9 For P_1, $\ddot{x}_1 = 0$, $\dot{x}_1 = u\cos\alpha$, $x_1 = ut\cos\alpha$, $\ddot{y}_1 = -g$, $\dot{y}_1 = u\sin\alpha - gt$, $y_1 = ut\sin\alpha - \tfrac{1}{2}gt^2$

P_2, $\ddot{x}_2 = 0$, $\dot{x}_2 = v\cos\beta$, $x_2 = vt\cos\beta$, $\ddot{y}_2 = -g$, $\dot{y}_2 = v\sin\beta - gt$, $y_2 = vt\sin\beta - \tfrac{1}{2}gt^2$

P_1 at greatest height when $\dot{y}_2 = 0$ **M1** $\Rightarrow t = \dfrac{u\sin\alpha}{g}$ **A1**

M1 Substd. into y_1 formula $\Rightarrow y_1 = h = \dfrac{u^2\sin^2\alpha}{2g}$ **A1**

$\Rightarrow u\sin\alpha = \sqrt{2gh}$ **A1** This may be implicit in following working ⑤

Note that if the two particles are at the same height at any two distinct times (one of which is $t = 0$ here), then their vertical speeds are the same throughout their motions. **E1**

Thus $u\sin\alpha = v\sin\beta$ **B1** **Somewhere**

ALT. P_1, P_2 at the same height at a common time $t = \tau \neq 0$, then

$u\tau\sin\alpha - \tfrac{1}{2}g\tau^2 = v\tau\sin\beta - \tfrac{1}{2}g\tau^2$ **E1** $\Rightarrow u\sin\alpha = v\sin\beta$ **B1** ②

$y_2 = 0, t \neq 0 \Rightarrow t = \dfrac{2v\sin\beta}{g}$ **M1 A1**

Collision at $x_2 = b \Rightarrow t = \dfrac{b}{v\cos\beta}$ **M1 A1**

Then $t\left(P_2\ \tfrac{1}{2}-\text{range}\right) < t(\text{collision}) < t\left(P_2\ \text{range}\right)$ **M1** or by distances

$\Rightarrow \dfrac{v\sin\beta}{g} < \dfrac{b}{v\cos\beta} < \dfrac{2v\sin\beta}{g}$ **A1** or $\tfrac{1}{2}.\dfrac{v^2\sin 2\beta}{g} < b < \dfrac{v^2\sin 2\beta}{g}$

$\Rightarrow \dfrac{v^2\sin\beta\cos\beta}{g} < b < \dfrac{2v^2\sin\beta\cos\beta}{g}$

$\Rightarrow \dfrac{(v\sin\beta)^2}{g}\cot\beta < b < \dfrac{2(v\sin\beta)^2}{g}\cot\beta$ **M1** relevant trig. work

M1 use of $u\sin\alpha = v\sin\beta$ **M1** use of $u\sin\alpha = \sqrt{2gh}$

$\Rightarrow \dfrac{2gh}{g}\cot\beta < b < \dfrac{4gh}{g}\cot\beta \Rightarrow 2h\cot\beta < b < 4h\cot\beta$ **A1** legit. (**ANSWER GIVEN**) ⑩

Particles at max. ht. simultaneously (see above reasoning) **M1**

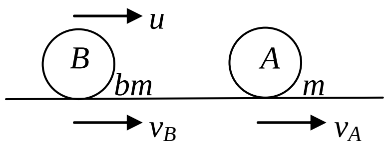
and would achieve max. ranges simultaneously also **M1**

$\Rightarrow 2h\cot\alpha < a < 4h\cot\alpha$ **A1** (**ANSWER GIVEN**) ③

Anyone who says "similarly" without explaining why … gets **0**

Those who do all the work again, give **M1** for clear intention to repeat it all, **M1** for *actually* doing it all again, and **A1** for legitimately obtaining given result.

10(i)

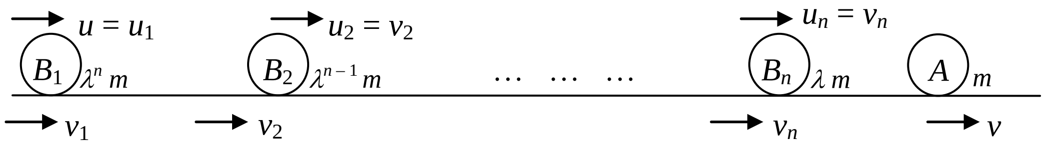


CLM→ $\quad b\cancel{m}u = b\cancel{m}v_B + \cancel{m}v_A$ **M1 A1**

NEL $\quad u = v_A - v_B$ **B1**

M1 Solving simultaneously: $v_A = \dfrac{2bu}{b+1}$ **A1** $\qquad v_B = \dfrac{(b-1)u}{b+1}$

Then $v_A = \left(\dfrac{2}{1+\frac{1}{b}}\right)u \to 2u-$ as $b \to \infty$, and $v_A < 2u$ always **E1** convincing ⑥



(ii) **M1** Using the results of (i), $\quad v_2 = u_2 = \left(\dfrac{2\lambda}{\lambda+1}\right)u$

M1 repeatedly $\qquad u_3 = \left(\dfrac{2\lambda}{\lambda+1}\right)u_2 = \left(\dfrac{2\lambda}{\lambda+1}\right)^2 u$

… … … …

M1 all the way down to u $\qquad u_n = \left(\dfrac{2\lambda}{\lambda+1}\right)u_{n-1} = \left(\dfrac{2\lambda}{\lambda+1}\right)^{n-1} u$

and $\qquad v = \left(\dfrac{2\lambda}{\lambda+1}\right)u_n = \left(\dfrac{2\lambda}{\lambda+1}\right)^n u$ **A1 A1**

Since $u_n = \dfrac{2\lambda}{\lambda+1} > 1$, as $\lambda > 1$ **E1**

it follows that v can be made as large as possible **E1** ⑦

In the case when $\lambda = 4$, $v = \left(\frac{8}{5}\right)^n u > 20u$ requires $n\log\left(\frac{8}{5}\right) > \log 20 \Rightarrow n > \dfrac{\log 20}{\log\left(\frac{8}{5}\right)}$ **M1 A1**

Now $\log 2 = 0.30103 \Rightarrow \log 8 = 3\log 2 = 0.90309$ **M1**

$\log 5 = \log 10 - \log 2 = 1 - 0.30103 = 0.69897$ **M1**

so that $\log\left(\frac{8}{5}\right) = \log 8 - \log 5 = 0.20412$

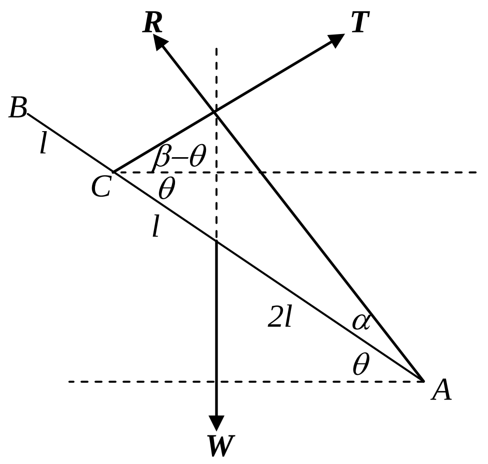
Also $\log 20 = \log 10 + \log 2 = 1 + 0.30103 = 1.30103$ **M1**

and $n > \dfrac{1.30103}{0.20412}$.

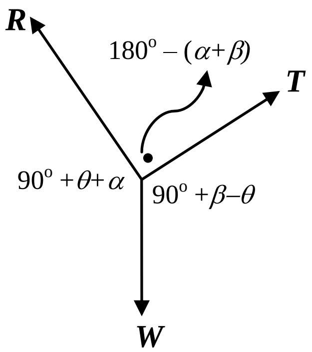
Since $6 \times 0.20412 = 1.22472$ and $7 \times 0.20412 = 1.42884$,

$n_{\min} = 7$ **A1** answer **E1** suitable justification ⑦

11



N.B. The three forces must be concurrent for equilibrium **M1**



Angles **A1** **A1**

By *Lami's Theorem* (or a triangle of forces and the *Sine Rule*): **M3**

$$\frac{T}{\sin(90^\circ+\theta+\alpha)}=\frac{R}{\sin(90^\circ+\beta-\theta)}=\frac{W}{\sin(180^\circ-[\alpha+\beta])}$$

$$\Rightarrow\ \frac{T}{\cos(\theta+\alpha)}=\frac{R}{\cos(\beta-\theta)}=\frac{W}{\sin(\alpha+\beta)}$$ **A1**

A↺ $W.2l\cos\theta = T.3l\sin\beta$ **M1 A1** ⑨

Then $T=\dfrac{2W\cos\theta}{3\sin\beta}=\dfrac{W\cos(\theta+\alpha)}{\sin(\alpha+\beta)}$ **M1**

$\Rightarrow$ $2\cos\theta\ \sin(\alpha+\beta) = 3\sin\beta\cos(\theta+\alpha)$ **M1**

$\Rightarrow$ $2\cos\theta\ \sin\alpha\ \cos\beta + 2\cos\theta\ \cos\alpha\ \sin\beta = 3\sin\beta\ \cos\theta\ \cos\alpha - 3\sin\beta\ \sin\theta\ \sin\alpha$ **M1**

Dividing by $\cos\theta\ \cos\alpha\ \cos\beta$ $\Rightarrow$ $2\tan\alpha + 2\tan\beta = 3\tan\beta - 3\tan\beta\ \tan\theta\ \tan\alpha$ **M1**

$\Rightarrow$ $2\tan\alpha + 3\tan\beta\ \tan\theta\ \tan\alpha = \tan\beta$

Dividing by $\tan\alpha\ \tan\beta$ **M1**

$\Rightarrow$ $2\cot\beta + 3\tan\theta = \cot\alpha$ **A1 (ANSWER GIVEN)** ⑥

$\theta = 30^\circ$, $\beta = 45^\circ$ $\Rightarrow$ $\cot\alpha = 2.1+3.\dfrac{1}{\sqrt{3}} = 2+\sqrt{3}$ **B1**

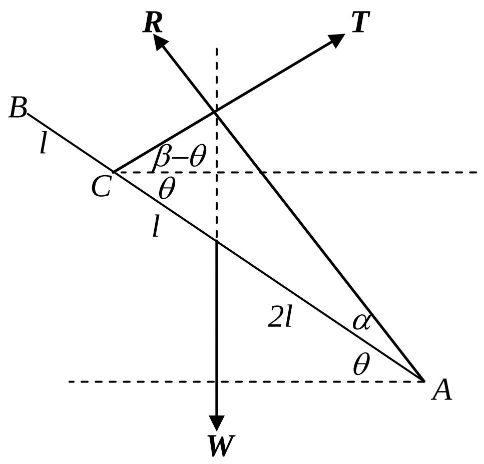
Now $\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A}$ $\Rightarrow$ $\frac{1}{\sqrt{3}}\left(1-t^2\right)=2t$ $\Rightarrow$ $0=t^2+2t\sqrt{3}-1$ **M1**

$\Rightarrow$ $t = \tan 15^\circ = \dfrac{-2\sqrt{3}\pm\sqrt{12+4}}{2}=-\sqrt{3}\pm 2$ **M1**

However, $\tan 15^\circ > 0$ since 15° is acute, so $\tan 15^\circ = 2-\sqrt{3}$ and $\cot 15^\circ = 2+\sqrt{3}$ **M1 A1**

ALT. $\tan(60^\circ-45^\circ) = \dfrac{\sqrt{3}-1}{1+\sqrt{3}}=\dfrac{\left(\sqrt{3}-1\right)^2}{3-1}=2-\sqrt{3}=\dfrac{1}{2+\sqrt{3}}$ or verification ⑤

11 ALTERNATIVE



M1 A1 A1 for relevant, correct angles for their working

$(\beta - \theta)$ & $(\alpha + \theta)$ below

<u>**Res.↑**</u> $T\sin(\beta - \theta) + R\sin(\alpha + \theta) = W$ **M1 A1**

<u>**Res.→**</u> $T\cos(\beta - \theta) = R\cos(\alpha + \theta)$ **M1 A1**

<u>**A↲**</u> $W.2l\cos\theta = T.3l\sin\beta$ **M1 A1** ⑨

Subst^{g}. to eliminate T 's (for instance): **M1**

$$T\sin(\beta - \theta) + \frac{T\cos(\beta-\theta)}{\cos(\alpha+\theta)}\sin(\alpha + \theta) = \frac{3T\sin\beta}{2\cos\theta}$$

$$\Rightarrow\ 2\cos\theta\left(\cos\alpha.\cos\theta - \sin\alpha.\sin\theta\right)\left(\sin\beta.\cos\theta - \cos\beta.\sin\theta\right)$$
$$+\ 2\cos\theta\left(\cos\beta.\cos\theta + \sin\beta.\sin\theta\right)\left(\sin\alpha.\cos\theta + \cos\alpha.\sin\theta\right)$$
$$= 3\sin\beta\left(\cos\alpha.\cos\theta - \sin\alpha.\sin\theta\right)$$

M1 Correct trig. expansions

Dividing by $\cos\theta\ \cos\alpha\ \cos\beta$ **M1**

$$\Rightarrow\ 2(\cos\theta - \tan\alpha.\sin\theta)(\tan\beta.\cos\theta - \sin\theta) + 2(\cos\theta + \tan\beta.\sin\theta)(\tan\alpha.\cos\theta + \sin\theta)$$
$$= 3\tan\beta(1 - \tan\alpha.\tan\theta)$$

M1 Multiplying out, cancelling and collecting up terms

M1 Dividing by $\tan\alpha\ \tan\beta$

$\Rightarrow\ 2\cot\beta + 3\tan\theta = \cot\alpha$ **A1 (ANSWER GIVEN)** ⑥

$\theta = 30^{o}$, $\beta = 45^{o}$ $\Rightarrow$ $\cot\alpha = 2.1 + 3.\dfrac{1}{\sqrt{3}} = 2 + \sqrt{3}$ **B1**

Now $\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$ $\Rightarrow$ $\frac{1}{\sqrt{3}}\left(1 - t^2\right) = 2t$ $\Rightarrow$ $0 = t^2 + 2t\sqrt{3} - 1$ **M1**

$\Rightarrow$ $t = \tan 15^{o} = \dfrac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = -\sqrt{3} \pm 2$ **M1**

However, $\tan 15^{o} > 0$ since 15^{o} is acute, so $\tan 15^{o} = 2 - \sqrt{3}$ and $\cot 15^{o} = 2 + \sqrt{3}$ **M1 A1**

ALT. $\tan(60^{o} - 45^{o}) = \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}} = \dfrac{\left(\sqrt{3}-1\right)^2}{3 - 1} = 2 - \sqrt{3} = \dfrac{1}{2 + \sqrt{3}}$ or verification ⑤

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12 Since the pdf is only non-zero between 0 & 1 and the area under its graph = 1 if a, b both < / > 1 then total area will be < / >1

M1 considering graph or ≡
M1 consideration of area … relative to 1 ②

(i) $1 = \int_0^1 f(x)\,dx = \int_0^k a\,dx + \int_k^1 b\,dx$ **M1** use of total prob. = 1

$= [ax]_0^k + [bx]_k^1 = ak + b - bk$ **M1** calculus used to find k

$\Rightarrow k = \dfrac{1-b}{a-b}$ **A1** ③

$E(X) = \int_0^1 x f(x)\,dx = \int_0^k ax\,dx + \int_k^1 bx\,dx$ **M1**

$= \left[\dfrac{ax^2}{2}\right]_0^k + \left[\dfrac{bx^2}{2}\right]_k^1 = \dfrac{ak^2}{2} + \dfrac{b}{2} - \dfrac{bk^2}{2}$

M1 use of k in terms of a, b $E(X) = \dfrac{b}{2} + \dfrac{(a-b)}{2} \times \left(\dfrac{1-b}{a-b}\right)^2 = \dfrac{ba - b^2 + 1 - 2b + b^2}{2(a-b)}$

$= \dfrac{1-2b+ab}{2(a-b)}$ **A1 (ANSWER GIVEN)** ③

(ii) If $ak \ge \frac{1}{2}$ (i.e. $M \in (0, k)$) **M1** recognition of this

then $\dfrac{a-ab}{a-b} \ge \frac{1}{2} \Rightarrow 2a - 2ab \ge a - b \Rightarrow a + b \ge 2ab$ **B1** correct condition confirmed

and $aM = \frac{1}{2}$ or $M = \dfrac{1}{2a}$ **A1 (ANSWER GIVEN)** ③

If $ak \le \frac{1}{2}$ (i.e. $M \in (k, 1)$) **M1** recognition that this ≡ $a + b \le 2ab$

then $ak + (M-k)b = \frac{1}{2}$ or $(1-M)b = \frac{1}{2}$ **M1** $\Rightarrow M = 1 - \dfrac{1}{2b}$ **A1** ③

(iii) If $a + b \ge 2ab$, then $\mu - M = \dfrac{1-2b+ab}{2(a-b)} - \dfrac{1}{2a}$ **M1** applying correct case

$= \dfrac{a - 2ab + a^2b - a + b}{2a(a-b)} = \dfrac{b(1-a)^2}{2a(a-b)}$ **M1** single fraction, fact^g. & compl^g. the sq.

or equivalent (inequalities) method

> 0 **A1** correctly concluded ③

If $a + b \le 2ab$, then $\mu - M = \dfrac{1-2b+ab}{2(a-b)} - 1 + \dfrac{1}{2b}$ **M1** applying correct case

$= \dfrac{b - 2b^2 + ab^2 - 2ab + 2b^2 + a - b}{2b(a-b)} = \dfrac{a(1-b)^2}{2b(a-b)}$ **M1** sing. frac., fact^g. & compl^g. the sq. (or ≡)

> 0 **A1** correctly concluded ③

13 (i) $P(W_{PPQ}) = P(W_P\, W_Q\, –) + P(L_P\, W_Q\, W_P)$ **M1** A sum of 2(3) probs or ≡ product

$= p\,.\,q.\,1 + (1 – p)qp = pq(2 – p)$ **A1**

$P(W_{PQQ}) = pq(2 – q)$ similarly **B1** ***ft***

$P(W_{PPQ}) – P(W_{PQQ}) = pq(q – p)$ **M1** Or comparing two sides of a relevant inequality (Ditto throughout qn.)

> 0 since $q > p \Rightarrow P(W_{PPQ}) > P(W_{PQQ})$ for all p, q

and "Ros plays Pardeep twice" is always her best strategy **A1** ⑤

(ii) **SI:** $P(W_1) = P(W_Q\, W_P\, ––) + P(W_Q\, L_P\, W_P\, –) + P(W_Q\, L_P\, L_P\, W_P)$ **M1** cases

$= pq + pq(1 – p) + pq(1 – p)^2$ **A1** unsimplified $= pq(3 – 3p + p^2)$

SIII: $p(W_3) = pq(3 – 3q + q^2)$ similarly **B1** ***ft***

SII: $p(W_2) = p(W_P\, W_Q\, ––) + p(L_P\, W_P\, W_Q\, –) + p(W_P\, L_Q\, W_Q\, –) + p(L_P\, W_P\, L_Q\, W_Q)$ **M1** cases

$= pq + pq(1 – p) + pq(1 – q) + pq(1 – p)(1 – q)$ **A1** unsimplified

$= pq(4 – 2p – 2q + pq)$ or $pq(2 – p)(2 – q)$ ⑤

$P(W_1) – P(W_3) = pq(q – p)\big(3-[p+q]\big) > 0$ since $q > p$ and $p + q < 2 < 3$

so that **SI** is always better than **S3** **B1** ①

$P(W_1) – P(W_2) = pq\left(p^2 - p - 1 - pq + 2q\right)$ **M1**

$= pq\big((2-p)(q-p)-(1-p)\big)$ **M1**

> 0 whenever $q - p > \dfrac{1-p}{2-p} = 1 - \dfrac{1}{2-p}$ **A1** [arrangements with > one q term not helpful]

Now $p + \frac{1}{2} < q < 1 \Rightarrow 0 < p < \frac{1}{2} \Rightarrow \frac{1}{3} < 1 - \dfrac{1}{2-p} < \frac{1}{2}$,

so that **SI** always better than **SII** when $q - p > \frac{1}{2}$. **E1**

<u>**ALT.**</u> Setting $q = p + \frac{1}{2} + \varepsilon$ where $\varepsilon > 0$ gives

$P(W_1) – P(W_2) = p\left(p+\tfrac{1}{2}+\varepsilon\right)\left(p^2 - p - 1 - p^2 - \tfrac{1}{2}p - p\varepsilon + 2p + 1 + 2\varepsilon\right)$

$= p\left(p+\tfrac{1}{2}+\varepsilon\right)\left(\tfrac{1}{2}p + (2-p)\varepsilon\right) > 0$ since all terms positive ④

$P(W_1) - P(W_2) > < 0 \Leftrightarrow q - p > < \dfrac{1-p}{2-p}$ **M1** Some clear method for deciding

Take $p = \frac{1}{4}, q = \frac{1}{2} \Rightarrow q - p = \frac{1}{4} < \frac{1}{2}$ and $\dfrac{1-p}{2-p} = \frac{3}{7} > \frac{1}{4}$ so **SII** is better than **SI** **M1 A1**

Take $p = \frac{1}{4}, q = \frac{3}{4} - \varepsilon \Rightarrow q - p = \frac{1}{2} - \varepsilon < \frac{1}{2}$ and $\dfrac{1-p}{2-p} = \frac{3}{7}$

so choosing $\varepsilon < \frac{3}{7} - \frac{1}{2} = \frac{1}{14}$ (say $\frac{1}{16}$) will give **M1**

$p = \frac{1}{4}, q = \frac{11}{16}$ and $q - p = \frac{7}{16} > \dfrac{1-p}{2-p} = \frac{3}{7}$ so **SI** is better than **SII** **A1**

For the most part, candidates are just picking values of *a* and *b* and subst[g]. into
SI : $pq(3 - 3p + p^2)$ and **SII** : $pq(2 - p)(2 - q)$

If they pick an *a* and a *b* and then do nothing with them, they score M0.
To score the M1, they must show that $q - p < \frac{1}{2}$ and attempt to work out the two probs.
To score the A1, *they* must demonstrate the result. Also, their numerical working must be both visible and correct

⑤

[I think that $q - p > k$ has $k = \frac{1}{2}$ as the least positive k which *always* gives **SI** better than **SII**]