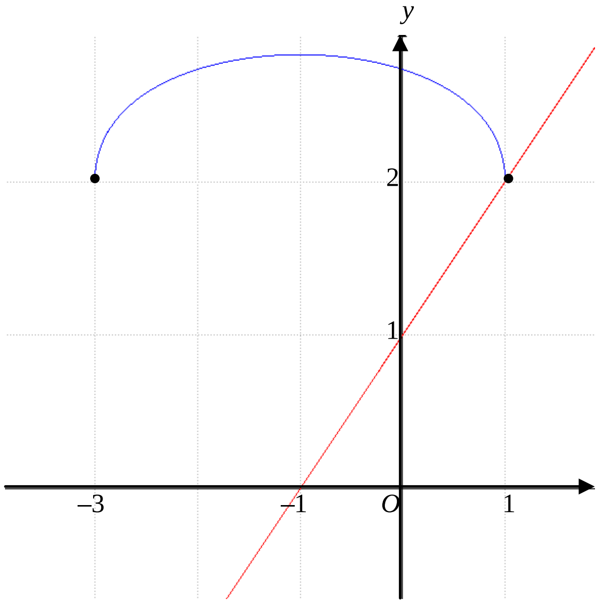
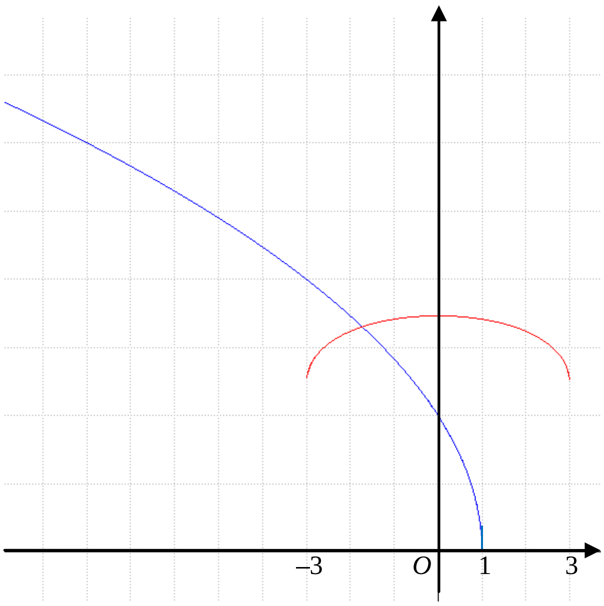
SII 2011 Q1



Domain $-3 \le x \le 1$	**B1**
$(-3, 2)$ and $(1, 2)$ on graph	**B1**
y–intercept at $\left(0,\ 1+\sqrt{3}\right)$	**B1**
Symmetry in the line $x = -1$	**B1**
Correct shape of curve (ignore gradient at endpoints, but no vertex at peak)	**B1**
Max. at $\left(-1,\ 2\sqrt{2}\right)$	**B1**
$y = x + 1$ correctly drawn on diagram	**B1**
Noting that curve & line cross only once so equation only has one root	**E1**
$x = 1$	**B1**



Half-parabola $y = 2\sqrt{1-x}$ from $(1, 0)$	**B1**
Dome-shaped curve for $y = \sqrt{3+x} + \sqrt{3-x}$ in $[-3, 3]$	**B1**
Noting that equation only has one root	**E1**
Squaring both sides:	**M1**
$4-4x = 3+x+2\sqrt{9-x^2}+3-x$	
$\Rightarrow \sqrt{9-x^2} = -(2x+1)$	**A1**
Squaring again after suitable rearrangement:	
$9-x^2 = 4x^2+4x+1$	**M1**
$\Rightarrow 0 = 5x^2+4x-8$	**A1**
Solving a quadratic equation	**M1**
$x = \dfrac{-4 \pm \sqrt{16+4.5.8}}{10} = -\tfrac{2}{5}\left(1 \pm \sqrt{11}\right)$	**A1**
$x = -\tfrac{2}{5}\left(1+\sqrt{11}\right)$ or exact ≡	**A1**
Valid reason given for choosing this root; e.g. from graph, intersection is at negative x	**E1**

SII 2011 Q2

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000   (Sorry, no marks)

$x + y = k$, $(x + y)(x^2 - xy + y^2) = kz^3$ — **M1** Substituting in

$\Rightarrow \; x^2 - (k-x)x + (k-x)^2 - z^3 = 0$

$\Rightarrow \; 3x^2 - 3kx + k^2 - z^3 = 0$ **(∗)** — **A1** Correct, identifiable quadratic

$$x = \frac{3k \pm \sqrt{9k^2 - 12(k^2 - z^3)}}{6} = \tfrac{1}{2}\left\{k \pm \sqrt{\frac{4z^3 - k^2}{3}}\right\}$$

For x real, we need $4z^3 - k^2 \geq 0$ i.e. $z^3 \geq \frac{1}{4}k^2$ — **M1** Considering discriminant; **A1 Given Answer**

For x integer, we need $\dfrac{4z^3 - k^2}{3}$ a perfect square — **E1 Given answer** Explained

For distinct positive roots (N.B. one root is x, the other is y)

we need $k - \sqrt{\dfrac{4z^3 - k^2}{3}} > 0$ i.e. $3k^2 > 4z^3 - k^2$ i.e. $z^3 < k^2$ — **M1 A1 Given Answer**

Alternatively, $y = k - x$ in **(∗)** $\Rightarrow \; z^3 = k^2 - 3xy < k^2$ (since $x, y > 0$) etc.

When $k = 20$, $100 \leq z^3 < 400 \Rightarrow z = 5, 6, 7$ — **M1** Using given results to get a suitable small set of values of z

with $\dfrac{4z^3 - k^2}{3} = \frac{100}{3}$, 88, 324 = 18^2 — **M1** Using other given condition to test these z's

Thus $z = 7$ gives $(x, y) = (1, 19)$ — **A1** i.e. $20 = \left(\dfrac{1}{7}\right)^3 + \left(\dfrac{19}{7}\right)^3$

NB $\dfrac{4z^3 - k^2}{3} = \begin{cases} (k - 2x)^2 \\ (y - x)^2 \end{cases}$ gives 2 of the results directly (**M1 A1 E1**)

$< k^2$ gives the 3rd (**M1 A1**)

$x + y = z^2$, $(x + y)(x^2 - xy + y^2) = kz.z^2$ **M1** Substituting in

$\Rightarrow\ x^2 - (z^2 - x)x + (z^2 - x)^2 - kz = 0$

$\Rightarrow\ 3x^2 - 3z^2x + z^4 - kz = 0$ **(∗)** **A1** Correct, identifiable quadratic

$$x = \frac{3z^2 \pm \sqrt{9z^4 - 12(z^4 - kz)}}{6} = \tfrac{1}{2}\left\{z^2 \pm \sqrt{\frac{4kz - z^4}{3}}\right\}$$

For x real, we need $4kz - z^4 \ge 0$ i.e. $z^3 \le 4k$ **M1** Considering discriminant; **A1**

and $\dfrac{4kz - z^4}{3}$ a perfect square **E1** Noted

and $z^2 - \sqrt{\dfrac{4kz - z^4}{3}} > 0$ i.e. $3z^4 > 4kz - z^4$ i.e. $z^3 > k$ **B1**

OR by alternate "squares" methods, as before

When $k = 19$, $19 < z^3 \le 76 \Rightarrow z = 3$ or 4 **M1** Using given results to get a suitable small set of values of z

with $\dfrac{z(76 - z^3)}{3} = 49 = 7^2$ or $16 = 4^2$ **M1** Using other given condition to test them

Thus $z = 3$ gives $(x, y) = (1, 8)$ **A1** i.e. $19 = \left(\dfrac{1}{3}\right)^3 + \left(\dfrac{8}{3}\right)^3$

$z = 4$ gives $(x, y) = (6, 10)$ **A1** i.e. $19 = \left(\dfrac{3}{2}\right)^3 + \left(\dfrac{5}{2}\right)^3$

<u>Additional Note</u>

$x + y = kz$, $(x + y)(x^2 - xy + y^2) = kz.z^2 \Rightarrow x^2 - (kz - x)x + (kz - x)^2 - z^2 = 0$

$\Rightarrow\ 3x^2 - 3kzx + z^2(k^2 - 1) = 0$

$$x = \frac{3kz \pm \sqrt{9k^2z^2 - 12z^2(k^2 - 1)}}{6} = \tfrac{1}{6}z\left\{3k \pm \sqrt{12 - 3k^2}\right\}$$

requiring $12 - 3k^2 \ge 0$ i.e. $k^2 \le 4 \Rightarrow k = 1$ or 2

$k = 1$: $x^3 + y^3 = z^3$ has NO solutions by *Fermat's Last Theorem*

$k = 2$: $x^3 + y^3 = 2z^3$ has (trivially) infinitely many solutions $x = y = z$

SII 2011 Q3

$f'(x) = \cos x - \{x.-\sin x + \cos x\} = x \sin x$ **M1** Product Rule; **A1**

≥ 0 for $x \in [0, \tfrac{1}{2}\pi]$ **B1** Noted that f is increasing (larger interval ok)

and since f(0) = 0 (and f increasing)

$f(x) = \sin x - x \cos x \geq 0$ for $0 \leq x \leq \tfrac{1}{2}\pi$ **E1** Fully explained/noted

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For $0 \leq x < 1$, $\frac{d}{dx}(\arcsin x) \geq \frac{d}{dx}(x)$ **M1** for using $1 = \frac{d}{dx}(x)$

$\Rightarrow \frac{d}{dx}(\arcsin x - x) \geq 0$ **M1** Combining the two sides

$\Rightarrow$ $f(x) = \arcsin x - x$ an increasing fn. **E1** Fully explained

and since f(0) = 0 (and f increasing)

$f(x) = \arcsin x - x \geq 0$ for $0 \leq x < 1$ **M1**

i.e. $\arcsin x \geq x$ for $0 \leq x < 1$ **E1** Fully explained

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$g(x) = \frac{x}{\sin x} \Rightarrow g'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$ **B1**

> 0 for $0 < x < \tfrac{1}{2}\pi$ **E1** that g is an increasing fn. from (a)

Let $u = \arcsin x$. Then $u \geq x$ (for $0 < x < 1$) **M1** by (b)

$\Rightarrow g(u) \geq g(x)$ since $g'(x) \geq 0$ **M1**

$\Rightarrow \frac{\arcsin x}{x} \geq \frac{x}{\sin x}$ (for $0 < x < 1$) **A1 Answer Given**

$g(x) = \frac{\tan x}{x}$, $g'(x) = \frac{x\sec^2 x - \tan x}{x^2} = \frac{2x - \sin 2x}{2x^2\cos^2 x}$ **M1**

and examine $f(x) = 2x - \sin 2x$, $f'(x) = 2 - 2\cos 2x \geq 0$ in $[0, \tfrac{1}{2}\pi]$ $\Rightarrow$ f increasing, **M1**

$\Rightarrow$ since f(0) = 0 that $g'(x) \geq 0 \Rightarrow$ g (strictly) increasing [ignore f(0) = 0] **E1**

Given $\frac{d}{dx}(\arctan x) \leq \frac{d}{dx}(x) \Rightarrow \frac{d}{dx}(x - \arctan x) \geq 0$ **M1** for using $1 = \frac{d}{dx}(x)$ etc.

Let $u = \arctan x$. Then $x \geq u$ (for $0 < x < \tfrac{1}{2}\pi$)

$\Rightarrow g(x) \geq g(u)$ **M1**

$\Rightarrow \frac{\tan x}{x} \geq \frac{x}{\arctan x}$ (for $0 < x < \tfrac{1}{2}\pi$) **A1 Answer Given**

SII 2011 Q4

Using $\sin A = \cos(90^\circ - A)$ to get $\theta = 360n \pm (90^\circ - 4\theta)$	**M1** (or ≡ Gen. Soln. for sine)
$\Rightarrow\ 5\theta = 360n + 90^\circ$ or $3\theta = 360n + 90^\circ$	**M1** either case
$\Rightarrow\ \theta = 72n + 18^\circ \Rightarrow \theta = 18^\circ, 90^\circ, 162^\circ$	**A1**
or $\theta = 120n + 30^\circ \Rightarrow \theta = 30^\circ, 150^\circ$	**A1**

$c = 2.2sc.(1 - 2s^2)$	**M1** double-angle formulae, twice
($c \neq 0$ for $\theta = 18^\circ$) $\Rightarrow\ 1 = 4s(1 - 2s^2)$ or $8s^3 - 4s + 1 = 0$	**M1** cubic formed
$\Rightarrow\ (2s - 1)(4s^2 + 2s - 1) = 0$	**M1** cubic factorisation attempted
($c \neq \frac{1}{2}$ for $\theta = 18^\circ$) $\Rightarrow\ s = \dfrac{-2 \pm \sqrt{20}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$	**M1** quadratic solution method
θ acute $\Rightarrow\ s = \sin 18^\circ > 0 \Rightarrow \sin 18^\circ = \dfrac{\sqrt{5} - 1}{4}$	**A1 GIVEN ANSWER** legitimately obtained
Explanation of $c \neq 0$, $s \neq \frac{1}{2}$ and $s > 0$	**E1** All 3 must appear somewhere

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$4s^2 + 1 = 16s^2(1 - s^2 \Rightarrow 0 = 16s^4 - 12s^2 + 1$	**M1** double-angles; **A1** correct quartic
$s^2 = \dfrac{12 \pm \sqrt{80}}{32} = \dfrac{3 \pm \sqrt{5}}{8}$	**M1**
$= \dfrac{6 \pm 2\sqrt{5}}{16} = \left(\dfrac{\sqrt{5} \pm 1}{4}\right)^2$	**M1** Method to find an exact square-root
$\Rightarrow\ \sin x = \pm\left(\dfrac{\sqrt{5} \pm 1}{4}\right)$	**A1** Must be *four* answers

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Noting $\sin^2 x + \frac{1}{4} = \sin^2 2x$ from (ii) with $x = 3\alpha = 18^\circ$	
works provided $5\alpha = 30^\circ \Rightarrow \alpha = 6^\circ$	**M1 A1**
Also, $\sin x = -\left(\dfrac{\sqrt{5} - 1}{4}\right) \Rightarrow 3\alpha = 180^\circ + 18^\circ = 198^\circ$	**M1**
can also work, since $5\alpha = 330^\circ$ also has $\sin 5\alpha = -\frac{1}{2}$	**E1** Must be mentioned explicitly
$\Rightarrow\ \alpha = 66^\circ$	**A1**

NB: $\alpha = 45^\circ$ works also, but does not follow from a “hence” argument

SII 2011 Q5

Let $M = OA \cap BC$. Then $\mathbf{m} = m\mathbf{a}$, **M1** $\overrightarrow{BM} = m\mathbf{a} - \mathbf{b}$ **A1**

and $(m\mathbf{a} - \mathbf{b}) \,.\, \mathbf{a} = 0$ **M1** $\Rightarrow m = \dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{a}\bullet\mathbf{a}}$

Then $\mathbf{c} = \overrightarrow{OB} + 2\overrightarrow{BM} = \mathbf{b} + 2m\mathbf{a} - 2\mathbf{b}$ **M1** $\Rightarrow \mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$ where $\lambda = 2m = 2\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{a}\bullet\mathbf{a}}\right)$ **A1**

ALT. OA is the bisector of $\angle BOC$ and $OB = OC$

$\Rightarrow A$ is on the diagonal OA' of \\gm $OBA'C \Rightarrow \mathbf{b} + \mathbf{c} = \lambda\mathbf{a}$ **M1 M1 A1**

$BC \perp^{r} OA \Rightarrow (\mathbf{b} - \mathbf{c}) \,.\, \mathbf{a} = 0$ **M1** $\Rightarrow (2\mathbf{b} - \lambda\mathbf{a}) \,.\, \mathbf{a} = 0 \Rightarrow \lambda = 2\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{a}\bullet\mathbf{a}}\right)$ **A1**

Similarly (replacing **a** by **b** and **b** by **c** in the above) **M1**

$\mathbf{d} = k\mathbf{b} - \mathbf{c}$ where $k = 2\left(\dfrac{\mathbf{b}\bullet\mathbf{c}}{\mathbf{b}\bullet\mathbf{b}}\right)$ **M1**

$= 2\left(\dfrac{\mathbf{b}\bullet\lambda\mathbf{a} - \mathbf{b}\bullet\mathbf{b}}{\mathbf{b}\bullet\mathbf{b}}\right) = 2\lambda\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{b}\bullet\mathbf{b}}\right) - 2$ **M1**

$\Rightarrow \mathbf{d} = \left(2\lambda\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{b}\bullet\mathbf{b}}\right) - 2\right)\mathbf{b} - (\lambda\mathbf{a} - \mathbf{b})$ **M1**

$= \mu\mathbf{b} - \lambda\mathbf{a}$ where $\mu = 2\lambda\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{b}\bullet\mathbf{b}}\right) - 1$ or $4\left(\dfrac{[\mathbf{a}\bullet\mathbf{b}]^2}{[\mathbf{a}\bullet\mathbf{a}][\mathbf{b}\bullet\mathbf{b}]}\right) - 1$ **A1**

Now $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} \parallel \overrightarrow{AD} = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$

$\Leftrightarrow t(\mathbf{b} - \mathbf{a}) = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$ for some $t\ (\neq 0)$ **M1**

Comparing coeffts. of **a** and **b** **M1** $\Rightarrow (t =)\ \mu = \lambda + 1$ **A1**

$\lambda = -\frac{1}{2} \Rightarrow \mu = \frac{1}{2}$ **B1** $\Rightarrow D$ is the midpoint of AB **B1**

$\mu = \frac{1}{2} \Rightarrow \frac{1}{2} = 4\left(\dfrac{[\mathbf{a}\bullet\mathbf{b}]^2}{[\mathbf{a}\bullet\mathbf{a}][\mathbf{b}\bullet\mathbf{b}]}\right) - 1 = 4\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{ab}\right)^2 - 1$ **M1 M1 A1**

Use of $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{b}}{ab}$ **M1** to get $\cos\theta = -\sqrt{\dfrac{3}{8}}$ **A1**

[**NB** : **a . b** has the same sign as λ]

ALT

$\mu = \frac{1}{2} \Rightarrow 4ab\cos\theta = -a^2$ and $2ab\cos\theta = -3a^2 \Rightarrow a = b\sqrt{6}$ and $a, b > 0 \Rightarrow \cos\theta = -\sqrt{\dfrac{3}{8}}$

M1 **A1** **A1** **M1** **A1**

SII 2011 Q6

$I=\int[f'(x)]^2[f(x)]^n\,dx=\int[f'(x)]\times\{[f'(x)][f(x)]^n\}dx$ **M1** Splitting **M1** Parts

$=f'(x)\times\frac{1}{n+1}[f(x)]^{n+1}-\int\left([f''(x)]\times\frac{1}{n+1}[f(x)]^{n+1}\right)dx$ **A1**

Using $f''(x)=kf(x)f'(x)$, $I=f'(x)\times\frac{1}{n+1}[f(x)]^{n+1}-\int\left(kf'(x)\times\frac{1}{n+1}[f(x)]^{n+2}\right)dx$ **M1**

$=f'(x)\times\frac{1}{n+1}[f(x)]^{n+1}-\frac{1}{(n+1)(n+3)}\times k[f(x)]^{n+3}$ $(+\,C)$ **A1**

$f(x)=\tan x\Rightarrow f'(x)=\sec^2x$ and $f''(x)=2\sec^2x\tan x=kf(x)f'(x)$ with $k=2$ **M1 A1 E1**

$$\Rightarrow I=\frac{\sec^2x\tan^{n+1}x}{n+1}-\frac{2\tan^{n+3}x}{(n+1)(n+3)}$$ **B1**

Differentiating this gives **M1**

$$\frac{dI}{dx}=\frac{1}{n+1}\left(\sec^2x.(n+1)\tan^nx.\sec^2x+2\sec x.\sec x\tan x.\tan^{n+1}x\right)$$

$$-\frac{1}{(n+1)(n+3)}\left(2(n+3)\tan^{n+2}x.\sec^2x\right)$$

$=\sec^4x\tan^nx=(f'(x))^2\times(f(x))^n$ as required **A1 E1**

NB This 4-mark chunk may be done in reverse as an integration

$\int\frac{\sin^4x}{\cos^8x}dx=\int\sec^4x\tan^4x\,dx=\frac{\sec^2x\tan^5x}{5}-\frac{2\tan^7x}{35}+C$ **M1 A1**

$f(x)=\sec x+\tan x\Rightarrow f'(x)=\sec x\tan x+\sec^2x=\sec x(\sec x+\tan x)$ **M1**

and $f''(x)=\sec^2x(\sec x+\tan x)+\sec x\tan x(\sec x+\tan x)$ **A1**

$=\sec x(\sec x+\tan x)^2=kf(x)f'(x)$ with $k=1$ **E1**

Then $\int\sec^2x(\sec x+\tan x)^6dx=\int\{\sec x(\sec x+\tan x)\}^2\times(\sec x+\tan x)^4dx$ **M1**

$=\frac{\sec x(\sec x+\tan x)^6}{5}-\frac{(\sec x+\tan x)^7}{35}+C$ **A1 A1**

NB Lack of "+ C" not penalised throughout

SII 2011 Q7

$\sum_{r=0}^{n} b_r = \left(1+\lambda+\lambda^2+\ldots+\lambda^n\right)-\left(1+\mu+\mu^2+\ldots+\mu^n\right)$ **M1**

$= \frac{\lambda^{n+1}-1}{\lambda-1}-\frac{\mu^{n+1}-1}{\mu-1}$ **M1** S_∞(GP) used at least once

$= \frac{1}{\sqrt{2}}\left(\lambda^{n+1}-1+\mu^{n+1}-1\right)$ since $\lambda-1=\sqrt{2}$ and $\mu-1=-\sqrt{2}$ **M1**

$= \frac{1}{\sqrt{2}}a_{n+1}-\sqrt{2}$ **A1 Answer Given**

$\sum_{r=0}^{n} a_r = \frac{\lambda^{n+1}-1}{\sqrt{2}}-\frac{\mu^{n+1}-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}b_{n+1}$ **B1** May be just stated/observed

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$\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \sum_{m=0}^{2n}\left(\frac{1}{\sqrt{2}}b_{m+1}\right) = \frac{1}{\sqrt{2}}\sum_{m=0}^{2n+1} b_m$ since $b_0=0$ **M1 A1**

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}a_{2n+2}-\sqrt{2}\right)$ from (i) **M1** Use of earlier result

$= \frac{1}{2}\left(\lambda^{2n+2}+\mu^{2n+2}-2\right)$ **M1**

$= \frac{1}{2}\left(\left[\lambda^{n+1}\right]^2-2[\lambda\mu]^{n+1}+\left[\mu^{n+1}\right]^2\right)$ since $\lambda\mu=-1$ and $n+1$ is even when n is odd **E1**

$= \frac{1}{2}\left(b_{n+1}\right)^2$ when n is odd **A1 Answer Given**

However, when n is even, $n+1$ is odd and $\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \frac{1}{2}\left(b_{n+1}\right)^2-2$ or $\frac{1}{2}\left(a_{n+1}\right)^2$ **B1**

$\left(\sum_{r=0}^{n} a_r\right)^2 = \frac{1}{2}\left(b_{n+1}\right)^2$

and $\sum_{r=0}^{n} a_{2r+1} = \left(\lambda+\lambda^3+\lambda^5+\ldots+\lambda^{2n+1}\right)+\left(\mu+\mu^3+\mu^5+\ldots+\mu^{2n+1}\right)$ **M1**

$= \frac{\lambda\left(\lambda^{2n+2}-1\right)}{\lambda^2-1}+\frac{\mu\left(\mu^{2n+2}-1\right)}{\mu^2-1}$ **M1** S_∞(GP) used at least once

Now $\lambda^2-1=3+2\sqrt{2}-1=2\left(1+\sqrt{2}\right)=2\lambda$ and $\mu^2-1=3-2\sqrt{2}-1=2\left(1-\sqrt{2}\right)=2\mu$ **M1 A1**

so $\sum_{r=0}^{n} a_{2r+1} = \frac{1}{2}\left(\lambda^{2n+2}+\mu^{2n+2}-2\right) = \frac{1}{2}\left(b_{n+1}\right)^2$ when n is odd **M1**

and $= \frac{1}{2}\left(b_{n+1}\right)^2-2$ when n is even **M1**

Thus $\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 0$ when n is odd / $= 2$ when n is even **A1** **<u>A1</u>** **2nd Answer Given**

SII 2011 Q8

The string leaves the circle at $C(-\cos\theta, \sin\theta)$ **B1**

Arc $AC = \pi - t = \theta$ (so $\cos\theta = -\cos t$ and $\sin\theta = \sin t$) **M1**

Then $B = (-\cos\theta + t\sin\theta, \sin\theta + t\cos\theta) = (\cos t + t\sin t, \sin t - t\cos t)$ **A1 Answer Given**

$\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t$ by the Product Rule **M1 A1**

= 0 **M1** when $t = 0$, $(x, y) = (1, 0)$ or $t = \frac{1}{2}\pi$, $(x, y) = \left(\frac{1}{2}\pi, 1\right)$. This is $x_{\max}$ so $t_0 = \frac{1}{2}\pi$ **A1**

Area under curve and above x-axis is

$A = \int_{\pi}^{\frac{1}{2}\pi} y\frac{dx}{dt}\,dt$ **B1** Including limits $= \int_{\pi}^{\frac{1}{2}\pi} (\sin t - t\cos t)t\cos t\,dt$ **M1**

$= \int_{\frac{1}{2}\pi}^{\pi} -\frac{1}{2}t\sin 2t\,dt + \int_{\frac{1}{2}\pi}^{\pi} \frac{1}{2}t^2(1+\cos 2t)\,dt$ **M1**

$\int_{\frac{1}{2}\pi}^{\pi} -\frac{1}{2}t\sin 2t\,dt = \left[\frac{1}{4}t\cos 2t\right]_{\frac{1}{2}\pi}^{\pi} - \int_{\frac{1}{2}\pi}^{\pi}\frac{1}{4}\cos 2t\,dt = \left[\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t\right]_{\frac{1}{2}\pi}^{\pi} = \frac{3\pi}{8}$ **M1 M1 A1**

and

$\int_{\frac{1}{2}\pi}^{\pi} \frac{1}{2}t^2\,dt = \left[\frac{1}{6}t^3\right]_{\frac{1}{2}\pi}^{\pi} = \frac{7\pi^3}{48}$ **B1**

and

$\int_{\frac{1}{2}\pi}^{\pi} \frac{1}{2}t^2\cos 2t\,dt = \left[\frac{1}{4}t^2\sin 2t\right]_{\frac{1}{2}\pi}^{\pi} - \int_{\frac{1}{2}\pi}^{\pi}\frac{1}{2}t\sin 2t\,dt = 0 - -\frac{3\pi}{8} = \frac{3\pi}{8}$ using previous answer **M1 A1**

Thus $A = \frac{7\pi^3}{48} + \frac{3\pi}{4}$ **A1**

Using limits $\frac{1}{2}\pi$ and 0 gives $-\frac{\pi^3}{48} + \frac{\pi}{4}$ **B1**

Total area swept out by string is then $\frac{7\pi^3}{48} + \frac{3\pi}{4} + -\frac{\pi^3}{48} + \frac{\pi}{4} \quad -\frac{\pi}{2}$ (area inside semi-circle) **M1**

$= \frac{\pi^3}{6}$ **A1**

SII 2011 Q9

CLM $3mu = 2mV_A + mV_B$ **B1**

NEL $e.3u = V_B - V_A$ **B1**

Solving simultaneously for V_A and V_B **M1**

$V_A = u(1 - e)$, $V_B = u(1 + 2e)$ **A1 A1**

Vel. B after collision with wall is $|f V_B|$ **B1**

CLM (away from wall) $fmV_B - 2mV_A = 2mW_A - mW_B$ **M1 A1** (W_B –ve since towards wall)

NEL $W_A + W_B = e(V_A + f V_B)$ **B1**

Subst^g. for V_A & V_B from before in *both* equations **M1**

$2W_A - W_B = u\{f(1+2e) - 2(1-e)\}$ **A1**

$W_A + W_B = eu\{(1-e) + f(1+2e)\}$ **A1**

Solving simultaneously for W_A and W_B **M1**

($W_A = \frac{1}{3}u\{f(1+2e)(1+e) - (2-e)(1-e)\}$ not required)

$W_B = \frac{1}{3}u\{2(1-e^2) - f(1-4e^2)\}$ **A1 Answer Given**

If $e = \frac{1}{2}$, $W_B = \frac{1}{3}u\{2(\frac{3}{4}) - f(0)\} = \frac{1}{2}u > 0$ (May be incorporated into one of the other cases)

If $\frac{1}{2} < e < 1$, $W_B = \frac{1}{3}u\{2(1-e^2) + f(4e^2 - 1)\} > 0$ for all e, f since each term in the bracket is +ve

If $0 < e < \frac{1}{2}$, $1 - e^2 > \frac{3}{4}$ and $W_B > \frac{1}{3}u\{\frac{3}{2} - f(1-4e^2)\} > \frac{1}{3}u\{\frac{3}{2} - 1\times 1\} > 0$

Attempt to show $W_B > 0$ for some values of e, f **M1**

Splitting into suitable cases **M1**

Cases as above **B1** **B1** **M1 A1** However done

SII 2011 Q10

$\dot{y} = u\sin\theta - gt = 0 \;\Rightarrow\; t = \dfrac{u\sin\theta}{g}$ substd. into $y = ut\sin\theta - \frac{1}{2}gt^2 \;\Rightarrow\; H = \dfrac{u^2\sin^2\theta}{2g}$ **M1 A1**

This may just be quoted

$l = \frac{1}{2}H = \dfrac{u^2\sin^2\theta}{4g}$ **B1**

$l = ut\sin\theta - \frac{1}{2}gt^2$ **M1** using vertical distance expression for l

$gt^2 - (2u\sin\theta)t + H = 0$ **M1** forming a quadratic in t

$t = \dfrac{2u\sin\theta \pm \sqrt{4u^2\sin^2\theta - 4gH}}{2g}$ **M1** good solving attempt

$= \dfrac{2\sqrt{2gH} \pm \sqrt{8gH - 4gH}}{2g}$ **M1** using $u\sin\theta = \sqrt{2gH}$

$= \dfrac{1}{g}\left(\sqrt{2gH} \pm \sqrt{gH}\right) = \sqrt{\dfrac{H}{g}}\left(\sqrt{2} - 1\right)$ **A1 Answer Given**

explained by observing that we want the first time when an unimpeded P is at this height **E1**

For P vertically, $v = \dot{y} = u\sin\theta - g\sqrt{\dfrac{H}{g}}\left(\sqrt{2} - 1\right)$

$= \sqrt{2gH} - \sqrt{gH}\left(\sqrt{2} - 1\right) = \sqrt{gH}$ or $\dfrac{u\sin\theta}{\sqrt{2}}$ **M1 A1**

Thus, common speed after string goes taut, by CLM, is

$v = \frac{1}{2}\sqrt{gH}$ or $\dfrac{u\sin\theta}{2\sqrt{2}}$ **M1 A1**

Consider now the projectile R,

with initial velocity components $u\cos\theta \rightarrow$ and $\dfrac{u\sin\theta}{2\sqrt{2}} \uparrow$ **M1**

$y = \dfrac{u\sin\theta}{2\sqrt{2}}t - \dfrac{g}{2}t^2 = 0$ $(t \neq 0)$ at Range, when $t = \dfrac{u\sin\theta}{g\sqrt{2}}$ **M1 A1**

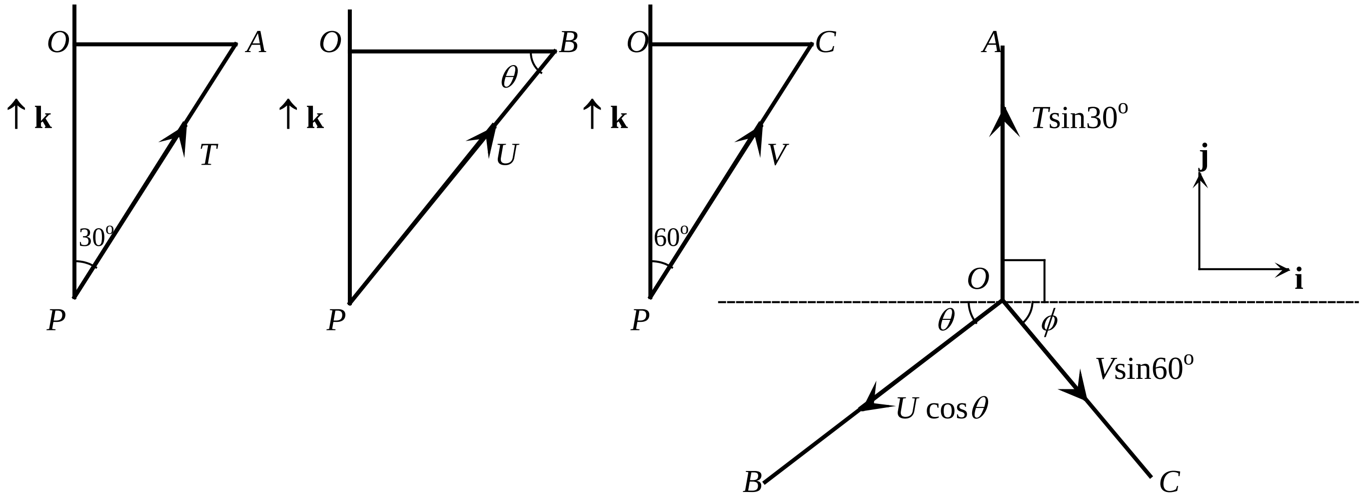
Then $D = x_1 + x_2$ where $x_1 = u\cos\theta\dfrac{u\sin\theta}{g\sqrt{2}}\left(\sqrt{2} - 1\right)$ and $x_2 = u\cos\theta\dfrac{u\sin\theta}{g\sqrt{2}}$ **M1** 2 distances

$= \dfrac{u^2\sin\theta\cos\theta}{g}$ **A1**

$D = H \Rightarrow \tan\theta = 2$ **M1** Must involve cancelling trig. terms **A1 CAO**

NB Throughout, results may be in terms of g & H rather than u and θ.

__SII 2011 Q11__



$\tan\theta = \sqrt{2} \;\Rightarrow\; \sin\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \dfrac{1}{\sqrt{3}}$ **B1**

$\tan\phi = \dfrac{\sqrt{2}}{4} \;\Rightarrow\; \sin\phi = \dfrac{1}{3}$ and $\cos\phi = \dfrac{2\sqrt{2}}{3}$ **B1**

Vector in direction PB is $-(U\cos\theta).\cos\theta\,\mathbf{i} - (U\cos\theta).\sin\theta\,\mathbf{j} + U\sin\theta\,\mathbf{k}$ **M1 A1 A1 A1**

$\underline{\boldsymbol{T}}_B = \left(-\dfrac{1}{3}\mathbf{i} - \dfrac{\sqrt{2}}{3}\mathbf{j} + \dfrac{\sqrt{2}}{\sqrt{3}}\mathbf{k}\right)U$ and the bracketed vector has magnitude 1

This must be explicitly verified **A1 Answer Given**

$\underline{\boldsymbol{T}}_A = T\sin 30°\,\mathbf{j} + T\cos 30°\,\mathbf{k} = \dfrac{1}{2}T\left(\mathbf{j} + \sqrt{3}\,\mathbf{k}\right)$ **B1**

$\underline{\boldsymbol{T}}_C = V\sin 60°\cos\phi\,\mathbf{i} - V\sin 60°\sin\phi\,\mathbf{j} + V\cos 60°\,\mathbf{k} = \dfrac{1}{2}V\left(\dfrac{2\sqrt{2}}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \mathbf{k}\right)$ **B1 B1**

$\underline{\boldsymbol{W}} = -W\,\mathbf{k}$ **B1**

$\underline{\boldsymbol{T}}_A + \underline{\boldsymbol{T}}_B + \underline{\boldsymbol{T}}_C + \underline{\boldsymbol{W}} = \underline{\mathbf{0}}$ **M1** May be implied by zero components

Comparing terms : (**i**) $0 - \dfrac{1}{3}U + \dfrac{\sqrt{6}}{3}V = 0$ **M1**

$\Rightarrow\; U = V\sqrt{6}$ **A1 Answer Given**

(**j**) $\dfrac{1}{2}T - \dfrac{\sqrt{2}}{3}U - \dfrac{\sqrt{3}}{6}V = 0$ **A1**

Use of $U = V\sqrt{6} \;\Rightarrow\; T = \dfrac{5\sqrt{3}}{3}V$ **A1**

(**k**) $\dfrac{\sqrt{3}}{2}T + \dfrac{\sqrt{6}}{3}U + \dfrac{1}{2}V = W$ **A1**

Use of $U = V\sqrt{6}$ and $T = \dfrac{5\sqrt{3}}{3}V \;\Rightarrow\; T = \dfrac{W\sqrt{3}}{3},\; U = \dfrac{W\sqrt{6}}{5},\; V = \dfrac{W}{5}$ **A1 A1 A1**

SII 2011 Q12

P(re-match) = P(XYX) + P(YXY) = $p(1-p)^2 + (1-p)^3 = (1-p)^2$ **B1**

P(Y wins directly) = P(YY) + P(XYY)

$= (1-p)p + p(1-p)p = p(1-p)(1+p)$ or $p(1-p^2)$ **B1**

P(Y wins) = $w = p(1-p^2) + w(1-p)^2$ **M1** recurrently defined or via S_∞(GP)

$\Rightarrow w = \dfrac{p(1-p^2)}{1-(1-p)^2} = \dfrac{p(1-p^2)}{(1-(1-p))(1+(1-p))} = \dfrac{p(1-p^2)}{p(2-p)}$ **M1** rearranging for w

$= \dfrac{1-p^2}{2-p}$ for $p \neq 0$ **A1 GIVEN ANSWER** legitimately obtained

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$w - \frac{1}{2} = \dfrac{2(1-p^2)-(2-p)}{2(2-p)} = \dfrac{p(1-2p)}{2(2-p)}$ **M1 A1**

Since $2-p > 0$, $w - \frac{1}{2}$ has the same sign as $1-2p$ and hence as $\frac{1}{2} - p$ **M1**

Hence, $w > \frac{1}{2}$ if $p < \frac{1}{2}$ and $w < \frac{1}{2}$ if $p > \frac{1}{2}$ **E1** both correctly concluded

[May be done by calculus: **M1 A1** then **M1 E1** for the explanation]

$\dfrac{dw}{dp} = \dfrac{(2-p)(-2p)-(1-p^2)(-1)}{(2-p)^2}$ **M1 A1** correct unsimplified

$= \dfrac{dw}{dp} = \dfrac{1}{(2-p)^2}\left(p^2-4p+1\right) = \dfrac{1}{(2-p)^2}\left([2-p]^2-3\right)$ **M1 A1**

$\dfrac{dw}{dp} > 0$ for $0 < p < 2-\sqrt{3}$ and $\dfrac{dw}{dp} < 0$ for $2-\sqrt{3} < p \leq 1$ **M1 A1** considering sign of $\dfrac{dw}{dp}$

E1 Justification

$p = \frac{2}{3} \Rightarrow w = \frac{5}{12}$, $1 - w = \frac{7}{12}$ and so $k = \frac{7}{5} \times$ £1 = £1-40 (to balance the game)

or by "expected gain" approach **M1 A1** $k = 1.4$

When $p = 0$, results run YXY ... re-match ... YXY ... re-match **E1**

... and the match never ends **B1**

SII 2011 Q13

Skewness is a measure of a distribution's *lack of symmetry* **B1**

$$E\left[(X-\mu)^3\right] = E\left[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right]$$ **B1** Correct binomial expansion

$$= E\left[X^3\right] - 3\mu E\left[X^2\right] + 3\mu^2 E[X] - \mu^3$$ **M1** Use of distributivity

$$= E\left[X^3\right] - 3\mu\left(\sigma^2 + \mu^2\right) + 3\mu^2.\mu - \mu^3$$ **M1** Use of both E[X] = μ & E[X^2] = $\sigma^2 + \mu^2$

$$= E\left[X^3\right] - 3\mu\sigma^2 - \mu^3$$ **A1 Answer Given**

$$E[X] = \int_0^1 2x^2\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \tfrac{2}{3} = \mu$$ **B1**

$$E\left[X^2\right] = \int_0^1 2x^3\,dx = \left[\frac{1}{2}x^4\right]_0^1 = \tfrac{1}{2} \Rightarrow \sigma^2 = \tfrac{1}{18}$$ **B1**

$$E\left[X^3\right] = \int_0^1 2x^4\,dx = \left[\frac{2}{5}x^5\right]_0^1 = \tfrac{2}{5}$$ **B1**

$$\Rightarrow \gamma = \frac{\frac{2}{5} - 3.\frac{2}{3}.\frac{1}{18} - \frac{8}{27}}{\frac{1}{18\sqrt{18}}} = -\frac{2\sqrt{2}}{5}$$ **M1 A1 Answer Given**

$$F(x) = \int_0^x 2x\,dx = x^2 \quad (0 \le x \le 1)$$ **B1**

$$\Rightarrow F^{-1}(x) = \sqrt{x} \quad (0 \le x \le 1)$$ **B1 B1**

$$\Rightarrow D = \frac{F^{-1}\left(\frac{9}{10}\right) - 2F^{-1}\left(\frac{1}{2}\right) + F^{-1}\left(\frac{1}{10}\right)}{F^{-1}\left(\frac{9}{10}\right) - F^{-1}\left(\frac{1}{10}\right)} = \frac{\frac{3}{\sqrt{10}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}}}$$ **M1** All correctly substituted

$$= \frac{3 - 2\sqrt{5} + 1}{3 - 1} = \frac{4 - 2\sqrt{5}}{2} = 2 - \sqrt{5}$$ **A1 Answer Given**

M is given by $\int_0^M 2x\,dx = \tfrac{1}{2}$ **M1**

$\Rightarrow M^2 = \tfrac{1}{2} \Rightarrow M = \tfrac{1}{\sqrt{2}}$ **A1** OR by $M = F^{-1}(\tfrac{1}{2}) = \tfrac{1}{\sqrt{2}}$

Then $P = \dfrac{3\left(\frac{2}{3} - \frac{1}{\sqrt{2}}\right)}{\frac{1}{3\sqrt{2}}} = 6\sqrt{2} - 9$ **B1**

RTP $D > P > \gamma$ i.e. $2 - \sqrt{5} > 6\sqrt{2} - 9 > -\tfrac{2}{5}\sqrt{2}$ **B1 B1**

1st B1 for $D > P$; 2nd for $P > \gamma$ May use inequality arguments (e.g. squaring) or use of approximations to $\sqrt{2}$, $\sqrt{5}$ – be strict on improper rounding

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