STEP II 2012 Q1

$(1-x^6)^{-2} = 1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots$ **B1** Enough terms to see what's going on

General term $(n+1)\,x^{6n}$ **B1** (Implies previous **B1**'s work) **2 marks**

(i) $(1-x^3)^{-1} = (1+x^3+x^6+x^9+\ldots)$ **B1**

The x^{24} term in $(1-x^6)^{-2}(1-x^3)^{-1} = (1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots)(1+x^3+x^6+x^9+\ldots)$

comes from $1.x^{24} + 2x^6.x^{18} + 3x^{12}.x^{12} + 4x^{18}.x^6 + 5x^{24}.1$

M1

Coefft. of x^{24} is $1 + 2 + 3 + 4 + 5 = 15$ **A1** **3 marks**

Coefft. of x^n is $\begin{cases} 0 & n = 6k + \{1, 2, 4, 5\} \\ \frac{1}{2}(k+1)(k+2) & n = 6k+3 \\ \frac{1}{2}(k+1)(k+2) & n = 6k \end{cases}$

B1 B1 B1 One each correct **3 marks**

(ii) $\mathrm{f}(x) = (1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots)(1+x^3+x^6+x^9+\ldots)(1+x+x^2+x^3+\ldots)$

The x^{24} term comes from

$$\begin{aligned} &1.1.5x^{24} + 1.x^6.4x^{18} + 1.x^{12}.3x^{12} + 1.x^{18}.2x^6 + 1.x^{24}.1 \\ &+ x^3.x^3.4x^{18} + x^3.x^9.3x^{12} + x^3.x^{15}.2x^6 + x^3.x^{21}.1 \\ &+ x^6.1.4x^{18} + x^6.x^6.3x^{12} + x^6.x^{12}.2x^6 + x^6.x^{18}.1 \\ &+ x^9.x^3.3x^{12} + x^9.x^9.2x^6 + x^9.x^{15}.1 \\ &+ x^{12}.1.3x^{12} + x^{12}.x^6.2x^6 + x^{12}.x^{12}.1 \\ &+ x^{15}.x^3.2x^6 + x^{15}.x^9.1 \\ &+ x^{18}.1.2x^6 + x^{18}.x^6.1 \\ &+ x^{21}.x^3.1 \\ &+ x^{24}.1.1 \end{aligned}$$

Coefft. of x^{24} is $15 + 2 \times (10 + 6 + 3 + 1) = 55$

M1 First **M** mark for keeping one term fixed from any bracket

M1 Second **M** mark for 2nd bracket

M1 Third **M** mark for fully correct method

A1 ***Answer Given*** **4 marks**

Alternatively, the sum is simply

$5 \times 1 + 4 \times 3 + 3 \times 5 + 2 \times 7 + 1 \times 9 = 5 + 12 + 15 + 14 + 9 = 55.$ **(∗)**

Note that, because every non-multiple-of-3 power in bracket 3 is redundant, the x^{24} term comes from considering $\mathrm{f}(x) = (1-x^6)^{-2}(1-x^3)^{-2} = (1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots)(1+2x^3+3x^6+4x^9+\ldots)$.

Again, every non-multiple-of-6 power in *this* 2nd bracket is also redundant, one might consider only

$$\mathrm{f}(x) = (1+3x^6+5x^{12}+7x^{18}+9x^{24}+\ldots)(1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots)$$

from which the coefft. of x^{24} is simply calculated as $1 \times 5 + 3 \times 4 + 5 \times 3 + 7 \times 2 + 9 \times 1 = 55$, exactly as in **(∗)**. The result **(∗)**, in some form or another – i.e. working from the 3rd bracket – gives the way of working out the coefficient of x^{6n} for any non-negative integer n. It is immediately obvious that it is $\sum_{r=0}^{n}(n+1-r)(2r+1)$ which turns out to be the same as $\sum_{r=1}^{n+1} r^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

The proof of this result could be by induction or direct manipulation of the standard results for Σr and Σr^2. However, I very much doubt any candidate will approach it in this general way and I am not presently requiring proofs of such results are not required.

The coefft. of x^{25} is 55 **B1**

This is the same as for x^{24}, since the extra x only arises from replacing 1 by x, x^3 by x^4, etc., in the first bracket's term (at each step of the working) and the coefficients are equal in each case.

E1 Credible reasoning (there must be some)
Or similar working repeated **2 marks**

In the case when $n = 11$, the coefficient of x^{66} is gained from

$12x^{66} \times$ the no. of ways of getting x^0 from the first two brackets **M1**

$+ 11x^{60} \times$ the no. of ways of getting x^6 from the first two brackets **M1**

$+ 10x^{54} \times$ the no. of ways of getting x^{12} from the first two brackets **M1**

$+ \ldots$

$+ 2x^6 \times$ the no. of ways of getting x^{60} from the first two brackets

$+ 1x^0 \times$ the no. of ways of getting x^{66} from the first two brackets **M1** all the way down

$= 12 \times 1 + 11 \times 3 + 10 \times 5 + \ldots + 2 \times 21 + 1 \times 23$ **M1**

$= 12 + 33 + 50 + 63 + 72 + 77 + 78 + 75 + 68 + 57 + 42 + 23$

$= 650$ **A1** **6 marks**

STEP II 2012 Q2

$\mathrm{p}(\mathrm{q}(x))$ has degree mn **B1** **1 mark**

(i) $\mathrm{Deg}[\mathrm{p}(x)] = n \Rightarrow \mathrm{Deg}[\mathrm{p}(\mathrm{p}(x))] = n^2$ & $\mathrm{Deg}[\mathrm{p}(\mathrm{p}(\mathrm{p}(x)))] = n^3$ **B1** Noted somewhere

$\mathrm{Deg}[\mathrm{LHS}] \le \max(n^3, n)$. RHS of degree 1.

Therefore LHS not constant (nb $n \ge 0$) so $n = 1$ and $\mathrm{p}(x)$ is linear.

E1 Essentially correct reasoning **2 marks**

Setting $\mathrm{p}(x) = ax + b$

$\Rightarrow \mathrm{p}(\mathrm{p}(x)) = a(ax + b) + b = a^2x + (a + 1)b$ **M1**

& $\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) = a[a^2x + (a + 1)b] + b = a^3x + (a^2 + a + 1)b$ **M1** Doesn't have to be correct yet

Then $a^3x + (a^2 + a + 1)b - 3ax - 3b + 2x \equiv 0$

$(a^3 - 3a + 2)x + (a^2 + a - 2)b \equiv 0$ **M1** Equating **both** coefft. of x and constant terms to zero

$(a - 1)(a^2 + a - 2)x + (a^2 + a - 2)b \equiv 0$ **M1** Factorising

$(a^2 + a - 2)[(a - 1)x + b] \equiv 0$

$(a + 2)(a - 1)[(a - 1)x + b] \equiv 0$

We have, then, that $a = -2$ or 1. **A1** Both a values correct

In either case, b takes any (arbitrary) value

Solutions are thus $\mathrm{p}_1(x) = -2x + b$ and $\mathrm{p}_2(x) = x + b$ **A1 A1** Must be arbitrary b.

Give one **A1** if any one correct, but not both **A1**s if extra answers appear. **7 marks**

(ii) Deg[RHS] = 4 while Deg[LHS] $\le \max\left(n^2, 2n, n\right)$, **E1** Supporting reasoning

so it follows that $n = 2$ and $p(x)$ is quadratic. **B1** Noted or implied anywhere

Setting $p(x) = ax^2 + bx + c$, we have

$2p(p(x)) = 2a(ax^2 + bx + c)^2 + 2b(ax^2 + bx + c) + 2c$ **M1**

$= 2a\left\{a^2x^4 + 2abx^3 + 2acx^2 + b^2x^2 + 2bcx + c^2\right\} + 2b(ax^2 + bx + c) + 2c$

$3(p(x))^2 = 3\left[a^2x^4 + 2abx^3 + \left(2ac + b^2\right)x^2 + 2bcx + c^2\right]$ **M1**

& $-4p(x) = -4ax^2 - 4bx - 4c$

LHS = $\left(2a^3 + 3a^2\right)x^4 + \left(4a^2b + 6ab\right)x^3 + \left(2ab^2 + 4a^2c + 2ab + 3b^2 + 6ac - 4a\right)x^2$
$+ \left(4abc + 2b^2 + 6bc - 4b\right)x + \left(2ac^2 + 2bc + 2c + 3c^2 - 4c\right)$

RHS = x^4

Equating terms: **M1**

x^4) $2a^3 + 3a^2 - 1 = 0 \Rightarrow (a+1)^2(2a-1) \Rightarrow a = -1$ or $\frac{1}{2}$ **A1** From correct terms

x^3) $2ab(2a + 3) = 0 \Rightarrow b = 0$ **A1** From correct terms

x^2) $2a(2ac + 3c - 2) = 0 \Rightarrow c = 2$ when $a = -1$ **A1** i.e. $p_1(x) = -x^2 + 2$

OR $c = \frac{1}{2}$ when $a = \frac{1}{2}$ **A1** i.e. $p_2(x) = \frac{1}{2}\left(x^2 + 1\right)$

x^1) $2b(2ac + b + 3c - 2) = 0$ Checks

x^0) $c(2ac + 3c - 2) = 0$ Checks **E1** Both checks must be visible **10 marks**

STEP II 2012 Q3

$t = \sqrt{x^2+1} + x \Rightarrow \dfrac{1}{t} = \sqrt{x^2+1} - x$ and $x = \frac{1}{2}\left(t - \dfrac{1}{t}\right)$ **M1**

OR via $(t-x)^2 = x^2 + 1 \Rightarrow t^2 - 2tx = 1 \Rightarrow x = \dfrac{t^2-1}{2t}$ or $x = \frac{1}{2}t - \frac{1}{2}t^{-1}$ **A1**

$dx = \left(\frac{1}{2} + \frac{1}{2}t^{-2}\right) dt$ **B1**

Also $x : (0, \infty) \to t : (1, \infty)$ **B1** Limits dealt with at some stage

so that $\displaystyle\int_0^\infty f\left(\sqrt{x^2+1} + x\right) dx = \int_1^\infty f(t) \times \frac{1}{2}\left(1 + \frac{1}{t^2}\right) dt$ **M1** Full substitution

$= \displaystyle\frac{1}{2}\int_1^\infty f(x)\left(1 + \frac{1}{x^2}\right) dx$ **A1** ***Answer Given*** **6 marks**

$I_1 = \int_0^\infty \frac{1}{\left(\sqrt{x^2+1}+x\right)^2}\ \mathrm{d}x$ i.e. $\mathrm{f}(x) = \frac{1}{x^2}$ **M1**

NB Qn. says "Hence" so alternative methods not accepted unless they are using the previous substitution again

$= \tfrac{1}{2}\int_1^\infty \left(1+\frac{1}{x^2}\right).\frac{1}{x^2}\ \mathrm{d}x = \tfrac{1}{2}\int_1^\infty \left(x^{-2}+x^{-4}\right)\ \mathrm{d}x$ **A1**

$= \tfrac{1}{2}\left[-\frac{1}{x}-\frac{1}{3x^3}\right]_1^\infty$ **A1** Integration correct

$= \tfrac{1}{2}\left(0+1+\tfrac{1}{3}\right)=\tfrac{2}{3}$ **A1** **4 marks**

$x = \tan\theta \Rightarrow \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$ **B1** Qn. says to use this substn. so alternative methods not accepted

$\sqrt{1+x^2} = \sec\theta$ **B1** Used at any stage

Limits: $\left(0,\tfrac{1}{2}\pi\right)\to\left(0,\infty\right)$ **B1**

$I_2 = \int_0^{\frac{1}{2}\pi} \frac{1}{\left(1+\sin\theta\right)^3}\ \mathrm{d}\theta = \int_0^{\frac{1}{2}\pi}\left(\frac{\sec\theta}{\sec\theta+\tan\theta}\right)^3\ \mathrm{d}\theta$ **M1**

$= \int_0^{\frac{1}{2}\pi} \frac{\sec\theta}{\left(\sec\theta+\tan\theta\right)^3}.\sec^2\theta\ \mathrm{d}\theta = \int_0^\infty \frac{\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)^3}\ \mathrm{d}x$ **A1**

$\mathrm{f}(t) = \frac{\tfrac{1}{2}\left(t+\frac{1}{t}\right)}{t^3} = \frac{t^2+1}{2t^4}$ **M1**

$= \tfrac{1}{2}\int_1^\infty \left(\frac{t^2+1}{t^2}\right).\left(\frac{t^2+1}{2t^4}\right)\ \mathrm{d}t$ **A1**

$= \tfrac{1}{4}\int_1^\infty \left(t^{-2}+2t^{-4}+t^{-6}\right)\ \mathrm{d}t$ **A1**

$= \tfrac{1}{4}\left[-\frac{1}{t}-\frac{2}{3t^3}-\frac{1}{5t^5}\right]_1^\infty$ **A1** Integration correct

$= \tfrac{1}{4}\left(0+1+\tfrac{2}{3}+\tfrac{1}{5}\right) \quad = \tfrac{7}{15}$ **A1** **10 marks**

STEP II 2012 Q4

(i) $n, k > 1 \Rightarrow n^{k+1} > n^k$ and $k + 1 > k$ so $(k+1)\times n^{k+1} > k \times n^k$ **E1**

$\Rightarrow \dfrac{1}{(k+1)n^{k+1}} < \dfrac{1}{kn^k}$ **E1** (since all terms > 0)

2 marks

$\ln\left(1+\dfrac{1}{n}\right) = \dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \dfrac{1}{4n^4} + \dfrac{1}{5n^5} - \ldots$ valid as $0 < \dfrac{1}{n} < 1$ **E1**

$= \dfrac{1}{n} - \left(\dfrac{1}{2n^2} - \dfrac{1}{3n^3}\right) - \left(\dfrac{1}{4n^4} - \dfrac{1}{5n^5}\right) - \ldots$ **M1**

$< \dfrac{1}{n}$ since each bracketed term is positive **A1**

by the previous result **E1** Justification that each bracket positive

$\Rightarrow 1 + \dfrac{1}{n} < e^{\frac{1}{n}} \Rightarrow \left(1+\dfrac{1}{n}\right)^n < e$ **B1** **5 marks**

ALT. Max 4/5 for non-"Hence" methods; e.g. using the exponential series

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(ii) $\ln\left(\dfrac{2y+1}{2y-1}\right) = \ln\left(1+\dfrac{1}{2y}\right) - \ln\left(1-\dfrac{1}{2y}\right)$ **M1** Log. work

$= \left(\dfrac{1}{2y} - \dfrac{1}{2(2y)^2} + \dfrac{1}{3(2y)^3} - \dfrac{1}{4(2y)^4} + \dfrac{1}{5(2y)^5} - \ldots\right) -$

$\left(-\dfrac{1}{2y} - \dfrac{1}{2(2y)^2} - \dfrac{1}{3(2y)^3} - \dfrac{1}{4(2y)^4} - \dfrac{1}{5(2y)^5} - \ldots\right)$

M1 A1 A1 Use of log. series (given in question); 1st correct; 2nd correct

$= 2\left(\dfrac{1}{2y} + \dfrac{1}{3(2y)^3} + \dfrac{1}{5(2y)^5} + \ldots\right)$ **A1** All correct

$> \dfrac{1}{y}$ (since following terms all positive) **A1**

Series valid for $0 < \dfrac{1}{2y} < 1$ i.e. $y > \frac{1}{2}$ **E1** **7 marks**

Then $\ln\left(\dfrac{2y+1}{2y-1}\right)^y > 1$ **B1**

Setting $y = n + \frac{1}{2}$ **M1** $\Rightarrow \ln\left(\dfrac{2n+2}{2n}\right)^{n+\frac{1}{2}} > 1 \Rightarrow \left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}} > e$ **A1 A1** **4 marks**

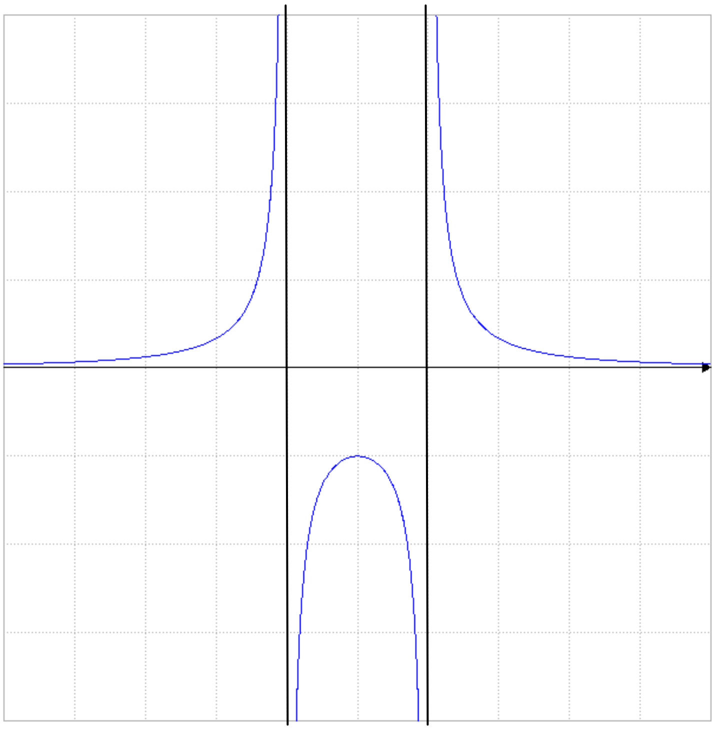
(iii) As $n \to \infty$, $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}} = \left(1+\dfrac{1}{n}\right)^n \times \left(1+\dfrac{1}{n}\right)^{\frac{1}{2}} \to \left(1+\dfrac{1}{n}\right)^n \times 1+$ **E1**

i.e. $\to \left(1+\dfrac{1}{n}\right)^n$ from above and e is squeezed into the same limit from both above and below

E1 Informal explanation only required **2 marks**

STEP II 2012 Q5

(i)



G1 Vertical asymptotes at $x = a - 1$ and $x = a + 1$

G1 Horizontal asymptote the x-axis

G1 Symmetry in $x = a$

G1 Three branches:
LH and RH branches $\approx 1/x^2$;
middle branch $\cap$ –shaped
(with max. at $x \approx a$)

Ignore the position of the y-axis

4 marks

(ii) $$g'(x) = \frac{-2}{\left[(x-a)^2 - 1\right]^2\left[(x-b)^2 - 1\right]^2}\left\{(x-b)\left[(x-a)^2 - 1\right] + (x-a)\left[(x-b)^2 - 1\right]\right\}$$

M1 Differentiated (may be done implicitly after "logging")

Setting the numerator = 0 **M1**

$$(x-a)(x-b)\left[x-a+x-b\right] + \left[x-a+x-b\right] = 0$$

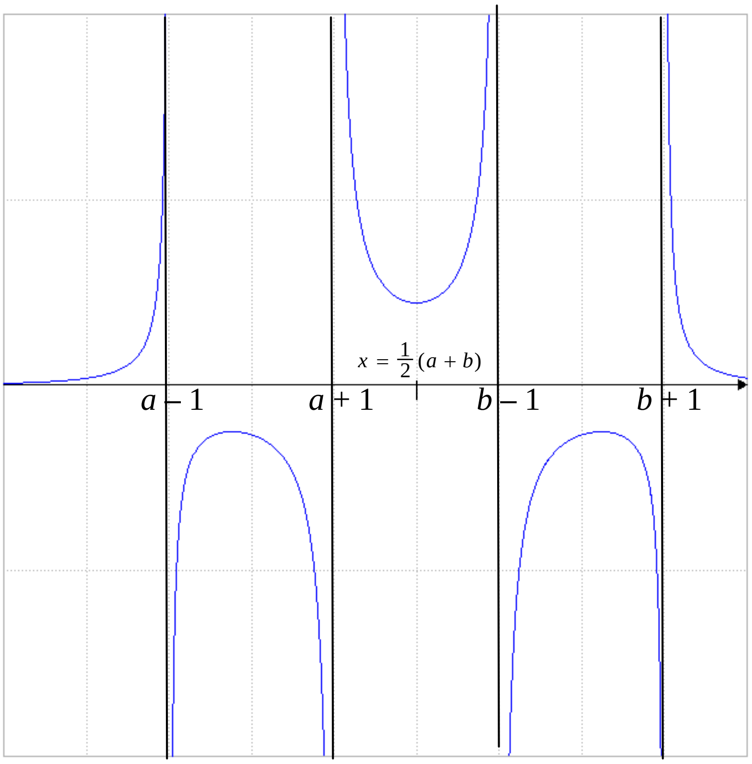
M1 Sensible factorisation attempt: $(2x - a - b)\left(x^2 - (a+b)x + (ab-1)\right) = 0$

$x = \tfrac{1}{2}(a+b)$ **A1** or $x = \dfrac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4}}{2}$

A1 A1 Any sensible form: e.g. $x = \tfrac{1}{2}\left\{a + b \pm \sqrt{(b-a)^2 + 4}\right\}$

6 marks

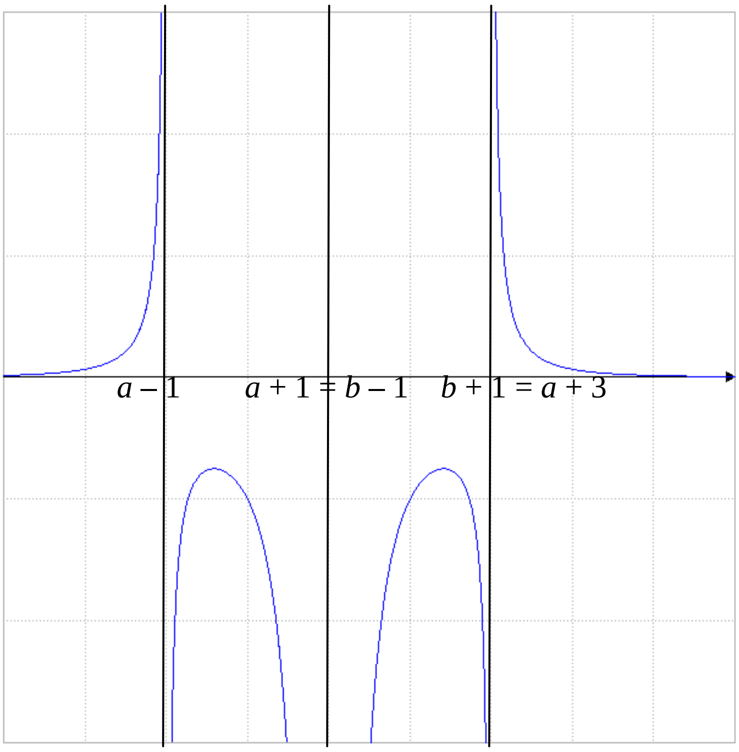
Case 1 $b > a + 2$ (i.e. $a + 1 < b - 1$)



G1	Five branches
G1	4 vertical asymptotes (with correct coordinates)
G1	LH & RH branches correctly placed
G1	Middle three branches correctly placed
G1	All correct

5 marks

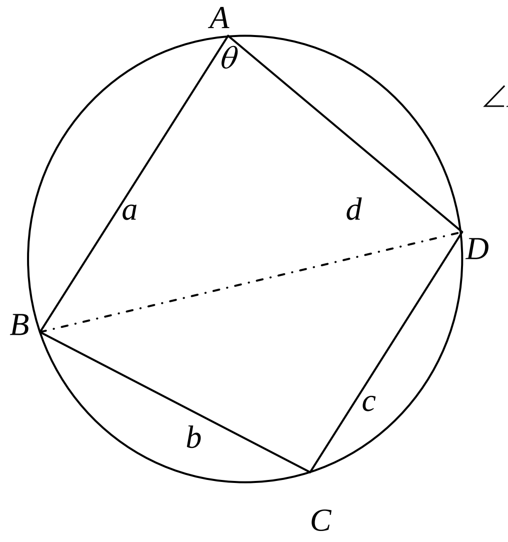
Case 2 $b = a + 2$ (i.e. $a + 1 = b - 1$)



G1	Four branches
G1	3 vertical asymptotes (with correct coordinates)
G1	LH & RH branches correctly placed
G1	Middle two branches correctly placed
G1	All correct

5 marks

STEP II 2012 Q6



$\angle BCD = \pi - \theta$ *(Opp. ∠s cyclic quad.)* **B1** Noted or used (possibly implicitly)

<u>Cosine Rule in ΔBAD</u>: **M1**

$BD^2 = a^2 + d^2 - 2ad\cos\theta$ **A1**

<u>Cosine Rule in ΔBCD</u>: **M1**

$BD^2 = b^2 + c^2 + 2bc\cos\theta$ **A1**

Equating for BD^2 **M1**

Identifying $\cos\theta$ **M1**

$= \dfrac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}$ **A1** **8 marks**

$Q = \tfrac{1}{2}ad\sin\theta + \tfrac{1}{2}bc\sin\theta$ **M1** Since $\sin(\pi - \theta) = \sin\theta$

$\Rightarrow \sin\theta = \dfrac{2Q}{ad + bc}$ or $\dfrac{4Q}{2(ad + bc)}$ **A1** **2 marks**

Use of $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \dfrac{16Q^2}{4(ad + bc)^2} + \dfrac{\left(a^2 - b^2 - c^2 + d^2\right)^2}{4(ad + bc)^2} = 1$ **M1**

$\Rightarrow 16Q^2 = 4(ad + bc)^2 - \left(a^2 - b^2 - c^2 + d^2\right)^2$ **A1** ***Answer Given***

2 marks

$16Q^2 = \left(2ad + 2bc - a^2 + b^2 + c^2 - d^2\right)\left(2ad + 2bc + a^2 - b^2 - c^2 + d^2\right)$ **M1** Use of the *difference of* 2 *squares* factorisation

$= \left([b + c]^2 - [a - d]^2\right)\left([a + d]^2 - [b - c]^2\right)$ **M1** Completing squares

$= \left([b + c] - [a - d]\right)\left([b + c] + [a - d]\right)\left([a + d] - [b - c]\right)\left([a + d] + [b - c]\right)$

M1 Use of the *difference of* 2 *squares* factorisation in both brackets

$= (b + c + d - a)(a + b + c - d)(a + c + d - b)(a + b + d - c)$ **A1**

Splitting the 16 into four 2's (one per bracket) and using $2s = a + b + c + d$ **M1**

$\Rightarrow Q^2 = \dfrac{(2s - 2a)}{2}\dfrac{(2s - 2b)}{2}\dfrac{(2s - 2c)}{2}\dfrac{(2s - 2d)}{2} = (s - a)(s - b)(s - c)(s - d)$ **A1** ***Answer Given***

6 marks

For a triangle (guaranteed cyclic) let $d \to 0$ (**Or** $s - d \to s$ **Or** let $D = A$) **E1**

$\Delta = \sqrt{s(s - a)(s - b)(s - c)}$ with or without explanation **B1** **2 marks**

STEP II 2012 Q7

Centroid, G, has p.v. $\mathbf{g} = \tfrac{1}{3}\left(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\right)$

$\overrightarrow{GX_1} = \mathbf{x}_1 - \mathbf{g} = \tfrac{1}{3}\left(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3\right)$ **M1 A1**

and so $\overrightarrow{GY_1} = -\tfrac{1}{3}\lambda_1\left(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3\right)$ (where $\lambda_1 > 0$) **B1**

Then $\overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \tfrac{1}{3}\left(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\right) - \tfrac{1}{3}\lambda_1\left(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3\right)$ **M1**

$= \tfrac{1}{3}\left([1-2\lambda]\mathbf{x}_1 + [1+\lambda_1](\mathbf{x}_2 + \mathbf{x}_3)\right)$ **A1** **5 marks**

Circle centre O, radius 1 has equation $|\,\mathbf{x}\,|^2 = 1$ or $\mathbf{x}\,.\,\mathbf{x} = 1$ noted at any stage **B1**

Since $\overrightarrow{OY_1}\bullet\overrightarrow{OY_1} = 1$, we have **M1**

(Note: using $\alpha = \mathbf{x}_2 \bullet \mathbf{x}_3$, $\beta = \mathbf{x}_3 \bullet \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \bullet \mathbf{x}_2$)

$1 = \tfrac{1}{9}\left\{(1-2\lambda_1)^2 + 2(1+\lambda_1)^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1\bullet(\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2\mathbf{x}_2\bullet\mathbf{x}_3\right\}$ **A1**

$\Rightarrow \quad 9 = 1 - 4\lambda_1 + 4\lambda_1{}^2 + 2 + 4\lambda_1 + 2\lambda_1{}^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1\bullet(\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2\mathbf{x}_2\bullet\mathbf{x}_3$ **A1**

$\Rightarrow \quad 0 = -3(1-\lambda_1)(1+\lambda_1) + (1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1\bullet(\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)^2\mathbf{x}_2\bullet\mathbf{x}_3$ **A1**

As $\lambda_1 > 0$, $0 = -3(1-\lambda_1) + (1-2\lambda_1)\mathbf{x}_1\bullet(\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)\mathbf{x}_2\bullet\mathbf{x}_3$ **M1**

$\Rightarrow \quad 0 = -3 + 3\lambda_1 + \left(\mathbf{x}_1\bullet\mathbf{x}_2 + \mathbf{x}_2\bullet\mathbf{x}_3 + \mathbf{x}_3\bullet\mathbf{x}_1\right) + \lambda_1(\mathbf{x}_2\bullet\mathbf{x}_3) - 2\lambda_1\left(\mathbf{x}_1\bullet\mathbf{x}_2 + \mathbf{x}_1\bullet\mathbf{x}_3\right)$ **M1**

$\Rightarrow \lambda_1 = \dfrac{3-(\alpha+\beta+\gamma)}{3+\alpha-2\beta-2\gamma}$ **A1** ***Answer Given*** **8 marks**

Similarly, $\lambda_2 = \dfrac{3-(\alpha+\beta+\gamma)}{3+\beta-2\alpha-2\gamma}$ and $\lambda_3 = \dfrac{3-(\alpha+\beta+\gamma)}{3+\gamma-2\alpha-2\beta}$ **B1 B1**

$\dfrac{GX_i}{GY_i} = \dfrac{1}{\lambda_i}$ noted or used **B1**

$\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} + \dfrac{1}{\lambda_3}$ **M1**

$= \dfrac{9 + (\alpha+\beta+\gamma) - 4(\alpha+\beta+\gamma)}{3-(\alpha+\beta+\gamma)}$ **M1**

$= \dfrac{9 - 3(\alpha+\beta+\gamma)}{3-(\alpha+\beta+\gamma)} = 3$ **M1 A1** Factorising leading to ***Given Answer*** **7 marks**

NB This result generalises to n points on a circle: $\displaystyle\sum_{i=1}^{n}\frac{GX_i}{GY_i} = n$.

STEP II 2012 Q8

$\beta - \alpha > q\,(>0) \;\Rightarrow\; \beta^2 - 2\alpha\beta + \alpha^2 > q^2$ **M1**

$\Rightarrow\; \alpha^2 + \beta^2 - q^2 > 2\alpha\beta \;\Rightarrow\; \dfrac{\alpha^2+\beta^2-q^2}{\alpha\beta} > 2 \;\Rightarrow\; \dfrac{\alpha^2+\beta^2-q^2}{\alpha\beta} - 2 > 0$

A1 ***Answer Given*** ($\alpha, \beta > 0$ given) **2 marks**

$u_{n+1} = \dfrac{u_n{}^2 - q^2}{u_{n-1}}$ etc. $\Rightarrow\; u_n{}^2 - u_{n+1}u_{n-1} = q^2 = u_{n+1}{}^2 - u_{n+2}u_n$ **M1** Equating for q^2

$\Rightarrow\; u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$ **A1** Re-arranged & factorised to get ***Given Answer*** **2 marks**

Then $\dfrac{u_n + u_{n+2}}{u_{n+1}} = \dfrac{u_{n-1} + u_{n+1}}{u_n}$ **B1**

which $\Rightarrow\; \dfrac{u_{n-1} + u_{n+1}}{u_n}$ is constant (independent of n) **M2**

Calling this constant p gives $u_{n+1} - pu_n + u_{n-1} = 0$, as required **A1** ***Answer Given***

$u_2 = \dfrac{\beta^2 - q^2}{\alpha}$ **B1** Anywhere

$p = \dfrac{u_0 + u_2}{u_1} = \dfrac{\alpha + \dfrac{\beta^2 - q^2}{\alpha}}{\beta}$ **M1** Use of first terms (even if not proved this is *always* the constant p)

$= \dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta}$ **A1** **7 marks**

ALTERNATIVE METHOD

$u_2 = \gamma = \dfrac{\beta^2 - q^2}{\alpha} = p\beta - \alpha \;\Leftrightarrow\; p = \dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta}$ **M1 A1**

$u_3 = \dfrac{\gamma^2 - q^2}{\beta} = p\gamma - \beta \;\Leftrightarrow\; p = \dfrac{\gamma^2 + \beta^2 - q^2}{\beta\gamma}$ **M1 A1**

$= \dfrac{\left(\dfrac{\beta^2 - q^2}{\alpha}\right)^2 + \beta^2 - q^2}{\beta\left(\dfrac{\beta^2 - q^2}{\alpha}\right)}$ **M1** $= \dfrac{(\beta^2 - q^2)^2 + \alpha^2(\beta^2 - q^2)}{\alpha\beta(\beta^2 - q^2)} = \dfrac{\beta^2 - q^2 + \alpha^2}{\alpha\beta}$ **A1** since

$\beta^2 - q^2 \neq 0$ as u_2 non-zero (given). Since p is consistent for any chosen α, β, the proof follows inductively on any two consecutive terms of the sequence. **E1** **7 marks**

If $\beta > \alpha + q$, $u_{n+1} - u_n = (p-1)u_n - u_{n-1} = \left(\dfrac{\beta^2 + \alpha^2 - q^2}{\alpha\beta} - 1\right)u_n - u_{n-1}$

M1 Considering $u_{n+1} - u_n$; **M1** using p

$> (2-1)u_n - u_{n-1}$ by the initial result **M1**

$> u_n - u_{n-1}$

Hence, if $u_n - u_{n-1} > 0$ then so is $u_{n+1} - u_n$ **A1** Valid conclusion

Since $\beta > \alpha$, $u_2 - u_1 > 0$ and proof follows by induction **E1** **5 marks**

If $\beta = \alpha + q$ then $p = 2$ **B1**

and $u_{n+1} - u_n = u_n - u_{n-1}$ so that the sequence is an AP **B1**

Also, $u_0 = \alpha$, $u_1 = \alpha + q$, $u_2 = \alpha + 2q$, … $\Rightarrow$ common difference is q
M1 A1 (which is still a strictly increasing sequence since $q > 0$ given) **4 marks**

STEP II 2012 Q9

Use of $x = ut\cos\alpha$ **M1** When $x = a$, $t = \dfrac{a}{u\cos\alpha}$ **A1**

$y = 2h - ut\sin\alpha - \frac{1}{2}gt^2$ **(✳) B1**

Subst^g. in their t at $x = a$ into their y **M1** $y = 2h - a\tan\alpha - \dfrac{ga^2}{2u^2}\sec^2\alpha$

A1 May still be $\cos^2\alpha$ at this stage

Use of their $y > h$ **M1** $2h - a\tan\alpha - \dfrac{ga^2}{2u^2}\sec^2\alpha > h \Rightarrow h - a\tan\alpha > \dfrac{ga^2}{2u^2}\sec^2\alpha$

$\Rightarrow \dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}$ **A1** ***Answer Given*** **7 marks**

Setting $y = 0$ in **(✳)** and writing it as a quadratic in t **M1**

$$t = \frac{-2u\sin\alpha + \sqrt{4u^2\sin^2\alpha + 16gh}}{2g}$$

A1 No need to mention that the negative root is inappropriate; allow ± for now

Setting $x = u\cos\alpha \times$ their t **M1**

Setting their $x < b$ **M1** $u\cos\alpha\left(\dfrac{\sqrt{u^2\sin^2\alpha + 4gh} - u\sin\alpha}{g}\right) < b$

$\Rightarrow \sqrt{u^2\sin^2\alpha + 4gh} < \dfrac{bg}{u\cos\alpha} + u\sin\alpha$ **A1** ***Answer Given*** **5 marks**

M1 Dividing throughout by $\cos\alpha$: $\sqrt{u^2\tan^2\alpha + 4gh\sec^2\alpha} < \dfrac{bg\sec^2\alpha}{u} + u\tan\alpha$

M1 Squaring (both sides positive):

$$u^2\tan^2\alpha + 4gh\sec^2\alpha < \frac{b^2g^2\sec^4\alpha}{u^2} + 2bg\sec^2\alpha\tan\alpha + u^2\tan^2\alpha$$

M1 Cancelling $u^2\tan^2\alpha$ both sides & dividing by $g\sec^2\alpha$: $4h < \dfrac{b^2g\sec^2\alpha}{u^2} + 2b\tan\alpha$

M1 Using first result, $\dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$, in here: $4h < 2b\tan\alpha + b^2g\sec^2\alpha \times \dfrac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}$

M1 Cancelling and multiplying by a^2: $4a^2h < 2a^2b\tan\alpha + 2b^2(h - a\tan\alpha)$

M1 Re-arranging for $\tan\alpha$: $2ab(b-a)\tan\alpha < 2h\left(b^2 - 2a^2\right)$

A1 Answer $\tan\alpha < \dfrac{h\left(b^2 - 2a^2\right)}{ab(b-a)}$ legitimately obtained ***Answer Given***

E1 Explanation that $b > a$ (other side of net) (else direction of inequality would reverse)

ALTERNATIVE

M1 Squaring $\sqrt{u^2\sin^2\alpha + 4gh} < \dfrac{bg}{u\cos\alpha} + u\sin\alpha$:

$$u^2\sin^2\alpha + 4gh < \frac{b^2g^2\sec^2\alpha}{u^2} + 2bg\tan\alpha + u^2\sin^2\alpha$$

M1 Cancelling $u^2\sin^2\alpha$ both sides & dividing by g: $4h < \dfrac{b^2g\sec^2\alpha}{u^2} + 2b\tan\alpha$

M2 Re-arranging for $\dfrac{1}{u^2}$: $\dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{1}{u^2}$

M1 Using first result, $\dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$, in here: $\dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$

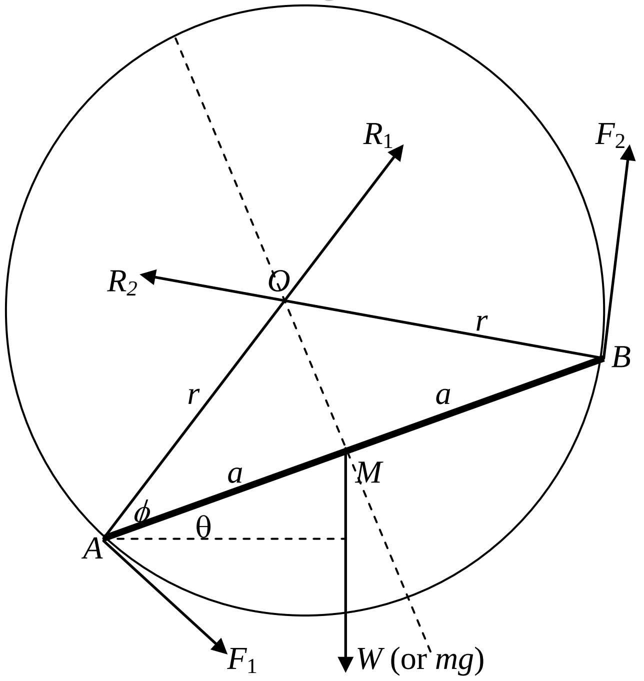
M1 Re-arranging for $\tan\alpha$: $ab(b-a)\tan\alpha < h\left(b^2 - 2a^2\right)$

A1 Answer $\tan\alpha < \dfrac{h\left(b^2 - 2a^2\right)}{ab(b-a)}$ legitimately obtained ***Answer Given***

E1 Explanation that $b > a$ (other side of net) (else direction of inequality would reverse)

8 marks

STEP II 2012 Q10



<u>Moments about M</u> :

$R_1\, a \sin\phi = R_2\, a \sin\phi + F_1\, a \cos\phi + F_2\, a \cos\phi$

M1 Four terms

A1 Correct magnitudes

A1 Correct signs

<u>Friction Law</u> : $F_1 = \mu R_1$ and $F_2 = \mu R_2$ **B1**

Dividing by $\cos\phi$ and re-arranging **M1**

$R_1 \tan\phi = R_2 \tan\phi + \mu R_1 + \mu R_2$

$\Rightarrow (R_1 - R_2)\tan\phi = \mu(R_1 + R_2)$ **A1**

Answer Given

6 marks

<u>Moments about O</u> : $\mu(R_1 - R_2)\, r = W r \sin\phi \sin\theta$ **M1 A1 A1** Method; LHS; RHS

<u>Resolving // AB</u> : $(R_1 - R_2)\cos\phi + \mu(R_1 + R_2)\sin\phi = W\sin\theta$ **M1 A1 A1** Method; LHS; RHS

(Give one **A1** here if all correct apart from a – sign)

<u>Resolving $\perp^r$ AB</u> : $(R_1 + R_2)\sin\phi - \mu(R_1 - R_2)\cos\phi = W\cos\theta$ **M1 A1 A1** Method; LHS; RHS

(Give one **A1** here if all correct apart from a – sign)

NB – Only two of these are required (give for their two best efforts) **6 marks**

M1 Dividing these last two eqns: $$\tan\theta = \frac{(R_1 - R_2)\cos\phi + \mu(R_1 + R_2)\sin\phi}{(R_1 + R_2)\sin\phi - \mu(R_1 - R_2)\cos\phi}$$

M1 Use of first result, $\mu(R_1 + R_2) = (R_1 - R_2)\tan\phi$:

$$\tan\theta = \frac{(R_1 - R_2)\cos\phi + (R_1 - R_2)\tan\phi\sin\phi}{(R_1 - R_2)\dfrac{\tan\phi}{\mu}\sin\phi - \mu(R_1 - R_2)\cos\phi}$$

A1 $$\tan\theta = \frac{\cos\phi + \tan\phi\sin\phi}{\dfrac{\tan\phi}{\mu}\sin\phi - \mu\cos\phi}$$ No need to note that $R_1 \neq R_2$

M1 Multiplying throughout by $\mu\cos\phi$: $$\tan\theta = \frac{\mu\left(\cos^2\phi + \sin^2\phi\right)}{\sin^2\phi - \mu^2\cos^2\phi} = \frac{\mu}{1 - \cos^2\phi - \mu^2\cos^2\phi}$$

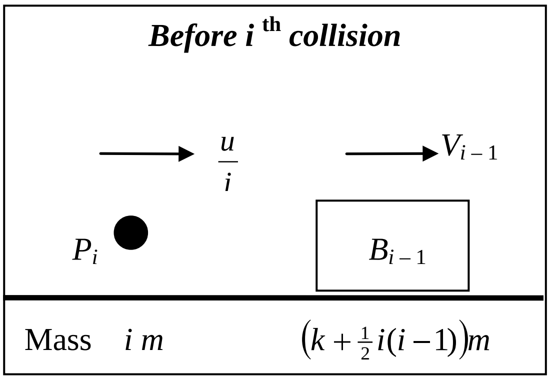
M1 Use of $\cos\phi = \dfrac{a}{r}$

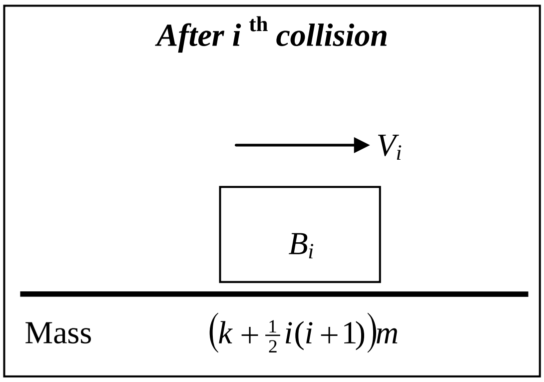
A1 $$\tan\theta = \frac{\mu}{1 - \dfrac{a^2}{r^2} - \mu^2\left(\dfrac{a^2}{r^2}\right)} = \frac{\mu r^2}{r^2 - a^2\left(1 + \mu^2\right)}$$ ***Answer Given*** **6 marks**

M1 $$\tan\lambda = \mu = \left(\frac{R_1 - R_2}{R_1 + R_2}\right)\tan\phi$$ from the first result

$< \tan\phi \Rightarrow \lambda < \phi$ **A1** ***Answer Given*** **2 marks**

STEP II 2012 Q11





Mass of block before/after **B1 B1** Allow unsimplified for now

CLM → $\quad m\,u + M\,V_{i-1} = (M + im)\,V_i$ **M1 A1** This done at any stage (NB $V_0 = 0$)

M1 A1 This done at a general stage

$$V_1 = \frac{u}{k+1},\ V_2 = \frac{2u}{k+1+2},\ V_3 = \frac{3u}{k+1+2+3},\ \ldots$$ **A1** ≥ 3 terms (or alt. method for generalising)

$$V_n = \frac{nu}{k+\frac{1}{2}n(n+1)} = \frac{2nu}{2k+n(n+1)}$$ **A1** General term correct ***Answer Given*** **8 marks**

ALTERNATIVE

CLM → for *all* particles $\quad mu + 2m\left(\frac{u}{2}\right) + 3m\left(\frac{u}{3}\right) + \ldots\ldots + nm\left(\frac{u}{n}\right) = \left(k+\frac{1}{2}n(n+1)\right)mV$ **M2 A2 B1**

A1 Method; LHS; Final total mass; RHS **M1** Re-arranging for V **A1** for $V_n = \dfrac{2nu}{2k+n(n+1)}$

Last collision occurs when $V_n \ge \dfrac{u}{n+1}$ **M1**

i.e. $\dfrac{2nu}{N(N+1)+n(n+1)} \ge \dfrac{u}{n+1}$ **M1** Use of given k in first result

$\Rightarrow\ 2n(n+1) \ge N(N+1) + n(n+1)\ \Rightarrow\ n(n+1) \ge N(N+1)$ **M1** Re-arrangement

$\Rightarrow$ there are N collisions **A1** **4 marks**

Total KE of all the P_i's is $\sum_{i=1}^{N} \frac{1}{2}(i\,m)\left(\frac{u}{i}\right)^2 = \frac{1}{2}mu^2 \sum_{i=1}^{N} \frac{1}{i}$ **M1 A1**

Final KE of the block is $\frac{1}{2}N(N+1)mV_N{}^2$ **M1 A1** Correct final mass

$= \frac{1}{2}N(N+1)m\left(\frac{u}{N+1}\right)^2 = \frac{1}{2}mu^2\left(\frac{N}{N+1}\right)$ **A1** Correct final speed

Loss in KE is the difference: $\frac{1}{2}mu^2 \sum_{i=1}^{N} \frac{1}{i} - \frac{1}{2}mu^2\left(\frac{N}{N+1}\right)$ **M1**

Use of $\dfrac{N}{N+1} = 1 - \dfrac{1}{N+1}$ **M1**

Loss in KE = $\frac{1}{2}mu^2\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - 1 + \frac{1}{N+1}\right) = \frac{1}{2}mu^2 \sum_{i=2}^{N+1}\left(\frac{1}{i}\right)$ **A1** ***Answer Given***

8 marks

STEP II 2012 Q12

P(light on) = $p \times \frac{3}{4} \times \frac{1}{2} + (1 - p) \times \frac{1}{4} \times \frac{1}{2}$ $= \frac{1}{8}(1+2p)$

M1 A1 **M1** Complementary probs. **A1**

Then P(Hall | on) = $\dfrac{\frac{1}{8}(1-p)}{\frac{1}{8}(1+2p)}$ **M1** Conditional prob. **B1** Numr. correct **A1** ***ft*** Denomr.

$= \dfrac{(1-p)}{(1+2p)}$ **A1** ***Answer Given*** **8 marks**

Recognition of B(7, p_1) for some prob. p_1 **M1**

Modal value 3 $\Rightarrow$ $\binom{7}{2}(p_1)^2(1-p_1)^5 < \binom{7}{3}(p_1)^3(1-p_1)^4$ *and* $\binom{7}{4}(p_1)^4(1-p_1)^3 < \binom{7}{3}(p_1)^3(1-p_1)^4$

B1 B1 B1 one for each correct binomial term (unsimplified)
M1 M1 for each correct inequality clearly stated (with some attempt to do something with them)

M1 for using $p_1 = \dfrac{(1-p)}{(1+2p)}$

M1 for using numerical binomial coeffts. and correct powers of (their) p_1 and ($1 - p_1$):

$21\left(\dfrac{1-p}{1+2p}\right)^2\left(\dfrac{3p}{1+2p}\right)^5 < 35\left(\dfrac{1-p}{1+2p}\right)^3\left(\dfrac{3p}{1+2p}\right)^4 \Rightarrow 3(3p) < 5(1-p) \Rightarrow p < \frac{5}{14}$ **M1** Cancelling **A1** correct RH half of the inequality ***Answer Given***

$35\left(\dfrac{1-p}{1+2p}\right)^4\left(\dfrac{3p}{1+2p}\right)^3 < 35\left(\dfrac{1-p}{1+2p}\right)^3\left(\dfrac{3p}{1+2p}\right)^4 \Rightarrow (1-p) < (3p) \Rightarrow p > \frac{1}{4}$ **M1** Cancelling **A1** correct LH half of the inequality ***Answer Given*** **12 marks**

STEP II 2012 Q13

P(no supermarkets) = $\mathrm{e}^{-k\pi y^2}$ **M1** A single Poisson term **A1** Correct **2 marks**

$\mathrm{P}(Y < y) = 1 - \mathrm{e}^{-k\pi y^2}$ **M1 A1**

Differentiating w.r.t. y **M1** $\Rightarrow$ $\mathrm{f}(y) = 2k\pi\, y\, \mathrm{e}^{-k\pi y^2}$ **A1** ***Answer Given*** **4 marks**

$\mathrm{E}(Y) = \int_0^\infty 2k\pi\, y^2\, \mathrm{e}^{-k\pi y^2}\, \mathrm{d}y$ **M1** *Integration by Parts* attempted **M1**

Writing $2k\pi\, y^2\, \mathrm{e}^{-k\pi y^2}$ as $y\left(2k\pi\, y\, \mathrm{e}^{-k\pi y^2}\right)$ **M1**

$$\left[y\left(\mathrm{e}^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty \mathrm{e}^{-k\pi y^2}\,\mathrm{d}y \quad \textbf{A1} \qquad = 0 + \int_0^\infty \mathrm{e}^{-k\pi y^2}\,\mathrm{d}y$$

M1 Use of the substitution $x = y\sqrt{2k\pi}$ $\quad = \dfrac{1}{\sqrt{2k\pi}}\displaystyle\int_0^\infty \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x$ **A1**

$$= \frac{1}{\sqrt{2k\pi}}\sqrt{\frac{\pi}{2}} = \frac{1}{2\sqrt{k}}$$ **A1** (by the given result) **7 marks**

$\mathrm{E}(Y^2) = \int_0^\infty 2k\pi\, y^3\, \mathrm{e}^{-k\pi y^2}\, \mathrm{d}y$ **M1** *Integration by Parts* attempted **M1**

Writing $2k\pi\, y^3\, \mathrm{e}^{-k\pi y^2}$ as $y^2\left(2k\pi\, y\, \mathrm{e}^{-k\pi y^2}\right)$ **M1**

$$\left[y^2\left(\mathrm{e}^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty 2y\, \mathrm{e}^{-k\pi y^2}\,\mathrm{d}y \quad \textbf{A1}$$

$$= 0 + \frac{1}{k\pi}\int_0^\infty 2k\pi\, y\, \mathrm{e}^{-k\pi y^2} = \frac{-1}{k\pi}\left[\mathrm{e}^{-k\pi y^2}\right]_0^\infty$$ **M1** Use of a previous result (or a substitution)

$\Rightarrow$ $\mathrm{E}(Y^2) = \dfrac{1}{k\pi}$ **A1**

$\Rightarrow$ $\mathrm{Var}(Y) = \dfrac{1}{k\pi} - \dfrac{1}{4k} = \dfrac{4-\pi}{4k\pi}$ **A1** ***Answer Given*** **7 marks**
