4). As K is arbitrary, we conclude $u = 0$ a.e. on I. □

LEMMA 7.12 (DBR in L^p)

Let $I = (a,b)$, possibly unbounded. Let $u \in L^1_{loc}(I)$ be such that

$$\int_I u v \, dx = 0 , \quad \forall v \in C_c(I) \text{ such that } \int_I v \, dx = 0.$$

Then $u = c$ a.e. on I, for some constant $c \in \mathbb{R}$.

Proof Let $\psi \in C_c(I)$ s.t. $\int_I \psi \, dx = 1$ (e.g. the bump function of REMARK 3.2, suitably rescaled).
Let $w \in C_c(I)$ be arbitrary and set

$$h(x) := w(x) - \left(\int_I w \, dx \right) \psi(x)$$

Then $h \in C_c(I)$, since $w, \psi \in C_c(I)$. Also $\int_I h \, dx = 0$ as $\int_I \psi \, dx = 1$. Then by assumption $\int_I u h \, dx = 0$. Thus

$$0 = \int_I u h \, dx = \int_I u w \, dx - \int_I w \, dx \cdot \int_I u \psi \, dx = \int_I (u - c) w \, dx \Rightarrow \int_I (u-c) w \, dx = 0$$

with $c := \int_I u \psi \, dx$. As c does not depend on w, and $w \in C_c(I)$ is arbitrary, by FLCV LEMMA 7.11 we conclude $u = c$ a.e. on I. $\square$

Similarly to the regular DBR Lemma, we have an alternative version:

LEMMA 7.13 (DBR in L^p - Alternative version)

Let $I = (a,b)$, possibly unbounded. Let $u \in L^1_{loc}(I)$ be such that

$$\int_I u \dot{v} \, dx = 0, \quad \forall v \in C^1_c(I).$$

Then $u = c$ a.e. on I, for some constant $c \in \mathbb{R}$.

Proof Let $w \in C_c(I)$ be such that $\int_I w \, dx = 0$. Set $W(x) := \int_a^x w(t) \, dt$. Then $W \in C^1(I)$ by the Fundamental Theorem of Calculus and $\dot{W}(x) = w(x)$. Moreover W is compactly supported in I, since w is compactly supported in I and $\int_I w \, dx = 0$. Thus $W \in C^1_c(I)$ and by assumption we get $\int_I u \dot{W} \, dx = 0$. As $\dot{W} = w$, we get

$$\int_I u w \, dx = 0, \quad \forall w \in C_c(I) \text{ s.t. } \int_I w \, dx = 0.$$

We can now apply LEMMA 7.12 and conclude $u = c$ a.e. on I, for some $c \in \mathbb{R}$. $\square$

SOBOLEV SPACES

MOTIVATION Let $I=(a,b)\subseteq\mathbb{R}$. If $u\in C^1_{pw}[a,b]$ then "morally" $u\in C^1[a,b]$ with the exception of a few points, in which u' is discontinuous.

The idea is that, if we integrate u', the Lebesgue integral does not see those exceptional points, as that set has zero measure. Now, if $\varphi\in C^1_c(a,b)$, we get, integrating by parts,

$$\int_a^b u\varphi'\,dx = \underbrace{u\varphi\Big|_a^b}_{\substack{=0 \text{ as} \\ \varphi(a)=\varphi(b)=0}} - \int_a^b u'\varphi\,dx$$

Thus for $u\in C^1_{pw}[a,b]$ and $\varphi\in C^1_c(a,b)$ we get

$$\int_a^b u\varphi'\,dx = -\int_a^b u'\varphi\,dx$$

Note that the above expression makes sense also if u,u' only belong to $L^1(a,b)$. This motivates the following definition.

DEFINITION 7.14 (Sobolev Space)

Let $I=(a,b)$ be an interval, possibly unbounded. Let $1\le p\le+\infty$. The **SOBOLEV SPACE** $W^{1,p}(I)$ is defined as

$$W^{1,p}(I):=\Big\{u\in L^p(I)\ \Big|\ \exists\, g\in L^p(I) \text{ s.t. } \underbrace{\int_I u\varphi'\,dx=-\int_I g\varphi\,dx}_{\text{"INTEGRATION BY PARTS"}},\ \forall\varphi\in C^1_c(I)\Big\}.$$

REMARK If $u\in W^{1,p}(I)$ then the function g is UNIQUE

Proof Assume $g, h \in L^p(I)$ both satisfy the "integration by parts" formula, i.e.,

$$\int_I u\dot{\varphi}\, dx = -\int_I g\varphi \quad , \quad \forall \varphi \in C_c^1(I)$$

$$\int_I u\dot{\varphi}\, dx = -\int_I h\varphi \quad , \quad \forall \varphi \in C_c^1(I)$$

Then $\int_I (g-h)\varphi\, dx = 0$, $\forall \varphi \in C_c^1(I)$. By FLCV LEMMA 7.11

we get $g = h$ a.e. on I. Thus g and h are the same L^p function, as they belong to the same equivalence class.

NOTATION ① If $u \in W^{1,p}(I)$ and g is the function from the definition, we denote

$$\dot{u} := g$$

and call $\dot{u}$ the WEAK DERIVATIVE of u.

② For $p = 2$ we denote $H^1(I) := W^{1,2}(I)$.

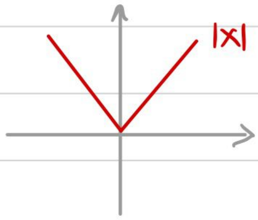
EXAMPLE 7.15 ① If $u \in C^1(I) \cap L^p(I)$ then $u \in W^{1,p}(I)$ and the weak derivative coincides with the classical derivative

② If I is bounded then $C^1(\overline{I}) \subset W^{1,p}(I)$ for all $1 \le p \le +\infty$

③ If $u \in C^1_{pw}(I) \cap L^p(I)$ then $u \in W^{1,p}(I)$ and the weak derivative coincides with the classical derivative in the points of differentiability of u (Note that this makes sense, since the weak derivative needs only to be defined almost everywhere, and the set of points where u is not differentiable is finite $\Rightarrow$ it has zero measure)

For example consider $I = (-1,1)$, $u(x) := |x|$. Then $u \in C^1_{pw}(I)$ with

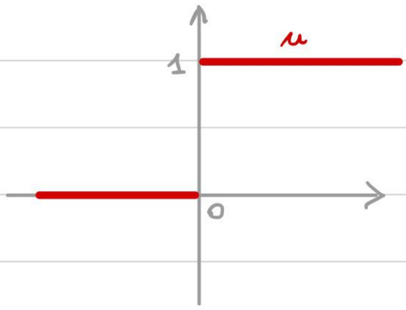
$$\dot{u}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



It is easy to check that this is the weak derivative of u.

④ Functions with JUMPS do not belong to $W^{1,p}(I)$. For example consider $I := (-1,1)$ and

$$u(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$



The pointwise derivative of u is the function $\dot{u}(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Thus we also have that $\dot{u} \in L^p(I)$. However it is easy to show that $\dot{u}$ is NOT the weak derivative of u. Moreover one can show that u does not admit any weak derivative, i.e., $u \notin W^{1,p}(I)$, for any $1 \leq p \leq +\infty$.

NOTATION

① The space $W^{1,p}(I)$ is equipped with the NORM

$$\boxed{\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|\dot{u}\|_{L^p}.}$$

② If $1 \leq p < +\infty$, $W^{1,p}(I)$ can be equipped with the EQUIVALENT NORM

$$\boxed{\|u\|_{W^{1,p}} := \left(\|u\|^p_{L^p} + \|\dot{u}\|^p_{L^p} \right)^{1/p}.}$$

③ The space $H^1(I)$ can be equipped with the INNER PRODUCT

$$\boxed{\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2}}$$

The induced norm is

$$\boxed{\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2 \right)^{1/2}}$$

[Checking the above statements is straightforward, using that $\|u\|_{L^p}$ is a norm on L^p and that $\langle u, v \rangle_{L^2}$ is an inner product on L^2]

PROPOSITION 7.16 Let $I \subseteq \mathbb{R}$ be open, bounded or unbounded. Then:

1. $W^{1,p}(I)$ is a BANACH SPACE for $1 \leq p \leq +\infty$.

2. $W^{1,p}(I)$ is REFLEXIVE for $1 < p < +\infty$.

3. $W^{1,p}(I)$ is SEPARABLE for $1 \leq p < +\infty$.

4. $H^1(I)$ is a SEPARABLE HILBERT space.

Proof (1) We need to prove that $W^{1,p}(I)$ is complete. So let $\{u_n\} \subseteq W^{1,p}(I)$ be a Cauchy sequence. As

$$\|u\|_{L^p} \leq \|u\|_{W^{1,p}} \quad \text{and} \quad \|\dot{u}\|_{L^p} \leq \|u\|_{W^{1,p}} \quad , \forall u \in W^{1,p}(I)$$

we have that $\{u_n\}, \{\dot{u}_n\}$ are Cauchy sequences in $L^p(I)$. As $L^p(I)$ is complete, there $\exists\, u, g \in L^p(I)$ s.t.

$$(*) \qquad u_n \to u \quad , \quad \dot{u}_n \to g \quad \text{strongly in } L^p(I).$$

By definition of $W^{1,p}$ we have

$$(**) \qquad \int_I u_n \dot{\varphi} \, dx = - \int_I \dot{u}_n \varphi \, dx \quad , \quad \forall \varphi \in C_c^1(I) \, , \quad \forall n \in \mathbb{N}$$

This is $L^p(I)^*$.
See THEOREM 6.14

As $u_n \to u$ strongly, then $u_n \rightharpoonup u$ weakly in $L^p(I)$. Since $C_c(I) \subset L^{p'}(I)$ we get

$$\int_I u_n \dot{\varphi}\, dx \to \int_I u \varphi \, dx \qquad \text{as } n \to +\infty$$

Similarly

$$\int_I \dot{u}_n \varphi \, dx \to \int_I g \varphi \, dx \qquad \text{as } n \to +\infty$$

Then we can pass to the limit in (**) and obtain

$$\int_I u \dot{\varphi} = - \int_I g \varphi \, dx \quad , \ \forall \varphi \in C_c^1(I).$$

This shows $u \in W^{1,p}(I)$ with weak derivative $\dot{u} = g$. Then by (*) we conclude $\| u_n - u \|_{W^{1,p}} \to 0$, showing completeness.

(2) Recall that $L^p(I)$ is REFLEXIVE for $1 < p < +\infty$ (THEOREM 6.14). Then it is easy to check that $E := L^p(I) \times L^p(I)$ is reflexive. Let

$$T : W^{1,p}(I) \to E$$
$$u \mapsto (u, \dot{u})$$

One can check that T is an isometry. Since $W^{1,p}(I)$ is Banach, it follows that $T(W^{1,p}(I)) \subseteq E$ is a closed subspace. Since closed subspaces of reflexive spaces are reflexive, we conclude.

(3) $L^p(I)$ is separable for all $1 \leq p < +\infty$ (THEOREM 6.15). Thus $E := L^p(I) \times L^p(I)$ is separable (immediate check). Consider T as above. As any SUBSET of a separable space is separable, from the inclusion $T(W^{1,p}(I)) \subseteq E$ we conclude.

(4) Follows from (1) and (3). $\square$

In the above proof, point 1, we showed a general fact which is worthy of its own numbered Remark.

REMARK 7.17 Let $\{u_n\} \subseteq W^{1,p}(I)$ be such that

$$\begin{cases} u_n \to u & \text{in } L^p(I) \\ \dot{u}_n \to g & \text{in } L^p(I) \end{cases}$$

Then $u_n \to u$ in $W^{1,p}(I)$ and $u \in W^{1,p}(I)$, with $\dot{u} = g$.

A similar Remark holds also for weak convergence.

REMARK 7.18 Let $\{u_n\} \subseteq H^1(I)$ be such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } L^2(I) \\ \dot{u}_n \rightharpoonup g & \text{weakly in } L^2(I) \end{cases}$$

Then $u \in H^1(I)$ with $\dot{u} = g$ in the weak sense and $u_n \rightharpoonup u$ in $H^1(I)$.

Proof As $u_n \in H^1(I)$ then

$$(*) \quad \int_I u_n \dot{\varphi} \, dx = - \int_I \dot{u}_n \varphi \, dx \quad , \quad \forall \varphi \in C^1_c(I) \quad , \quad \forall n \in \mathbb{N} .$$

As $L^2(I)$ is Hilbert, we have that $L^2(I)^* = L^2(I)$. Since $C^1_c(I) \subseteq L^2(I)$, by the weak convergence $u_n \rightharpoonup u$, we get

$$\int_I u_n \dot{\varphi} \, dx \to \int_I u \dot{\varphi} \, dx .$$

Similarly, as $\dot{u}_n \rightharpoonup g$, we get

$$\int_I \dot{u}_n \varphi \, dx \longrightarrow \int_I g \varphi \, dx .$$

Then we can pass to the limit in ⊛ and get that $\dot{u} = g$ in the weak sense.
As $g \in L^2(I)$, we get $u \in H^1(I)$. If $v \in H^1(I)$ we get

$$\langle u_n, v \rangle_{H^1} = \langle u_n, v \rangle_{L^2} + \langle \dot{u}_n, \dot{v} \rangle_{L^2} \to \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2} = \langle u, v \rangle_{H^1}$$

showing that $u_n \rightharpoonup u$ weakly in $H^1(I)$. □

LESSON 8 - 5 MAY 2021

We now prove one of the main results on 1-dimensional Sobolev functions, namely, that they are CONTINUOUS and they are PRIMITIVES of L^p functions.

THEOREM 7.19 Let $I = (a,b)$ be bounded or unbounded, and $1 \le p \le +\infty$.
Let $u \in W^{1,p}(I)$. Then there $\exists\, \tilde{u} \in C(I)$ s.t.

$$u = \tilde{u} \quad \text{a.e. on } I$$

and

$$⊛ \qquad \tilde{u}(x) - \tilde{u}(y) = \int_y^x \dot{u}(t)\, dt \,, \qquad \forall\, x, y \in I.$$

(Generalized Fundamental Thm of Calculus)

NOTE Theorem 7.19 is saying that if $u \in W^{1,p}(I)$ then $\exists\, \tilde{u}$ continuous in the same equivalence class of u. We call $\tilde{u}$ the CONTINUOUS REPRESENTATIVE of u, and in the future we just denote it by u (Notice that the continuous representative is unique, by ⊛).

During the proof of THEOREM 7.19 we need the following lemma.

LEMMA 7.20 $I = (a,b)$, $g \in L^1_{loc}(I)$. Fix $y_0 \in I$ and define

$$u(x) := \int_{y_0}^x g(t)\, dt \,, \qquad \forall x \in I.$$

Then $u \in C(I)$ and $\dot{u} = g$ in the weak sense.

Proof of LEMMA 7.20 The fact that u is continuous follows by DOMINATED CONVERGENCE. Indeed, for $x \in I$,

$$(*) \quad |u(x+\varepsilon) - u(x)| \leq \int_x^{x+\varepsilon} |g(t)|\,dt = \int_K \chi_{[x,x+\varepsilon]}(t)\,|g(t)|\,dt,$$

where K is any compact set such that $[x, x+\varepsilon] \subset K$, $\forall\, 0<\varepsilon<1$. Now $\chi_{[x,x+\varepsilon]}\, g \to 0$ a.e. as $\varepsilon \to 0$, and $|\chi_{[x,x+\varepsilon]}\, g| \leq |g|$,

with $g \in L^1_{loc}(I)$. Thus $g \in L^1(K)$, and by dominated convergence we conclude that the RHS of $(*)$ goes to 0 as $\varepsilon \to 0$, showing continuity.

We now show that $\dot{u} = g$ in the weak sense. Thus let $\varphi \in C^1_c(I)$. Consider $\psi(t,x) := u(x)\dot{\varphi}(t)$. Clearly $\psi \in L^1(I \times I)$, being $u, \dot{\varphi}$ continuous. Then we can apply FUBINI'S THEOREM 6.10 to get:

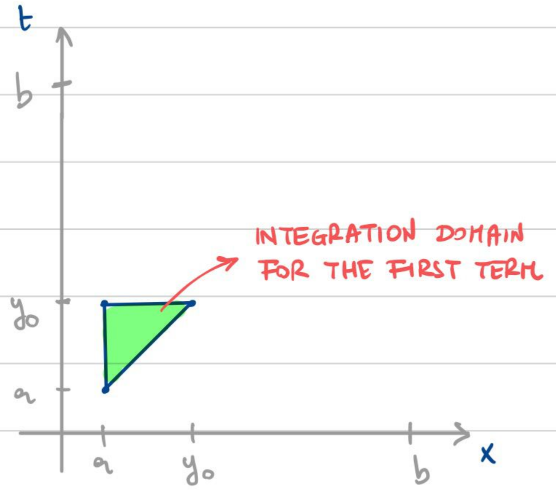
$$\int_a^b u\,\dot{\varphi}\,dx = \int_a^b \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx \qquad \text{(definition of } u\text{)}$$

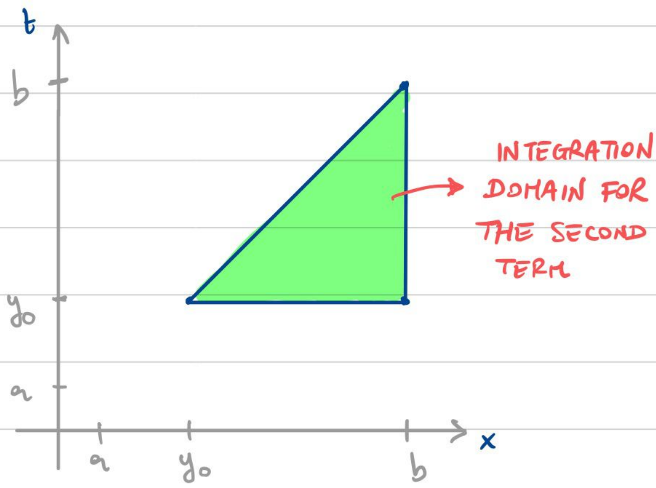
Splitting integral WRT x

$$= \int_a^{y_0} \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx$$

Reversing integration values of inner integral for the FIRST TERM

$$= -\int_a^{y_0} \left\{ \int_x^{y_0} g(t)\,\dot{\varphi}(x)\,dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx$$





We can write the integration domains normal WRT to t, and apply FUBINI:

$$\int_a^b u \dot{\varphi}\, dx = -\int_a^{y_0} \left\{ \int_x^{y_0} g(t)\, \dot{\varphi}(x)\, dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\, \dot{\varphi}(x)\, dt \right\} dx$$

FUBINI $\rightarrow$
$$= -\int_a^{y_0} \left\{ \int_a^t g(t)\, \dot{\varphi}(x)\, dx \right\} dt + \int_{y_0}^b \left\{ \int_t^b g(t)\, \dot{\varphi}(x)\, dx \right\} dt$$

TAKE $g(t)$ OUT $\rightarrow$
$$= -\int_a^{y_0} g(t) \left\{ \int_a^t \dot{\varphi}(x)\, dx \right\} dt + \int_{y_0}^b g(t) \left\{ \int_t^b \dot{\varphi}(x)\, dx \right\} dt$$

$$= -\int_a^{y_0} g(t)\, [\varphi(t) - \varphi(a)]\, dt + \int_{y_0}^b g(t)\, [\varphi(b) - \varphi(t)]\, dt$$

$\varphi(a) = \varphi(b) = 0$, since φ is COMPACTLY SUPPORTED $\rightarrow$
$$= -\int_a^{y_0} g(t)\varphi(t)\, dt - \int_{y_0}^b g(t)\varphi(t)\, dt = -\int_a^b g(t)\varphi(t)\, dt,$$

showing that $\dot{u} = g$ in the weak sense and concluding. $\square$

<u>Proof of THEOREM 7.19</u> Fix $y_0 \in I$ arbitrary and define

$$\hat{u}(x) := \int_{y_0}^x \dot{u}(t)\, dt, \quad \forall x \in I.$$

Since $\dot{u} \in L^p(I)$ (as $u \in W^{1,p}(I)$), then $\dot{u} \in L^1_{loc}(I)$. We can then apply LEMMA 7.20 to infer that $\hat{u} \in C(I)$ and $(\hat{u})' = \dot{u}$ in the weak sense, i.e.

$$\text{(*)} \quad \int_a^b \hat{u}\, \dot{\varphi}\, dx = -\int_a^b \dot{u}\, \varphi\, dx, \quad \forall \varphi \in C^1_c(I)$$

On the other hand $u \in W^{1,p}(I)$, so that by definition

$$\int_a^b u \dot{\varphi}\, dx = -\int_a^b \dot{u} \varphi \, dx , \quad \forall \varphi \in C_c^1(I).$$

by (*) we then get

$$\int_a^b (\hat{u} - u) \dot{\varphi} \, dx = 0 , \quad \forall \varphi \in C_c^1(I).$$

We can then apply DBR LEMMA 7.13 to get that $\exists\, c \in \mathbb{R}$ s.t. $u = \hat{u} + c$ a.e. in I. Thus the continuous representative is $\tilde{u} := \hat{u} + c$. The second part of the statement follows by definition of $\tilde{u}$. □

REMARK 7.21 Lemma 7.20 implies that, if $g \in L^p(I)$ and its primitive u also belongs to $L^p(I)$, then $u \in W^{1,p}(I)$.

With similar ideas, we can prove the following proposition.

PROPOSITION 7.22 Let $I = (a,b)$ be bounded or unbounded, $1 \le p \le +\infty$. Assume that $u \in W^{1,p}(I)$ is s.t. $\dot{u} \in C(I)$. Then $u \in C^1(I)$.

Proof Define $U(x) := \int_a^x \dot{u}(t)\, dt$. As $\dot{u}$ is continuous, by the Fundamental Theorem of Calculus we have that $U \in C^1(I)$ and $\dot{U} = \dot{u}$. Let $\varphi \in C_c^1(I)$. Integrating by parts:

$$\int_a^b U \dot{\varphi}\, dx = \underbrace{U\varphi \Big|_a^b}_{\substack{=0 \text{ as} \\ \varphi(a) = \varphi(b) = 0}} - \int_a^b \dot{U} \varphi \, dx \underset{\dot{U} = \dot{u}}{=} -\int_a^b \dot{u}\varphi\, dx \underset{\text{Definition of weak derivative}}{=} \int_a^b u \dot{\varphi}\, dx$$

Thus

$$\int_a^b (U - u)\dot{\varphi}\, dx = 0 , \quad \forall \varphi \in C_c^1(I).$$

By DBR LEMMA 7.13 we get $u = U + c$ for some $c \in \mathbb{R}$. As $U \in C^1(I) \Rightarrow u \in C^1(I)$. □

HÖLDER REGULARITY

We can actually improve on THEOREM 7.19 by showing Hölder regularity for Sobolev functions. We recall that u is **α-HÖLDER** for some $0<\alpha<1$ if $\exists\, C>0$ s.t.

$$|u(x)-u(y)| \leq C\, |x-y|^{\alpha} \quad , \quad \forall\, x,y \in I$$

We denote the space of α-Hölder functions by $C^{0,\alpha}(I)$.

THEOREM 7.23 Let $I=(a,b)$ be bounded or unbounded. Let $1<p\leq +\infty$ and $u\in W^{1,p}(I)$. Then $u \in C^{0,1-1/p}(I)$, with

$$|u(x)-u(y)| \leq \|\dot{u}\|_{L^p}\, |x-y|^{1-1/p} \quad , \qquad \forall\, x,y\in I.$$

Proof By THEOREM 7.19 we have that u is continuous and

$$u(x)-u(y) = \int_y^x \dot{u}(t)\,dt \quad , \quad \forall\, x,y \in I.$$

Then for $y>x$,

$$|u(x)-u(y)| \leq \int_x^y |\dot{u}(t)|\,dt$$

(Hölder inequality) $\leq \left(\int_x^y |\dot{u}(t)|^p\,dt\right)^{1/p} \left(\int_x^y 1^{p'}\,dt\right)^{1/p'}$ ($p'=\frac{p}{p-1}$ Hölder conjugate)

$$= \left(\int_a^b |\dot{u}(t)|^p\,dt\right)^{1/p} (y-x)^{1/p'}$$

$$= \|\dot{u}\|_{L^p}\, |y-x|^{1-1/p}$$

If $x>y$ we conclude with the same argument. $\square$

WARNING THEOREM 7.23 does not hold for $p=1$.

DENSITY OF SMOOTH FUNCTIONS

Our goal is to prove the following theorem.

THEOREM 7.24 Let $1 \leq p < +\infty$, $u \in W^{1,p}(I)$ for $I = (a,b)$ bounded or unbounded. Then $\exists \{u_n\} \subseteq C_c^\infty(\mathbb{R})$ s.t.

$$u_n|_I \to u \quad \text{strongly in } W^{1,p}(I).$$

WARNING The above differs from the density result for L^p functions COROLLARY 7.10:
If $u \in L^p(I)$, $\exists \{u_n\} \subseteq C_c^\infty(I)$ s.t. $u_n \to u$ strongly in $L^p(I)$.

In order to prove the above theorem we need an extension result.

LEMMA 7.25 Let $I = (a,b)$ be bounded or unbounded, $1 \leq p \leq +\infty$.
There $\exists$ a linear continuous operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ called EXTENSION OPERATOR such that:

a) $Pu|_I = u \quad , \quad \forall u \in W^{1,p}(I)$

b) $\|Pu\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(I)} \quad , \quad \forall u \in W^{1,p}(I)$

c) $\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C \|u\|_{W^{1,p}(I)} \quad , \quad \forall u \in W^{1,p}(I)$

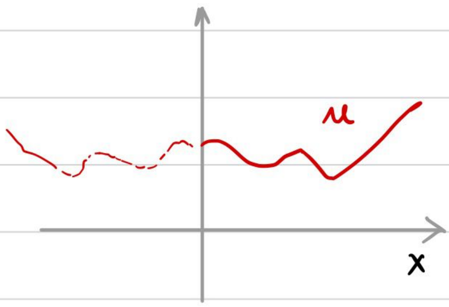
where C depends only on $|I|$: $C = 4$ in (b) and $C = 4\left(1 + \frac{1}{|I|}\right)$ in (c).

Proof of LEMMA 7.25 We have two cases:

1) If I is unbounded, then by translation it is sufficient to consider either $I = (0, +\infty)$, or $I = (-\infty, 0)$.
2) If I is bounded, then by translation and scaling it is sufficient to consider $I = (0,1)$.

CASE 1 Let $I = (0, +\infty)$. If $u \in W^{1,p}(I)$, we extend u by REFLECTION:

$$(Pu)(x) := u^*(x) := \begin{cases} u(x) & \text{if } x > 0 \\ u(-x) & \text{if } x < 0 \end{cases}$$



Clearly $u^*|_I = u$, so that (a) holds. Also

$$\|u^*\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} |u^*|^p \, dx = 2 \int_0^{+\infty} |u|^p \, dx$$

then

$$\circledast \quad \|u^*\|_{L^p(\mathbb{R})} \leq 2^{1/p} \|u\|_{L^p(I)} \leq 2 \|u\|_{L^p(I)},$$

showing (b). Now define

$$g(x) := \begin{cases} \dot{u}(x) & \text{for a.e. } x > 0 \\ -\dot{u}(-x) & \text{for a.e. } x < 0 \end{cases}$$

(use the definition of u^*, doing separately the cases $x > 0$ and $x < 0$)

Clearly $g \in L^p(\mathbb{R})$. Also, by using THEOREM 7.19, it is easy to check that

$$u^*(x) - u^*(0) = \int_0^x g(t) \, dt, \qquad \forall x \in \mathbb{R}.$$

Hence $u^* \in W^{1,p}(\mathbb{R})$ by REMARK 7.21, with $(u^*)' = g$ in the weak sense. Finally

$$\|(u^*)'\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} |g(t)|^p \, dt = 2 \int_0^{+\infty} |\dot{u}(t)|^p \, dt = 2 \|\dot{u}\|^p_{L^p(I)},$$

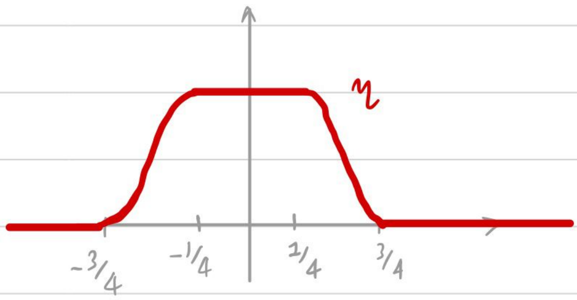
so that

$$\|(u^*)'\|_{L^p(\mathbb{R})} \leq 2^{1/p} \|\dot{u}\|_{L^p(I)} \leq 2 \|\dot{u}\|_{L^p(I)}.$$

Together with $\circledast$, this implies (c). The case $I = (-\infty, 0)$ is the same.

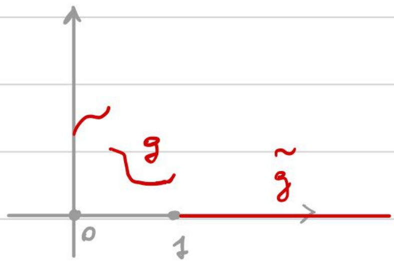
CASE 2 Let $I = (0,1)$. Let $\eta \in C_c^1(\mathbb{R})$ be a cut-off such that

- $0 \le \eta \le 1$ in $\mathbb{R}$
- $\eta(x) = 1$ for all $x \in [-1/4, 1/4]$
- $\eta(x) = 0$ for all $x \in \mathbb{R} \setminus (-3/4, 3/4)$.



For $g \in L^p(0,1)$, define

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } x \in (0,1) \\ 0 & \text{if } x \ge 1 \end{cases}$$



If $u \in W^{1,p}(0,1)$, we claim that

$$\circledast \qquad \eta \tilde{u} \in W^{1,p}(0,+\infty) \quad \text{and} \quad (\eta \tilde{u})' = \eta' \tilde{u} + \eta \cdot \widetilde{(u')} \quad \text{weakly.}$$

Indeed, let $\varphi \in C_c^1(0,+\infty)$. As both φ and η are regular, we have $(\eta\varphi)' = \eta'\varphi + \eta\varphi'$. Then

$$\int_0^{+\infty} (\eta \tilde{u})\varphi' \, dx = \int_0^1 (\eta u)\varphi' \, dx \qquad \left(\begin{array}{l}\text{since } \tilde{u} = 0 \text{ if } x \ge 1 \text{ and } \tilde{u} = u \\ \text{for } 0 < x < 1\end{array}\right)$$

$$= \int_0^1 u(\eta\varphi)' \, dx - \int_0^1 u \eta' \varphi \, dx \qquad (\text{using } (\eta\varphi)' = \eta'\varphi + \eta\varphi')$$

$$= -\int_0^1 u' \eta \varphi \, dx - \int_0^1 u \eta' \varphi \, dx \qquad \left(\begin{array}{l}\text{since } \eta\varphi \in C_c^1(0,1) \\ \text{and } u \in W^{1,p}(0,1)\end{array}\right)$$

$$= -\int_0^1 [u'\eta + u\eta'] \varphi \, dx$$

$$= -\int_0^{+\infty} [\widetilde{(u')}\eta + \tilde{u}\eta'] \varphi \, dx \qquad \left(\begin{array}{l}\text{since extending } u \text{ and} \\ u' \text{ to zero does not} \\ \text{alter the integral}\end{array}\right)$$

Showing that $(\eta\tilde{u})' = \eta'\tilde{u} + \eta\widetilde{(u')}$ weakly.

Clearly $\eta\tilde{u} \in L^p(0,+\infty)$. Also, by using the formula just proven, $(\eta\tilde{u})' \in L^p(0,+\infty)$. Then $\eta\tilde{u} \in W^{1,p}(0,+\infty)$ and $\circledast$ is proven.

$$(u)^*(x) = \begin{cases} u(x) & \text{if } x>0 \\ u(-x) & \text{if } x<0 \end{cases}$$

We can now define the extension operator $P : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$. First define $P_1 : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ by setting $P_1 u := (\eta\tilde{u})^*$, with $*$ being the operator from CASE 1, that is, we first extend u to $(0,+\infty)$ by setting u to 0 in $[1,+\infty)$, then we multiply by η and extend $\eta\tilde{u}$ to $(-\infty,0)$ by reflection. By the properties of $*$ we know that

$$\|(\eta\tilde{u})^*\|_{L^p(\mathbb{R})} \leq 2\,\|\eta\tilde{u}\|_{L^p(0,+\infty)}, \qquad \left\|\left[(\eta\tilde{u})^*\right]'\right\|_{L^p(\mathbb{R})} \leq 2\,\|(\eta\tilde{u})'\|_{L^p(0,+\infty)}$$

Now

$$\|\eta\tilde{u}\|_{L^p(0,+\infty)} \leq \|\eta\|_{L^\infty(0,+\infty)}\,\|\tilde{u}\|_{L^p(0,+\infty)} = \|u\|_{L^p(0,1)}$$

since $0 \leq \eta \leq 1$ and $\tilde{u} = 0$ in $(1,+\infty)$. Moreover, by (*),

$$\|(\eta\tilde{u})'\|_{L^p(0,+\infty)} \overset{(*)}{\leq} \|\eta'\tilde{u}\|_{L^p(0,+\infty)} + \|\eta(\tilde{u}')\|_{L^p(0,+\infty)}$$

(since $u=0$, $(\tilde{u})'=0$ in $(1,+\infty)$)
$$\leq \|\eta'\|_{L^\infty(I)}\,\|u\|_{L^p(I)} + \|\eta\|_{L^\infty(0,+\infty)}\,\|u'\|_{L^p(I)}$$

$$\leq C\,\|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$$

with $C := \|\eta'\|_{L^\infty(I)}$. In total, we have

$$(**) \qquad \|P_1 u\|_{L^p(\mathbb{R})} \leq 2\,\|u\|_{L^p(I)}, \qquad \|P_1 u\|_{W^{1,p}(\mathbb{R})} \leq 2(1+C)\,\|u\|_{W^{1,p}(I)}$$

Also notice that $(P_1 u)|_I = \eta u$. Now define $P_2 : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ in the following way: $P_2 u$ is defined by

- Extending $(1-\eta)u$ to 0 in $(-\infty,0)$, obtaining a map defined in $(-\infty,1]$;
- Then extend to the whole $\mathbb{R}$ by reflection around 1.

In a similar way one can check that

$$(\ast\ast\ast) \quad \|P_2 u\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)}, \qquad \|P_2 u\|_{W^{1,p}(\mathbb{R})} \leq 2(1+C)\|u\|_{W^{1,p}(I)}$$

and that $(P_2 u)|_I = (1-\eta)u$. Finally we define $P: W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ by

$$Pu := P_1 u + P_2 u.$$

By $(\ast\ast)$ - $(\ast\ast\ast)$ we have that P satisfies (b), (c). Moreover

$$(Pu)|_I = (P_1 u)|_I + (P_2 u)|_I = \eta u + (1-\eta)u = u,$$

so that also (a) holds, concluding. □

Another result needed to prove THEOREM 7.24 is the following:

LEMMA 7.26 Let $\rho \in L^1(\mathbb{R})$, $u \in W^{1,p}(\mathbb{R})$ with $1 \leq p \leq +\infty$. Then $\rho * u \in W^{1,p}(\mathbb{R})$ and $(\rho * u)' = \rho * u'$ in the weak sense.

Proof Assume first that ρ is compactly supported, so that $\rho \in L^1_{loc}(\mathbb{R})$. By THEOREM 7.2 we have $\rho * u \in L^p(\mathbb{R})$. Let $\varphi \in C^1_c(\mathbb{R})$. One can check that

$$(\ast) \quad \int_{\mathbb{R}} (\rho * u)\varphi' \, dx = \int_{\mathbb{R}} u(\check{\rho} * \varphi') \, dx, \qquad \check{\rho}(x) := \rho(-x).$$

Now, as $\check{\rho} \in L^1(\mathbb{R})$ and $\varphi \in C^1_c(\mathbb{R})$, by THEOREM 7.6 we have $\check{\rho} * \varphi \in C^1(\mathbb{R})$ and $\check{\rho} * \varphi' = (\check{\rho} * \varphi)'$. Moreover $\check{\rho} * \varphi$ is compactly supported, as $\operatorname{supp}(\check{\rho} * \varphi) \subset \overline{\operatorname{supp}\check{\rho} + \operatorname{supp}\varphi}$ by PROPOSITION 7.4, and ρ, φ are compactly supported. Therefore $\check{\rho} * \varphi \in C^1_c(\mathbb{R})$ and by $(\ast)$

use $\check{\rho} * \varphi' = (\check{\rho} * \varphi)'$

$$\int_{\mathbb{R}} (\rho * u)\varphi' \, dx \overset{(\ast)}{=} \int_{\mathbb{R}} u(\check{\rho} * \varphi') \, dx = \int_{\mathbb{R}} u(\check{\rho} * \varphi)' \, dx$$

(As $\check{\rho} * \varphi$ is a test function and $u \in W^{1,p}(\mathbb{R})$) →

$$= -\int_{\mathbb{R}} u'(\check{\rho} * \varphi) \, dx = -\int_{\mathbb{R}} (\rho * u')\varphi \, dx$$

Use $(\ast)$ with u replaced by u'

Thus $(\rho * u)' = \rho * u'$ in the weak sense. As $u' \in L^p(\mathbb{R})$, by THEOREM 7.2 we get $\rho * u' \in L^p(\mathbb{R})$, showing that $\rho * u \in W^{1,p}(\mathbb{R})$.

If ρ is not compactly supported, by COROLLARY 7.10 we can find a sequence $\{\rho_n\} \subseteq C_c(\mathbb{R})$ s.t. $\rho_n \to \rho$ strongly in $L^1(\mathbb{R})$. Note that what we proved so far holds for ρ_n, so that

$$(**) \qquad \rho_n * u \in W^{1,p}(\mathbb{R}), \qquad (\rho_n * u)' = \rho_n * u' \quad \text{weakly}, \qquad \forall n \in \mathbb{N}.$$

By Young's inequality we have

$$\| \rho_n * u - \rho * u \|_{L^p} \leq \| \rho_n - \rho \|_{L^1} \; \| u \|_{L^p} \to 0 \quad \text{as} \quad n \to +\infty$$

(as $\rho_n \to \rho$ in L^1)

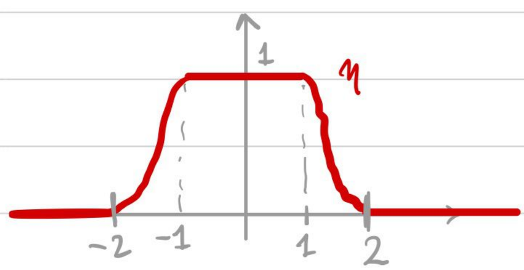
$$\| \rho_n * u' - \rho * u' \|_{L^p} \leq \| \rho_n - \rho \|_{L^1} \; \| u' \|_{L^p} \to 0 \quad \text{as} \quad n \to +\infty$$

This means $\rho_n * u \to \rho * u$ and $(\rho_n * u)' \overset{(**)}{=} \rho_n * u' \to \rho * u'$ strongly in $L^p(\mathbb{R})$. Since $\rho * u' \in L^p(\mathbb{R})$ by THEOREM 7.2, we can invoke REMARK 7.17 to conclude that $\rho * u \in W^{1,p}(I)$, with weak derivative $(\rho * u)' = \rho * u'$. $\square$

Proof of THEOREM 7.24 Let $I \subseteq \mathbb{R}$ be open, bounded or unbounded. We need to show that for $u \in W^{1,p}(I)$ there $\exists \{u_n\} \subseteq C_c^\infty(\mathbb{R})$ s.t. $(u_n)|_I \to u$ strongly in $W^{1,p}(I)$.

First, let $\tilde{u} := Pu$ be the extension to $\mathbb{R}$ of u given by LEMMA 7.25. In particular

$$\tilde{u}|_I = u, \qquad \| \tilde{u} \|_{W^{1,p}(\mathbb{R})} \leq C \, \| u \|_{W^{1,p}(I)}.$$



Let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off s.t.

$$0 \leq \eta \leq 1, \quad \eta(x) = 1 \text{ for } x \in [-1,1], \quad \eta(x) = 0 \text{ for } x \in \mathbb{R} \setminus (-2,2).$$

Define $\eta_n(x) := \eta\left(\frac{x}{n}\right)$. Note that $\eta_n \to 1$ pointwise. Therefore $\eta_n \tilde{u} \to \tilde{u}$ a.e. in $\mathbb{R}$.

Since $|\eta_n \tilde{u}| \leq |\tilde{u}|$ and $\tilde{u} \in L^p(\mathbb{R})$, by Dominated Convergence (THEOREM 6.7) we get

$$(*) \quad \eta_n \tilde{u} \to \tilde{u} \quad \text{strongly in } L^p(\mathbb{R}).$$

Let $\rho_n \in C_c^\infty(\mathbb{R})$ be a sequence of mollifiers. Define $u_n := \eta_n \cdot (\rho_n * \tilde{u})$. Notice that $\rho_n * \tilde{u} \in C^\infty(\mathbb{R})$ by THEOREM 7.6 (indeed note that $\tilde{u} \in L^p(\mathbb{R})$ and so $u \in L^1_{loc}(\mathbb{R})$). Since $\eta_n \in C_c^\infty(\mathbb{R})$, it follows that $u_n \in C_c^\infty(\mathbb{R})$. We will show that $(u_n)|_I \to u$ strongly in $W^{1,p}(I)$. First note that

$$u_n - \tilde{u} = \eta_n (\rho_n * \tilde{u}) - \tilde{u} = \eta_n [\rho_n * \tilde{u} - \tilde{u}] + \eta_n \tilde{u} - \tilde{u}$$

Since $\|\eta_n\|_{L^\infty(\mathbb{R})} \leq 1$, we get

$$\|u_n - \tilde{u}\|_{L^p(\mathbb{R})} \leq \underbrace{\|\rho_n * \tilde{u} - \tilde{u}\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by THEOREM 7.9}}} + \underbrace{\|\eta_n \tilde{u} - \tilde{u}\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by } (*)}} \to 0 \quad \text{as } n \to +\infty$$

so $u_n \to \tilde{u}$ strongly in $L^p(\mathbb{R})$. In particular $u_n \to u$ strongly in $L^p(I)$. Also

$$u_n' \underset{\substack{\text{def + classical} \\ \text{derivation of product}}}{=} \eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u})' \underset{\substack{\text{LEMMA 7.26 to} \\ \text{differentiate } \rho_n * \tilde{u}: \\ (\rho_n * \tilde{u})' = \rho_n * \tilde{u}' \text{ weakly}}}{=} \eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u}')$$

Note To differentiate $\rho_n * \tilde{u}$ we could also use THEOREM to get $(\rho_n * \tilde{u})' = \rho_n' * \tilde{u}$. However this term would be useless in our proof, because we need $\tilde{u}'$ to appear.

Therefore

$$\|u_n' - \tilde{u}'\|_{L^p(\mathbb{R})} = \|\eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u}') - \tilde{u}'\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n' (\rho_n * \tilde{u})\|_{L^p(\mathbb{R})} + \|\eta_n (\rho_n * \tilde{u}') - \tilde{u}'\|_{L^p(\mathbb{R})}$$

(add subtract $\eta_n \tilde{u}'$ and use Δ inequality) $\rightarrow$

$$\leq \underbrace{\|\eta_n' (\rho_n * \tilde{u})\|_{L^p(\mathbb{R})}}_{:= I_1} + \underbrace{\|\eta_n [(\rho_n * \tilde{u}') - \tilde{u}']\|_{L^p(\mathbb{R})}}_{:= I_2} + \underbrace{\|\eta_n \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}}_{:= I_3}$$

We now estimate I_1, I_2, I_3 separately.

- For I_1, notice that, as $\eta_n(x) := \eta\left(\frac{x}{n}\right)$, then $\eta_n'(x) = \frac{1}{n}\eta'\left(\frac{x}{n}\right)$.
Setting $C := \|\eta'\|_{L^\infty(\mathbb{R})}$ we get

$$I_1 = \|\eta_n'(\rho_n * \tilde{u})\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n'\|_{L^\infty(\mathbb{R})} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})}$$

$$\leq \frac{C}{n} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})}$$

By Youngs' inequality we have

As $\|\rho_n\|_{L^1(\mathbb{R})} = 1$ by properties of mollifiers

$$\|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \leq \|\rho_n\|_{L^1(\mathbb{R})} \|\tilde{u}'\|_{L^p(\mathbb{R})} = \|\tilde{u}'\|_{L^p(\mathbb{R})}$$

so that

$$I_1 \leq \frac{C}{n} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \leq \frac{C}{n} \|\tilde{u}'\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } n \to +\infty.$$

- For I_2,

$$I_2 = \|\eta_n \left[(\rho_n * \tilde{u}') - \tilde{u}'\right]\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n\|_{L^\infty(\mathbb{R})} \|\rho_n * \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}$$

(Since $\|\eta_n\|_{L^\infty(\mathbb{R})} = 1$) $$= \underbrace{\|\rho_n * \tilde{u}' - \tilde{u}'\|}_{\text{This goes to } 0 \text{ by THEOREM 7.9, as } \tilde{u}' \in L^1_{loc}(\mathbb{R})}{}_{L^p(\mathbb{R})} \to 0 \quad \text{as } n \to +\infty.$$

• For I_3: Recall that $\eta_n \to 1$ pointwise in $\mathbb{R}$. Thus $\eta_n \tilde{u}' \to \tilde{u}'$ a.e. in $\mathbb{R}$. Also $|\eta_n \tilde{u}'| \leq |\tilde{u}'|$ as $|\eta_n| \leq 1$. Then we can apply DOMINATED CONVERGENCE to get

$$\eta_n \tilde{u}' \to \tilde{u}' \quad \text{strongly in } L^p(\mathbb{R}),$$

which implies

$$I_3 = \| \eta_n \tilde{u}' - \tilde{u}' \|_{L^p(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \to +\infty$$

In total, we just proved that

$$\| u_n' - \tilde{u}' \|_{L^p(\mathbb{R})} \leq I_1 + I_2 + I_3 \to 0 \quad \text{as } n \to +\infty,$$

that is, $u_n' \to \tilde{u}'$ strongly in $L^p(\mathbb{R})$. In particular,

$$(**) \quad u_n' \to \tilde{u}'|_I \quad \text{strongly in } L^p(I)$$

Now recall that we had

$$(***) \quad u_n \to u \quad \text{strongly in } L^p(I)$$

Note that $\{u_n\} \subseteq W^{1,p}(I)$, as $\{u_n\} \subseteq C_c^\infty(\mathbb{R})$. Therefore, as $(**)$, $(***)$ hold we can apply REMARK 7.17 and conclude

$$u_n \to u \quad \text{strongly in } W^{1,p}(I),$$

ending the proof. $\square$

LESSON 9 - 12 MAY 2021

SOBOLEV EMBEDDING

DEFINITION X, Y normed spaces, $X \subseteq Y$. We say that

1. X EMBEDS continuously in Y, in symbols $X \hookrightarrow Y$, if the identity $i:(X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous, i.e. if $\exists\, C>0$ s.t.

$$\|u\|_Y \leq C \|u\|_X, \quad \forall u \in X.$$

2. The embedding $X \hookrightarrow Y$ is COMPACT if the identity $i: X \to Y$ is a continuous compact operator, i.e.,

$$\text{If } B \subseteq X \text{ is norm bounded WRT } \|\cdot\|_X \Rightarrow \overline{B}^{\|\cdot\|_Y} \text{ is compact WRT } \|\cdot\|_Y.$$

THEOREM 7.27 (Sobolev embedding)

Let $I \subseteq \mathbb{R}$ be open. There $\exists\, C>0$, depending only on $|I|$, s.t.

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I), \quad 1 \leq p \leq +\infty.$$

Thus $W^{1,p}(I) \hookrightarrow L^\infty(I)$. If in addition I is <u>BOUNDED</u>:

(a) The embedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is COMPACT $\forall\, 1 < p \leq +\infty$,

(b) The embedding $W^{1,1}(I) \hookrightarrow L^q(I)$ is COMPACT $\forall\, 1 \leq q < +\infty$,

(c) The embedding $W^{1,p}(I) \hookrightarrow L^p(I)$ is COMPACT $\forall\, 1 \leq p \leq +\infty$.

In order to prove THEOREM 7.27 we need two auxiliary results:

THEOREM 7.28 **(ASCOLI - ARZELÀ)**

Let (K, d) be a compact metric space, and consider $C(K)$ i.e. the set of continuous functions $u: K \to \mathbb{R}$. Let $A \subseteq C(K)$ and suppose that:

1. A is **BOUNDED**: i.e. $M > 0$ s.t. $\|u\|_\infty \leq M$ for all $u \in A$
2. A is **EQUI-CONTINUOUS**: i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$d(x_1, x_2) < \delta \Rightarrow |u(x_1) - u(x_2)| < \varepsilon \quad , \quad \forall u \in A.$$

Then the closure of A in $C(K)$ is COMPACT.

(This theorem should already be well-known in euclidean spaces. For a proof of the metric case, see the book by RUDIN.)

For the next result, recall the notation: if $u: \mathbb{R} \to \mathbb{R}$, $h \in \mathbb{R}$, the TRANSLATION operator T_h is defined by $(T_h u)(x) := u(x+h)$.

THEOREM 7.29 **(Characterization of Sobolev Functions)**

Let $1 < p < +\infty$ and $u \in L^p(\mathbb{R})$. They are equivalent:

(a) $u \in W^{1,p}(\mathbb{R})$

(b) It holds $\|T_h u - u\|_{L^p(\mathbb{R})} \leq \|u'\|_{L^p(\mathbb{R})} |h|$, $\forall h \in \mathbb{R}$.

Moreover, the implication (a) $\Rightarrow$ (b) is also true for $p = 1$.

(The proof of the above theorem will be left for the exercises course)

Proof of THEOREM 7.27 We start by showing the embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$. WLOG we can suppose $I = \mathbb{R}$, otherwise we can use the extension operator of THEOREM 7.24. Also, the embedding is trivial for $p = +\infty$. Hence assume $1 \leq p < +\infty$. Define $G(s) := |s|^{p-1}s$. Let $u \in C_c^1(\mathbb{R})$ and set

$$w := G(u).$$

Clearly $w \in C_c^1(\mathbb{R})$, with (w is compactly supported since $u \in C_c^1(\mathbb{R})$ and $G(0) = 0$)

$$w' = G'(u)u' = p|u|^{p-1}u'$$

Therefore for $x \in \mathbb{R}$,

$$G(u(x)) = w(x) = \int_{-\infty}^{x} w'(s)\,ds \quad \text{(by the Fundamental Theorem of Calculus, since } w \in C_c^1(\mathbb{R})\text{)}$$

(*)

$$= \int_{-\infty}^{x} p\,|u(s)|^{p-1} u'(s)\,ds$$

Now $|G(u)| = |u|^p$, thus, by (*) and Hölder's inequality,

$$|u(x)|^p = |G(u(x))| \overset{(*)}{\leq} \int_{-\infty}^{x} p\,|u(s)|^{p-1}\,|u'(s)|\,ds$$

(Since integrand is non-negative) →

$$\leq p \int_{\mathbb{R}} |u(s)|^{p-1}\,|u'(s)|\,ds$$

(Hölder WRT p, p')

$$\leq p \left(\int_{\mathbb{R}} |u|^{p'(p-1)}\,ds \right)^{1/p'} \left(\int_{\mathbb{R}} |u'|^p\,ds \right)^{1/p}$$

(Recall $p' = \frac{p}{p-1}$. Then $p'(p-1) = p$ and $\frac{1}{p'} = \frac{p-1}{p}$)

$$= p\,\|u\|_{L^p}^{p-1}\,\|u'\|_{L^p}$$

Therefore

$$|u(x)| \leq p^{1/p}\,\|u\|_{L^p}^{1/p'}\,\|u'\|_{L^p}^{1/p}, \quad \forall x \in \mathbb{R}.$$

Recall Young's Inequality for real numbers: $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$, $\forall a,b \ge 0$. Apply it to $a = \|u'\|_{L^p}^{1/p}$, $b = \|u\|_{L^p}^{1/p'}$ to get

$$|u(x)| \le p^{1/p} \|u\|_{L^p}^{1/p'} \|u'\|_{L^p}^{1/p}$$

$$\text{(Young)} \quad \le p^{1/p} \left\{ \frac{\|u\|_{L^p}}{p'} + \frac{\|u'\|_{L^p}}{p} \right\}$$

$$\text{(Since } p, p' \ge 1\text{)} \quad \le p^{1/p} \left\{ \|u\|_{L^p} + \|u'\|_{L^p} \right\} = p^{1/p} \|u\|_{W^{1,p}}$$

Taking the supremum for $x \in \mathbb{R}$ and noting that $p^{1/p} \le e^{1/e}$ $\forall p \ge 1$, we get

$$(**) \qquad \|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}, \quad \forall u \in C_c^1(\mathbb{R})$$

with $C := e^{1/e}$. Suppose now that $u \in W^{1,p}(\mathbb{R})$. By THEOREM 7.24 there $\exists \{u_n\} \subseteq C_c^1(\mathbb{R})$ s.t. $u_n \to u$ strongly in $W^{1,p}(\mathbb{R})$. By applying $(**)$ to $(u_n - u_m) \in C_c^1(\mathbb{R})$ we have

$$\|u_n - u_m\|_{L^\infty} \le C \|u_n - u_m\|_{W^{1,p}} \to 0 \quad \text{as } n, m \to +\infty,$$

Since u_n is convergent in $W^{1,p}(\mathbb{R})$ and so it is Cauchy, being $W^{1,p}$ complete (see PROPOSITION 7.16). Therefore $\{u_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$.
As $L^\infty(\mathbb{R})$ is complete, we conclude the $\exists$ of $\tilde{u} \in L^\infty(\mathbb{R})$ s.t. $u_n \to \tilde{u}$ strongly in $L^\infty(\mathbb{R})$.
Recalling that $u_n \to u$ in $W^{1,p}(\mathbb{R})$, we immediately conclude that $\tilde{u} = u$.
By $(**)$ we have

$$\|u_n\|_{L^\infty} \le C \|u_n\|_{W^{1,p}}, \quad \forall n \in \mathbb{N}$$

Since $u_n \to u$ in $L^\infty(\mathbb{R})$ and in $W^{1,p}(\mathbb{R})$, we can pass to the limit as $n \to +\infty$ in the above and obtain our thesis:

$$\|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}, \quad \forall u \in W^{1,p}(\mathbb{R}).$$

(a) Assume I bounded. We need to prove that the embedding

$$W^{1,p}(I) \hookrightarrow C(\bar{I})$$

is compact, for all $1 < p \leq +\infty$. Therefore let $B \subseteq W^{1,p}(I)$ be a bounded set, so that there $\exists\, M > 0$ s.t.

$$\|u\|_{W^{1,p}} \leq M, \qquad \forall u \in B.$$

By the embedding we just proved, it follows that

$$\|u\|_{L^\infty} \leq C\, \|u\|_{W^{1,p}} \leq CM, \qquad \forall\, u \in B.$$

Recalling that $W^{1,p}(I) \subseteq C(\bar{I})$ (see THEOREM 7.19), we get $\|u\|_\infty = \|u\|_{L^\infty}$, so that

$$(*) \qquad \|u\|_\infty \leq CM, \qquad \forall u \in B.$$

Moreover, by THEOREM 7.23 we have $W^{1,p}(I) \subseteq C^{0,1-1/p}(I)$, if $p > 1$, with

$$|u(x) - u(y)| \leq \|\dot{u}\|_{L^p}\, |x-y|^{1-1/p}, \qquad \forall x, y \in \bar{I}.$$

As $\|\dot{u}\|_{L^p} \leq M$ for all $u \in B$, we conclude that

$$(**) \qquad |u(x) - u(y)| \leq M\, |x-y|^{1-1/p}, \qquad \forall x, y \in \bar{I}, \quad \forall u \in B$$

which shows that the family $B \subseteq C(\bar{I})$ is EQUI-CONTINUOUS. As $(*)$ - $(**)$ hold, we can apply the ASCOLI - ARZELÀ THEOREM 7.28 with $K = \bar{I}$, to conclude that $\bar{B}$ is compact in $C(\bar{I})$ (where the closure is taken WRT the uniform norm in $C(\bar{I})$). Thus, (a) is established.

(b) Let I be bounded, $1 \leq q < +\infty$. We need to prove that the embedding

$$W^{1,1}(I) \hookrightarrow L^q(I)$$

is compact. So let $B \subseteq W^{1,1}(I)$ be a bounded set, i.e.

$$\|u\|_{W^{1,1}(I)} \leq M, \quad \forall u \in B.$$

Let $P: W^{1,1}(I) \to W^{1,1}(\mathbb{R})$ be the extension operator from LEMMA 7.25. By the properties of P, the set $P(B)$ is bounded in $W^{1,1}(\mathbb{R})$, and also $P(B)|_I = B$, where

$$P(B)|_I := \{ u : I \to \mathbb{R} \mid \exists v \in P(B) \text{ s.t. } v|_I = u \}.$$

By the embedding $W^{1,1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we already proved, we have that $P(B)$ is also bounded in $L^\infty(\mathbb{R})$. Then, for $u \in P(B)$ we have

$$\int_{\mathbb{R}} |u|^q \, dx = \int_{\mathbb{R}} |u|^{q-1} |u| \, dx \leq \|u\|_{L^\infty}^{q-1} \|u\|_{L^1}, \qquad \forall u \in P(B)$$

showing that $P(B)$ is also bounded in $L^q(\mathbb{R})$, i.e., $\exists M > 0$ s.t.

$$(*) \qquad \|u\|_{L^q(\mathbb{R})} \leq M, \qquad \forall u \in P(B).$$

We now check that

$$(**) \qquad \lim_{|h| \to 0} \sup_{u \in P(B)} \| \tau_h u - u \|_{L^q(\mathbb{R})} = 0 .$$

Indeed, by THEOREM 7.29 (implication (a) $\Rightarrow$ (b) with $p=1$) we have

$$(***) \qquad \| \tau_h u - u \|_{L^1(\mathbb{R})} \leq \| \dot{u} \|_{L^1(\mathbb{R})} |h| \leq C|h|, \qquad \forall u \in P(B)$$

Since $P(B)$ is bounded in $W^{1,1}(\mathbb{R})$

Therefore, for $u \in P(B)$,

$$\| \tau_h u - u \|^q_{L^q(\mathbb{R})} = \int_{\mathbb{R}} |\tau_h u - u|^q \, dx = \int_{\mathbb{R}} |\tau_h u - u|^{q-1} \, |\tau_h u - u| \, dx$$

$$\leq \| \tau_h u - u \|^{q-1}_{L^\infty(\mathbb{R})} \; \| \tau_h u - u \|_{L^1(\mathbb{R})}$$

$$\left(\| \tau_h u - u \|_{L^\infty} \leq 2 \| u \|_{L^\infty} \right) \quad \leq 2^{q-1} \| u \|^{q-1}_{L^\infty(\mathbb{R})} \; \| \tau_h u - u \|_{L^1(\mathbb{R})}$$

$$\left(\begin{array}{l} \text{By } (**{*}) \text{ and the fact} \\ \text{that } P(B) \text{ is bounded in } L^\infty(\mathbb{R}) \end{array} \right) \quad \leq 2^{q-1} \, \tilde{C} \, C \, |h|$$

showing (**). Since (*) - (**) hold, and I bounded, $q \neq +\infty$, we can apply FRÉCHET-KOLMOGOROV THEOREM 6.17 to conclude that the closure of $P(B)|_I$ is compact in $L^q(I)$. Recalling that $P(B)|_I = B$, we have that the closure of B is compact in $L^q(I)$.

(c) Let $I \subseteq \mathbb{R}$ be bounded. We are left to show that the embedding

$$(*) \qquad W^{1,p}(I) \hookrightarrow L^p(I)$$

is compact for every $1 \leq p \leq +\infty$. Indeed, for $p = 1$, (*) is just a special case of (b) with $q = 1$. Instead, for $1 < p \leq +\infty$, (*) follows from the compact embedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ of point (a), and from the fact that uniform convergence implies L^p convergence. □

REMARK 7.30

We want to discuss (without proof) what happens in the cases left out from THEOREM 7.27.

1. For the compact embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$, I bounded, $1 < p \leq +\infty$:

   - Let I be bounded. We have that $W^{1,1}(I)$ embeds into $C(\bar{I})$ (by THEOREM 7.9), but the embedding is in general NOT compact.

   - What kind of compactness can we expect in this case? The answer is as follows: Let $I \subseteq \mathbb{R}$ be open (bounded or unbounded). If $\{u_n\} \subseteq W^{1,1}(I)$ is bounded, there exists a subsequence u_{n_k} s.t. $u_{n_k}(x)$ converges pointwise for all $x \in \bar{I}$

     (this is called HELLY'S SELECTION THEOREM)

2. Concerning the embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$ for all $1 \leq p \leq +\infty$:

   - When I is unbounded, the above embedding is NEVER COMPACT

   - Assume I unbounded and $1 < p \leq +\infty$. If $\{u_n\} \subseteq W^{1,p}(I)$ is bounded, then $\exists\, u \in W^{1,p}(I)$ and a subsequence s.t. $u_{n_k} \to u$ in $L^\infty(J)$ for every $J \subseteq I$ bounded.

3. Let I be unbounded. Then $W^{1,p}(I) \hookrightarrow L^q(I)$ for all $q \in [p, \infty]$. However, in general, $W^{1,p}(I)$ does NOT embed into $L^q(I)$ if $q \in [1, p)$.

We want to explicitely state a Corollary of THEOREM 7.27 regarding weak convergence. To this end, we first recall the general definition of compact operator.

DEFINITION Let X, Y be normed spaces, and $T \in \mathcal{J}(X,Y)$. We say that T is **COMPACT** if it holds:

$$B \subseteq X \text{ bounded WRT } \|\cdot\|_X \Rightarrow \overline{T(B)}^{\|\cdot\|_Y} \text{ compact WRT } \|\cdot\|_Y$$

PROPOSITION 7.31 Let X, Y be normed spaces, and $T \in \mathcal{J}(X,Y)$ be compact. It holds:

$$x_n \rightharpoonup x_0 \text{ weakly in } X \Rightarrow Tx_n \to Tx_0 \text{ strongly in } Y$$

Proof. Assume $x_n \rightharpoonup x_0$ weakly in X. Since X is a normed space, we have that $\{x_n\}$ is bounded WRT $\|\cdot\|$.
Thus, by definition of compact operator, $\overline{\{Tx_n\}}^{\|\cdot\|_Y}$ is compact WRT $\|\cdot\|_Y$. Therefore, as $\{Tx_n\} \subseteq \overline{\{Tx_n\}}^{\|\cdot\|_Y}$, there $\exists$ a subsequence and $y \in Y$ s.t.

$$(*) \quad Tx_{n_k} \to y \quad \text{strongly in } Y.$$

Now, we know that $x_n \rightharpoonup x_0$ and T continuous. Thus (easy check)

$$(**) \quad Tx_n \rightharpoonup Tx_0 \quad \text{weakly in } Y.$$

Since $(*)$ holds, and strong convergence implies weak convergence, we get $Tx_{n_k} \rightharpoonup y$ weakly in Y. By $(**)$ and uniqueness of the weak limit we get $y = Tx_0$. Therefore $(*)$ reads

$$(***) \quad Tx_{n_k} \to Tx_0 \quad \text{strongly in } Y.$$

To conclude, we use the following standard fact:

FACT (X,τ) topological space, $\{x_n\} \subseteq X$. Suppose that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a subsequence $\{x_{n_{k_j}}\}$ such that

$$x_{n_{k_j}} \to x_0 \quad \text{as } j \to +\infty,$$

for some $x_0 \in X$ which does not depend on the subsequence $\{x_{n_k}\}$ chosen. Then $x_n \to x_0$.

Therefore, reasoning as above, we could have started from an arbitrary subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$, and shown that $\exists$ $\{Tx_{n_{k_j}}\}$ such that

$$Tx_{n_{k_j}} \to Tx_0 \text{ strongly in } Y, \text{ as } j \to +\infty.$$

Since the limit does not depend on the chosen subsequence $\{Tx_{n_k}\}$, we conclude that $Tx_n \to Tx_0$ strongly in Y. $\square$

COROLLARY 7.32 Let $I=(a,b)$ be bounded, and $1 \le p < +\infty$. If

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(a,b)$$

(i.e., $u_n \rightharpoonup u$, $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^p(a,b)$), then

$$u_n \to u \text{ strongly in } L^p(a,b).$$

Proof By point (c) of THEOREM 7.27 we have that $W^{1,p}(a,b) \hookrightarrow L^p(a,b)$ is compact for every $1 \le p \le +\infty$. The thesis follows by applying PROPOSITION 7.31. $\square$