1. $\frac{dz}{dx} = y^{n-1}\frac{dy}{dx}\left(n\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}\right)$

$\frac{dz}{dx} = ny^{n-1}\frac{dy}{dx}\left(\frac{dy}{dx}\right)^2 + y^n 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}$ **M1 expansion**

$z = y^n\left(\frac{dy}{dx}\right)^2$ **M1 attempt to obtain z by integration A1** **(3)**

(these marks can be awarded by implication if result used in specific case anywhere else in question)

(i) $\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = \sqrt{y}$

So if $n = 1,\ z = y\left(\frac{dy}{dx}\right)^2$ and $\frac{dz}{dx} = \sqrt{y}\frac{dy}{dx}$ **M1 choice of n value, B1 for z result, M1 attempt to apply stem, A1 for doing so correctly**

Thus $z = \frac{2}{3}\ y^{\frac{3}{2}} + c$ **M1 attempt to integrate A1**

Hence $y\left(\frac{dy}{dx}\right)^2 = \frac{2}{3}\ y^{\frac{3}{2}} + c$ **M1 equating z expressions**

Given that $y = 1$, $\frac{dy}{dx} = 0$, when $x = 0$, **M1 evaluation of constant**

$0 = \frac{2}{3} + c$, $\left(\frac{dy}{dx}\right)^2 = \frac{2}{3}\left(y^{\frac{1}{2}} - \frac{1}{y}\right)$

$\left(\frac{dx}{dy}\right)^2 = \frac{3}{2}\ \frac{y}{y^{\frac{3}{2}}-1}$

Hence, $\frac{dx}{dy} = \sqrt{\frac{3}{2}}\ \frac{y^{\frac{1}{2}}}{\left(y^{\frac{3}{2}}-1\right)^{\frac{1}{2}}}$

So $x = \sqrt{\frac{3}{2}}\frac{4}{3}\left(y^{\frac{3}{2}} - 1\right)^{\frac{1}{2}} + c'$ and as $y = 1$ when $x = 0$, $c' = 0$

Thus $x^2 = \frac{8}{3}\left(y^{\frac{3}{2}} - 1\right)$ and so $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$ **M1 rearrangement, integration, finding constant (*) A1 cso** **(10)**

(ii) $\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} = -y^2$

So $-2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2y^2$ and if $n = -2,\ z = y^{-2}\left(\frac{dy}{dx}\right)^2$ and $\frac{dz}{dx} = y^{-3}\frac{dy}{dx}\ 2y^2$ **M1 attempting to arrange into form to use stem, B1 for z, B1 for $\frac{dz}{dx}$**

$\frac{dz}{dx} = \frac{2}{y}\frac{dy}{dx}$ which means $z = 2\ln y + c$

$y^{-2}\left(\frac{dy}{dx}\right)^2 = 2\ln y + c$ **M1 A1** and as $y = 1$ when $\frac{dy}{dx} = 0$, $c = 0$

$\frac{dx}{dy} = \frac{1}{\sqrt{2}} \frac{1}{y} \frac{1}{\sqrt{\ln y}}$ so $x = (2 \ln y)^{\frac{1}{2}} + c'$ and as $y = 1$ when $x = 0$, $c' = 0$

$x^2 = 2 \ln y$ and so $y = e^{\frac{1}{2}x^2}$ **M1 A1ft** **(7)**

2. (i) $(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$

$=(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$ **B1**

$=(1-x^4)(1+x^4)\ldots(1+x^{2^n})$ **B1**

$=(1-x^{2^{n+1}})$ **B1** ` **(3)**

Thus $(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})=\frac{1-x^{2^{n+1}}}{1-x}$ **M1**

$\frac{1}{1-x}=(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})+\frac{x^{2^{n+1}}}{1-x}$ **(*) A1** **(2)**

$\ln\frac{1}{1-x}=\ln\left((1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})+\frac{x^{2^{n+1}}}{1-x}\right)$

As $|x|<1$, letting $n\to\infty$, $-\ln(1-x)=\ln\left((1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})\ldots\right)$ **M1**

$-\ln(1-x)=\ln(1+x)+\ln(1+x^2)+\ln(1+x^4)+\cdots$ **M1**

$\ln(1-x)=-\sum_{r=0}^{\infty}\ln(1+x^{2^r})$ **(*) A1** **(3)**

Differentiating with respect to *x* gives $\frac{-1}{1-x}=-\left\{\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots\right\}$, **M1**

That is $\frac{1}{1-x}=\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots$ **(*) A1** **(2)**

(ii) Replacing x by x^3 in $\ln(1-x)=-\sum_{r=0}^{\infty}\ln(1+x^{2^r})$ **M1**

$\ln(1-x^3)=-\sum_{r=0}^{\infty}\ln(1+(x^3)^{2^r})$ **A1**

$\ln[(1-x)(1+x+x^2)]=-\sum_{r=0}^{\infty}\ln\left[(1+x^{2^r})\left(1-x^{2^r}+\left(x^{2^r}\right)^2\right)\right]$ **M1 LHS factor M1 RHS factor**

So $\ln(1-x)+\ln(1+x+x^2)=-\sum_{r=0}^{\infty}\ln(1+x^{2^r})-\sum_{r=0}^{\infty}\ln\left(1-x^{2^r}+\left(x^{2^r}\right)^2\right)$ **M1**

Subtracting the result $\ln(1-x)=-\sum_{r=0}^{\infty}\ln(1+x^{2^r})$, **M1**

$\ln(1+x+x^2)=-\sum_{r=0}^{\infty}\ln\left(1-x^{2^r}+\left(x^{2^r}\right)^2\right)$ **M1 A1**

and differentiating with respect to *x* gives,

$\frac{1+2x}{1+x+x^2}=-\sum_{r=0}^{\infty}\frac{-2^r x^{2^r-1}+2^{r+1}x^{2^{r+1}-1}}{1-x^{2^r}+\left(x^{2^r}\right)^2}$ **M1**

$\frac{1+2x}{1+x+x^2}=\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\cdots$ **(*)** **A1** **(10)**

Alternatively, following a similar argument to part (i),

$$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right) \qquad \textbf{M1 A1}$$

$$=(1+x^2+x^4)(1-x^2+x^4)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right) \qquad \textbf{A1}$$

$$=(1+x^4+x^8)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right)$$

$$=\left(1+x^{2^{n+1}}+x^{2^{n+2}}\right) \qquad \textbf{A1}$$

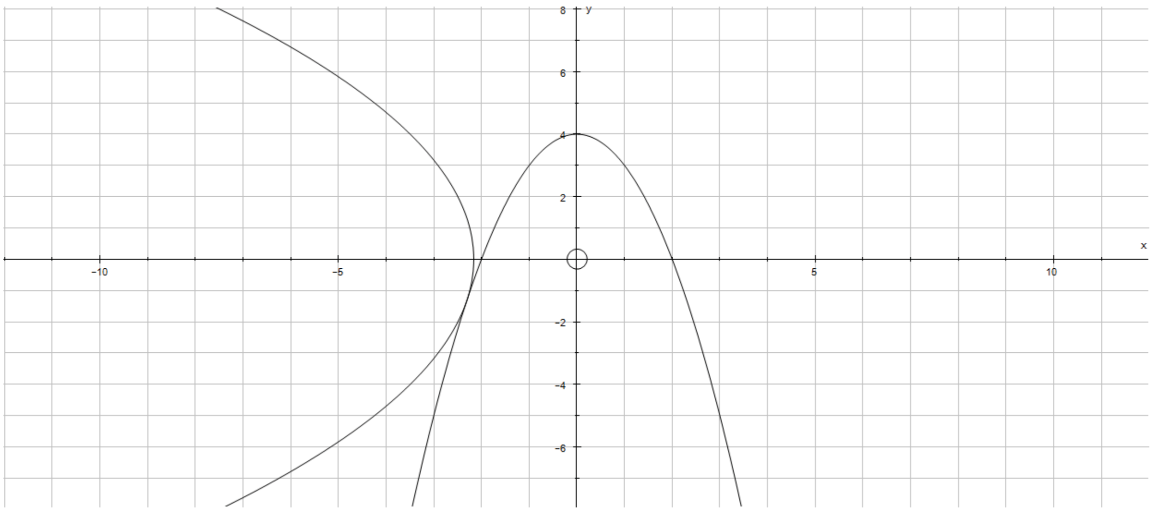
Thus $(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right)=\frac{\left(1+x^{2^{n+1}}+x^{2^{n+2}}\right)}{(1+x+x^2)}$ **M1**

and hence, letting $n\to\infty$, $\ln(1+x+x^2)=-\sum_{r=0}^{\infty}\ln\left(1-x^{2^r}+\left(x^{2^r}\right)^2\right)$ **M1 M1 A1**

then final **M1 A1** as before.

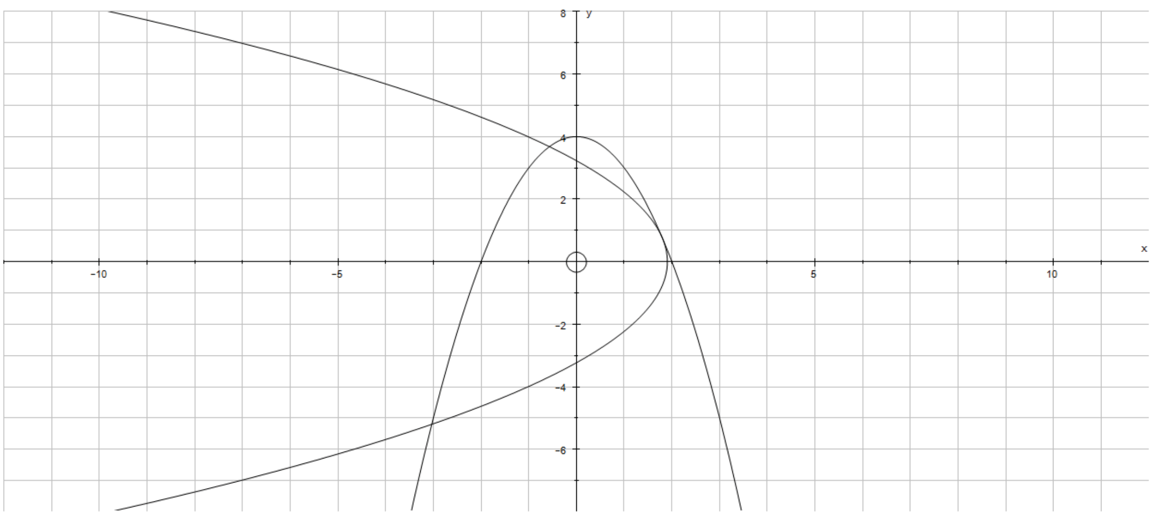
3. (i) **G1** for $y = 4 - x^2$ curve **B1** for all intercepts $(\pm 2,0), (0,4), \left(\frac{k}{m}, 0\right)$ at least once and $\left(0, \pm\sqrt{k}\right)$ at least once in b,c, or d

(a)



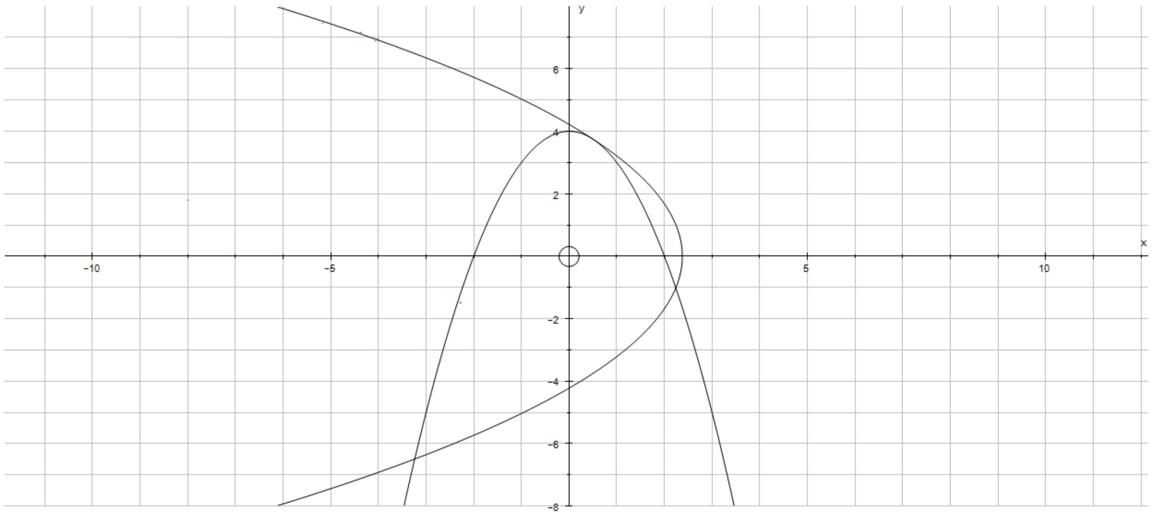
G1 correct look

(b)



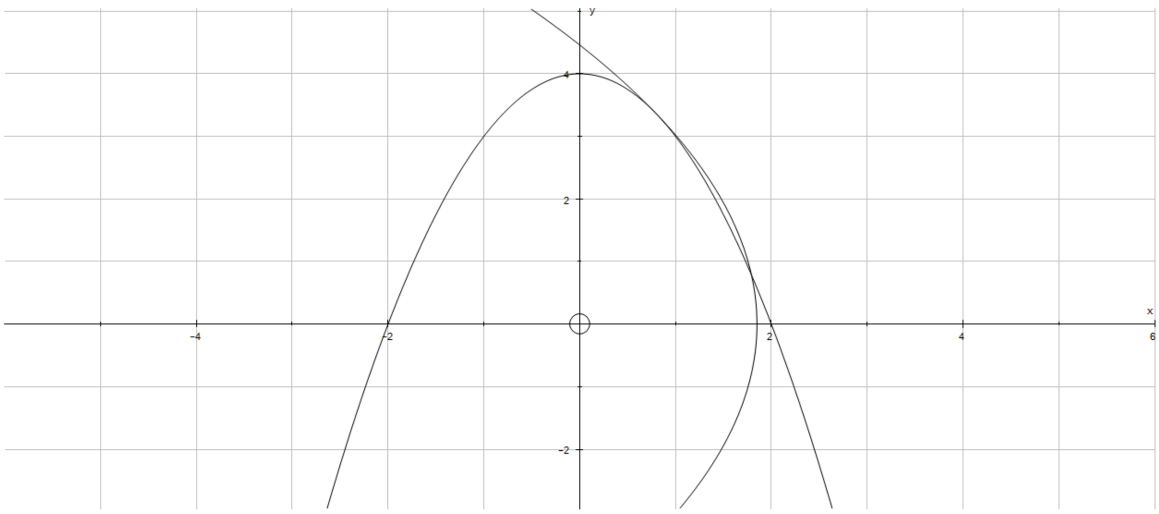
G1 correct look

(c)



G1 correct look

(d)



G1 correct look – note correct intersection and touching arrangement **(6)**

(ii) $y = 4 - x^2$ and $12x = k - y^2$

So $12x = k - (4 - x^2)^2$ at any intersection. **M1**

$12x = k - 16 + 8x^2 - x^4$

$x^4 - 8x^2 + 12x + 16 - k = 0$ **(*)** **A1**

If the curves touch, they have equal gradients.

$\frac{dy}{dx} = -2x$ and $12 = -2y\frac{dy}{dx}$ thus $-2x = \frac{-6}{y}$ **M1**

i.e. $xy = 3$ so as $x = a$, $y = 4 - a^2$

$a(4 - a^2) = 3$

$a^3 - 4a + 3 = 0$ **M1** **(*) A1**

(These three marks can be obtained by considering properties of the roots of the quartic, where the roots are a,a,c,d)

$(a-1)(a^2 + a - 3) = 0$

So $a = 1$ or $a = \frac{-1\pm\sqrt{13}}{2}$ **M1 A1**

As $x^4 - 8x^2 + 12x + 16 - k = 0$, $a^4 - 8a^2 + 12a + 16 - k = 0$, so $k = a^4 - 8a^2 + 12a + 16$

$k = aa^3 - 8a^2 + 12a + 16 = a(4a - 3) - 8a^2 + 12a + 16 = -4a^2 + 9a + 16$ **M1 (*) A1 (9)**

If $a = 1$, $k = 21$, $\frac{k}{m} = \frac{21}{12} < 2$ which is case (d). **B1**

$x^4 - 8x^2 + 12x - 5 = 0$, $(x-1)^2(x^2 + 2x - 5) = 0 \implies x = 1,\ -1 \pm \sqrt{6}$

If $a = \frac{-1+\sqrt{13}}{2}$, $k = -(-1+\sqrt{13})^2 + 9\left(\frac{-1+\sqrt{13}}{2}\right) + 16 = \frac{13}{2}\sqrt{13} - \frac{5}{2}$, so, $\frac{k}{m} < 2$ which is case (d).

B2

$x^4 - 8x^2 + 12x + \frac{37}{2} - \frac{13}{2}\sqrt{13} = 0$, $\left(x - \frac{-1+\sqrt{13}}{2}\right)^2 \left(x^2 + (\sqrt{13} - 1)x + \frac{-3\sqrt{13}+5}{2}\right) = 0$
$\Rightarrow x = \frac{-1+\sqrt{13}}{2}$, etc

If $a = \frac{-1-\sqrt{13}}{2}$, $k = -(-1-\sqrt{13})^2 + 9\left(\frac{-1-\sqrt{13}}{2}\right) + 16 = \frac{-13}{2}\sqrt{13} - \frac{5}{2}$, so, $k < 0$ which is case (a)

B2

(5)

4. (i) $\sum_{n=1}^{\infty}\frac{n+1}{n!}=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{n!}=e+e-1=2e-1$

M1 for splitting into two sums, A1 for one sum correctly stated, (*) A1 for result **(3)**

$\sum_{n=1}^{\infty}\frac{(n+1)^2}{n!}=\sum_{n=1}^{\infty}\frac{n(n-1)+3n+1}{n!}=e+3e+e-1=5e-1$

M1 for re-writing in given form, M1 A1 for dealing with first part correctly , (*) A1 for result **(4)**

Alternative M1 for simple expansion, M1 A1 for obtaining $\sum_{n=1}^{\infty}\frac{n^2}{n!}=e+e$**, (*) A1 for result**

$\sum_{n=1}^{\infty}\frac{(2n-1)^3}{n!}=\sum_{n=1}^{\infty}\frac{8n(n-1)(n-2)+12n(n-1)+2n-1}{n!}=8e+12e+2e-(e-1)=21e+1$

M1 for re-writing in given form, M1 A1 for dealing with first part correctly , (*) A1 for result **(4)**

Alternative M1 for simple expansion, M1 A1 for obtaining $\sum_{n=1}^{\infty}\frac{n^3}{n!}=5e$**, (*) A1 for result**

(ii) $\sum_{n=0}^{\infty}\frac{(n^2+1)2^{-n}}{(n+1)(n+2)}=\sum_{n=0}^{\infty}\left\{\frac{(n+1)(n+2)2^{-n}}{(n+1)(n+2)}-\frac{3(n+2)2^{-n}}{(n+1)(n+2)}+\frac{5\times 2^{-n}}{(n+1)(n+2)}\right\}$ **M1A1**

$=\sum_{n=0}^{\infty}\left\{2^{-n}-3\,\frac{2^{-n}}{n+1}+5\,\frac{2^{-n}}{(n+1)(n+2)}\right\}$

$\sum_{n=0}^{\infty}r^n=\frac{1}{1-r}$, $\sum_{n=0}^{\infty}\frac{r^{n+1}}{n+1}=-\ln(1-r)$, $\sum_{n=0}^{\infty}\frac{r^{n+2}}{(n+1)(n+2)}=-r\ln(1-r)+r+\ln(1-r)$

B1 **B1** **M1A1 (by integration of previous result)**

$\sum_{n=0}^{\infty}\frac{(n^2+1)2^{-n}}{(n+1)(n+2)}=2-3\times 2\times -\ln\left(\frac{1}{2}\right)+5\times 4\times\left(-\frac{1}{2}\ln\left(\frac{1}{2}\right)+\frac{1}{2}+\ln\left(\frac{1}{2}\right)\right)$ **M1 A1 (permit one error)**

$=2-6\ln 2+10\ln 2+10-20\ln 2=12-16\ln 2$ **A1** **(9)**

Alternatively, using partial fractions

$\frac{n^2+1}{(n+1)(n+2)}=1+\frac{2}{n+1}-\frac{5}{n+2}$ **M1 A1 (permit one error)A1**

$\sum_{n=0}^{\infty}r^n=\frac{1}{1-r}$, $\sum_{n=0}^{\infty}\frac{r^{n+1}}{n+1}=-\ln(1-r)$, **B1 B1**

so $\sum_{n=0}^{\infty}\frac{(n^2+1)2^{-n}}{(n+1)(n+2)}=\sum_{n=0}^{\infty}2^{-n}+2\sum_{n=0}^{\infty}\frac{2^{-n}}{n+1}-5\sum_{n=0}^{\infty}\frac{2^{-n}}{n+2}$ **M1**

$=2-4\ln\frac{1}{2}+20\left(\ln\frac{1}{2}+\frac{1}{2}\right)=12-16\ln 2$ **M1 A1 (permit one error)A1**

5. (i) (a) Integer rational points $(1,0)$ $(0,1)$ etc **B1**

Non-integer rational points $\left(\frac{3}{5},\frac{4}{5}\right),\left(\frac{5}{13},\frac{12}{13}\right)$ etc **B1** **(2)**

(b) IR point on $x^2+y^2=2$ $(1,1)$ etc **B1**

$\left(\cos\theta+\sqrt{m}\sin\theta\right)^2+\left(\sin\theta-\sqrt{m}\cos\theta\right)^2$

$=\cos^2\theta+2\sqrt{m}\cos\theta\sin\theta+m\sin^2\theta+\sin^2\theta-2\sqrt{m}\sin\theta\cos\theta+m\ \cos^2\theta=1+m$

M1 expansion and use of $\sin^2\theta+\cos^2\theta=1$ **A1**

Let $m=1$, $\cos\theta=\frac{3}{5}$ NIR point $\left(\frac{7}{5},\frac{1}{5}\right)$ **M1 A1** **(5)**

Other likely choices, $\cos\theta=\frac{4}{5}$ NIR point $\left(\frac{7}{5},\frac{-1}{5}\right)$, $\cos\theta=\frac{5}{13}$ NIR point $\left(\frac{17}{13},\frac{7}{13}\right)$

(ii) (a) I2R point on $x^2+y^2=3$ let $m=2$, $\cos\theta=1$ so $\left(1,-\sqrt{2}\right)$ etc. **B1**

NI2R point on $x^2+y^2=3$ let $m=2$, $\cos\theta=\frac{3}{5}$ so $\left(\frac{3}{5}+\frac{4}{5}\sqrt{2},\frac{4}{5}-\frac{3}{5}\sqrt{2}\right)$ etc. **M1 A1 (3)**

(b) $\left(a\cos\theta+b\sqrt{m}\sin\theta\right)^2+\left(a\sin\theta-b\sqrt{m}\cos\theta\right)^2=a^2\cos^2\theta+2ab\sqrt{m}\cos\theta\sin\theta+$
$mb^2\sin^2\theta+a^2\sin^2\theta-2ab\sqrt{m}\sin\theta\cos\theta+mb^2\cos^2\theta=a^2+mb^2$ **M1 A1**

With $m=2$, $a^2+2b^2=11$ **B1**

Let $a=3$, $b=1$, $\cos\theta=\frac{3}{5}$ NI2R point $\left(\frac{9}{5}+\frac{4}{5}\sqrt{2},\frac{12}{5}-\frac{3}{5}\sqrt{2}\right)$ **M1 A1 (5)**

(c) $\left(a\cosh\theta+\sqrt{m}\sinh\theta\right)^2-\left(a\sinh\theta+\sqrt{m}\cosh\theta\right)^2=a^2-m$ **M1 A1**

With $m=2$, $a^2-m=7$ so $a=3$ **M1**

If $\cosh\theta=\frac{13}{5}$, $\sinh\theta=\frac{12}{5}$ then a NI2R point is $\left(\frac{39}{5}+\frac{12}{5}\sqrt{2},\frac{36}{5}+\frac{13}{5}\sqrt{2}\right)$ **M1 A1 (5)**

Alternatively,

$\left(a\sec\theta+\sqrt{m}\ \tan\theta\right)^2-\left(a\tan\theta+\sqrt{m}\ \sec\theta\right)^2=a^2-m$

With $m=2$, $a=3$, if $\sec\theta=\frac{13}{5}$, $\tan\theta=\frac{12}{5}$ gives the same result.

6. $(x+iy)^2+p(x+iy)+1=0$

Equating real and imaginary parts, $x^2-y^2+px+1=0$, $2xy+py=0 \Rightarrow (2x+p)y=0$ **M1**

So either $p=-2x$ or $y=0$ in which case $x^2+px+1=0 \Rightarrow p=-\frac{x^2+1}{x}$ **(*) A1**

$y=0$, $p=-\frac{x^2+1}{x}$ represents the real axis with all values of x possible except zero, **B1**

or $p=-2x$ so $x^2-y^2+px+1=0$ becomes $x^2+y^2=1$ which represents a circle centred on the origin radius 1. **M1 A1 (5)**

$pz^2+z+1=0$, so $p(x+iy)^2+(x+iy)+1=0$

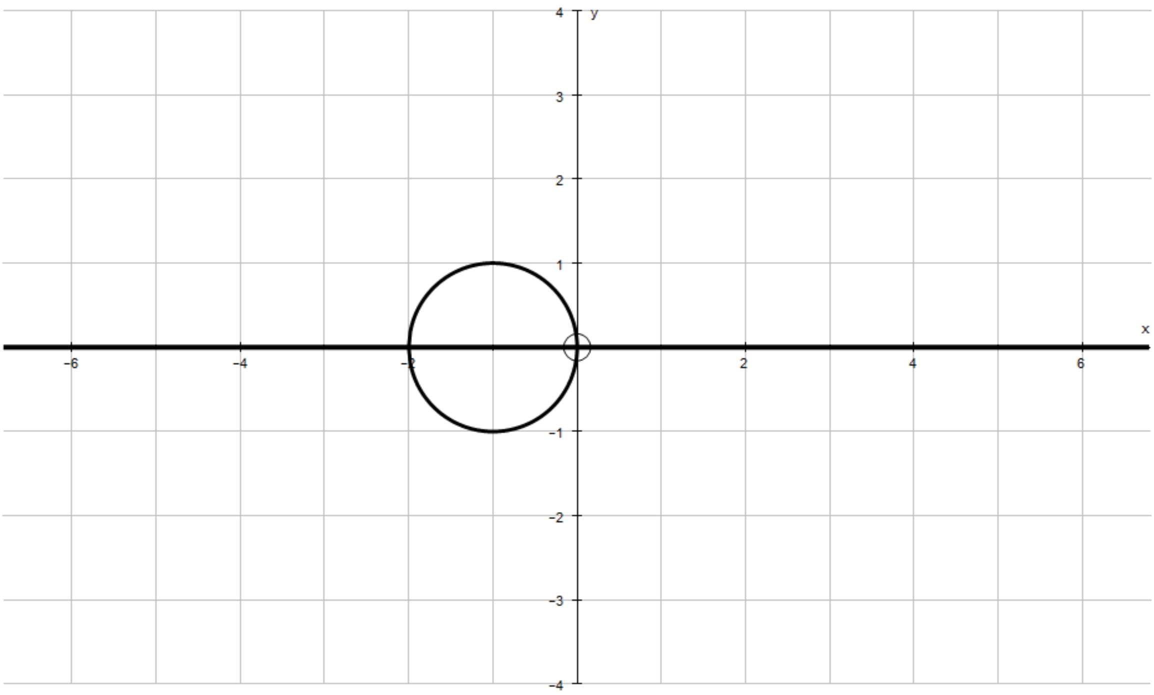
Equating real and imaginary parts, $px^2-py^2+x+1=0$, $2pxy+y=0 \Rightarrow (2px+1)y=0$

M1 A1

So either $p=-\frac{1}{2x}$ or $y=0$ in which case $px^2+x+1=0 \Rightarrow p=-\frac{x+1}{x^2}$ **A1**

$y=0$, $p=-\frac{x+1}{x^2}$ represents the real axis with all values of x possible except zero, **B1**

or $p=-\frac{1}{2x}$ so $px^2-py^2+x+1=0$ becomes $x^2-y^2=2x(x+1)$ i.e. $x^2+y^2+2x=0$ which represents a circle centred on $(-1,0)$ radius 1. **M1**



G1 **(6)**

$pz^2 + p^2z + 2 = 0$, so $p(x+iy)^2 + p^2(x+iy) + 2 = 0$

Equating real and imaginary parts, $px^2 - py^2 + p^2x + 2 = 0$,

and $2pxy + p^2y = 0 \Rightarrow (2x+p)py = 0$ **M1 A1**

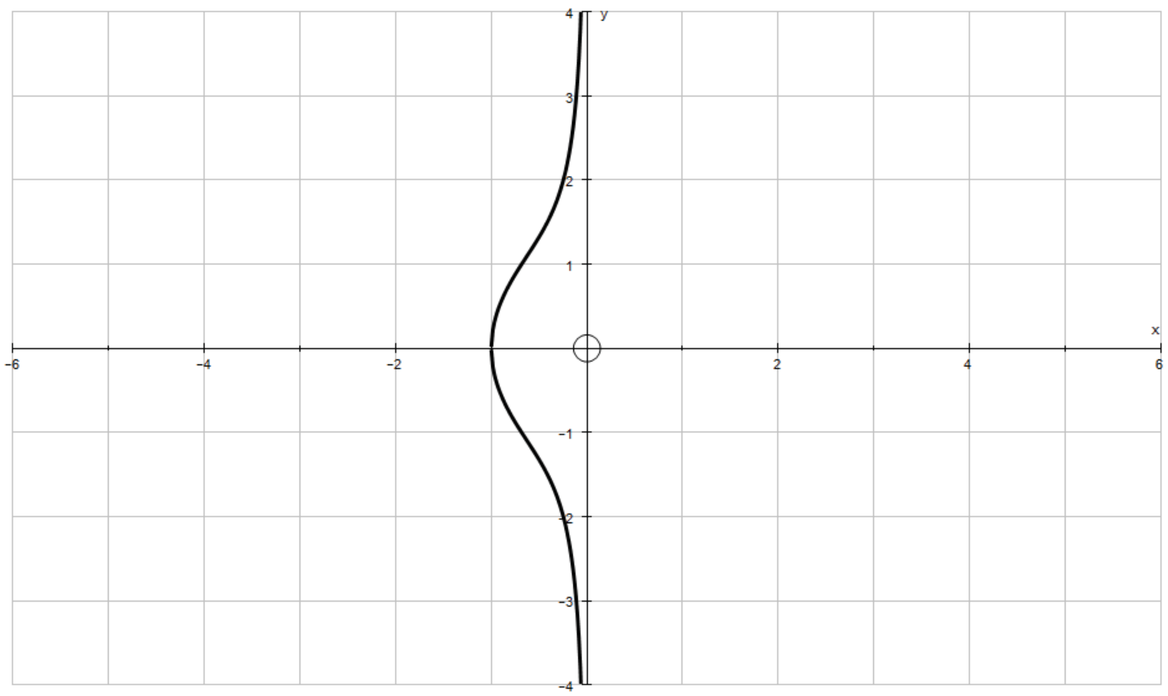
So either $y = 0$ in which case

$px^2 + p^2x + 2 = 0 \Rightarrow p = \frac{-x^2 \pm \sqrt{x^4 - 8x}}{2x}$ representing all points on the real axis $x < 0$ and $x \geq 2$

M1 **A1**

or $p = -2x$ so $px^2 - py^2 + p^2x + 2 = 0$ becomes $-2x^3 + 2xy^2 + 4x^3 + 2 = 0$

i.e. $x^3 + xy^2 + 1 = 0$, $y^2 = -\frac{x^3+1}{x}$ **M1A1**



plus the parts of the real axis G2 for curve G1 for complete graph

7. $\dot{y} = -2(y - z) \Longrightarrow \ddot{y} = -2(\dot{y} - \dot{z})$

As $\dot{z} = -\dot{y} - 3z$, $\ddot{y} = -2(\dot{y} - \dot{z}) = -2(\dot{y} + \dot{y} + 3z)$ **M1 diff M1 sub for $\dot{z}$**

As $\dot{y} = -2(y - z), -2z = -2y - \dot{y}$ **M1 making "z" subject**

So $\ddot{y} = -2(\dot{y} + \dot{y} + 3z) = -2\dot{y} - 2\dot{y} - 6y - 3\dot{y}$ **M1 sub**

Thus $\ddot{y} + 7\dot{y} + 6y = 0$ **A1**

AQE $m^2 + 7m + 6 = 0$ which has solutions $m = -1, -6$ **M1 A1**

Thus $y = Ae^{-t} + Be^{-6t}$

$z = y + \frac{1}{2}\dot{y} = Ae^{-t} + Be^{-6t} - \frac{1}{2}Ae^{-t} - 3Be^{-6t} = \frac{1}{2}Ae^{-t} - 2Be^{-6t}$ **M1 A1 (both y and z) (9)**

(i) $z(0) = 0 \implies \frac{1}{2}A - 2B = 0$ and $y(0) = 5 \implies A + B = 5$ and so $A = 4, B = 1$ **M1 obtain both equations M1 solve simultaneously**

So $z_1(t) = 2e^{-t} - 2e^{-6t}$ **A1** **(3)**

(ii) (i) $z(0) = c \implies \frac{1}{2}A - 2B = c$ and (i) $z(1) = c \implies \frac{A}{2e} - \frac{2B}{e^6} = c$ **M1**

So $A - 4B = 2c$ and $A - \frac{4B}{e^5} = 2ec$, thus $4B\left(1 - \frac{1}{e^5}\right) = 2c(e - 1)$ **M1**

So $B = \frac{c}{2}\frac{e^5(e-1)}{(e^5-1)}$

Also $(e^5 - 1)A = 2c(e^6 - 1)$ so $A = \frac{2c(e^6-1)}{(e^5-1)}$ **M1**

So $z_2(t) = \frac{c(e^6-1)}{(e^5-1)}e^{-t} - \frac{ce^5(e-1)}{(e^5-1)}e^{-6t}$ **A1 (4)**

(iii) $\sum_{n=-\infty}^{0} z_1(t - n) = z_1(t) + z_1(t + 1) + z_1(t + 2) + \cdots$

$= 2e^{-t} - 2e^{-6t} + 2e^{-(t+1)} - 2e^{-6(t+1)} + 2e^{-(t+2)} - 2e^{-6(t+2)} + \cdots$ **M1**

$= \frac{2e^{-t}}{1-e^{-1}} - \frac{2e^{-6t}}{1-e^{-6}} = \frac{2e}{e-1}e^{-t} - \frac{2e^6}{e^6-1}e^{-6t} = \frac{c(e^6-1)}{(e^5-1)}e^{-t} - \frac{ce^5(e-1)}{(e^5-1)}e^{-6t}$ if $c = \frac{2e}{e-1}\frac{(e^5-1)}{(e^6-1)}$ as required.

M1 using GP sum A1 **A1 value of c (any correct equivalent) (4)**

8. (i) $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ **B1**

from using $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

So $F_0 F_3 - F_1 F_2 = 0 \times 2 - 1 \times 1 = -1$ and $F_2 F_5 - F_3 F_4 = 1 \times 5 - 2 \times 3 = -1$ and thus **M1**

$F_0 F_3 - F_1 F_2 = F_2 F_5 - F_3 F_4$ **(*) A1** **(3)**

(ii) $[(F_n F_{n+3} - F_{n+1} F_{n+2}) - (F_{n-2} F_{n+1} - F_{n-1} F_n)]$ **M1**

$= F_n(F_{n+3} + F_{n-1}) - F_{n+1}(F_{n+2} + F_{n-2})$ **M1**

$= F_n\big((F_{n+2} + F_{n+1}) + (F_{n+1} - F_n)\big) - F_{n+1}\big((F_{n+1} + F_n) + (F_n - F_{n-1})\big)$ **M1**

$= F_n F_{n+2} - {F_n}^2 - {F_{n+1}}^2 + F_{n+1} F_{n-1}$

$= F_n(F_{n+2} - F_n) - F_{n+1}(F_{n+1} - F_{n-1})$ **M1**

$= F_n F_{n+1} - F_{n+1} F_n = 0$ **M1**

$F_1 F_4 - F_2 F_3 = 1 \times 3 - 1 \times 1 = 1$ and $F_0 F_3 - F_1 F_2 = -1$ **M1**

So $F_n F_{n+3} - F_{n+1} F_{n+2} = 1$ if *n* is odd, and $F_n F_{n+3} - F_{n+1} F_{n+2} = -1$ if *n* is even.

A1 **A1** **(8)**

An alternative approach relies on proving $(F_n F_{n+3} - F_{n+1} F_{n+2}) = -(F_{n-1} F_{n+2} - F_n F_{n+1})$ **by, for example considering** $(F_n F_{n+3} - F_{n+1} F_{n+2}) + (F_{n-1} F_{n+2} - F_n F_{n+1})$ **M1**

$= F_n(F_{n+3} - F_{n+1}) + F_{n+2}(F_{n-1} - F_{n+1})$ **M1**

$= F_n F_{n+2} + F_{n+2} \times -F_n = 0$ **M1 A1 A1 then last three marks as in other method, except only value needs to be obtained as a starting point.**

(A variety of methods may appear. The first M1 can be for considering a suitable expression which we require to show is zero, or right at the end obtaining the "next term". Two marks are for using Fibonacci in two places, and two for simplifying algebra. Every solution will need very careful checking as it is easy to hide inability to complete this part in dense algebra.)

(iii) $\tan\left(\tan^{-1}\left(\frac{1}{F_{2r+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2r+2}}\right)\right) = \dfrac{\frac{1}{F_{2r+1}} + \frac{1}{F_{2r+2}}}{1 - \frac{1}{F_{2r+1} F_{2r+2}}} = \dfrac{F_{2r+2} + F_{2r+1}}{F_{2r+1} F_{2r+2} - 1} = \dfrac{F_{2r+3}}{F_{2r} F_{2r+3}} = \dfrac{1}{F_{2r}}$

M1 A1 for use of compound angle formula M1 for simplifying algebra and M1 for use of recurrence relation and part (ii)

and so $\tan^{-1}\left(\frac{1}{F_{2r}}\right) = \tan^{-1}\left(\frac{1}{F_{2r+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2r+2}}\right)$ **(*) A1** **(5)**

$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2r+1}}\right) = \sum_{r=1}^{\infty}\left(\tan^{-1}\left(\frac{1}{F_{2r}}\right) - \tan^{-1}\left(\frac{1}{F_{2r+2}}\right)\right) = \tan^{-1}\left(\frac{1}{F_2}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

M1 use of previous result M1 method of differences A1 **A1** **(4)**

9. (i) Supposing m_1 accelerates downwards at a, then

$m_1 g - T_1 = m_1 a$, $T_1 r - T_2 r = I\alpha$, $T_2 - m_2 g = m_2 a$ (equations 1,2,3) **B1 B1 B1**

and $\alpha = \frac{a}{r}$ **B1**

so we have

$m_1 g - T_1 = m_1 a$

$T_1 - T_2 = I\frac{a}{r^2}$ (equation 4) **M1**

$T_2 - m_2 g = m_2 a$

Adding all three of these gives $m_1 g - m_2 g = \left(m_1 + m_2 + \frac{I}{r^2}\right) a$ (equation 5) **M1**

Subtracting the third equation from the first $T_1 + T_2 - m_1 g - m_2 g = m_2 a - m_1 a$ **M1**

Resolving vertically for the pulley, $P + Mg = Mg + T_1 + T_2$, and so $P = T_1 + T_2$ **B1**

Thus $P - (m_1 + m_2)g = (m_2 - m_1)a$ and $(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{r^2}\right) a$

As $(m_2 - m_1)\left(m_1 + m_2 + \frac{I}{r^2}\right) a = \left(m_1 + m_2 + \frac{I}{r^2}\right)(P - (m_1 + m_2)g) = (m_2 - m_1)(m_1 - m_2)g$

M1

then $m_1 + m_2 + \frac{I}{r^2} = \frac{(m_1 - m_2)^2 g}{(m_1 + m_2)g - P}$ (equation 6)

$\frac{I}{r^2} = \frac{(m_1 - m_2)^2 g}{(m_1 + m_2)g - P} - (m_1 + m_2) = \frac{(m_1 + m_2)P - (m_1 + m_2)^2 g + (m_1 - m_2)^2 g}{(m_1 + m_2)g - P}$

So $\frac{I}{r^2} = \frac{(m_1 + m_2)P - 4m_1 m_2 g}{(m_1 + m_2)g - P}$ and thus $I = \frac{((m_1 + m_2)P - 4m_1 m_2 g)r^2}{(m_1 + m_2)g - P}$ **M1 (*) A1 (11)**

There are numerous routes through the algebra. Effectively, there are 5 equations need to be written down for the variables T_1, T_2 , a , α , and I - each a B1. Eliminating or substituting for 4 of the variables – each M1, then making I the subject M1, and A1 for the correct result.

(ii) Equation 2 becomes $T_1 r - T_2 r - C = I\alpha$

and so equation 4 becomes $T_1 - T_2 - \frac{C}{r} = I\frac{a}{r^2}$ **M1**

and equation 5 becomes $m_1 g - m_2 g - \frac{C}{r} = \left(m_1 + m_2 + \frac{I}{r^2}\right) a$ **M1**

and, in turn, equation 6 becomes $m_1 + m_2 + \frac{I}{r^2} = \frac{(m_1 - m_2)\left((m_1 - m_2)g - \frac{C}{r}\right)}{(m_1 + m_2)g - P}$ **M1**

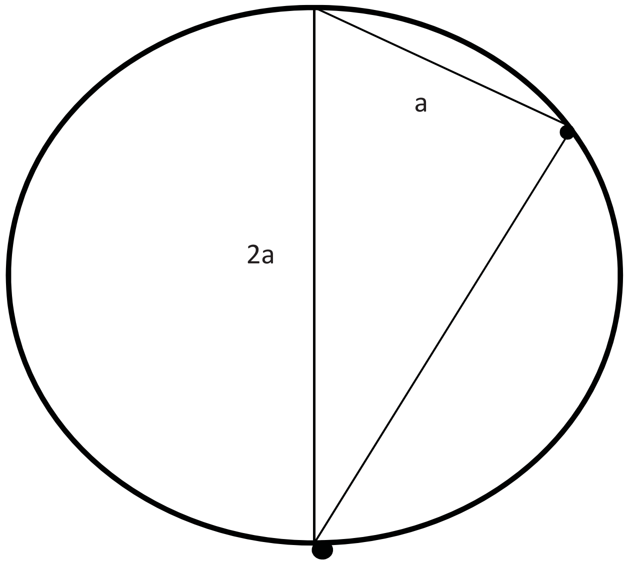
Thus $I = \frac{((m_1 + m_2)P - 4m_1 m_2 g)r^2 - (m_1 - m_2)Cr}{(m_1 + m_2)g - P}$, which is smaller than (*) **M1A1 for I, B1 for smaller provided it has been substantiated by writing I in a sensible way, or an argument that holds together. (6)**

As $m_1 g - m_2 g - \frac{C}{r} = \left(m_1 + m_2 + \frac{I}{r^2}\right) a$ **M1**

$m_1 g - m_2 g - \frac{C}{r} > 0$ as $a > 0$ **M1**

so $C < (m_1 - m_2) r g$ **(*) A1**

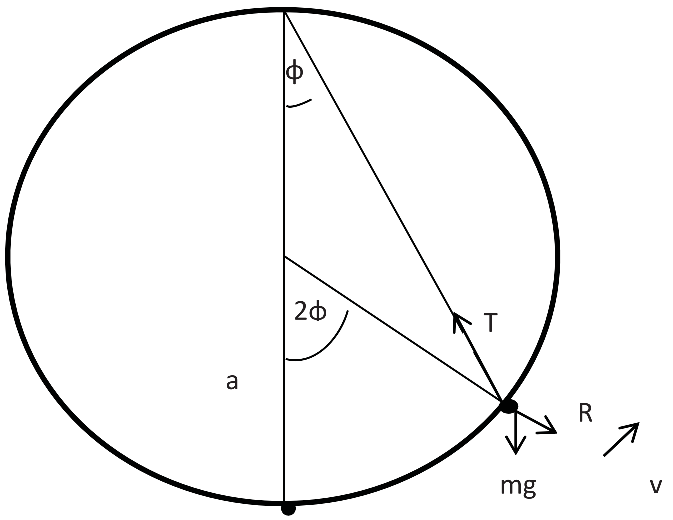
10. Using the centre of the hoop as the zero PE level, conserving energy from the initial position to the position where the ring comes to rest



$-mga + \frac{\lambda a^2}{2a} = mg\frac{a}{2}$ so $\lambda = 3mg$

M1 conserving energy with GPE and EPE included, B1 for $\frac{a}{2}$ above centre or equivalent, A1 for fully correct energy equation, and A1 for λ **(4)**

Conserving energy from the initial position to the general position,



$-mga + \frac{\lambda a^2}{2a} = \frac{\lambda\,(2a\cos\phi - a)^2}{2a} - mga\cos 2\phi + \frac{1}{2}mv^2$

M1 for conserving energy, B1 for ϕ , A2 (1 off per error)

Thus $\frac{mv^2}{a} = \lambda - 2mg + 2mg\cos 2\phi - \lambda(2\cos\phi - 1)^2$

$= 3mg - 2mg + 4mg\cos^2\phi - 2mg - 3mg(4\cos^2\phi - 4\cos\phi + 1)$ **M1 for double angle formula (may be gained later), M1 for simplifying**

$= mg(-4 + 12\cos\phi - 8\cos^2\phi)$ **A1** **(7)**

Resolving radially inwards,

$-R + T\cos\phi - mg\cos 2\phi = m\frac{v^2}{a}$ **M1 for resolving, A2 (1 off per error)**

So $R = -mg(-4 + 12\cos\phi - 8\cos^2\phi) - mg\cos 2\phi + \frac{\lambda(2a\cos\phi - a)}{a}\cos\phi$

M1 for Hooke's law for T, M1 for substituting for $\frac{v^2}{a}$

$= mg(4 - 12\cos\phi + 8\cos^2\phi - 2\cos^2\phi + 1 + 6\cos^2\phi - 3\cos\phi)$ **M1 simplification**

$= mg(12\cos^2\phi - 15\cos\phi + 5)$ as required. **(*) A1** **(7)**

$R = 12mg\left(\cos^2\phi - \frac{5}{4}\cos\phi + \frac{5}{12}\right) = 12mg\left(\left(\cos\phi - \frac{5}{8}\right)^2 + \frac{5}{192}\right) > 0$ so R is non-zero.

Alternatively, discriminant $= 15^2 - 4 \times 12 \times 5 = -15$ so R is non-zero.

M1 for completing the square or finding discriminant, A1 correctly done including conclusion (2)

11. If potential energy zero level is taken to be at P, then the initial potential energy is $-2Mg\frac{L}{2}$. **B1**

When the particle has fallen a distance *x*, the longer, stationary piece of rope is of length $L+\frac{1}{2}x$ and the shorter, moving piece has length $L-\frac{1}{2}x$.

The kinetic energy of the particle is $\frac{1}{2}mv^2$ **B1**

The potential energy of the particle is $-mgx$ **B1**

The potential energy of the part of the stationary piece of string of length *x* is $-\frac{x}{2L}2Mg\frac{x}{2}$

The potential energy of the remaining piece of (doubled up) string is

$$-\left(1-\frac{x}{2L}\right)2Mg\left(x+\frac{1}{2}\left(L-\frac{1}{2}x\right)\right)$$ **M1 A1 A1**

See below for alternative.

The kinetic energy of the shorter moving piece is $\frac{1}{2}\frac{L-\frac{1}{2}x}{2L}\ 2M\ v^2$ **B1**

Thus $\frac{1}{2}mv^2+\frac{1}{2}\frac{L-\frac{1}{2}x}{2L}\ 2M\ v^2-mgx-\frac{x}{2L}2Mg\frac{x}{2}-\left(1-\frac{x}{2L}\right)2Mg\left(x+\frac{1}{2}\left(L-\frac{1}{2}x\right)\right)=-2Mg\frac{L}{2}$ **M1**

So $v^2\left(m+M-\frac{xM}{2L}\right)=g\left(2mx+\frac{Mx^2}{L}+4Mx-2\frac{Mx^2}{L}+2ML-Mx-Mx+\frac{Mx^2}{2L}-2ML\right)$

$$v^2\left(mL+ML-\frac{xM}{2}\right)=g\left(2mxL-\frac{Mx^2}{2}+2MLx\right)$$

$$v^2=\frac{2gx\left(mL+ML-\frac{Mx}{4}\right)}{\left(mL+ML-\frac{xM}{2}\right)}$$ **M1 (*) A1** **(10)**

as required.

Alternative

The potential energy of the part of the string of length $L+\frac{1}{2}x$ is $-\frac{L+\frac{1}{2}x}{2L}2Mg\frac{L+\frac{1}{2}x}{2}$

The potential energy of the part of the string of length $L-\frac{1}{2}x$ is $-\frac{L-\frac{1}{2}x}{2L}2Mg\frac{L+\frac{3}{2}x}{2}$

M1 A1 A1

Differentiating with respect to t,

$$2v\frac{dv}{dt} = \frac{\left(mL + ML - \frac{xM}{2}\right)2gv\left(mL + ML - \frac{Mx}{2}\right) - 2gx\left(mL + ML - \frac{Mx}{4}\right) \times \frac{-Mv}{2}}{\left(mL + ML - \frac{xM}{2}\right)^2}$$

M1 A1 (LHS) A2 (-1 per error) (RHS)

or alternatively, differentiating with respect to x giving $2v\frac{dv}{dx} = 2a$

So

$$\frac{dv}{dt} = \frac{g\left(\left(mL + ML - \frac{xM}{2}\right)\left(mL + ML - \frac{Mx}{2}\right) - x\left(mL + ML - \frac{Mx}{4}\right) \times \frac{-M}{2}\right)}{\left(mL + ML - \frac{xM}{2}\right)^2}$$

M1

$$\frac{dv}{dt} = g + \frac{g\frac{M}{2}x\left(mL + ML - \frac{Mx}{4}\right)}{\left(mL + ML - \frac{xM}{2}\right)^2}$$

$$\frac{dv}{dt} = g + \frac{Mgx\left(mL + ML - \frac{Mx}{4}\right)}{2\left(mL + ML - \frac{xM}{2}\right)^2}$$

(*) A1 (6)

As $0 < x \leq 2L$, $ML - \frac{Mx}{4} \geq \frac{ML}{2}$ **M1 A1**

and as $Mgx > 0$ and the denominator is twice a square **E1**

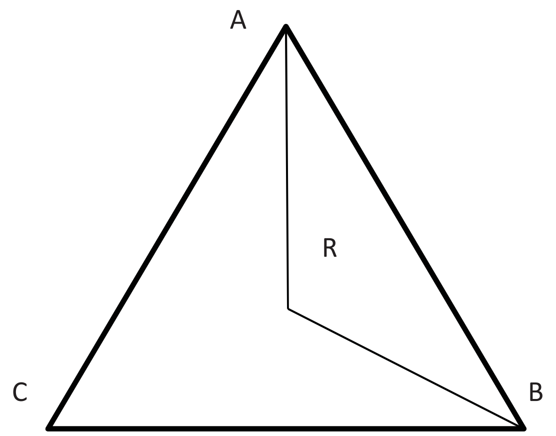
$\frac{Mgx\left(mL+ML-\frac{Mx}{4}\right)}{2\left(mL+ML-\frac{xM}{2}\right)^2} > 0$ and thus $\frac{dv}{dt} > g$ as required. **E1** **(4)**

12. (i) Area ABP + Area BCP + Area CAP = Area ABC **E1**

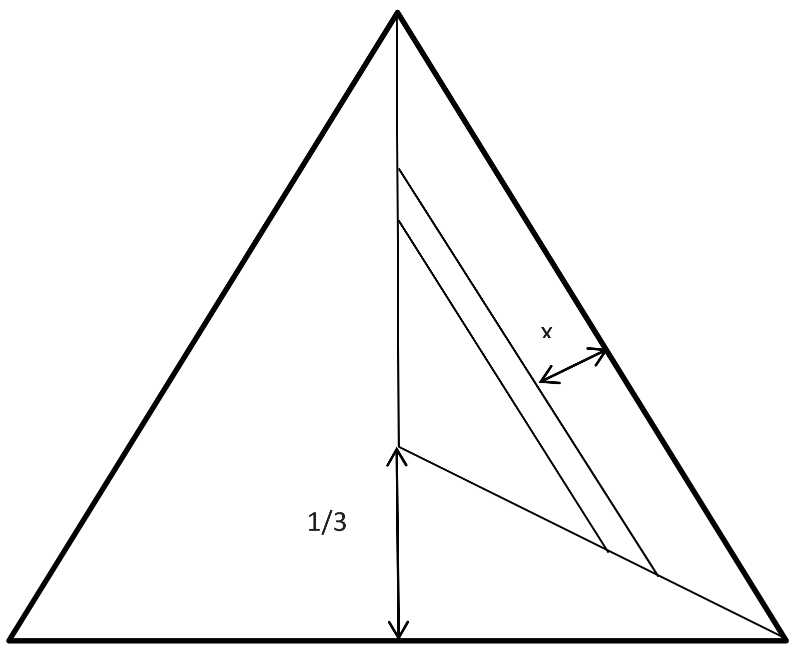
If AB (which = BC = CA) = d (where in fact $1+\frac{d^2}{4}=d^2$)

then $\frac{1}{2}x_1 d+\frac{1}{2}x_2 d+\frac{1}{2}x_3 d=\frac{1}{2}\times 1\times d$ **M1**

so $x_1+x_2+x_3=1$ **(*) A1 (3)**



P lies in the region R if $X_1=min(X_1,X_2,X_3)$ **B1**



Without loss of generalisation, considering the case $X_1=min(X_1,X_2,X_3)$, **E1**

the probability of a random point being in the element shown is proportional to its area. The element's length, by similar triangles is proportional to $\left(\frac{1}{3}-x\right)$, so $f(x)\propto\left(\frac{1}{3}-x\right)$, **E1** i.e. $f(x) = k\left(\frac{1}{3} - x\right)$

$\int_0^{\frac{1}{3}} f(x)dx = 1$ so $\int_0^{\frac{1}{3}} k\left(\frac{1}{3} - x\right)dx = 1$ **M1**

$k\left[-\frac{1}{2}\left(\frac{1}{3} - x\right)^2\right]_0^{\frac{1}{3}} = 1$ **M1**

$\frac{k}{18} = 1$ **A1**

So $f(x) = 18\left(\frac{1}{3} - x\right) = 6(1 - 3x)$ for $0 \le x \le \frac{1}{3}$ and $f(x) = 0$ otherwise. **(*) A1 (7)**

See below for alternative

$$E(X) = \int_0^{\frac{1}{3}} xf(x)dx = \int_0^{\frac{1}{3}} 6x(1 - 3x)dx = [3x^2 - 6x^3]_0^{\frac{1}{3}} = \frac{1}{9}$$

M1 **A1** **A1** **A1** **(4)**

(ii) The results become $x_1 + x_2 + x_3 + x_4 = 1$, **B1**

$g(x) = k'\left(\frac{1}{4} - x\right)^2$, **M1**

$k'\left[-\frac{1}{3}\left(\frac{1}{4} - x\right)^3\right]_0^{\frac{1}{4}} = 1$, $\frac{k'}{192} = 1$ **A1 A1**

$$E(X) = \int_0^{\frac{1}{4}} 12x(1 - 4x)^2 dx = [6x^2 - 32x^3 + 48x^4]_0^{\frac{1}{4}} = \frac{1}{16}$$

M1 **A1** **(6)**

Alternative method for pdf in (i)

Without loss of generalisation, consider the case $X_1 = min(X_1, X_2, X_3)$, **E1**

the probability of a random point being in the triangle height $\left(\frac{1}{3} - x\right)$ is proportional to its area

so $P(X > x) \propto \left(\frac{1}{3} - x\right)^2$ **E1**

$P(X > 0) = 1$ so $P(X > x) = 9\left(\frac{1}{3} - x\right)^2$ **M1**

Thus $F(x) = 1 - 9\left(\frac{1}{3} - x\right)^2$ **A1**

and so $f(x) = F'(x) = 18\left(\frac{1}{3} - x\right) = 6(1 - 3x)$ **M1 A1**

13. (i)

$$P(z < Z < z + \delta z | a < Z < b) = \frac{P(z < Z < z + \delta Z)}{P(a < Z < b)}$$

M1

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-z^2/_2}\, \delta z}{\Phi(b) - \Phi(a)}$$

M1 A1

$$\therefore E(Z) = \frac{\int_a^b z e^{-z^2/_2}\, dz}{\sqrt{2\pi}\left(\Phi(b) - \Phi(a)\right)}$$

M1

$$= \frac{\left[-e^{-z^2/_2}\right]_a^b}{\sqrt{2\pi}\left(\Phi(b) - \Phi(a)\right)}$$

A1

$$= \frac{e^{-a^2/_2} - e^{-b^2/_2}}{\sqrt{2\pi}\left(\Phi(b) - \Phi(a)\right)}$$

(*) A1 (6)

(ii)

$$E(X|X > 0) = E(\sigma Z + \mu | \sigma Z + \mu > 0)$$

M1

$$= \mu + E(\sigma Z | \sigma Z + \mu > 0)$$

M1

$$= \mu + \sigma\, E\left(Z \middle| Z > -\frac{\mu}{\sigma}\right)$$

M1 (*) A1 (4)

$$m = E\,(|X|) = E(X|X > 0)\, P(X > 0) + E(-X|X < 0)\, P(X < 0)$$

M1

$$= \left(\mu + \sigma\, E\left(Z \middle| Z > -\frac{\mu}{\sigma}\right)\right) P\left(Z > -\frac{\mu}{\sigma}\right) - \left(E(X|X < 0)\right) P\left(Z < -\frac{\mu}{\sigma}\right)$$

M1

$$= \left(\mu + \sigma\left(\frac{e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi}\left(1-\Phi\left(\frac{-\mu}{\sigma}\right)\right)}\right)\right)\left(1-\Phi\left(\frac{-\mu}{\sigma}\right)\right) - \left(\mu + \sigma E\left(Z|Z<-\frac{\mu}{\sigma}\right)\right)\Phi\left(\frac{-\mu}{\sigma}\right)$$

M1 A1

$$= \left(\mu + \sigma\left(\frac{e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi}\left(1-\Phi\left(\frac{-\mu}{\sigma}\right)\right)}\right)\right)\left(1-\Phi\left(\frac{-\mu}{\sigma}\right)\right) - \left(\mu + \sigma\left(\frac{-e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}\right)\right)\Phi\left(\frac{-\mu}{\sigma}\right)$$

$$= \mu\left(1-2\Phi\left(\frac{-\mu}{\sigma}\right)\right) + 2\sigma\frac{e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi}} = \mu\left(1-2\Phi\left(\frac{-\mu}{\sigma}\right)\right) + \sqrt{\frac{2}{\pi}}\, e^{-\mu^2/2\sigma^2}$$ **M1 (*) A1 (6)**

$$Var(X) = E(X^2) - \mu^2 = \sigma^2$$

so $E(X^2) = \mu^2 + \sigma^2$ **M1**

$$Var(|X|) = E(|X|^2) - (E|X|)^2 = E(X^2) - m^2 = \mu^2 + \sigma^2 - m^2$$

M1 **A1** **A1 (4)**