LESSON 1 - 3 MARCH 2021

CALCULUS OF VARIATIONS: The study of minimization problems

X = set, $F : X \to \mathbb{R}$ (often $\mathbb{R} \cup \{\pm\infty\}$) function

We want to solve:

$$\min\{F(u) \mid u \in X\} \quad , \quad \operatorname{argmin}\{F(u) \mid u \in X\}$$

This is a real number, called MINIMUM

These are elements of X, called MINIMIZERS

We will look at the following classes of methods to study minimization problems:

- INDIRECT METHODS
- DIRECT METHODS
- RELAXATION
- Γ - CONVERGENCE

Let us see some basic examples to see what the above words mean:

EXAMPLE 1 $F : \mathbb{R} \to \mathbb{R}$, $F(x) = x^2 - 4x$. What is $\min\{F(x) \mid x \in \mathbb{R}\}$?

INDIRECT METHOD: Find candidate minimizers. If $\hat{x}$ is min. then $F'(\hat{x}) = 0$. Now $F'(x) = 2x - 4 = 0$ iff $x = 2$

So $\hat{x} = 2$ is a candidate minimizer and minimum value is $F(2) = -4$

CLAIM $\overset{1}{x} = 2$ is the UNIQUE minimizer of F. Thus

$$\min_{\mathbb{R}} F = -4 \quad , \quad \operatorname{argmin}_{\mathbb{R}} F = \{2\} \subseteq \mathbb{R}$$

Proof We need to show that

1) $F(x) \geq F(2) \quad \forall x \in \mathbb{R} \quad (\hat{x} = 2$ is minimizer$)$
2) $F(x) > F(2) \quad \forall x \in \mathbb{R} \setminus \{2\} \quad (\overset{1}{x} = 2$ is unique min.$)$

1) $F(x) \geq F(2) \iff x^2 - 4x \geq -4 \iff (x-2)^2 \geq 0 \iff x = 2$
2) $F(x) > F(2) \iff (x-2)^2 > 0 \iff x \neq 2$ $\square$

EXAMPLE 2 DIRECT METHOD: proving existence of a minimizer by general results

Ex: $F: \mathbb{R} \to \mathbb{R}$ continuous and coercive, i.e.,

$$\lim_{|x| \to +\infty} F(x) = +\infty$$

Then $\exists$ minimizer by Weierstrass Theorem

Example 3 RELAXATION: This technique is relevant when a minimizer does not exist, e.g.,

$$(*) \quad \min\{(x^2-2)^2 \mid x \in \mathbb{Q}\}$$

Solution of $(*)$ would be $\overset{1}{x} = \pm\sqrt{2}$ which is not in $\mathbb{Q}$. So in this case there is no minimum. But we are left with 2 questions

1) What is

$$\inf \{ (x^2-2)^2 \mid x \in \mathbb{Q} \} \quad ?$$

2) If $\{x_n\}$ is MINIMIZING SEQUENCE, i.e.

$$F(x_n) \to \inf \{ F(x) \mid x \in \mathbb{Q} \}, \quad F(x) = (x^2-2)^2$$

what can we say about accumulation points of $\{x_n\}$?

Answer: 1) As we guessed, min over $\mathbb{R}$ would be $x^1 = \pm\sqrt{2}$, so

$$\inf \{ (x^2-2)^2 \mid x \in \mathbb{Q} \} = F(\pm\sqrt{2}) = 0$$

2) $x_n \to \sqrt{2}$ OR $x_n \to -\sqrt{2}$ (up to subsequences)

Relaxation is useful to treat problems such as $(*)$. To ensure that a minimizer exists one could, for example,

- Extend F over some set $\hat{X}$ with $\hat{X} \supseteq X$ ($\hat{X} = \mathbb{R}$ for $(*)$)
- Change F so that a minimizer is more likely to exist

EXAMPLE 4 Γ-CONVERGENCE: We have a family of problems

$$\min \{ F_n(x) \mid x \in X \}, \quad F_n : X \to \mathbb{R}, \quad n \in \mathbb{N}$$

What happens as $n \to +\infty$? We hope to find $F_\infty : X \to \mathbb{R}$ such that

1) $\min \{ F_n \mid x \in X \} \to \min \{ F_\infty \mid x \in \mathbb{R} \}$ as $n \to +\infty$

2) If $x_n \in \operatorname{argmin} \{ F_n \mid x \in X \}$ then $x_n \to x_\infty$ with

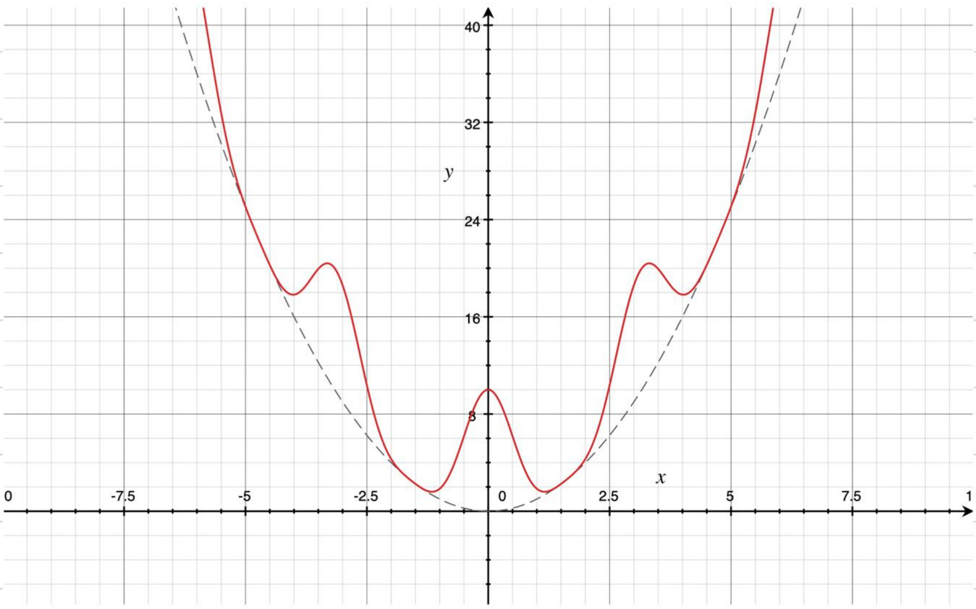
$$x_\infty \in \operatorname{argmin} \{ F_\infty \mid x \in X \}$$

F_∞ is the Γ-limit of $\{F_n\}$ as $n \to +\infty$

For example consider

$$m_n = \min\{F_n(x) \mid x \in \mathbb{R}\}, \quad F_n(x) = x^2 + n\cos^4 x$$

What is the limit of m_n?



Dashed $y = x^2$

Red F_n, $n = 10$

- F_n is sum of two positive terms
- x^2 small $\iff x \simeq 0$
- $n\cos^4 x$ small $\iff \cos x \simeq 0$

True when $x = \pm\frac{\pi}{2}$

Indeed one has

1) $m_n \to \left(\frac{\pi}{2}\right)^2$ as $n \to +\infty$

2) $\{x_n\}$ minimizing sequence converges (up to subsequences) to $\pm\pi/2$.

3) The Γ-limit is

$$F_\infty(x) = \begin{cases} x^2 & \text{if } \cos x = 0 \\ +\infty & \text{otherwise} \end{cases}$$

INTEGRAL FUNCTIONALS

This course mainly focusses on integral functionals

$X =$ some functions space, e.g.,

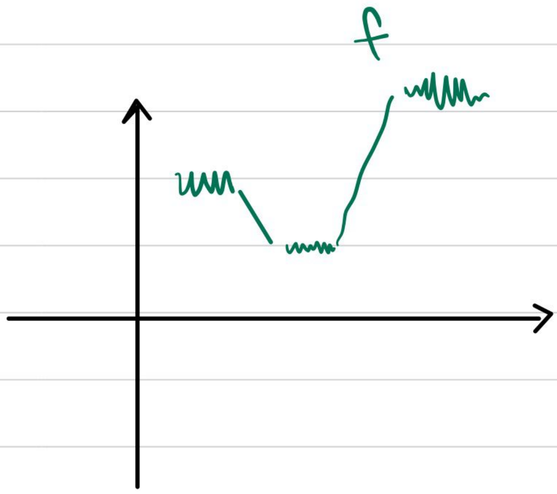
$$X = C^k[a,b] = \{ u : [a,b] \to \mathbb{R} \mid u \text{ } k\text{-times continuously differentiable} \}$$

and $F : X \to \mathbb{R}$ is of the form

$$F(u) := \int_a^b L(x, u(x), u'(x), \dots, u^{(k)}(x))\, dx$$

with $L : [a,b] \times \mathbb{R}^k \to \mathbb{R}$ LAGRANGIAN

Typically $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, p)$



EXAMPLE 1 (DENOISING)

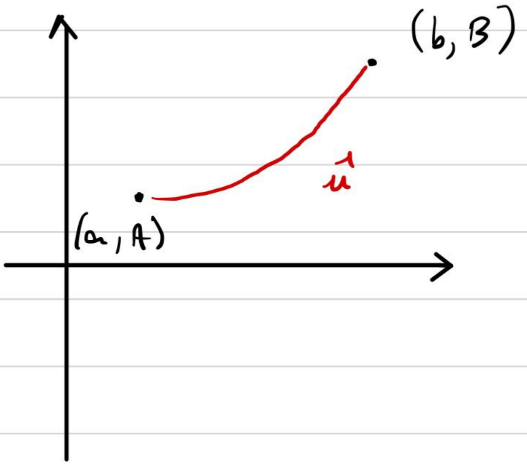
We receive a signal $f : [0,1] \to \mathbb{R}$ which we want to denoise

This task is achieved by solving

$$\hat{u} \in \operatorname{argmin} \left\{ \int_0^1 u'^2 + (u-f)^2\, dx \;\middle|\; u \in C^1[0,1] \right\}$$

Note
- u' penalizes oscillations
- $u - f$ penalizes discrepancy from the noisy signal f

Example 2 (Hanging Rope) Find the profile of a rope hanging at (a, A), (b, B)



The energy of a profile $u : [a,b] \to \mathbb{R}$ is modelled by

$$E(u) = \int_a^b \dot{u}^2 + u \, dx \quad , \; u(a) = A, u(b) = B$$

$\dot{u}^2$: elastic energy

u: potential energy

<u>Note</u> 1) u can be negative which lowers E

2) Due to boundary conditions, if $u \ll 0$ then $\dot{u} \gg 0 \Rightarrow E$ higher

The solution

$$\hat{u} \in \operatorname{argmin} \{ E(u) \mid u \in C^1[a,b], u(a) = A, u(b) = B \}$$

will be a balance between (1) and (2)

PROBLEMS WE WILL NOT TALK ABOUT

1) GEOMETRIC FUNCTIONALS:

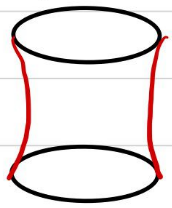
- <u>DIDO'S PROBLEM</u>: $\min \{ \text{Area}(\partial V) \mid V \subseteq \mathbb{R}^3, \text{Vol}(V) = 1 \}$

Intuitively the sol is a sphere. However proving it when not requiring regularity on V requires advanced tools (Geometric Measure Theory)

- PLATEAU'S PROBLEM : Given γ collection of curves in $\mathbb{R}^3$ find

$$\min \{ \text{Area}(S) \mid S \subseteq \mathbb{R}^3 \text{ surface}, \ \partial S = \gamma \}$$

γ



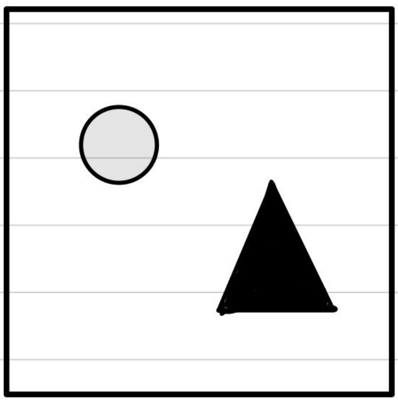
$\overset{1}{S}$ CATENOID is solution

Again this requires GMT

2) IMAGING FUNCTIONALS : used for tasks such as DEnoising, Segmentation, reconstruction of medical data. Usually

$$u : \Omega \to \mathbb{R} \quad , \ \Omega \subseteq \mathbb{R}^2, \mathbb{R}^3$$

Ω



u encodes gray-scale value of pixels of a picture in the frame Ω

Example : To segment the image at the left, i.e. find contours of shapes within it, one could minimize the MUMFORD-SHAH functional:

$$F(u, K) = \underbrace{\int_{\Omega \setminus K} |\nabla u|^2 dx}_{(I)} + \underbrace{\int_{\Omega} |u - f|^2 dx}_{(II)} + \underbrace{\text{Length}(K)}_{(III)}$$

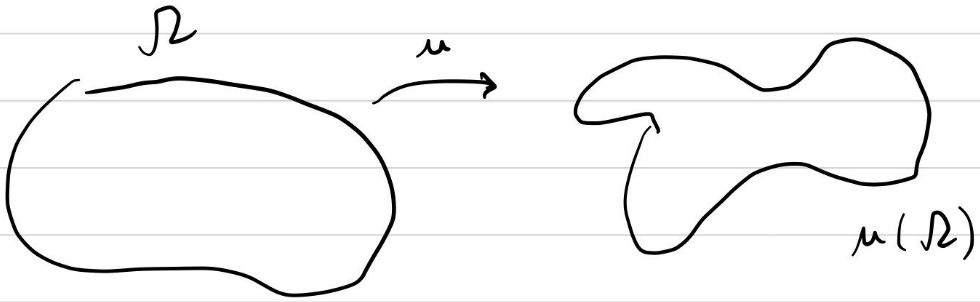
- f is the original picture, generally noisy. Want to clean f and detect edges K within it
- (I) : enforces smoothness of u outside of K (we don't want to pay energy for the natural transitions)

- (II) : Enforces the clean image u to be close to the original f
- (III) : Forces short contours

A solution is then

$$(\hat{u}, \hat{K}) \in \operatorname{argmin} \{ F(u, K) \mid K \subseteq \Omega \text{ compact}, \ u \in C^1(\Omega \setminus K) \}$$

3. VECTORIAL PROBLEMS : For example in materials science $\Omega \subseteq \mathbb{R}^3$ represents the reference configuration of an elastic body, $u : \Omega \to \mathbb{R}^3$ is a deformation



The elastic energy of the deformed configuration is

$$E(u) = \int_\Omega W(\nabla u) \, dx \quad , \quad W : \mathbb{R}^{3\times 3} \to [0, +\infty]$$

In this case the problem of minimizing E is vectorial, and the analysis requires advanced tools (quasi-convexity, etc)

An equilibrium configuration given boundary data $g : \partial\Omega \to \mathbb{R}^3$ is

$$\min \{ E(u) \mid u \in C^1(\Omega; \mathbb{R}^3), \ u = g \text{ on } \partial\Omega \}$$

BASIC FUNCTIONAL ANALYSIS (Revision)

REFERENCE: J. B. CONWAY
"A COURSE IN FUNCTIONAL ANALYSIS"
SECOND EDITION, SPRINGER, 1997

METRIC SPACE X set, $d: X \times X \to [0, +\infty)$. We say that d is a **METRIC** over X if

- $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x)$, $\forall x, y \in X$ (Symmetric)
- $d(x,y) \leq d(x,z) + d(y,z)$, $\forall x, y, z \in X$ (Triangle inequality)

The pair (X, d) is called a **Metric Space**

CONVERGENCE For $\{x_n\} \subseteq X$ we say that $x_n \to x_0$ if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_0) < \varepsilon \text{ if } n \geq N_\varepsilon$$

CAUCHY SEQUENCE $\{x_n\} \subseteq X$ is a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_m) < \varepsilon \text{ if } n, m \geq N_\varepsilon$$

COMPLETENESS A metric space (X, d) is complete if every Cauchy sequence $\{x_n\} \subseteq X$ converges to some $x_0 \in X$.

Topology generated by d (X, d) metric space. Define

$$\tau := \{ A \subseteq X \mid \forall x \in A, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A \}$$

with $B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$. Then τ is a **TOPOLOGY** over X. The sets in τ are called OPEN. $A \subseteq X$ is CLOSED if $A^c := X \setminus A$ is open.

NOTATION (X, τ) topological space, $A \subseteq X$. We denote by

- $\mathring{A}$ the **INTERIOR** of A: $\mathring{A} = \bigcup \{ O \mid O \subseteq A, O \text{ open} \}$
- $\bar{A}$ the **CLOSURE** of A: $\bar{A} = \bigcap \{ C \mid A \subseteq C, C \text{ closed} \}$

In other words:

- $\mathring{A}$ is the largest open set contained in A
- $\bar{A}$ is the smallest closed set which contains A

DENSITY (X, d) metric space. $D \subseteq X$ is DENSE in X if $\bar{D} = X$

SEPARABILITY (X, d) metric space is SEPARABLE if $\exists$ a COUNTABLE set $D \subseteq X$ which is dense, i.e., $\bar{D} = X$

LIMITS (X, d_X), (Y, d_Y) metric spaces, $U \subseteq X$ open, $F: U \to Y$, $x_0 \in U$. We say that $F(x) \to L$ as $x \to x_0$, in symbols

$$\lim_{x \to x_0} F(x) = L,$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(F(x), L) < \varepsilon$ if $d_X(x, x_0) < \delta$

CONTINUITY (X, d_X), (Y, d_Y) metric spaces, $U \subseteq X$ open, $F: U \to Y$. We say that F is continuous at $x_0 \in U$ if $F(x) \to F(x_0)$ as $x \to x_0$. F is continuous in U if it is continuous $\forall x_0 \in U$.

NORMED SPACE X vector space over $\mathbb{R}$, $\|\cdot\| : X \to [0, +\infty)$. We say that $\|\cdot\|$ is a norm over X if

- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \, \|x\|$, $\forall \lambda \in \mathbb{R}$, $x \in X$ (1-homogeneous)
- $\|x + y\| \le \|x\| + \|y\|$, $\forall x, y \in X$ (subadditive)

The pair $(X, \|\cdot\|)$ is called normed space

REMARK $(X, \|\cdot\|)$ normed space. Then $d(x,y) = \|x - y\|$ is a metric over X. In particular X is a topological space with τ induced by d. By convention all the topological notions in X are given WRT τ.

BANACH SPACE $(X, \|\cdot\|)$ normed space is BANACH if (X, d) with $d(x,y) = \|x-y\|$ is complete.

LINEAR OPERATORS X, Y normed spaces, $T: X \to Y$. We say that

- T is **LINEAR** if $T(\lambda x + y) = \lambda Tx + Ty$, $\forall \lambda \in \mathbb{R}, x, y \in X$
- T is **BOUNDED** if

$$\sup_{\|x\|_X \leq 1} \|Tx\|_Y < +\infty$$

FACT Let $T: X \to Y$ be linear. Then

$$T \text{ is continuous} \iff T \text{ is bounded}$$

NOTATION $\mathcal{L}(X, Y) := \{ T: X \to Y \mid T \text{ linear bounded} \}$

$X^* := \mathcal{L}(X, \mathbb{R})$ **DUAL SPACE** of X

REMARK 1) $\mathcal{L}(X, Y)$ is a vector space over $\mathbb{R}$, with operations

$$(\alpha T_1 + T_2)(x) := \lambda T_1 x + T_2 x, \quad \forall \alpha \in \mathbb{R}, T_1, T_2 \in \mathcal{L}(X, Y)$$

2) $\mathcal{L}(X, Y)$ is a normed space with norm

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

3) If Y is Banach then $\mathcal{L}(X, Y)$ is Banach

4) X normed space $\Rightarrow$ X^* Banach space

CONVERGENCES $(X, \|\cdot\|)$ normed space $\{x_n\} \subseteq X$, $x_0 \in X$, $\{\varphi_n\} \subseteq X^*$, $\varphi_0 \in X^*$,

1) $x_n \to x_0$ **STRONGLY** if $\|x_n - x_0\|_X \to 0$ as $n \to +\infty$
2) $x_n \rightharpoonup x_0$ **WEAKLY** if $\varphi(x_n) \to \varphi(x_0)$, $\forall \varphi \in X^*$
3) $\varphi_n \overset{*}{\rightharpoonup} \varphi_0$ **WEAKLY*** if $\varphi_n(x) \to \varphi_0(x)$, $\forall x \in X$
4) $\varphi_n \to \varphi_0$ **STRONGLY** if $\|\varphi_n - \varphi_0\|_{X^*} \to 0$

NOTE

- $x_n \to x_0 \implies x_n \rightharpoonup x_0$
- The reverse implication is not true. For example let

$$X = \ell^p := \left\{ (x_1, x_2, \dots, x_n, \dots) \;\middle|\; \sum_{j=1}^{+\infty} |x_j|^p < +\infty \right\}$$

with $1 < p < +\infty$. Recall that $(X, \|\cdot\|)$ is a normed space with

$$\|x\| := \left(\sum_{j=1}^{+\infty} |x_j|^p \right)^{1/p}$$

Let $e_j := (0, \dots, 0, \underset{\substack{\uparrow \\ j\text{-th position}}}{1}, 0, \dots)$. Then $e_j \rightharpoonup 0$ but

$\|e_j\| = 1 \not\to 0$.

DEFINITION $(X, \|\cdot\|)$ normed space, $K \subseteq X$, $\tilde{K} \subseteq X^*$

1) K is **COMPACT** if $\forall \{x_n\} \subseteq K$, $\exists\, x_0 \in K$ s.t. $x_{n_k} \to x_0$ along some subsequence
2) K is **SEQUENTIALLY WEAKLY COMPACT** if $\forall \{x_n\} \subseteq K$, $\exists\, x_0 \in K$ s.t. $x_{n_k} \rightharpoonup x_0$ along some subsequence
3) $\tilde{K}$ is **SEQUENTIALLY WEAKLY* COMPACT** if $\forall \{\varphi_n\} \subseteq \tilde{K}$, $\exists\, \varphi_0 \in \tilde{K}$ s.t. $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi_0$ along some subsequence.

WARNING If (X, τ) is a topological space then $K \subseteq X$ is compact if any OPEN COVER of K admits a FINITE SUBCOVER.
If the topology τ is metrizable (e.g. metric or normed spaces) then SEQUENTIAL COMPACTNESS is equivalent to COMPACTNESS.

However, if X is normed space, then the weak topology on X and weak* topology on X^* are NOT metrizable in general. Thus, in general WEAK (WEAK*) COMPACTNESS and WEAK (WEAK*) SEQUENTIAL COMPACTNESS are not equivalent. With additional assumptions, however, they are the same:

1) If X is Banach then WEAK SEQUENTIAL COMPACTNESS and WEAK COMPACTNESS are equivalent (EBERLEIN - SMULIAN THEOREM)

2) If X is a SEPARABLE BANACH space then WEAK* SEQUENTIAL COMPACTNESS and WEAK* COMPACTNESS are equivalent

THEOREM (BANACH - ALAOGLU) $(X, \|\cdot\|)$ normed space. Denote by $B := \{\varphi \in X^* \mid \|\varphi\|_{X^*} \leq 1\}$ the closed unit ball of X^*:

1) Then B is WEAKLY* COMPACT
2) If in addition X is BANACH and SEPARABLE then B is also SEQUENTIALLY WEAKLY* COMPACT

There is a corollary of Banach-Alaoglu concering the weak compactness of the unit ball of X. For that we need the following definition

REFLEXIVITY $(X, \|\cdot\|)$ normed space. Consider X^* and its dual WRT to the strong norm of X^*, i.e., $X^{**} := (X^*, \|\cdot\|_{X^*})^*$

Define the **CANONICAL EMBEDDING**

$$J : X \to X^{**} \quad \text{s.t.} \quad J(x)(\varphi) := \varphi(x), \quad x \in X, \varphi \in X^*$$

We have $\|J(x)\|_{X^{**}} = \|x\|_X$. We say that X is **REFLEXIVE** if J is surjective, i.e., if

$$X^{**} = \{ J(x), \ x \in X \}$$

COROLLARY (of BANACH-ALAOGLU)

$(X, \|\cdot\|)$ normed space. Define $B := \{ x \in X \mid \|x\| \leq 1 \}$.

1) If X is reflexive then B is WEAKLY COMPACT
2) If X is reflexive and Banach then B is WEAKLY SEQUENTIALLY COMPACT

As a consequence of the PRINCIPLE OF UNIFORM BOUNDEDNESS (PUB) (see book of Conway), we have.

PROPOSITION $(X, \|\cdot\|)$ Banach space

1) If $\{x_n\} \subseteq X$ is s.t. $x_n \rightharpoonup x_0$ then $\sup_n \|x_n\| < +\infty$
2) If $\{\varphi_n\} \subseteq X^*$ is s.t. $\varphi_n \overset{*}{\rightharpoonup} \varphi_0$ then $\sup_n \|\varphi_n\|_{X^*} < +\infty$

Another important notion needed throughout the course is the one of lower semicontinuity.

DEFINITION (X, d) metric space, $F: X \to \mathbb{R}$. We say that F is LOWER SEMICONTINOUS at $x_0 \in X$ if

$$F(x_0) \leq \liminf_{n \to +\infty} F(x_n),$$

for all $\{x_n\} \subseteq X$ s.t. $x_n \to x_0$.

DEFINITION $(X, \|\cdot\|)$ normed space, $F: X \to \mathbb{R}$, $G: X^* \to \mathbb{R}$.

1) F is (SEQUENTIALLY) WEAKLY LOWER SEMICONTINOUS at $x_0 \in X$ if

$$F(x_0) \leq \liminf_{n \to +\infty} F(x_n)$$

for all $\{x_n\} \subseteq X$ s.t. $x_n \rightharpoonup x_0$.

2) G is (SEQUENTIALLY) WEAKLY* LOWER SEMICONTINOUS at $\varphi_0 \in X^*$ if

$$G(\varphi_0) \leq \liminf_{n \to +\infty} G(\varphi_n)$$

for all $\{\varphi_n\} \subseteq X^*$ s.t. $\varphi_n \overset{*}{\rightharpoonup} \varphi_0$.

PROPOSITION $(X, \|\cdot\|)$ normed space. Then

1) The norm $\|\cdot\|$ is WEAKLY SEQUENTIALLY LOWER SEMICONTINOUS, i.e.,

$$x_n \rightharpoonup x_0 \Rightarrow \|x_0\| \leq \liminf_{n \to +\infty} \|x_n\|$$

2) The norm $\|\cdot\|_{X^*}$ is WEAKLY* SEQUENTIALLY LOWER SEMICONTINOUS, i.e.,

$$\varphi_n \overset{*}{\rightharpoonup} \varphi_0 \Rightarrow \|\varphi_0\|_{X^*} \leq \liminf_{n \to +\infty} \|\varphi_n\|_{X^*}$$

LESSON 2 - 10 MARCH 2021

HILBERT SPACES

HILBERT $\subseteq$ BANACH $\subseteq$ COMPLETE METRIC $\subseteq$ TOPOLOGICAL

INNER PRODUCT SPACES Let H be a real vector space. A function $\langle \cdot , \cdot \rangle : H \times H \to \mathbb{R}$ is an INNER PRODUCT if

- $\langle x, y \rangle = \langle y, x \rangle \quad , \forall x, y \in H$ (Symmetric)
- $\langle \lambda x + \mu y , z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle , \forall \lambda, \mu \in \mathbb{R}, x, y \in H$ (Bilinear)
- $\langle x, x \rangle \geq 0 \; , \forall x$ and $\langle x, x \rangle = 0$ iff $x = 0$ (Positive definite)

The pair $(H, \langle \cdot , \cdot \rangle)$ is called an INNER PRODUCT SPACE

REMARK $(H, \langle \cdot , \cdot \rangle)$ inner product space. Then $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm on X.

CAUCHY-SCHWARZ INEQUALITY H inner prod space. Then $|\langle x, y \rangle| \leq \|x\| \, \|y\|$

HILBERT SPACE $(H, \langle \cdot , \cdot \rangle)$ inner product space. We say that H is a HILBERT SPACE if $(H, \| \cdot \|)$ is COMPLETE, with $\|x\| = \sqrt{\langle x, x \rangle}$

BASIS H Hilbert. A set of elements $\{e_n\}_{n \in \mathbb{N}} \subseteq H$ is called BASIS if

- $\langle e_i, e_j \rangle = 0$ for all $i \neq j$, $\langle e_i, e_i \rangle = 1 \quad \forall i$
- $\operatorname{span}\{e_j\}$ is dense in H

where for a set $A \subseteq H$ we define $\operatorname{span} A = \left\{ \sum_{j=1}^{n} \lambda_j x_j \;\middle|\; \lambda_j \in \mathbb{R}, x_j \in A, n \in \mathbb{N} \right\}$

THEOREM If H is Hilbert separable then $\exists \{e_n\} \subseteq H$ basis

NOTATION Given a basis $\{e_n\}$ and $x \in H$ we define $x_k := \langle x, e_k \rangle$ k-th coordinate of x WRT $\{e_n\}$

PROPOSITION H Hilbert with basis $\{e_n\}$. Then

1) $$\|v\|^2 = \sum_{j=1}^{+\infty} v_n^2 \quad , \quad v_n := \langle v, e_n \rangle$$

2) $$\langle v, w \rangle = \sum_{j=1}^{+\infty} v_n w_n$$

For a separable Hilbert space there is a natural correspondence between H and ℓ^2. Thus we can think of H as $\mathbb{R}^\infty$. To make this statement precise we need the following definition

(Recall: $\ell^2 = \{ (x_1, x_2, \dots) \mid \sum_{j=1}^{+\infty} |x_j|^2 < +\infty \}$. This is normed by $\|x\| = \left(\sum_{j=1}^{+\infty} |x_j|^2\right)^{1/2}$ and is Hilbert with inner product $\langle x, y \rangle = \sum_{j=1}^{+\infty} x_j y_j$)

DEF X normed space, $\{x_n\} \subseteq X$. We say that

$$\sum_{n=1}^{+\infty} x_n = x_0$$

if $S_n \to x_0$, where $S_n := \sum_{j=1}^{n} x_j$ partial sums.

THEOREM H Hilbert with basis $\{e_n\}$. Let $\{v_n\} \subseteq \mathbb{R}$. Then

$$\sum_{n=1}^{+\infty} v_n e_n \text{ converges in } H \iff \sum_{n=1}^{+\infty} v_n^2 \text{ converges in } \mathbb{R}$$

In particular $H \cong \ell^2$, with isomorphism $v \in H \mapsto (v_1, v_2, \dots, v_n, \dots) \in \ell^2$

Another nice aspect of Hilbert spaces is that $H = H^*$ dual space.

THEOREM (RIESZ) H Hilbert. Define the map $\Phi : H \to H^*$

$$\Phi(x)(z) := \langle x, z \rangle, \quad \forall z \in H$$

Then Φ is invertible and $\|\Phi(x)\|_{H^*} = \|x\|_H$.

Thus $H \cong H^*$ isomorphic.

In particular H can be identified with H^*. Thus weak* and weak topologies coincide, and we can characterize weak convergence by

$$x_n \rightharpoonup x_0 \iff \langle x_n, z\rangle \to \langle x_0, z\rangle, \ \forall z \in H.$$

FURTHER PROPERTIES OF WEAK CONVERGENCE IN HILBERT

PROP H Hilbert with basis $\{e_n\}$. If $x_n \rightharpoonup x_0$ then

$$(x_n)_k \to (x_0)_k, \quad \forall\, k \in \mathbb{N}$$

WARNING We know that $x_n \rightharpoonup x_0$ does not imply $x_n \to x_0$. However it is also NOT true that $\|x_n\| \to \|x_0\|$ (i.e. the norm is not weakly continuous).

However, the following proposition relating strong convergence to weak convergence holds.

PROP H Hilbert. Then

$$x_n \longrightarrow x_0 \iff x_n \rightharpoonup x_0 \text{ and } \|x_n\| \to \|x_0\|$$

NOTE: This is not saying that $\|x_n - x_0\| \to 0$

Another useful proposition is that weak convergence can be tested against a dense subset

PROP H Hilbert. Assume that $\{x_n\} \subseteq H$ is bounded, i.e. $\sup_n \|x_n\| < +\infty$. Suppose that $W \subseteq H$ is s.t. $\overline{\operatorname{span} W} = H$ and

$$\langle x_n, w\rangle \to \langle x_0, w\rangle, \quad \forall w \in W.$$

Then $x_n \rightharpoonup x_0$.

COROLLARY H Hilbert with basis $\{e_n\}$. Let $\{x_n\}$ be BOUNDED. Then if

$$(x_n)_k \to (x_0)_k \quad , \forall k \in \mathbb{N}$$

we have $x_n \to x_0$.

[Proof: take $W = \{e_n\}$]

EXAMPLE $X = C[a,b]$ with $\|u\|_\infty := \max_{x \in [a,b]} |u(x)|$

$Y = C^1[a,b]$ with $\|u\|_1 := \|u\|_\infty + \|u'\|_\infty$

Then $(X, \|\cdot\|_\infty)$ and $(Y, \|\cdot\|_1)$ are Banach spaces, but not Hilbert spaces.

Hint to show this: in an inner product space the parallelogram law holds:

$$\|x+y\|^2 + \|y-x\|^2 = 2(\|x\|^2 + \|y\|^2)$$
$\forall x, y \in H$

1. CALCULUS IN NORMED SPACES

Reference : S. KESAVAN
"NONLINEAR FUNCTIONAL ANALYSIS, A FIRST COURSE"
HINDUSTAN BOOK AGENCY, 2004

Throughout this section X, Y are real NORMED SPACES, $U \subseteq X$ is OPEN and $F : U \to Y$ is a given function.

GOAL: Construct a theory of differentiation for maps $F : U \to Y$

DEFINITION 1.1 We say that F is **FRÉCHET DIFFERENTIABLE** at $u_0 \in U$ if $\exists\, A_{u_0} \in \mathcal{L}(X, Y)$ s.t.

$$(*) \qquad \lim_{\|v\|_X \to 0} \frac{\| F(u_0 + v) - F(u_0) - A_{u_0}(v) \|_Y}{\|v\|_X} = 0 \,.$$

REMARK If F is diff. at u_0 then $A_{u_0} \in \mathcal{L}(X, Y)$ satisfying $(*)$ is UNIQUE (Check it by exercise)

NOTATION If F is diff. at $u_0 \in U$ we call A_{u_0} the **Fréchet derivative** (or just derivative) of F at u_0. We denote

$$F'(u_0) := A_{u_0} \in \mathcal{L}(X, Y)$$

NOTE This generalizes diff. for maps $F : \mathbb{R}^n \to \mathbb{R}^m$. In this case the differential is $F'(u)(v) = DF(u)\, v$, with $DF(u) \in \mathbb{R}^{m \times n}$ matrix of partial derivatives of F, i.e.

$$[DF(u)]_{ij} = \frac{\partial F_i}{\partial x_j}(u) \,, \qquad F = (F_1, \dots, F_m)$$

DEFINITION 1.2 Assume that F is diff. $\forall u_0 \in U$. Then we can define the map

$$F' : U \to \mathcal{L}(X,Y)$$
$$u \mapsto F'(u)$$

If F' is continuous we say that $F \in C^1(U,Y)$

WRT norm on X and operator norm on $\mathcal{L}(X,Y)$

In words we say F is continuously diff.

EXAMPLES The most common examples throughout the course will be real valued functions, i.e., $Y = \mathbb{R}$

1) X normed, $U \subseteq X$ open, $F: U \to \mathbb{R}$. If F is diff at $u_0 \in U$ then $F'(u_0) \in \mathcal{L}(X, \mathbb{R}) = X^*$

Then if F diff in U, the derivative defines an application $F' : U \to X^*$ $(u \mapsto F'(u) \in X^*)$

2) H Hilbert, $U \subseteq H$ open, $F: U \to \mathbb{R}$. If F is diff. at $u_0 \in U$ then $F'(u_0) \in \mathcal{L}(H, \mathbb{R})$. By Riesz's Thm $\exists !$ $z_0 \in H$ s.t.

$$F'(u_0)(w) = \langle z_0, w \rangle \quad , \forall w \in H$$

We denote $z_0 := \text{grad}\, F(u_0)$ (gradient of F at u_0).

PROPOSITION 1.3 Assume that F is diff at $u_0 \in U$. Then F is continuous at u_0.

Proof Introduce the notation $o(\|v\|_X)$ for a quantity such that

$$\frac{o(\|v\|_X)}{\|v\|_X} \to 0 \text{ as } \|v\|_X \to 0$$

, Since U is open and $u_0 \in U$ then $\exists \varepsilon > 0$ such that $B_\varepsilon(u_0) \subseteq U$. Let $v \in B_\varepsilon(0)$ so that $u_0 + v \in B_\varepsilon(u_0) \subseteq U$: then

$$\begin{aligned}
\| F(u_0+v) - F(u_0)\|_Y &\leq \| F(u_0+v) - F(u_0) - A_{u_0}(v)\|_Y + \| A_{u_0}(v)\|_Y \\
&\leq o(\|v\|_X) + \|A_{u_0}\|_{\mathcal{L}(X,Y)} \|v\|_X \quad \text{(since } A_{u_0} \in \mathcal{L}(X,Y)\text{)} \\
&= o(\|v\|_X) \longrightarrow 0 \text{ as } \|v\|_X \to 0
\end{aligned}$$

□

THEOREM 1.4 (CHAIN RULE) X, Y, Z normed, $U \subseteq X$, $V \subseteq Y$ open, $F: U \to V$, $G: V \to Z$. Assume F is diff at $u_0 \in U$ and G is diff. at $v_0 := F(u_0) \in V$. Then $G \circ F : U \to Z$ is diff. at u_0 with

$$(G \circ F)'(u_0) = G'(v_0) \circ F'(u_0) \in \mathcal{L}(X, Z)$$

composition of linear continuous operators in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$

The proof is very simple, and we thus omit it. If you are interested you can find it in the book of KESAVAN, PROPOSITION 1.1.1 page 7

DEFINITION 1.5 We say that F is GÂTEAUX DIFFERENTIABLE at $u_0 \in U$ in the DIRECTION $v \in X$ if

$$F_g'(u_0)(v) := \lim_{t \to 0} \frac{F(u_0 + tv) - F(u_0)}{t} \in Y,$$

Here $t \in \mathbb{R}$

i.e., if the above limit exists.

NOTE The Gâteaux derivative (G. derivative) generalizes the directional derivative for maps $F: \mathbb{R}^n \to \mathbb{R}$.
In this case we have

$$F'_G(u)(v) = \frac{\partial F}{\partial v}(u) = \nabla F(u_0) \cdot v$$

If F diff at u

Scalar product in $\mathbb{R}^n$

WARNING $F'_G(u): X \to Y$ is always linear. But in general we don't have that $F'_G(u) \in \mathcal{L}(X,Y)$, as it happens for Fréchet derivatives.

PROPOSITION 1.6 If F diff. at $u_0 \in U$ then F is G. diff at u_0 in every direction v and

$$F'_G(u_0)(v) = F'(u_0)(v)$$

Gâteaux derivative can be computed by Frechet derivative, and viceversa

Proof

$$\frac{F(u_0+tv) - F(u_0)}{t} =$$

$$= \underbrace{\frac{F(u_0+tv) - F(u_0) - F'(u_0)(tv)}{t\|v\|_X}}_{\to 0 \text{ as } t \to 0,} \|v\|_X + \frac{F'(u_0)(tv)}{t}$$

since F is diff. at u_0

(note this is converging in Y to 0)

$$= o(t) + F'(u_0)(v) \to F'(u_0)(v) \text{ as } t \to 0$$

$\square$

Here we used that $F'(u_0)$ is lin. operator, so $F'(u_0)(tv) = tF'(u_0)(v)$

WARNING The converse of PROP 1.6 does not hold, i.e.

$$\text{Gâteaux Diff.} \not\Rightarrow \text{Fréchet Diff.}$$

For example take $F: \mathbb{R}^2 \to \mathbb{R}$

$$F(x,y) := \begin{cases} \dfrac{x^5}{(y-x^2)^2+x^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

It is easy to check that $F'_G(0)(v) = 0$, $\forall v \in \mathbb{R}$. So F is G-diff at $0=(0,0)$ in every direction. Thus, if F was Fréchet diff we would have (by PROP 1.6) that $F'(0)(v) = 0$. Thus by def of derivative

$$(*) \qquad \lim_{\|v\|_{\mathbb{R}^2} \to 0} \frac{|F(v) - F(0)|}{\|v\|_{\mathbb{R}^2}} = 0$$

However if in the above limit we consider $v \in \{y = x^2\}$ we obtain

$$\lim_{\substack{\|v\|_{\mathbb{R}^2} \to 0 \\ v \in \{y=x^2\}}} \frac{|F(v) - F(0)|}{\|v\|_{\mathbb{R}^2}} = 1,$$

which contradicts $(*)$. Thus F is not Fréchet diff at 0.

REMARK PROPOSITION 1.6 is very useful to guess the Fréchet derivative of a function $F: U \to Y$, as the Gâteaux derivative can be computed via a formula

EXAMPLE $X = C[0,1]$, with norm $\|\cdot\|_\infty$. Define $F : X \to \mathbb{R}$ by

$$F(u) = \int_0^1 \sin(u(x))\, dx$$

What could be the derivative of F? Let us compute the Gâteaux derivative at u in the direction v:

$$\lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} = \lim_{t\to 0} \int_0^1 \frac{\sin(u+tv) - \sin(u)}{t}\, dx$$

From the chain Rule (THEOREM 1.4) we have:

$$\lim_{t\to 0} \frac{\sin(u(x) + tv(x)) - \sin(u(x))}{t} = \cos(u(x))\, v(x) \quad , \text{ uniformly in } x \in [0,1]$$

Thus we can pass the limit under the integral and obtain

$$F_g'(u)(v) = \int_0^1 \cos(u(x))\, v(x)\, dx . \qquad \left(F_g'(u) \text{ is linear!} \right)$$

If F is Fréchet diff. then by PROPOSITION 1.6 we must have $F'(u) = F_g'(u)$. So we guess that $F_g'(u)$ is the Fréchet derivative of F at u. Indeed notice that $F_g'(u) \in \mathcal{L}(X, \mathbb{R})$, since

$$|F_g'(u)(v)| \leq \|v\|_\infty \int_0^1 |\cos(u(x))|\, dx \leq \|v\|_\infty \Rightarrow \sup_{\|v\|_X \leq 1} |F_g(u)(v)| < +\infty .$$

Moreover it is easy to see that

$$\lim_{\|v\|_X \to 0} \frac{|F(u+v) - F(u) - F_g'(u)(v)|}{\|v\|_X} = 0$$

showing that F is Fréchet diff. at u with $F'(u)(v) = \int_0^1 \cos(u(x))\, v(x)\, dx$.

QUESTION Assume that $F : U \to Y$ is Gâteaux differentiable. Under which assumptions on F are we guaranteed Fréchet differentiability?

To answer the above question, we need the following theorem.

THEOREM 1.7 (MEAN VALUE)

Suppose F is G. diff. in U in every direction. Let $x_1, x_2 \in U$ be such that the segment

$$[x_1, x_2] := \{ x_1 + t(x_2 - x_1),\ t \in [0,1] \} \subseteq U$$

Assume also that $F_g'(u) : X \to Y$ is s.t. $F_g'(u) \in \mathcal{L}(X,Y)$, $\forall u \in U$.
Then

$$\| F(x_2) - F(x_1) \|_Y \leq \sup_{u \in [x_1, x_2]} \| F_g'(u) \|_{\mathcal{L}(X,Y)} \| x_2 - x_1 \|_X$$

Proof If $F(x_1) = F(x_2)$ then the thesis is trivial. Thus assume that $F(x_1) \neq F(x_2)$. We now employ the following:

FACT (COROLLARY OF HAHN-BANACH) Y normed space, $z \in Y$, $z \neq 0$. Then $\exists\, \Lambda \in Y^*$ s.t.

$$\| \Lambda \|_{Y^*} = 1 \quad \text{and} \quad \Lambda(z) = \| z \|_Y .$$

Thus let $\Lambda \in Y^*$ be such that $\Lambda(F(x_2) - F(x_1)) = \| F(x_2) - F(x_1) \|_Y$ and $\| \Lambda \|_{Y^*} = 1$.
Define the segment function $\alpha : [0,1] \to U$ by $\alpha(t) := x_1 + t(x_2 - x_1)$.
Consider the map $H : [0,1] \to \mathbb{R}$ defined by

$$H := \Lambda \circ F \circ \alpha .$$

The thesis is obtained by applying the classical Mean Value Theorem to H.
Thus, all we need to show is that H is differentiable.

WARNING: It would be tempting to apply the Chain Rule of Theorem 1.4 to H. However F is only Gâteaux differentiable, and in general the Chain Rule does not apply in this case.

We will check by hand that H is differentiable. Thus let $t \in [0,1]$, and $\tau \neq 0$ be such that $(t+\tau) \in [0,1]$. We have

$$(*) \qquad \frac{H(t+\tau) - H(t)}{\tau} = \Lambda \left[\frac{F(\alpha(t+\tau)) - F(\alpha(t))}{\tau} \right] \qquad \text{(being } \Lambda \text{ linear)}$$

Note that $F(\alpha(t+\tau)) = F(\alpha(t) + \tau(x_2 - x_1))$. Since F is gâteaux diff. at $\alpha(t)$ we then get

$$\frac{F(\alpha(t+\tau)) - F(\alpha(t))}{\tau} = \frac{F(\alpha(t) + \tau(x_2-x_1)) - F(\alpha(t))}{\tau} \longrightarrow F_g'(\alpha(t))(x_2-x_1)$$

as $\tau \to 0$. Note that by definition the above convergence is WRT the norm of Y. As $\Lambda \in Y^*$ is continuous, by taking the limit as $\tau \to 0$ in $(*)$ we get

$$H'(t) = \lim_{\tau \to 0} \frac{H(t+\tau) - H(t)}{\tau} = \Lambda \left[F_g'(\alpha(t))(x_2 - x_1) \right].$$

In particular H is diff. in $[0,1]$. Therefore we can apply the Mean Value Theorem to find $\xi \in (0,1)$ such that

$$(**) \qquad H(1) - H(0) = H'(\xi) \qquad \left(= H'(\xi)(1-0) \right)$$

Now

$$\begin{aligned} H(1) - H(0) &= \Lambda[F(\alpha(1))] - \Lambda[F(\alpha(0))] \\ &= \Lambda[F(x_2)] - \Lambda[F(x_1)] \\ &= \Lambda[F(x_2) - F(x_1)] \\ &= \| F(x_2) - F(x_1) \|_Y \end{aligned}$$

by the properties of Λ.

On the other hand, as we computed, we have

$$H'(\xi) = \Lambda\left[\, F'_g(\alpha(\xi))\,(x_2 - x_1)\,\right]$$

and so

$$|H'(\xi)| \leq \|\Lambda\|_{Y^*}\;\|F'_g(\alpha(\xi))\|_{\mathcal{L}(X,Y)}\;\|x_2 - x_1\|_X$$

(Using that Λ and $F'_g(\alpha(\xi))$ are linear and bounded)

$$\leq \sup_{u\in[x_1,x_2]} \|F'_g(u)\|_{\mathcal{L}(X,Y)}\;\|x_2 - x_1\|_X$$

(Using that $\|\Lambda\|_{Y^*}=1$ and that $\alpha(\xi)\in[x_1,x_2]$)

From $**$ we then get

$$\|F(x_2) - F(x_1)\|_Y = H(1) - H(0) = H'(\xi)$$

$$\leq \sup_{u\in[x_1,x_2]} \|F'_g(u)\|_{\mathcal{L}(X,Y)}\;\|x_2 - x_1\|_X \qquad \square$$

COROLLARY 1.8 (of MEAN VALUE) Make the same assumptions of Theorem 1.7. Then

$$\|F(x_2) - F(x_1) - F'_g(x_1)(x_2 - x_1)\|_Y \leq \sup_{u\in[x_1,x_2]} \|F'_g(u) - F'_g(x_1)\|_{\mathcal{L}(X,Y)}\;\|x_2 - x_1\|_X$$

Proof Define $H : U \to Y$ by $H(u) := F(u) - F'_g(x_1)(u)$. Note that $F'_g(x_1) \in \mathcal{L}(X,Y)$ by assumption. In particular $F'_g(x_1) : X \to Y$ is Fréchet differentiable with derivative constantly equal to itself.

(Check it by exercise: X, Y normed spaces, $T \in \mathcal{L}(X,Y)$. Then T is Fréchet diff with $T'(u) = T$, $\forall u \in X$).

Therefore H is Gâteaux diff. with $H'_g(u) = F'_g(u) - F'_g(x_1)$. Thus $H'_g(u) \in \mathcal{L}(X,Y)$ for all $u \in U$. Thus H satisfies assumptions of THEOREM 1.7. Applying THM 1.7 to H we conclude. $\square$

We are finally ready to answer our question:

"When does Gâteaux diff. imply Fréchet diff.?"

THEOREM 1.9 Assume that $F: U \to Y$ is Gâteaux diff. at every point of U and in every direction. Also suppose that $F'_g(u) \in \mathcal{L}(X,Y)$ for all $u \in U$. Define the map

$$F'_g : U \longrightarrow \mathcal{L}(X,Y)$$
$$u \longmapsto F'_g(u)$$

If F'_g is continuous at u_0 then F is Fréchet diff at u_0 and

continuity is WRT norm on X and operator norm on $\mathcal{L}(X,Y)$

$$F'(u_0)(v) = F'_g(u_0)(v), \quad \forall v \in X$$

Proof Apply COROLLARY 1.8 to points $x_1 := u_0$, $x_2 := u_0 + v$. Since U is open, for v s.t. $\|v\|_X$ is sufficiently small we have $[x_1, x_2] \subseteq U$. By COROLLARY 1.8 we have

$$(*) \quad \| F(u_0+v) - F(u_0) - F'_g(u_0)(v) \|_Y \leq \sup_{u \in [x_1,x_2]} \| F'_g(u) - F'_g(u_0) \|_{\mathcal{L}(X,Y)} \|v\|_X$$

Recall that $[x_1, x_2] = [u_0, u_0+v]$. As F'_g is continuous at u_0 we have

$$\sup_{u \in [x_1,x_2]} \| F'_g(u) - F'_g(u_0) \|_{\mathcal{L}(X,Y)} \longrightarrow 0 \quad \text{as} \quad \|v\|_X \to 0$$

(in practice this implies $[x_1, x_2] \to \{u_0\}$)

Therefore, dividing $(*)$ by $\|v\|_X$ and taking the limit as $\|v\|_X \to 0$ concludes.

$\square$

LESSON 3 - 17 MARCH 2021

HIGHER ORDER DERIVATIVES

Let X, Y be normed spaces, $U \subseteq X$ open, $F : U \to Y$. Suppose that F is Fréchet diff. in U. Then we can define the map

$$F' : U \longrightarrow \mathcal{L}(X, Y)$$
$$u \longmapsto F'(u)$$

Now we can try and differentiate the above expression. This would yield a second derivative.

DEFINITION 1.10 Assume that F is diff. in U. We say that F is twice FRÉCHET diff. at $u_0 \in U$ if $F' : U \to \mathcal{L}(X, Y)$ is FRÉCHET diff. at u_0, i.e., if $\exists\, A_{u_0} \in \mathcal{L}(X, \mathcal{L}(X, Y))$ s.t.

$$\lim_{\|v\|_X \to 0} \frac{\| F'(u_0 + v) - F'(u_0) - A_{u_0}(v) \|_{\mathcal{L}(X,Y)}}{\|v\|_X} = 0$$

NOTATION The second Fréchet derivative is unique, if it exists. We denote it by $F''(u_0) := A_{u_0} \in \mathcal{L}(X, \mathcal{L}(X, Y))$

REMARK 1.11 Introduce the set

$$\mathcal{L}_2(X, Y) := \{ T : X \times X \to Y \mid T \text{ is bilinear, continuous} \}$$

where by continuous we mean $\exists\, M > 0$ s.t.

$$\| T(u, v) \|_Y \leq M \|u\|_X \|v\|_X \,, \quad \forall u, v \in X$$

Then one can show that

$$\mathcal{L}(X, \mathcal{L}(X, Y)) \cong \mathcal{L}_2(X, Y)$$

topologically and as sets (simple exercise)

Therefore $\mathcal{L}_2(X,Y)$ is naturally a normed space (and Banach if Y is Banach) with the norm

$$\|T\|_{\mathcal{L}_2(X,Y)} := \sup_{\|u\|_X, \|v\|_X \le 1} \|T(u,v)\|_Y$$

DEFINITION 1.12 Let $F: U \to Y$ and assume that F is twice Fréchet diff. at each point of U. Thus we can define

$$F'': U \to \mathcal{L}_2(X,Y)$$
$$u \mapsto F''(u)$$

If F'' is continuous we say that $F \in C^2(U,Y)$

WRT norm on X and operator norm on $\mathcal{L}_2(X,Y)$

In words we say that F is twice continuously diff. in U

THEOREM 1.13

Assume $F: U \to Y$ is twice Fréchet diff at $u_0 \in U$. In particular we have $F''(u_0) \in \mathcal{L}_2(X,Y)$ bilinear and continuous. Then $F''(u_0)$ is also SYMMETRIC, i.e.,

$$F''(u_0)(v,w) = F''(u_0)(w,v) \quad , \forall\, v,w \in X$$

COMMENT ON THE PROOF The proof of THM 1.13 is quite long, and I decided to skip it. The interested reader can find it in the book of HENRI CARTAN – "CALCUL DIFFERENTIEL", 1967 (IN FRENCH) in Theorem 5.1.1 at page 65.

The main ideas of the proof are the following. Introduce the map

$$A(v,w) := \left[F(u_0 + v + w) - F(u_0+v) - F(u_0+w) + F(u_0) \right] \in Y$$

for $v, w \in X$ with $\|v\|_X, \|w\|_X$ sufficiently small, so that $u_0 + v + w \in U$ and A is well defined.

Notice that A is symmetric. One can show that

$$(*) \quad \| A(v,w) - F''(u_0)(v,w) \|_Y = o\left(\left(\|v\|_X + \|w\|_X \right)^2 \right)$$

(The above is not difficult to show, but it would require further analysis which is outside the scope of this course)

As A is symmetric we can swap v and w in $(*)$ to obtain

$$\| A(v,w) - F''(u_0)(w,v) \|_Y = o\left(\left(\|v\|_X + \|w\|_X \right)^2 \right)$$

Therefore by triangle ineq. and the above estimates we get

$$\| F''(u_0)(v,w) - F''(u_0)(w,v) \|_Y = o\left(\left(\|v\|_X + \|w\|_X \right)^2 \right)$$

Thus $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\| F''(u_0)(v,w) - F''(u_0)(w,v) \|_Y \leq \varepsilon \left(\|v\|_X + \|w\|_X \right)^2, \quad \text{if } \left(\|v\|_X + \|w\|_X \right) < \delta$$

If $v, w \in X$ are arbitrary, we can find $\lambda \neq 0$ s.t. $\|\lambda v\|_X + \|\lambda w\|_X < \delta$. Applying the above ineq. to $\lambda v, \lambda w$ yields

$$\lambda^2 \| F''(u_0)(v,w) - F''(u_0)(w,v) \|_Y \leq \lambda^2 \varepsilon \left(\|v\|_X + \|w\|_X \right)^2, \quad \forall v, w \in X$$

As $\lambda \neq 0$ and ε is arbitrary, we conclude. $\square$

NOTE THEOREM 1.13 is a generalization of the classical SCHWARZ THEOREM on second derivatives of maps $F : \mathbb{R}^2 \to \mathbb{R}$. Indeed, if F is $C^2(\mathbb{R}^2)$ then

$$F'(u) = \nabla F(u) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$$

where the application is given by

$$F'(u)(v) = \nabla F(u) \cdot v, \quad \nabla F(u) = \left(\frac{\partial F}{\partial x_1}(u), \frac{\partial F}{\partial x_2}(u) \right)$$

Thus $F' : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2, \mathbb{R})$. Then $F''(u) \in \mathcal{L}(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2, \mathbb{R})) = \mathcal{L}_2(\mathbb{R}^2, \mathbb{R})$ with application given by

$$F''(u)(v,w) = v^T \nabla^2 F(u)\, w, \quad \nabla^2 F(u) = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} \end{pmatrix}$$

Therefore $F''(u)$ is symmetric $\iff \dfrac{\partial^2 F}{\partial x_1 \partial x_2} = \dfrac{\partial^2 F}{\partial x_2 \partial x_1}$, which is true by the classical Schwarz Theorem.

MORE DERIVATIVES! If $F: U \to Y$ is twice diff. in U then we can define

$$F'': U \longrightarrow \mathcal{L}(X, \mathcal{L}(X,Y))$$
$$u \longmapsto F''(u)$$

The function F'' can be in turn differentiated again, with

$$F'''(u) \in \mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X,Y)))$$

This procedure can be of course iterated, as F''' defines

$$F''': U \longrightarrow \mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X,Y)))$$

In general we have that

$$\underbrace{\mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X, \ldots\ldots \mathcal{L}(X,Y))))}_{n \text{ times}} \cong \mathcal{L}_n(X,Y)$$

where

$$\mathcal{L}_n(X,Y) := \{ T : \overbrace{X \times \cdots \times X}^{n \text{ times}} \to Y \mid T \text{ } n\text{-linear, bounded} \}$$

meaning that $\exists M > 0$ s.t. $\| T(u_1, \ldots, u_n) \|_Y \le M \|u_1\|_X \cdots \|u_n\|_X$.
The space $\mathcal{L}_n(X,Y)$ is normed by

$$\|T\|_{\mathcal{L}_n(X,Y)} := \sup_{\|u_i\|_X \le 1} \| T(u_1, \ldots, u_n)\|_Y$$

and $\mathcal{L}_n(X,Y)$ is Banach if Y is Banach. In particular the n-th Fréchet derivative is s.t.

$$F^{(n)}(u) \in \mathcal{L}_n(X,Y)$$

THEOREM 1.14 (TAYLOR FORMULA) (For a proof see book by CARTAN page 75)

X, Y Banach, $U \subseteq X$ open, $F: U \to Y$ $(n-1)$-times diff in U and n-times diff at $u_0 \in U$. Then

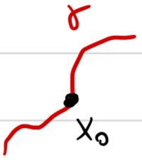
$$F(u_0 + v) = F(u_0) + F'(u_0)(v) + \frac{1}{2} F''(u_0)(v,v) + \cdots + \frac{1}{n!} F^{(n)}(u_0)\underbrace{(v, \ldots, v)}_{n\text{-times}} + o(\|v\|_X^n)$$

where $\dfrac{o(\|v\|_X^n)}{\|v\|_X^n} \to 0$ as $\|v\|_X \to 0$.

2. FIRST VARIATION (INDIRECT METHOD)

IDEA The FIRST VARIATION represents the DERIVATIVE of a functional $F: X \to \mathbb{R}$ with X = set

DEFINITION 2.1 Let $x_0 \in X$. A VARIATION at x_0 is a curve $\gamma: [-\delta, \delta] \to X$ s.t. $\gamma(0) = x_0$ (for some $\delta > 0$).



Note this is a scalar function of real variable

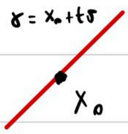
Consider the composition $\psi: [-\delta, \delta] \to \mathbb{R}$, $\psi(t) := F(\gamma(t))$.
If ψ is diff. at $t=0$ we denote

$$\boxed{\delta F(x_0, \gamma) := \psi'(0)}$$

and call $\delta F(\gamma, x_0)$ the FIRST VARIATION of F at x_0 along γ.

NOTE We are not assuming any regularity on F or structure on X

EXAMPLE 2.2 1) $X = \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$ fixed and $F: \mathbb{R}^d \to \mathbb{R}$ differentiable.
For $v \in \mathbb{R}^d$ consider the variation $\gamma(t) = x_0 + tv$, s.t. $\gamma(0) = x_0$.



Then

$$\delta F(x_0, \gamma) := \frac{d}{dt} F(x_0 + tv) \Big|_{t=0} = \nabla F(x_0) \cdot v$$

is the directional derivative of F at x_0 in direction v

2) X normed space, $x_0 \in X$, $F: X \to \mathbb{R}$ Gâteaux diff. at x_0 in every direction. Let $v \in X$ and consider the variation $\gamma(t) := x_0 + tv$.
Then

$$\delta F(x_0, \gamma) := (F \circ \gamma)'(0) = \lim_{t \to 0} \frac{F(\gamma(t)) - F(\gamma(0))}{t} =$$

$$= \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t} = F_g'(x_0)(v),$$

The first variation is just the Gâteaux derivative of F at x_0 in direction v.

PROPOSITION 2.3 X = Set, $F : X \to \mathbb{R}$. Assume that $x_0 \in X$ is a minimizer for F, that is,

$$F(x) \geq F(x_0) \quad , \forall x \in X$$

Let $\gamma : [-\delta, \delta] \to X$ s.t. $\gamma(0) = x_0$. If $\psi = F \circ \gamma$ is differentiable at $t=0$ then

$$\boxed{\delta F(x_0, \gamma) = 0 .}$$

Proof By assumption ψ is diff at $t=0$. Therefore

$$\psi'(0) = \lim_{t \to 0} \frac{\psi(t) - \psi(0)}{t} = \lim_{t \to 0^+} \frac{\psi(\gamma(t)) - \psi(\gamma(0))}{t} = \lim_{t \to 0^+} \frac{F(\gamma(t)) - F(x_0)}{t} \geq 0$$

so that $\delta F(x_0, \gamma) \geq 0$. Similarly:

$$\psi'(0) = \lim_{t \to 0^-} \frac{\psi(t) - \psi(0)}{t} = \lim_{t \to 0^-} \frac{F(\gamma(t)) - F(x_0)}{t} \leq 0$$

So $\delta F(x_0, \gamma) \leq 0 \implies \delta F(x_0, \gamma) = 0$. $\square$

THE CASE OF AFFINE SPACES

REMINDER Let V real vector space. A set X is called an AFFINE SPACE with REFERENCE SPACE V if it is defined the addition operation

$$+ : X \times V \to X$$

with properties:

1) $x + 0 = x$, $\forall x \in X$, where $0 \in V$ is the zero in V
2) $x + (v + w) = (x + v) + w$, $\forall x \in X$, $\forall v, w \in V$
3) For every $x \in X$ the map $V \to X$, $v \mapsto x + v$ is a bijection

From the above reminder, the only important part for us is that:
If X is affine space with ref vector space V, then

$$\boxed{x \in X,\ v \in V \Rightarrow x + v \in X}$$

DEFINITION 2.4 X affine space with reference V. Let $x_0 \in X$. A **VARIATION** at x_0 in direction $v \in V$ is a curve $\gamma(t) := x_0 + tv \in X$

(Thus we restrict ourselves to the case of straight line variations)

REMARK 2.5 Notice that $\gamma : \mathbb{R} \to X$ by def. of affine space (i.e. γ is defined for all $t \in \mathbb{R}$). If $\psi := F \circ \gamma$ is diff at $t=0$ then

$$\psi'(0) = \lim_{t \to 0} \frac{\psi(t) - \psi(0)}{t} = \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t}.$$

Therefore the **FIRST VARIATION** reads

$$\boxed{\delta F(x_0, v) := \delta F(x_0, \gamma) = \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t}}$$

DEFINITION 2.6 X affine space over V, $F : X \to \mathbb{R}$. If $\delta F(v, x_0)$ exists then

$$\boxed{\delta F(x_0, v) = 0}$$

is called **EULER-LAGRANGE EQUATION (ELE)**. If $x_0 \in X$ is such that (ELE) holds for all $v \in V$ then x_0 is called a **CRITICAL POINT OF F** (or **STATIONARY** POINT)

REMARK 2.7 X affine space over V, $F : X \to \mathbb{R}$ s.t. $\delta F(x_0, v)$ exists $\forall v \in V$. If x_0 minimizes F then x_0 is a CRITICAL POINT, i.e.

$$\delta F(x_0, v) = 0, \quad \forall v \in V$$

Proof Apply PROPOSITION 2.3 to $\gamma(t) := x_0 + tv$. $\square$

THREE EXAMPLES

Let $a<b$, and $A<B$. Consider the set

$$X = \{ u \in C^1[a,b] \mid u(a) = A, \; u(b) = B \}$$

Then X is an affine space with reference vector space

$$V = \{ u \in C^1[a,b] \mid u(a) = 0, \; u(b) = 0 \}$$

We consider functionals in integral form:

$$u \in X \longmapsto \int_a^b L(x, u(x), u'(x)) \, dx$$

for LAGRANGIANS $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Specifically, consider $F, G, H : X \to \mathbb{R}$ defined by

$$F(u) := \int_a^b |\dot{u}(x)|^2 \, dx , \quad G(u) := \int_a^b |u(x)| \, dx , \quad H(u) = \int_a^b |\dot{u}(x)|^{1/2} dx$$

GOAL: We want to solve

$$\min_{u \in X} F(u) , \qquad \min_{u \in X} G(u), \qquad \min_{u \in X} H(u)$$

and possibly characterize the solutions (if they exist!)

MINIMIZATION FOR F

By REMARK 2.7 we know that minimizers solve (ELE). Therefore, let us compute the first variation of F.

To do that, we compute the Fréchet derivative of F (This is actually not needed. The Gâteaux derivative of F would be sufficient, as seen in EXAMPLE 2.2 point (2). However we compute the Fréchet derivative as an exercise.)

PROPOSITION 2.8 Set $\tilde{X} := C^1[a,b]$ with norm $\|u\|_{C^1} := \|u\|_\infty + \|\dot{u}\|_\infty$. Extend F by

$$F(u) = \int_a^b |\dot{u}(x)|^2\,dx, \qquad u \in \tilde{X}$$

Then F is Fréchet differentiable in $\tilde{X}$ with $F'(u) \in \mathcal{L}(\tilde{X}, \mathbb{R})$ given by

$$F'(u)(v) = 2\int_a^b \dot{u}(x)\,\dot{v}(x)\,dx, \qquad \forall v \in \tilde{X}$$

Proof Let us start by computing the Gâteaux derivative of F at $u \in \tilde{X}$ in direction $v \in \tilde{X}$. We have

$$F(u+tv) = \int_a^b |\dot{u} + t\dot{v}|^2\,dx = \int_a^b |\dot{u}|^2\,dx + 2t\int_a^b \dot{u}\,\dot{v}\,dx + t^2\int_a^b |\dot{v}|^2\,dx$$

Therefore

$$F_G'(u)(v) = \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} =$$

$$= 2\int_a^b \dot{u}\,\dot{v}\,dx + \lim_{t\to 0} t\int_a^b |\dot{v}|^2\,dx$$

$$= 2\int_a^b \dot{u}\,\dot{v}\,dx$$

Now notice that $F_G'(u) \in \mathcal{L}(\tilde{X}, \mathbb{R})$: In fact

$$|F_G'(u)(v)| \le 2\int_a^b |\dot{u}|\,|\dot{v}|\,dx \le 2\|v\|_{C^1}\int_a^b |\dot{u}|\,dx$$

so that

$$\sup_{\|v\|_{C^1}\le 1} |F_G'(u)(v)| \le 2\int_a^b |\dot{u}|\,dx < +\infty$$

and so $F_G'(u) \in \mathcal{L}(\tilde{X}, \mathbb{R})$. Note that this is true for all $u \in \tilde{X}$. Consider

$$F_G' : \tilde{X} \longrightarrow \mathcal{L}(\tilde{X}, \mathbb{R})$$
$$u \longmapsto F_G'(u)$$

Then F'_g is continuous in $\tilde{X}$. Indeed, let $u, v \in \tilde{X}$:

$$\| F'_g(u) - F'_g(v) \|_{\mathcal{L}(\tilde{X}, \mathbb{R})} = \sup_{\|w\|_{C^1} \leq 1} \left| F'_g(u)(w) - F'_g(v)(w) \right|$$

$$\leq 2 \sup_{\|w\|_{C^1} \leq 1} \int_a^b |\dot{u} - \dot{v}| \, |\dot{w}| \, dx$$

$$\leq 2 \int_a^b |\dot{u} - \dot{v}| \, dx \leq 2(b-a) \|u - v\|_{C^1}$$

Showing continuity. We can then apply THEOREM 1.9 and conclude that F is Fréchet diff. in $\tilde{X}$ with $F'(u) = F'_g(u)$. $\square$

Since X affine space, REMARK 2.5 tells us that for $u_0 \in X$ we have

$$\delta F(u_0, v) = \lim_{t \to 0} \frac{F(u_0 + tv) - F(u_0)}{t}, \quad \forall v \in V$$

if the limit exists. Note that $X, V \subseteq \tilde{X}$. Therefore, as we just computed the Fréchet derivative of F (PROPOSITION 2.8), we get

$$F'(u_0)(v) = F'_g(u_0)(v) = \lim_{t \to 0} \frac{F(u_0 + tv) - F(u_0)}{t}$$

and so in particular

$$\boxed{\delta F(u_0, v) = 2 \int_a^b \dot{u}(x) \dot{v}(x) \, dx, \quad \forall u_0 \in X, v \in V}$$

We now look for solutions to (ELE) in order to find STATIONARY POINTS.

Thus assume $u_0 \in X$ is a min. of F over X. By REMARK 2.7 we have

$$\boxed{\delta F(u_0, v) = 0 \,, \quad \forall v \in V} \qquad \text{(ELE)}$$

Assuming also that $u_0 \in C^2[a,b]$ we get

$$0 = \delta F(u_0, v) = 2 \int_a^b \dot{u}_0 \dot{v} \, dx$$

(Integrate by parts WRT to $(\dot{u} v)' = \ddot{u} v + \dot{u}\dot{v}$)

$$= 2 \Big[\dot{u}_0 v \Big]_{x=a}^{x=b} - 2 \int_a^b \ddot{u}_0 v \, dx$$

(as $v(a) = v(b) = 0$)

$$= -2 \int_a^b \ddot{u}_0 v \, dx$$

Thus

$$(*) \quad \int_a^b \ddot{u}_0 v \, dx = 0 \,, \quad \forall v \in V$$

It looks like $(*)$ can hold iff $\ddot{u}_0 \equiv 0$. Let's say this is true (it actually is true and we will show it soon). Therefore, as $u_0(a) = A$, $u_0(b) = B$, we have

$$\ddot{u}_0 \equiv 0 \implies u_0 \text{ is straight line connecting } (a,A) \text{ and } (b,B)$$

WARNING This does not prove that u_0 minimizes F over X. We just proved that:

"If $u_0 \in C^2[a,b]$ is a min. of F over $X \Rightarrow u_0$ straight line"

PROPOSITION 2.9 Let $u_0 \in X$ be a straight line. Then u_0 is the unique solution to

$$\min_{u \in X} F(u) \,.$$

Recall: $X = \{ u \in C^1[a,b] \mid u(a) = A, u(b) = B \}$, $F(u) := \int_a^b |\dot{u}|^2 \, dx$, $A < B$

Proof Let $w \in X$ be arbitrary. We need to prove:

1) $F(u_0) \leq F(w), \ \forall w \in X$ (thus u_0 is a minimizer)

2) $F(u_0) = F(w) \iff u_0 = w$ (thus u_0 is unique minimizer)

Let us show (1): As $u_0, w \in X$ then $v := w - u_0 \in V$, since $v(a) = v(b) = 0$.

$$F(w) = F(u_0 + v) = \int_a^b |\dot{u}_0 + \dot{v}|^2 dx$$

$$= \int_a^b |\dot{u}_0|^2 dx + 2\int_a^b \dot{u}_0 \dot{v} \, dx + \int_a^b |\dot{v}|^2 dx$$

$$= F(u_0) + 2\int_a^b \dot{u}_0 \dot{v} \, dx + F(v)$$

Now u_0 is actually $C^2[a,b]$ (being a straight line). As $v(a) = v(b) = 0$, we can proceed as above (integrating by parts to obtain)

$$\int_a^b \dot{u}_0 \dot{v} \, dx = \underbrace{\left[\dot{u}_0 v \right]_a^b}_{=0 \text{ as } v(a)=v(b)=0} - \underbrace{\int_a^b \ddot{u}_0 v \, dx}_{=0 \text{ as } \ddot{u}_0 = 0, \text{ since } u_0 \text{ is a line}} = 0$$

Thus

$$F(w) = F(u_0) + F(v) \geq F(u_0),$$ (Since $F \geq 0$ by definition)

showing (1). Let us prove (2): We know that

$$F(w) = F(u_0) + F(v)$$

So $F(w) = F(u_0)$ iff $F(v) = 0$. By def of F this is true iff $\dot{v} \equiv 0$.
Thus $v \equiv$ constant. Since $v(a) = v(b) = 0 \implies v \equiv 0$. Recalling that $v = w - u_0$, we infer $w \equiv u_0$, as claimed. $\square$

Recall: $X = \{ u \in C^1[a,b] \mid u(a)=A,\ u(b)=B \}$

MINIMIZATION FOR G

$G(u) := \int_a^b |\dot{u}|\, dx$, $A<B$

PROPOSITION 2.10 We have that

$$(*) \qquad \min_{u \in X} G(u) = B - A$$

and the minimum exists. Moreover the only solutions to $(*)$ are the monotonic functions, that is, $u_0 \in X$ with $\dot{u}_0 \geq 0$.

Proof Let $u \in X$ be arbitrary. Then

$$G(u) = \int_a^b |\dot{u}(x)|\, dx \geq \left| \int_a^b \dot{u}(x)\, dx \right| = |u(b) - u(a)| = B - A$$

Hence

$$(LB) \qquad G(u) \geq B - A, \qquad \forall u \in X$$

This lower bound is achieved by u_0 straight line between (a,A), (b,B). Indeed,

$$G(u_0) = \int_a^b |\dot{u}_0(x)|\, dx = \int_a^b \frac{B-A}{b-a}\, dx = B - A$$

Therefore u_0 solves $(*)$, as (LB) implies

$$(**) \qquad G(u) \geq G(u_0) = B - A, \quad \forall u \in X.$$

In particular $(*)$ holds.

Assume now that $w_0 \in X$ solves $(*)$. Thus $G(w_0) = B - A$. Since (as above)

$$G(w_0) = \int_a^b |\dot{w}_0(x)|\, dx \geq \left| \int_a^b \dot{w}_0(x)\, dx \right| = B - A,$$

we have $G(w_0) = B - A$ iff

$$\int_a^b |\dot{w}_0|\, dx = \left| \int_a^b \dot{w}_0(x)\, dx \right|.$$

But this is true iff $\operatorname{sign}(\dot{w}_0)$ is constant $\iff \dot{w}_0 \geq 0$ (as $A<B$). $\square$

Recall: $X = \{ u \in C^1[a,b] \mid u(a) = A,\ u(b) = B \}$

MINIMIZATION FOR H

$$H(u) := \int_a^b \sqrt{|\dot u|}\, dx \quad , \quad A < B$$

PROPOSITION 2.11 We have that

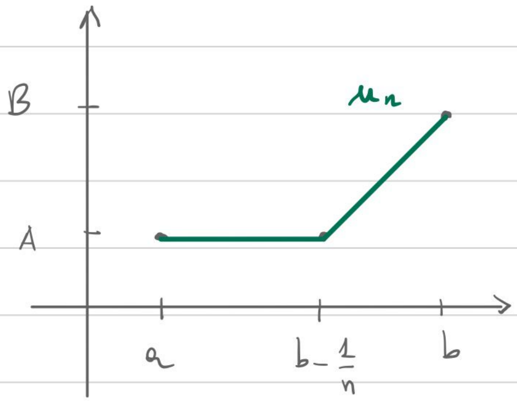
$$(*) \qquad \min_{u \in X} H(u)$$

has no solutions, and

$$(**) \qquad \inf_{u \in X} H(u) = 0 .$$

Proof Let us first show $(**)$. As $H \geq 0$, then $\inf_X H \geq 0$. Thus we need to find $\{u_n\} \subseteq X$ s.t. $H(u_n) \to 0$, so that $\inf H = 0$ follows.

Define $u_n : [a,b] \to \mathbb{R}$ as in the picture below



That is:

- $u_n \equiv A$ in $[a, b - \frac{1}{n}]$
- u_n straight line in $[b - \frac{1}{n}, B]$, so that

  $u_n(b - \frac{1}{n}) = A$ and $u_n(b) = B$

Then we have

$$H(u_n) = \int_a^b \sqrt{|\dot u_n|}\, dx = \int_{b-1/n}^{b} \sqrt{|\dot u_n|}\, dx$$

As $\dot u_n \equiv 0$ in $[a, b - \frac{1}{n})$ and $\dot u_n \equiv \frac{B-A}{b-(b-1/n)} = n(B-A)$ in $(b - 1/n, b]$

$$= \int_{b-1/n}^{b} \sqrt{n(B-A)}\, dx =$$

$$= \frac{1}{\sqrt{n}} \sqrt{B-A} \longrightarrow 0 \quad \text{as } n \to +\infty$$

PROBLEM: This almost shows (**). The only issue is that $u_n \notin C^1[a,b]$, as $\dot{u}_n$ has a jump at $x = b - 1/n$.

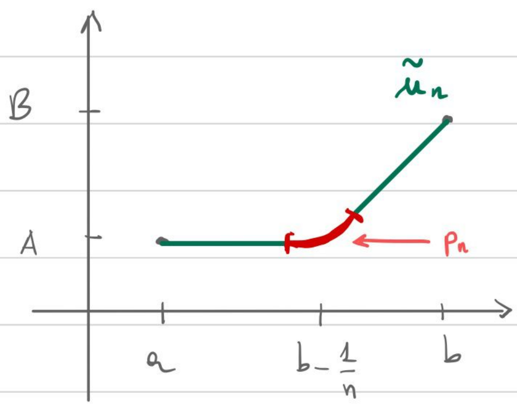
FIX: Smooth u_n around $x = b - 1/n$. For example one could define

$$\tilde{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in [a, b - \frac{2}{n}] \cup [b - \frac{1}{2n}, b] \\ p_n(x) & \text{if } x \in [b - \frac{2}{n}, b - \frac{1}{2n}] \end{cases}$$

where p_n is a polynomial such that

$$\begin{cases} p_n(b - \frac{2}{n}) = u_n(b - \frac{2}{n}), & \dot{p}_n(b - 2/n) = \dot{u}_n(b - 2/n) \\ p_n(b - 1/2n) = u_n(b - 1/2n), & \dot{p}_n(b - 1/2n) = \dot{u}_n(b - 1/2n) \end{cases}$$

so that $\tilde{u}_n \in C^1[a,b]$ and so $\tilde{u}_n \in X$ is admissible. This would look like



Since the region where we changed u_n is infinitesimal as $n \to +\infty$, we get

$$H(\tilde{u}_n) = H(u_n) + o(1) \to 0 \quad \text{as } n \to +\infty$$

(Showing these details will be left as an exercise in the Ex. course)

showing (**). We are left to show that (*) admits no minimizers. Assume by contradiction that the infimum is achieved by some $u_0 \in X$. As (**) holds we get $H(u_0) = 0$, which is possible iff $u_0 \equiv$ constant. However, since $u_0(a) = A$ and $u_0(b) = B$, and since we are assuming $A \neq B$, we get a contradiction. $\square$

Summary We considered functionals on $X = \{u \in C^1(a,b) \mid u(a) = A, u(b) = B\}$,

$$F(u) = \int_a^b \dot{u}^2 \, dx \quad , \quad G(u) = \int_a^b |\dot{u}| \, dx \quad , \quad H(u) = \int_a^b \sqrt{|\dot{u}|} \, dx$$

For these the solutions were as follows:

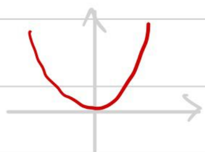
$\min_{u \in X} F(u)$ — UNIQUE MINIMIZER: u_0 STRAIGHT LINE BETWEEN $(a,A),(b,B)$

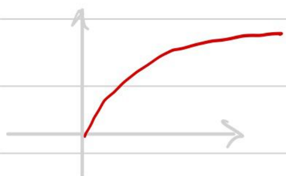
$\min_{u \in X} G(u)$ — INFINITELY MANY MINIMIZERS: ALL THE MONOTONIC FUNCTIONS $\dot{u} \geq 0$

$\min_{u \in X} H(u)$ — NO MINIMIZERS

To understand what is going on, let us considers the LAGRANGIANS associated to F, G, H

$$L_F(x,s,p) = p^2 \quad , \quad L_G(x,s,p) = |p| \quad , \quad L_H(x,s,p) = |p|^{1/2}$$





We observe:

- L_F is STRICTLY CONVEX in p $\rightsquigarrow$ $\exists!$ minimizer $u_0 \in C^\infty[a,b]$ (smooth)
- L_G is CONVEX in p, but NOT STRICTLY $\rightsquigarrow$ $\exists$ minimizer, but no uniqueness, no smooth
- L_H is NOT CONVEX in p $\rightsquigarrow$ Non Existence and no regularity

LESSON 4 - 24 MARCH 2021

3. FUNDAMENTAL LEMMAS

We now prove two fundamental Lemmas which will be ubiquitous throughout the course (we already used one of them after in the example of F, right before PROPOSITION 2.9).

DEFINITION 3.1 Let $u : (U \subseteq \mathbb{R}) \to \mathbb{R}$. The SUPPORT of u is the set

$$\operatorname{supp} u := \overline{\{x \in U \mid u(x) \neq 0\}}$$

We define the space of SMOOTH COMPACTLY SUPPORTED functions on (a,b) as

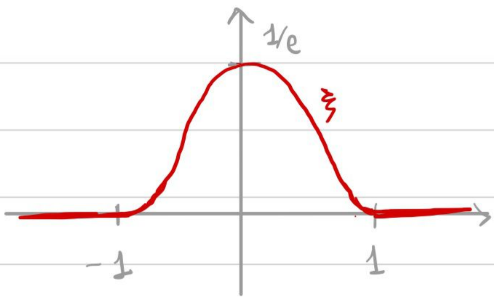
$$C_c^\infty(a,b) := \{ u \in C^\infty(a,b) \mid \operatorname{supp} u \text{ is COMPACT} \}$$

In other words, $u \in C_c^\infty(a,b)$ iff $\exists \; [c,d] \subseteq (a,b)$ s.t. $\operatorname{supp} u \subseteq [c,d]$, i.e., $u \equiv 0 \quad (a,b) \setminus [c,d]$.

REMARK 3.2 We can construct $u \in C_c^\infty(a,b)$ having PRESCRIBED support in some interval $[c,d] \subseteq (a,b)$, and having the same sign, i.e., either $u \geq 0$ or $u \leq 0$.

To do that, consider the BUMP FUNCTION

$$\xi(x) := \begin{cases} \exp\left(-\dfrac{1}{1-x^2}\right), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}$$



Then $\xi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \xi \subseteq [-1,1]$ and $\xi > 0$ in $(-1,1)$.

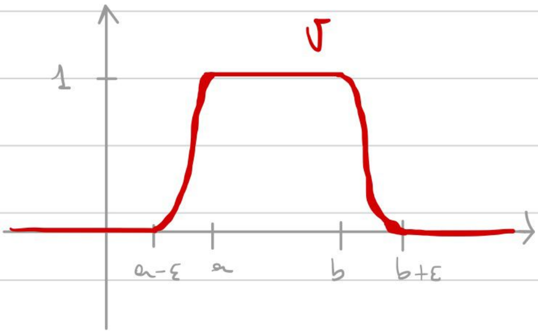
For $x_0 \in \mathbb{R}$, $r > 0$ fixed define

$$(\ast) \qquad u(x) := \xi\left(\frac{x - x_0}{r}\right)$$

Then $u \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} u \subseteq [x_0 - r, x_0 + r]$ and $u > 0$ in $(x_0 - r, x_0 + r)$

(To get $u < 0$ just consider $-\xi$ in the definition $(\ast)$)

REMARK 3.3 Using the function ξ at REMARK 3.2 and CONVOLUTIONS, it is possible to construct $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$ and



$$\varphi(x) = \begin{cases} 1 & x \in [a,b] \\ 0 \quad \text{if} & x \notin [a-\varepsilon, b+\varepsilon] \end{cases}$$

where a, b and $\varepsilon > 0$ can be chosen arbitrarily. Such φ is called CUT-OFF function

(We omit the proof of this fact for the moment. It will be left as an exercise in the EXERCISES COURSE).

LEMMA 3.4 (FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS) (FLCV)

Let $f : [a,b] \to \mathbb{R}$ be continuous. Assume that

$$\int_a^b f(x)\, \varphi(x)\, dx = 0, \qquad \forall \varphi \in C_c^\infty(a,b)$$

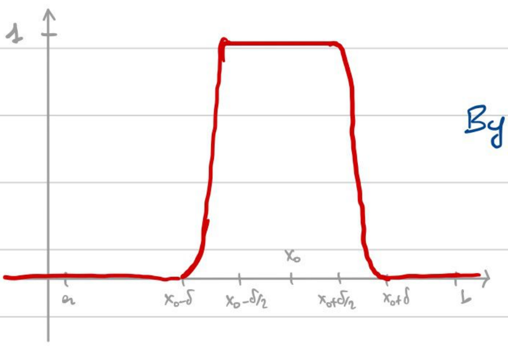
Then $f \equiv 0$.

We give 2 proofs of this Lemma, to show different and interesting techniques:

PROOF 1 OF LEMMA 3.4 (By contradiction)

Assume by contradiction that $f \not\equiv 0$. Then WLOG $\exists\, x_0 \in (a,b)$ such that $f(x_0) > 0$. By continuity also $\exists\, \delta > 0$ s.t.

$$f(x) \geq \frac{f(x_0)}{2}, \quad \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq [a,b]$$



By REMARK 3.3 $\exists\, \varphi \in C_c^\infty(\mathbb{R})$ s.t. $0 \leq \varphi \leq 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in [x_0 - \delta/2, x_0 + \delta/2] \\ 0 & \text{for } x \notin [x_0 - \delta, x_0 + \delta] \end{cases}$$

Thus by assumption we have

$$\int_a^b f(x)\,v(x)\,dx = 0.$$

On the other hand,

$$\int_a^b f(x)v(x)\,dx = \int_{x_0-\delta}^{x_0+\delta} f(x)v(x)\,dx \geq \int_{x_0-\delta/2}^{x_0+\delta/2} f(x)v(x)\,dx \geq \frac{f(x_0)}{2}\delta > 0$$

As $v=0$ outside of $[x_0-\delta, x_0+\delta]$

As $v\geq 0$ always, while $f \geq \frac{f(x_0)}{2} > 0$ in $[x_0-\delta, x_0+\delta]$

Since $v \equiv 1$ and $f(x) \geq f(x_0)/2$ here

which is a contradiction. $\square$

Before proceeding with the second proof of LEMMA 3.4, we make the following remark (a proof of which is left for the exercises course)

REMARK 3.5 Let $v : [a,b] \to \mathbb{R}$ continuous. There exists a sequence $\{v_n\} \subseteq C_c^\infty(a,b)$ s.t.

1) $\{v_n\}$ is uniformly bounded, i.e., $\exists M>0$ s.t.

$$\sup_n \|v_n\|_\infty \leq M$$

2) For each $K \subseteq [a,b]$ compact we have that $v_n \to v$ uniformly on K, that is,

$$\lim_{n\to+\infty} \sup_{x\in K} |v_n(x) - v(x)| = 0.$$

PROOF 2 OF LEMMA 3.4 (By Density)

We claim the following

$$(*) \quad \int_a^b f(x)v(x) = 0, \ \forall v \in C_c^\infty(a,b) \Rightarrow \int_a^b f(x)v(x) = 0, \ \forall v \in C(a,b)$$

Notice that if $(*)$ holds then the thesis of Lemma 3.4 follows: indeed, as we are assuming that f satisfies

$$\int_a^b f(x)\varphi(x)\,dx = 0 \quad , \forall \varphi \in C_c^\infty(a,b) ,$$

then by $(*)$ we get that

$$\int_a^b f(x)\varphi(x)\,dx = 0 \quad , \forall \varphi \in C(a,b) .$$

Thus we can choose $\varphi = f$ in the above (as f is continuous by assumption) and obtain

$$\int_a^b |f|^2\,dx = 0 \quad \Rightarrow f \equiv 0 ,$$

which concludes the proof.

Thus, we are left to show $(*)$. To this end, fix $\varphi \in C(a,b)$. By REMARK 3.5 $\exists \{\varphi_n\} \subseteq C_c^\infty(a,b)$ s.t. $\{\varphi_n\}$ is unif. bounded and $\varphi_n \to \varphi$ uniformly on each $K \subset (a,b)$ compact. As φ_n is smooth, by assumption we have

$$(**) \qquad \int_a^b f(x)\varphi_n(x)\,dx = 0 \quad , \forall \ n \in \mathbb{N} .$$

On the other hand, let $K \subset (a,b)$ be compact. Then

$$\left| \int_a^b f\varphi_n\,dx - \int_a^b f\varphi\,dx \right| \leq \|f\|_\infty \int_a^b |\varphi_n - \varphi|\,dx =$$

$$= \|f\|_\infty \left(\int_K |\varphi_n - \varphi|\,dx + \int_{K^c} |\varphi_n - \varphi|\,dx \right) \qquad \left(K^c := (a,b) \setminus K \right)$$

Now the first integral:

$$\int_K |v_n - v|\,dx \leq |K| \sup_{x\in K} |v_n(x) - v(x)| \to 0 \quad \text{as } n\to+\infty$$

by the properties of v_n. For the second integral we have:

$$\int_{K^c} |v_n - v|\,dx \leq |K^c| \left(\|v_n\|_\infty + \|v\|_\infty \right) \leq |K^c| \left(M + \|v\|_\infty \right)$$

In total,

$$\limsup_{n\to+\infty} \left| \int_a^b f v_n\,dx - \int_a^b f v\,dx \right| \leq \|f\|_\infty |K^c| \left(M + \|v\|_\infty \right).$$

Now, remember that $K \subset (a,b)$ is an arbitrary compact set. Thus $|K^c|$ is as small as we wish, from which we infer

$$\int_a^b f v_n\,dx \to \int_a^b f v\,dx \quad \text{as } n\to+\infty$$

Since $(**)$ holds, we conclude that $\int_a^b f v\,dx = 0$, and the CLAIM is proven. $\square$

The second proof immediately suggests possible generalizations of LEMMA 3.4, which will allow us to test f against a smaller set of functions.

REMARK 3.6 Assume that $f \in C(a,b)$ satisfies

$$\int_a^b f(x)\,v(x)\,dx = 0, \quad \forall\, v \in V$$

where $V \subset C(a,b)$ is some set. Then

1) By linearity of the integral we have

$$\int_a^b f(x)\, v(x)\, dx = 0 \quad , \quad \forall\, v \in \operatorname{span} V$$

2) By a density argument similar to the one of PROOF 2 of LEMMA 3.4 we have

$$\int_a^b f(x)\, v(x)\, dx = 0 \quad , \quad \forall\, v \in \overline{V} ,$$

where the closure is taken WRT the uniform convergence of bounded sequences on compact sets $K \subset (a,b) \setminus \{x_1, .., x_N\}$ where the collection of points $\{x_1, .., x_N\}$ is FINITE.

As a consequence of REMARK 3.6, and following the arguments of PROOF 2 of LEMMA 3.4 we get:

LEMMA 3.7 (Generalized FLCV)

Let $f \in C(a,b)$, $V \subset C(a,b)$ such that $\overline{\operatorname{span} V} = C(a,b)$, where the closure is as in REMARK 3.6 point (2), i.e.,

$$\overline{\operatorname{span} V} := \Big\{ v \in C(a,b) \ \Big|\ \exists \{v_n\} \subseteq \operatorname{span} V, \text{ with } \sup_n \|v_n\|_\infty < +\infty \text{ and } v_n \to v \text{ uniformly on each compact } K \subset (a,b) \setminus I \Big\}$$

with $I := \{x_1, .., x_N\}$ is a fixed finite collection of points. Then

$$\int_a^b f(x)\, v(x) = 0 \ , \quad \forall\, v \in V \ \Rightarrow\ f \equiv 0 \ .$$

We now state and prove a second "fundamental" lemma, which again will be very useful in the rest of the course.

LEMMA 3.8 (DU BOIS REYMOND) (DBR Lemma)

Let $f \in C(a,b)$ and assume that

Zero average function

$$(*) \quad \int_a^b f(x) v(x)\, dx = 0, \quad \forall v \in C_c^\infty(a,b) \text{ s.t. } \int_a^b v(x)\, dx = 0.$$

Then $f \equiv c$ for some $c \in \mathbb{R}$.

Proof The idea is to apply the FLCV (LEMMA 3.4). Thus let $v \in C_c^\infty(a,b)$. It would be nice if we could use

$$\tilde{v}(x) := v(x) - \frac{1}{b-a} \int_a^b v(y)\, dy$$

as a test function in $(*)$, seeing that $\int_a^b \tilde{v}(x)\, dx = 0$. However $\tilde{v}$ is not compactly supported.

To make this attempt rigorous, take $w \in C_c^\infty(a,b)$ s.t. $\int_a^b w(x)\, dx = 1$, and define

$$\varphi(x) := v(x) - w(x) \int_a^b v(y)\, dy$$

Then $\varphi \in C_c^\infty(a,b)$ and $\int_a^b \varphi(x)\, dx = 0$. By using φ as a test function in $(*)$ we get

$$0 = \int_a^b f(x)\varphi(x)\, dx = \int_a^b f(x) v(x)\, dx - \int_a^b f(x) w(x) \left(\int_a^b v(y)\, dy \right) dx$$

$$= \int_a^b f(x) v(x)\, dx - c \int_a^b v(x)\, dx,$$

where $c := \int_a^b f(x) w(x)\, dx$

Thus

$$0 = \int_a^b f(x)v(x)\,dx - c\int_a^b v(x)\,dx$$

$$= \int_a^b \left[f(x) - c\right] v(x)\,dx$$

Since this is true for all $v \in C_c^\infty(a,b)$, by FLCV LEMMA 3.4 we conclude $f - c \equiv 0 \Rightarrow f \equiv c$. $\square$

A simple (but useful) equivalent formulation of the DBR Lemma is the following one.

LEMMA 3.9 (DBR - Second formulation)

Let $f \in C(a,b)$ and assume that

$$(*) \quad \int_a^b f(x)\dot{v}(x)\,dx = 0, \quad \forall v \in C_c^\infty(a,b)$$

Then $f \equiv c$ for some $c \in \mathbb{R}$.

Proof For $v \in C_c^\infty(a,b)$ we have

$$(**) \quad \int_a^b v(x)\,dx = 0 \iff \exists\, w \in C_c^\infty(a,b) \text{ s.t. } \dot{w} = v$$

Indeed, if $w \in C_c^\infty(a,b)$ is s.t. $\dot{w} = v$, then

$$\int_a^b v(x)\,dx = \int_a^b \dot{w}(x)\,dx = w(b) - w(a) = 0 \qquad \text{(w is compactly supported)}$$

Conversely, assume $\int_a^b v(x)\,dx = 0$, and let $\varepsilon > 0$ be s.t

$\operatorname{supp} v \subset [a+\varepsilon, b-\varepsilon]$ (since v is compactly supported)

For $x \in [a,b]$ define

$$w(x) := \int_a^x v(y)\,dy$$

Then $\dot{w} = v$, and in particular $w \in C^\infty(a,b)$. Moreover

$$w(x) = \int_a^x v(y)\,dy = 0 \quad \text{if} \quad x \in [a, a+\varepsilon]$$

as $v \equiv 0$ in $[a, a+\varepsilon]$, while

We are assuming this

$$w(x) = \int_a^x v(y)\,dy = \int_a^b v(y)\,dy = 0$$

if $x \in [b-\varepsilon, b]$, as the whole support of v is in $[a, b-\varepsilon]$.

Thus (**) is proven. Now assume that (*) holds. Let $v \in C_c^\infty(a,b)$ be such that $\int_a^b v(x)\,dx = 0$. Then by (**) $\exists\, w \in C_c^\infty(a,b)$ s.t. $\dot{w} = v$. Therefore, by (*), we have $\int_a^b f(x)\,\dot{w}(x) = 0$. Then, as $\dot{w} = v$,

$$\int_a^b f(x)\,v(x) = \int_a^b f(x)\,\dot{w}(x) = 0$$

As v is arbitrary, then $f \equiv c$ by DBR LEMMA 3.8. $\square$

As for the FLCV, also in the DBR lemma we can test f against a smaller set of functions, since the DBR can also be proven with a density argument (very similar to PROOF 2 of LEMMA 3.4). Such argument makes use of the following remark (Again, left for the exercise course)

REMARK 3.10 Let $v \in C(a,b)$ with $\int_a^b v(x)\,dx = 0$. Then $\exists$ $\{v_n\} \subseteq C_c^\infty(a,b)$ such that

1) $\sup_n \|v_n\|_\infty \leq M$, for some $M>0$

2) $v_n \to v$ uniformly on compact sets $K \subset (a,b)$

3) $\int_a^b v_n(x)\,dx = 0$, $\forall n \in \mathbb{N}$.

We have the following alternative proof of the DBR LEMMA 3.8.

ALTERNATIVE PROOF OF LEMMA 3.8 (by density)

By proceeding exactly as in PROOF 2 of LEMMA 3.4 (using REMARK 3.10 in place of REMARK 3.5) we can show that

$$\boxed{\begin{gathered}\int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in C_c^\infty(a,b) \text{ with } \int_a^b v(x)\,dx = 0 \\ \Downarrow \\ \int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in C(a,b) \text{ with } \int_a^b v(x)\,dx = 0\end{gathered}} \quad (*)$$

Now the thesis of LEMMA 3.8 follows immediately by $(*)$. Indeed, assume that $f \in C(a,b)$ is such that

$$\int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in C_c^\infty(a,b) \text{ with } \int_a^b v(x)\,dx = 0.$$

As v has zero average, then also $f+c$ for any $c \in \mathbb{R}$ satisfies the above.

Thus, by $(*)$,

$(**)$ $\int_a^b [f(x)+c]\, v(x)\,dx = 0, \quad \forall v \in C(a,b)$ with $\int_a^b v(x)\,dx = 0$

In particular, take $c = -\frac{1}{b-a}\int_a^b f(x)\,dx$, so that $\int_a^b f+c = 0$.

Thus, we can test $(**)$ against $v := f+c$ to get $\int_a^b (f+c)^2 = 0$
$\Rightarrow$ $f \equiv -c$. $\square$

Following a similar reasoning to the one in REMARK 3.6, and arguments similar to the ones contained in the above proof, we can obtain a generalized version of the DBR Lemma (which we state without proof).

LEMMA 3.11 (Generalized DBR)

Consider the space

$$V = \left\{ v \in C(a,b) \;\middle|\; \int_a^b v(x)\,dx = 0 \right\}$$

Assume that $F \subseteq V$ is such that $\overline{\operatorname{span} F} = V$, where $\overline{\operatorname{span} V}$ is

$$\overline{\operatorname{span} V} := \left\{ v \in C(a,b) \;\middle|\; \exists \{v_n\} \subseteq \operatorname{span} V, \text{ with } \sup_n \|v_n\|_\infty < +\infty \text{ and } v_n \to v \text{ uniformly on each compact } K \subset (a,b) \setminus I \right\}$$

with $I := \{x_1, \dots, x_N\}$ is a fixed finite collection of points. Let $f \in C(a,b)$. If

$$\int_a^b f(x)\, v(x)\,dx = 0, \quad \forall v \in F$$

then $f \equiv c$ for some $c \in \mathbb{R}$.

BOUNDARY CONDITIONS (By Examples)

EXAMPLE 1 (DIRICHLET BOUNDARY CONDITIONS)

$$F(u) = \int_0^1 \dot{u}^2 + u^2 \, dx \quad \text{with} \quad u \in X,$$

$$X := \{ u \in C^1[0,1] \mid u(0) = \alpha, u(1) = \beta \}$$

We want to find solutions to

$$\min_{u \in X} F(u).$$

Let us start by computing the first variation. Thus let

$$V = \{ v \in C^1[0,1] \mid v(0) = v(1) = 0 \}$$

So that X is an affine space over V. For $u \in X$, $v \in V$ we get

$$F(u+tv) = \int_0^1 (\dot{u} + t\dot{v})^2 + (u+tv)^2 \, dx =$$

$$= \int_0^1 \dot{u}^2 + 2t \int_0^1 \dot{u}\dot{v} + t^2 \int_0^1 \dot{v}^2 +$$

$$\int_0^1 u^2 + 2t \int_0^1 uv + t^2 \int_0^1 v^2$$

$$= F(u) + t^2 F(v) + 2t \int_0^1 (uv + \dot{u}\dot{v}) \, dx$$

Therefore

$$\delta F(u,v) = \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} =$$

$$= \lim_{t\to 0} \; tF(v) + 2\int_0^1 (uv + \dot{u}\dot{v})\,dx$$

$$= 2\int_0^1 (uv + \dot{u}\dot{v})\,dx$$

Therefore the EULER-LAGRANGE EQUATION reads

$$(*) \qquad \boxed{\int_0^1 (uv + \dot{u}\dot{v})\,dx = 0, \quad \forall v \in V}$$

Assuming that $u \in C^2[0,1]$, we can integrate $(*)$ by parts to obtain

$$(**) \qquad \boxed{\int_0^1 (-\ddot{u} + u)\, v \,dx = 0, \quad \forall v \in V}$$

where we used $v(0) = v(1) = 0$.

<u>NOTATION</u>

- $(*)$ is called 1st INTEGRAL FORM OF (ELE)
- $(**)$ is called 2nd INTEGRAL FORM OF (ELE)

Thus, if u is minimum of F and $u \in C^2[0,1]$, then u solves $(**)$.
As $C_c^\infty(0,1) \subseteq V$, we can apply FLCV (LEMMA 3.4) to $(**)$ and obtain

$$-\ddot{u} + u \equiv 0$$

Recalling that u satisfies BC, we then need to solve the ORDINARY DIFFERENTIAL EQUATION (ODE)

$$\begin{cases} \ddot{u}(x) = u(x), & \forall x \in (0,1) \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{DIRICHLET BOUNDARY CONDITIONS (DBC)}$$

Now this is solved by

$$\circledast \qquad u(x) = A\cosh(x) + B\sinh(x)$$

for appropriate A, B (as well known from basic analysis courses).

WARNING Recall that this just proves that if $u \in C^2[0,1]$ is a minimizer for F in X, then u is of the form $\circledast$. Showing that u as in $\circledast$ is actually a minimum requires a proof (energy estimates)

EXAMPLE 2 (DBC and NEUMANN BOUNDARY CONDITION (NBC))

Same functional F from the previous example, but defined on

$$X = \{ u \in C^1[0,1] \mid u(0) = \alpha \}$$

Note: we do not assign a condition for $u(1)$.

Let us compute the first variation. This time the reference vector space is

$$V = \{ v \in C^1[0,1] \mid v(0) = 0 \}.$$

Note that, as a consequence of the def. of X, we do not need to assign conditions on $v(1)$.

As before, the first variation at $u \in X$ along the direction $v \in V$ is

$$\delta F(u,v) = 2\int_0^1 (uv + \dot{u}\dot{v})\,dx$$

Assuming $u \in C^2[0,1]$ and integrating by parts:

$$\delta F(u,v) = 2\int_0^1 uv\,dx + \underbrace{2\,\dot{u}v\Big|_0^1}_{} - 2\int_0^1 \ddot{u}v\,dx$$

This time this term is not zero, but it is equal to $2\,\dot{u}(1)\,v(1)$

Thus the 2nd integral form of (ELE) is

(ELE) $$\boxed{\int_0^1 (-\ddot{u}+u)\,v\,dx + \dot{u}(1)\,v(1) = 0, \quad \forall v \in V}$$

Thus if $u \in C^2[0,1]$ and u minimizes F in X, then (ELE) holds.
How do we proceed? We cannot apply FLCV or DBR straightforwardly.
So we proceed in 2 steps:

- Step 1: Consider only test function $v \in V$ such that $v(1)=0$.
  In this case (ELE) reads

$$\int_0^1 (-\ddot{u}+u)\,v\,dx = 0, \quad \forall v \in C^1[0,1] \text{ s.t. } v(0)=v(1)=0$$

In particular (as in EXAMPLE 1) we can apply FLCV to get

$$-\ddot{u}+u \equiv 0$$

and hence the ODE

$$\begin{cases} \ddot{u}(x) = u(x) & , \forall x \in (0,1) \\ u(0) = \alpha \end{cases}$$

• <u>Step 2</u>: Now we know that $\ddot{u} \equiv u$. Therefore (ELE) becomes

$$\dot{u}(1)\, v(1) = 0 \quad , \quad \forall v \in V$$

Thus, by testing against $v \in V$ s.t. $v(1) \neq 0$ we get

$$\dot{u}(1) = 0$$

In total, we found that u solves

$$\begin{cases} \ddot{u}(x) = u(x) & , \forall x \in (0,1) \\ u(0) = \alpha & \text{(DIRICHLET BOUNDARY CONDITION)} \\ \dot{u}(1) = 0 & \text{(NEUMANN BOUNDRY CONDITION NBC)} \end{cases}$$

<u>NOTICE</u>: By not imposing a DIRICHLET BOUNDARY CONDITION on $u(1)$ for $u \in X$, we see that minimizers must satisfy a homogeneous condition on $\dot{u}(1)$.

This will be true in general. Also note that the NBC is of one less order than the highest derivative appearing in F.

EXAMPLE 3 (NEUMANN BOUNDARY CONDITIONS - NBC)

F as before but $X := C^1[0,1]$, with no additional conditions.

Note that in this case it is trivially true that $u \equiv 0$ minimizes F. However, for instructive purposes, let us ignore this fact and proceed with our usual method.

This time the ref. vector space is $V = C^1[0,1]$. The first variation is always the same,

$$\delta F(u,v) = 2\int_0^1 (uv + \dot{u}\dot{v})\,dx.$$

Assuming that $u \in C^2[0,1]$ minimizes F on X, and integrating by parts

(ELE) $$\boxed{\int_0^1 (-\ddot{u} + u)\,v\,dx + \dot{u}(1)v(1) - \dot{u}(0)v(0) = 0, \quad \forall v \in V}$$

We now proceed in 2 steps:

- Step 1: Test (ELE) against $v \in C_c^\infty(0,1) \subseteq V$, so that

$$\int_0^1 (-\ddot{u} + u)\,v\,dx = 0, \quad \forall v \in C_c^\infty(0,1)$$

Thus FLCV implies

$$-\ddot{u} + u \equiv 0$$

• Step 2: Since $-\ddot{u}+u\equiv 0$, (ELE) becomes

$$\textcircled{*} \quad \dot{u}(1)v(1)-\dot{u}(0)v(0)=0, \qquad \forall v\in V$$

Testing $\textcircled{*}$ against $v\in V$ s.t. $v(0)\neq 0$, $v(1)=0$ yields

$$\dot{u}(0)=0$$

Testing $\textcircled{*}$ against $v\in V$ s.t. $v(0)=0$, $v(1)\neq 0$ yields

$$\dot{u}(1)=0$$

In total, u solves

$$\begin{cases} \ddot{u}(x)=u(x) & , \quad x\in(0,1) \\ \dot{u}(0)=0 \\ \dot{u}(1)=0 \end{cases}$$

NEUMANN BOUNDARY CONDITIONS (NBC)

EXAMPLE 4 (PERIODIC BOUNDARY CONDITIONS – PBC)

F as before, but

$$X=\{u\in C^1[0,1] \mid u(0)=u(1)\}$$

(Also now the solution is trivially $u\equiv 0$. But let's ignore this).

Note X is vector space, so we can take $V=X$. The first variation δF is the same. Assuming $u\in C^2[0,1]$ minimizes F on X and integrating by parts:

(ELE) $$\boxed{\int_0^1 (-\ddot{u}+u)v\,dx + v(0)\{\dot{u}(1)-\dot{u}(0)\} = 0, \quad \forall v \in V}$$

where we used that $v(0)=v(1)$. We proceed in 2 steps:

- Step 1 As usual, we can test against all $v \in C_c^\infty(0,1) \subseteq V$ and get
$$-\ddot{u}+u \equiv 0$$

- Step 2: We know that
$$v(0)\{\dot{u}(1)-\dot{u}(0)\}=0, \quad \forall v \in V$$
Testing against $v \in V$ with $v(0) \neq 0$ (and $v(1)=v(0)$) we conclude
$$\dot{u}(0)=\dot{u}(1)$$

Recalling that $u(0)=u(1)$ as $u \in X$, we then get

$$\begin{cases} \ddot{u}(x)=u(x), & \forall x \in (0,1) \\ u(0)=u(1) \\ \dot{u}(0)=\dot{u}(1) \end{cases}$$

PERIODIC BOUNDARY CONDITIONS (PBC)

EXAMPLES F the same, $X=\{u \in C^1[0,1] \mid u(1)=u(0)+2\}$

X is not a vector space. It is however affine space over

$$V=\{C^1[0,1] \mid v(0)=v(1)\}$$

By very similar calculations to the previous 4 examples, we get that if $u \in C^2[0,1]$ minimizes F over X, then

$$\begin{cases} \ddot{u}(x) = u(x), & \forall x \in (0,1) \\ u(1) = u(0) + 5 & \text{(This was enforced in } X\text{)} \\ \dot{u}(0) = \dot{u}(1) & \text{(NBC / PBC)} \end{cases}$$

EXAMPLE 6 (TOO MANY BOUNDARY CONDITIONS!)

F the same,

$$X = \{ u \in C^1[0,1] \mid u(\tfrac{1}{2}) = \alpha \}.$$

X is affine over $V = \{ v \in C^1[0,1] \mid v(\tfrac{1}{2}) = 0 \}$. If $u \in C^2[0,1]$ minimizes F over X, we integrate by parts to find

(ELE) $$\boxed{\int_0^1 (-\ddot{u} + u) v \, dx + \dot{u}(1) v(1) - \dot{u}(0) v(0) = 0, \quad \forall v \in V}$$

- Step 1: Define

$$W := \{ v \in C^1[0,1] \mid v(0) = v(\tfrac{1}{2}) = v(1) = 0 \} \subseteq V$$

By (ELE) we have

(*) $$\int_0^1 (-\ddot{u} + u) v \, dx = 0, \quad \forall v \in W$$

Now notice that $\overline{\operatorname{span} W} = C(0,1)$, where the closure is taken WRT the uniform convergence on compact subsets of $[0,1] \setminus \{\frac{1}{2}\}$. Then we can apply the GENERALIZED FLCV (LEMMA 3.7) to (*) and infer

$$\begin{cases} -\ddot{u} + u \equiv 0 \\ u(\frac{1}{2}) = \alpha \quad \text{(this is from } u \in X\text{)} \end{cases}$$

- Step 2: As $-\ddot{u} + u \equiv 0$, from (ELE) we get

$$\dot{u}(1)v(1) - \dot{u}(0)v(0) = 0 \quad , \quad \forall v \in V$$

Now just take $v \in V$ s.t. $v(1) = 0$, $v(0) \neq 0$ and $\tilde{v} \in V$ s.t. $\tilde{v}(1) \neq 0$, $\tilde{v}(0) = 0$ and obtain

$$\dot{u}(1) = \dot{u}(0) = 0.$$

In total, u solves

$$\text{(ODE)} \begin{cases} \ddot{u}(x) = u(x) & , \ x \in (0,1) \\ u(1/2) = \alpha \\ \dot{u}(0) = \dot{u}(1) = 0 \end{cases}$$

As the ODE is of order 2 and we get 3 pointwise conditions, it is very unlikely that (ODE) admits a solution.

Notice that solvin (ODE) is equivalent to solving 2 seperate ODES and then hoping that the solutions can be glued at 1/2 in a C^2 way where the two ODEs are

$$\text{(P1)} \begin{cases} \ddot{u} = u & \text{in } (0, 1/2) \\ \dot{u}(0) = 0 \\ u(1/2) = \alpha \end{cases} \quad , \quad \text{(P2)} \begin{cases} \ddot{u} = u & \text{in } (1/2, 1) \\ \dot{u}(1) = 0 \\ u(\frac{1}{2}) = \alpha \end{cases}$$

So there are two possibilities:

1) (ODE) admits a solution u $\Rightarrow$ with energy arguments we show that u minimizes F over X.

2) (ODE) does not admit a solution. Thus

$$\min_{u \in X} F(u)$$

admits no minimizer

$$\Downarrow$$

We solve (P1) and (P2), say with solutions $u_1 \in C^1[0, 1/2]$, $u_2 \in C^1[1/2, 1]$ respectively. Then

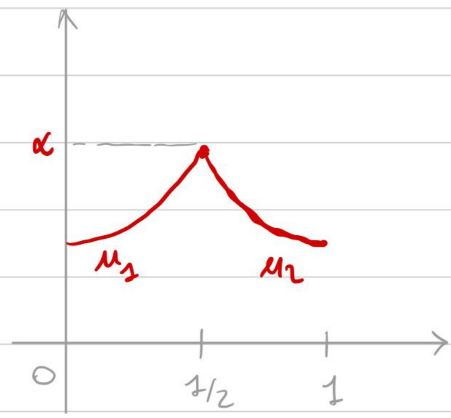
$$\hat{u}(x) := \begin{cases} u_1(x) & \text{if } x \in [0, 1/2] \\ u_2(x) & \text{if } x \in [1/2, 1] \end{cases}$$

DOES NOT BELONG to $C^1[0,1]$ (otherwise it would be a minimum).

One can show that

$$\inf_{u \in X} F(u) = F(\hat{u}) \leftarrow$$

(Note $F(\hat{u})$ is well defined by splitting the integral)



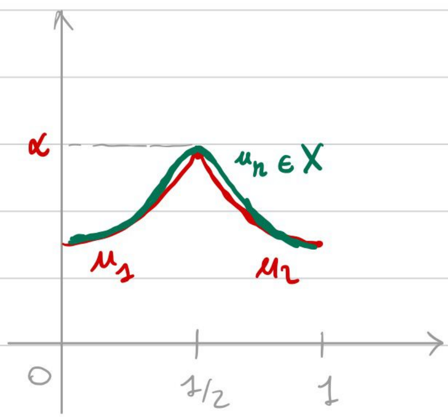
Idea of the proof:

1. Show that $F(u) \geq F(\hat{u})$ for all $u \in X$, by the usual energy estimates

2. Construct $\{u_n\} \subseteq X$ s.t. $u_n \to u$ uniformly on each $K \subseteq [0,1] \setminus \{1/2\}$ compact and

$$F(u_n) \to F(\hat{u})$$

This is done in the usual way: ROUNDING the corner of $\hat{u}$ at $x = 1/2$.



LESSON 5 - 14 APRIL 2021

4. THE EULER-LAGRANGE EQUATION

After the many examples seen so far, we look at the general theory for the minimization of integral functionals

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx$$

where $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, p)$, is the LAGRANGIAN and $u : [a,b] \to \mathbb{R}$. We want to make sufficient assumptions on L so that F admits the first variation δF in some appropriate domain of definition. Specifically, we have:

THEOREM 4.1 Suppose that $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and continuously partially differentiable WRT to the variables s, p.
Let $X \subseteq C^1([a,b])$ be an affine space, with reference vector space $V \subseteq C^1[a,b]$.
Define $F : X \to \mathbb{R}$ by setting

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx$$

Then F is gâteaux differentiable at all points $u \in X$ and all directions $v \in V$, with

$$F_g'(u)(v) = \int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v}\, dx$$

with $L_s := \partial_s L$, $L_p := \partial_p L$. In particular $\delta F(u, v)$ exists, with

$$\boxed{\delta F(u, v) = \int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v}\, dx}$$

NOTE: Here $C^1[a,b]$ is equipped with the norm $\|u\| := \|u\|_\infty + \|u'\|_\infty$.

Proof Let $u \in X$, $v \in V$. As X is affine space over V, then $u + tv \in X$, $\forall t \in \mathbb{R}$. Then

$$(*) \quad \frac{F(u+tv) - F(u)}{t} = \int_a^b \underbrace{\frac{L(x, u+tv, \dot{u}+t\dot{v}) - L(x,u,\dot{u})}{t}}_{=: \Lambda(t,x)} dx$$

Now suppose $|t| \leq \varepsilon$. Then

$$\Lambda(t,x) = \frac{1}{t} \int_0^t \left\{ \frac{d}{d\tau} L(x, u(x) + \tau v(x), \dot{u}(x) + \tau \dot{v}(x)) \right\} d\tau$$

As L diff. in s, p $\rightarrow$
$$= \frac{1}{t} \int_0^t \left\{ L_s(x, u+\tau v + \dot{u} + \tau \dot{v}) v + L_p(x, u+\tau v, \dot{u} + \tau \dot{v}) \dot{v} \right\} d\tau$$

Adding and subtracting $\rightarrow$
$$= L_s(x, u, \dot{u}) v + L_p(x,u,\dot{u}) \dot{v}$$
$$+ \frac{1}{t} \int_0^t \left\{ L_s(x, u+\tau v, \dot{u} + \tau \dot{v}) v - L_s(x,u,\dot{u}) v \right\} d\tau \quad (=: R_1(t,x))$$
$$+ \frac{1}{t} \int_0^t \left\{ L_p(x, u+\tau v, \dot{u} + \tau \dot{v}) \dot{v} - L_p(x,u,\dot{u}) \dot{v} \right\} d\tau \quad (=: R_2(t,x))$$

Thus, by $(*)$,

$$\frac{F(u+tv) - F(u)}{t} = \int_a^b \Lambda(t,x)\, dx$$
$$= \int_a^b L_s(x,u,\dot{u}) v + L_p(x,u,\dot{u}) \dot{v}\, dx + \int_a^b R_1(t,x)\, dx + \int_a^b R_2(t,x) dx$$

To see that F is Gâteaux diff it is sufficient to show that

$$\lim_{t \to 0} \int_a^b R_j(t,x) dx = 0, \quad \text{for } j = 1,2.$$

To this end, notice that, as $u, v \in C^1[a,b]$, then

$$K := \{ (x, u(x) + \tau v(x), \dot{u}(x) + \tau \dot{v}(x)) \mid x \in [a,b], |\tau| \leq \varepsilon/2 \}$$

is compact in $[a,b] \times \mathbb{R} \times \mathbb{R}$. As L_s is continuous on $[a,b] \times \mathbb{R} \times \mathbb{R}$, then in particular it is uniformly continuous on K (continuous on compact $\Rightarrow$ u.c.)
Then $\forall\ \tilde{\varepsilon} > 0$, $\exists\ \delta > 0$ s.t.

$$\left| L_s(x, u(x) + \tau v(x), \dot{u}(x) + \tau \dot{v}(x)) - L_s(x, u(x), \dot{u}(x)) \right| < \tilde{\varepsilon} \quad (*)$$

for all $x \in [a,b]$ and $|\tau| \leq \frac{\varepsilon}{2}$, such that $|\tau| (|v(x)| + |\dot{v}(x)|) < \delta$.
The last condition is fullfilled for τ s.t.

$$|\tau| < \min \left\{ \frac{\varepsilon}{2}, \frac{\delta}{\|v\|} \right\} \quad (B)$$

Therefore let $\hat{\varepsilon} > 0$ be arbitrary and fix $\tilde{\varepsilon} > 0$ s.t.

$$\tilde{\varepsilon} < \frac{\hat{\varepsilon}}{\|v\|} \quad (**)$$

Let also $\tilde{\delta} := \min \{ \varepsilon/2, \frac{\delta}{\|v\|} \}$. Then for $|t| < \delta$ we have

$$|R_1(t,x)| \leq \frac{1}{|t|} \int_0^t | L_s(x, u + \tau v, \dot{u} + \tau \dot{v}) - L_s(x, u, \dot{u}) | \, d\tau \, |v(x)|$$

$$\left(\begin{array}{c} \text{by } (*) \text{, as} \\ |\tau| \leq |t| < \delta \text{, so} \\ \text{that (B) holds} \end{array} \right) \leq \frac{\|v\|}{|t|} \int_0^t \tilde{\varepsilon} \, d\tau = \|v\| \tilde{\varepsilon} < \hat{\varepsilon} \quad (\text{by } (**))$$

As $\hat{\varepsilon}$ is arbitrary, and $\tilde{\delta}$ does not depend on x, we conclude

$$\lim_{t \to 0} \sup_{x \in [a,b]} |R_1(t,x)| = 0 .$$

By similar arguments also $\lim_{t\to 0} \sup_{x\in[a,b]} |R_2(t,x)| = 0$.

Then $\int_a^b R_i(t,x)\,dx \to 0$ as $t\to 0$. Taking the limit in

$$\frac{F(u+tv)-F(u)}{t} = \int_a^b L_s(x,u,\dot{u})v + L_p(x,u,\dot{u})\dot{v}\,dx + \int_a^b R_1(t,x)\,dx + \int_a^b R_2(t,x)\,dx$$

yields that

$$F_g'(u)(v) = \int_a^b L_s(x,u,\dot{u})v + L_p(x,u,\dot{u})\dot{v}\,dx,$$

as claimed. Now just recall that for affine spaces which are also normed,

$$\delta F(u,v) := \lim_{t\to 0} \frac{F(u+tv)-F(u)}{t}$$

(see REMARK 2.5). This concludes. □

DEFINITION 4.2 In the setting of THEOREM 4.1, we call

$$(*) \quad \boxed{\delta F(u,v) = \int_a^b L_s(x,u,\dot{u})v + L_p(x,u,\dot{u})\dot{v}\,dx}$$

the FIRST INTEGRAL FORM of the FIRST VARIATION.

CASE OF DIRICHLET BOUNDARY CONDITIONS

Assume $L : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ continuous, and continuously diff. in s,p. Let

$$X = \{ u\in C^1[a,b] \mid u(a)=\alpha,\ u(b)=\beta \}$$

which is an affine space over

$$V = \{ v\in C^1[a,b] \mid v(a)=v(b)=0 \}.$$

Then we can apply THEOREM 4.1, and $\delta F(u,v)$ is given by $(*)$.

Assume also that

$$L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}), \qquad u \in C^2[a,b] \cap X$$

Then the second term in $\circledast$ can be integrated by parts:

$$\int_a^b L_p(x,u,\dot{u})\,\dot{v}\,dx = L_p(x,u,\dot{u})\,v\Big|_a^b - \int_a^b (L_p(x,u,\dot{u}))'\,v(x)\,dx$$

$$= -\int_a^b (L_p(x,u,\dot{u}))'\,v(x)\,dx \qquad (\text{as } v(a)=v(b)=0)$$

Therefore $\circledast$ reads

$$(**) \qquad \boxed{\delta F(u,v) = \int_a^b \left\{ L_s(x,u,\dot{u}) - (L_p(x,u,\dot{u}))' \right\} v(x)\,dx}$$

Note that, in the above assumptions, we can explicitly compute

$$(L_p(x,u,\dot{u}))' = L_{px}(x,u,\dot{u}) + L_{ps}(x,u,\dot{u})\,\dot{u} + L_{pp}(x,u,\dot{u})\,\ddot{u}$$

DEFINITION 4.3 $(**)$ is called the SECOND INTEGRAL FORM of the FIRST VARIATION

Assume in addition that u minimizes F over X.

Then by REMARK 3.7 we know that $\delta F(u,v)=0$, $\forall v \in V$. Note that $C_c^\infty(a,b) \subseteq V$. Hence we can apply the FLCV (LEMMA 3.4) to $(**)$ (equated to zero), and obtain

$$(***) \qquad \boxed{[L_p(x,u,\dot{u})]' = L_s(x,u,\dot{u})}$$

Note that, in addition to $(***)$, u satisfies also the DIRICHLET BC imposed in X, that is,

$$u(a) = \alpha, \quad u(b) = \beta$$

DEFINITION 4.4 (***) is called **EULER-LAGRANGE EQUATION in DIFFERENTIAL FORM.**

We therefore have proven the following theorem.

THEOREM 4.5 Let $L: [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and continuously partially differentiable in s, p.
Define

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha, u(b) = \beta \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = v(b) = 0 \}$$

Define the functional $F: X \to \mathbb{R}$ s.t.

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

(1) If $u \in X$ minimizes F over X, then u solves the **ELE in INTEGRAL FORM**:

$$\int_a^b L_s(x, u, \dot{u}) \, v + L_p(x, u, \dot{u}) \, \dot{v} \; dx = 0$$

for all $v \in V$.

(2) Assume in addition $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$.
If $u \in X \cap C^2[a,b]$ minimizes F over X, then u solves the **ELE in DIFFERENTIAL FORM**:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x)) \Big] = L_s(x, u(x), \dot{u}(x)), & \forall x \in (a,b) \\ u(a) = \alpha, \quad u(b) = \beta \end{cases}$$

THE CASE OF NEUMANN BOUNDARY CONDITIONS

Again, suppose $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously diff. in s, p. Define

$$X = \{ u \in C^1[a,b] \mid u(a) = \alpha \}$$

which is affine over

$$V = \{ v \in C^1[a,b] \mid v(a) = 0 \}.$$

We can then apply THEOREM 4.1 to obtain the FIRST INTEGRAL FORM of the FIRST VARIATION:

$$(*) \qquad \boxed{\delta F(u,v) = \int_a^b L_s(x,u,\dot{u})\, v + L_p(x,u,\dot{u})\, \dot{v} \; dx}$$

Assume in addition that

$$L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}), \qquad u \in C^2[a,b] \cap X.$$

Then the second term in $(*)$ can be integrated by parts

$$\int_a^b L_p(x,u,\dot{u})\, \dot{v}\, dx = L_p(x,u,\dot{u})\, v \Big|_a^b - \int_a^b (L_p(x,u,\dot{u}))'\, v(x)\, dx$$

$$= L_p(b, u(b), \dot{u}(b))\, v(b) - \int_a^b \left[L_p(x,u,\dot{u}) \right]' v \, dx$$

obtaining the SECOND INTEGRAL FORM of the FIRST VARIATION:

$$(**) \qquad \boxed{\delta F(u,v) = \int_a^b \left\{ L_s(x,u,\dot{u}) - \left[L_p(x,u,\dot{u}) \right]' \right\} v \, dx + L_p(b,u(b),\dot{u}(b))\, v(b)}$$

Assume now that u is also a minimizer. Then by REMARK 3.7 we have $\delta F(u,v)=0$, $\forall v\in V$. In particular we can test for

$$v \in V \quad \text{s.t} \quad v(b)=0$$

to obtain

$$\int_a^b \left\{ L_s(x,u,\dot{u}) - \left[L_p(x,u,\dot{u})\right]' \right\} v \, dx = 0, \quad \forall v \in C^1[a,b] \text{ s.t. } v(a)=v(b)=0.$$

Then by FLCV we obtain the EULER-LAGRANGE EQUATION in DIFF. FORM:

$$(***) \qquad \boxed{\left[L_p(x,u,\dot{u})\right]' = L_s(x,u,\dot{u})}$$

Now, the first boundary condition to pair to (***) is already given in X:

$$u(a)=\alpha$$

For the second BC, just test (**) against $v\in V$ s.t. $v(b)\neq 0$, and recall (***), to get

$$L_p(b,u(b),\dot{u}(b))\,v(b)=0, \ \forall v\in V \quad \xRightarrow[\text{s.t. } v(b)\neq 0]{\text{Taking } v\in V} \quad L_p(b,u(b),\dot{u}(b))=0$$

which is a NEUMANN BOUNDARY CONDITION.

NOTE If we took $X=V=C^1[a,b]$ in the above example, we would have obtained that minimizers $u\in C^2[a,b]\cap X$ satisfy (***) with two NEUMANN BC

$$L_p(a,u(a),\dot{u}(a)) = L_p(b,u(b),\dot{u}(b)) = 0$$

To summarize, we have proven the following Theorem:

THEOREM 4.6 Let $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and continuously partially differentiable WRT s, p.
Define sets

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = 0 \}$$

Define $F : X \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx$$

(1) Suppose u minimizes F over X. Then u solves the ELE in INTEGRAL FORM:

$$\int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v}\; dx = 0, \quad \forall v \in V$$

(2) Suppose in addition $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, and that $u \in X \cap C^2[a,b]$ minimizes F over X. Then u solves ELE in DIFFERENTIAL FORM:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x)) \Big] = L_s(x, u(x), \dot{u}(x)), & \forall x \in (a,b) \\[2ex] u(a) = \alpha, \quad L_p(b, u(b), \dot{u}(b)) = 0 \end{cases}$$

ELE IN ERDMANN FORM

Consider the special case of Lagrangians not depending on x, i.e.,

$$F(u) = \int_a^b L(u, \dot{u})\, dx \quad , \quad L : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

with $F : X \to \mathbb{R}$, $X \subseteq C^1[a,b]$ affine space over $V \subseteq C^1[a,b]$.

As done previously, if $L \in C^2(\mathbb{R}\times\mathbb{R})$ and $u \in C^2[a,b] \cap X$ minimizes F over X, the ELE reads

$$(*) \qquad \left[L_p(u, \dot{u}) \right]' = L_s(u, \dot{u}) \quad , \quad \forall\, x \in (a,b)$$

Multiplying by $\dot{u}$ yields

$$(**) \qquad \left[L_p(u, \dot{u}) \right]' \dot{u} = L_s(u, \dot{u})\, \dot{u}$$

Now the LHS is

$$\left[L_p(u, \dot{u}) \right]' \dot{u} = \left[L_p(u, \dot{u})\, \dot{u} \right]' - L_p(u, \dot{u})\, \ddot{u}$$

so that, from $(**)$

$$\left[L_p(u, \dot{u})\, \dot{u} \right]' \overset{\text{by } (**)}{=} L_s(u, \dot{u})\, \dot{u} + L_p(u, \dot{u})\, \ddot{u} \overset{\text{by direct calculation}}{=} \left[L(u, \dot{u}) \right]'$$

Therefore

$$\boxed{L_p(u, \dot{u})\, \dot{u} = L(u, \dot{u}) + \text{constant}}$$

ELE in ERDMANN FORM

REMARK 4.5 ELE and ELE-ERDMANN are not equivalent. It holds:

1. If u satisfies ELE $\Rightarrow$ u satisfies ELE-ERDMANN

   (we just proved this)

2. If u satisfies ELE-ERDMANN $\Rightarrow$ u satisfies ELE in the points $x \in [a,b]$ s.t. $\dot{u}(x) \neq 0$

   (To show this, just go backwards in the above calculation)

ELE FOR GENERAL LAGRANGIANS

HIGHER ORDER

$X \subseteq C^k[a,b]$ affine space over $V \subseteq C^k[a,b]$, $L : [a,b] \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$

$L = L(x, s, p_1, \dots, p_k)$, $F : X \to \mathbb{R}$ defined by:

$$F(u) := \int_a^b L(x, u, \dot{u}, \ddot{u}, \dots, u^{(k)})\, dx$$

Assume L is continuous and continuously differentiable WRT $s, p_1, \dots, p_k$.

Analogously to THEOREM 4.1, one can compute the Gâteaux derivative of F and obtain the FIRST INTEGRAL FORM of the FIRST VARIATION

$$(*) \quad \boxed{\delta F(u,v) = \int_a^b L_s(x, u, \dots, u^{(k)})\, v + \sum_{i=1}^{k} L_{p_i}(x, u, \dots, u^{(k)})\, v^{(i)}\, dx}$$

Assume now that $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}^k)$, $u \in C^{k+1}[a,b]$, and $v \in V$ is s.t. $v^{(i)}(a) = v^{(i)}(b) = 0$ for all $i = 0, .., k-1$. Integrating (*) by parts we get the SECOND INTEGRAL FORM of the FIRST VARIATION

$$\boxed{\delta F(u,v) = \int_a^b \left\{ L_s(x,u,..,u^{(k)}) + \sum_{i=1}^{k} (-1)^i \frac{d^i}{dx^i} L_{p_i}(x,u,..,u^{(k)}) \right\} v \, dx}$$

Finally, if in addition u is a minimizer, then $\delta F(u,v) = 0$, and by the FLCV we get the ELE in DIFFERENTIAL FORM

$$\boxed{\sum_{i=1}^{k} (-1)^{i+1} \frac{d^i}{dx^i} L_{p_i}(x,u,..,u^{(k)}) = L_s(x,u,..,u^{(k)})} \quad , \forall x \in (a,b)$$

MORE UNKNOWNS

$X \subseteq C^1[a,b]$ affine space over $V \subseteq C^1[a,b]$, $L : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$,

$L = L(x, s_1, .., s_k, p_1, .., p_k)$, $F : \underbrace{X \times \cdots \times X}_{k \text{ times}} \to \mathbb{R}$ defined by

$$F(u_1, .., u_k) := \int_a^b L(x, u_1, .., u_k, \dot{u}_1, .., \dot{u}_k) \, dx$$

Assume L is continuous and continuously differentiable in $s_1, .., s_k, p_1, .., p_k$.

Analogously to THEOREM 4.1, one can compute the Gâteaux derivative of F and obtain the FIRST INTEGRAL FORM of the FIRST VARIATION

$$\boxed{\delta F(u,v) = \int_a^b \sum_{i=1}^{k} \left[L_{s_i}(x,u,\dot u)\, v_i + L_{p_i}(x,u,\dot u)\, \dot v_i \right] dx} \quad (*)$$

where $u = (u_1, \dots, u_k) \in X^k$, $v = (v_1, \dots, v_k) \in X^k$.

Suppose in addition that $L \in C^2([a,b] \times \mathbb{R}^k \times \mathbb{R}^k)$, $u_i \in C^2[a,b] \cap X$ and that $v_i \in V$ are s.t. $v_i(a) = v_i(b) = 0$, for all $i = 1, \dots, k$. Then we can integrate $(*)$ by parts to get the SECOND INTEGRAL FORM of the FIRST VARIATION

$$\boxed{\delta F(u,v) = \int_a^b \sum_{i=1}^{k} \left[L_{s_i}(x,u,\dot u) - L_{p_i}(x,u,\dot u)' \right] v_i \, dx}$$

Finally, taking $u \in X^k$ minimum of F and $v_1 \in C_c^\infty(a,b)$, $v_2 = v_3 = \dots = v_k = 0$ and applying FLCV, we get

$$L_{p_1}(x,u,\dot u)' = L_{s_1}(x,u,\dot u)$$

Similarly, by taking the other components of v to be zero, except for one, we obtain the ELE in DIFFERENTIAL FORM

$$\boxed{L_{p_i}(x,u,\dot u)' = L_{s_i}(x,u,\dot u), \quad i = 1, \dots, k} \quad , \forall x \in (a,b)$$

which in this case is a SYSTEM of k **ODEs** of ORDER 2.

5. SUFFICIENT CONDITIONS FOR MINIMALITY

LESSON 6
21 APRIL 2021

So far we have shown that solutions to a minimization problem for integral functionals also solve the associated EULER-LAGRANGE EQUATION.

QUESTION: Are solutions to (ELE) minimizers? If YES, how do we prove it?

To answer the above, we will analyze 4 methods:

WARNING: This is in general not true: we can have $\bar{u}$ sol. to (ELE) but not minimizer

(1) CONVEXITY
(2) TRIVIAL LEMMA
} NOW

(3) CALIBRATIONS
(4) WEIERSTRASS FIELDS
} LATER in the course (if we have time!)

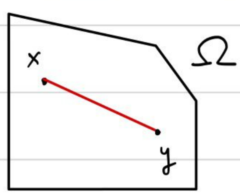
(1) CONVEXITY

If the Lagrangian $L = L(x,s,p)$ is convex in s,p, we will prove that solutions to (ELE) are minimizers.

DEFINITION 5.1

Let $\Omega \subseteq \mathbb{R}^d$. We say that Ω is convex if

$$\lambda x + (1-\lambda) y \in \Omega, \quad \forall\, x,y \in \Omega,\ \lambda \in [0,1].$$

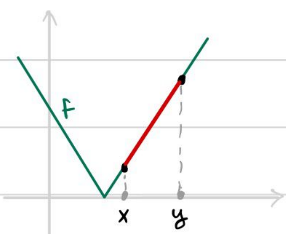


CONVEX SET Ω

Let $f : \Omega \to \mathbb{R}$, with Ω convex. We say that f is convex if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x,y \in \Omega$, $\lambda \in [0,1]$.

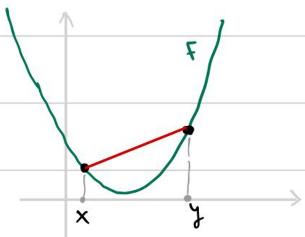


f CONVEX BUT NOT STRICTLY

We say that f is STRICTLY CONVEX if

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y),$$

for all $x,y \in \Omega$, s.t. $x \neq y$ and $\lambda \in (0,1)$.



f STRICTLY CONVEX

For regular convex functions the following result holds:

THEOREM 5.2 Let $\Omega \subseteq \mathbb{R}^d$ be open convex, $f: \Omega \to \mathbb{R}$, $f \in C^1(\Omega)$.
Then

1) f is convex iff

(f above tangent planes) $\rightsquigarrow$ $$f(y) \geq f(x) + \nabla f(x) \cdot (y-x), \quad \forall x, y \in \Omega$$

2) f is strictly convex iff

$$f(y) > f(x) + \nabla f(x) \cdot (y-x), \quad \forall x, y \in \Omega, \ x \neq y$$

Assume in addition that $f \in C^2(\Omega)$

3) f is convex iff the HESSIAN $\nabla^2 f$ is POSITIVE SEMI-DEFINITE, i.e.,

$$y^T \nabla^2 f(x) \, y \geq 0, \quad \forall x \in \Omega, \ y \in \mathbb{R}^d$$

4) Assume $\nabla^2 f$ is POSITIVE DEFINITE, i.e.,

$$y^T \nabla^2 f(x) \, y > 0, \quad \forall x \in \Omega, \ y \in \mathbb{R}^d \setminus \{0\}$$

Then f is strictly convex.

(The proof is standard, from analysis courses. See B. DACOROGNA - "INTRODUCTION TO THE CALCULUS OF VARIATIONS", IMPERIAL COLLEGE PRESS, 2004 - THEOREM 1.5)

WARNING: The converse of (4) does not hold.

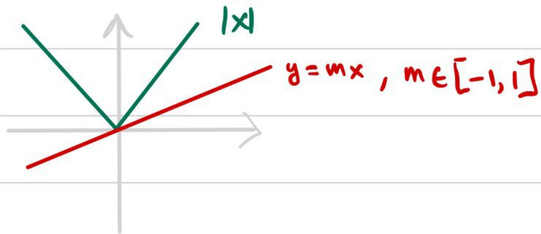
If instead we have no regularity, then we get:

THEOREM 5.3 Let $\Omega \subseteq \mathbb{R}^d$ be convex, and $f: \Omega \to \mathbb{R}$ convex. Let $\bar{x} \in \Omega$. Then $\exists\, m \in \mathbb{R}^d$ s.t.

$$f(y) \geq f(\bar{x}) + m \cdot (y - \bar{x}), \quad \forall\, y \in \Omega$$

(Proof is omitted. This result is saying that if f convex then $\partial f(\bar{x}) \neq \emptyset$, i.e., the SUBDIFFERENTIAL of f at $\bar{x}$ is non-empty. For a proof see R.T. ROCKAFELLAR – "CONVEX ANALYSIS", PRINCETON UNIVERSITY PRESS, 1970 – THEOREM 23.4)

NOTE: For $f: [a,b] \to \mathbb{R}$ one can take $m \in [f'_-(\bar{x}), f'_+(\bar{x})]$ left and right derivatives.



APPLICATION TO COV

Let $X := \{ u \in C^1[a,b] \mid u(a) = \alpha,\ u(b) = \beta \}$, $V := \{ v \in C^1[a,b] \mid v(a) = v(b) = 0 \}$.
Let $L: [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,p)$ and define $F: X \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx.$$

THEOREM 5.4 Suppose $L \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ and let $\bar{u} \in X$ be a solution to ELE in INTEGRAL FORM, i.e.,

$$(*) \quad \int_a^b L_s(x, \bar{u}, \bar{u}')\, v + L_p(x, \bar{u}, \bar{u}')\, v' \, dx = 0, \quad \forall\, v \in V.$$

(1) If $(s,p) \mapsto L(x,s,p)$ is CONVEX for all $x \in [a,b]$ fixed, then $\bar{u}$ is a minimizer for F in X.

(2) If $(s,p) \mapsto L(x,s,p)$ is STRICTLY CONVEX for all $x \in [a,b]$, then $\bar{u}$ is the unique minimizer of F in X.

NOTE: If u solves ELE in DIFFERENTIAL FORM then it solves ELE in INTEGRAL FORM

<u>Proof</u> (1) Let $w \in X$ be arbitrary and set $v := w - \bar{u}$.
Then $v \in V$ i.e. $v(a) = v(b) = 0$. We have

$$F(w) = F(\bar{u} + v) = \int_a^b L(x, \bar{u} + v, \bar{u}' + v')\, dx$$

As L is C^1 and is convex in s, p, we can apply Theorem 5.2 and obtain

$$L(x, s+\tilde{s}, p+\tilde{p}) \geq L(x, s, p) + L_s(x,s,p)\,\tilde{s} + L_p(x,s,p)\,\tilde{p}, \quad \begin{array}{l} \forall s, \tilde{s}, p, \tilde{p} \in \mathbb{R} \\ \forall x \in [a,b] \end{array}$$

Apply the above with $s = \bar{u}$, $\tilde{s} = v$, $p = \bar{u}'$, $\tilde{p} = v'$,

$$F(w) = \int_a^b L(x, \bar{u} + v, \bar{u}' + v')\, dx \geq$$

$$\geq \int_a^b L(x, \bar{u}, \bar{u}')\, dx + \underbrace{\int_a^b L_s(x, \bar{u}, \bar{u}')\, v + L_p(x, \bar{u}, \bar{u}')\, v'\, dx}_{=0 \text{ by } (*) \text{, since } v \in V}$$

$$= F(\bar{u})$$

showing that $\bar{u}$ minimizes F over X.

(2) Assume $\bar{u}$ and $\hat{u}$ both minimize F over X. Set $m := \min\{F(u) \mid u \in X\}$.
Therefore $F(\hat{u}) = F(\bar{u}) = m$, and also $F(u) \geq m$, $\forall u \in X$.

Define $w := \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}$, so $w \in X$. By convexity of L (just using the definition)

$$L(x, w, w') = L\left(x, \tfrac{1}{2}\bar{u} + \tfrac{1}{2}\hat{u}, \tfrac{1}{2}\bar{u}' + \tfrac{1}{2}\hat{u}'\right)$$

$$\leq \tfrac{1}{2} L(x, \bar{u}, \bar{u}') + \tfrac{1}{2} L(x, \hat{u}, \hat{u}')$$

Integrating the above inequality we obtain

$$m \leq F(w) = \int_a^b L(x, w, w')\,dx \leq$$

Since $w \in X$

$$\leq \frac{1}{2}\int_a^b L(x, \bar{u}, \bar{u}')\,dx + \frac{1}{2}\int_a^b L(x, \hat{u}, \hat{u}')\,dx =$$

$$= \frac{1}{2} F(\bar{u}) + \frac{1}{2} F(\hat{u})$$

$$= \frac{1}{2} m + \frac{1}{2} m = m$$

Thus, all the inequalities in the above chain are actually equalities, and we get

$$(*) \quad \int_a^b \left\{ \frac{1}{2} L(x, \bar{u}, \bar{u}') + \frac{1}{2} L(x, \hat{u}, \hat{u}') - L\left(x, \tfrac{1}{2}\bar{u} + \tfrac{1}{2}\hat{u}, \tfrac{1}{2}\bar{u}' + \tfrac{1}{2}\hat{u}'\right) \right\} dx = 0$$

Now, by convexity of L, the INTEGRAND in $(*)$ is always ≥ 0. Hence, by continuity, we conclude that

$$L\left(x, \tfrac{1}{2}\bar{u} + \tfrac{1}{2}\hat{u}, \tfrac{1}{2}\bar{u}' + \tfrac{1}{2}\hat{u}'\right) = \frac{1}{2} L(x, \bar{u}, \bar{u}') + \frac{1}{2} L(x, \hat{u}, \hat{u}'), \quad \forall x \in [a,b]$$

Since L is STRICTLY CONVEX, the above is possible iff $\bar{u}(x) = \hat{u}(x)$ and $\bar{u}'(x) = \hat{u}'(x)$, $\forall x \in [a,b]$.

Thus $\bar{u} \equiv \hat{u}$ and the minimizer is unique. $\square$

EXAMPLE Let $L : \mathbb{R} \to \mathbb{R}$, $L = L(p)$. Assume $L \in C^2(\mathbb{R})$. Define

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha, u(b) = \beta \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = 0, v(b) = 0 \}$$

Consider $F : X \to \mathbb{R}$ defined by

$$F(u) := \int_a^b L(\dot{u})\, dx$$

We can then write ELE in DIFFERENTIAL FORM:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x))\Big] = L_s(x, u(x), \dot{u}(x)) & , \quad \forall x \in (a,b) \\ u(a) = \alpha , \quad u(b) = \beta \end{cases}$$

which in this case reads

$$\text{(ELE)} \quad \begin{cases} \dfrac{d}{dx}\Big[L'(\dot{u}(x))\Big] = 0 & , \quad \forall x \in (a,b) \\ u(a) = \alpha , \quad u(b) = \beta . \end{cases}$$

Now, the above ODE implies that

$$\boxed{L'(\dot{u}) = \text{CONSTANT}}$$

Therefore the straight line

$$\bar{u}(x) := \frac{\beta - \alpha}{b - a}(x - a) + \alpha$$

is ALWAYS a solution to (ELE).

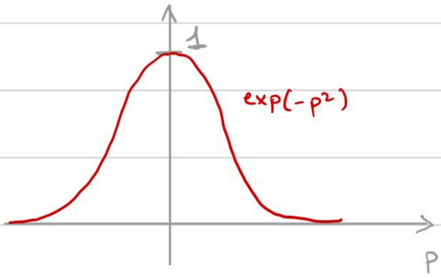
QUESTION When does $\bar{u}$ also solve

$$\text{(P)} \qquad F(\bar{u}) = \min \{ F(u) \mid u \in X \} \qquad ?$$

CASE 1 Assume L CONVEX. As $\bar{u}$ solves (ELE), in particular it solves ELE in INTEGRAL FORM. Then by THEOREM 5.4 we have that $\bar{u}$ solves (P).

CASE 2 If we do not assume convexity, then in general $\bar{u}$ DOES NOT solve (P). For example let

$$L(p) := \exp(-p^2)$$



Let us consider the case with zero Dirichlet conditions, i.e.,

$$X = V = \{ u \in C^1[0,1] \mid u(0) = u(1) = 0 \}.$$

Note that in this setting our straight line is $\bar{u} \equiv 0$. Then $\bar{u}$ solves (ELE), but is it solution to (P) ?

Clearly L is not convex, so THEOREM 5.4 cannot be applied.

FACT The minimization problem (P) has NO SOLUTION and

$$m := \inf \{ F(u) \mid u \in X \} = 0$$

(This will be left as an exercise)

Therefore $\bar{u} \equiv 0$ solves (ELE) but DOES NOT solve (P).

② TRIVIAL LEMMA

Given $\bar{u}$ solution to ELE, we want to know if $\bar{u}$ is also a minimizer. A possible way to answer this question is given by the following Lemma.

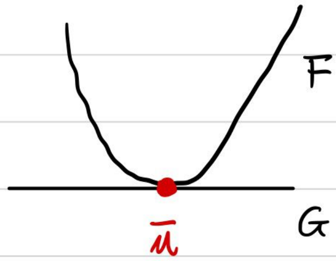
LEMMA 5.5 (LEMMA TRIVIAL)

Let X be a set and $F, G : X \to \mathbb{R}$ functionals. Assume that

(i) $F(u) \geq G(u)$, $\forall u \in X$

(ii) $\bar{u} \in X$ is a minimizer for G on X

(iii) $F(\bar{u}) = G(\bar{u})$.



Then $\bar{u}$ is a minimizer for F. If in addition $\bar{u}$ is the unique minimizer of G, then $\bar{u}$ is the unique minimizer of F.

Proof Let $u \in X$ be arbitrary. Then

$$F(u) \overset{(i)}{\geq} G(u) \overset{(ii)}{\geq} G(\bar{u}) \overset{(iii)}{=} F(\bar{u}),$$

showing that $\bar{u}$ minimizes F.

Assume now that $\bar{u}$ is the unique minimizer of G. Then, for the first part of the statement, we know that $\bar{u}$ also minimizes F. Suppose that $\bar{w} \in X$ is another minimizer for F. Then

$$G(\bar{w}) \underset{(i)}{\leq} F(\bar{w}) \underset{\substack{\text{minimality of } \bar{u} \\ \text{and } \bar{w} \text{ for } F}}{=} F(\bar{u}) \overset{(iii)}{=} G(\bar{u})$$

Thus $G(\bar{w}) = G(\bar{u})$, being $\bar{u}$ minimizer for G. $\Rightarrow \bar{u} = \bar{w}$ as the minimizer of G is unique. $\square$

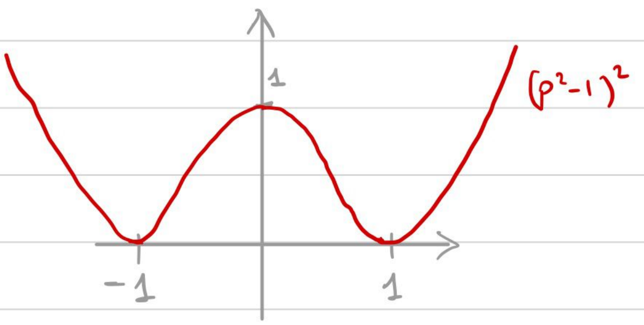
COMMENT: The above lemma requires to find a functional G satisfying (i),(ii),(iii). This is not always obvious. However in the future we will see a systematic way to construct G from F.

EXAMPLE 5.6 $\quad X = \{ u \in C^1[0,1] \mid u(0) = 1 \,,\; u(1) = 3 \}$

$F : X \to \mathbb{R}$ defined by

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 \, dx$$

Note that the Lagrangian is $L = L(p) = (p^2 - 1)^2$, which is not convex. Such Lagrangian is very typical and is named DOUBLE WELL.



DOUBLE WELL

The ELE for the minimum problem associated to F is

$$\begin{cases} \dfrac{d}{dx} L_p(\dot{u}) = 0 \,, & \forall x \in (0,1) \\ u(0) = 1 \,,\; u(1) = 3 \end{cases}$$

From $L_p(\dot{u})' = 0$ we deduce $L_p(\dot{u}) = \text{CONSTANT}$. Therefore the line

$$\bar{u}(x) := 2x + 1$$

Satisfies the BOUNDARY CONDITIONS and ELE,

NOTE If L was CONVEX we could have concluded that $\bar{u}$ minimizes F, by THEOREM 5.4. However L is not convex, so we need to proceed differently.

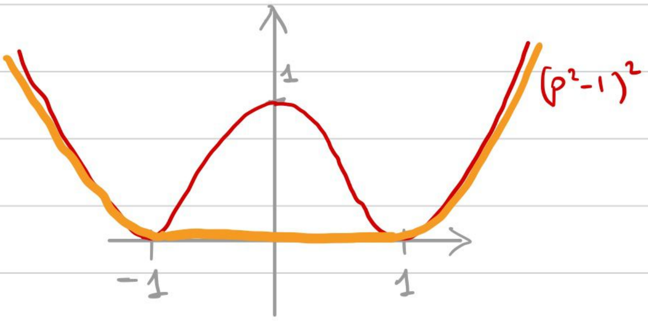
CLAIM $\bar{u}$ is the unique minimizer of F in X.

Proof We make use of the TRIVIAL LEMMA. We need to find G satisfying the assumptions.

IDEA: Find a Lagrangian $\hat{L}$ such that $\hat{L} \leq L$ and that the functional

$$G(u) := \int_0^1 \hat{L}(\dot{u})\, dx \quad , \quad u \in X$$

is likely to admit unique minizer. Ideally we also want $\hat{L}$ to be CONVEX, so we can apply THEOREM 5.4 to G.



A good idea is then to convexify L, by setting

$$\hat{L}(p) := \begin{cases} L(p) & \text{if } |p| \geq 1 \\ 0 & \text{if } |p| \leq 1 \end{cases}$$

Notice that $\hat{L}$ is convex. We now verify (i), (ii), (iii) from LEMMA 5.5:

(i) $F(u) \leq G(u)$, $\forall u \in X$: True because $\hat{L} \leq L$ pointwise.

(ii) $\bar{u}$ minimizes G: True because $\hat{L}$ depends only on p. Therefore the line $\bar{u}$ is solution of ELE for G:

$$\begin{cases} \dfrac{d}{dx} \hat{L}_p(\bar{u}') = 0 \\ \bar{u}(0) = 1, \quad \bar{u}(1) = 2 \end{cases}$$

Therefore $\bar{u}$ minimizes G by THEOREM 5.4, as $\hat{L}$ is CONVEX.

(iii) $F(\bar{u}) = G(\bar{u})$: True because $\bar{u}' \equiv 2$, and $\hat{L}(2) = L(2)$ by definition.

Therefore $\bar{u}$ minimizes F by LEMMA TRIVIAL 5.5.

Also note that $\hat{L}$ is STRICTLY CONVEX in a neighborhood of $p=2$ (that is, in a neighborhood of $\bar{u}' \equiv 2$). Thus (by a slightly more general version of THEOREM 5.4 we conclude that $\bar{u}$ is the unique minimizer of G.
By Lemma 5.5 we then have that $\bar{u}$ is the unique minimizer of F. □

EXAMPLE 5.7 (VARIATION ON EXAMPLE 5.6)

Let us consider the same Lagrangian $L(p) = (p^2 - 1)^2$ as in EXAMPLE 5.6

However this time we look for a minimum of F over the set

$$X = \{ u \in C^1[a,b] \mid u(0) = 0 \ , \ u(1) = 0 \}$$

Note : The only difference is we have changed the DIRICHLET BC

Let's try to show that the line passing through $(0,0)$ and $(1,0)$, i.e.,

$$\bar{u}(x) \equiv 0$$

(which solves ELE associated to F) is a minimizer for F.
We immediately see that the above strategy fails, because by definition

$$\hat{L}(0) = 0 \ , \quad \text{while} \quad L(0) = 1$$

Thus (iii) does not hold and we cannot apply LEMMA 5.5 to F, G and $\bar{u}$.

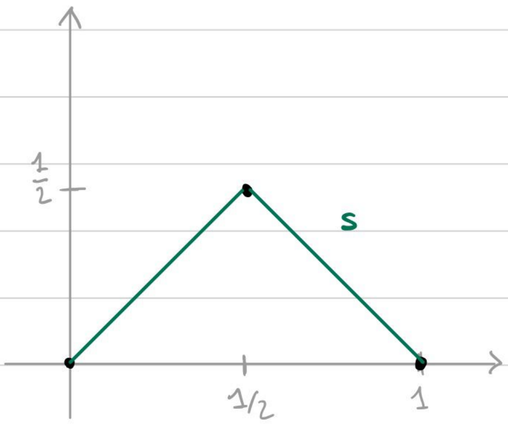
What is going on ?

Since the constraints are $u(0)=0$, $u(1)=0$, then $\bar{u}$ is NOT a minimizer for F. More in general:

CLAIM F admits no minimizer in X. Moreover

$$m := \inf \{ F(u) \mid u \in X \} = 0$$

Proof The idea is that, since $L(1) = L(-1) = 0$, we can construct a function $\tilde{s} \in X$ s.t. $|\tilde{s}'| \approx 1$ and so $F(\tilde{s}) \approx 0$. This is possible because the points $(0,0)$, $(1,0)$ are sufficiently close. To construct $\tilde{s}$, define $s : [0,1] \to \mathbb{R}$ by



$$s(x) := \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ -x+1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Notice that $s(0) = 0$, $s(1) = 0$ and $s' \in \{-1, 1\}$. Thus $F(s) = 0$.

The only problem is that s is not C^1. However, we can "ROUND the CORNER" at $x = 1/2$ by paying a small amount of energy (see WORKSHEET 3)
Thus we can define $\tilde{s} : [0,1] \to \mathbb{R}$ s.t.

$$\tilde{s} \in C^1[0,1], \quad \tilde{s}(0) = \tilde{s}(1) = 0, \quad \tilde{s}'(x) = \pm 1 \text{ for } x \in [0,1] \setminus I, \quad F(\tilde{s}) = \varepsilon$$

with $I = (1/2 - \delta, 1/2 + \delta)$ for some $\delta > 0$ and $\varepsilon > 0$ arbitrary. Then $\tilde{s} \in X$, so that $m \le F(\tilde{s}) = \varepsilon$. As ε is arbitrary, we conclude $m = 0$ (as $F \ge 0$).

Finally, to see that the infimum is not attained, if it existed $\bar{u} \in X$ s.t. $F(\bar{u}) = 0$, then in particular

$$F(\bar{u}) = \int_0^1 ((\bar{u}')^2 - 1)^2 \, dx = 0 \implies \bar{u}' \in \{-1, 1\} \text{ for all } x \in [0,1].$$

However, as $\bar{u}'$ is continuous, we can only have $\bar{u}' \equiv 1$, or $\bar{u}' \equiv -1$, which are not possible since we must have $\bar{u}(0) = \bar{u}(1) = 0$ by the DIRICHLET BC. Thus F admits no minimizer. $\square$

NOTE In general if we define $X := \{ u \in C^1[0,1] \mid u(0) = \alpha,\ u(1) = \beta \}$ and

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 \, dx, \qquad u \in X.$$

then

- If $|\beta - \alpha| > 1$, then the unique minimizer of F is the straight line

$$\bar{u}(x) = (\beta - \alpha) x + \alpha$$

  which can be shown as in EXAMPLE 5.6.

- If $|\beta - \alpha| \le 1$ then F admits no minimizers and the infimum is 0. This can be shown by adapting the arguments of EXAMPLE 5.7.

SUMMARY OF INDIRECT METHOD

Given a minimization problem, the strategy is as follows:

1. Finding necessary conditions for minimality: ELE + BC
2. Solve ELE + BC (This is possible in very few cases: linear differential equations and not much more)

3. Prove that STATIONARY POINTS found in 2 are minimizers:

   → Using CONVEXITY
   → Using TRIVIAL LEMMA

6. L^p SPACES REVISION

REFERENCE W. RUDIN – "REAL AND COMPLEX ANALYSIS"
Mc GRAW – HILL, 2001

MEASURE THEORY

σ-Algebra Let Ω be a SET. Denote by $P(\Omega)$ the set of all subsets of Ω. A collection $\mathcal{A} \subseteq P(\Omega)$ is called a σ-ALGEBRA if

(1) $\emptyset \in \mathcal{A}$

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, where $A^c := \Omega \setminus A$

(3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n=1}^{+\infty} A_n \in \mathcal{A}$

The sets in $\mathcal{A}$ are called MEASURABLE.
The pair $(\Omega, \mathcal{A})$ is called MEASURE SPACE

NOTATION If $\mathcal{G} \subseteq P(\Omega)$ is a collection of sets, we denote by $\sigma(\mathcal{G})$ the smallest σ-algebra on Ω containing $\mathcal{G}$, that is,

$$\sigma(\mathcal{G}) := \bigcap \{\mathcal{A} \subseteq P(\Omega) \mid \mathcal{A} \text{ is } \sigma\text{-algebra}, \mathcal{G} \subseteq \mathcal{A}\}$$

BOREL SETS If τ is a topology over Ω, we call $\sigma(\tau)$ the BOREL σ-algebra. The elements of $\sigma(\tau)$ are called BOREL SETS

MEASURES A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a MEASURE if

(1) $\mu(\emptyset) = 0$

COUNTABLY ADDITIVE → (2) $\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} \mu(A_n)$ whenever $\{A_n\} \subseteq \mathcal{A}$ and they are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ if $i \neq j$

The triple $(\Omega, \mathcal{A}, \mu)$ is called MEASURABLE SPACE

TERMINOLOGY Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space.

- μ is called **COMPLETE** if for all $E \in \mathcal{A}$ s.t $\mu(E)=0$, then every $F \subseteq E$ satisfies $F \in \mathcal{A}$.

- μ is **σ-FINITE** if $\exists \{\Omega_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ s.t.

$$\Omega = \bigcup_{n=1}^{+\infty} \Omega_n \quad \text{and} \quad \mu(\Omega_n) < +\infty, \ \forall n \in \mathbb{N}.$$

- μ is **FINITE** if $\mu(\Omega) < +\infty$

- The sets $E \in \mathcal{A}$ s.t. $\mu(E)=0$ are called **NULL SETS**

- We say that a property holds **μ-ALMOST EVERYWHERE** in Ω (abbreviated in μ-a.e.) if $\exists$ $E \in \mathcal{A}$ s.t. $\mu(E)=0$ and the property holds for all $x \in \Omega \setminus E$.

OUTER MEASURES Ω set. A set map $\mu^*: P(\Omega) \to [0,+\infty]$ is called **OUTER MEASURE** if

(a) $\mu^*(\emptyset)=0$

Monotonic $\to$ (b) $\mu^*(E) \leq \mu^*(F)$ for all $E \subseteq F \subseteq \Omega$

Sub-additive $\to$ (c) $\mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(E_n)$, for all $\{E_n\}_{n\in\mathbb{N}} \subseteq \Omega$

To construct an outer measure we usually start with a family $\mathcal{G} \subseteq P(\Omega)$ of elementary sets (e.g. cubes in $\mathbb{R}^d$), for which we have a desired notion of measure $\rho: \mathcal{G} \to [0,+\infty]$.

PROPOSITION 6.1 Let $\Omega \neq \emptyset$, $\mathcal{G} \subseteq P(\Omega)$, $\rho : \mathcal{G} \to [0, +\infty]$. Assume that

- $\emptyset \in \mathcal{G}$ and $\rho(\emptyset) = 0$,
- $\exists \{\Omega_n\}_{n\in\mathbb{N}} \subseteq \mathcal{G}$ s.t. $\Omega = \bigcup_{n=1}^{+\infty} \Omega_n$

Define $\mu^* : P(\Omega) \to [0, +\infty]$ by

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{+\infty} \rho(E_n) \;\middle|\; \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{G},\ E \subseteq \bigcup_{n=1}^{+\infty} E_n \right\}$$

Then μ^* is an OUTER MEASURE.

The problem with outer measures is that they are not additive on disjoint sets. To solve this problem, we restrict μ^* on a smaller collection of sets $\mathcal{A}^* \subseteq P(\Omega)$:

μ^*-MEASURABLE SETS Given $\mu^* : P(\Omega) \to [0, +\infty]$ outer measure, we say that $E \subseteq \Omega$ is μ^*-MEASURABLE if

$$\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c), \quad \forall F \subseteq \Omega$$

THEOREM 6.2 (CARATHÉODORY)

Let $\Omega \neq \emptyset$ and let $\mu^* : P(\Omega) \to [0, +\infty]$ be an outer measure. Define

$$\mathcal{A}^* := \{ E \subseteq \Omega \mid E \text{ is } \mu^*\text{-measurable} \}.$$

Then $\mathcal{A}^*$ is a σ-algebra and $\mu^* : \mathcal{A}^* \to [0, +\infty]$ is a COMPLETE MEASURE.

THE LEBESGUE MEASURE

On $\mathbb{R}^d$ we can construct a particular measure called the LEBESGUE MEASURE.

For $x \in \mathbb{R}^d$, $r > 0$, define $Q(x,r) := x + \left(-\frac{r}{2}, \frac{r}{2}\right)^d$, the CUBE of side length r centered at x. Introduce the collection of cubes $\mathcal{G} \subseteq P(\mathbb{R}^d)$ as

$$\mathcal{G} := \{ Q(x,r) \mid x \in \mathbb{R}^d, r > 0 \} \cup \{\emptyset\}$$

and $\rho : \mathcal{G} \to [0, +\infty)$ s.t. $\rho(\emptyset) := 0$ and $\rho(Q(x,r)) := r^d$. We can then define $\mathcal{L}_0^d : P(\mathbb{R}^d) \to [0, +\infty]$ as

$$\mathcal{L}_0^d(E) := \inf \left\{ \sum_{i=1}^{+\infty} r_i^d \;\middle|\; E \subseteq \bigcup_{i=1}^{+\infty} Q(x_i, r_i) \right\}$$

$$= \inf \left\{ \sum_{i=1}^{+\infty} \rho(E_i) \;\middle|\; \{E_i\} \subseteq \mathcal{G},\ E \subseteq \bigcup_{i=1}^{+\infty} E_i \right\}$$

i.e., cover E with cubes and sum up the volumes (counting overlapping). Then take the smallest outcome.

By PROPOSITION 6.1 we have that $\mathcal{L}_0^d$ is an outer measure, called the LEBESGUE OUTER MEASURE. It can be shown that

- $\mathcal{L}_0^d(Q(x,r)) = r^d$
- $\mathcal{L}_0^d$ is TRANSLATION INVARIANT:

$$\mathcal{L}_0^d(x+E) = \mathcal{L}_0^d(E), \quad \forall x \in \mathbb{R}^d, E \subseteq \mathbb{R}^d$$

Define

$$\mathcal{L}^* := \{ E \subseteq \mathbb{R}^d \mid E \text{ is } \mathcal{L}_0^d\text{-measurable} \}$$

Then by THEOREM 6.2 we have that:

(1) $\mathcal{L}^*$ is a σ-algebra, called the σ-ALGEBRA of LEBESGUE MEASURABLE SETS

(2) $\mathcal{L}_0^d$ restricted to $\mathcal{L}^*$ is a COMPLETE MEASURE. We denote it by $\mathcal{L}^d$ and call it the d-DIMENSIONAL LEBESGUE MEASURE

Notice that $\mathcal{L}^d$ is <u>not</u> FINITE ($\mathcal{L}(\mathbb{R}^d) = +\infty$) but it is σ-FINITE, since

$$\mathbb{R}^d = \bigcup_{n=1}^{+\infty} Q(0,n) \quad \text{and} \quad \mathcal{L}^d(Q(0,n)) = n^d < +\infty .$$

Moreover, if we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra of $\mathbb{R}^d$ WRT the Euclidean topology, we have

$$\boxed{\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{L}^*}$$

i.e., all Borel sets of $\mathbb{R}^d$ are Lebesgue measurable.

WARNING The inclusion $\mathcal{B}(\mathbb{R}^d) \subsetneq \mathcal{L}^*$ is STRICT: $\exists$ sets in $\mathcal{L}^*$ which is not Borel measurable. Thus $\mathcal{L}^d$ restricted to $\mathcal{B}(\mathbb{R}^d)$ is <u>not</u> COMPLETE.

WARNING There exist sets $E \subseteq \mathbb{R}^d$ which are <u>NOT</u> Lebesgue measurable.

INTEGRABILITY

On a measurable space $(\Omega, \mathcal{A}, \mu)$ we can define the notion of integrability.

MEASURABLE FUNCTIONS

Let X, Y be non-empty sets, $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras on X and Y respectively. A function $u : X \to Y$ is **MEASURABLE** if

$$u^{-1}(E) \in \mathcal{A} \quad \text{for all } E \in \mathcal{B}.$$

If X, Y are topological spaces and $\mathcal{A}, \mathcal{B}$ are Borel σ-algebras then measurable functions are called **BOREL FUNCTIONS**.

REMARK 6.3

1. If $(X, \mathcal{A})$ is a measurable space and $u : X \to \mathbb{R}$ with $\mathbb{R}$ equipped with the Borel σ-algebra, then

$$u \text{ is measurable} \iff u^{-1}((a, +\infty)) \in \mathcal{A} \text{ for every } a \in \mathbb{R}.$$

If instead $u : X \to [-\infty, +\infty]$ (always with Borel σ-algebra) then

$$u \text{ is measurable} \iff u^{-1}((a, +\infty]) \in \mathcal{A} \text{ for every } a \in \mathbb{R}.$$

2. If X, Y are topological spaces equipped with Borel σ-algebras then

$$u : X \to Y \text{ CONTINUOUS} \implies u \text{ Borel}$$

3. The composition of measurable functions is measurable. In particular if $(X, \mathcal{A})$ is a measure space and $u : X \to \mathbb{R}$ is measurable, then u^p, $|u|$, cu and

$$u^+ := \begin{cases} u & \text{if } u(x) \geq 0 \\ 0 & \text{if } u(x) < 0 \end{cases}, \quad u^- := \begin{cases} -u & \text{if } u(x) \leq 0 \\ 0 & \text{if } u(x) > 0 \end{cases}$$

are all measurable, for $p \geq 1$, $c \in \mathbb{R}$.

(4) Moreover if $v : X \to \mathbb{R}$ is measurable then $u+v$, uv, $\min\{u,v\}$, $\max\{u,v\}$ are measurable.

(5) Let $(X, \mathcal{A})$ be a measurable space and $u_n : X \to [-\infty, +\infty]$ be measurable. Then the functions

$$\sup_{n \in \mathbb{N}} u_n \quad , \quad \inf_{n \in \mathbb{N}} u_n \quad , \quad \liminf_{n \to +\infty} u_n \quad , \quad \limsup_{n \to +\infty} u_n$$

are measurable.

(6) Let $(X, \mathcal{A}, \mu)$ be a measurable space. Assume that μ is COMPLETE. If $u_n : X \to [-\infty, +\infty]$ are measurable and

$$u(x) := \lim_{n \to +\infty} u_n(x) \quad \text{exists for } \mu\text{- a.e. } x \in X$$

then u is measurable.

(7) Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure. Suppose $u : X \to Y$ is measurable. If $v : X \to Y$ is s.t.

$$u(x) = v(x) \quad \text{for } \mu\text{- a.e. } x \in X$$

then v is also measurable.

We are now ready to introduce integrals. For a measurable space $(X, \mathcal{A})$ and $E \subseteq X$ we define the CHARACTERISTIC FUNCTION of E as

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that χ_E is measurable if $E \in \mathcal{A}$.

SIMPLE FUNCTIONS $(X, \mathcal{A})$ measurable space. A **SIMPLE FUNCTION** is a measurable map $s : X \to \mathbb{R}$ such that $s(X)$ is finite, i.e., there exist disjoint sets $E_1, \dots, E_N \in \mathcal{A}$, $N \in \mathbb{N}$ and $c_1, \dots, c_N \in \mathbb{R}$ distinct, s.t.

$$(*) \qquad s(x) = \sum_{i=1}^{N} c_i \chi_{E_i}(x) \quad , \quad \forall x \in X.$$

THEOREM 6.4 $(X, \mathcal{A})$ measure space, $u : X \to [0, +\infty]$ measurable. Then there exists a sequence $\{s_n\}$ of SIMPLE FUNCTIONS s.t. $0 \leq s_1 \leq s_2 \leq \dots$ and $s_n(x) \to u(x)$ for all $x \in X$.

LEBESGUE INTEGRAL Let $(X, \mathcal{A}, \mu)$ be a measurable space. The LEBESGUE INTEGRAL is defined in 3 steps:

(1) Let $s \geq 0$ a step function of the form $(*)$. We define the LEBESGUE INTEGRAL of s on a set $E \in \mathcal{A}$ by

$$\boxed{\int_E s(x)\, d\mu(x) := \sum_{i=1}^{N} c_i\, \mu(E \cap E_i)}$$

where if $c_i = 0$ and $\mu(E \cap E_i) = +\infty$ we adopt the standard convention

$$c_i\, \mu(E \cap E_i) := 0.$$

(2) Let $u : X \to [0, +\infty]$ be a measurable function (note that $u \geq 0$). The LEBESGUE INTEGRAL of u over a set $E \in \mathcal{A}$ is defined as

$$\boxed{\int_E u(x)\, d\mu(x) := \sup \left\{ \int_E s\, d\mu \;\middle|\; s \text{ simple},\ 0 \leq s \leq u \right\}}$$

(This is well posed thanks to THEOREM 6.4)

(3) Let $u : X \to [-\infty, +\infty]$ be measurable. Note that $u = u^+ - u^-$ with $u^+, u^- \geq 0$. The LEBESGUE INTEGRAL of u over a set $E \in \mathcal{A}$ is defined

$$\boxed{\int_E u(x)\, d\mu := \int_E u^+\, d\mu - \int_E u^-\, d\mu}$$

If $\int_E u^+ d\mu$ and $\int_E u^- d\mu$ are FINITE then u is said to be LEBESGUE INTEGRABLE WRT μ.

<u>REMARK</u> Let $(X, \mathcal{A}, \mu)$ be a measurable space. Let $u, v : X \to [-\infty, +\infty]$ be measurable.

(1) If $0 \leq u \leq v$ then $\int_E u\, d\mu \leq \int_E v\, d\mu$, $\forall E \in \mathcal{A}$

(2) If $c \in [0, +\infty]$, then $\int_E cu\, d\mu = c \int_E u\, d\mu$, $\forall E \in \mathcal{A}$ $\quad (0 \cdot (\pm\infty) := 0)$

(3) Let $E \in \mathcal{A}$ and $u \geq 0$. Then $\int_E u\, d\mu = 0$ iff $u(x) = 0$ for μ-a.e. $x \in E$.

(4) If $E \in \mathcal{A}$ and $\mu(E) = 0$ then $\int_E u\, d\mu = 0$

(5) If $E \in \mathcal{A}$ then $\int_E u\, d\mu = \int_X \chi_E u\, d\mu$

(6) u is LEBESGUE INTEGRABLE iff $\int_E |u|\, d\mu < +\infty$ for all $E \in \mathcal{A}$.

(7) If u is LEBESGUE INTEGRABLE then

$$\mu(\{x \in X : |u(x)| = +\infty\}) = 0.$$

8) If u, v are integrable and $\alpha, \beta \in \mathbb{R}$ then $\alpha u + \beta v$ is integrable, and

$$\int_X (\alpha u + \beta v)\, d\mu = \alpha \int_X u \, d\mu + \beta \int_X v \, d\mu .$$

9) If u, v are integrable and $u = v$ μ-a.e. in X, then

$$\int_X u \, d\mu = \int_X v \, d\mu$$

10) If u is integrable then

$$\left| \int_X u \, d\mu \right| \leq \int_X |u| \, d\mu$$

UNFORGETTABLE THEOREMS

We recall a few theorems concerning the Lebesgue integral:

THEOREM 6.5 (MONOTONE CONVERGENCE)

Let $(X, \mathcal{A}, \mu)$ be a measurable space, and $u_n : X \to [0, +\infty]$ s.t.

1) u_n is measurable $\forall n \in \mathbb{N}$

2) $0 \leq u_1(x) \leq u_2(x) \leq \dots$ for all $x \in X$

3) $u_n(x) \to u(x)$ as $n \to +\infty$ for all $x \in X$

Then

$$\lim_{n \to +\infty} \int_X u_n \, d\mu = \int_X u \, d\mu$$

THEOREM 6.6 (FATOU'S LEMMA)

Let $(X, \mathcal{A}, \mu)$ be a measurable space. If $u_n : X \to [0, +\infty]$ is a sequence of measurable functions, then

$$\int_X \liminf_{n \to +\infty} u_n(x) \, d\mu(x) \leq \liminf_{n \to +\infty} \int_X u_n(x) \, d\mu(x)$$

THEOREM 6.7 (DOMINATED CONVERGENCE)

Let $(X, \mathcal{A}, \mu)$ be a measurable space and $u_n : X \to [-\infty, +\infty]$ a sequence of measurable functions. Suppose that:

1. $u_n(x) \to u(x)$ as $n \to +\infty$, for μ-a.e. $x \in X$
2. $\exists\, v$ Lebesgue integrable such that

$$|u_n(x)| \leq v(x), \quad \forall n \in \mathbb{N} \text{ and } \mu\text{-a.e. } x \in X.$$

Then u is Lebesgue integrable and

$$\lim_{n \to +\infty} \int_X |u_n - u| \, d\mu = 0$$

THEOREM 6.8 (JENSEN'S INEQUALITY)

Let $(X, \mathcal{A}, \mu)$ be measurable space, with $\mu(X) = 1$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. For all $u : X \to \mathbb{R}$ integrable we have

$$\varphi\left(\int_X u \, d\mu \right) \leq \int_X \varphi \circ u \, d\mu.$$

Finally we recall FUBINI'S and TONELLI'S THEOREMS. We first need:

PRODUCT MEASURE Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces. On the cartesian product $X_1 \times X_2$ define the PRODUCT σ-ALGEBRA

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \bigcap \left\{ \mathcal{A} \subseteq P(X_1 \times X_2) \mid \mathcal{A} \text{ is a } \sigma\text{-algebra}, (E_1 \times E_2) \in \mathcal{A}, \ \forall E_i \in \mathcal{A}_i \right\}$$

Thus $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest σ-algebra on $X_1 \times X_2$ containing all the sets of the form $E_1 \times E_2$ with $E_i \in \mathcal{A}_i$. Whenever μ_1, μ_2 are σ-FINITE, there exists a unique measure $\mu_1 \otimes \mu_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0, +\infty]$ such that

$$(\mu_1 \otimes \mu_2)(E_1 \times E_2) = \mu_1(E_1)\, \mu_2(E_2), \qquad \forall E_i \in \mathcal{A}_i$$

(it can be constructed via PROP 6.1 and THM 6.2). The measure $\mu_1 \otimes \mu_2$ is called PRODUCT MEASURE between μ_1 and μ_2.

NOTE For the Lebesgue measure it holds that $\mathcal{L}^{d_1} \otimes \mathcal{L}^{d_2} = \mathcal{L}^{d_1+d_2}$.

THEOREM 6.9 (TONELLI)

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces, with μ_1, μ_2 σ-finite.
Let $u : X_1 \times X_2 \to \mathbb{R}$ be measurable WRT $\mathcal{A}_1 \otimes \mathcal{A}_2$, and s.t.

(a) For μ_1-a.e. $x \in X_1$ the map $y \in X_2 \mapsto u(x,y)$ is measurable and it holds

$$\int_{X_2} |u(x,y)| \, d\mu_2(y) < +\infty \quad \text{for } \mu_1\text{-a.e. } x \in X_1$$

(b) $\displaystyle\int_{X_1} \left(\int_{X_2} |u(x,y)| \, d\mu_2(y) \right) d\mu_1(x) < +\infty$

Then u is INTEGRABLE WRT the product measure $\mu_1 \otimes \mu_2$.

THEOREM 6.10 (FUBINI)

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces, with μ_1, μ_2 σ-finite. Let $u: X_1 \times X_2 \to [-\infty, +\infty]$ be measurable WRT $\mathcal{A}_1 \otimes \mathcal{A}_2$ and integrable WRT $\mu_1 \otimes \mu_2$. Then

1. For μ_1-a.e. $x \in X_1$ the map $y \in X_2 \mapsto u(x,y)$ is measurable and

$$\int_{X_2} |u(x,y)| \, d\mu_2(y) < +\infty, \qquad \int_{X_1} \left\{ \int_{X_2} |u(x,y)| \, d\mu_2(y) \right\} d\mu_1(x) < +\infty$$

2. For μ_2 - a.e. $y \in X_2$ the map $x \in X_1 \mapsto u(x,y)$ is measurable and

$$\int_{X_1} |u(x,y)| \, d\mu_1(x) < +\infty, \qquad \int_{X_2} \left\{ \int_{X_1} |u(x,y)| \, d\mu_1(x) \right\} d\mu_2(y) < +\infty$$

3. The so-called FUBINI'S FORMULA holds:

$$\int_{X_1 \times X_2} |u(x,y)| \, d(\mu_1 \otimes \mu_2)(x,y) = \int_{X_1} \left\{ \int_{X_2} |u(x,y)| \, d\mu_2(y) \right\} d\mu_1(x)$$

$$= \int_{X_2} \left\{ \int_{X_1} |u(x,y)| \, d\mu_1(x) \right\} d\mu_2(y)$$

L^p SPACES

Let $(X, \mathcal{A}, \mu)$ be a measurable space. For $p \geq 1$ we set

$$L^p(X, \mu) := \left\{ u : X \to \mathbb{R} \mid u \text{ measurable}, \int_X |u|^p \, d\mu < +\infty \right\}$$

In other words, $u \in L^p(X, \mu)$ iff u is μ- INTEGRABLE.

For the case $p = +\infty$ we have an ad-hoc definition

$$L^\infty(X,\mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \ \exists C>0 \text{ s.t. } |u(x)| \le C \ \ \mu\text{-a.e. in } X \}$$

The condition $|u(x)| \le C$ for μ-a.e. $x \in X$ is called ESSENTIAL BOUNDEDNESS.

WARNING The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are <u>NOT</u> functions, but classes of equivalence of functions, where the equivalence is

$$u \sim v \iff u(x) = v(x) \ \text{ for } \ \mu\text{-a.e. } x \in X$$

This is not an issue, since in this case $\int_X u \, d\mu = \int_X v \, d\mu$.
Therefore $L^p(X,\mu)$ and $L^\infty(X,\mu)$ have to be understood as QUOTIENT SPACES WRT $\sim$

THEOREM 6.11 Let $1 \le p \le +\infty$ and define the CONJUGATE EXPONENT $p' := \frac{p}{p-1}$. If $u \in L^p(X,\mu)$, $v \in L^{p'}(X,\mu)$ then

$$uv \in L^1(X,\mu) \quad \text{and} \quad \underbrace{\|uv\|_1 \le \|u\|_p \, \|v\|_{p'}}_{\text{HÖLDER'S INEQUALITY}}$$

THEOREM 6.12 $L^p(X,\mu)$, $L^\infty(X,\mu)$ are Banach spaces with the norms

$$\|u\|_p := \left(\int_X |u|^p \, d\mu \right)^{1/p}, \quad u \in L^p(X,\mu)$$

$$\|u\|_\infty := \inf \{ C : |u(x)| \le C \ \ \mu\text{-a.e. in } X \}, \quad u \in L^\infty(X,\mu)$$

Moreover $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_X uv \, d\mu, \quad u, v \in L^2(X,\mu)$$

A standard corollary of the proof of THEOREM 6.12 is the following.

PROPOSITION 6.13 Let $\{u_n\} \subseteq L^p(X,\mu)$ and suppose $u_n \to u$ strongly. Then there exist a subsequence u_{n_k} and $h \in L^p(X,\mu)$ s.t.

(a) $u_{n_k}(x) \to u(x)$ as $k \to +\infty$ for μ-a.e. $x \in X$

(b) $\sup_k |u_{n_k}(x)| \le h(x)$ for μ-a.e. $x \in X$

THEOREM 6.14 (DUALITY)

Let $1 < p < +\infty$. Then $L^p(X,\mu)^* \cong L^{p'}(X,\mu)$, with isometry

$$L^{p'}(X,\mu) \longrightarrow L^p(X,\mu)^*$$
$$u \longmapsto \left(v \mapsto \int_X uv \, d\mu \right)$$

In particular, as $(p')' = p$, we have that $L^p(X,\mu)$ is REFLEXIVE.
Also $L^1(X,\mu)^* \cong L^\infty(X,\mu)$.

WARNING It is NOT TRUE that $L^\infty(X,\mu)^* \cong L^1(X,\mu)$.

We now recall a result about SEPARABILITY of L^p spaces. We need first the following definition

SEPARABLE MEASURE SPACE

Let $(X, \mathcal{A})$ be a SEPARABLE measure space, i.e., $\exists \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ s.t. $\sigma(\{E_n\}) = \mathcal{A}$, where

$$\sigma(\{E_n\}) := \bigcap \{ \mathcal{M} \mid \mathcal{M} \text{ is } \sigma\text{-algebra on } X, \ \{E_n\} \subseteq \mathcal{M} \},$$

i.e., $\sigma(\{E_n\})$ is the smallest σ-algebra on X which contains $\{E_n\}$.

EXAMPLE

- $\mathbb{R}^d$ is separable with the Borel σ-algebra.
- $(\mathbb{R}^d, \mathcal{I}^*)$ is separable, where $\mathcal{I}^*$ is the σ-algebra of Lebesgue measurable sets
- (X, d) separable metric space, τ_d topology induced by d. Then $(X, \sigma(\tau_d))$ is a separable measure space.

THEOREM 6.15 (SEPARABILITY) Let $(X, \mathcal{A}, \mu)$ be a SEPARABLE measure space. Then $L^p(X, \mu)$ equipped with the standard norm is SEPARABLE, for all $1 \leq p < +\infty$.
The space $L^\infty(X, \mu)$ is in general NOT separable.

We summarize the above results in a table

	REFLEXIVE	SEPARABLE	DUAL SPACE
L^p with $1 < p < +\infty$	YES	YES	$L^{p'}$
L^1	NO	YES	L^∞
L^∞	NO	NO	Strictly bigger than L^1

Finally we conclude with a useful density result:

THEOREM 6.16 Consider $(\mathbb{R}^d, \mathcal{I}^*, \mathcal{I}^d)$, where $\mathcal{I}^*$ is the LEBESGUE σ-algebra and $\mathcal{I}^d$ is the d-dimensional LEBESGUE MEASURE. Let $1 \leq p < +\infty$. Then $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, i.e.,

$$\forall u \in L^p(\mathbb{R}^d),\ \forall \varepsilon > 0,\ \exists v \in C_c(\mathbb{R}^d) \text{ s.t. } \|u - v\|_p \leq \varepsilon.$$

STRONG COMPACTNESS IN L^p

We conclude with a STRONG COMPACTNESS criterion for L^p spaces. To this end, given $f: \mathbb{R}^d \to \mathbb{R}$, $h \in \mathbb{R}^d$, we define the SHIFT of f by h as the function $\tau_h f: \mathbb{R}^d \to \mathbb{R}$

$$(\tau_h f)(x) := f(x+\tau), \qquad \forall x \in \mathbb{R}^d.$$

THEOREM 6.17 (FRÉCHET - KOLMOGOROV)

Let $1 \le p < +\infty$ and $A \subseteq L^p(\mathbb{R}^d)$. For a measurable set $\Omega \subseteq \mathbb{R}^d$ with finite measure, we denote by $A_{|\Omega}$ the restrictions to Ω of the functions in A, i.e.,

$$A_{|\Omega} = \{ v : \Omega \to \mathbb{R} \mid \exists\, u \in A \text{ s.t. } v = u_{|\Omega} \}.$$

Assume that

1. A is **BOUNDED**: i.e., $\exists\, M > 0$ s.t. $\|u\|_{L^p(\mathbb{R}^d)} \le M$, $\forall u \in A$

2. A is **EQUI-INTEGRABLE**: i.e.,

$$\lim_{|h| \to 0} \left\{ \sup_{u \in A} \| \tau_h u - u \|_{L^p(\mathbb{R}^d)} \right\} = 0.$$

Then the closure of $A_{|\Omega}$ in $L^p(\Omega)$ is COMPACT, i.e., if $\{u_n\} \subseteq \overline{A_{|\Omega}}$, $\exists\, \bar{u} \in \overline{A_{|\Omega}}$ and a subsequence n_k such that

$$u_{n_k} \to \bar{u} \quad \text{as } k \to +\infty \quad \text{strongly in } L^p(\Omega)$$

OTHER MEASURE THEORETIC RESULTS

THEOREM 6.18 (ABSOLUTE CONTINUITY OF LEBESGUE INTEGRAL)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in L^1(X;\mu)$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\mu(E) < \delta \quad \Rightarrow \quad \left| \int_E u \, d\mu \right| < \varepsilon .$$

THEOREM 6.19 (EGOROFF)

Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) < +\infty$.
Suppose $f_n : X \to \mathbb{R}$ is a sequence s.t.

$$f_n \to f \quad \text{a.e. in } X$$

Then $\forall \varepsilon > 0$, $\exists \ E_\varepsilon \in \mathcal{A}$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E_\varepsilon$, i.e.

$$\lim_{n \to +\infty} \ \sup_{x \in X \setminus E_\varepsilon} |f_n(x) - f(x)| = 0 .$$

THEOREM 6.20 (LUSIN)

Let $\Omega \subseteq \mathbb{R}^d$ with $|\Omega| < +\infty$. Let $u : \Omega \to \mathbb{R}$ be Lebesgue measurable.
Then $\forall \varepsilon > 0$, $\exists K \subseteq \Omega$ compact such that

$$|\Omega \setminus K| < \varepsilon \quad \text{and} \quad u_{|K} \text{ is CONTINUOUS.}$$

7. SOBOLEV SPACES

REFERENCE:
GIOVANNI LEONI – "A FIRST COURSE IN SOBOLEV SPACES"
AMERICAN MATHEMATICAL SOCIETY, 2017

So far our minimization problems were mostly set in C^1. Often a solution did not exist in C^1, however in many examples we saw that we could find an infimizing sequence converging to something piecewise C^1, i.e.,

$$C^1_{pw}[a,b] := \{ u \in C[a,b] \mid \exists \{x_0, \dots, x_N\} \subseteq [a,b] \text{ with } a = x_0 < x_1 < \dots < x_N = b, \; u \in C^1[x_i, x_{i+1}], \; i = 0, \dots, N-1 \}$$

However these functional spaces are not very convenient to work with, due to their lack of completeness WRT weaker norms (e.g. the L^p convergence). The default functional spaces for setting variational problems are (nowadays and in the past 60-70 years) SOBOLEV SPACES.

In order to define Sobolev spaces, we rely on previous knowledge about L^p spaces (LEBESGUE SPACES). A self-contained summary of definitions and properties can be found in SECTION 6 of these notes. (L^p SPACES REVISION)

Here we just recall the definition of L^p spaces, to establish some notation.

L^p SPACES Let $(X, \mathcal{A}, \mu)$ be a measurable space, where X set, $\mathcal{A}$ is σ-algebra over X and $\mu : \mathcal{A} \to [0, +\infty]$ is a measure.
For $1 \leq p < +\infty$ and $p = +\infty$ we set, respectively:

$$L^p(X, \mu) := \left\{ u : X \to \mathbb{R} \mid u \text{ measurable}, \; \int_X |u|^p \, d\mu < +\infty \right\}$$

$$L^\infty(X, \mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \; \exists C > 0 \text{ s.t. } |u(x)| \leq C \; \mu\text{-a.e. in } X \}$$

When we say μ-a.e. we mean that a certain property holds in $X \setminus E$, where $\mu(E) = 0$.

WARNING The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are NOT functions, but classes of equivalence of functions, where the equivalence is

$$u \sim v \iff u(x) = v(x) \quad \text{for } \mu\text{-a.e. } x \in X$$

This is not an issue, since in this case $\int_X u \, d\mu = \int_X v \, d\mu$.
Therefore $L^p(X,\mu)$ and $L^\infty(X,\mu)$ have to be understood as QUOTIENT SPACES WRT $\sim$

RECALL $L^p(X,\mu)$, $L^\infty(X,\mu)$ are Banach spaces with the norms

$$\|u\|_p := \left(\int_X |u|^p \, d\mu \right)^{1/p}, \quad u \in L^p(X,\mu), \quad 1 \leq p < +\infty,$$

$$\|u\|_\infty := \inf \{ C : |u(x)| \leq C \ \mu\text{-a.e. in } X \}, \quad u \in L^\infty(X,\mu)$$

Moreover $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_X u v \, d\mu, \quad u, v \in L^2(X,\mu)$$

NOTE In the following the definition of L^p will be employed in this setting:

- X will always be an OPEN SET OF $\mathbb{R}^d$
- $\mathcal{A}$ is the d-dimensional LEBESGUE σ-Algebra
- $\mu = dx = \mathcal{L}^d$ the d-dimensional LEBESGUE MEASURE

Thus we will always write $L^p(X)$ in place of $L^p(X,\mu)$, as there is no ambiguity.

We need to introduce versions of the FLCV and DBR Lemma for L^p functions.
For that we need tools to smoothen functions, i.e., convolutions.

CONVOLUTIONS

DEFINITION 7.1 Let $u, v : \mathbb{R} \to \mathbb{R}$. The CONVOLUTION between u and v, is defined as

$$\boxed{(u * v)(x) := \int_{\mathbb{R}} u(x-y)\, v(y)\, dy}$$

for all $x \in \mathbb{R}$ s.t. the RHS is FINITE.

REMARK It is immediate to check that, whenever the convolution is finite,

$$u * v = v * u \qquad \text{and} \qquad u * (v * w) = (u * v) * w$$

for $u, v, w : \mathbb{R} \to \mathbb{R}$.

The following Theorem gives a sufficient condition for $u * v$ to be well-defined.

THEOREM 7.2 (YOUNG)

Let $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$ for some $1 \le p \le +\infty$. Then for a.e. $x \in \mathbb{R}$ the map $y \mapsto u(x-y)\, v(y)$ is integrable, so that $u * v$ is finite. Moreover $u * v \in L^p(\mathbb{R})$, with

$$(*) \qquad \| u * v \|_p \le \| u \|_1 \, \| v \|_p \, .$$

Proof • $p = +\infty$: this is immediate, since for a.e. $x \in \mathbb{R}$

$$|(u * v)(x)| \le \int_{\mathbb{R}} |u(x-y)|\, |v(y)|\, dy \le \|v\|_\infty \int_{\mathbb{R}} |u(x-y)|\, dy = \|v\|_\infty \, \|u\|_1 \, .$$

Taking the essential supremum in the above inequality we obtain $(*)$

• $p=1$: Set $\psi(x,y) := u(x-y)v(y)$. For a.e. $y \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} |\psi(x,y)|\, dx = |v(y)| \int_{\mathbb{R}} |u(x-y)|\, dx = |v(y)|\, \|u\|_1 < +\infty$$

Integrating WRT x we get

$$(**) \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dx \right\} dy = \|v\|_1 \|u\|_1 < +\infty$$

Then ψ satisfies the assumptions of TONELLI'S THEOREM (THEOREM 6.9) and we infer $\psi \in L^1(\mathbb{R}\times\mathbb{R})$ (where $\mathbb{R}\times\mathbb{R}$ is equipped with the 2-dimensional Lebesgue measure). We can then apply FUBINI'S THEOREM (THEOREM 6.10) to get that

$$\int_{\mathbb{R}} |\psi(x,y)|\, dy < +\infty \quad \text{for a.e. } x \in \mathbb{R}.$$

and also

$$(***) \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dy \right\} dx \overset{\text{FUBINI}}{=} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dx \right\} dy \overset{(**)}{=} \|v\|_1 \|u\|_1$$

Therefore

$$\int_{\mathbb{R}} |(u*v)(x)|\, dx \overset{\text{def. of convolution}}{=} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} u(x-y)v(y)\, dy \right| dx \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |u(x-y)v(y)|\, dy \right\} dx$$

$$\text{def of } \psi \rightarrow \; = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dy \right\} dx \overset{(***)}{=} \|u\|_1 \|v\|_1 ,$$

which is exactly (*).

• $1 < p < +\infty$: The functions $|u|, |v|^p \in L^1(\mathbb{R})$. Thus, from the case $p=1$, we know that $y \mapsto |u(x-y)|\,|v(y)|^p$ belongs to $L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$.
In particular

$$|u(x-\cdot)|^{1/p}\,|v(\cdot)| \in L^p(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}.$$

Moreover, as $u \in L^1(\mathbb{R})$, we also have

$$|u(x-\cdot)|^{1/p'} \in L^{p'}(\mathbb{R}) \qquad \text{for a.e. } x \in \mathbb{R}$$

where we chose p' as the HÖLDER CONJUGATE, i.e.

$$p' := \frac{p}{p-1} \quad , \quad \text{so that} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

From HÖLDER INEQUALITY (THEOREM 6.11) we get, for a.e. $x \in \mathbb{R}$,

$$|(u * v)(x)| \le \int_{\mathbb{R}} |u(x-y)|\,|v(y)|\,dy$$

$$= \int_{\mathbb{R}} \underbrace{|u(x-y)|^{1/p'}}_{\in L^{p'}} \underbrace{|u(x-y)|^{1/p}\,|v(y)|}_{\in L^p}\,dy$$

(HÖLDER) $$\le \left(\int_{\mathbb{R}} |u(x-y)|\,dy\right)^{1/p'} \left(\int_{\mathbb{R}} |u(x-y)|\,|v(y)|^p\,dy\right)^{1/p}$$

$$= \|u\|_1^{1/p'} \cdot \left[(|u| * |v|^p)(x)\right]^{1/p}$$

Taking the p-power of the above we get

$$(**)\quad |(u*v)(x)|^p \leq \|u\|_1^{p/p'} \, (|u| * |v|^p)(x) \qquad \text{for a.e. } x \in \mathbb{R}$$

Now, as $|u|, |v|^p \in L^1(\mathbb{R})$, we can apply $(*)$ for the case $p=1$ to get:

$$(***)\quad \| \, |u| * |v|^p \, \|_1 \leq \|u\|_1 \, \| \, |v|^p \, \|_1 = \|u\|_1 \, \|v\|_p^p$$

By integrating $(**)$:

$$\int_{\mathbb{R}} |(u*v)(x)|^p \, dx \overset{(**)}{\leq} \|u\|_1^{p/p'} \int_{\mathbb{R}} | (|u| * |v|^p)(x) | \, dx$$

and so

$$= \|u\|_1^{p/p'} \, \| \, |u| * |v|^p \, \|_1$$

$$\overset{(***)}{\leq} \|u\|_1^{p/p'} \, \|u\|_1 \, \|v\|_p^p$$

As $\frac{p}{p'} + 1 = p \longrightarrow$ $\quad = \|u\|_1^p \, \|v\|_p^p \, .$

Taking the $\frac{1}{p}$-power of the above inequality yields $(*)$. $\square$

We now need the notion of **SUPPORT** for L^p functions. Indeed, as elements of L^p are actually equivalence classes, and thus defined a.e., the definition of support we used for continuous functions makes no sense:

EXAMPLE $\quad u := \chi_{\mathbb{Q}}$. As the Lebesgue measure of $\mathbb{Q}$ is zero, u belongs to the same equivalence class of $v \equiv 0$. Using the classical definition of support we get

$$\operatorname{supp} u = \overline{\{x \in \mathbb{R} \mid u(x) \neq 0\}} = \overline{\mathbb{Q}} = \mathbb{R} \quad , \text{ while } \quad \operatorname{supp} v = \emptyset \, .$$

LESSON 7 - 28 APRIL 2021

DEFINITION 7.3 (SUPPORT)

Let $u : \mathbb{R}^d \to \mathbb{R}$. Let $\{w_i\}_{i \in I}$ be the family of all open sets in $\mathbb{R}^d$ s.t.

$$u = 0 \text{ a.e. on } w_i \, , \quad \forall i \in I.$$

We define the support of u as

$$\operatorname{Supp} u := \mathbb{R}^d \setminus \bigcup_{i \in I} w_i \, .$$

REMARK The above definition makes sense, since it is possible to show that:

1. $u = 0$ a.e. on $\bigcup_{i \in I} w_i$
2. If $u_1 = u_2$ a.e. on $\mathbb{R}^d$, then $\operatorname{supp} u_1 = \operatorname{supp} u_2$.
3. If u is continuous, then DEFINITION 7.3 coincides with the classical one, i.e.,

$$\operatorname{Supp} u = \mathbb{R}^d \setminus \bigcup_{i \in I} w_i = \overline{\{ x \in \mathbb{R} \mid u(x) \neq 0 \}}$$

EXAMPLE Again consider $u := \chi_{\mathbb{Q}}$. As $\mathcal{L}(\mathbb{Q}) = 0$, we know that $u = 0$ a.e. on $\mathbb{R}$. Therefore $u = 0$ a.e. on w, for all $w \subseteq \mathbb{R}$ open. It follows that $\operatorname{supp} u = \mathbb{R} \setminus \bigcup_{i \in I} w_i = \mathbb{R} \setminus \mathbb{R} = \emptyset$. This coincides with $\operatorname{supp} v$, where $v \equiv 0$.

The support of a convolution can be estimated in the following way:

PROPOSITION 7.4 Let $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Then

$$(*) \quad \operatorname{Supp}(u * v) \subset \overline{\operatorname{Supp} u + \operatorname{supp} v}$$

The sum in $(*)$ is defined as $E + F := \{ x + y \, , \, x \in E, y \in F \}$, where $E, F \subseteq \mathbb{R}$ subsets.

(The proof of PROPOSITION 7.4 will be left as an Exercise in the EX. COURSE)

The main point of introducing convolutions is that they have a smoothing effect. To make this statement rigorous we need the definition of LOCAL INTEGRABILITY.

DEFINITION 7.5 Let $\Omega \subset \mathbb{R}^d$ be open. Let $1 \leq p \leq +\infty$. We say that $u : \Omega \to \mathbb{R}$ is **LOCALLY INTEGRABLE** on Ω, if

$$u \chi_K \in L^p(\Omega) \quad \text{for all} \quad K \subset \Omega, \ K \text{ compact}.$$

The space of locally integrable functions on Ω is denoted by $L^p_{loc}(\Omega)$

REMARK We have that $L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$, for all $\Omega \subseteq \mathbb{R}^d$ open

(This is not true for $L^p(\Omega)$ with Ω unbounded)

THEOREM 7.6 (Smoothing via convolutions)

(a) Let $u \in C_c(\mathbb{R})$, $v \in L^1_{loc}(\mathbb{R})$. Then $(u * v)(x)$ is well-defined $\forall x \in \mathbb{R}$ and

$$u * v \in C(\mathbb{R})$$

(b) Let $k \geq 1$, $u \in C^k_c(\mathbb{R})$, $v \in L^1_{loc}(\mathbb{R})$. Then $(u * v) \in C^k(\mathbb{R})$ and

$$\frac{d^k}{dx^k}[u * v] = u^{(k)} * v$$

In particular, if $u \in C^\infty_c(\mathbb{R})$ and $v \in L^1_{loc}(\mathbb{R})$, then $(u * v) \in C^\infty(\mathbb{R})$.

(The proof of this Theorem will be left as an ex. for the Exercises Course)

DEFINITION 7.7 **(MOLLIFIERS)** A sequence of **MOLLIFIERS** is any sequence $\rho_n : \mathbb{R} \to \mathbb{R}$ s.t.

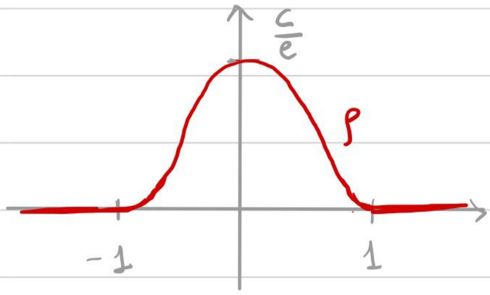
$$\rho_n \in C^\infty_c(\mathbb{R}), \quad \rho_n \geq 0, \quad \operatorname{supp} \rho_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right], \quad \int_{\mathbb{R}} \rho_n(x)\, dx = 1, \quad \forall n \in \mathbb{N}.$$

The most commonly used sequence of mollifiers is defined as follows:

EXAMPLE (STANDARD MOLLIFIERS)

Let $\rho : \mathbb{R} \to \mathbb{R}$ be as in REMARK 3.2, i.e.,

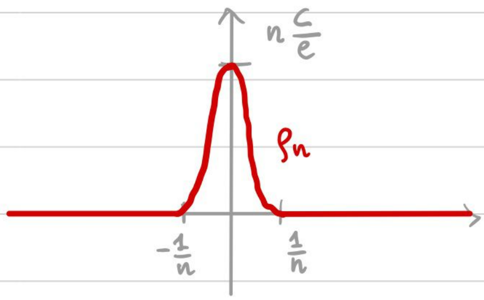
$$\rho(x) := \begin{cases} C \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



where $C \in \mathbb{R}$ is $C := \left(\int_{\mathbb{R}} \rho(x)\,dx \right)^{-1}$. Then $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$, $\operatorname{supp} \rho \subset [-1,1]$, $\int_{\mathbb{R}} \rho = 1$.

In particular $\rho_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\boxed{\rho_n(x) := n\,\rho(nx)}$$



is a sequence of mollifiers.

PROPOSITION 7.8 Let $u \in C(\mathbb{R})$ and $\{\rho_n\}$ be a sequence of mollifiers. Then $\rho_n * u \to u$ uniformly on compact sets, i.e. for each $K \subset \mathbb{R}$ compact we have

$$\lim_{n \to +\infty} \max_{x \in K} |(\rho_n * u)(x) - u(x)| = 0.$$

(Also the proof of this is left as exercise for the EX. COURSE)

THEOREM 7.9 Let $1 \leq p < +\infty$, $u \in L^p(\mathbb{R})$, $\{\rho_n\}$ a sequence of mollifiers. Then

$$\rho_n * u \to u \quad \text{STRONGLY in } L^p(\mathbb{R}).$$

<u>Proof</u> Let $\varepsilon > 0$. By THEOREM 6.16 $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$. Thus $\exists\, \tilde{u} \in C_c(\mathbb{R})$ such that

$$(*) \qquad \| u - \tilde{u} \|_p < \varepsilon .$$

By PROPOSITION 7.8 we have that $\rho_n * \tilde{u} \to \tilde{u}$ uniformly on compact sets. Moreover PROPOSITION 7.4 says that

$$\operatorname{supp}(\rho_n * \tilde{u}) \subset \overline{\operatorname{supp}\rho_n + \operatorname{supp}\tilde{u}} \overset{\text{by def of } \rho_n}{\subseteq} \left[-\tfrac{1}{n}, \tfrac{1}{n}\right] + \operatorname{supp}\tilde{u} \subseteq [-1,1] + \operatorname{supp}\tilde{u}$$

In particular

$$\begin{aligned} \operatorname{supp}(\rho_n * \tilde{u} - \tilde{u}) &\subset \left(\operatorname{supp}(\rho_n * \tilde{u}) \cup \operatorname{supp}\tilde{u} \right) \\ (**) \qquad &\subset \left(\{ [-1,1] + \operatorname{supp}\tilde{u} \} \cup \operatorname{supp}\tilde{u} \right) \\ &= [-1,1] + \operatorname{supp}\tilde{u} =: K \end{aligned}$$

and K is compact, as $\tilde{u}$ is compactly supported. Then

$$\begin{aligned} \int_{\mathbb{R}} |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \, dx &\overset{(**)}{=} \int_K |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \, dx \\ &\leq \max_{x \in K} |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \cdot |K| \end{aligned}$$

and the RHS goes to zero as $n \to +\infty$, since $\rho_n * \tilde{u} \to \tilde{u}$ uniformly on compact sets, and $|K| < +\infty$ being K bounded (as it is compact). Thus

$$(***) \qquad \rho_n * \tilde{u} \to \tilde{u} \quad \text{STRONGLY in } L^p(\mathbb{R})$$

Now notice that $\rho_n \in L^1(\mathbb{R})$ with $\|\rho_n\|_1 = 1$ by definition. Moreover $(u-\tilde{u}) \in L^p(\mathbb{R})$. Therefore by YOUNG INEQUALITY (THEOREM 7.2)

$$\text{(Y)} \quad \|\rho_n * (u-\tilde{u})\|_p \leq \|\rho_n\|_1 \|u-\tilde{u}\|_p = \|u-\tilde{u}\|_p$$

Finally, by adding and subtracting we can estimate

$$\|\rho_n * u - u\|_p \leq \|\rho_n * (u-\tilde{u})\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p + \|u-\tilde{u}\|_p$$

$$\overset{(Y)}{\leq} \|u-\tilde{u}\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p + \|u-\tilde{u}\|_p$$

$$= 2\|u-\tilde{u}\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p$$

Recalling that $\|u-\tilde{u}\|_p < \varepsilon$ and $\|\rho_n * \tilde{u} - \tilde{u}\|_p \to 0$ by (***), we get

$$0 \leq \limsup_{n \to +\infty} \|\rho_n * u - u\|_p \leq 2\varepsilon$$

As $\varepsilon > 0$ was arbitrary, we conclude $\|\rho_n * u - u\|_p \to 0$. $\square$

COROLLARY 7.10 Let $I \subset \mathbb{R}$ be open, $1 \leq p < +\infty$. Then $C_c^\infty(I)$ is dense in $L^p(I)$.

FLCV and DBR in L^p

We are now ready to prove L^p versions of the FLCV and DBR Lemma.

LEMMA 7.11 (FLCV in L^p)

Let $I \subset \mathbb{R}$ be open. Suppose $u \in L^1_{loc}(I)$ is s.t.

$$\int_I u v \, dx = 0, \qquad \forall\, v \in C_c^\infty(I).$$

Then $u = 0$ a.e. on I.

Proof Let $\psi \in L^\infty(\mathbb{R})$ be s.t. supp ψ is compact and contained in I. Let $\psi_n := \rho_n * \psi$, with ρ_n mollifier. Then

$$\operatorname{supp} \psi_n \subset \overline{\operatorname{supp} \rho_n + \operatorname{supp} \psi} \overset{\text{by def of } \rho_n}{=} \overline{\left[-\tfrac{1}{n}, \tfrac{1}{n}\right] + \operatorname{supp} \psi}$$

by PROPOSITION 7.4. Then there $\exists\, N \in \mathbb{N}$ s.t. $\operatorname{supp} \psi_n \subset I$ for all $n \geq N$. Moreover $\psi_n \in C^\infty(\mathbb{R})$ by THEOREM 7.6. Thus $\psi_n \in C_c^\infty(I)$ for all $n \geq N$. By assumption we get

$$(*) \qquad \int_I u \psi_n \, dx = 0, \qquad \forall\, n \geq N.$$

Notice that $\psi \in L^1(\mathbb{R})$, being compactly supported and in $L^\infty(\mathbb{R})$. Then by THEOREM 7.9 we have $\psi_n \to \psi$ strongly in $L^1(\mathbb{R})$. Therefore, up to subsequencies (not relabelled), we have $\psi_n \to \psi$ pointwise a.e. in $\mathbb{R}$ (see PROPOSITION 6.13). Also, by YOUNG'S INEQUALITY, we get $\|\psi_n\|_{L^\infty(\mathbb{R})} \leq \|\rho_n\|_{L^1(\mathbb{R})} \|\psi\|_{L^\infty(\mathbb{R})} = \|\psi\|_{L^\infty(\mathbb{R})}$, as $\|\rho_n\|_{L^1(\mathbb{R})} = 1$ by definition. Therefore

$$u\psi_n \to u\psi \quad \text{a.e. in } I, \qquad |u \psi_n| \leq \|\psi\|_{L^\infty(\mathbb{R})} |u| \in L^1(I)$$

Therefore we can invoke DOMINATED CONVERGENCE (THEOREM 6.7) to conclude

$$\int_I u \psi_n \, dx \to \int_I u \psi \, dx \qquad \text{as } n \to +\infty.$$

However $\int_I u \varphi_n \, dx = 0$ for n sufficiently large, by ($*$), and so

$$\int_I u \varphi \, dx = 0 .$$

Therefore we have proven that

$$(**) \qquad \int_I u \varphi \, dx = 0 \quad , \quad \forall \varphi \in L^\infty(I) \text{ s.t. } \operatorname{supp}\varphi \text{ is compact, } \operatorname{supp}\varphi \subset I$$

To obtain our thesis we now choose a function φ in ($**$) in a clever way:
Let $K \subset I$ be compact and define

$$\tilde{\varphi}(x) := \begin{cases} \operatorname{sign} u(x) & \text{if } x \in K \\ 0 & \text{if } x \in \mathbb{R} \setminus K \end{cases} \qquad \left(\operatorname{sign} a = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases} \right)$$

Therefore $\tilde{\varphi} \in L^\infty(\mathbb{R})$ and $\operatorname{supp}\tilde{\varphi} \subset K \subset I$. Then, from ($**$),

$$0 \overset{(**)}{=} \int_I u \tilde{\varphi} \, dx = \int_K u \operatorname{sign} u \, dx = \int_K |u| \, dx .$$

Then $u = 0$ a.e. on K by the properties of the Lebesgue integral (see the REMARK after THEOREM 6.4). As K is arbitrary, we conclude $u = 0$ a.e. on I. $\square$

LEMMA 7.12 (DBR in L^p)

Let $I = (a,b)$, possibly unbounded. Let $u \in L^1_{loc}(I)$ be such that

$$\int_I u v \, dx = 0 \quad , \quad \forall v \in C_c(I) \text{ such that } \int_I v \, dx = 0 .$$

Then $u = c$ a.e. on I, for some constant $c \in \mathbb{R}$.

Proof Let $\psi \in C_c(I)$ s.t. $\int_I \psi \, dx = 1$ (e.g. the bump function of REMARK 3.2, suitably rescaled).
Let $w \in C_c(I)$ be arbitrary and set

$$h(x) := w(x) - \left(\int_I w \, dx \right) \psi(x)$$

Then $h \in C_c(I)$, since $w, \psi \in C_c(I)$. Also $\int_I h \, dx = 0$ as $\int_I \psi \, dx = 1$. Then by assumption $\int_I u h \, dx = 0$. Thus

$$0 = \int_I u h \, dx = \int_I u w \, dx - \int_I w \, dx \cdot \int_I u \psi \, dx = \int_I (u - c) w \, dx \Rightarrow \int_I (u-c) w \, dx = 0$$

with $c := \int_I u \psi \, dx$. As c does not depend on w, and $w \in C_c(I)$ is arbitrary, by FLCV LEMMA 7.11 we conclude $u = c$ a.e. on I. $\square$

Similarly to the regular DBR Lemma, we have an alternative version:

LEMMA 7.13 (DBR in L^p - Alternative version)

Let $I = (a,b)$, possibly unbounded. Let $u \in L^1_{loc}(I)$ be such that

$$\int_I u \dot{v} \, dx = 0 \,, \quad \forall v \in C^1_c(I) .$$

Then $u = c$ a.e. on I, for some constant $c \in \mathbb{R}$.

Proof Let $w \in C_c(I)$ be such that $\int_I w \, dx = 0$. Set $W(x) := \int_a^x w(t) \, dt$. Then $W \in C^1(I)$ by the Fundamental Theorem of Calculus and $\dot{W}(x) = w(x)$. Moreover W is compactly supported in I, since w is compactly supported in I and $\int_I w \, dx = 0$. Thus $W \in C^1_c(I)$ and by assumption we get $\int_I u \dot{W} \, dx = 0$. As $\dot{W} = w$, we get

$$\int_I u w \, dx = 0 \,, \quad \forall w \in C_c(I) \text{ s.t. } \int_I w \, dx = 0.$$

We can now apply LEMMA 7.12 and conclude $u = c$ a.e. on I, for some $c \in \mathbb{R}$. $\square$

SOBOLEV SPACES

MOTIVATION Let $I = (a,b) \subseteq \mathbb{R}$. If $u \in C^1_{pw}[a,b]$ then "morally" $u \in C^1[a,b]$ with the exception of a few points, in which u' is discontinuous.

The idea is that, if we integrate u', the Lebesgue integral does not see those exceptional points, as that set has zero measure. Now, if $\varphi \in C^1_c(a,b)$, we get, integrating by parts,

$$\int_a^b u\varphi'\,dx = \underbrace{u\varphi\Big|_a^b}_{\substack{=0 \text{ as} \\ \varphi(a)=\varphi(b)=0}} - \int_a^b u'\varphi\,dx$$

Thus for $u \in C^1_{pw}[a,b]$ and $\varphi \in C^1_c(a,b)$ we get

$$\int_a^b u\varphi'\,dx = -\int_a^b u'\varphi\,dx$$

Note that the above expression makes sense also if u, u' only belong to $L^1(a,b)$. This motivates the following definition.

DEFINITION 7.14 (Sobolev Space)

Let $I = (a,b)$ be an interval, possibly unbounded. Let $1 \le p \le +\infty$. The SOBOLEV SPACE $W^{1,p}(I)$ is defined as

$$W^{1,p}(I) := \Big\{ u \in L^p(I) \;\Big|\; \exists\, g \in L^p(I) \text{ s.t. } \underbrace{\int_I u\varphi'\,dx = -\int_I g\varphi\,dx}_{\text{"INTEGRATION BY PARTS"}},\ \forall \varphi \in C^1_c(I) \Big\}.$$

REMARK If $u \in W^{1,p}(I)$ then the function g is UNIQUE

Proof Assume $g, h \in L^p(I)$ both satisfy the "integration by parts" formula, i.e.,

$$\int_I u\dot{\varphi}\, dx = -\int_I g\varphi \quad , \quad \forall \varphi \in C_c^1(I)$$

$$\int_I u\dot{\varphi}\, dx = -\int_I h\varphi \quad , \quad \forall \varphi \in C_c^1(I)$$

Then $\int_I (g-h)\varphi\, dx = 0 \quad , \forall \varphi \in C_c^1(I)$. By FLCV LEMMA 7.11

we get $g = h$ a.e. on I. Thus g and h are the same L^p function, as they belong to the same equivalence class.

NOTATION ① If $u \in W^{1,p}(I)$ and g is the function from the definition, we denote

$$\dot{u} := g$$

and call $\dot{u}$ the WEAK DERIVATIVE of u.

② For $p=2$ we denote $H^1(I) := W^{1,2}(I)$.

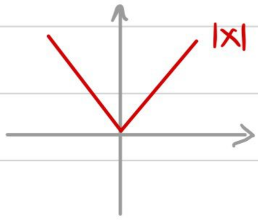
EXAMPLE 7.15 ① If $u \in C^1(I) \cap L^p(I)$ then $u \in W^{1,p}(I)$ and the weak derivative coincides with the classical derivative

② If I is bounded then $C^1(\overline{I}) \subset W^{1,p}(I)$ for all $1 \leq p \leq +\infty$

③ If $u \in C^1_{pw}(I) \cap L^p(I)$ then $u \in W^{1,p}(I)$ and the weak derivative coincides with the classical derivative in the points of differentiability of u (Note that this makes sense, since the weak derivative needs only to be defined almost everywhere, and the set of points where u is not differentiable is finite $\Rightarrow$ it has zero measure)

For example consider $I=(-1,1)$, $u(x):=|x|$. Then $u \in C^1_{pw}(I)$ with

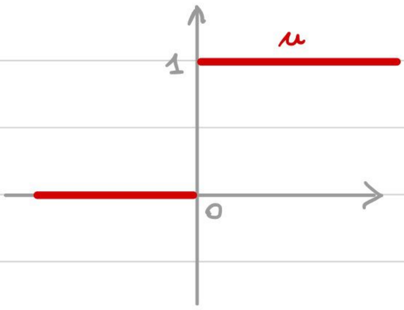
$$\dot{u}(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$



It is easy to check that this is the weak derivative of u.

④ Functions with JUMPS do not belong to $W^{1,p}(I)$. For example consider $I:=(-1,1)$ and

$$u(x) := \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$$



The pointwise derivative of u is the function $\dot{u}(x)=0$ for all $x \in \mathbb{R}\setminus\{0\}$. Thus we also have that $\dot{u} \in L^p(I)$. However it is easy to show that $\dot{u}$ is NOT the weak derivative of u. Moreover one can show that u does not admit any weak derivative, i.e., $u \notin W^{1,p}(I)$, for any $1 \le p \le +\infty$.

NOTATION

① The space $W^{1,p}(I)$ is equipped with the NORM

$$\boxed{\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|\dot{u}\|_{L^p} .}$$

② If $1 \le p < +\infty$, $W^{1,p}(I)$ can be equipped with the EQUIVALENT NORM

$$\boxed{\|u\|_{W^{1,p}} := \left(\|u\|^p_{L^p} + \|\dot{u}\|^p_{L^p} \right)^{1/p}.}$$

③ The space $H^1(I)$ can be equipped with the INNER PRODUCT

$$\boxed{\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2}}$$

The induced norm is

$$\boxed{\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2 \right)^{1/2}}$$

[Checking the above statements is straightforward, using that $\|u\|_{L^p}$ is a norm on L^p and that $\langle u, v \rangle_{L^2}$ is an inner product on L^2]

PROPOSITION 7.16 Let $I \subseteq \mathbb{R}$ be open, bounded or unbounded. Then:

1. $W^{1,p}(I)$ is a BANACH SPACE for $1 \leq p \leq +\infty$.

2. $W^{1,p}(I)$ is REFLEXIVE for $1 < p < +\infty$.

3. $W^{1,p}(I)$ is SEPARABLE for $1 \leq p < +\infty$.

4. $H^1(I)$ is a SEPARABLE HILBERT space.

Proof ① We need to prove that $W^{1,p}(I)$ is complete. So let $\{u_n\} \subseteq W^{1,p}(I)$ be a Cauchy sequence. As

$$\|u\|_{L^p} \leq \|u\|_{W^{1,p}} \quad \text{and} \quad \|\dot{u}\|_{L^p} \leq \|u\|_{W^{1,p}} \quad , \forall u \in W^{1,p}(I)$$

we have that $\{u_n\}, \{\dot{u}_n\}$ are Cauchy sequences in $L^p(I)$. As $L^p(I)$ is complete, there $\exists\, u, g \in L^p(I)$ s.t.

$$(*) \qquad u_n \to u \quad , \quad \dot{u}_n \to g \quad \text{strongly in } L^p(I).$$

By definition of $W^{1,p}$ we have

$$(**) \qquad \int_I u_n \dot{\varphi} \, dx = - \int_I \dot{u}_n \varphi \, dx \quad , \quad \forall \varphi \in C_c^1(I) \, , \quad \forall n \in \mathbb{N}$$

This is $L^p(I)^*$.
See THEOREM 6.14

As $u_n \to u$ strongly, then $u_n \rightharpoonup u$ weakly in $L^p(I)$. Since $C_c(I) \subset L^{p'}(I)$ we get

$$\int_I u_n \dot{\varphi}\, dx \to \int_I u \varphi\, dx \qquad \text{as } n \to +\infty$$

Similarly

$$\int_I \dot{u}_n \varphi\, dx \to \int_I g \varphi\, dx \qquad \text{as } n \to +\infty$$

Then we can pass to the limit in (**) and obtain

$$\int_I u \dot{\varphi} = -\int_I g \varphi\, dx \quad , \quad \forall\, \varphi \in C_c^1(I).$$

This shows $u \in W^{1,p}(I)$ with weak derivative $\dot{u} = g$. Then by (*) we conclude $\|u_n - u\|_{W^{1,p}} \to 0$, showing completeness.

2. Recall that $L^p(I)$ is REFLEXIVE for $1 < p < +\infty$ (THEOREM 6.14). Then it is easy to check that $E := L^p(I) \times L^p(I)$ is reflexive. Let

$$\begin{aligned} T : W^{1,p}(I) &\to E \\ u &\mapsto (u, \dot{u}) \end{aligned}$$

One can check that T is an isometry. Since $W^{1,p}(I)$ is Banach, it follows that $T(W^{1,p}(I)) \subseteq E$ is a closed subspace. Since closed subspaces of reflexive spaces are reflexive, we conclude.

3. $L^p(I)$ is separable for all $1 \leq p < +\infty$ (THEOREM 6.15). Thus $E := L^p(I) \times L^p(I)$ is separable (immediate check). Consider T as above. As any SUBSET of a separable space is separable, from the inclusion $T(W^{1,p}(I)) \subseteq E$ we conclude.

4. Follows from 1 and 3. $\square$

In the above proof, point 1, we showed a general fact which is worthy of its own numbered Remark.

REMARK 7.17 Let $\{u_n\} \subseteq W^{1,p}(I)$ be such that

$$\begin{cases} u_n \to u & \text{in } L^p(I) \\ \dot{u}_n \to g & \text{in } L^p(I) \end{cases}$$

Then $u_n \to u$ in $W^{1,p}(I)$ and $u \in W^{1,p}(I)$, with $\dot{u} = g$.

A similar Remark holds also for weak convergence.

REMARK 7.18 Let $\{u_n\} \subseteq H^1(I)$ be such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } L^2(I) \\ \dot{u}_n \rightharpoonup g & \text{weakly in } L^2(I) \end{cases}$$

Then $u \in H^1(I)$ with $\dot{u} = g$ in the weak sense and $u_n \rightharpoonup u$ in $H^1(I)$.

Proof As $u_n \in H^1(I)$ then

$$(*) \quad \int_I u_n \dot{\varphi} \, dx = - \int_I \dot{u}_n \varphi \, dx \, , \quad \forall \varphi \in C^1_c(I) \, , \quad \forall n \in \mathbb{N}.$$

As $L^2(I)$ is Hilbert, we have that $L^2(I)^* = L^2(I)$. Since $C^1_c(I) \subseteq L^2(I)$, by the weak convergence $u_n \rightharpoonup u$, we get

$$\int_I u_n \dot{\varphi} \, dx \to \int_I u \dot{\varphi} \, dx \, .$$

Similarly, as $\dot{u}_n \rightharpoonup g$, we get

$$\int_I \dot{u}_n \varphi \, dx \longrightarrow \int_I g \varphi \, dx \, .$$

Then we can pass to the limit in (*) and get that $\dot{u} = g$ in the weak sense. As $g \in L^2(I)$, we get $u \in H^1(I)$. If $v \in H^1(I)$ we get

$$\langle u_n, v \rangle_{H^1} = \langle u_n, v \rangle_{L^2} + \langle \dot{u}_n, \dot{v} \rangle_{L^2} \to \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2} = \langle u, v \rangle_{H^1}$$

showing that $u_n \rightharpoonup u$ weakly in $H^1(I)$. □

LESSON 8 - 5 MAY 2021

We now prove one of the main results on 1-dimensional Sobolev functions, namely, that they are CONTINUOUS and they are PRIMITIVES of L^p functions.

THEOREM 7.19 Let $I = (a,b)$ be bounded or unbounded, and $1 \leq p \leq +\infty$. Let $u \in W^{1,p}(I)$. Then there $\exists\, \tilde{u} \in C(I)$ s.t.

$$u = \tilde{u} \quad \text{a.e. on } I$$

and

$$(*) \qquad \tilde{u}(x) - \tilde{u}(y) = \int_y^x \dot{u}(t)\,dt, \qquad \forall\, x, y \in I.$$

(Generalized Fundamental Thm of Calculus)

NOTE Theorem 7.19 is saying that if $u \in W^{1,p}(I)$ then $\exists\, \tilde{u}$ continuous in the same equivalence class of u. We call $\tilde{u}$ the CONTINUOUS REPRESENTATIVE of u, and in the future we just denote it by u (Notice that the continuous representative is unique, by (*)).

During the proof of THEOREM 7.19 we need the following lemma.

LEMMA 7.20 $I = (a,b)$, $g \in L^1_{loc}(I)$. Fix $y_0 \in I$ and define

$$u(x) := \int_{y_0}^x g(t)\,dt, \qquad \forall x \in I.$$

Then $u \in C(I)$ and $\dot{u} = g$ in the weak sense.

Proof of LEMMA 7.20 The fact that u is continuous follows by DOMINATED CONVERGENCE. Indeed, for $x \in I$,

$$(*) \quad |u(x+\varepsilon) - u(x)| \leq \int_x^{x+\varepsilon} |g(t)|\, dt = \int_K \chi_{[x,x+\varepsilon]}(t) |g(t)|\, dt,$$

where K is any compact set such that $[x, x+\varepsilon] \subset K$, $\forall\, 0 < \varepsilon < 1$. Now $\chi_{[x,x+\varepsilon]} g \to 0$ a.e. as $\varepsilon \to 0$, and $|\chi_{[x,x+\varepsilon]} g| \leq |g|$,

with $g \in L^1_{loc}(I)$. Thus $g \in L^1(K)$, and by dominated convergence we conclude that the RHS of $(*)$ goes to 0 as $\varepsilon \to 0$, showing continuity.

We now show that $\dot{u} = g$ in the weak sense. Thus let $\varphi \in C^1_c(I)$. Consider $\psi(t,x) := u(x)\dot{\varphi}(t)$. Clearly $\psi \in L^1(I \times I)$, being $u, \dot{\varphi}$ continuous. Then we can apply FUBINI'S THEOREM 6.10 to get:

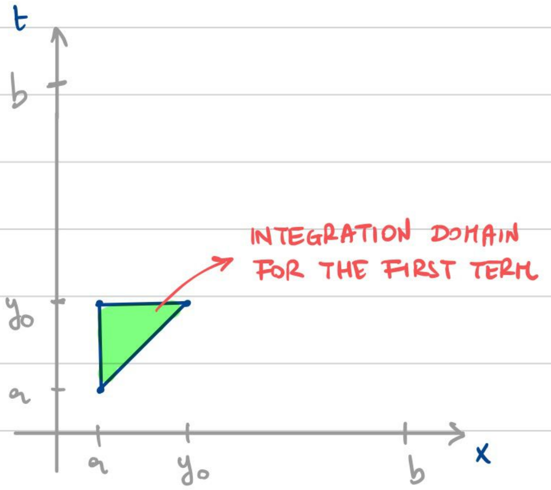
$$\int_a^b u \dot{\varphi}\, dx = \int_a^b \left\{ \int_{y_0}^x g(t)\dot{\varphi}(x)\, dt \right\} dx \qquad \text{(definition of } u\text{)}$$

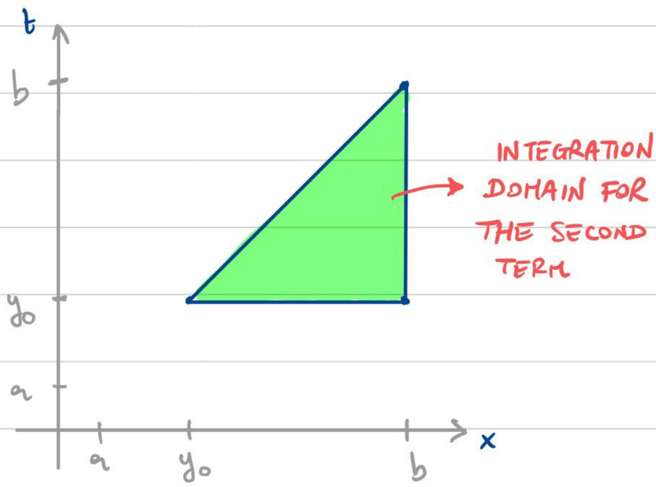
Splitting integral WRT x →

$$= \int_a^{y_0} \left\{ \int_{y_0}^x g(t)\dot{\varphi}(x)\, dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\dot{\varphi}(x)\, dt \right\} dx$$

Reversing integration values of inner integral for the FIRST TERM →

$$= -\int_a^{y_0} \left\{ \int_x^{y_0} g(t)\dot{\varphi}(x)\, dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\dot{\varphi}(x)\, dt \right\} dx$$





We can write the integration domains normal WRT to t, and apply FUBINI:

$$\int_a^b u\dot{\varphi}\,dx = -\int_a^{y_0}\left\{\int_x^{y_0} g(t)\dot{\varphi}(x)\,dt\right\}dx + \int_{y_0}^b\left\{\int_{y_0}^x g(t)\dot{\varphi}(x)\,dt\right\}dx$$

FUBINI $\longrightarrow$
$$= -\int_a^{y_0}\left\{\int_a^t g(t)\dot{\varphi}(x)\,dx\right\}dt + \int_{y_0}^b\left\{\int_t^b g(t)\dot{\varphi}(x)\,dx\right\}dt$$

TAKE $g(t)$ OUT $\rightarrow$
$$= -\int_a^{y_0} g(t)\left\{\int_a^t \dot{\varphi}(x)\,dx\right\}dt + \int_{y_0}^b g(t)\left\{\int_t^b \dot{\varphi}(x)\,dx\right\}dt$$

$$= -\int_a^{y_0} g(t)\,[\varphi(t)-\varphi(a)]\,dt + \int_{y_0}^b g(t)\,[\varphi(b)-\varphi(t)]\,dt$$

$\varphi(a)=\varphi(b)=0$, since φ is COMPACTLY SUPPORTED $\longrightarrow$
$$= -\int_a^{y_0} g(t)\varphi(t)\,dt - \int_{y_0}^b g(t)\varphi(t)\,dt = -\int_a^b g(t)\varphi(t)\,dt,$$

showing that $\dot{u}=g$ in the weak sense and concluding. $\square$

Proof of THEOREM 7.19 Fix $y_0 \in I$ arbitrary and define

$$\hat{u}(x) := \int_{y_0}^x \dot{u}(t)\,dt, \quad \forall x \in I.$$

Since $\dot{u}\in L^p(I)$ (as $u \in W^{1,p}(I)$), then $\dot{u} \in L^1_{loc}(I)$. We can then apply LEMMA 7.20 to infer that $\hat{u}\in C(I)$ and $(\hat{u})' = \dot{u}$ in the weak sense, i.e.

$$(*) \quad \int_a^b \hat{u}\dot{\varphi}\,dx = -\int_a^b \dot{u}\varphi\,dx, \quad \forall \varphi \in C^1_c(I)$$

On the other hand $u \in W^{1,p}(I)$, so that by definition

$$\int_a^b u \dot{\varphi}\, dx = -\int_a^b \dot{u} \varphi\, dx, \quad \forall \varphi \in C_c^1(I).$$

by (*) we then get

$$\int_a^b (\hat{u} - u)\dot{\varphi}\, dx = 0, \quad \forall \varphi \in C_c^1(I).$$

We can then apply DBR LEMMA 7.13 to get that $\exists\, c \in \mathbb{R}$ s.t. $u = \hat{u} + c$ a.e. in I. Thus the continuous representative is $\tilde{u} := \hat{u} + c$. The second part of the statement follows by definition of $\tilde{u}$. $\square$

REMARK 7.21 Lemma 7.20 implies that, if $g \in L^p(I)$ and its primitive u also belongs to $L^p(I)$, then $u \in W^{1,p}(I)$.

With similar ideas, we can prove the following proposition.

PROPOSITION 7.22 Let $I = (a,b)$ be bounded or unbounded, $1 \le p \le +\infty$. Assume that $u \in W^{1,p}(I)$ is s.t. $\dot{u} \in C(I)$. Then $u \in C^1(I)$.

Proof Define $U(x) := \int_a^x \dot{u}(t)\,dt$. As $\dot{u}$ is continuous, by the Fundamental Theorem of Calculus we have that $U \in C^1(I)$ and $\dot{U} = \dot{u}$. Let $\varphi \in C_c^1(I)$. Integrating by parts:

$$\int_a^b U \dot{\varphi}\, dx = \underbrace{U\varphi \Big|_a^b}_{\substack{=0 \text{ as} \\ \varphi(a) = \varphi(b) = 0}} - \int_a^b \dot{U} \varphi\, dx \underset{\dot{U} = \dot{u}}{=} -\int_a^b \dot{u}\varphi\, dx \underset{\text{Definition of weak derivative}}{=} \int_a^b u \dot{\varphi}\, dx$$

Thus

$$\int_a^b (U - u)\dot{\varphi}\, dx = 0, \quad \forall \varphi \in C_c^1(I).$$

By DBR LEMMA 7.13 we get $u = U + c$ for some $c \in \mathbb{R}$. As $U \in C^1(I) \Rightarrow u \in C^1(I)$. $\square$

HÖLDER REGULARITY

We can actually improve on THEOREM 7.19 by showing Hölder regularity for Sobolev functions. We recall that u is **α - HÖLDER** for some $0<\alpha<1$ if $\exists\, c>0$ s.t.

$$|u(x)-u(y)| \leq c\, |x-y|^{\alpha} \quad , \quad \forall\, x,y \in I$$

We denote the space of α-Hölder functions by $C^{0,\alpha}(I)$.

THEOREM 7.23 Let $I=(a,b)$ be bounded or unbounded. Let $1<p\leq +\infty$ and $u \in W^{1,p}(I)$. Then $u \in C^{0,1-1/p}(I)$, with

$$|u(x)-u(y)| \leq \|\dot{u}\|_{L^p}\, |x-y|^{1-1/p} \quad , \qquad \forall\, x,y \in I .$$

Proof By THEOREM 7.19 we have that u is continuous and

$$u(x)-u(y) = \int_y^x \dot{u}(t)\,dt \quad , \quad \forall\, x,y \in I .$$

Then for $y>x$,

$$|u(x)-u(y)| \leq \int_x^y |\dot{u}(t)|\,dt$$

(Hölder inequality) $$\leq \left(\int_x^y |\dot{u}(t)|^p\,dt\right)^{1/p} \left(\int_x^y 1^{p'}\,dt\right)^{1/p'}$$ $\left(p'=\frac{p}{p-1} \text{ Hölder conjugate}\right)$

$$= \left(\int_a^b |\dot{u}(t)|^p\,dt\right)^{1/p} (y-x)^{1/p'}$$

$$= \|\dot{u}\|_{L^p}\, |y-x|^{1-1/p}$$

If $x>y$ we conclude with the same argument. □

WARNING THEOREM 7.23 does not hold for $p=1$.

DENSITY OF SMOOTH FUNCTIONS

Our goal is to prove the following theorem.

THEOREM 7.24 Let $1 \leq p < +\infty$, $u \in W^{1,p}(I)$ for $I = (a,b)$ bounded or unbounded. Then $\exists \{u_n\} \subseteq C_c^\infty(\mathbb{R})$ s.t.

$$u_n|_I \to u \quad \text{strongly in } W^{1,p}(I).$$

WARNING The above differs from the density result for L^p functions COROLLARY 7.10: If $u \in L^p(I)$, $\exists \{u_n\} \subseteq C_c^\infty(I)$ s.t. $u_n \to u$ strongly in $L^p(I)$.

In order to prove the above theorem we need an extension result.

LEMMA 7.25 Let $I = (a,b)$ be bounded or unbounded, $1 \leq p \leq +\infty$. There $\exists$ a linear continuous operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ called EXTENSION OPERATOR such that:

a) $Pu|_I = u \quad , \quad \forall u \in W^{1,p}(I)$

b) $\|Pu\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(I)} \quad , \quad \forall u \in W^{1,p}(I)$

c) $\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C \|u\|_{W^{1,p}(I)} \quad , \quad \forall u \in W^{1,p}(I)$

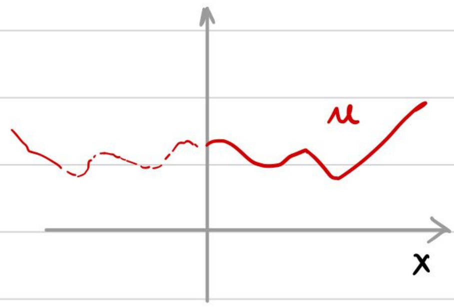
where C depends only on $|I|$: $C = 4$ in (b) and $C = 4\left(1 + \frac{1}{|I|}\right)$ in (c).

Proof of LEMMA 7.25 We have two cases:

1) If I is unbounded, then by translation it is sufficient to consider either $I = (0, +\infty)$, or $I = (-\infty, 0)$.
2) If I is bounded, then by translation and scaling it is sufficient to consider $I = (0, 1)$.

CASE 1 Let $I = (0, +\infty)$. If $u \in W^{1,p}(I)$, we extend u by REFLECTION:

$$(Pu)(x) := u^*(x) := \begin{cases} u(x) & \text{if } x > 0 \\ u(-x) & \text{if } x < 0 \end{cases}$$



Clearly $u^*|_I = u$, so that (a) holds. Also

$$\|u^*\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} |u^*|^p \, dx = 2 \int_0^{+\infty} |u|^p \, dx$$

then

$$(*) \quad \|u^*\|_{L^p(\mathbb{R})} \leq 2^{1/p} \|u\|_{L^p(I)} \leq 2 \|u\|_{L^p(I)},$$

showing (b). Now define

$$g(x) := \begin{cases} \dot{u}(x) & \text{for a.e. } x > 0 \\ -\dot{u}(-x) & \text{for a.e. } x < 0 \end{cases}$$

(use the definition of u^*, doing separately the cases $x > 0$ and $x < 0$)

Clearly $g \in L^p(\mathbb{R})$. Also, by using THEOREM 7.19, it is easy to check that

$$u^*(x) - u^*(0) = \int_0^x g(t) \, dt, \qquad \forall x \in \mathbb{R}.$$

Hence $u^* \in W^{1,p}(\mathbb{R})$ by REMARK 7.21, with $(u^*)' = g$ in the weak sense. Finally

$$\|(u^*)'\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} |g(t)|^p \, dt = 2 \int_0^{+\infty} |\dot{u}(t)|^p \, dt = 2 \|\dot{u}\|^p_{L^p(I)},$$

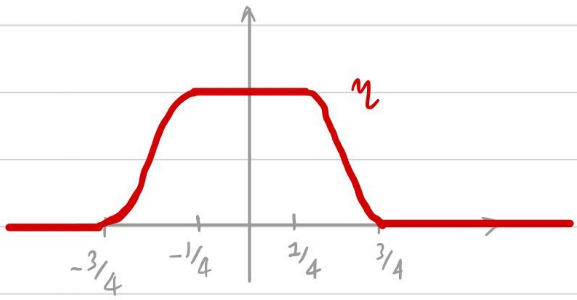
so that

$$\|(u^*)'\|_{L^p(\mathbb{R})} \leq 2^{1/p} \|\dot{u}\|_{L^p(I)} \leq 2 \|\dot{u}\|_{L^p(I)}.$$

Together with $(*)$, this implies (c). The case $I = (-\infty, 0)$ is the same.

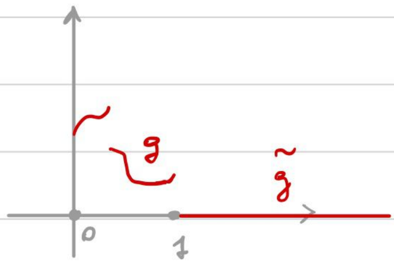
<u>CASE 2</u> Let $I = (0,1)$. Let $\eta \in C^1_c(\mathbb{R})$ be a cut-off such that

- $0 \le \eta \le 1$ in $\mathbb{R}$
- $\eta(x) = 1$ for all $x \in [-1/4, 1/4]$
- $\eta(x) = 0$ for all $x \in \mathbb{R} \setminus (-3/4, 3/4)$.



For $g \in L^p(0,1)$, define

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } x \in (0,1) \\ 0 & \text{if } x \ge 1 \end{cases}$$



If $u \in W^{1,p}(0,1)$, we claim that

$$(*) \qquad \eta \tilde{u} \in W^{1,p}(0,+\infty) \quad \text{and} \quad (\eta \tilde{u})' = \eta' \tilde{u} + \eta \cdot \widetilde{(u')} \quad \text{weakly.}$$

Indeed, let $\varphi \in C^1_c(0,+\infty)$. As both φ and η are regular, we have $(\eta\varphi)' = \eta'\varphi + \eta\varphi'$.
Then

$$\int_0^{+\infty} (\eta \tilde{u})\,\varphi'\,dx = \int_0^1 (\eta u)\,\varphi'\,dx \qquad \left(\begin{array}{l}\text{since } \tilde{u} = 0 \text{ if } x \ge 1 \text{ and } \tilde{u} = u \\ \text{for } 0 < x < 1\end{array}\right)$$

$$= \int_0^1 u\,(\eta\varphi)'\,dx - \int_0^1 u\,\eta'\varphi\,dx \qquad (\text{using } (\eta\varphi)' = \eta'\varphi + \eta\varphi')$$

$$= -\int_0^1 u'\,\eta\varphi\,dx - \int_0^1 u\,\eta'\varphi\,dx \qquad \left(\begin{array}{l}\text{since } \eta\varphi \in C^1_c(0,1) \\ \text{and } u \in W^{1,p}(0,1)\end{array}\right)$$

$$= -\int_0^1 [u'\eta + u\eta']\,\varphi\,dx$$

$$= -\int_0^{+\infty} [\widetilde{(u')}\,\eta + \tilde{u}\,\eta']\,\varphi\,dx \qquad \left(\begin{array}{l}\text{since extending } u \text{ and} \\ u' \text{ to zero does not} \\ \text{alter the integral}\end{array}\right)$$

Showing that $(\eta\tilde{u})' = \eta'\tilde{u} + \eta\,\widetilde{(u')}$ weakly.
Clearly $\eta\tilde{u} \in L^p(0,+\infty)$. Also, by using the formula just proven, $(\eta\tilde{u})' \in L^p(0,+\infty)$. Then $\eta\tilde{u} \in W^{1,p}(0,+\infty)$ and $(*)$ is proven.

$$(u)^*(x) = \begin{cases} u(x) & \text{if } x>0 \\ u(-x) & \text{if } x<0 \end{cases}$$

We can now define the extension operator $P: W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$. First define $P_1 : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ by setting $P_1 u := (\eta\tilde{u})^*$, with $*$ being the operator from CASE 1, that is, we first extend u to $(0,+\infty)$ by setting u to 0 in $[1,+\infty)$, then we multiply by η and extend $\eta\tilde{u}$ to $(-\infty,0)$ by reflection. By the properties of $*$ we know that

$$\|(\eta\tilde{u})^*\|_{L^p(\mathbb{R})} \leq 2\,\|\eta\tilde{u}\|_{L^p(0,+\infty)}, \qquad \left\|\left[(\eta\tilde{u})^*\right]'\right\|_{L^p(\mathbb{R})} \leq 2\,\|(\eta\tilde{u})'\|_{L^p(0,+\infty)}$$

Now

$$\|\eta\tilde{u}\|_{L^p(0,+\infty)} \leq \|\eta\|_{L^\infty(0,+\infty)}\,\|\tilde{u}\|_{L^p(0,+\infty)} = \|u\|_{L^p(0,1)}$$

since $0 \leq \eta \leq 1$ and $\tilde{u}=0$ in $(1,+\infty)$. Moreover, by (*),

$$\|(\eta\tilde{u})'\|_{L^p(0,+\infty)} \overset{(*)}{\leq} \|\eta'\tilde{u}\|_{L^p(0,+\infty)} + \|\eta(\tilde{u}')\|_{L^p(0,+\infty)}$$

(since $u=0$, $(\tilde{u})'=0$ in $(1,+\infty)$)

$$\leq \|\eta'\|_{L^\infty(I)}\,\|u\|_{L^p(I)} + \|\eta\|_{L^\infty(0,+\infty)}\,\|u'\|_{L^p(I)}$$

$$\leq C\,\|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$$

with $C := \|\eta'\|_{L^\infty(I)}$. In total, we have

$$(**) \qquad \|P_1 u\|_{L^p(\mathbb{R})} \leq 2\,\|u\|_{L^p(I)}, \qquad \|P_1 u\|_{W^{1,p}(\mathbb{R})} \leq 2(1+C)\,\|u\|_{W^{1,p}(I)}$$

Also notice that $(P_1 u)|_I = \eta u$. Now define $P_2 : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ in the following way: $P_2 u$ is defined by

- Extending $(1-\eta)u$ to 0 in $(-\infty,0)$, obtaining a map defined in $(-\infty,1]$;
- Then extend to the whole $\mathbb{R}$ by reflection around 1.

In a similar way one can check that

$$(***) \quad \| P_2 u \|_{L^p(\mathbb{R})} \le 2 \| u \|_{L^p(I)}, \qquad \| P_2 u \|_{W^{1,p}(\mathbb{R})} \le 2(1+C) \| u \|_{W^{1,p}(I)}$$

and that $(P_2 u)|_I = (1-\eta) u$. Finally we define $P : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ by

$$Pu := P_1 u + P_2 u .$$

By (**)-(***) we have that P satisfies (b), (c). Moreover

$$(Pu)|_I = (P_1 u)|_I + (P_2 u)|_I = \eta u + (1-\eta) u = u ,$$

so that also (a) holds, concluding. □

Another result needed to prove THEOREM 7.24 is the following:

LEMMA 7.26 Let $\rho \in L^1(\mathbb{R})$, $u \in W^{1,p}(\mathbb{R})$ with $1 \le p \le +\infty$. Then $\rho * u \in W^{1,p}(\mathbb{R})$ and $(\rho * u)' = \rho * u'$ in the weak sense.

Proof Assume first that ρ is compactly supported, so that $\rho \in L^1_{loc}(\mathbb{R})$. By THEOREM 7.2 we have $\rho * u \in L^p(\mathbb{R})$. Let $\varphi \in C^1_c(\mathbb{R})$. One can check that

$$(*) \quad \int_{\mathbb{R}} (\rho * u) \varphi' \, dx = \int_{\mathbb{R}} u (\check{\rho} * \varphi') \, dx , \qquad \check{\rho}(x) := \rho(-x) .$$

Now, as $\check{\rho} \in L^1(\mathbb{R})$ and $\varphi \in C^1_c(\mathbb{R})$, by THEOREM 7.6 we have $\check{\rho} * \varphi \in C^1(\mathbb{R})$ and $\check{\rho} * \varphi' = (\check{\rho} * \varphi)'$. Moreover $\check{\rho} * \varphi$ is compactly supported, as $\operatorname{supp}(\check{\rho} * \varphi) \subset \overline{\operatorname{supp} \check{\rho} + \operatorname{supp} \varphi}$ by PROPOSITION 7.4, and ρ, φ are compactly supported. Therefore $\check{\rho} * \varphi \in C^1_c(\mathbb{R})$ and by (*)

use $\check{\rho} * \varphi' = (\check{\rho} * \varphi)'$

$$\int_{\mathbb{R}} (\rho * u) \varphi' \, dx \overset{(*)}{=} \int_{\mathbb{R}} u (\check{\rho} * \varphi') \, dx = \int_{\mathbb{R}} u (\check{\rho} * \varphi)' \, dx$$

(As $\check{\rho} * \varphi$ is a test function and $u \in W^{1,p}(\mathbb{R})$) →

$$= - \int_{\mathbb{R}} u' (\check{\rho} * \varphi) \, dx = - \int_{\mathbb{R}} (\rho * u') \varphi \, dx$$

Use (*) with u replaced by u'

Thus $(\rho * u)' = \rho * u'$ in the weak sense. As $u' \in L^p(\mathbb{R})$, by THEOREM 7.2 we get $\rho * u' \in L^p(\mathbb{R})$, showing that $\rho * u \in W^{1,p}(\mathbb{R})$.

If ρ is not compactly supported, by COROLLARY 7.10 we can find a sequence $\{\rho_n\} \subseteq C_c(\mathbb{R})$ s.t. $\rho_n \to \rho$ strongly in $L^1(\mathbb{R})$. Note that what we proved so far holds for ρ_n, so that

$$(**) \qquad \rho_n * u \in W^{1,p}(\mathbb{R}), \qquad (\rho_n * u)' = \rho_n * u' \quad \text{weakly}, \qquad \forall n \in \mathbb{N}.$$

By Young's inequality we have

$$\| \rho_n * u - \rho * u \|_{L^p} \leq \| \rho_n - \rho \|_{L^1} \, \| u \|_{L^p} \to 0 \quad \text{as } n \to +\infty$$

(as $\rho_n \to \rho$ in L^1)

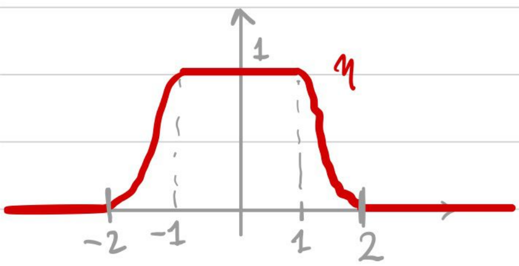
$$\| \rho_n * u' - \rho * u' \|_{L^p} \leq \| \rho_n - \rho \|_{L^1} \, \| u' \|_{L^p} \to 0 \quad \text{as } n \to +\infty$$

This means $\rho_n * u \to \rho * u$ and $(\rho_n * u)' \overset{(**)}{=} \rho_n * u' \to \rho * u'$ strongly in $L^p(\mathbb{R})$. Since $\rho * u' \in L^p(\mathbb{R})$ by THEOREM 7.2, we can invoke REMARK 7.17 to conclude that $\rho * u \in W^{1,p}(I)$, with weak derivative $(\rho * u)' = \rho * u'$. $\square$

Proof of THEOREM 7.24 Let $I \subseteq \mathbb{R}$ be open, bounded or unbounded. We need to show that for $u \in W^{1,p}(I)$ there $\exists \{u_n\} \subseteq C_c^\infty(\mathbb{R})$ s.t. $(u_n)|_I \to u$ strongly in $W^{1,p}(I)$.

First, let $\tilde{u} := Pu$ be the extension to $\mathbb{R}$ of u given by LEMMA 7.25. In particular

$$\tilde{u}|_I = u, \qquad \| \tilde{u} \|_{W^{1,p}(\mathbb{R})} \leq C \| u \|_{W^{1,p}(I)}.$$



Let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off s.t.

$$0 \leq \eta \leq 1, \quad \eta(x) = 1 \text{ for } x \in [-1,1], \quad \eta(x) = 0 \text{ for } x \in \mathbb{R} \setminus (-2,2).$$

Define $\eta_n(x) := \eta\left(\frac{x}{n}\right)$. Note that $\eta_n \to 1$ pointwise. Therefore $\eta_n \tilde{u} \to \tilde{u}$ a.e. in $\mathbb{R}$.

Since $|\eta_n \tilde{u}| \leq |\tilde{u}|$ and $\tilde{u} \in L^p(\mathbb{R})$, by Dominated Convergence (THEOREM 6.7) we get

$$(*) \quad \eta_n \tilde{u} \to \tilde{u} \text{ strongly in } L^p(\mathbb{R}).$$

Let $\rho_n \in C_c^\infty(\mathbb{R})$ be a sequence of mollifiers. Define $u_n := \eta_n \cdot (\rho_n * \tilde{u})$. Notice that $\rho_n * \tilde{u} \in C^\infty(\mathbb{R})$ by THEOREM 7.6 (indeed note that $\tilde{u} \in L^p(\mathbb{R})$ and so $u \in L^1_{loc}(\mathbb{R})$). Since $\eta_n \in C_c^\infty(\mathbb{R})$, it follows that $u_n \in C_c^\infty(\mathbb{R})$. We will show that $(u_n)_{|I} \to u$ strongly in $W^{1,p}(I)$. First note that

$$u_n - \tilde{u} = \eta_n (\rho_n * \tilde{u}) - \tilde{u} = \eta_n [\rho_n * \tilde{u} - \tilde{u}] + \eta_n \tilde{u} - \tilde{u}$$

Since $\|\eta_n\|_{L^\infty(\mathbb{R})} \leq 1$, we get

$$\|u_n - \tilde{u}\|_{L^p(\mathbb{R})} \leq \underbrace{\|\rho_n * \tilde{u} - \tilde{u}\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by THEOREM 7.9}}} + \underbrace{\|\eta_n \tilde{u} - \tilde{u}\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by } (*)}} \to 0 \quad \text{as } n \to +\infty$$

so $u_n \to \tilde{u}$ strongly in $L^p(\mathbb{R})$. In particular $u_n \to u$ strongly in $L^p(I)$. Also

$$u_n' \underset{\substack{\text{def + classical} \\ \text{derivation of product}}}{=} \eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u})' \underset{\substack{\text{LEMMA 7.26 to} \\ \text{differentiate } \rho_n * \tilde{u}: \\ (\rho_n * \tilde{u})' = \rho_n * \tilde{u}' \text{ weakly}}}{=} \eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u}')$$

$\boxed{\text{Note}}$ To differentiate $\rho_n * \tilde{u}$ we could also use THEOREM to get $(\rho_n * \tilde{u})' = \rho_n' * \tilde{u}$. However this term would be useless in our proof, because we need $\tilde{u}'$ to appear.

Therefore

$$\|u_n' - \tilde{u}'\|_{L^p(\mathbb{R})} = \|\eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u}') - \tilde{u}'\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n' (\rho_n * \tilde{u})\|_{L^p(\mathbb{R})} + \|\eta_n (\rho_n * \tilde{u}') - \tilde{u}'\|_{L^p(\mathbb{R})}$$

(add subtract $\eta_n \tilde{u}'$ and use Δ inequality) $\to$

$$\leq \underbrace{\|\eta_n' (\rho_n * \tilde{u})\|_{L^p(\mathbb{R})}}_{:= I_1} + \underbrace{\|\eta_n [(\rho_n * \tilde{u}') - \tilde{u}']\|_{L^p(\mathbb{R})}}_{:= I_2} + \underbrace{\|\eta_n \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}}_{:= I_3}$$

We now estimate I_1, I_2, I_3 separately.

- For I_1, notice that, as $\eta_n(x) := \eta\left(\frac{x}{n}\right)$, then $\eta_n'(x) = \frac{1}{n}\eta'\left(\frac{x}{n}\right)$.
  Setting $C := \|\eta'\|_{L^\infty(\mathbb{R})}$ we get

$$
\begin{aligned}
I_1 &= \|\eta_n'(\rho_n * \tilde{u})\|_{L^p(\mathbb{R})} \\
&\leq \|\eta_n'\|_{L^\infty(\mathbb{R})} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \\
&\leq \frac{C}{n} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})}
\end{aligned}
$$

By Youngs' inequality we have

As $\|\rho_n\|_{L^1(\mathbb{R})} = 1$ by properties of mollifiers

$$
\|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \leq \|\rho_n\|_{L^1(\mathbb{R})} \|\tilde{u}'\|_{L^p(\mathbb{R})} = \|\tilde{u}'\|_{L^p(\mathbb{R})}
$$

so that

$$
I_1 \leq \frac{C}{n} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \leq \frac{C}{n} \|\tilde{u}'\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } n \to +\infty.
$$

- For I_2,

$$
\begin{aligned}
I_2 &= \|\eta_n \left[(\rho_n * \tilde{u}') - \tilde{u}'\right]\|_{L^p(\mathbb{R})} \\
&\leq \|\eta_n\|_{L^\infty(\mathbb{R})} \|\rho_n * \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}
\end{aligned}
$$

(Since $\|\eta_n\|_{L^\infty(\mathbb{R})} = 1$)

$$
= \underbrace{\|\rho_n * \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by THEOREM 7.9,} \\ \text{as } \tilde{u}' \in L^1_{loc}(\mathbb{R})}} \to 0 \quad \text{as } n \to +\infty.
$$

• For I_3: Recall that $\eta_n \to 1$ pointwise in $\mathbb{R}$. Thus $\eta_n \tilde{u}' \to \tilde{u}'$ a.e. in $\mathbb{R}$. Also $|\eta_n \tilde{u}'| \leq |\tilde{u}'|$ as $|\eta_n| \leq 1$. Then we can apply DOMINATED CONVERGENCE to get

$$\eta_n \tilde{u}' \to \tilde{u}' \quad \text{strongly in } L^p(\mathbb{R}),$$

which implies

$$I_3 = \| \eta_n \tilde{u}' - \tilde{u}' \|_{L^p(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \to +\infty$$

In total, we just proved that

$$\| u_n' - \tilde{u}' \|_{L^p(\mathbb{R})} \leq I_1 + I_2 + I_3 \to 0 \quad \text{as } n \to +\infty,$$

that is, $u_n' \to \tilde{u}'$ strongly in $L^p(\mathbb{R})$. In particular,

$$(**) \quad u_n' \to \tilde{u}'|_I \quad \text{strongly in } L^p(I)$$

Now recall that we had

$$(***) \quad u_n \to u \quad \text{strongly in } L^p(I)$$

Note that $\{u_n\} \subseteq W^{1,p}(I)$, as $\{u_n\} \subseteq C_c^\infty(\mathbb{R})$. Therefore, as $(**)$, $(***)$ hold we can apply REMARK 7.17 and conclude

$$u_n \to u \quad \text{strongly in } W^{1,p}(I),$$

ending the proof. $\square$

LESSON 9 - 12 MAY 2021

SOBOLEV EMBEDDING

DEFINITION X, Y normed spaces, $X \subseteq Y$. We say that

1. X EMBEDS continuously in Y, in symbols $X \hookrightarrow Y$, if the identity $i:(X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous, i.e. if $\exists\, C > 0$ s.t.

$$\|u\|_Y \leq C \|u\|_X, \quad \forall u \in X.$$

2. The embedding $X \hookrightarrow Y$ is COMPACT if the identity $i: X \to Y$ is a continuous compact operator, i.e.,

$$\text{If } B \subseteq X \text{ is norm bounded WRT } \|\cdot\|_X \Rightarrow \overline{B}^{\|\cdot\|_Y} \text{ is compact WRT } \|\cdot\|_Y.$$

THEOREM 7.27 (Sobolev embedding)

Let $I \subseteq \mathbb{R}$ be open. There $\exists\, C > 0$, depending only on $|I|$, s.t.

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I), \quad 1 \leq p \leq +\infty.$$

Thus $W^{1,p}(I) \hookrightarrow L^\infty(I)$. If in addition I is BOUNDED:

(a) The embedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is COMPACT $\forall\, 1 < p \leq +\infty$,

(b) The embedding $W^{1,1}(I) \hookrightarrow L^q(I)$ is COMPACT $\forall\, 1 \leq q < +\infty$,

(c) The embedding $W^{1,p}(I) \hookrightarrow L^p(I)$ is COMPACT $\forall\, 1 \leq p \leq +\infty$.

In order to prove THEOREM 7.27 we need two auxiliary results:

THEOREM 7.28 (ASCOLI - ARZELÀ)

Let (K, d) be a compact metric space, and consider $C(K)$ i.e. the set of continuous functions $u: K \to \mathbb{R}$. Let $A \subseteq C(K)$ and suppose that:

1. A is BOUNDED: i.e. $M > 0$ s.t. $\|u\|_\infty \leq M$ for all $u \in A$
2. A is EQUI-CONTINUOUS: i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$d(x_1, x_2) < \delta \Rightarrow |u(x_1) - u(x_2)| < \varepsilon, \quad \forall u \in A.$$

Then the closure of A in $C(K)$ is COMPACT.

(This theorem should already be well-known in euclidean spaces. For a proof of the metric case, see the book by RUDIN.)

For the next result, recall the notation: if $u: \mathbb{R} \to \mathbb{R}$, $h \in \mathbb{R}$, the TRANSLATION operator T_h is defined by $(T_h u)(x) := u(x+h)$.

THEOREM 7.29 (Characterization of Sobolev Functions)

Let $1 < p < +\infty$ and $u \in L^p(\mathbb{R})$. They are equivalent:

(a) $u \in W^{1,p}(\mathbb{R})$

(b) It holds $\|T_h u - u\|_{L^p(\mathbb{R})} \leq \|u'\|_{L^p(\mathbb{R})} |h|$, $\forall h \in \mathbb{R}$.

Moreover, the implication (a) $\Rightarrow$ (b) is also true for $p = 1$.

(The proof of the above theorem will be left for the exercises course)

Proof of THEOREM 7.27 We start by showing the embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$. WLOG we can suppose $I = \mathbb{R}$, otherwise we can use the extension operator of THEOREM 7.24. Also, the embedding is trivial for $p = +\infty$. Hence assume $1 \le p < +\infty$. Define $G(s) := |s|^{p-1} s$. Let $u \in C_c^1(\mathbb{R})$ and set

$$w := G(u).$$

Clearly $w \in C_c^1(\mathbb{R})$, with (w is compactly supported since $u \in C_c^1(\mathbb{R})$ and $G(0) = 0$)

$$w' = G'(u)u' = p|u|^{p-1} u'$$

Therefore for $x \in \mathbb{R}$,

$$G(u(x)) = w(x) = \int_{-\infty}^{x} w'(s)\, ds$$ (by the Fundamental Theorem of Calculus, since $w \in C_c^1(\mathbb{R})$)

(*) $$= \int_{-\infty}^{x} p\,|u(s)|^{p-1} u'(s)\, ds$$

Now $|G(u)| = |u|^p$, thus, by (*) and Hölder's inequality,

$$|u(x)|^p = |G(u(x))| \overset{(*)}{\le} \int_{-\infty}^{x} p\,|u(s)|^{p-1} |u'(s)|\, ds$$

(Since integrand is non-negative) → $$\le p \int_{\mathbb{R}} |u(s)|^{p-1} |u'(s)|\, ds$$

(Hölder WRT p, p') $$\le p \left(\int_{\mathbb{R}} |u|^{p'(p-1)} ds \right)^{1/p'} \left(\int_{\mathbb{R}} |u'|^p ds \right)^{1/p}$$

(Recall $p' = \frac{p}{p-1}$. Then $p'(p-1) = p$ and $\frac{1}{p'} = \frac{p-1}{p}$) $$= p \,\|u\|_{L^p}^{p-1} \|u'\|_{L^p}$$

Therefore

$$|u(x)| \le p^{1/p} \|u\|_{L^p}^{1/p'} \|u'\|_{L^p}^{1/p}, \quad \forall x \in \mathbb{R}.$$

Recall Young's Inequality for real numbers: $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$, $\forall a,b \ge 0$. Apply it to $a = \|u'\|_{L^p}^{1/p}$, $b = \|u\|_{L^p}^{1/p'}$ to get

$$|u(x)| \le p^{1/p} \|u\|_{L^p}^{1/p'} \|u'\|_{L^p}^{1/p}$$

$$(\text{Young}) \quad \le p^{1/p} \left\{ \frac{\|u\|_{L^p}}{p'} + \frac{\|u'\|_{L^p}}{p} \right\}$$

$$(\text{Since } p, p' \ge 1) \quad \le p^{1/p} \left\{ \|u\|_{L^p} + \|u'\|_{L^p} \right\} = p^{1/p} \|u\|_{W^{1,p}}$$

Taking the supremum for $x \in \mathbb{R}$ and noting that $p^{1/p} \le e^{1/e}$ $\forall p \ge 1$, we get

$$(**) \qquad \|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}, \quad \forall u \in C_c^1(\mathbb{R})$$

with $C := e^{1/e}$. Suppose now that $u \in W^{1,p}(\mathbb{R})$. By THEOREM 7.24 there $\exists \{u_n\} \subseteq C_c^1(\mathbb{R})$ s.t. $u_n \to u$ strongly in $W^{1,p}(\mathbb{R})$. By applying $(**)$ to $(u_n - u_m) \in C_c^1(\mathbb{R})$ we have

$$\|u_n - u_m\|_{L^\infty} \le C \|u_n - u_m\|_{W^{1,p}} \to 0 \quad \text{as } n, m \to +\infty,$$

Since u_n is convergent in $W^{1,p}(\mathbb{R})$ and so it is Cauchy, being $W^{1,p}$ complete (see PROPOSITION 7.16). Therefore $\{u_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$.
As $L^\infty(\mathbb{R})$ is complete, we conclude the $\exists$ of $\tilde{u} \in L^\infty(\mathbb{R})$ s.t. $u_n \to \tilde{u}$ strongly in $L^\infty(\mathbb{R})$.
Recalling that $u_n \to u$ in $W^{1,p}(\mathbb{R})$, we immediately conclude that $\tilde{u} = u$.
By $(**)$ we have

$$\|u_n\|_{L^\infty} \le C \|u_n\|_{W^{1,p}}, \quad \forall n \in \mathbb{N}$$

Since $u_n \to u$ in $L^\infty(\mathbb{R})$ and in $W^{1,p}(\mathbb{R})$, we can pass to the limit as $n \to +\infty$ in the above and obtain our thesis:

$$\|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}, \quad \forall u \in W^{1,p}(\mathbb{R}).$$

(a) Assume I bounded. We need to prove that the embedding

$$W^{1,p}(I) \hookrightarrow C(\bar{I})$$

is compact, for all $1 < p \leq +\infty$. Therefore let $B \subseteq W^{1,p}(I)$ be a bounded set, so that there $\exists\, M > 0$ s.t.

$$\|u\|_{W^{1,p}} \leq M, \qquad \forall u \in B.$$

By the embedding we just proved, it follows that

$$\|u\|_{L^\infty} \leq C\, \|u\|_{W^{1,p}} \leq CM, \qquad \forall\, u \in B.$$

Recalling that $W^{1,p}(I) \subseteq C(\bar{I})$ (see THEOREM 7.19), we get $\|u\|_\infty = \|u\|_{L^\infty}$, so that

$$(*) \qquad \|u\|_\infty \leq CM, \qquad \forall u \in B.$$

Moreover, by THEOREM 7.23 we have $W^{1,p}(I) \subseteq C^{0,1-1/p}(I)$, if $p > 1$, with

$$|u(x) - u(y)| \leq \|\dot{u}\|_{L^p}\, |x-y|^{1-1/p}, \qquad \forall x, y \in \bar{I}.$$

As $\|\dot{u}\|_{L^p} \leq M$ for all $u \in B$, we conclude that

$$(**) \qquad |u(x) - u(y)| \leq M\, |x-y|^{1-1/p}, \qquad \forall x,y \in \bar{I}, \quad \forall u \in B$$

which shows that the family $B \subseteq C(\bar{I})$ is EQUI-CONTINUOUS. As $(*)$ - $(**)$ hold, we can apply the ASCOLI - ARZELÀ THEOREM 7.28 with $K = \bar{I}$, to conclude that $\bar{B}$ is compact in $C(\bar{I})$ (where the closure is taken WRT the uniform norm in $C(\bar{I})$). Thus, (a) is established.

(b) Let I be bounded, $1 \leq q < +\infty$. We need to prove that the embedding

$$W^{1,1}(I) \hookrightarrow L^q(I)$$

is compact. So let $B \subseteq W^{1,1}(I)$ be a bounded set, i.e.

$$\|u\|_{W^{1,1}(I)} \leq M, \quad \forall u \in B.$$

Let $P: W^{1,1}(I) \to W^{1,1}(\mathbb{R})$ be the extension operator from LEMMA 7.25. By the properties of P, the set $P(B)$ is bounded in $W^{1,1}(\mathbb{R})$, and also $P(B)|_I = B$, where

$$P(B)|_I := \{ u : I \to \mathbb{R} \mid \exists v \in P(B) \text{ s.t. } v|_I = u \}.$$

By the embedding $W^{1,1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we already proved, we have that $P(B)$ is also bounded in $L^\infty(\mathbb{R})$. Then, for $u \in P(B)$ we have

$$\int_{\mathbb{R}} |u|^q \, dx = \int_{\mathbb{R}} |u|^{q-1} |u| \, dx \leq \|u\|_{L^\infty}^{q-1} \|u\|_{L^1}, \qquad \forall u \in P(B)$$

showing that $P(B)$ is also bounded in $L^q(\mathbb{R})$, i.e., $\exists M > 0$ s.t.

$$(*) \qquad \|u\|_{L^q(\mathbb{R})} \leq M, \qquad \forall u \in P(B).$$

We now check that

$$(**) \qquad \lim_{|h| \to 0} \sup_{u \in P(B)} \|\tau_h u - u\|_{L^q(\mathbb{R})} = 0.$$

Indeed, by THEOREM 7.29 (implication (a) $\Rightarrow$ (b) with $p=1$) we have

$$(***) \qquad \|\tau_h u - u\|_{L^1(\mathbb{R})} \leq \|\dot{u}\|_{L^1(\mathbb{R})} |h| \leq C|h|, \qquad \forall u \in P(B)$$

Since $P(B)$ is bounded in $W^{1,1}(\mathbb{R})$

Therefore, for $u \in P(B)$,

$$\| \tau_h u - u \|^q_{L^q(\mathbb{R})} = \int_{\mathbb{R}} | \tau_h u - u |^q \, dx = \int_{\mathbb{R}} | \tau_h u - u |^{q-1} \, | \tau_h u - u | \, dx$$

$$\leq \| \tau_h u - u \|^{q-1}_{L^\infty(\mathbb{R})} \, \| \tau_h u - u \|_{L^1(\mathbb{R})}$$

$$\left(\| \tau_h u - u \|_{L^\infty} \leq 2 \| u \|_{L^\infty} \right) \leq 2^{q-1} \| u \|^{q-1}_{L^\infty(\mathbb{R})} \, \| \tau_h u - u \|_{L^1(\mathbb{R})}$$

$$\left(\begin{array}{l} \text{By } *** \text{ and the fact} \\ \text{that } P(B) \text{ is bounded in } L^\infty(\mathbb{R}) \end{array} \right) \leq 2^{q-1} \tilde{C} \, C \, |h|$$

showing $**$. Since $*$ - $**$ hold, and I bounded, $q \neq +\infty$, we can apply FRÉCHET-KOLMOGOROV THEOREM 6.17 to conclude that the closure of $P(B)_{|I}$ is compact in $L^q(I)$. Recalling that $P(B)_{|I} = B$, we have that the closure of B is compact in $L^q(I)$.

(c) Let $I \subseteq \mathbb{R}$ be bounded. We are left to show that the embedding

$$* \qquad W^{1,p}(I) \hookrightarrow L^p(I)$$

is compact for every $1 \leq p \leq +\infty$. Indeed, for $p = 1$, $*$ is just a special case of (b) with $q = 1$. Instead, for $1 < p \leq +\infty$, $*$ follows from the compact embedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ of point (a), and from the fact that uniform convergence implies L^p convergence. $\square$

REMARK 7.30

We want to discuss (without proof) what happens in the cases left out from THEOREM 7.27.

1. For the compact embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$, I bounded, $1 < p \leq +\infty$:

   - Let I be bounded. We have that $W^{1,1}(I)$ embeds into $C(\bar{I})$ (by THEOREM 7.9), but the embedding is in general NOT compact.

   - What kind of compactness can we expect in this case? The answer is as follows: Let $I \subseteq \mathbb{R}$ be open (bounded or unbounded). If $\{u_n\} \subseteq W^{1,1}(I)$ is bounded, there exists a subsequence u_{n_k} s.t. $u_{n_k}(x)$ converges pointwise for all $x \in \bar{I}$

     (this is called HELLY'S SELECTION THEOREM)

2. Concerning the embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$ for all $1 \leq p \leq +\infty$:

   - When I is unbounded, the above embedding is NEVER COMPACT

   - Assume I unbounded and $1 < p \leq +\infty$. If $\{u_n\} \subseteq W^{1,p}(I)$ is bounded, then $\exists\, u \in W^{1,p}(I)$ and a subsequence s.t. $u_{n_k} \to u$ in $L^\infty(J)$ for every $J \subseteq I$ bounded.

3. Let I be unbounded. Then $W^{1,p}(I) \hookrightarrow L^q(I)$ for all $q \in [p, \infty]$. However, in general, $W^{1,p}(I)$ does NOT embed into $L^q(I)$ if $q \in [1, p)$.

We want to explicitely state a Corollary of THEOREM 7.27 regarding weak convergence. To this end, we first recall the general definition of compact operator.

DEFINITION Let X, Y be normed spaces, and $T \in \mathcal{L}(X,Y)$. We say that T is **COMPACT** if it holds:

$$B \subseteq X \text{ bounded WRT } \|\cdot\|_X \Rightarrow \overline{T(B)}^{\|\cdot\|_Y} \text{ compact WRT } \|\cdot\|_Y$$

PROPOSITION 7.31 Let X, Y be normed spaces, and $T \in \mathcal{L}(X,Y)$ be compact. It holds:

$$x_n \rightharpoonup x_0 \text{ weakly in } X \Rightarrow Tx_n \to Tx_0 \text{ strongly in } Y$$

Proof Assume $x_n \rightharpoonup x_0$ weakly in X. Since X is a normed space, we have that $\{x_n\}$ is bounded WRT $\|\cdot\|$.
Thus, by definition of compact operator, $\overline{\{Tx_n\}}^{\|\cdot\|_Y}$ is compact WRT $\|\cdot\|_Y$. Therefore, as $\{Tx_n\} \subseteq \overline{\{Tx_n\}}^{\|\cdot\|_Y}$, there $\exists$ a subsequence and $y \in Y$ s.t.

$$(*) \quad Tx_{n_k} \to y \quad \text{strongly in } Y.$$

Now, we know that $x_n \rightharpoonup x_0$ and T continuous. Thus (easy check)

$$(**) \quad Tx_n \rightharpoonup Tx_0 \quad \text{weakly in } Y.$$

Since $(*)$ holds, and strong convergence implies weak convergence, we get $Tx_{n_k} \rightharpoonup y$ weakly in Y. By $(**)$ and uniqueness of the weak limit we get $y = Tx_0$. Therefore $(*)$ reads

$$(***) \quad Tx_{n_k} \to Tx_0 \quad \text{strongly in } Y.$$

To conclude, we use the following standard fact:

FACT (X,τ) topological space, $\{x_n\} \subseteq X$. Suppose that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a subsequence $\{x_{n_{k_j}}\}$ such that

$$x_{n_{k_j}} \to x_0 \quad \text{as } j \to +\infty,$$

for some $x_0 \in X$ which does not depend on the subsequence $\{x_{n_k}\}$ chosen. Then $x_n \to x_0$.

Therefore, reasoning as above, we could have started from an arbitrary subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$, and shown that $\exists\ \{Tx_{n_{k_j}}\}$ such that

$$Tx_{n_{k_j}} \to Tx_0 \text{ strongly in } Y, \text{ as } j \to +\infty.$$

Since the limit does not depend on the chosen subsequence $\{Tx_{n_k}\}$, we conclude that $Tx_n \to Tx_0$ strongly in Y. $\square$

COROLLARY 7.32 Let $I=(a,b)$ be bounded, and $1 \le p < +\infty$. If

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(a,b)$$

(i.e., $u_n \rightharpoonup u$, $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^p(a,b)$), then

$$u_n \to u \text{ strongly in } L^p(a,b).$$

Proof By point (c) of THEOREM 7.27 we have that $W^{1,p}(a,b) \hookrightarrow L^p(a,b)$ is compact for every $1 \le p \le +\infty$. The thesis follows by applying PROPOSITION 7.31. $\square$

LESSON 10 - 19 MAY 2021

HIGHER ORDER SOBOLEV SPACES

We can of course generalize the definition of Sobolev function to higher order derivatives:

DEFINITION Let $k \geq 2$ be an integer, $1 \leq p \leq +\infty$. Let $I \subseteq \mathbb{R}$ be an open set. We define

$$W^{k,p}(I) := \{ u \in W^{k-1,p}(I) \mid u' \in W^{k-1,p}(I) \}.$$

For $p=2$ we set

$$H^k(I) := W^{k,2}(I).$$

REMARK $u \in W^{k,p}(I)$ iff $\exists \; g_1, \dots, g_k \in L^p(I)$ s.t.

$$\int_I u \varphi^{(j)} \, dx = (-1)^j \int_I g_j \varphi \, dx \,, \quad \forall \varphi \in C_c^\infty(I) \,, \; \forall j = 1, \dots, k \,,$$

i.e. u admits weak derivatives up to order k. (easy check)

NOTATION In view of the above remark, and due to the uniqueness of weak derivatives, if $u \in W^{k,p}(I)$ we denote by

$$u^{(j)} := g_j \,, \quad j = 1, \dots, k$$

the j-th weak derivative.

PROPOSITION 7.33 Let $I \subseteq \mathbb{R}$ be open, $k \geq 2$ be an integer, $1 \leq p \leq +\infty$. Then, the space $W^{k,p}(I)$ is Banach with the norm

$$\boxed{\|u\|_{W^{k,p}} := \|u\|_{L^p} + \sum_{j=1}^{k} \|u^{(j)}\|_{L^p}}$$

Moreover $H^k(I)$ is Hilbert with the inner product

$$\boxed{\langle u, v \rangle_{H^k} := \langle u, v \rangle_{L^2} + \sum_{j=1}^{k} \langle u^{(j)}, v^{(j)} \rangle_{L^2}}$$

(The proof is obtained following the lines of the proof of PROPOSITION 7.16)

REMARK $I \subseteq \mathbb{R}$ open, $k \geq 2$, $1 \leq p \leq +\infty$. Then $W^{k,p}(I) \subseteq C^{k-1}(\bar{I})$.

(Proof is consequence of THEOREM 7.19. For example, for $k=2$ we have that if $u \in W^{2,p}(I)$, then by definition $u' \in W^{1,p}(I)$.
As $W^{1,p}(I) \subset C(\bar{I})$ by THM 7.19, we get that $u' \in C(\bar{I})$. Therefore

$$u \in W^{1,p}(I), \quad u' \in C(I)$$

and thus, by PROPOSITION 7.22 we get $u \in C^1(\bar{I})$, concluding that

$$W^{2,p}(I) \subset C^1(\bar{I}).$$

Similarly, one can conclude the other cases.).

THE SPACE $W_0^{1,p}$

When dealing with Dirichlet type boundary conditions, it is useful to introduce the space $W_0^{1,p}$, which will be the space of functions $u \in W^{1,p}$ s.t. $u=0$ on ∂I.

DEFINITION Let $I \subseteq \mathbb{R}$ be open, $1 \leq p < +\infty$. The space $W_0^{1,p}(I)$ is defined as the CLOSURE of $C_c^1(I)$ in $W^{1,p}(I)$. We denote

$$H_0^1(I) := W_0^{1,2}(I).$$

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, while $H_0^1(I)$ is equipped with the inner product of $H^1(I)$.

REMARK
- $W_0^{1,p}$ is a SEPARABLE BANACH space
- $W_0^{1,p}$ is REFLEXIVE for $1 < p < +\infty$
- H_0^1 is a SEPARABLE HILBERT space

(These follow from PROPOSITION 7.16 and the fact that $W_0^{1,p}$ is closed by definition)

REMARK By THEOREM 7.24 we know that $C_c^1(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$. Therefore

$$W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R}).$$

THEOREM 7.34 Let $I \subseteq \mathbb{R}$ be open, $1 \leq p < +\infty$. They are equivalent:

(a) $u \in W_0^{1,p}(I)$

(b) $u = 0$ on ∂I

We only prove the easy implication of THEOREM 7.34, that is, (a) $\Rightarrow$ (b).

Proof

(a) $\Rightarrow$ (b): By definition, if $u \in W_0^{1,p}(I)$ there $\exists \{u_n\} \subseteq C_c^1(I)$ s.t. $u_n \to u$ strongly in $W^{1,p}(I)$. By the embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$ (THEOREMS 7.19 and 7.27) we get that $u_n \to u$ uniformly in $\bar{I}$. As $u_n = 0$ on ∂I we then conclude $u = 0$ on ∂I.

(b) $\Rightarrow$ (a): See THEOREM 8.12 in BREZIS – "Functional Analysis, Sobolev Spaces and PDE", SPRINGER 2011. $\square$

POINCARÉ INEQUALITIES

THEOREM 7.35 (POINCARÉ INEQUALITY)

Let $I=(a,b)$ be bounded, $1 \le p < +\infty$. There $\exists$ $C>0$ (depending only on $|I|$) s.t.

$$\|u\|_{W^{1,p}(I)} \le C \, \|u'\|_{L^p(I)} , \quad \forall u \in W_0^{1,p}(I).$$

In particular $\|u\|_{W^{1,p}(I)}$ and $\|u'\|_{L^p(I)}$ are equivalent norms on $W_0^{1,p}(I)$.

We give two proofs: the first one is more direct, while the second one is more abstract, but useful for proving generalizations.

WARNING The Poincaré Inequality does not hold in $W^{1,p}(a,b)$ (think of constants)

Proof 1 Let $u \in W_0^{1,p}(a,b)$. As $u(a)=0$ by THEOREM 7.34, we get

$$|u(x)| = |u(x) - u(a)|$$

(Here use THEOREM 7.19) $\longrightarrow$
$$= \left| \int_a^x u'(x)\, dx \right| \le \|u'\|_{L^1(a,b)}$$

Therefore $\|u\|_{L^\infty(a,b)} \le \|u'\|_{L^1(a,b)}$. Then

$$(*) \quad \|u\|^p_{L^p(a,b)} = \int_a^b |u|^p \, dx \le (b-a) \, \|u\|^p_{L^\infty(a,b)} \le (b-a) \, \|u'\|^p_{L^1(a,b)}$$

By Hölder's inequality we get

$$(**) \quad \|u'\|_{L^1(a,b)} \le \left(\int_a^b |u'|^p \, dx \right)^{1/p} \left(\int_a^b 1^{p'} \, dx \right)^{1/p'}$$

$$= \|u'\|_{L^p(a,b)} \, (b-a)^{1/p'}$$

Thus, by (*),

$$\|u\|_{L^p(a,b)} \overset{(*)}{\leq} (b-a)^{1/p} \|u'\|_{L^1(a,b)}$$

Since $\frac{1}{p} + \frac{1}{p'} = 1$

$$\overset{(**)}{\leq} (b-a)^{1/p + 1/p'} \|u'\|_{L^p(a,b)} = (b-a) \|u'\|_{L^p(a,b)}$$

Now

$$\|u\|_{W^{1,p}(a,b)} = \|u\|_{L^p(a,b)} + \|u'\|_{L^p(a,b)} \leq (b-a+1) \|u'\|_{L^p(a,b)}$$

therefore we conclude setting $C := b-a+1 = |I|+1$. □

Proof 2 Assume by contradiction that the inequality does not hold. Then we can find a sequence $\{u_n\} \subseteq W^{1,p}_0(a,b)$ s.t.

$$(*) \qquad \|u_n\|_{L^p} \geq n \|u_n'\|_{L^p}, \qquad \forall n \in \mathbb{N}.$$

As the norm is homogeneous, up to rescaling u_n by $\|u_n\|_{L^p}$, we can assume that $\|u_n\|_{L^p} = 1$, $\forall n \in \mathbb{N}$. Then, from (*), we get

$$(**) \qquad \|u_n\|_{L^p} = 1, \qquad \|u_n'\|_{L^p} \leq \frac{1}{n}, \qquad \forall n \in \mathbb{N}.$$

In particular $\{u_n\}$ is bounded in $W^{1,p}(a,b)$. By the SOBOLEV EMBEDDING THM 7.27 (point (c)) we know that $W^{1,p}(a,b) \hookrightarrow L^p(a,b)$ compactly. Thus $\overline{\{u_n\}}$ is compact in $L^p(a,b)$. In particular $\{u_n\}$ admits a subsequence s.t.

$$u_{n_k} \to u \text{ strongly in } L^p(a,b).$$

Moreover, from (**) we know that $\|u_{n_k}'\|_{L^p} \leq \frac{1}{n_k}$, $\forall k \in \mathbb{N}$. Therefore

$$u_{n_k}' \to 0 \text{ strongly in } L^p(a,b).$$

Thus, from REMARK 7.17 we conclude that $u_{n_k} \to u$ strongly in $W^{1,p}(a,b)$, with $u' = 0$ in the weak sense.

Therefore, by definition of weak derivative, we get

$$\int_a^b u\,\varphi'\,dx = -\int_a^b u'\varphi\,dx \overset{u'=0}{=} 0 \quad , \qquad \forall\, \varphi \in C_c^1(a,b)$$

and the DBR LEMMA 7.13 implies that $u = c$ a.e. on (a,b), for some $c \in \mathbb{R}$.

Now recall that $W_0^{1,p}(a,b)$ is closed by definition, therefore, as $u_{n_k} \to u$ in $W^{1,p}(a,b)$, and $\{u_n\} \subseteq W_0^{1,p}(a,b)$, we get that $u \in W_0^{1,p}(a,b)$.

By THEOREM 7.34 we then have $u(a) = u(b) = 0$. Since $u = c$, this implies $c = 0$ and

$$u \equiv 0 .$$

However, taking the limit as $k \to +\infty$ in the first condition in (**) gives

$$\|u\|_{L^p} = 1 ,$$

which is a contradiction, as $u \equiv 0$. $\square$

When dealing with BC which are more general than homogeneous Dirichlet BC, the above Poincaré inequality is useless.

Therefore we look for a more general version. In order to do that, notice that the Poincaré Inequality

$$\|u\|_{W^{1,p}(I)} \leq C\,\|u'\|_{L^p(I)} \quad , \qquad \forall\, u \in W_0^{1,p}(I)$$

holds because non-zero constant functions do not belong to $W_0^{1,p}$.

This simple observation motivates the following generalization of THEOREM 7.35.

THEOREM 7.36 (GENERALIZED POINCARÉ INEQUALITY)

Let $I=(a,b)$ be bounded, $1\le p<+\infty$. Let $V\subseteq W^{1,p}(I)$ be a SUBSPACE s.t.

(i) V is closed in $W^{1,p}(I)$

(ii) If $u\in V$ is constant, then $u\equiv 0$.

Then there $\exists\ C>0$

$$\|u\|_{W^{1,p}(I)} \le C\,\|u'\|_{L^p(I)}, \quad \forall u\in V.$$

In particular $\|u\|_{W^{1,p}(I)}$ and $\|u'\|_{L^p(I)}$ are equivalent norms on V.

(The proof of THEOREM 7.36 can be obtained following the lines of PROOF 2 of THEOREM 7.35. It is left for exercise in the Exercises Course).

EXAMPLE 7.37 We give some examples of subspaces $V\subseteq W^{1,p}$ satisfying the assumptions of THEOREM 7.36:

- $V=\{u\in W^{1,p}(a,b) \mid u(p)=0\}$ for $p\in[a,b]$ fixed
  (V is closed by the embedding $W^{1,p}(a,b)\hookrightarrow C[a,b]$)

- $V=\{u\in W^{1,p}(a,b) \mid \int_a^b u\,dx=0\}$

- $V=\{u\in W^{1,p}(a,b) \mid \int_E u\,dx=0\}$, for $E\subseteq[a,b]$ with $|E|>0$

8. EULER-LAGRANGE EQUATION, SOBOLEV CASE

We now analyze variational problems in Sobolev space. First we generalize the following theorems we proved in the C^1 setting: consider the spaces

$$X = \{u \in C^1[a,b] \mid u(a) = \alpha,\ u(b) = \beta\}, \qquad V = \{u \in C^1[a,b] \mid u(a) = u(b) = 0\},$$

the Lagrangian $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, \xi)$, and the functional

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx, \qquad u \in X.$$

① **THEOREM 4.5:** L continuous and C^1 WRT s, ξ.

1) If u_0 minimizes F over X then u_0 solves

(INTEGRAL ELE) $$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0, \quad \forall v \in V}$$

2) If $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ minimizes F over X, then u_0 solves

(ELE) $$\boxed{\begin{cases} \dfrac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0)\right] = L_s(x, u_0, \dot{u}_0), & \forall x \in [a,b] \\ u_0(a) = \alpha,\ u_0(b) = \beta \end{cases}}$$

② **THEOREM 5.4:** $L \in C^1$, $u_0 \in X$ solution to (INTEGRAL ELE).

1) If L is CONVEX in s, ξ then u_0 is minimizer of F.

2) If L is STRICTLY CONVEX in s, ξ, then u_0 is the UNIQUE minimizer of F.

We start by relaxing the assumptions on L, by just requiring measurability. Precisely, we will require L to be a Carathéodory function:

DEFINITION 8.1 $\Omega \subseteq \mathbb{R}^d$ open, $L: \Omega \times \mathbb{R}^n \to \mathbb{R}$. We say that L is a CARATHÉODORY FUNCTION if

1) $y \mapsto L(x,y)$ is continuous for a.e. $x \in \Omega$,

2) $x \mapsto L(x,y)$ is Lebesgue measurable for all $y \in \mathbb{R}^n$.

NOTATION Let $L: (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$. Whenever we say that the Lagrangian L is Carathéodory we mean that

$$\Omega = (a,b), \quad d = 1, \; n = 2 \quad \text{and} \quad y = (s,\xi)$$

in DEFINITION 8.1.

EXAMPLE $L: (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $L(x,s,\xi) := \alpha(x) + g(s,\xi)$ is Carathéodory if $\alpha: (0,1) \to \mathbb{R}$ is measurable and $g: \mathbb{R}^2 \to \mathbb{R}$ is continuous.

PROPOSITION 8.2 Let $\Omega \subseteq \mathbb{R}^d$ be open, $L: \Omega \times \mathbb{R}^n \to \mathbb{R}$ Carathéodory, $u: \Omega \to \mathbb{R}^n$ measurable. Then $g: \Omega \to \mathbb{R}$ defined by

$$g(x) := L(x, u(x))$$

is measurable.

(Proof is omitted. It is obvious by approximation by step functions – see PROPOSITION 3.7 in the book by Dacorogna).

WEAK EULER-LAGRANGE EQUATION

Let $p \geq 1$, $a<b$, and define the space

$$X := \{ u \in W^{1,p}(a,b) \mid u(a)=\alpha, \; u(b)=\beta \}$$

Note
- X is well-defined, since $W^{1,p}$ functions are continuous by THEOREM 7.19 (so $u(a)$ and $u(b)$ make sense)
- X is an AFFINE space with reference vector space $W_0^{1,p}(a,b)$ (since functions in $W_0^{1,p}(a,b)$ vanish on a,b, by THEOREM 7.34).

Let $L: (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F: W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

WEAK DERIVATIVE

The Sobolev version of THEOREM 4.5 is as follows:

ASSUMPTION 8.3 Assume L, L_s, L_ξ are Carathéodory functions.
Suppose that either of the following holds:

(H1) $\forall R>0$, $\exists$ $\alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $p' := \frac{p}{p-1}$, $\beta = \beta(R)$ such that $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

$$|L(x,s,\xi)|, \; |L_s(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$$

$$|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1}$$

(H2) $\forall R>0$, $\exists$ $\alpha_1 \in L^1(a,b)$, $\beta = \beta(R)$ such that $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

$$|L(x,s,\xi)|, \; |L_s(x,s,\xi)|, \; |L_\xi(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$$

THEOREM 8.4 Suppose the above ASSUMPTION 8.3 holds.
Let $u_0 \in X$ be a minimizer for F over X.

1) If (H1) holds then u_0 satisfies the weak form of ELE

(W-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx = 0, \quad \forall v \in W_0^{1,p}(a,b)}$$

2) If (H2) holds then u_0 satisfies the weaker form of ELE

(W'-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx = 0, \quad \forall v \in C_c^\infty(a,b)}$$

3) If in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ then u_0 satisfies the classical ELE

(ELE)
$$\boxed{\frac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0)\right] = L_s(x, u_0, \dot{u}_0), \quad \forall x \in [a,b]}$$

Proof **Step 1. F is well-defined:** let $u \in W^{1,p}(a,b)$. Then both u and $\dot{u}$ are measurable. Since L is Carathéodory, by PROPOSITION 8.2 we get that $g : (a,b) \to \mathbb{R}$ defined by

$$g(x) := L(x, u(x), \dot{u}(x))$$

is measurable. Thus g can be integrated, with the integral possibly being unbounded.

Next we need to show that F is bounded.

Since $W^{1,p}(a,b) \hookrightarrow L^\infty(a,b)$ (THEOREM 7.27), we get $u \in L^\infty(a,b)$. Therefore

$$|u(x)| \leq \|u\|_\infty \quad \text{a.e. in } (a,b).$$

Choose $R = \|u\|_\infty$ in (H1) or (H2), so that there exist $\alpha_1 \in L^1(a,b)$, $\beta = \beta(R)$ s.t.

$$(*) \qquad |L(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p, \quad \forall x \in (a,b),\ |s| \leq \|u\|_\infty,\ \xi \in \mathbb{R}.$$

Thus $(x, u(x), \dot{u}(x)) \in (a,b) \times [-\|u\|_\infty, \|u\|_\infty] \times \mathbb{R}$, and

$$|F(u)| \leq \int_a^b |L(x, u(x), \dot{u}(x))| \, dx$$

$\rightarrow \alpha_1 \in L^1,\ \dot{u} \in L^p$

$$\overset{(*)}{\leq} \int_a^b \alpha_1(x)\, dx + \beta \int_a^b |\dot{u}|^p \, dx < +\infty$$

showing that F is well-defined.

Step 2. Gâteaux derivative of F:

CASE OF (H1) : Assume (H1). We show that for every $u \in W^{1,p}$ the functional F is Gâteaux differentiable in every direction $v \in W^{1,p}$, by proving that

$$(**) \qquad \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \, dx$$

Since we are assuming that L_s, L_ξ are Carathéodory, this means that the maps

$$(s,\xi) \mapsto L_s(x,s,\xi) \quad , \quad (s,\xi) \mapsto L_\xi(x,s,\xi)$$

are continuous for a.e $x \in (a,b)$ fixed. Therefore we can apply the standard chain rule to conclude that the map

$$t \longmapsto L(x, u+t\varepsilon v, \dot{u}+\varepsilon t \dot{v})$$

is differentiable, with

$$\frac{d}{dt}\left\{ L(x, u+t\varepsilon v, \dot{u}+\varepsilon t\dot{v}) \right\} = \varepsilon L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v + \varepsilon L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, \dot{v} \tag{$**$}$$

Now set

$$g(x,\varepsilon) := \int_0^1 L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v + L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, \dot{v} \; dt$$

Then

$$\frac{1}{\varepsilon}\left\{ F(u+tv) - F(u) \right\} \overset{\text{def of } F}{=} \frac{1}{\varepsilon}\int_a^b \left\{ L(x, u+tv, \dot{u}+t\dot{v}) - L(x,u,\dot{u}) \right\} dx$$

$$\text{(Fundamental Thm of Calculus)} \quad = \frac{1}{\varepsilon}\int_a^b \left[\int_0^1 \frac{d}{dt}\left\{ L(x, u+t\varepsilon v, \dot{u}+\varepsilon t\dot{v}) \right\} dt \right] dx$$

$$\text{(by } ** \text{ and def of } g\text{)} \quad = \int_a^b g(x,\varepsilon)\, dx$$

In order to prove (**) it is then sufficient to show that

(C) $$\lim_{\varepsilon\to 0}\int_a^b g(x,\varepsilon)\,dx = \int_a^b \underbrace{L_s(x,u,\dot u)\,v + L_\xi(x,u,\dot u)\,\dot v}_{=\,g(x,0)\ \text{by definition of } g}\,dx$$

IDEA To show (C) we use DOMINATED CONVERGENCE: i.e., we need to show

(A) $\lim_{\varepsilon\to 0} g(x,\varepsilon) = g(x,0)$ for a.e. $x\in(a,b)$

(B) $\sup_{0<\varepsilon<1} |g(x,\varepsilon)| \leq |\Lambda(x)|$ for a.e. $x\in(a,b)$, for some $\Lambda\in L^1(a,b)$

To do that, first notice that by the embedding $W^{1,p}(a,b)\hookrightarrow L^\infty(a,b)$ we get $u+\varepsilon t v\in L^\infty(a,b)$ for all $\varepsilon>0$, $t\in[0,1]$.
In particular, for $0<\varepsilon<1$, $t\in[0,1]$ we get

(B) $|u(x)+\varepsilon t v(x)| \leq \|u\|_\infty + \|v\|_\infty$ a.e. on (a,b).

Thus set $R:=\|u\|_\infty+\|v\|_\infty$ in (H1), to obtain the existence of $\alpha_1\in L^1(a,b)$, $\alpha_2\in L^{p'}(a,b)$, $\beta=\beta(R)$ s.t.

(1) $|L(x,s,\xi)| \leq \alpha_1(x)+\beta|\xi|^p$

(2) $|L_s(x,s,\xi)| \leq \alpha_1(x)+\beta|\xi|^p$, $\forall\, x\in(a,b)$, $|s|\leq R$, $\xi\in\mathbb{R}$

(3) $|L_\xi(x,s,\xi)| \leq \alpha_2(x)+\beta|\xi|^{p-1}$

We now show Ⓐ : need DOMINATED CONVERGENCE, as $g(x,\varepsilon)$ is itself an integral.
For a.e. $x \in (a,b)$ we know that the maps

$$(s,\xi) \mapsto L_s(x,s,\xi), \qquad (s,\xi) \mapsto L_\xi(x,s,\xi)$$

are continuous (as L_s, L_ξ Carathéodory). Therefore

$$\lim_{\varepsilon \to 0} \left\{ L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon \dot{v})\, v + L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v} \right\} =$$

$$= L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\,\dot{v}$$

for all $t \in [0,1]$ and a.e. $x \in (0,1)$.
Moreover, as $u+t\varepsilon v$ satisfies (B), we can invoke (2) to get

$$|L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v| \overset{(2)}{\leq} \left[\alpha_1(x) + \beta\, |\dot{u}+t\varepsilon\dot{v}|^p\right] |v|$$

$$\left(\begin{array}{l}\text{as } \varepsilon, t \in (0,1), \text{ and using} \\ (a+b)^p \leq 2^{p-1}(a^p+b^p) \text{ for } p \geq 1\end{array}\right) \leq \left[\alpha_1(x) + \beta\, 2^{p-1}(|\dot{u}|^p + |\dot{v}|^p)\right] |v|, \quad \forall t \in [0,1]$$

and the RHS belongs to $L^1(0,1)$ since x is fixed.
Similarly, using (3), one also shows that

$$|L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v}| \leq C(x), \quad \forall t \in [0,1]$$

so that $C(x) \in L^1(0,1)$, being a constant (x is fixed). Then by DOMINATED CONVERGENCE

$$\lim_{\varepsilon \to 0} g(x,\varepsilon) = \lim_{\varepsilon\to 0} \int_0^1 L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v + L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v}\, dt$$

$$= \int_0^1 L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\,\dot{v}\, dt = g(x,0)$$

showing Ⓐ.

We now prove Ⓑ : we need to estimate $g(x,\varepsilon)$:

$$|g(x,\varepsilon)| \leq \int_0^1 |L_s(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, v|\, dt + \int_0^1 |L_\xi(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, \dot v|\, dt$$

For the first integral we use (2):

$$\int_0^1 |L_s(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, v|\, dt \overset{(2)}{\leq} \int_0^1 \left[\alpha_1(x) + \beta |\dot u + t\varepsilon \dot v|^p\right] |v(x)|\, dt$$

(as $\varepsilon, t \in (0,1)$ and using $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $p \geq 1$)

$$\leq \int_0^1 \left[\alpha_1(x) + \beta 2^{p-1}\left(|\dot u|^p + |\dot v|^p\right)\right] |v(x)|\, dt$$

(as nothing depends on t anymore)

$$= \left[\alpha_1(x) + \beta 2^{p-1}\left(|\dot u|^p + |\dot v|^p\right)\right] |v(x)|$$

(since $v \in W^{1,p} \hookrightarrow L^\infty$)

$$\leq \underbrace{\left[\alpha_1(x) + \beta 2^{p-1}\left(|\dot u|^p + |\dot v|^p\right)\right]}_{\in L^1(a,b) \text{ since } \alpha_1 \in L^1(a,b),\ \dot u, \dot v \in L^p(a,b)} \|v\|_\infty$$

For the second integral we use (3):

$$\int_0^1 |L_\xi(x, u+t\varepsilon v, \dot u + t\varepsilon \dot v)\, \dot v|\, dt \overset{(3)}{\leq} \int_0^1 \left[\alpha_2(x) + \beta |\dot u + \varepsilon t \dot v|^{p-1}\right] |\dot v|\, dt$$

(the first term does not depend on t) →

$$= \underbrace{\alpha_2(x)\, |\dot v(x)|}_{\substack{\in L^1(a,b) \text{ by Hölder} \\ \text{as } \alpha_2 \in L^{p'},\ \dot v \in L^p}} + \underbrace{\beta |\dot v(x)| \int_0^1 |\dot u + \varepsilon t \dot v|^{p-1}\, dt}_{\substack{\text{This one is estimated} \\ \text{separately below}}}$$

$$\beta |\dot v(x)| \int_0^1 |\dot u + \varepsilon t \dot v|^{p-1}\, dt \leq \sup_{t\in[0,1]} \underbrace{\beta |\dot v(x)|\, |\dot u(x) + \varepsilon t \dot v(x)|^{p-1}}_{\substack{\in L^1(a,b) \text{ by Hölder, as} \\ |\dot u + \varepsilon t \dot v(x)|^{p-1} \in L^{p'}(a,b) \text{ since } \dot u + \varepsilon t \dot v \in L^p}}$$

Thus, $\exists\, \Lambda \in L^1(a,b)$ s.t. $|g(x,\varepsilon)| \leq \Lambda(x)$ for a.e. $x \in (a,b)$, $\forall\, 0<\varepsilon<1$, showing Ⓑ.

Using the same argument of PROPOSITION 2.3 it is immediate to check that the above implies

$$F_g'(u_0)(v) = 0 .$$

Therefore we conclude that

$$\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx = 0 \qquad , \qquad \forall v \in W_0^{1,p}(a,b)$$

proving that u_0 solves (W-ELE).

- Assume (H2). For what already proved, F is gâteaux differentiable at u_0 $\forall$ directions in $C^\infty(a,b)$, with

$$F_g'(u_0)(v) = \int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx \quad , \quad \forall v \in C^\infty(a,b)$$

Let $v \in C_c^\infty(a,b)$ be arbitrary. Thus $u_0 + \varepsilon v \in X$, $\forall \varepsilon \in \mathbb{R}$ (as $v(a) = v(b) = 0$). Since u_0 is a minimizer, as above we can show $F_g'(u_0)(v) = 0$, i.e.

$$\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \, dx = 0 \qquad , \qquad \forall v \in C_c^\infty(a,b)$$

proving that u_0 solves (W'-ELE).

Then (C) follows by DOMINATED CONVERGENCE, showing that F is gâteaux diff. at each $u \in W^{1,p}(a,b)$ with

$$F'_g(u)(v) = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \, dx \quad , \quad \forall v \in W^{1,p}(a,b)$$

CASE OF (H2) : Assume (H2). By similar arguments we can show that F is Gâteaux differentiable for every $u \in W^{1,p}$, in every direction $v \in C^\infty(a,b)$, with

$$F'_g(u)(v) = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \, dx$$

The difference WRT the case of (H1) is that now the bound on L_ξ is different, but since $v \in C^\infty(a,b)$ (not $v \in W^{1,p}$ as in the previous case) all the estimates work.

Step 3. Show ELE : Suppose now that $u_0 \in X$ minimizes F over X.

- Assume (H1). For what already proved, F is gâteaux differentiable at u_0 $\forall$ directions in $W^{1,p}(a,b)$, with

$$F'_g(u_0)(v) = \int_a^b L_s(x,u_0,\dot{u}_0)\, v + L_\xi(x,u_0,\dot{u}_0)\, \dot{v} \, dx \quad , \quad \forall v \in W^{1,p}(a,b)$$

Let $v \in W^{1,p}_0(a,b)$ be arbitrary. Thus $u_0 + \varepsilon v \in X$, $\forall \varepsilon \in \mathbb{R}$ (as $v(a) = v(b) = 0$)
Since u_0 is a minimizer, we get

$$F(u_0) \le F(u_0 + \varepsilon v)$$

- Assume that in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$. Since L satisfies at least one between (H1) and (H2) by assumption, we deduce that u_0 solves either (W-ELE) or (W'-ELE). In both cases, we have

$$\int_a^b L_s(x,u_0,\dot{u}_0)\, v + L_\xi(x,u_0,\dot{u}_0)\, \dot{v} \, dx = 0 \qquad , \quad \forall v \in C_c^\infty(a,b)$$

As u_0 and L are C^2, we can integrate by parts the above, and use that $v(a) = v(b) = 0$ to get

$$\int_a^b \left\{ L_s(x,u_0,\dot{u}_0) - \left[L_\xi(x,u_0,\dot{u}_0) \right]' \right\} v \, dx = 0 \qquad , \quad \forall v \in C_c^\infty(a,b)$$

By the standard FLCV LEMMA 3.4 we deduce (ELE) $\square$

LESSON 11 - 26 MAY 2021

OTHER BOUNDARY CONDITIONS

So far we only dealt with Dirichlet Boundary Conditions. What about other BC? For example one could set the problem in

$$\{ u \in W^{1,p}(a,b) \mid u(a) = \alpha \}$$

This space is well-defined, since Sobolev functions are continuous (THEOREM 7.19).

REMARK Inspecting the proof of THEOREM 8.4 we notice:

- Step 1 - F is well-defined: here we only used the growth assumptions on L to prove that $|F(u)| < +\infty$ $\forall\, u \in W^{1,p}(a,b)$

- Step 2 - Gâteaux derivative: Here we used (H1) and (H2) separately:

  - Assuming (H1) we proved that F is Gâteaux differentiable at each $u \in W^{1,p}(a,b)$, in every direction $v \in W^{1,p}(a,b)$

  - Assuming (H2) we proved that F is Gâteaux differentiable at each $u \in W^{1,p}(a,b)$, in every direction $v \in C^\infty(a,b)$

- Step 3 - Showing ELE: Here we used the BC

  - Assuming (H1), we chose the variations $v \in W_0^{1,p}(a,b)$. This allowed to deduce (W-ELE), since F was diff. in every direction in $W^{1,p}$

  - Assuming (H2), we chose the variations $v \in C_c^\infty(a,b)$. This allowed to deduce (W'-ELE), since F was diff. in every direction in C^∞

Therefore, we deduce the following general result.

THEOREM 8.5 Let $p \geq 1$, $a<b$. Let $X \subseteq W^{1,p}(a,b)$ be an AFFINE SPACE with reference vector space $V \subseteq W^{1,p}(a,b)$.

Suppose $L:(a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies ASSUMPTION 8.3.
Define $F: W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx .$$

Let $u_0 \in X$ be a minimizer for F over X. Then:

1) If (H1) holds then u_0 satisfies the weak form of ELE

(W-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0, \quad \forall v \in V}$$

2) If (H2) holds then u_0 satisfies the weaker form of ELE

(W'-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0, \quad \forall v \in C^\infty(a,b) \cap V}$$

3) If in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ then u_0 satisfies the classical ELE

(ELE)
$$\begin{cases} \dfrac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0)\right] = L_s(x, u_0, \dot{u}_0), & \forall x \in [a,b] \\ \text{BC from the integration by parts of (W'-ELE)} & \end{cases}$$

SUFFICIENT CONDITIONS FOR MINIMALITY

We now address the generalization of THEOREM 5.4. Let us recall the setting:

Let $p \geq 1$, $a < b$, and let $X \subseteq W^{1,p}(a,b)$ be an affine space over $V \subseteq W^{1,p}(a,b)$.

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, \xi)$ and define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx$$

The Sobolev version of THEOREM 5.4 (with general BC) is as follows:

THEOREM 8.6 Suppose L satisfies Assumption 8.3.

Assume $u_0 \in X$ solves (W-ELE) or (ELE) in THEOREM 8.5.

1) If

$(s, \xi) \longmapsto L(x, s, \xi)$ is CONVEX for a.e. $x \in (a,b)$

then u_0 is a minimizer for F on X.

2) If

$(s, \xi) \longmapsto L(x, s, \xi)$ is STRICTLY CONVEX for a.e. $x \in (a,b)$

then u_0 is the UNIQUE minimizer for F on X.

(The proof carries out exactly like the one of THEOREM 5.4, with straightforward changes. THEOREM 5.2 can be used because L_s, L_ξ are Carathéodory. Hence L is C^1 WRT (s, ξ), for a.e. $x \in (a,b)$ fixed).

9. DIRECT METHOD

GOAL The Direct Method is used to prove existence of minimizers

We state a very general version of the direct method, for functionals $F: X \to \mathbb{R}$ with X space with a notion of convergence.

DEFINITION 9.1 Let X be a set, and $Seq(X)$ the set of all sequences in X:

$$Seq(X) := \{ f: \mathbb{N} \to X \}.$$

A NOTION OF CONVERGENCE on X is a subset N of

$$\underbrace{Seq(X)}_{\text{contains sequences}} \times \underbrace{X}_{\text{contains limits}}$$

NOTATION Thus a notion of convergence is a list of sequences with corresponding limit. Therefore, whenever we say that $\{x_n\} \subseteq X$ converges to $x_0 \in X$, in symbols $x_n \to x_0$, we mean that $(\{x_n\}, x_0) \in N$ notion of convergence.

EXAMPLES

- X = topological space. A notion of convergence is for example the list of all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x_0$ WRT to τ, for some $x_0 \in X$
- The above example contains all the well-known cases: Metric spaces, Normed spaces with weak or strong convergence, Hilbert spaces, $\mathbb{R}^d$.

DEFINITION 9.2 X space with notion of convergence. We say that $K \subseteq X$ is (sequentially) compact if every sequence $\{x_n\} \subseteq K$ admits a subsequence such that $x_{n_k} \to x_0$ with $x_0 \in K$.

DEFINITION 9.3 X space with notion of convergence. A function $f: X \to \mathbb{R}$ is

- **CONTINUOUS** if for all sequences $x_n \to x_0$ we have

$$f(x_n) \to f(x_0).$$

- **LOWER SEMICONTINUOUS (LSC)** if for all $x_n \to x_0$ we have

$$f(x_0) \leq \liminf_{n \to +\infty} f(x_n).$$

THEOREM 9.4 (DIRECT METHOD)

X space with notion of convergence, $f: X \to \mathbb{R}$. Assume that

$$\left.\begin{array}{l} \text{(i) } X \text{ is compact} \\ \\ \text{(ii) } f \text{ is LSC} \end{array}\right\} \text{WRT the SAME notion of convergence}$$

Then the problem

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution: i.e., $\exists \, \hat{x} \in X$ s.t. $f(\hat{x}) = I$.

Proof Exactly like the Weierstraß Theorem of Analysis 1.

By the properties of the infimum $\exists$ infimizing sequence $\{y_n\} \subseteq f(X)$ s.t.

$$y_n \to I.$$

Note that, a priori, $I \in [-\infty, +\infty]$.

By def. of image $\exists \{x_n\} \subseteq X$ s.t. $y_n = f(x_n)$, $\forall n \in \mathbb{N}$. Thus

$$f(x_n) \to I.$$

As X is compact, there $\exists$ a subsequence s.t. $x_{n_k} \to \hat{x}$ for some $\hat{x} \in X$. Then

$$I \leq f(\hat{x}) \leq \liminf_{k \to +\infty} f(x_{n_k}) = \lim_{n \to +\infty} f(x_n) = I$$

I is inf — f is LSC and $x_{n_k} \to \hat{x}$ — $f(x_n)$ is convergent

Thus $f(\hat{x}) = I$, concluding that $\hat{x}$ is a minimizer and that I is finite. $\square$

REMARK The direct method is deceivingly simple. The highly non-trivial task is finding a notion of convergence on X s.t. (i)-(ii) hold. Note that:

- If we have many convergent sequences then (i) is easy and (ii) hard
- If we have few convergent sequences then (i) is hard and (ii) easy

Therefore (i) and (ii) are in competition, and and finding a notion of convergence s.t. both hold is delicate.

Let us now see some variants of the direct method.

DEFINITION 9.5 X space with notion of convergence. A function $f: X \to \mathbb{R}$ is **COERCIVE** if $\exists K \subseteq X$ compact s.t.

$$\inf \{ f(x) \mid x \in K \} = \inf \{ f(x) \mid x \in X \}$$

EXAMPLE $X = \mathbb{R}$, $f(x) = x^2$. Then f is coercive (i.e. $K = [-1, 1]$)

THEOREM 9.6 X space with notion of convergence, $f: X \to \mathbb{R}$ s.t.

(i) f is COERCIVE
(ii) f is LSC

Then

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution.

Proof As f is coercive then by definition $\exists K \subseteq X$ compact s.t.

$$(*) \quad I = \inf \{ f(x) \mid x \in K \}$$

As K is compact and f is LSC on K (as f is LSC on X), we can apply THEOREM 9.4 to obtain $\hat{x} \in K$ s.t. $f(\hat{x}) = \inf \{ f(x) \mid x \in K \}$. Thus, by $(*)$, $f(\hat{x}) = I$ and we conclude. $\square$

THEOREM 9.7 X space with notion of convergence, $f: X \to \mathbb{R}$ s.t.

(i) $\exists M \geq 0$, $\exists K \subseteq X$ compact s.t. $\{x \in X \mid f(x) \leq M\} \neq \emptyset$ and

$$\{x \in X \mid f(x) \leq M\} \subseteq K$$

(ii) f is LSC

Then

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution $\hat{x} \in X$ s.t. $f(\hat{x}) \leq M$.

Proof We want to show that $\tilde{K} := \{ x \in X \mid f(x) \le M \}$ is compact. So let $\{x_n\} \subseteq \tilde{K}$.
As $\tilde{K} \subseteq K$ and K is compact, there $\exists$ a subsequence and $x_0 \in K$ s.t.

$$x_{n_k} \to x_0 .$$

Since $\{x_{n_k}\} \subseteq \tilde{K}$ and f is LSC, we get

$$f(x_0) \le \liminf_{n \to +\infty} f(x_{n_k}) \le M$$

f LSC and $x_{n_k} \to x_0$ As $\{x_{n_k}\} \subseteq \tilde{K}$

proving that $x_0 \in \tilde{K}$ and so that $\tilde{K}$ is compact. We now have two cases:

- $I = M$: then by def. of $\tilde{K}$ and of infimum

$$\tilde{K} = \{ x \in X \mid f(x) \le I \} = \{ x \in X \mid f(x) = I \}.$$

Thus $\tilde{K}$ is exactly set of minimizers. Since $\tilde{K} \ne \emptyset$ by assumption, we conclude.

- $I < M$: Let $\{x_n\} \subseteq X$ be an infimizing sequence, i.e., such that

$$f(x_n) \to I .$$

Since $I < M$, we conclude that $\{x_n\} \subseteq \tilde{K}$. (upon discarding a finite number of indices). As $\tilde{K}$ is compact, $\exists$ a subsequence and $\hat{x} \in \tilde{K}$ s.t. $x_{n_k} \to \hat{x}$. Now

$$I \le f(\hat{x}) \le \liminf_{k \to +\infty} f(x_{n_k}) = \lim_{n \to +\infty} f(x_n) = I ,$$

def of inf $x_{n_k} \to \hat{x}$ and f is LSC   as $f(x_n)$ is convergent

Thus $f(\hat{x}) = I$ and we conclude that $\hat{x}$ is a minimizer. $\square$

DIRECT METHOD – ACTION PLAN

Given $F : X \to \mathbb{R}$, the minimization problem

$$(P) \qquad \inf\{F(x) \mid x \in X\}$$

can be studied in the following way:

1. **WEAK FORMULATION**: Extend F to a functional $\hat{F} : \hat{X} \to \mathbb{R}$ with $X \subseteq \hat{X}$, i.e., to a larger space (Typically $\hat{X}$ will by a SOBOLEV SPACE, rather than the usual C^1, C^∞ or C^1_{pw})

2. **COMPACTNESS**: Prove that the sublevel sets of $\hat{F}$ are compact WRT some appropriate notion of convergence on $\hat{X}$

3. **LOWER SEMICONTINUITY**: Prove that $\hat{F}$ is LSC WRT the same notion of convergence of point 2.

4. **REGULARITY**: At this point one can apply THEOREM 9.7 and conclude the $\exists$ of a solution $\bar{x} \in \hat{X}$ to

$$\inf\{\hat{F}(x) \mid x \in \hat{X}\}$$

The last step consists in showing that

$$\bar{x} \text{ is more regular, i.e., } \bar{x} \in X$$

Note that, as $\hat{F} = F$ on X, this immediately implies that $\bar{x}$ solves the original minimization problem (P)

EXAMPLE 9.8 Set $X = \{ u \in C^1[0,1] \mid u(0)=0, \; u(1)=1 \}$ and

$$F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx \,, \qquad u \in X.$$

Note that the Lagrangian appearing in F is non-linear. Thus the associated ELE is hard (maybe impossible) to solve explicitly).

We then resort to our ACTION PLAN for the DIRECT METHOD:

① **WEAK FORMULATION**: We extend F to the larger space

$$\hat{X} := \{ u \in H^1(0,1) \mid u(0)=0, \quad u(1)=1 \}.$$

Note that $\hat{X}$ is well-defined, since H^1 functions are continuous by THEOREM 7.19. Therefore the Dirichlet Boundary conditions appearing in $\hat{X}$ make sense.

The extension of F to $\hat{X}$ is trivially defined by

$$\hat{F}(u) := \int_0^1 \underbrace{\dot{u}^2}_{\text{WEAK DERIVATIVE}} + \sin(u^5)\, dx \,, \qquad \forall u \in H^1(0,1),$$

Note that $\hat{F}$ is well-defined, since

- $\dot{u} \in L^2(0,1)$ as $u \in H^1(0,1)$
- $\sin(u^5) \in L^1(0,1)$ as $H^1(0,1) \hookrightarrow L^\infty(0,1)$ by the SOBOLEV EMBEDDING THEOREM 7.27 (or, more simply, because $|\sin x| \leq 1$)

Moreover $\hat{F} = F$ on X, since if $u \in C^1[0,1]$, then its weak derivative $\dot{u}$ coincides a.e. with the classical derivative.

(2) COMPACTNESS : We need to show that there $\exists\ M>0$ s.t. the sublevel

$$K := \{ u \in \hat{X} \mid \hat{F}(u) \leq M \}$$

is non-empty, and compact WRT some notion of convergence on $H^1(0,1)$.
Clearly we can choose $M := F(u)$ with $u(x) := x$, so that $K \neq \emptyset$.

As notion of convergence we take the weak convergence on H^1. We have to show that K is compact. Hence assume that $\{u_n\} \subseteq K$, that is,

$$\{u_n\} \subseteq \hat{X} \quad \text{and} \quad \hat{F}(u_n) \leq M, \qquad \forall\ n \in \mathbb{N}.$$

As $|\sin x| \leq 1$, we get

$$\int_0^1 \dot{u}_n^2\, dx - 1 \leq \int_0^1 \dot{u}_n^2 + \sin(u^5)\, dx = \hat{F}(u_n) \leq M \quad \Rightarrow \quad \boxed{\|\dot{u}_n\|_{L^2} \leq \sqrt{M+1}}$$

Thus $\{\dot{u}_n\}$ is uniformly bounded in $L^2(0,1)$. Since $L^2(0,1)$ is Hilbert separable, by Banach-Alaoglu Theorem we conclude that there $\exists$ a subseq. and $\hat{v} \in L^2(0,1)$ s.t.

$$\dot{u}_n \rightharpoonup \hat{v} \quad \text{weakly in } L^2(0,1).$$

Moreover by the Hölder estimate of THEOREM 7.23 (with $p=2$) we get

$$(*) \quad |u_n(x) - u_n(y)| \leq \|u_n'\|_{L^2} |x-y|^{1/2}$$
$$\leq \sqrt{M+1}\, |x-y|^{1/2}, \qquad \forall x,y \in [0,1],$$

showing that $\{u_n\}$ is EQUI-CONTINUOUS.

Using the boundary condition $u_n(0)=0$ we also get

$$|u_n(x)| = |u_n(x) - u_n(0)| \overset{\circledast}{\leq} \sqrt{M+1} \quad , \qquad \forall x \in [0,1],$$

showing that $\{u_n\}$ is UNIFORMLY BOUNDED in $C[0,1]$. Therefore we can apply ASCOLI-ARZELA' THEOREM 7.28 to conclude that $\overline{\{u_n\}}$ is COMPACT in $C[0,1]$. Then $\exists$ a subsequence and $\hat{u} \in C[0,1]$ s.t.

$$u_{n_k} \to \hat{u} \quad \text{uniformly in } [0,1].$$

In particular $u_{n_k} \to \hat{u}$ strongly in $L^2(0,1)$ and so

$$u_{n_k} \rightharpoonup \hat{u} \quad \text{weakly in } L^2(0,1).$$

Recalling that $\dot{u}_{n_k} \rightharpoonup \hat{v}$ weakly in $L^2(0,1)$, by REMARK 7.18 we get that

$$u_{n_k} \rightharpoonup \hat{u} \text{ weakly in } H^1(0,1) \quad , \quad \text{with } \hat{u}' = \hat{v} \text{ in the weak sense.}$$

In particular $\hat{u} \in H^1(0,1)$, and $\hat{u}(0)=0$, $\hat{u}(1)=1$ by the uniform convergence. Thus $\hat{u} \in \hat{X}$. As norms are weakly lower semicontinuous, we get that

$$\int_0^1 (\hat{u}')^2 \, dx \leq \liminf_{k \to +\infty} \int_0^1 u_{n_k}' \, dx$$

Also, since $u_{n_k} \to \hat{u}$ uniformly,

$$\lim_{k \to +\infty} \int_0^1 \sin(u_n^s) \, dx = \int_0^1 \sin(\hat{u}^s) \, dx$$

Therefore

$$\hat{F}(\hat{u}) = \int_0^1 (\hat{u}')^2 + \sin(\hat{u}^s) \, dx \leq \liminf_{k \to +\infty} \int_0^1 \dot{u}_n^2 + \sin(u_n^s) \, dx = \liminf_{k \to +\infty} \hat{F}(u_n) \leq M$$

showing that $\hat{F}(\hat{u}) \leq M$. Thus $\hat{u} \in K$, proving that K is weakly compact.

③ LOWER SEMICONTINUITY :

(Here we could conclude with the same arguments of point ②. But it is instructive to make a separate argument.)

We need to prove that $\hat{F}$ is lower semicontinuous WRT the weak convergence of H^1, that is,

$$(S) \quad u_n \to u \text{ weakly in } H^1(0,1) \quad \Rightarrow \quad \hat{F}(u) \leq \liminf_{n\to+\infty} \hat{F}(u_n)$$

From the SOBOLEV EMBEDDING THEOREM 7.27 we have $H^1(0,1) \hookrightarrow C[0,1]$ compactly. Now recall that COMPACT OPERATORS transform weakly convergent sequences into strongly convergent sequences (PROPOSITION 7.31). Therefore

$$u_n \to u \text{ weakly in } H^1(0,1) \quad \Rightarrow \quad u_n \to u \text{ uniformly in } [0,1]$$

From the weak lower semicontinuity of the norm, we obtain

$$\int_0^1 \dot{u}^2\,dx \leq \liminf_{n\to+\infty} \int_0^1 \dot{u}_n^2\,dx \qquad \text{(since } \dot{u}_n \to \dot{u} \text{ weakly in } L^2(0,1))$$

Moreover, as $u_n \to u$ uniformly, we also have

$$\int_0^1 \sin(u^5)\,dx = \lim_{n\to+\infty} \int_0^1 \sin(u_n^5)\,dx.$$

Thus $\hat{F}(u) \leq \liminf_{n\to+\infty} \hat{F}(u_n)$, and (S) is proven.

Therefore, by THEOREM 9.7 we conclude the existence of $\bar{u} \in \hat{X}$ s.t.

$$\boxed{\hat{F}(\bar{u}) = \inf\{\hat{F}(u) \mid u \in \hat{X}\}}$$

④ REGULARITY:

We wish to show that $\bar{u} \in \hat{X}$ actually belongs to X, so that it automatically solves the original problem

$$F(\bar{u}) = \inf \{ F(u) \mid u \in X \}$$

CLAIM All minimizers of $\hat{F}$ in $\hat{X}$ belong to $C^\infty(0,1)$.

HOW TO PROVE IT

- 4.1: WRITE THE WEAK ELE FOR $\hat{F}$
- 4.2: SHOW THAT u_0 IS CONTINUOUS (where u_0 is minimizer)
- 4.3: BOOTSTRAP ARGUMENT

Proof of Claim Let $u_0 \in \hat{X}$ be a minimizer for $\hat{F}$. We want to apply THEOREM 8.4 (with $p=2$) to derive the ELE.

Since u_0 is not regular for now, we can only hope that either the WEAK ELE, or worse the VERY WEAK ELE, hold. So let us check ASSUMPTION 8.3.
In our case the Lagrangian is

$$L(x,s,\xi) = \xi^2 + \sin(s^5)$$

- L is C^∞, therefore L, L_s, L_ξ are Carathéodory functions
- We check (H1): we need to show that $\forall R>0$, $\exists\, \alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $\beta = \beta(R)$ s.t.

(1) $|L(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$

(2) $|L_s(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$, $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

(3) $|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1}$

Notice that in our case $a=0$, $b=1$ and $p=2$, so that $p'=2$.
Let us check (1):

$$|L(x,s,\xi)| = |\xi^2 + \sin(s^5)| \leq 1 + \xi^2 \quad , \quad \forall x \in (0,1),\ s,\xi \in \mathbb{R}$$

Therefore it looks like we can choose $\alpha_1 \equiv 1$ independent on x (since the RHS does not depend on x) and, $\beta \equiv 1$ independent on R (since the estimate holds for all $s \in \mathbb{R}$).

Let us see if this choice of α_1 and β works for (2):

$$|L_s(x,s,\xi)| = |5s^4 \cos(s^5)| \leq 5|s|^4$$

This estimate can resemble (2) only if we assume $|s| \leq R$, in which case we get

$$|L_s(x,s,\xi)| \leq 5R^4 \quad , \quad \forall x \in (0,1),\ |s| \leq R,\ \xi \in \mathbb{R}.$$

This is saying that we should have $\alpha_1 \equiv 5R^4$ and $\beta \equiv 0$.

Let us look into (3):

$$|L_\xi(x,s,\xi)| = 2|\xi| \quad , \quad \forall x \in (0,1),\ s \in \mathbb{R},\ \xi \in \mathbb{R}$$

Therefore (3) is satisfied for $\alpha_2 \equiv 0$ and $\beta \equiv 2$.

Then, it is immediate to check that L satisfies (1), (2), (3) for

$$\alpha_1(x) \equiv \max\{1, 5R^4\} \quad , \quad \alpha_2(x) \equiv 0 \quad , \quad \beta = 2$$

Since $\alpha_1, \alpha_2 \in L^1(0,1)$, we get that (H1) holds.

Therefore L satisfies ASSUMPTION 8.3, and since $u_0 \in X$ is a minimizer of $\hat{F}$ over $\hat{X}$, by THEOREM 8.4 we get that u_0 satisfies the WEAK ELE

$$\int_0^1 L_s(x,u,\dot{u})\,v + L_\xi(x,u,\dot{u})\,\dot{v}\,dx = 0, \quad \forall\, v \in W_0^{1,2}(0,1),$$

which in our case reads

$$\int_0^1 5u_0^4 \cos(u_0^5)\,v + 2\dot{u}_0\,\dot{v}\,dx = 0, \qquad \forall\, v \in W_0^{1,2}(0,1).$$

Rearranging we get

$$\text{(W-ELE)} \qquad \int_0^1 \underbrace{2\dot{u}_0}_{f}\,\underbrace{\dot{v}}_{\dot{\varphi}}\,dx = -\int_0^1 \underbrace{5u_0^4\cos(u_0^5)}_{g}\,\underbrace{v}_{\varphi}\,dx, \qquad \forall\, v \in W_0^{1,2}(0,1).$$

Recalling that $C_c^1(0,1) \subseteq W_0^{1,2}(0,1)$, (W-ELE) is saying that $f := 2\dot{u}_0$ is weakly differentiable, with weak derivative given by $g := 5u_0^4\cos(u_0^5)$, that is

$$(*) \qquad (2\dot{u}_0)' = 5u_0^4\cos(u_0^5) \qquad \text{weakly}$$

We use $(*)$ to prove regularity of u_0. Note that

$$\int_0^1 |g|^2\,dx \underset{|\cos x| \le 1}{\le} 25\int_0^1 |u_0|^8\,dx$$

and the RHS is finite, since $u_0 \in W^{1,2}(0,1)$ and $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1)$ continuously by THEOREM 7.27.

This shows $g \in L^2(0,1)$. But then

$$\dot f = g \text{ weakly},\ g \in L^2 \Rightarrow f \in W^{1,2}(0,1) \underset{f = 2\dot u_0}{\Rightarrow} \dot u_0 \in W^{1,2}(0,1)$$

But

$$\dot u_0 \in W^{1,2}(0,1) \overset{\text{THM 7.19}}{\Rightarrow} \dot u_0 \in C[0,1]$$

(Note: here $\dot u_0$ is still a weak derivative)

Now, by PROPOSITION 7.22 we have that, as $u_0 \in W^{1,2}$ and the weak derivative $\dot u_0$ is continuous, then

$$\boxed{u_0 \in C^1[0,1]}$$

Then

$$u_0 \text{ is } C^1 \Rightarrow g = 5u_0^4 \cos(u_0^5) \text{ is } C^1 \Rightarrow g \in C^0$$

(As $(2\dot u_0)' = g$ weakly) $\Rightarrow$ $2\dot u_0$ has continuous weak derivative

$$(\text{PROP 7.22}) \Rightarrow \dot u_0 \in C^1 \Rightarrow u_0 \in C^2$$

this is true because $\dot u_0$ is a classical derivative

Now that we proved $u_0 \in C^2$, we can employ the BOOTSTRAP argument.

BOOTSTRAP: Since now we know $u_0 \in C^2$, the relationship

$$(2\dot{u}_0)' = 5u_0^4 \cos(u_0^5) \qquad \text{weakly}$$

holds in the classical sense (the weak derivative of a diff. function is just the classical derivative), i.e.,

$$(**) \qquad 2\ddot{u}_0 = 5u_0^4 \cos(u_0^5) \qquad , \qquad \forall x \in [0,1].$$

Then, as $u_0 \in C^2$, the RHS of $(**)$ belongs to C^2, and so

$$\ddot{u}_0 \in C^2 \Rightarrow u_0 \in C^4$$

Again, as $u_0 \in C^4$, the RHS of $(**)$ belongs to C^4, and so

$$\ddot{u}_0 \in C^4 \Longrightarrow u_0 \in C^6$$

Proceeding with the botstrap argument we conclude that

$$\ddot{u}_0 \in C^k \Longrightarrow u_0 \in C^{k+2}$$

and therefore $u_0 \in C^\infty(0,1)$. $\square$

To summarize, in EXAMPLE 9.8 we proved the following:

PROPOSITION Let $X := \{ u \in C^1[0,1] \mid u(0)=0, \; u(1)=1 \}$ and

$$F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx .$$

Then F admits a minimizer $\bar{u} \in X \cap C^\infty(0,1)$.

NOTE: The remarkable feature of this ACTION PLAN for the DIRECT METHOD is that we never tried to solve the ELE equation, but just use abstract arguments to prove $\exists$ of a minimizer, and then the structure of ELE to recover Regularity.

LESSON 12 - 2 JUNE 2021

GENERAL EXISTENCE RESULT IN SOBOLEV

Let $p>1$, $a<b$, and consider the space

$$X := \{ u \in W^{1,p}(a,b) \mid u(a)=\alpha, \ u(b)=\beta \}$$

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F : W^{1,p}(a,b) \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

THEOREM 9.9 Let $p>1$. Assume L is a Carathéodory function. Suppose that the following conditions hold:

(M1) $\xi \mapsto L(x,s,\xi)$ is CONVEX for a.e. $x \in (a,b)$ and $\forall s \in \mathbb{R}$.

(M2) $\exists \ q \in [1,p)$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ s.t.

$$L(x,s,\xi) \geq \alpha_1 |\xi|^p + \alpha_2 |s|^q + \alpha_3 , \quad \text{for a.e. } x \in (a,b), \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}$$

Set

$$m := \inf \{ F(u) \mid u \in X \} .$$

1. If $m < +\infty$, then $\exists \ u_0 \in X$ which minimizes F over X.

2. If in addition $(s,\xi) \mapsto L(x,s,\xi)$ is STRICTLY CONVEX for a.e. $x \in (a,b)$, then the minimizer is UNIQUE.

REMARK Assumptions (M1) - (M2) in THEOREM 9.9 cannot be weakened.
I will leave some exercises for the Exercise Course to show this claim.

Proof of THEOREM 9.9

Step 1. F is well-defined: Let $u \in W^{1,p}(a,b)$. The map

$$x \mapsto L(x, u(x), \dot{u}(x))$$

is measurable by PROPOSITION 8.2, since L is Carathéodory and $u, \dot{u}$ are measurable. Therefore $x \mapsto L(x, u(x), \dot{u}(x))$ can be integrated and $F(u)$ is well-defined, possibly being infinite.

Step 2. F is weakly LSC:

The proof of weak LSC is very difficult under the assumptions given: see THEOREMS 3.30, 4.1 in B. DACOROGNA - "DIRECT METHODS IN THE CALCULUS OF VARIATIONS", SPRINGER, 2008.

Instead, we prove LSC under much stronger assumptions, just to give an idea of what lies behind it.

Just for this step, assume then that

- $L \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$
- $(s, \xi) \mapsto L(x, s, \xi)$ is convex for every $x \in [a,b]$.
- $\exists \ \beta \geq 0$ s.t.

(Note that this implies (M1))

$$|L_s(x,s,\xi)| \, , \, |L_\xi(x,s,\xi)| \leq \beta \left(1 + |s|^{p-1} + |\xi|^{p-1}\right) \, , \quad \forall \, x \in [a,b], \ s, \xi \in \mathbb{R}.$$

We now show that F is weakly LSC, that is,

$$u_n \to u_0 \text{ weakly in } W^{1,p}(a,b) \implies F(u_0) \leq \liminf_{n \to +\infty} F(u_n)$$

Indeed, since L is C^1 and convex WRT (s,ξ), by THEOREM 5.2 we get

$$(*) \quad L(x, u_n(x), \dot{u}_n(x)) \geq L(x, u_0(x), \dot{u}_0(x)) + L_s(x, u_0, \dot{u}_0)(u_n - u_0) + L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)$$

Notice that

$$(**) \quad L_s(x, u_0, \dot{u}_0),\ L_\xi(x, u_0, \dot{u}_0) \in L^{p'}(a,b)$$

since

$$\int_a^b |L_s(x, u_0, \dot{u}_0)|^{p'} dx \overset{\text{ASSUMPTION}}{\leq} \beta^{p'} \int_a^b (1 + |u_0|^{p-1} + |\dot{u}_0|^{p-1})^{p'} dx$$

$$\left(\begin{array}{l} p' = \frac{p}{p-1} \text{ and} \\ (a+b)^{p'} \leq 2^{p'-1}(a^{p'} + b^{p'}) \end{array} \right) \leq \beta^{p'} C \int_a^b |u_0|^p + |\dot{u}_0|^p \, dx$$

$$= \beta^{p'} C \, \|u_0\|^p_{W^{1,p}} < +\infty$$

The same calculation shows that also $L_\xi(x, u_0, \dot{u}_0) \in L^{p'}(a,b)$.
Then, since $u_n, u_0 \in W^{1,p}(a,b)$, from $(**)$ and Hölder's inequality we get

$$L_s(x, u_0, \dot{u}_0)(u_n - u_0),\ L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0) \in L^1(a,b).$$

Therefore we can integrate (*) to get

$$F(u_n) \geq F(u_0) + \int_a^b L_s(x, u_0, \dot{u}_0)(u_n - u_0)\,dx$$

(**)

$$+ \int_a^b L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)\,dx$$

Now, $u_n \rightharpoonup u_0$ weakly in $W^{1,p}(a,b)$. In particular $u_n \rightharpoonup u_0$, $\dot{u}_n \rightharpoonup \dot{u}_0$ weakly in $L^p(a,b)$. Since (**) holds, by definition of weak convergence we get

$$\int_a^b L_s(x, u_0, \dot{u}_0)(u_n - u_0)\,dx\,, \quad \int_a^b L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)\,dx \to 0$$

as $n \to +\infty$. Taking the liminf in (**) yields weak LSC for F.

Step 3. F has COMPACT sublevels :

We are going to prove this part with the original assumptions. So fix $M \in \mathbb{R}$ and let

$$K := \{ u \in X \mid F(u) \leq M \}$$

From (M2) we deduce that $\exists$ M such that $K \neq \emptyset$.

By (M2) we have

$$F(u) \overset{(M2)}{\geq} \alpha_1 \|\dot{u}\|_{L^p}^p + \alpha_2 \|u\|_{L^q}^q + \alpha_3 (b-a)$$

(*)

$$\geq \alpha_1 \|\dot{u}\|_{L^p}^p - |\alpha_2| \|u\|_{L^q}^q - |\alpha_3| (b-a)$$

By Hölder inequality we get

Hölder with exponents $p/q > 1$, $(p/q)' = \frac{p}{p-q}$

$$\|u\|^q_{L^q} = \int_a^b |u|^q \, dx \le \left(\int_a^b |u|^p \, dx \right)^{q/p} \left(\int_a^b 1^{p/p-q} \, dx \right)^{\frac{p-q}{p}}$$

$$= \|u\|^q_{L^p} \, (b-a)^{\frac{p-q}{p}}$$

Then from (*)

$$F(u) \overset{(*)}{\geq} \alpha_1 \, \|\dot{u}\|^p_{L^p} - |\alpha_2| \, \|u\|^q_{L^q} - |\alpha_3| \, (b-a)$$

(**)

$$\geq \alpha_1 \, \|\dot{u}\|^p_{L^p} - C_1 \, \|u\|^q_{L^p} - C_2$$

for some $C_1, C_2 \in \mathbb{R}$. Moreover, if $u \in X$, we have

$$|u(x)| = | u(a) - u(a) + u(x) |$$

(as $u(a) = \alpha$) $\quad \le \alpha + |u(x) - u(a)|$

(THEOREM 7.23, as $p > 1$) $\quad \le \alpha + \|\dot{u}\|_{L^p} \, |x-a|^{1-1/p}$

$$\le \alpha + \|\dot{u}\|_{L^p} \, |b-a|^{1-1/p}$$

and so, integrating the above,

$$\|u\|^q_{L^p} \le c \left\{ 1 + \|\dot{u}\|^q_{L^p} \right\}, \qquad \forall x \in X.$$

Using (**) and the above, we then get some $C_1, C_2 \in \mathbb{R}$ s.t.

$$F(u) \geq \alpha_1 \, \|\dot{u}\|^p_{L^p} - C_1 \, \|\dot{u}\|^q_{L^p} - C_2$$

Now let $\{u_n\} \subseteq K$. Then

$$\underbrace{\alpha_1 \|\dot{u}_n\|^p_{L^p} - C_1 \|\dot{u}_n\|^q_{L^p} - C_2}_{\text{Polynomial in } \|\dot{u}_n\|_{L^p}} \overset{\text{Estimate above}}{\leq} F(u_n) \underset{\text{Since } \{u_n\} \subseteq K}{\leq} M$$

As $p > q \geq 1$, we deduce that $\|\dot{u}_n\|_{L^p}$ must be bounded uniformly, i.e.

$$(**) \qquad \|\dot{u}_n\|_{L^p} \leq \tilde{C}, \qquad \forall n \in \mathbb{N}.$$

Since we already proved that

$$\|u\|^q_{L^p} \leq c\{1 + \|\dot{u}\|^q_{L^p}\}, \qquad \forall u \in X,$$

from $(**)$ we get

$$\|u_n\|_{W^{1,p}} \leq \tilde{C}, \qquad \forall n \in \mathbb{N}.$$

Recalling that $W^{1,p}$ is a REFLEXIVE BANACH space for $1 < p < +\infty$ (PROPOSITION 7.16) from BANACH - ALAOGLU we conclude the existence of $u_0 \in W^{1,p}(a,b)$ s.t.

$$u_{n_k} \rightharpoonup u_0 \quad \text{weakly in} \quad W^{1,p}(a,b),$$

along some subsequence. By weak LSC of F we also get

$$F(u_0) \leq \liminf_{k \to +\infty} F(u_{n_k}) \underset{\text{As } \{u_n\} \subseteq K}{\leq} M$$

Finally, from the COMPACT embedding $W^{1,p}(a,b) \hookrightarrow C[a,b]$ for $p > 1$ (THEOREM 7.27) we get, by PROPOSITION 7.31,

$$u_{n_k} \to u_0 \quad \text{uniformly in } [a,b].$$

Since $\{u_n\} \subseteq X$, and so $u_n(a) = \alpha$, $u_n(b) = \beta$ $\forall n \in \mathbb{N}$, we conclude

$$u_0(a) = \alpha, \quad u_0(b) = \beta$$

showing that $u_0 \in X$. In total $u_0 \in K$, proving that K is compact.

Step 4. Existence of a minimizer :

So far we have shown that:

- F is weakly LSC in $W^{1,p}(a,b)$
- $\exists\, M \in \mathbb{R}$ s.t.

$$K := \{ u \in X \mid F(u) \le M \}$$

is non-empty and weakly compact in X.

Thus by the DIRECT METHOD (THEOREM 9.7) we conclude the existence of $\hat{u} \in X$ s.t.

$$F(\hat{u}) = \inf \{ F(u) \mid u \in X \}.$$

Step 5. Uniqueness: Usual stuff: follows as in the proof of THEOREM 5.4, with straightforward adaptations. $\square$

10. RELAXATION

LESSON 13 - 9 JUNE 2021

LSC ENVELOPE

NOTATION In the following we denote the extended real numbers by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

DEFINITION 10.1 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. We say that f is LOWER SEMICONTINUOUS (LSC) at $x_0 \in X$ if

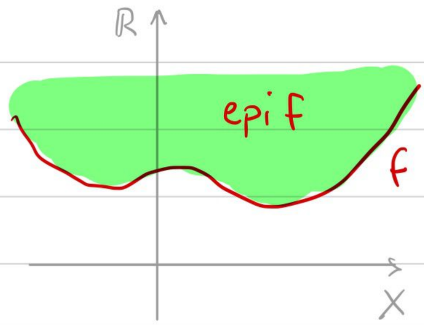
$$x_n \to x_0 \text{ in } (X, d) \Rightarrow f(x_0) \leq \liminf_{n \to +\infty} f(x_n)$$

PROPOSITION 10.2 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. They are equivalent:

1. f is LSC

2. For all $x \in X$ it holds

$$f(x) \leq \liminf_{y \to x} f(y)$$



3. The epigraph of f

$$\operatorname{epi} f := \{ (x, y) \in X \times \mathbb{R} \mid y \geq f(x) \}$$

is closed in $X \times \mathbb{R}$.

4. For all $M \in \mathbb{R}$ the sublevel

$$\{ x \in X \mid f(x) \leq M \}$$

is closed in X.

(Proof is easy, but omitted)

PROPOSITION 10.3 (Sup of LSC is LSC)

(X, d) metric space, I arbitrary set of indices, $f_i : X \to \overline{\mathbb{R}}$ LSC for all $i \in I$. Then $f : X \to \overline{\mathbb{R}}$ defined by

$$f(x) := \sup \{ f_i(x) \mid i \in I \}$$

is LSC.

Proof Let $x_n \to x_0$ in X. Then

$$\liminf_{n \to +\infty} f(x_n) \geq \liminf_{n \to +\infty} f_i(x_n) \geq f_i(x_0)$$

As f is defined as the supremum   As f_i is LSC

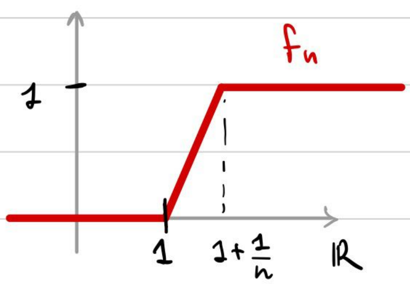
Taking the supremum for $i \in I$ allows to conclude. $\square$

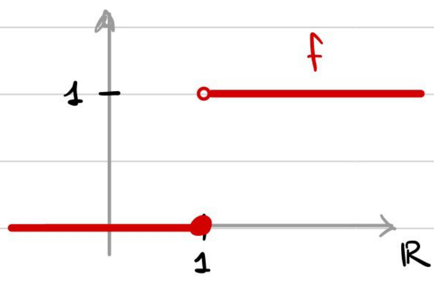
REMARK Let $f_i : X \to \mathbb{R}$ be a family of CONTINUOUS functions for $i \in I$. Then

$$(*) \quad f(x) := \sup \{ f_i(x) \mid x \in X \}$$

is in general only LSC.

For example consider f_n as in the picture. Clearly f defined by $(*)$ is not continuous.





$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

DEFINITION 10.4 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ a function.
The **LSC ENVELOPE** of f is the function $\hat{f}: X \to \overline{\mathbb{R}}$ defined by:

$$\hat{f}(x) := \sup \{ g(x) \mid g: X \to \overline{\mathbb{R}} \text{ is LSC}, \ g \leq f \text{ on } X \}$$

REMARK

1. The LSC ENVELOPE is well-defined, since we can always consider $g \equiv -\infty$. Thus the class in which we take the sup is non-empty.

2. The LSC envelope $\hat{f}$ is LSC (by PROPOSITION 10.3)

NOTE The LSC envelope is not straightforward to compute. For this reason we introduce a more practical notion of envelope (called RELAXATION). Eventually we will prove that the two notions coincide.

RELAXATION

DEFINITION 10.5 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ a function.
The **RELAXATION** of f is the function $\overline{f}: X \to \overline{\mathbb{R}}$ defined by

$$(*) \qquad \overline{f}(x) := \inf \{ \liminf_{n \to +\infty} f(x_n) \mid \{x_n\} \subseteq X, \ x_n \to x \}$$

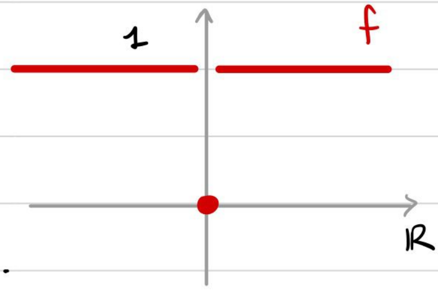
WARNING The relaxation in $(*)$ is **NOT** equivalent to

$$\overline{f}(x) \neq \liminf_{y \to x} f(y)$$

This is because the above limit does not allow to take $y \equiv x$, whereas in $(*)$ we can take $x_n \equiv x$.

For example, consider $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



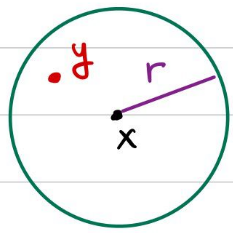
Then the relaxation is $\overline{f}(x) \equiv f(x)$.
However

$$\lim_{y \to 0} f(y) = 1$$

GOAL We aim to prove that LSC ENVELOPE and RELAXATION coincide.

LEMMA 10.6 (X, d) metric space, $f : X \to \mathbb{R}$ a function. Then

$$\boxed{\begin{gathered} \forall x \in X,\ \forall r > 0,\ \forall \varepsilon > 0,\ \exists y \in X \text{ s.t.} \\ d(x,y) \leq r \quad \text{and} \quad f(y) \leq \overline{f}(x) + \varepsilon \end{gathered}}$$



Proof Fix $x \in X$, $r > 0$ and $\varepsilon > 0$. By definition of Relaxation and of infimum, $\exists \{x_n\} \subseteq X$ s.t.

$$(*) \qquad x_n \to x \qquad \text{and} \qquad \liminf_{n \to +\infty} f(x_n) \leq \overline{f}(x) + \frac{\varepsilon}{2}$$

By the properties of liminf $\exists$ a subsequence $\{x_{n_k}\}$ s.t.

$$\liminf_{n \to +\infty} f(x_n) = \lim_{k \to +\infty} f(x_{n_k}).$$

From (*) we get

$$x_{n_k} \to x \qquad \text{and} \qquad \lim_{k \to +\infty} f(x_{n_k}) \le \overline{f}(x) + \varepsilon/2 .$$

Therefore, $\exists\, N \in \mathbb{N}$ sufficiently large such that

$$d(x_N, x) < r, \qquad f(x_N) \le \overline{f}(x) + \varepsilon .$$

Setting $y := x_N$ yields the thesis. □

DEFINITION 10.7 (RECOVERY SEQUENCE)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$. We say that $\{x_n\} \subseteq X$ is a RECOVERY SEQUENCE for f at $x \in X$ if

$$x_n \to x \qquad \text{and} \qquad \overline{f}(x) = \lim_{n \to +\infty} f(x_n)$$

LEMMA 10.8 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. For all $x \in X$ there exists a Recovery Sequence $\{x_n\} \subseteq X$.

Proof Use LEMMA 10.6 with $\varepsilon = 1/n$, $r = 1/n$ to find $y_n \in X$ s.t.

$$d(x, y_n) < \frac{1}{n}, \qquad f(y_n) \le \overline{f}(x) + \frac{1}{n}, \qquad \forall n \in \mathbb{N}.$$

Therefore $y_n \to x$ and

$$(*) \qquad \limsup_{n \to +\infty} f(y_n) \le \limsup_{n \to +\infty} \overline{f}(x) + \frac{1}{n} = \overline{f}(x).$$

On the other hand

$$\bar{f}(x) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n \to +\infty} f(x_n) \;\middle|\; \{x_n\} \subseteq X, \; x_n \to x \right\}.$$

$$(\text{Since } y_n \to x) \to \; \le \liminf_{n\to+\infty} f(y_n) \le \limsup_{n \to +\infty} f(y_n) \overset{(*)}{\le} \bar{f}(x)$$

showing that $\bar{f}(x) = \lim_{n \to +\infty} f(y_n)$. Thus $\{y_n\}$ is Recovery Sequence for f at x. □

PROPOSITION 10.9 (Equivalence of LSC ENVELOPE and RELAXATION)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$ function. We have

1. $\hat{f}$ is LSC and $\hat{f}(x) \le f(x) \quad \forall x \in X$,
2. $\bar{f}$ is LSC and $\bar{f}(x) \le f(x) \quad \forall x \in X$,
3. $\bar{f}(x) = \hat{f}(x), \quad \forall x \in X$.

Proof ① $\hat{f}$ is the supremum of LSC functions, hence it is LSC by PROP 10.3. The inequality is obvious by definition of $\hat{f}$.

② We first show the inequality: Consider the sequence $\bar{x}_n \equiv x$. Then

$$\bar{f}(x) \overset{\text{def}}{:=} \inf \left\{ \liminf_{n \to +\infty} f(x_n) \;\middle|\; \{x_n\} \subseteq X, \; x_n \to x \right\}$$

$$\le \liminf_{n \to +\infty} f(\bar{x}_n) \underset{\bar{x}_n \equiv x}{=} f(x).$$

We show that $\bar{f}$ is LSC. So let $x_n \to x_0$ be arbitrary. We want to prove

$$f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

$\forall n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = 1/n$, $\varepsilon = 1/n$ to find $y_n \in X$ s.t.

$$(*) \qquad d(x_n, y_n) < \frac{1}{n}, \qquad f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}, \qquad \forall n \in \mathbb{N}$$

Since $x_n \to x_0$, the first condition implies $y_n \to x_0$. Therefore

$$\bar{f}(x_0) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n\to+\infty} f(z_n) \mid \{z_n\} \subseteq X,\ z_n \to x_0 \right\}$$

(As $y_n \to x_0$) $\to$ $$\leq \liminf_{n\to+\infty} f(y_n) \overset{(*)}{\leq} \liminf_{n\to+\infty} \left[\bar{f}(x_n) + \frac{1}{n} \right]$$

(Property of liminf) $$\leq \liminf_{n\to+\infty} \bar{f}(x_n) + \liminf_{n\to+\infty} \frac{1}{n} = \liminf_{n\to+\infty} \bar{f}(x_n),$$

showing that $\bar{f}$ is LSC.

3 • $\bar{f}(x) \geq \hat{f}(x)$: Let $x_n \to x$ be arbitrary. Then by 1

$$\liminf_{n\to+\infty} f(x_n) \underset{f \geq \hat{f}}{\geq} \liminf_{n\to+\infty} \hat{f}(x_n) \underset{\hat{f} \text{ is LSC}}{\geq} \hat{f}(x)$$

Taking the infimum for all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x$, we obtain the thesis.

• $\hat{f}(x) \geq \bar{f}(x)$: $\bar{f}$ is LSC and $\bar{f} \leq f$ by ②, thus

$$\hat{f}(x) \overset{\text{def of } \hat{f}}{=} \sup\{ g(x) \mid g : X \to \overline{\mathbb{R}} , \ g \text{ LSC}, \ g \leq f \text{ on } X \} \overset{\text{As } \bar{f} \text{ is competitor}}{\geq} \bar{f}(x)$$

$\square$

NOTE In the following $\bar{f}$ and $\hat{f}$ will be used interchangeably, depending on which is the most convenient.

RELATIONSHIP BETWEEN inf / min f AND inf / min $\bar{f}$

The next proposition shows why RELAXATION and LSC ENVELOPE are useful.

PROPOSITION 10.10 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$ function. Then

$$\boxed{\inf_{x \in X} f(x) = \inf_{x \in X} \bar{f}(x) = \inf_{x \in X} \hat{f}(x)}$$

Proof Since $\bar{f} = \hat{f}$ by PROPOSITION 10.9, we only need to show the first equality.

$\boxed{\geq}$ This is clear, since $f \geq \bar{f}$ by PROPOSITION 10.9

$\boxed{\leq}$ Let $\{x_n\}$ be an infimizing sequence for $\bar{f}$, i.e.,

$$\overline{f}(x_n) \to \inf_{x\in X} \overline{f}(x).$$

For all $n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = 1$, $\varepsilon = 1/n$, so that $\exists$ $\{y_n\} \subseteq X$ s.t.

$$(*) \quad d(x_n, y_n) < 1 \quad \text{and} \quad f(y_n) \leq \overline{f}(x_n) + \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Then

$$\inf_{x\in X} \overline{f}(x) \overset{\{x_n\} \text{ is infimizing}}{=} \lim_{n\to+\infty} \overline{f}(x_n) \overset{\{\overline{f}(x_n)\} \text{ is convergent}}{=} \liminf_{n\to+\infty} \overline{f}(x_n)$$

$$\left(\text{As } \frac{1}{n} \to 0\right) \to \quad = \liminf_{n\to+\infty} \left[\overline{f}(x_n) + \frac{1}{n}\right]$$

$$\overset{(*)}{\geq} \liminf_{n\to+\infty} f(y_n) \underset{\text{def of inf}}{\geq} \inf_{x\in X} f(x)$$

$\square$

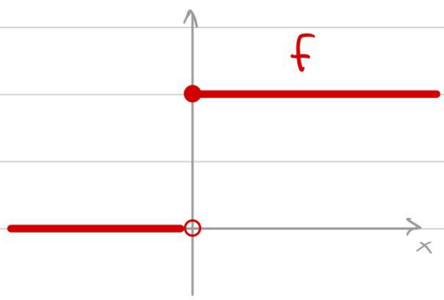
WARNING The statement of PROPOSITION 10.10 only holds on the whole X. In general one has

$$\inf_{x\in A} f(x) > \inf_{x\in A} \overline{f}(x)$$

for $A \subset X$.

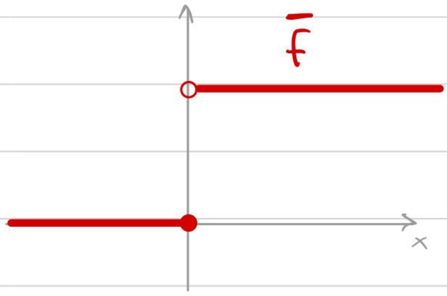
For example consider $X = \mathbb{R}$ and

$$f(x) := \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$



Then

$$\bar{f}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



For $A = [0, +\infty)$ we have

$$\inf_{x \in A} f(x) = 1 \quad , \quad \inf_{x \in A} \bar{f}(x) = 0$$

However the thesis of PROPOSITION 10.10 holds when A is open:

PROPOSITION 10.11 (X, d) metric space, $f : X \to \bar{\mathbb{R}}$ function, $A \subset X$ open. Then

$$\boxed{\inf_{x \in A} f(x) = \inf_{x \in A} \bar{f}(x) = \inf_{x \in A} \hat{f}(x)}$$

Proof Since $\bar{f} = \hat{f}$ by PROPOSITION 10.9, we only need to show the first equality.

$\boxed{\geq}$ This is clear, since $f \geq \bar{f}$ by PROPOSITION 10.9

$\boxed{\leq}$ Let $\{x_n\}$ be an infimizing sequence for $\bar{f}$ over A, i.e., $\{x_n\} \subseteq A$ and

$$\bar{f}(x_n) \to \inf_{x \in A} \bar{f}(x) .$$

Since A is open, $\forall n \in \mathbb{N}$, $\exists\, r_n > 0$ s.t. $B_{r_n}(x_n) \subset A$.

For all $n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = r_n$, $\varepsilon = 1/n$, so that $\exists$ $\{y_n\} \subseteq X$ s.t.

$$(*) \qquad d(x_n, y_n) < r_n \qquad \text{and} \qquad f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}.$$

The first condition tells us that $y_n \in B_{r_n}(x_n)$, so that $\{y_n\} \subset A$. Then

$$\inf_{x \in A} \bar{f}(x) \overset{\{x_n\} \text{ is infimizing}}{=} \lim_{n \to +\infty} \bar{f}(x_n) \overset{\{\bar{f}(x_n)\} \text{ is convergent}}{=} \liminf_{n \to +\infty} \bar{f}(x_n)$$

$$\left(\text{As } \tfrac{1}{n} \to 0\right) \to \quad = \liminf_{n \to +\infty} \left[\bar{f}(x_n) + \frac{1}{n}\right]$$

$$\overset{(*)}{\geq} \liminf_{n \to +\infty} f(y_n) \geq \inf_{x \in A} f(x)$$

def of inf, since $\{y_n\} \subset A$

$\square$

Now recall the definition of COERCIVE function (DEFINITION 9.5)

DEFINITION X space with notion of convergence. A map $f: X \to \overline{\mathbb{R}}$ is COERCIVE if $\exists$ $K \subset X$ compact s.t.

$$\inf_{x \in X} f(x) = \inf_{x \in K} f(x)$$

For coercive functions on metric space, the following holds:

PROPOSITION 10.12 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ COERCIVE. Then $\bar{f}$ admits minimum over X and

$$\boxed{\inf_{x\in X} f(x) = \min_{x\in X} \bar{f}(x)}$$

WARNING PROP 10.12 is saying that if f is COERCIVE then the minimum of $\bar{f}$ exists and is equal to the infimum of f.
It is NOT saying that f admits minimum. This is false in general.

Proof As f coercive, $\exists\, K \subset X$ compact s.t.

$$\inf_{x\in X} f(x) = \inf_{x\in K} f(x).$$

By PROPOSITION 10.9 we have that $\bar{f}$ is LSC. As K is compact, from THEOREM 9.4 (DIRECT METHOD) we have that $\bar{f}$ admits minimum on K, i.e.,

$$(*) \qquad \inf_{x\in K} \bar{f}(x) = \min_{x\in K} \bar{f}(x)$$

We CLAIM that $\bar{f}$ admits minimum over X, with

$$(**) \qquad \min_{x\in X} \bar{f}(x) = \min_{x\in K} \bar{f}(x)$$

Let $y\in X$ be arbitrary, and let $\{y_n\}\subset X$ be a RECOVERY SEQUENCE for f at y (which $\exists$ by LEMMA 10.8), i.e.,

$$\bar{f}(y) = \lim_{n\to+\infty} f(y_n)$$

Then

$$\bar{F}(y) \overset{\text{Recovery}}{=} \lim_{n \to +\infty} f(y_n) \overset{\text{Def of inf}}{\geq} \inf_{x \in X} f(x) \overset{\text{Coercivity of } f}{=} \inf_{x \in K} f(x)$$

$$\underset{\left(\substack{f \geq \bar{f} \text{ by} \\ \text{PROP } 10.9}\right)}{\geq} \inf_{x \in K} \bar{f}(x) \overset{*}{=} \min_{x \in K} \bar{f}(x)$$

Since y was arbitrary, we get

$$\inf_{x \in X} \bar{f}(x) \geq \min_{x \in K} \bar{f}(x)$$

The reverse inequality is obvious, as $K \subset X$. We conclude $(**)$. Therefore

$$\inf_{x \in X} f(x) \underset{\text{PROP 10.10}}{=} \inf_{x \in X} \bar{f}(x) \underset{(**)}{=} \min_{x \in X} \bar{f}(x)$$

$\square$

PROPOSITION 10.13 (Behavior of infimizing sequences)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$. Suppose that $\{x_n\} \subseteq X$ is s.t.

$$x_n \to x_0 \quad \text{and} \quad f(x_n) \to \inf_{x \in X} f(x) \qquad (\text{i.e. } \{x_n\} \text{ infimizing for } f)$$

Then x_0 is a minimizer for $\bar{f}$, i.e.,

$$\bar{f}(x_0) = \inf_{x \in X} \bar{f}(x)$$

Proof

$$\inf_{x\in X} \bar{f}(x) \underset{\text{def of inf}}{\leq} \bar{f}(x_0) \underset{\substack{\text{def of LSC envelope,}\\ \text{as } x_n \to x_0}}{\leq} \liminf_{n\to+\infty} f(x_n) \underset{\text{as } f(x_n) \text{ convergent}}{=} \lim_{n\to+\infty} f(x_n)$$

$$\underset{\text{assumption}}{=} \inf_{x\in X} f(x) \underset{\text{PROP 10.10}}{=} \inf_{x\in X} \bar{f}(x) \Rightarrow x_0 \in \underset{x\in X}{\operatorname{arg\,min}}\, \bar{f}(x)$$

$\square$

COROLLARY 10.14 (X,d) metric space, $f: X \to \bar{\mathbb{R}}$. Assume that $\exists M>0$ and $K \subseteq X$ compact s.t.

$$\tilde{K} := \{x\in X \mid f(x) < M\} \neq \emptyset \quad \text{and} \quad \tilde{K} \subseteq K.$$

If $\{x_n\} \subseteq X$ is infimizing for f, i.e.,

$$f(x_n) \to \inf_{x\in X} f(x)$$

then $\exists$ subsequence and $x_0 \in X$ s.t.

$$x_{n_k} \to x_0 \quad \text{and} \quad x_0 \in \underset{x\in X}{\operatorname{arg\,min}}\, \bar{f}(x).$$

Proof Since $\tilde{K} \neq \emptyset$, it means that $I < M$, where $I := \inf\{f(x) \mid x\in X\}$. As $f(x_n) \to I$, we then conclude that $\exists N \in \mathbb{N}$ s.t.

$$x_n \in \tilde{K}, \quad \forall n \geq N.$$

As $\tilde{K} \subseteq K$ and K is compact, then $\exists x_0 \in K$ and a subsequence s.t. $x_{n_k} \to x_0$. We then conclude from PROP 10.13, since $\{x_{n_k}\}$ is an infimizing sequence for f. $\square$

COMPUTING THE RELAXATION

We will see 2 strategies to compute the relaxation.

PROPOSITION 10.15 (STRATEGY 1)

(X, d) metric space, $F: X \to \overline{\mathbb{R}}$. Suppose that $g: X \to \overline{\mathbb{R}}$ is s.t.

1 (liminf inequality) For all $x \in X$ and $x_n \to x$ it holds

$$g(x) \leq \liminf_{n \to +\infty} F(x_n)$$

2 (limsup inequality) For all $x \in X$, $\exists\ x_n \to x$ s.t.

$$\limsup_{n \to +\infty} F(x_n) \leq g(x)$$

Then $g = \overline{F}$.

NOTE If 1 and 2 hold, then the limsup in 2 is actually a limit.

Proof $\boxed{g \leq \overline{F}}$ Let $x_n \to x$ be arbitrary. By 1 we have

$$g(x) \leq \liminf_{n \to +\infty} F(x_n)$$

Since $\{x_n\}$ is arbitrary, taking the infimum over all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x$, we get $g \leq \overline{F}$.

$\boxed{\bar{f} \leq g}$ Conversely, let $\{x_n\}$ be the sequence existing by ②. Then

$$\bar{f}(x) \underset{\substack{\uparrow \\ \text{def of } \bar{f} \\ \text{since } x_n \to x}}{\leq} \liminf_{n \to +\infty} f(x_n) \underset{\substack{\uparrow \\ \text{properties of} \\ \liminf / \limsup}}{\leq} \limsup_{n \to +\infty} f(x_n) \underset{\substack{\uparrow \\ \text{by ②}}}{\leq} g(x)$$

showing that $\bar{f} \leq g$ and concluding. □

We now look at a second strategy to compute the relaxation.

DEFINITION 10.16 (ENERGY DENSE SUBSETS)

(X, d) metric space, $f : X \to \bar{\mathbb{R}}$. A subset $D \subseteq X$ is ENERGY DENSE WRT f if

$$\forall x \in X, \quad \exists \{x_n\} \subseteq D \text{ s.t. } \quad x_n \to x \quad \text{and} \quad f(x_n) \to f(x).$$

REMARK ① Suppose $f : X \to \mathbb{R}$ is continuous. Then $D \subseteq X$ is Energy Dense WRT to f iff it is Dense.

② $D \subseteq X$ is Energy Dense WRT f iff

$$\{(x, f(x)), \; x \in D\} \subseteq X \times \bar{\mathbb{R}}$$

is dense in $X \times \bar{\mathbb{R}}$.

LEMMA 10.17 (X, d) metric space, $\varphi, \psi : X \to \overline{\mathbb{R}}$. Let $D \subseteq X$. Suppose that

(i) $\varphi(x) \leq \psi(x) \quad \forall x \in D$

(ii) D is Energy Dense WRT ψ

(iii) φ is LSC

Then

$$\varphi(x) \leq \psi(x), \quad \forall x \in X.$$

Proof Let $x \in X$. By (ii) there $\exists \{x_n\} \subseteq D$ s.t. $x_n \to x$ and $\psi(x_n) \to \psi(x)$.

Then

$$\underset{\substack{\uparrow \\ \varphi \text{ is LSC and} \\ x_n \to x}}{\varphi(x) \leq} \liminf_{n \to +\infty} \varphi(x_n) \underset{\substack{\uparrow \\ \text{By (i), since } \{x_n\} \subseteq D}}{\leq} \liminf_{n \to +\infty} \psi(x_n) \underset{\substack{\uparrow \\ \text{As } \psi(x_n) \to \psi(x)}}{=} \psi(x)$$

$\square$

PROPOSITION 10.18 (STRATEGY 2)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$. Suppose that $g : X \to \overline{\mathbb{R}}$ satisfies

1. g is LSC
2. $g(x) \leq f(x), \quad \forall x \in X$
3. $\exists D \subseteq X$ Energy Dense WRT g, s.t.

$$\forall x \in D, \ \exists \{x_n\} \subseteq X \text{ s.t. } x_n \to x \text{ and } \limsup_{n \to +\infty} f(x_n) \leq g(x)$$

Then $\overline{f} = g$.

<u>Proof</u> $\boxed{g \leq \bar{f}}$ Let $x_n \to x$ be arbitrary. Then

$$\liminf_{n \to +\infty} f(x_n) \overset{2}{\geq} \liminf_{n \to +\infty} g(x_n) \overset{1}{\geq} g(x)$$

Taking the infimum for all $x_n \to x$, we conclude $\bar{f} \geq g$.

$\boxed{\bar{f} \leq g}$ Set $\varphi := \bar{f}$, $\psi := g$. Let us verify the assumptions of LEMMA 10.17:

(i) $\varphi(x) \leq \psi(x) \quad \forall x \in D$ (TRUE because of 3 and definition of $\bar{f}$)

(ii) D is Energy Dense WRT ψ (TRUE: it is assumed in 3)

(iii) φ is LSC (TRUE because $\varphi = \bar{f}$ and $\bar{f}$ is LSC by PROP 10.9)

Therefore by LEMMA 10.17 we have that $\varphi \leq \psi$ on X, i.e. $\bar{f} \leq g$ on X. $\square$

EXTENSION BY RELAXATION : CONVEX CASE

LESSON 14
16 JUNE 2021

Setting : $(\hat{X}, d)$ metric space , $X \subseteq \hat{X}$ and $f : X \to \overline{\mathbb{R}}$.

QUESTION Find $\hat{f} : \hat{X} \to \overline{\mathbb{R}}$ which extends f in a meaningful way.

IDEA Extend f on $\hat{X}$ by setting

$$(*) \qquad \hat{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in X \\ +\infty & \text{if} \quad x \in \hat{X} \setminus X \end{cases}$$

Then consider $\hat{f} := \overline{f}$. In the following f is always extended according to $(*)$

EXAMPLE $\hat{X} = L^2(a,b)$, $X = C^1[a,b]$, $F : X \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b \dot{u}^2 \, dx \quad , \quad \forall \, u \in X .$$

Extend F to $+\infty$ on $\hat{X} \setminus X$. Then set $\hat{F} := \overline{F}$. (relax in L^2)

CLAIM We have that

weak derivative

$$\hat{F}(u) = G(u) := \begin{cases} \int_a^b \dot{u}^2 \, dx & \text{if} \quad u \in H^1(a,b) \\ +\infty & \text{if} \quad u \in L^2 \setminus H^1 \end{cases}$$

(proof left as exercise. One can employ STRATEGY 2 in this case)

IN GENERAL We want to compute relaxation for $F : C^1[a,b] \to \overline{\mathbb{R}}$

$$F(u) := \int_a^b \psi(\dot{u})\, dx \, , \qquad \psi : \mathbb{R} \to \mathbb{R} \, , \quad \psi = \psi(\xi)$$

Under some assumptions the relaxation of F in $L^p(a,b)$ is given by

$$\hat{F} : L^p(a,b) \to \overline{\mathbb{R}} \, , \qquad \hat{F}(u) := \begin{cases} \int_a^b \psi(\dot{u})\, dx \, , & \text{if } u \in W^{1,p}(a,b) \\ +\infty & \text{otherwise in } L^p(a,b) \end{cases}$$

THEOREM 10.19 Consider F, $\hat{F}$ as above. Assume $\psi : \mathbb{R} \to \mathbb{R}$ is

1. Convex

2. $\exists\, A > 0$, $B \in \mathbb{R}$, $p \in (1, +\infty)$ s.t.

$$\psi(\xi) \geq A\, |\xi|^p - B \quad , \qquad \forall\, \xi \in \mathbb{R} .$$

Then $\overline{F} = \hat{F}$ in $L^p(a,b)$.

Proof We use STRATEGY 2 (PROP 10.18). We need to show that:

1. $\hat{F}$ is LSC in $L^p(a,b)$

2. $\hat{F}(u) \leq F(u)$, $\forall\, u \in L^p(a,b)$ (Here $F(u) := +\infty$ if $u \notin C^1[a,b]$)

3. $\exists\, D \subseteq L^p(a,b)$ Energy Dense WRT $\hat{F}$, s.t.

$$\forall\, u \in D \, , \; \exists\, \{u_n\} \subseteq C^1[a,b] \text{ s.t. } \underset{\text{strongly in } L^p(a,b)}{u_n \to u} \text{ and } \limsup_{n \to +\infty} F(u_n) \leq \hat{F}(u)$$

Checking 1: Need to show that if $u_n \to u$ in $L^p(a,b)$ then

$$(*) \qquad \hat{F}(u) \leq \liminf_{n \to +\infty} \hat{F}(u_n).$$

If RHS is $+\infty$ then $(*)$ is trivial. Then WLOG we can assume that

$$\hat{F}(u_n) \leq M, \qquad \forall n \in \mathbb{N}.$$

From the growth assumption on ψ we get

$$\int_a^b A |\dot{u}_n|^p - B \, dx \leq \hat{F}(u_n) \leq M$$

so that

$$\int_a^b |\dot{u}_n|^p \, dx \leq \frac{M + (b-a)B}{A}$$

proving that $\{\dot{u}_n\}$ is bounded in $L^p(a,b)$. Thus, up to subsequences

$$\dot{u}_n \rightharpoonup v \quad \text{weakly in } L^p(a,b)$$

As $u_n \to u$ strongly in $L^p(a,b)$, in particular we get

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(a,b)$$

Thus, from REMARK 7.18 (trivially adaptable to $W^{1,p}$ case) we get

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(a,b)$$

Now $(*)$ can be shown as in THEOREM 9.9 (if we assume ψ is C^1 and growth of ψ from above). In general, see THM 3.6 in BUTTAZZO, GIAQUINTA, HILDEBRANDT

Checking ②: This is obvious by definition of F, $\hat{F}$, and by the fact that weak derivatives coincide with classical ones for maps in $C^1[a,b]$.

Checking ③: Set

$$D := \{ u : [a,b] \to \mathbb{R} \mid u \text{ continuous and piecewise linear} \}$$

CLAIM D is Energy Dense WRT $\hat{F}$

[Given $\hat{u} \in L^p(a,b)$ we need to find $\{u_n\} \subseteq D$ s.t.

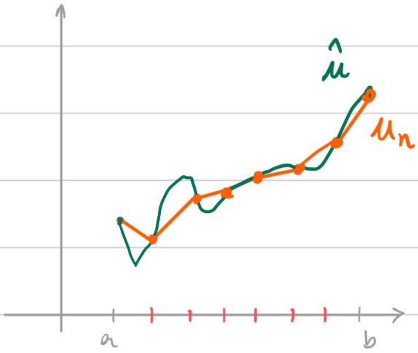
$$(*) \qquad u_n \to \hat{u} \text{ in } L^p \quad \text{and} \quad \hat{F}(u_n) \to \hat{F}(\hat{u})$$

- If $\hat{u} \notin W^{1,p}(a,b)$ then $\hat{F}(\hat{u}) = +\infty$. Now it is easy to approximate u in L^p with a sequence in D and obtain $(*)$ by LSC of $\hat{F}$:

$$+\infty = \hat{F}(\hat{u}) \leq \liminf_{n \to +\infty} \hat{F}(u_n) \implies \lim_{n \to +\infty} \hat{F}(u_n) = +\infty .$$

- If $\hat{u} \in W^{1,p}(a,b)$ and $\hat{F}(\hat{u}) = +\infty$, then proceed as above.

- If $\hat{u} \in W^{1,p}(a,b)$ and $\hat{F}(\hat{u}) < +\infty$: by THEOREM 7.19 we know that $\hat{u} \in C[a,b]$. Then construct u_n as in picture



Divide $[a,b]$ in sub-intervals I_i of amplitude $1/n$.
Define u_n by linear interpolation of values of $\hat{u}$ on the grid.

As the mesh-size goes to zero as $n \to +\infty$ and $\hat{u}$ is uniformly continuous in $[a,b]$ we get

$$u_n \to \hat{u} \quad \text{uniformly in } [a,b] \qquad \text{(easy check)}$$

Then in particular

$$u_n \to \hat{u} \quad \text{strongly in } L^p(a,b)$$

Moreover, it holds that

$$(**) \qquad \hat{F}(u_n) \leq \hat{F}(\hat{u}) \quad , \quad \forall n \in \mathbb{N} .$$

Indeed

$$\hat{F}(u_n) \overset{\text{def of } u_n}{=} \sum_{k=1}^{N} \int_{I_k} \psi(\dot{u}_n)\, dx$$

Now, consider the problem :

$$(P) \qquad \min \left\{ \int_{I_k} \psi(\dot{u})\, dx \;\middle|\; u \in W^{1,p}(I_k) \;,\; u_{|\partial I_k} = \hat{u}_{|\partial I_k} \right\}$$

Since $\psi = \psi(\xi)$, and ψ is convex, one immediately sees that the straight line solves (P) (by Jensen's Inequality THEOREM 6.8). Thus

$$(***) \quad \int_{I_k} \psi(\dot{u}_n)\, dx \leq \int_{I_k} \psi(\dot{u})\, dx \quad , \qquad \forall u \in W^{1,p}(I_k) \text{ s.t. } u_{|\partial I_k} = \hat{u}_{|\partial I_k}$$

Since $\hat{u}$ satisfies the Dirichlet BC, we get

$$\hat{F}(u_n) \overset{\text{def of } u_n}{=} \sum_{k=1}^{N} \int_{I_k} \Psi(\dot{u}_n)\,dx \overset{(**)}{\leq} \sum_{k=1}^{N} \int_{I_k} \Psi(\hat{u}')\,dx$$

$$= \int_a^b \Psi(\hat{u}')\,dx = \hat{F}(\hat{u})$$

so that (**) holds. Taking the limsup in (**) yields

(LS) $\displaystyle \limsup_{n \to +\infty} \hat{F}(u_n) \leq \hat{F}(\hat{u})$.

By (1) we know that $\hat{F}$ is LSC in $L^p(a,b)$, so that

(LI) $\displaystyle \hat{F}(\hat{u}) \leq \liminf_{n \to +\infty} \hat{F}(u_n)$

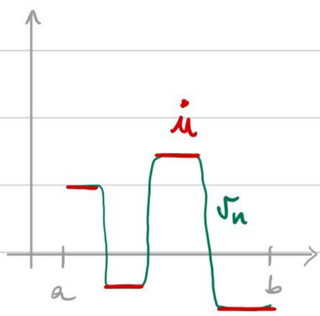
since $u_n \to u$ in $L^p(a,b)$ by construction

From (LS)–(LI) we conclude $\hat{F}(u_n) \to \hat{F}(\hat{u})$, and (*) follows.]

<u>CLAIM</u> $\forall u \in D$, $\exists \{u_n\} \subseteq C^1[a,b]$ s.t.

$u_n \to u$ in $L^p(a,b)$ and $\displaystyle \limsup_{n \to +\infty} F(u_n) \leq \hat{F}(u)$

[Usual approximation argument: for $u \in D$, we approximate $\dot{u}$ with some smooth v_n and then define u_n as the primitive of v_n.]



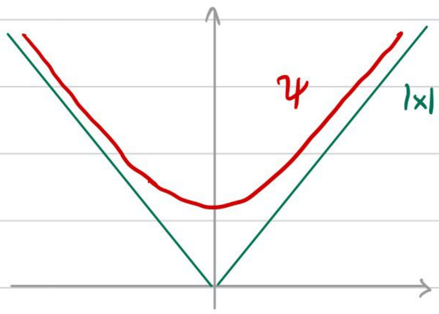
Therefore (1), (2), (3) hold and so PROP 10.18 implies $\bar{F} = \hat{F}$ in $L^p(a,b)$.

$\square$

WARNING The thesis of THEOREM 10.19 is FALSE for $p=1$.

[For example consider

$$\psi(\xi) := \sqrt{1+\xi^2}$$



which is CONVEX and such that

$$\psi(\xi) \geq |\xi| \quad , \qquad \forall \xi \in \mathbb{R} .$$

Consider the functional $F : C^1[-1,1] \to \mathbb{R}$

$$F(u) := \int_{-1}^{1} \sqrt{1+\dot{u}^2}\, dx$$

and $\hat{F} : W^{1,1}(-1,1) \to \mathbb{R}$

$$\hat{F}(u) := \begin{cases} \displaystyle\int_{-1}^{1} \sqrt{1+\dot{u}^2}\, dx & \text{if } u \in W^{1,1}(-1,1) \\ +\infty & \text{otherwise} \end{cases}$$

Then

$$\boxed{\overline{F} \neq \hat{F}}$$

In fact, let

$$\hat{u}(x) := \begin{cases} 0 & \text{if} \quad x \in [-1,0) \\ J & \text{if} \quad x \in [0,1] \end{cases} \quad , \qquad J \in \mathbb{R}$$

Then $\hat{F}(\hat{u}) = +\infty$, since $\hat{u} \notin W^{1,1}(-1,1)$. But $\overline{F}(\hat{u}) = |J|$.
In this case $\overline{F}$ is finite on the space $BV(-1,1)$.]

EXAMPLE Consider the functional of EXAMPLE 9.8 :

$$F : C^1[a,b] \to \mathbb{R} , \qquad F(u) := \int_a^b \dot{u}^2 + \sin(u^5)\, dx$$

We can write $F = G + H$ with

$$G(u) := \int_a^b \psi(\dot{u})\, dx , \qquad \psi(\xi) := \xi^2 , \qquad H(u) := \int_a^b \sin(u^5)\, dx$$

ψ is convex and satisfies the growth assumption of THEOREM 10.19 with $p=2$, $A=1$, $B=0$. Therefore the relaxation of G in $L^2(a,b)$ is

$$\bar{G}(u) = \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

Also notice that H is continuous in $L^2(a,b)$. Thus $\bar{H} = H$ (exercise)

Then one can prove (exercise)

$$\bar{F} = \bar{G} + H$$

showing that

$$\bar{F}(u) = \begin{cases} \int_a^b \dot{u}^2 + \sin(u^5)\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise in } L^2(a,b) \end{cases}$$

is the relaxation of F in $L^2(a,b)$, as anticipated in EXAMPLE 9.8.

EXTENSION BY RELAXATION : NON - CONVEX CASE

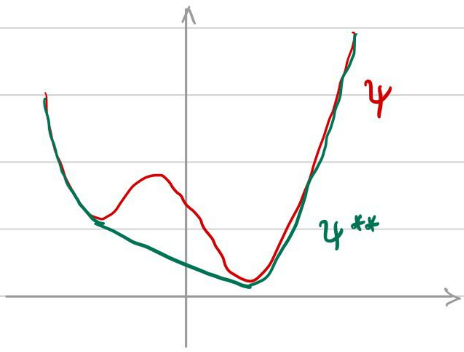
We want to generalize THEOREM 10.19 to NON-CONVEX Lagrangians.

DEFINITION 10.20 (CONVEX ENVELOPE)

Let $\psi : \mathbb{R} \to \mathbb{R}$. The CONVEX ENVELOPE of ψ is the map $\psi^{**} : \mathbb{R} \to \overline{\mathbb{R}}$

$$\boxed{\psi^{**}(x) := \sup \left\{ g(x) \mid g : \mathbb{R} \to \overline{\mathbb{R}} \text{ is convex}, \ g \leq \psi \text{ in } \mathbb{R} \right\}}$$

PROPOSITION 10.21 (PROPERTIES OF ψ^{**})



Let $\psi : \mathbb{R} \to \mathbb{R}$. Then

1. ψ^{**} is convex
2. $\psi^{**} \leq \psi$ on $\mathbb{R}$
3. The supremum in the definition of ψ^{**} is a maximum
4. We have

$$\psi^{**}(x) = \sup \left\{ mx + q \mid my + q \leq \psi(y), \ \forall y \in \mathbb{R} \right\}$$

i.e. ψ^{**} is the supremum of all lines below the graph of ψ.

(Proof is omitted)

THEOREM 10.22 Let $\psi : \mathbb{R} \to \mathbb{R}$ and define $F : C^1[a,b] \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b \psi(\dot{u})\, dx$$

Suppose $\exists\, p \in (1, +\infty)$ and $A > 0$, $B \in \mathbb{R}$ s.t.

$$(*) \qquad \psi(\xi) \geq A|\xi|^p - B, \qquad \forall\, \xi \in \mathbb{R}.$$

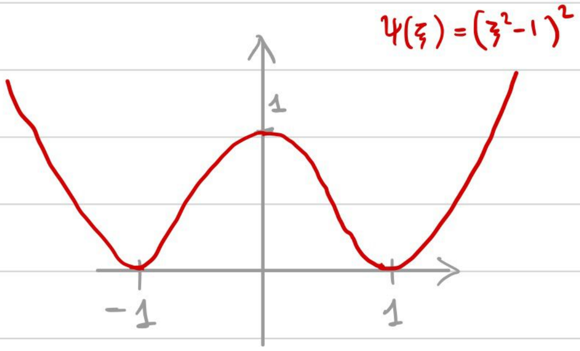
Then $\overline{F} = \hat{F}$ in $L^p(a,b)$, where

$$\hat{F}(u) := \begin{cases} \int_a^b \psi^{**}(\dot{u})\, dx, & \text{if } u \in W^{1,p}(a,b) \\ +\infty, & \text{otherwise in } L^p(a,b) \end{cases}$$

(Proof is omitted. It is similar to the proof of THEOREM 10.19)

EXAMPLE Consider the functional of EXAMPLE 5.6: $F : C^1[a,b] \to \mathbb{R}$,

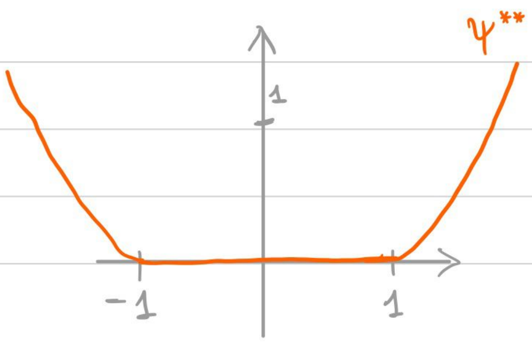
$$F(u) := \int_a^b (\dot{u}^2 - 1)^2\, dx$$



DOUBLE WELL

The Lagrangian is $\psi(\xi) := (\xi^2 - 1)^2$ which satisfies $(*)$ with $p = 4$ (for some $A > 0$, $B \in \mathbb{R}$). Thus THM 10.22 implies that the relaxation of F in $L^4(a,b)$ is given by

$$\overline{F}(u) = \begin{cases} \int_a^b \psi^{**}(\dot{u})\, dx & \text{if } u \in W^{1,4}(a,b) \\ +\infty & \text{otherwise in } L^4(a,b) \end{cases}$$



$$\psi^{**}(\xi) = \begin{cases} \psi(\xi) & \text{if } |\xi| > 1 \\ 0 & \text{if } |\xi| \leq 1 \end{cases}$$

11. GAMMA - CONVERGENCE

DEFINITION 11.1 (Γ- CONVERGENCE)

(X,d) metric space, $f_n, f : X \to \overline{\mathbb{R}}$. We say that $f_n \xrightarrow{\Gamma} f$, Γ-converges, if

① (Γ-liminf inequality) $\forall x \in X$, $\forall\ x_n \to x$ it holds

$$f(x) \leq \liminf_{n \to +\infty} f_n(x_n)$$

② (Γ-limsup inequality) $\forall x \in X$, $\exists\ x_n \to x$ such that

$$\limsup_{n \to +\infty} f_n(x_n) \leq f(x).$$

REMARK If ① and ② hold, then the limsup in ② is actually a limit.

NOTATION The sequences $\{x_n\}$ satisfying ② are called RECOVERY SEQUENCES

RELATIONSHIP WITH POINTWISE CONVERGENCE

PROPOSITION 11.2 (X,d) metric space, $f : X \to \overline{\mathbb{R}}$. Define $f_n := f$, $\forall n \in \mathbb{N}$. Then

$$f_n \xrightarrow{\Gamma} \overline{f}$$

Proof ① Γ-liminf inequality: Let $x_n \to x$. Then

$$\bar{f}(x) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n \to +\infty} f(z_n) \ \middle| \ z_n \to x \right\}$$

$$\leq \liminf_{n \to +\infty} f(x_n) \underset{f_n \equiv f}{=} \liminf_{n \to +\infty} f_n(x_n)$$

② Γ-limsup inequality: Let $x \in X$. By LEMMA 10.8 $\exists\, x_n \to x$ s.t.

$$\bar{f}(x) = \lim_{n \to +\infty} f(x_n)$$

As $f_n \equiv f$, we conclude. $\square$

REMARK PROPOSITION 11.2 implies that Γ-convergence is not related to pointwise convergence. Indeed if $f_n \equiv f$, $f \neq \bar{f}$ then

- $f_n \to f$ pointwise but $f_n \overset{\Gamma}{\not\to} f$ (because Γ-limit is unique)
- $f_n \overset{\Gamma}{\to} \bar{f}$ but $f_n \not\to \bar{f}$ pointwise (because pointwise limit is unique)

However, under additional assumptions, uniform conv. implies Γ-conv.

PROPOSITION 11.3 (X, d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$. Suppose:

(i) $f_n \to f$ uniformly on compact sets of X

(ii) f is LSC.

Then $f_n \overset{\Gamma}{\to} f$.

(Proof will be left as an exercise)

STABILITY PROPERTIES

We now investigate stability properties of Γ-conv. WRT continuous perturbations.

PROPOSITION 11.4 (Stability)

(X, d) metric space, $f_n, f : X \to \overline{\mathbb{R}}$ s.t. $f_n \xrightarrow{\Gamma} f$. Assume $g : X \to \mathbb{R}$ is continuous. Then

$$f_n + g \xrightarrow{\Gamma} f + g$$

(Proof is consequence of PROPOSITION 11.5 below)

A simple generalization of the above is the following.

PROPOSITION 11.5 (Stability)

(X, d) metric space, $f_n, f : X \to \overline{\mathbb{R}}$ s.t. $f_n \xrightarrow{\Gamma} f$. Assume $g_n, g : X \to \mathbb{R}$ are such that:

(i) $g_n \to g$ uniformly on compact sets of X,

(ii) g is continuous.

Then

$$f_n + g_n \xrightarrow{\Gamma} f + g$$

(Proof will be left as an exercise)

Γ-liminf and Γ-limsup

As usual with limits, they don't always exist. For this reason one introduces notions of Γ-liminf and Γ-limsup.

DEFINITION 11.6 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. We define

$$\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) := \inf \left\{ \liminf_{n\to+\infty} f_n(x_n) \;\middle|\; \{x_n\} \subseteq X,\ x_n \to x \right\}$$

$$\Gamma\text{-}\limsup_{n\to+\infty} f_n(x) := \inf \left\{ \limsup_{n\to+\infty} f_n(x_n) \;\middle|\; \{x_n\} \subseteq X,\ x_n \to x \right\}$$

PROPOSITION 11.7 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. Then Γ-liminf f_n and Γ-limsup f_n always exist and satisfy

$$\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) \le \Gamma\text{-}\limsup_{n\to+\infty} f_n(x), \qquad \forall x\in X.$$

Moreover $f_n \xrightarrow{\Gamma} f$ for some $f : X \to \overline{\mathbb{R}}$ if and only if

$$(*) \qquad \boxed{\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) = \Gamma\text{-}\limsup_{n\to+\infty} f_n(x), \qquad \forall x\in X.}$$

Proof The first part of the statement is trivial. Suppose now that $f_n \xrightarrow{\Gamma} f$. Let $x_n \to x$. Then by the Γ-liminf inequality

$$f(x) \le \liminf_{n\to+\infty} f_n(x_n)$$

and so, taking the inf for all $x_n \to x$ yields

$$(1) \qquad f(x) \le \Gamma\text{-}\liminf_{n\to+\infty} f_n(x), \qquad \forall x\in X$$

On the other hand the Γ-limsup inequality says there $\exists\, x_n \to x$ s.t.

$$\limsup_{n \to +\infty} f_n(x_n) \leq f(x)$$

By def. of Γ-limsup we get

$$② \qquad \Gamma\text{-}\limsup_{n \to +\infty} f_n(x) \leq f(x) \quad , \qquad \forall\, x \in X$$

Therefore from ①-② we infer

$$\Gamma\text{-}\limsup_{n \to +\infty} f_n(x) \leq \Gamma\text{-}\liminf_{n \to +\infty} f_n(x) \quad , \qquad \forall\, x \in X$$

As the other inequality was already proven in the first part of the statement, we conclude $(*)$.

Conversely, assume $(*)$ and set

$$f(x) := \Gamma\text{-}\liminf_{n \to +\infty} f_n(x) \overset{(*)}{=} \Gamma\text{-}\limsup_{n \to +\infty} f_n(x)$$

We want to prove that $f_n \xrightarrow{\Gamma} f$. So we need to check Γ-liminf and Γ-limsup inequalities:

- Γ-liminf ineq: Let $x_n \to x$. Then

$$\liminf_{n \to +\infty} f_n(x_n) \underset{\uparrow \\ \text{def of } \Gamma\text{-liminf } f_n}{\geq} \Gamma\text{-}\liminf_{n \to +\infty} f_n(x) \underset{\uparrow \\ \text{def of } f}{=} f(x)$$

• Γ-limsup ineq: Let $x \in X$. Since (*) holds, one can show that there there $\exists \{x_n\}$ s.t.

(3) $\quad x_n \to x \quad$ and $\quad \liminf_{n\to+\infty} f_n(x_n) = \limsup_{n\to+\infty} f_n(x_n) = f(x)$

concluding. $\square$

- Γ-limsup ineq: Let $x \in X$. Since (*) holds, one can show that there there $\exists \{x_n\}$ s.t.

$$③ \quad x_n \to x \quad \text{and} \quad \liminf_{n\to+\infty} f_n(x_n) = \limsup_{n\to+\infty} f_n(x_n) = f(x)$$

concluding. $\square$

LESSON 15 - 23 JUNE 2021

FUNDAMENTAL THEOREM OF Γ-CONVERGENCE

We now want to show that the Γ-limit captures the asymptotic behavior of minimizers for a sequence $f_n : X \to \overline{\mathbb{R}}$.

LEMMA 11.8 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$, $f_n \xrightarrow{\Gamma} f$. Then f is LSC.

Proof Assume $x_n \to x$. We need to show

$$f(x) \leq \liminf_{n\to+\infty} f(x_n).$$

Since $f_k \xrightarrow{\Gamma} f$ we know that for each x_n there $\exists$ a recovery sequence $\{y_k\}$ s.t.

$$\lim_{k\to+\infty} y_k = x_n \quad , \quad f(x_n) = \lim_{k\to+\infty} f_k(y_k)$$

Therefore by a diagonal argument we can find $\{\tilde{y}_n\}$ s.t.

$$(*) \quad d(\tilde{y}_n, x_n) < \frac{1}{n} \quad , \quad |f_n(\tilde{y}_n) - f(x_n)| < \frac{1}{n} \quad , \quad \forall n \in \mathbb{N}.$$

Since $x_n \to x$, the first condition implies $\tilde{y}_n \to x$. Therefore,

Γ- liminf inequality as $\tilde{y}_n \to x$ ... Second condition in (*) ... As $1/n \to 0$

$$f(x) \leq \liminf_{n\to+\infty} f_n(\tilde{y}_n) \leq \liminf_{n\to+\infty} \left[f(x_n) + 1/n \right] = \liminf_{n\to+\infty} f(x_n)$$

concluding that f is LSC. $\square$

PROPOSITION 11.9

(X, d) metric space, $f_n : X \to \overline{\mathbb{R}}$, $f_n \xrightarrow{\Gamma} f$.

1. Let $A \subseteq X$ be open. Then

$$\limsup_{n\to+\infty} \left\{ \inf_{x\in A} f_n(x) \right\} \leq \inf_{x\in A} f(x)$$

2. Let $K \subseteq X$ be compact. Then

$$\liminf_{n\to+\infty} \left\{ \inf_{x\in K} f_n(x) \right\} \geq \inf_{x\in K} f(x)$$

Proof 1. Fix $\varepsilon > 0$. By definition of inf there $\exists$ $\hat{x} \in A$ s.t.

$$(*) \qquad f(\hat{x}) \leq \inf_{x\in A} f(x) + \varepsilon$$

Let x_n be a recovery sequence for $\hat{x}$, i.e.

$$x_n \to \hat{x} \qquad \text{and} \qquad f(\hat{x}) \geq \limsup_{n\to+\infty} f_n(x_n)$$

Since A is open, $\hat{x} \in A$, and $x_n \to \hat{x}$, then $x_n \in A$ for $n >> 0$. Then

$$\inf_{x\in A} f(x) + \varepsilon \overset{(*)}{\geq} f(\hat{x}) \overset{x_n \text{ rec. seq. for } \hat{x}}{\geq} \limsup_{n\to+\infty} f_n(x_n) \overset{\text{As } x_n\in A \text{ for } n>>0}{\geq} \limsup_{n\to+\infty} \inf_{x\in A} f_n(x)$$

As ε is arbitrary, we conclude.

(2) Let $\{x_n\}\subseteq K$ be a sequence of quasi-minimizers, i.e.

$$(**) \qquad f_n(x_n) \leq \inf_{x\in K} f_n(x) + \frac{1}{n}, \qquad \forall\, n\in\mathbb{N}.$$

Since K is compact, up to extracting a subsequence, we can suppose

$$x_n \to \hat{x} \quad \text{for some } \hat{x}\in K.$$

Then

$$\inf_{x\in K} f(x) \overset{\text{As } \hat{x}\in K}{\leq} f(\hat{x}) \overset{\Gamma\text{-liminf ineq., as } x_n\to\hat{x}}{\leq} \liminf_{n\to+\infty} f_n(x_n)$$

$$\overset{(**)}{\leq} \liminf_{n\to+\infty} \left[\inf_{x\in K} f_n(x) + \frac{1}{n}\right] \overset{\text{as } 1/n\to 0}{=} \liminf_{n\to+\infty} \left[\inf_{x\in K} f_n(x)\right]$$

□

DEFINITION 11.10 (EQUICOERCIVITY)

(X,d) metric space, $f_n : X \to \mathbb{R}$. We say that $\{f_n\}$ is EQUICOERCIVE if $\exists\, K\subseteq X$ non empty and compact s.t.

$$\inf\{f_n(x) : x\in X\} = \inf\{f_n(x) : x\in K\}, \qquad \forall\, n\in\mathbb{N}.$$

(K is independent on n)

REMARK 11.11 (X, d) metric space, $f_n : X \to \mathbb{R}$. Suppose that there $\exists$ $M \in \mathbb{R}$ s.t. the set

$$\{ x \in X \mid f_n(x) \leq M, \ \forall n \in \mathbb{N} \}$$

is non empty and pre-compact. Then $\{f_n\}$ is EQUICOERCIVE.

Proof: Set $K := \{ x \in X \mid f_n(x) \leq M, \ \forall n \in \mathbb{N} \}$. This is non empty, compact, and satisfies the condition $\inf_{x \in X} f_n = \inf_{x \in K} f_n$ for all $n \in \mathbb{N}$. $\square$

We are finally able to prove the main result of this section.

THEOREM 11.12 (CONVERGENCE OF MINIMUMS AND MINIMIZERS)

(X, d) metric space, $f_n : X \to \overline{\mathbb{R}}$. Suppose that:

(i) $\{f_n\}$ is equicoercive WRT the compact set K

(ii) $f_n \xrightarrow{\Gamma} f$ for some $f : X \to \overline{\mathbb{R}}$

Then: 1. f admits minimum on X

2. As $n \to +\infty$ we have $\inf_{x \in X} f_n(x) \longrightarrow \min_{x \in X} f(x)$

3. Assume $\{x_n\}$ is a sequence of almost-minimizers, i.e.,

$$\lim_{n \to +\infty} \left\{ f_n(x_n) - \inf_{x \in X} f_n(x) \right\} = 0 .$$

Suppose that $x_{n_k} \to \hat{x}$. Then $\hat{x}$ is minimum for f over X.

Proof 1 By LEMMA 11.8 we know that the Γ-limit f is LSC.
Since K is compact, by the DIRECT METHOD (THM 9.4) there $\exists\ \hat{x} \in K$ s.t.

$$(K) \qquad f(\hat{x}) = \min_{x \in K} f(x)$$

(f admits minimum on K)

We claim that

$$(*) \qquad f(\hat{x}) = \min_{x \in X} f(x)$$

($\hat{x}$ minimizes f on X)

Indeed let $y \in X$ be arbitrary. Then there $\exists$ a recovery sequence $\{y_n\}$ s.t.

$$y_n \to y \qquad \text{and} \qquad f(y) = \lim_{n \to +\infty} f_n(y_n)$$

Then

$\{f_n(y_n)\}$ is convergent — def of inf

$$f(y) = \lim_{n \to +\infty} f_n(y_n) = \liminf_{n \to +\infty} f_n(y_n) \geq \liminf_{n \to +\infty} \inf_{x \in X} f_n(x)$$

$$= \liminf_{n \to +\infty} \left\{ \inf_{x \in K} f_n(x) \right\} \geq \inf_{x \in K} f(x) = f(\hat{x})$$

Equicoercivity

PROP 11.9, point 2 as K is compact — by (K)

and so $(*)$ holds.

2 We have:

By PROPOSITION 11.9 point 1, since X is open — Equicoercivity

$$\inf_{x \in X} f(x) \geq \liminf_{n \to +\infty} \left\{ \inf_{x \in X} f_n(x) \right\} = \liminf_{n \to +\infty} \left\{ \inf_{x \in K} f_n(x) \right\}$$

$$\geq \inf_{x \in K} f(x) = \min_{x \in X} f(x)$$

By PROPOSITION 11.9 point 2, as K is compact — by (K) and $(*)$

proving that

$$(**) \qquad \liminf_{n \to +\infty} \left\{ \inf_{x \in X} f_n(x) \right\} = \min_{x \in X} f(x)$$

Now let $\hat{x}$ be minimizer for f on X, which exists by point (1).
Let $\{x_n\}$ be a recovery sequence, i.e.

$$x_n \to \hat{x} \qquad \text{and} \qquad f(\hat{x}) = \lim_{n \to +\infty} f_n(x_n)$$

Clearly

$$\inf_{x \in X} f_n(x) \le f_n(x_n), \qquad \forall\, n \in \mathbb{N}$$

Taking the limsup in the above yields

$$(***) \qquad \limsup_{n \to +\infty} \left\{ \inf_{x \in X} f_n(x) \right\} \le \limsup_{n \to +\infty} f_n(x_n) \underset{\{x_n\} \text{ is recovery sequence}}{=} f(\hat{x}) \underset{\hat{x} \text{ is minimizer}}{=} \min_{x \in X} f(x)$$

Therefore

$$\min_{x \in X} f(x) \overset{(**)}{=} \liminf_{n \to +\infty} \left\{ \inf_{x \in X} f_n(x) \right\} \overset{\text{property of liminf / limsup}}{\le} \limsup_{n \to +\infty} \left\{ \inf_{x \in X} f_n(x) \right\} \overset{(***)}{\le} \min_{x \in X} f(x)$$

concluding.

③ Let $\{x_n\}$ be a sequence of quasi-minimizers s.t. $x_{n_k} \to \hat{x}$. Then

$$f(\hat{x}) \overset{\Gamma\text{-liminf inequality}}{\leq} \liminf_{k\to+\infty} f_{n_k}(x_{n_k}) = \liminf_{k\to+\infty} \Big\{ \underbrace{f_{n_k}(x_{n_k}) - \inf_{x\in X} f_{n_k}(x)}_{\to 0 \text{ by assumption}} + \inf_{x\in X} f_{n_k}(x) \Big\}$$

$$= \liminf_{k\to+\infty} \Big\{ \inf_{x\in X} f_{n_k}(x) \Big\} \underset{\text{point ② of this Theorem}}{=} \min_{x\in X} f(x)$$

showing that $\hat{x}$ minimizes f over X. □

EXAMPLE 11.13 Consider the functionals $F_n : C^1[0,1] \to \mathbb{R}$ defined by

$$F_n(u) := \int_0^1 n\dot{u}^2 + (u - \arctan x)^2 \, dx$$

QUESTION What is the limit of $M_n := \inf\{F_n(u) : u \in C^1[0,1]\}$.

Extend F_n to $L^2(0,1)$ by setting $F_n := +\infty$ in $L^2 \setminus C^1$. Thus

$$M_n = \inf\{F_n(u) \mid u \in L^2(0,1)\}.$$

CLAIM $F_n \xrightarrow{\Gamma} F$ in $L^2(0,1)$, with

$$F(u) := \begin{cases} \int_0^1 (u - \arctan x)^2 dx, & \text{if } u \text{ is constant} \\ +\infty, & \text{otherwise in } L^2(0,1) \end{cases}$$

Proof of CLAIM Note that $F_n = G_n + H$ with

$$G_n(u) := \begin{cases} \int_0^1 n \dot{u}^2 \, dx & \text{if } u \in C^1[0,1] \\ +\infty & \text{otherwise} \end{cases}, \qquad H(u) := \int_0^1 (u - \arctan x)^2 \, dx$$

Clearly H is continuous in $L^2(0,1)$. Therefore, by PROPOSITION 11.4, it is sufficient to compute the Γ-limit of G_n. We have that

$$G_n \xrightarrow{\Gamma} G, \qquad \text{with} \qquad G(u) := \begin{cases} 0, & \text{if } u \text{ is constant} \\ +\infty, & \text{otherwise in } L^2(0,1) \end{cases}$$

- Γ-liminf inequality: suppose $u_n \to u$ in $L^2(0,1)$. We need to show

$$(*) \qquad G(u) \leq \liminf_{n \to +\infty} G_n(u_n).$$

WLOG we can assume the RHS to be finite, so there $\exists$ a subsequence s.t.

$$G_{n_k}(u_{n_k}) \leq M, \qquad \forall \, k \in \mathbb{N}.$$

This means

$$\int_0^1 \dot{u}_{n_k}^2 \, dx \leq \frac{M}{n_k},$$

which implies

$$\dot{u}_{n_k} \to 0 \quad \text{strongly in } L^2(0,1).$$

Since we are assuming $u_n \to u$ strongly in $L^2(0,1)$, REMARK 7.17 implies

$$u_{n_k} \to u \quad \text{strongly in } W^{1,2}(0,1), \qquad \text{with } \dot{u} = 0 \quad \text{weakly}.$$

Thus $u \in W^{1,2}(0,1)$ with $\dot{u} \in C[0,1]$. Therefore $u \in C^1[0,1]$ by PROPOSITION 7.22. Hence the relationship $\dot{u}=0$ also holds in the classical sense (as the weak derivative of a differentiable function coincides with the classical one). Since $[0,1]$ is connected then

$$u \in C^1[0,1], \quad \dot{u}=0 \quad \Longrightarrow \quad u = \text{constant}$$

Thus $G(u)=0$ by definition and (*) holds (being $G_n \geq 0$).

- Γ-limsup inequality: Let $u \in L^2(0,1)$. We need to construct a recovery sequence.

  - If u is not constant, then $G(u)=+\infty$. Thus setting $u_n := u$, $\forall n \in \mathbb{N}$ we get $u_n \to u$ and, trivially,

  $$\limsup_{n \to +\infty} G_n(u_n) \leq +\infty = G(u).$$

  - If u is constant, then $G(u)=0$. Again set $u_n := u$, $\forall n \in \mathbb{N}$. Then $u_n \to u$. Moreover, as u is constant, then $u \in C^\infty[0,1]$ and $\dot{u}=0$. Therefore

  $$G_n(u_n) = G_n(u) = \int_0^1 n\, \dot{u}^2 \, dx = 0, \qquad \forall n \in \mathbb{N}$$

  and the Γ-limsup inequality trivially holds.

Then $G_n \xrightarrow{\Gamma} G$ and so $F_n = G_n + H \xrightarrow{\Gamma} G + H = F$, by PROPOSITION 11.4. $\square$

In order to apply THEOREM 11.12, we also need to show that the sequence of functionals F_n is EQUICOERCIVE in $L^2(0,1)$.

CLAIM $\{F_n\}$ is EQUICOERCIVE in $L^2(0,1)$.

Proof of Claim By REMARK 11.11 it is sufficient to show $\exists$ of $M \in \mathbb{R}$ s.t.

$$K := \{ u \in L^2(0,1) \mid F_n(u) \le M \}$$

is non-empty and pre-compact. First of all, note that

$$F_n(0) = \int_0^1 (\arctan x)^2 \, dx \le \left(\frac{\pi}{2}\right)^2 < 10 \qquad , \qquad \forall n \in \mathbb{N}.$$

We then choose $M := 10$, so that $K \ne \emptyset$. We are left to show that K is pre-compact in $L^2(0,1)$. Indeed,

$$F_n(u) \le 10 \overset{\text{def of } F_n}{\Longrightarrow} \begin{cases} \displaystyle\int_0^1 \dot{u}^2 \, dx \le \frac{10}{n} \\ \displaystyle\int_0^1 (u - \arctan x)^2 \, dx \le 10 \end{cases} \Rightarrow \|u\|_{W^{1,2}} \le C$$

for some $C > 0$ not depending on n and on u. Thus

$$K = \{ u \in L^2(0,1) \mid F_n(u) \le 10 \} \subseteq \tilde{K} := \{ u \in W^{1,2}(0,1) \mid \|u\|_{W^{1,2}} \le C \}$$

Note that $\tilde{K}$ is compact in $L^2(0,1)$, thanks to the compact embedding $W^{1,2}(0,1) \hookrightarrow L^2(0,1)$ of THEOREM 7.27.

Therefore K is pre-compact, since $\overline{K}$ is closed and contained in the compact $\tilde{K}$. $\square$

Thus we have shown

(i) $\{F_n\}$ is EQUICOERCIVE in $L^2(0,1)$

(ii) $F_n \xrightarrow{\Gamma} F$ in $L^2(0,1)$

From THEOREM 11.12 we then get

$$\inf_{u \in L^2(0,1)} F_n(u) \longrightarrow \min_{u \in L^2(0,1)} F(u),$$

that is,

$$M_n \to M := \min_{u \in L^2(0,1)} F(u)$$

Since $F(u) < +\infty$ if and only if u is constant, then

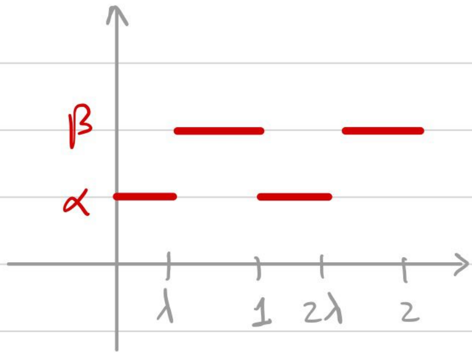
$$M = \min \left\{ \int_0^1 (\lambda - \arctan x)^2 \, dx \;\middle|\; \lambda \in \mathbb{R} \right\}$$

which can be computed explicitly.

APPLICATION: HOMOGENIZATION PROBLEMS

DEFINITION 11.14 For $\alpha, \beta \in \mathbb{R}$, $\lambda \in (0,1)$ define

$$A(x) := \begin{cases} \alpha & \text{if } x \in [0, \lambda) \\ \beta & \text{if } x \in [\lambda, 1) \end{cases}$$



Extend A to $\mathbb{R}$ by periodicity. For $n \in \mathbb{N}$ set

$$A_n(x) := A(nx)$$

FACT $A_n(x) \rightharpoonup \underbrace{\lambda\alpha + (1-\lambda)\beta}_{\text{Average of } A \text{ in } [0,1]}$ weakly in $L^p(a,b)$, $\forall\, 1 \leq p < +\infty$

Define $F_n : C^1[a,b] \to \mathbb{R}$ by

$$F_n(u) := \int_a^b A_n(x)\, \dot{u}(x)\, dx$$

Extend F_n to $+\infty$ on $L^2(a,b) \setminus C^1[a,b]$. Define $F : L^2(a,b) \to \overline{\mathbb{R}}$ by

$$F(u) := \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

One might expect $F_n \xrightarrow{\Gamma} cF$ with $c = \lambda\alpha + (1-\lambda)\beta$. However this is <u>FALSE</u>

THEOREM 11.15 Suppose $\alpha, \beta > 0$. Consider F_n, F as above. Then

$$F_n \xrightarrow{\Gamma} cF, \qquad c := \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}}$$

(Harmonic mean of A in $[0,1]$: $\int_0^1 \frac{1}{A(x)}\, dx$)

In order to prove the above, consider the following:

CELL - PROBLEM For $\ell > 0$, consider the problem:

$$\min \left\{ \int_0^1 A(x)\, \dot{u}^2 \, dx \;\middle|\; u(0)=0,\ u(1)=\ell \right\}$$

This is called cell-problem because A has only one oscillation in $[0,1]$

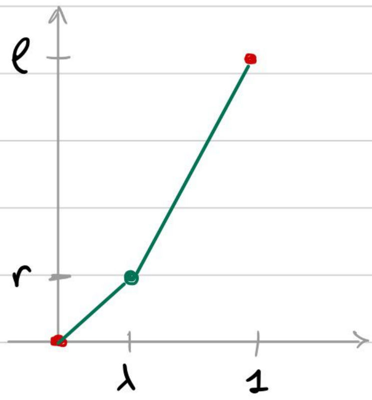
LEMMA 11.16 Let $\alpha, \beta > 0$. The cell-problem has solution

$$\min \left\{ \int_0^1 A(x)\, \dot{u}^2 \, dx \;\middle|\; u(0)=0,\ u(1)=\ell \right\} = c\ell^2, \qquad c := \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}}$$

Proof Since $A=\alpha$ in $[0,\lambda)$ and $A=\beta$ in $[\lambda,1)$, the separate problems in $[0,\lambda]$ and $[\lambda,1]$ become

$$\min \left\{ \alpha \int_0^\lambda \dot{u}^2 \, dx \;\middle|\; u(0)=0,\ u(\lambda)=r \right\}$$

$$\min \left\{ \beta \int_\lambda^1 \dot{u}^2 \, dx \;\middle|\; u(\lambda)=r,\ u(1)=\ell \right\}$$



We already know that the above problems are solved by straight lines u_1, u_2 respectively. In particular

$$\dot{u}_1 = \frac{r}{\lambda}, \qquad \dot{u}_2 = \frac{\ell - r}{1-\lambda}$$

Define

$$u_r(x) := \begin{cases} u_1 & \text{if } x \in [0,\lambda) \\ u_2 & \text{if } x \in [\lambda, 1) \end{cases}$$

It is easy to show that

$$(*)\quad \min\left\{ \int_0^1 A(x)\,\dot{u}^2\,dx \;\middle|\; u(0)=0,\ u(1)=\ell \right\} = \min\left\{ \int_0^1 A(x)\,\dot{u}_r^2\,dx \;:\; r\in\mathbb{R} \right\}$$

Now

$$\int_0^1 A(x)\,\dot{u}_r^2\,dx = \int_0^\lambda A(x)\,\dot{u}_1^2\,dx + \int_\lambda^1 A(x)\,\dot{u}_2^2\,dx = \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda}$$

so that

$$\min_{r\in\mathbb{R}} \int_0^1 A(x)\,\dot{u}_r^2\,dx = \min_{r\in\mathbb{R}} \left\{ \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} \right\}$$

Now

$$\alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} = Ar^2 + Br + C, \qquad \begin{cases} A = \dfrac{\alpha}{\lambda} + \dfrac{\beta}{1-\lambda} \\[2mm] B = -\dfrac{2\ell\beta}{1-\lambda} \\[2mm] C = \dfrac{\beta}{1-\lambda}\,\ell^2 \end{cases}$$

which is minimized at $r = -\dfrac{B}{2A}$. Substituting into Ar^2+Br+C, we obtain the minimum $-\dfrac{B^2}{4A}+C$, i.e.,

$$\min_{r\in\mathbb{R}} \left\{ \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} \right\} = -\frac{B^2}{4A} + C = \frac{\alpha\beta}{\lambda\beta + (1-\lambda)\alpha}\,\ell^2 = c\,\ell^2$$

Recalling $(*)$, we conclude. $\square$

REMARK Solving the cell-problem is equivalent to solving

$$\min\left\{ c\int_0^1 \dot{u}^2\,dx \;\middle|\; u(0)=0,\; u(1)=e \right\}, \qquad c := \frac{1}{\frac{\lambda}{\alpha}+\frac{1-\lambda}{\beta}}$$

(Indeed, the solution to the above is given by the straight line $u(x)=ex$, so that $c\int_0^1 \dot{u}^2 dx = ce^2$)

LEMMA 11.17 (RESCALED CELL – PROBLEM)

The rescaled cell-problem satisfies

Harmonic average of A

$$\min\left\{ \int_{k/n}^{\frac{k+1}{n}} \underbrace{A(nx)}\,\dot{u}^2\,dx \;\middle|\; u\left(\tfrac{k}{n}\right)=A,\; u\left(\tfrac{k+1}{n}\right)=B \right\} = cn\,(B-A)^2$$

$A(nx)$ values α in $\left[\frac{k}{n}, \frac{k}{n}+\frac{\lambda}{n}\right)$ and β in $\left[\frac{k}{n}+\frac{\lambda}{n}, \frac{k+1}{n}\right)$

Thus $A(nx)$ has only one oscillation in $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, and this is still a cell-problem

Proof Same as LEMMA 11.16. □

REMARK Solving the rescaled cell-problem is equivalent to solving

$$\min\left\{ c\int_{k/n}^{\frac{k+1}{n}} \dot{u}^2\,dx \;\middle|\; u\left(\tfrac{k}{n}\right)=A,\; u\left(\tfrac{k+1}{n}\right)=B \right\}, \qquad c := \frac{1}{\frac{\lambda}{\alpha}+\frac{1-\lambda}{\beta}}$$

(Indeed, the solution to the above is given by the straight line u with

$$\dot{u} = \frac{B-A}{\frac{k+1}{n}-\frac{k}{n}} = n(B-A)$$

so that

$$c\int_{k/n}^{\frac{k+1}{n}} \dot{u}^2\,dx = c\,n^2(B-A)^2\cdot\frac{1}{n} = cn\,(B-A)^2$$

)

Γ-LIMINF INEQUALITY FOR THEOREM 11.15

We now sketch the proof of the Γ-liminf inequality in THEOREM 11.15.
Let $u_n \to u$ strongly in $L^2(a,b)$. We need to prove that

$$(*) \qquad c\, F(u) \leq \liminf_{n \to +\infty} F_n(u_n)$$

WLOG assume RHS finite, i.e., $\exists$ a subsequence s.t.

$$F_{n_k}(u_{n_k}) \leq M \,, \qquad \forall\, k \in \mathbb{N}.$$

In particular $\{u_{n_k}\} \subseteq C^1[a,b]$ and

$$\int_a^b A_{n_k}(x)\, \dot{u}_{n_k}^2 \, dx \leq M \,, \qquad \forall\, k \in \mathbb{N}.$$

Now $A_{n_k} \geq \min\{\alpha, \beta\} > 0$, from which we deduce that $\{\dot{u}_{n_k}\}$ is bounded in $L^2(a,b)$. Thus, $\exists\, v \in L^2(a,b)$ s.t.

$$\dot{u}_{n_{k_j}} \rightharpoonup v \quad \text{weakly in } L^2(a,b)$$

As $u_n \to u$ in $L^2(a,b)$, from REMARK 7.18 we conclude that

$$u_{n_{k_j}} \rightharpoonup u \quad \text{weakly in } W^{1,2}(a,b)$$

Since the above limit does not depend on the subsequence, we get convergence along the whole sequence, i.e.,

$$(W) \qquad u_n \rightharpoonup u \quad \text{weakly in } W^{1,2}(a,b)$$

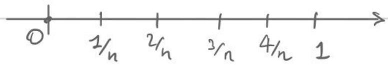
The compact embedding $W^{1,2}(a,b) \hookrightarrow C[a,b]$ (see THEOREM 7.27) implies

$$(U) \qquad u_n \to u \quad \text{uniformly in } [a,b] \,, \qquad \{u_n\}, u \text{ continuous}$$

• Step 1: Assume $u_n \to u$ uniformly in $[0,1]$.

We want to prove ⊛ localized to $[0,1]$ (i.e. $a=0$, $b=1$)

Divide $[0,1]$ in subintervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. Then



$$\int_0^1 A(nx)\, \dot{u}_n^2\, dx = \sum_{k=0}^{n-1} \underbrace{\int_{\frac{k}{n}}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2\, dx}$$

u_n is competitor for RESCALED CELL-PROBLEM WITH $A = u_n\left(\frac{k}{n}\right)$, $B = u_n\left(\frac{k+1}{n}\right)$

$$(\text{LEMMA } 11.17) \geq \sum_{k=0}^{n-1} c\, n \left[u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right) \right]^2$$

$$\left(n \sum \alpha_k^2 \geq \left(\sum \alpha_k\right)^2 \right) \geq c \left[\sum_{k=0}^{n-1} u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right) \right]^2$$

$$= c \left[u_n(1) - u_n(0) \right]^2$$

As $u_n \to u$ uniformly, we get

$$\liminf_{n \to +\infty} \int_0^1 A(nx)\, \dot{u}_n^2\, dx \geq \liminf_{n \to +\infty} c \left[u_n(1) - u_n(0) \right]^2$$

$$(u_n \to u \text{ uniformly}) = c \left[u(1) - u(0) \right]^2$$

NOTE The above would be ⊛ if $u(x) = mx + q$ line, since

$$c \int_0^1 \dot{u}^2\, dx = c\, m^2 = c \left[u(1) - u(0) \right]^2$$

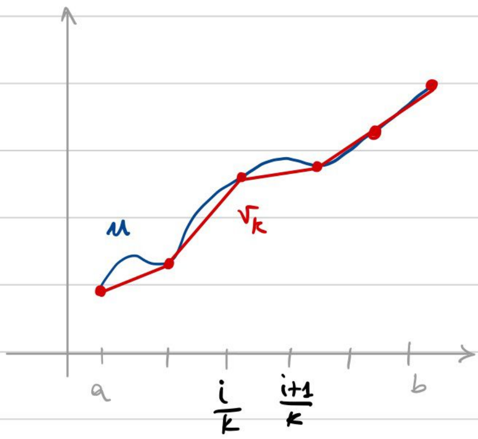
- **Step 2:** Assume $u \in L^2(\tilde{a}, \tilde{b})$ with $\tilde{a} < \tilde{b}$ arbitrary.
Suppose $u_n \to u$ uniformly in $[\tilde{a}, \tilde{b}]$. By a rescaling argument, and proceeding as in STEP 1, one can show

$$\text{(L)} \qquad \liminf_{n \to +\infty} \int_{\tilde{a}}^{\tilde{b}} A_n(x)\, \dot{u}_n^2 \, dx \geq c\, (\tilde{b} - \tilde{a}) \left[\frac{u(\tilde{b}) - u(\tilde{a})}{\tilde{b} - \tilde{a}} \right]^2$$

NOTE Again, (L) is (*) if $u(x) = mx + q$

- **Step 3:** General case. Assume $u_n \to u$ in $L^2(a,b)$. Then WLOG (W)-(U) hold and u_n, u are continuous.

Divide (a,b) in k equal parts, $k \in \mathbb{N}$ (suppose $\frac{b-a}{k} \in \mathbb{N}$)
Let v_k be the linear interpolation of u on the grid (recall u continuous)



By applying (L) to u_n on $\left[\frac{i}{k}, \frac{i+1}{k}\right]$ we get
(this is possible as $u_n \to u$ uniformly)

$$\liminf_{n \to +\infty} \int_{\frac{i}{k}}^{\frac{i+1}{k}} A(nx)\, \dot{u}_n^2 \, dx \overset{(L)}{\geq} c\, \frac{1}{k} \left(\frac{u\left(\frac{i+1}{k}\right) - u\left(\frac{i}{k}\right)}{\frac{1}{k}} \right)^2$$

$$= c \int_{\frac{i}{k}}^{\frac{i+1}{k}} \dot{v}_k^2 \, dx \qquad \left(\text{as } \dot{v}_k = \frac{u\left(\frac{i+1}{k}\right) - u\left(\frac{i}{k}\right)}{1/k} \text{ on } \left[\frac{i}{k}, \frac{i+1}{k}\right] \right)$$

Summing over i we find

$$(**) \qquad \liminf_{n \to +\infty} F_n(u_n) \geq c\, F(v_k), \qquad \forall\, k \in \mathbb{N}$$

Now, one can check that as the grid width goes to zero, we have

$$v_k \to u \quad \text{uniformly}, \quad \text{and} \quad F(v_k) \nearrow F(u)$$

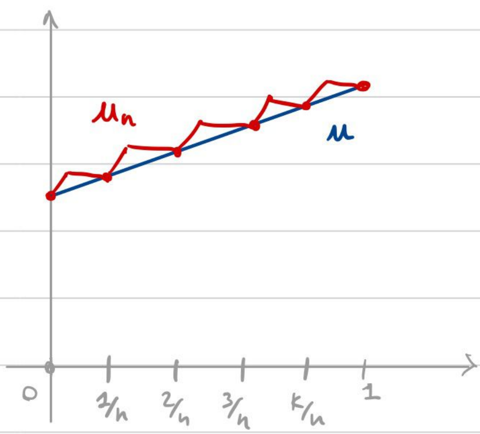
Thus, taking the sup for $k \in \mathbb{N}$ in (**) yields (*), concluding the Γ-liminf inequality.

Γ-LIMSUP INEQUALITY FOR THEOREM 11.15

We now sketch the proof of the Γ-limsup inequality in THEOREM 11.15

- **Step 1** : $u = \ell x + q$ in $[0,1]$. Then we can chose the recovery sequence u_n in the following way:

Divide $[0,1]$ in sub-intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. In each sub-int. u_n is the solution to the rescaled cell-problem with data $A = u\left(\frac{k}{n}\right)$, $B = u\left(\frac{k+1}{n}\right)$



In this way the energy in each $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ is

$$\int_{k/n}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx = c\, n \left[u\left(\frac{k+1}{n}\right) - u\left(\frac{k}{n}\right)\right]^2 = c \int_{k/n}^{\frac{k+1}{n}} \dot{u}^2 \, dx$$

u_n solution to rescaled cell-problem (LEMMA 11.17)

u straight line

and the total energy becomes :

$$F_n(u_n) = \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx$$

$$= c \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} \dot{u}^2 \, dx \quad = c\, F(u)$$

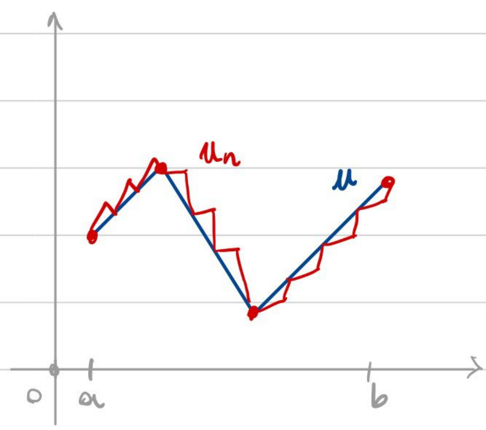
Thus $F_n(u_n) \to c\, F(u)$ trivially. One can also show that, as the grid refines,

$$u_n \to u \quad \text{uniformly in} \quad [0,1],$$

concluding the Γ-limsup inequality.

- **Step 2:** u piecewise affine in $[a,b]$.

To obtain u_n just divide $[a,b]$ into the sub-intervals in which u is affine and in each of those define u_n as in STEP 1.



- **Step 3:** Let $u \in L^2(a,b)$ be arbitrary.

REMARK 11.18 In general, it is sufficient to show the Γ-limsup inequality for elements in D, where $D \subseteq X$ is an energy dense set wRT the Γ-limit.

(This is easily proven with a diagonal argument)

Choose D as the set of PIECEWISE AFFINE FUNCTIONS. Then D is energy dense WRT cF (easy check). The Γ-limsup inequality holds in D by STEP 2. Therefore we conclude the Γ-limsup inequality in $L^2(a,b)$ by REMARK 11.18.

EXAM INFO

- ORAL EXAM ON TOPICS SEEN DURING THE COURSE
  (I WILL REFER TO THE SUMMARY ON THE COURSE WEB PAGE)

- PLEASE, SEND ME AN EMAIL WITH SUGGESTED DATES
  (silvio.fanzon@uni-graz.at)

- EXAM HELD ONLINE
  (IF OPTION IS AVAILABLE)