STEP Support Programme

2020 STEP 3 worked paper

General comments - In 2020 the STEP papers were delivered remotely, and were only available to students with offers involving STEP from Cambridge, Warwick or Imperial.

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen! There are often many ways to approach a STEP question. Your methods may be different to the ones shown here but correct maths done correctly (and explained fully, especially in the case of a "show that") always gets the marks.

You can find the examiners report and mark schemes for this paper from the Cambridge Assessment Admissions Testing website. These are the general comments for the STEP 2020 exam from the Examiner's report:

These are the general comments for the STEP 2020 exam from the Examiner's report: "*In spite of the change to criteria for entering the paper, there was still a very healthy number of candidates, and the vast majority handled the protocols for the online testing very well. Just over half the candidates attempted exactly six questions, and whilst about 10% attempted a seventh, hardly any did more than seven. With 20% attempting five questions, and 10% attempting only four, overall, there were very few candidates not attempting the target number. There was a spread of popularity across the questions, with no question attracting more than 90% of candidates and only one less than 10% every question received a good number of attempts. Likewise, there was a spread of success on the questions, though every question attracted at least one perfect solution.*"

Please send any corrections, comments or suggestions to step@maths.org.

Question 1

1 For non-negative integers a and b, let

$$\mathrm{I}(a,b) = \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \, \mathrm{d}x.$$

(i) Show that for positive integers a and b,

$$\mathrm{I}(a,b) = \frac{a}{a+b}\mathrm{I}(a-1,b-1).$$

(ii) Prove by induction on n that for non-negative integers n and m,

$$\int_0^{\frac{\pi}{2}} \cos^n x \cos(n+2m+1)x \, \mathrm{d}x = (-1)^m \frac{2^n n!(2m)!(n+m)!}{m!(2n+2m+1)!}.$$

Examiner's report

This was the most popular question, being attempted by about 90% with a fair degree of success: the mean score of about 63% made it the second best attempted question by a small margin.

In part **(i)**, nearly all candidates understood that they would need to use integration by parts and one (or more than one, for some methods) compound angle formula. However, there were numerous manipulative errors in the integration or differentiation of the components, and even sign errors in using compound angle formulae. There were a number of different correct approaches which could be used, but they were essentially very similar to one or other of the methods in the mark scheme.

Part **(ii)** was prescriptively worded, and it was a test of correctly expressed formalism. In spite of this, some candidates did not employ the principle of induction, some ignored that the induction was on n, and some overlooked 'non-negative' requiring the base case to be zero. Often the first component of proof by induction was omitted or incorrectly expressed. 'Assume (or suppose) the result is true for some particular k' would be an improvement on what quite a number wrote. The word 'assume (suppose)' was often not written by candidates and clearly the letter n could not be used for the assumption. Similarly, demonstrating that the base case works correctly needs to be thorough and with fully precise detail as patently it will work, and so a solution must be convincing.

Solution

(i) Note that we have $a \geqslant 1$ and $b \geqslant 1$ (a and b are positive in part **(i)**). Integrating by parts gives:

$$\begin{aligned}
\mathrm{I}(a,b) &= \left[\cos^a x \times \tfrac{1}{b}\sin bx\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -a\cos^{(a-1)} x \sin x \times \tfrac{1}{b}\sin bx \,\mathrm{d}x \\
&= 0 + \tfrac{a}{b}\int_0^{\frac{\pi}{2}} \cos^{(a-1)} x \sin x \times \sin bx \,\mathrm{d}x
\end{aligned}$$

We want to get this in terms of $\mathrm{I}(a-1,b-1) = \displaystyle\int_0^{\frac{\pi}{2}} \cos^{(a-1)} x \cos(b-1)x \,\mathrm{d}x$. We have the $\cos^{(a-1)} x$ term. Consider $\cos(b-1)x$:

$$\begin{aligned}
\cos(b-1)x &= \cos bx \cos x + \sin bx \sin x \\
\implies \sin bx \sin x &= \cos(b-1)x - \cos bx \cos x
\end{aligned}$$

Substituting for $\sin bx \sin x$ gives:

$$\begin{aligned}
\mathrm{I}(a,b) &= \tfrac{a}{b}\int_0^{\frac{\pi}{2}} \cos^{(a-1)} x \times \left[\cos(b-1)x - \cos bx \cos x\right] \mathrm{d}x \\
&= \tfrac{a}{b}\int_0^{\frac{\pi}{2}} \cos^{(a-1)} x \cos(b-1)x \,\mathrm{d}x - \tfrac{a}{b}\int_0^{\frac{\pi}{2}} \cos^a x \cos bx \,\mathrm{d}x \\
\mathrm{I}(a,b) &= \tfrac{a}{b}\mathrm{I}(a-1,b-1) - \tfrac{a}{b}\mathrm{I}(a,b) \\
b\mathrm{I}(a,b) &= a\mathrm{I}(a-1,b-1) - a\mathrm{I}(a,b) \\
(a+b)\mathrm{I}(a,b) &= a\mathrm{I}(a-1,b-1) \\
\mathrm{I}(a,b) &= \frac{a}{a+b}\mathrm{I}(a-1,b-1)
\end{aligned}$$

(ii) Here n is "non-negative", which means that $n \geqslant 0$. Taking the base case as $n = 0$ (and noting that $0! = 1$) gives:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^n x \cos(n+2m+1)x \,\mathrm{d}x &= (-1)^m \frac{2^n n!(2m)!(n+m)!}{m!(2n+2m+1)!} \\
n = 0 \implies \quad \int_0^{\frac{\pi}{2}} \cos(2m+1)x \,\mathrm{d}x &= (-1)^m \frac{(2m)!(m)!}{m!(2m+1)!} \\
\left[\frac{1}{2m+1}\sin(2m+1)x\right]_0^{\frac{\pi}{2}} &= (-1)^m \frac{(2m)!\cancel{(m)!}}{\cancel{m!}(2m+1)!} \\
\frac{1}{2m+1}\sin\left[(2m+1)\frac{\pi}{2}\right] &= \frac{(-1)^m}{2m+1} \\
\frac{1}{2m+1}\sin\left[m\pi + \frac{\pi}{2}\right] &= \frac{(-1)^m}{2m+1}
\end{aligned}$$

which is true for all non-negative integers m.

For the induction step, start by assuming that the statement is true when $n = k$, so we have:

$$\int_0^{\frac{\pi}{2}} \cos^k x \cos(k+2m+1)x \,\mathrm{d}x = (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!}$$

The left hand side has the same form as in part **(i)**, so we can write this as:

$$\mathrm{I}(k, k+2m+1) = (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!}$$

Considering $n = k+1$ we have:

$$\int_0^{\frac{\pi}{2}} \cos^{k+1} x \cos\big[(k+1)+2m+1\big]x \,\mathrm{d}x = \mathrm{I}(k+1, k+2m+2)$$

Using the result in part **(i)** we have:

$$\begin{aligned}
\mathrm{I}(k+1, k+2m+2) &= \frac{k+1}{(k+1)+(k+2m+2)}\mathrm{I}(k, k+2m+1) \\
&= \frac{k+1}{2k+2m+3}\mathrm{I}(k, k+2m+1) \\
&= \frac{k+1}{2k+2m+3} \times (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!} \\
&= (-1)^m \frac{2^k (k+1)!(2m)!(k+m)!}{m!(2k+2m+1)! \times (2k+2m+3)} \\
&= (-1)^m \frac{2^k (k+1)!(2m)!(k+m)!(2k+2m+2)}{m!(2k+2m+1)!(2k+2m+2)(2k+2m+3)} \\
&= (-1)^m \frac{2^k (k+1)!(2m)!(k+m)!(k+m+1) \times 2}{m!(2k+2m+3)!} \\
&= (-1)^m \frac{2^{k+1} (k+1)!(2m)!(k+m+1)!}{m!(2k+2m+3)!} \\
&= (-1)^m \frac{2^{k+1} (k+1)!(2m)![(k+1)+m]!}{m![2(k+1)+2m+1]!}
\end{aligned}$$

which is the required for when $n = k+1$. Hence if it is true for $n = k$ then it is true for $n = k+1$, and since it is true when $n = 0$ it is true for all integers $n \geqslant 0$.

Question 2

2 The curve C has equation $\sinh x + \sinh y = 2k$, where k is a positive constant.

(i) Show that the curve C has no stationary points and that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ at the point (x, y) on the curve if and only if

$$1 + \sinh x \sinh y = 0.$$

Find the co-ordinates of the points of inflection on the curve C, leaving your answers in terms of inverse hyperbolic functions.

(ii) Show that if (x, y) lies on the curve C and on the line $x + y = a$, then

$$\mathrm{e}^{2x}(1 - \mathrm{e}^{-a}) - 4k\mathrm{e}^{x} + (\mathrm{e}^{a} - 1) = 0$$

and deduce that $1 < \cosh a \leqslant 2k^2 + 1$.

(iii) Sketch the curve C.

Examiner's report

This was both the fourth most popular and successful question being attempted by 84% with a mean score of about 55%. Most generally performed much better in parts **(i)** and **(ii)** than in in **(iii)**.

In part **(i)**, most successfully showed that were no stationary points and obtained the given result. Likewise, generally they found the points of inflection although a few struggled to do so. In part **(ii)**, almost all candidates obtained the required equation and then noticed that it was a quadratic in e^x. Then they usually noticed that the discriminant being non-negative gave the higher bound for $\cosh a$. A surprising number seemed not to notice there was a strict lower bound to deduce, and, as a consequence, did not subsequently appreciate that a was non-zero. Given the amount of information obtained in parts **(i)** and **(ii)**, there was frequently a reluctance to apply this to part **(iii)**. For example, although stationary points usually appeared, points of inflection often did not.

Even fewer candidates used **(ii)** to deduce that the graph has to lie between the lines $x + y = 0$ and $x + y = \cosh^{-1}(2k^2 + 1)$. It is expected that candidates should observe that the graph is symmetrical in the line $y = x$, that the two bounding lines should be labelled with their equations, and that the coordinates of the intercepts with the coordinate axes, the points of inflection and the point where it touches $x + y = \cosh^{-1}(2k^2 + 1)$ should be written in on the sketch.

Solution

(i) Differentiating with respect to x gives:

$$\cosh x + \cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{*}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\cosh x}{\cosh y}$$

In order for there to be a stationary point we would need $\cosh x = 0$, which is not possible as $\cosh x \geqslant 1$.

Differentiating (*) with respect to x gives:

$$\sinh x + \sinh y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \cosh y \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

and so:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{\sinh x}{\cosh y} - \frac{\sinh y}{\cosh y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\ &= -\frac{\sinh x}{\cosh y} - \frac{\sinh y}{\cosh y}\left(\frac{-\cosh x}{\cosh y}\right)^2\end{aligned}$$

We have a point of inflection only if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, and in this case there are no stationary points, so in this situation there is a point of inflection if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. Hence there is a point of inflection if and only if:

$$\begin{aligned}\cosh^2 y \sinh x + \sinh y \cosh^2 x &= 0 \\ (1 + \sinh^2 y)\sinh x + \sinh y(1 + \sinh^2 x) &= 0 \\ \sinh^2 y \sinh x + \sinh^2 x \sinh y + \sinh x + \sinh y &= 0 \\ \sinh y \sinh x(\sinh y + \sinh x) + (\sinh x + \sinh y) &= 0 \\ (\sinh x + \sinh y)(\sinh y \sinh x + 1) &= 0\end{aligned}$$

We are told that curve C has equation $\sinh x + \sinh y = 2k$, where $k > 0$, so we cannot have $\sinh x + \sinh y = 0$. Hence we have a point of inflection on the curve if and only if $\sinh y \sinh x + 1 = 0$.

Substituting in $\sinh y = 2k - \sinh x$ into $\sinh y \sinh x + 1 = 0$ gives:

$$\begin{aligned}(2k - \sinh x)\sinh x + 1 &= 0 \\ \sinh^2 x - 2k \sinh x - 1 &= 0\end{aligned}$$

Which has solutions $\sinh x = \dfrac{2k \pm \sqrt{4k^2+4}}{2} = k \pm \sqrt{k^2+1}$.

When $\sinh x = k + \sqrt{k^2+1}$ then:

$$\begin{aligned}\sinh y &= \frac{-1}{k+\sqrt{k^2+1}}\\ &= \frac{-(k-\sqrt{k^2+1})}{(k+\sqrt{k^2+1})(k-\sqrt{k^2+1})}\\ &= \frac{-(k-\sqrt{k^2+1})}{\cancel{k^2}-(\cancel{k^2}+1)}\\ &= k-\sqrt{k^2+1}\end{aligned}$$

and when $\sinh x = k - \sqrt{k^2+1}$:

$$\begin{aligned}\sinh y &= \frac{-1}{k-\sqrt{k^2+1}}\\ &= \frac{-(k+\sqrt{k^2+1})}{(k-\sqrt{k^2+1})(k+\sqrt{k^2+1})}\\ &= \frac{-(k+\sqrt{k^2+1})}{\cancel{k^2}-(\cancel{k^2}+1)}\\ &= k+\sqrt{k^2+1}\end{aligned}$$

So the points of inflection are:

$$\begin{aligned}&\left(\sinh^{-1}\left(k+\sqrt{k^2+1}\right),\ \sinh^{-1}\left(k-\sqrt{k^2+1}\right)\right)\\ \text{and}\quad &\left(\sinh^{-1}\left(k-\sqrt{k^2+1}\right),\ \sinh^{-1}\left(k+\sqrt{k^2+1}\right)\right)\end{aligned}$$

(ii) If we have $x+y=a$ then $y=a-x$ and so:

$$\begin{aligned}\sinh x + \sinh y &= 2k\\ \frac{e^x - e^{-x}}{2} + \frac{e^{a-x}-e^{-(a-x)}}{2} &= 2k\\ e^x - e^{-x} + e^{a-x} - e^{x-a} &= 4k\\ e^{2x} - 1 + e^a - e^{2x-a} &= 4ke^x\\ e^{2x} - e^{2x-a} + e^a - 1 &= 4ke^x\\ e^{2x}(1-e^{-a}) + (e^a-1) - 4ke^x &= 0\end{aligned}$$

Which can be rearrange to get $e^{2x}(1-e^{-a}) - 4ke^x + (e^a-1) = 0$. This is a quadratic equation in e^x, and if this is going to have real solutions then we need:

$$\begin{aligned}\text{“}b^2-4ac\text{”} &\geqslant 0\\ (-4k)^2 - 4(1-e^{-a})(e^a-1) &\geqslant 0\\ 16k^2 &\geqslant 4(1-e^{-a})(e^a-1)\\ 4k^2 &\geqslant e^a + e^{-a} - 1 - e^{-a}e^a\\ 4k^2 &\geqslant 2\cosh a - 2\\ 2k^2+1 &\geqslant \cosh a\end{aligned}$$

For the other part of the inequality we have $\cosh a \geqslant 1$, and $\cosh a = 1$ if and only if $a=0$. If we have $a=0$ then $x=-y$ which when substituted into $\sinh x + \sinh y = 2k$ becomes $\sinh(-y) + \sinh y = 2k \implies 0 = 2k$, which is contradiction as k is positive.

Hence $\cosh a > 1$ and so we have $1 < \cosh a \leqslant 2k^2+1$.

(iii) We know that this curve in symmetric in $y = x$ (as the equation $\sinh x + \sinh y = 2k$ is symmetric in y and x). We also know that there are no turning points, and that there are two points of inflection.

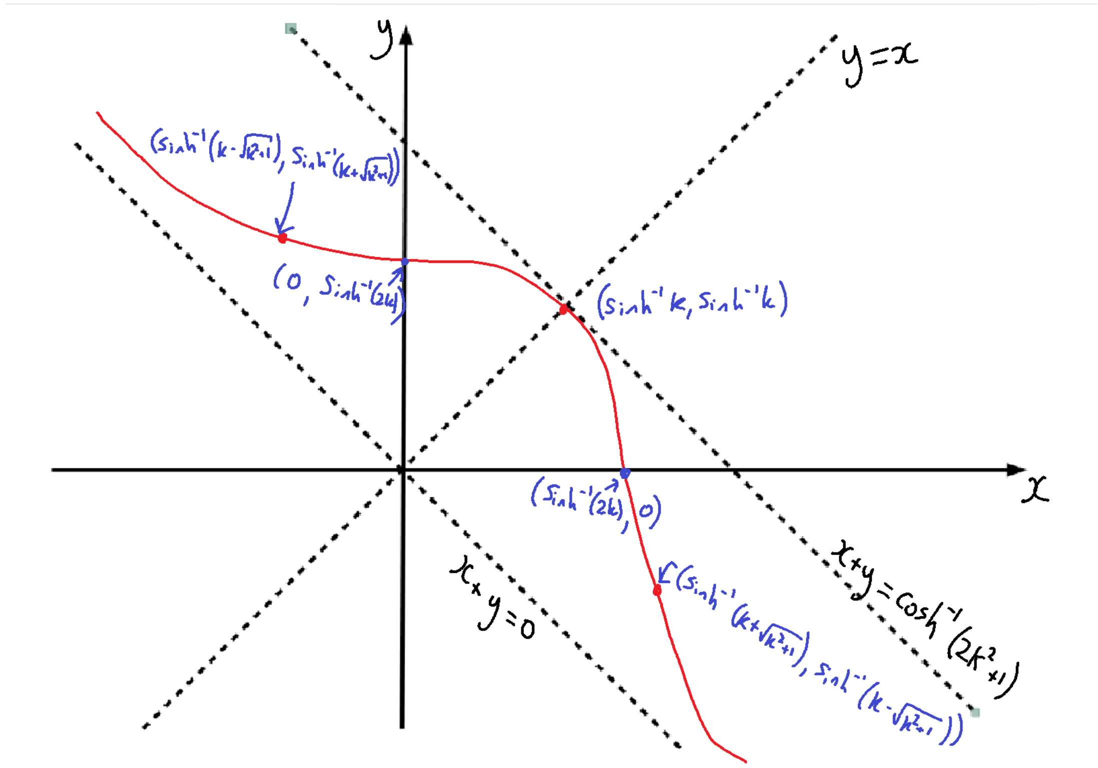
As $x \to +\infty$ we have $\sinh y = 2k - \sinh x$, so $\sinh y \to -\infty$ and hence $y \to -\infty$. Similarly as $y \to +\infty$, $x \to -\infty$.

We have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-\cosh x}{\cosh y}$, as so as $x \to +\infty, y \to -\infty$ or $x \to -\infty, y \to +\infty$ we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} \to -1$. We also know that the gradient is always negative, and on the line $y = x$ the gradient is equal to -1.

From part **(ii)** we know that if C is going to intersect with the line $x + y = a$ then we have $1 < \cosh a \leqslant 2k^2 + 1$, which means that $0 < a \leqslant \cosh^{-1}(2k^2 + 1)$. This means that curve lies between the lines $x + y = 0$ and $x + y = \cosh^{-1}(2k^2 + 1)$.

Since the curve is symmetrical about $y = x$ it would be good to find where the curve intersects this line. Substituting in $x = y$ into $\sinh x + \sinh y = 2k$ gives $2 \sinh x = 2k \implies \sinh x = k$ and so the curve passes through $(\sinh^{-1} k, \sinh^{-1} k)$.

The graph should look something like the one below. Here is a neater version.



I would have probably have written next to the above graph "No stationary points" just to make it very clear to the examiner that, even though my graph looks a little flat where it crosses the y axis, I did not intend any stationary points.

The way I have drawn the graph, it looks as if the "bulge" hits the point where the lines $y = x$ and $x + y = \cosh^{-1}(2k^2 + 1)$ meet. Given the amount of work already done to sketch the graph I don't think you would have been required to prove this. My working to shown that this was true is below:

Substituting the values $x = y = \sinh^{-1} k$ into $x + y = a$ gives:

$$\begin{aligned}
2\sinh^{-1} k &= a \\
\sinh^{-1} k &= \frac{a}{2} \\
k &= \sinh\left(\frac{a}{2}\right) \\
2k^2 + 1 &= 2\sinh^2\left(\frac{a}{2}\right) + 1 \\
2k^2 + 1 &= \cosh(a) \qquad \text{using } \cosh 2A = 2\sinh^2 A + 1 \\
\cosh^{-1}(2k^2 + 1) &= a
\end{aligned}$$

and so the point where the curve meets the line $y = x$ is also the point where it touches the upper boundary line $x + y = \cosh^{-1}(2k^2 + 1)$.

Question 3

3 Given distinct points A and B in the complex plane, the point G_{AB} is defined to be the centroid of the triangle ABK, where the point K is the image of B under rotation about A through a clockwise angle of $\frac{1}{3}\pi$.

Note: if the points P, Q and R are represented in the complex plane by p, q and r, the centroid of triangle PQR is defined to be the point represented by $\frac{1}{3}(p+q+r)$.

(i) If A, B and G_{AB} are represented in the complex plane by a, b and g_{ab}, show that

$$g_{ab} = \frac{1}{\sqrt{3}}(\omega a + \omega^* b),$$

where $\omega = \mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$.

(ii) The quadrilateral Q_1 has vertices A, B, C and D, in that order, and the quadrilateral Q_2 has vertices G_{AB}, G_{BC}, G_{CD} and G_{DA}, in that order. Using the result in part **(i)**, show that Q_1 is a parallelogram if and only if Q_2 is a parallelogram.

(iii) The triangle T_1 has vertices A, B and C and the triangle T_2 has vertices G_{AB}, G_{BC} and G_{CA}. Using the result in part **(i)**, show that T_2 is always an equilateral triangle.

Examiner's report

This was the second least popular pure question, but many candidates produced good solutions to it, including some very elegant ones, and the mean score was just shy of half marks. The most successful candidates were those confident in manipulating terms of the form $\mathrm{e}^{\mathrm{i}\theta}$. Candidates demonstrating a good knowledge of $g_{bc} - g_{ab}$ classical geometry also did well. Several candidates abandoned their attempts after part **(i)** or **(ii)**.

Part **(i)** was generally well answered. The most common mistakes were to rotate anticlockwise rather than clockwise, or to omit the "$+a$" from their expression for k. Some candidates (including many of those with the wrong angle) still achieved the required result with no or incorrect working; as the required result was in the question, candidates could not be rewarded without correct justification.

In part **(ii)** most candidates attempted to show both implications at the same time, which tended to be more successful if candidates started with Q_2 being a parallelogram. Despite the question stating the order of the vertices, some candidates used an incorrect direction for one of their lines.

Most candidates used the fact that $ABCD$ is a parallelogram $\implies b-a=c-d$ (or an equivalent), though a few candidates tried to use the condition that pairs of opposite sides are parallel (not appreciating the fact that if one set of sides are parallel and equal in length then the shape must be a parallelogram).

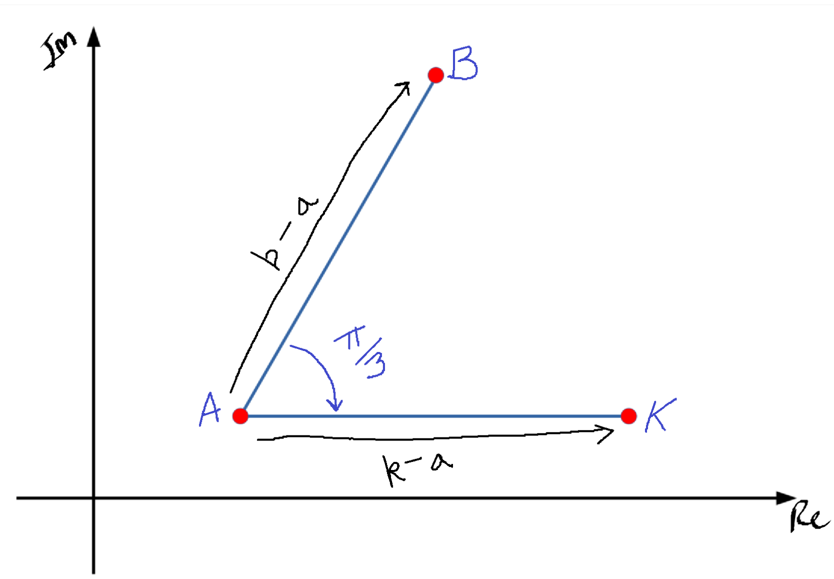
Some candidates did not appreciate that they had to state that $\omega - \omega^* \neq 0$ before being able to deduce $b - a = c - d$ from the corresponding result for Q_2. Instead, these candidates either simply cancelled $\omega - \omega^*$ without justification, or attempted (unsuccessfully) to argue that the coefficients of ω and ω^* could be directly equated. In part **(iii)**, some candidates stopped their attempts after finding an expression for $g_{BC} - g_{AB}$ (or similar), but most of those who attempted this part went on to produce good solutions.

There were some elegant and creative solutions to this part, but the most common approach was to try to show that $G_{CA}G_{AB}$ is a rotation of $G_{BC}G_{AB}$ by $\frac{\pi}{3}$ about G_{AB}. A common error here was instead to try to show that $G_{CA}G_{AB}$ is a rotation of $G_{CA}G_{AB}$ by $-\frac{\pi}{3}$ about G_{AB} (sometimes arising from an incorrect labelling of T_2).

Another error which arose in several attempts was to try and compute $|g_{AB} - g_{BC}|$ by treating a, b and c as real numbers. Whilst this approach led to an expression which was totally symmetric in a, b and c, leading these candidates to 'conclude' that T_2 was equilateral, very little credit could be gained for such an approach. Only one candidate attempted to prove part **(iii)** independently of part **(i)**.

Solution

A diagram of the situation described in the stem is a good starting point:



Since K is the image of B rotated by $\frac{\pi}{3}$ (or 60°) about A this means that the triangle ABK is an equilateral triangle.

(i) To find g_{ab} we can use $\frac{1}{3}(a + b + k)$, (where k is the complex number representing K). To obtain g_{ab} in the required form we need to find k in terms of a and b.

To rotate by an angle θ *anti-clockwise* we can multiply by $\mathrm{e}^{\mathrm{i}\theta}$. This gives $b - a = (k - a)\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$. Rearranging gives $k = (b - a)\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} + a$.

We have:

$$\begin{aligned} g_{ab} &= \tfrac{1}{2}(a+b+k) \\ &= \tfrac{1}{2}\left[a+b+(b-a)\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}+a\right] \\ &= \tfrac{1}{3}\left[a(2-\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}})+b(1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}})\right] \end{aligned}$$

We have $\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} = \cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$. Substituting this gives:

$$\begin{aligned} g_{ab} &= a\left(\frac{2-\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)}{3}\right) + b\left(\frac{1+\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)}{3}\right) \\ &= a\left(\frac{3+\mathrm{i}\sqrt{3}}{6}\right) + b\left(\frac{3-\mathrm{i}\sqrt{3}}{6}\right) \end{aligned}$$

We also have $\omega = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{\mathrm{i}}{2}$ and $\omega^* = \frac{\sqrt{3}}{2} - \frac{\mathrm{i}}{2}$. Going back to our expression for g_{ab} we have:

$$\begin{aligned} g_{ab} &= a\left(\frac{\sqrt{3}(\sqrt{3}+\mathrm{i})}{6}\right) + b\left(\frac{\sqrt{3}(\sqrt{3}-\mathrm{i})}{6}\right) \\ &= \frac{\sqrt{3}}{3}\left[a\left(\frac{\sqrt{3}+\mathrm{i}}{2}\right) + b\left(\frac{\sqrt{3}-\mathrm{i}}{2}\right)\right] \\ &= \frac{1}{\sqrt{3}}\left(a\omega + b\omega^*\right) \end{aligned}$$

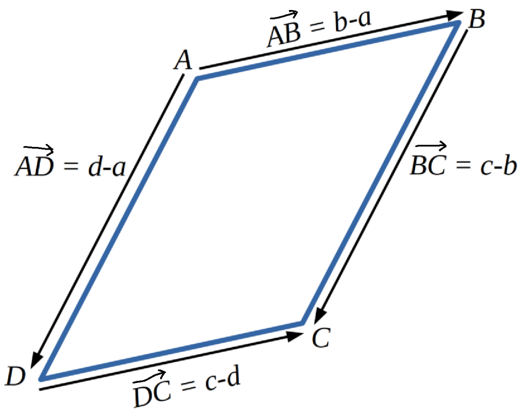
as required.

(ii) The vertices of Q_2 are given by:

$$\begin{aligned} g_{ab} &= \frac{1}{\sqrt{3}}(\omega a + \omega^* b) \\ g_{bc} &= \frac{1}{\sqrt{3}}(\omega b + \omega^* c) \\ g_{cd} &= \frac{1}{\sqrt{3}}(\omega c + \omega^* d) \\ g_{da} &= \frac{1}{\sqrt{3}}(\omega d + \omega^* a) \end{aligned}$$

If $ABCD$ is a parallelogram then the opposite sides are parallel and apposite sides are the same length. This means that we have $\overrightarrow{AB} = \overrightarrow{DC}$ (you need to be careful to get the direction of the vectors the same, the vector $\overrightarrow{CD}$ has the same magnitude as $\overrightarrow{AB}$, but it is in the opposite direction). We also have $\overrightarrow{AD} = \overrightarrow{BC}$. It is actually only necessary to show that one of these pairs of vectors is the same as this will imply that the other pair is the same (if one pair of sides is parallel and the same length, then the other pair will be parallel and the same length).

It can be helpful to draw a picture of the parallelogram:



Hence, if Q_1 is a parallelogram then we have $b - a = c - d$ and also $d - a = c - b$.

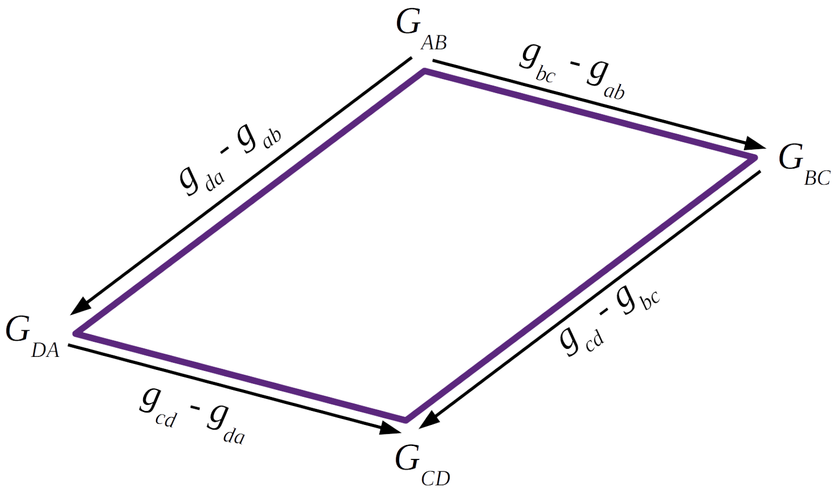
Consider one of the sides of Q_2:

$$\begin{aligned}
g_{bc} - g_{ab} &= \frac{1}{\sqrt{3}}(\omega b + \omega^* c) - \frac{1}{\sqrt{3}}(\omega a + \omega^* b) \\
&= \frac{1}{\sqrt{3}}\Big(\omega(b-a) + \omega^*(c-b)\Big) \\
&= \frac{1}{\sqrt{3}}\Big(\omega(c-d) + \omega^*(d-a)\Big) \qquad \text{as } Q_1 \text{ is a parallelogram} \\
&= \frac{1}{\sqrt{3}}(\omega c + \omega^* d) - \frac{1}{\sqrt{3}}(\omega d + \omega^* a) \\
&= g_{cd} - g_{da}
\end{aligned}$$

So we have $g_{bc} - g_{ab} = g_{cd} - g_{da}$ and so one pair of opposite sides are parallel and equal in length, hence Q_2 is a parallelogram.

If you want to, you can rearrange to give $g_{da} - g_{ab} = g_{cd} - g_{bc}$ and hence we have both pairs of opposite sides being equal and parallel.

Again, it can be helpful to draw a picture of the parallelogram:



It wasn't necessary to draw these parallelograms, but it helped me to understand what was going on, and to have my negative signs in the correct places.

So far we have shown that Q_1 is a parallelogram $\implies$ Q_2 is a parallelogram. Now we have to work in the opposite direction.

If Q_2 is a parallelogram then $g_{bd} - g_{ab} = g_{cd} - g_{da}$, which means we have:

$$\begin{aligned}\frac{1}{\sqrt{3}}(\omega b + \omega^* c) - \frac{1}{\sqrt{3}}(\omega a + \omega^* b) &= \frac{1}{\sqrt{3}}(\omega c + \omega^* d) - \frac{1}{\sqrt{3}}(\omega d + \omega^* a) \\ (\omega b + \omega^* c) - (\omega a + \omega^* b) &= (\omega c + \omega^* d) - (\omega d + \omega^* a) \\ (\omega + \omega^*)b - (\omega + \omega^*)a &= (\omega + \omega^*)c - (\omega + \omega^*)d\end{aligned}$$

Since we know that $\omega + \omega^* = \sqrt{3}$ (i.e. it is not 0) we can divide throughout by $(\omega + \omega^*)$ to get:

$$\begin{aligned}b - a &= c - d \quad \text{and rearranging we also have} \\ d - a &= c - b\end{aligned}$$

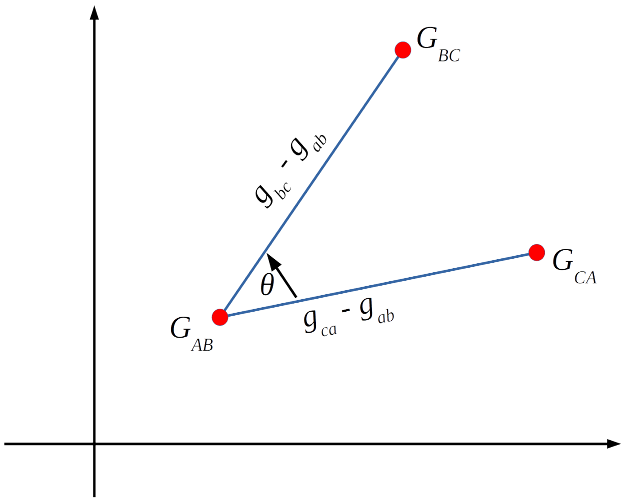
Hence we have $\overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{AB} = \overrightarrow{DC}$, and so this means that Q_1 is a parallelogram.

Hence we have:

$$Q_1 \text{ is a parallelogram} \iff Q_2 \text{ is a parallelogram}$$

(iii) In order to show that T_2 is an equilateral, we can show that one edge is a rotation of another edge by $\frac{\pi}{3}$ about the vertex that connects them (like the picture at the start of the solution with A, B and K).

Lets assume that $g_{bc} - g_{ab}$ is a rotation of θ anti-clockwise of the line $g_{ca} - g_{ab}$ about the point g_{ab}.



This means we have:

$$g_{bc} - g_{ab} = e^{i\theta}(g_{ca} - g_{ab}) .$$

T_2 will be equilateral if, and only if, $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$.

Using the result from part **(i)** we have:

$$\begin{aligned} g_{bc} - g_{ab} &= e^{i\theta}(g_{ca} - g_{ab}) \\ \frac{1}{\sqrt{3}}\Big[(\omega b + \omega^* c) - (\omega a + \omega^* b)\Big] &= \frac{e^{i\theta}}{\sqrt{3}}\Big[(\omega c + \omega^* a) - (\omega a + \omega^* b)\Big] \\ (\omega - \omega^*)b + \omega^* c - \omega a &= e^{i\theta}\Big[(\omega^* - \omega)a - \omega^* b + \omega c\Big] \end{aligned}$$

From earlier we know that:

$$\omega - \omega^* = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = i$$

and we also have:

$$\begin{aligned} \omega \times \omega^* &= e^{i\frac{\pi}{6}} \times e^{-i\frac{\pi}{6}} = 1 \\ \omega^3 &= e^{i\frac{\pi}{2}} = i \\ -\omega^* &= -e^{i\frac{\pi}{6}} = \omega^5 \end{aligned}$$

This last result is most easily derived from drawing a sketch of ω, ω^* and $-\omega^*$ in the Argand plane. The reason for finding these is that I want to try and force a factor of ω^2 out somewhere!

Using these gives:

$$
\begin{aligned}
\mathrm{i}b + \omega^* c - \omega a &= \mathrm{e}^{\mathrm{i}\theta}\Big[-\mathrm{i}a - \omega^* b + \omega c \Big] \\
\mathrm{e}^{\mathrm{i}\theta} &= \frac{\mathrm{i}b + \omega^* c - \omega a}{-\mathrm{i}a - \omega^* b + \omega c} \\
\mathrm{e}^{\mathrm{i}\theta} &= \frac{\mathrm{i}b + \omega^* c - \omega a}{-\omega^3 a + \omega^5 b + \omega(\omega\omega^*)c} \\
\mathrm{e}^{\mathrm{i}\theta} &= \frac{\mathrm{i}b + \omega^* c - \omega a}{\omega^2(-\omega a + \omega^3 b + \omega^* c)} \\
\mathrm{e}^{\mathrm{i}\theta} &= \frac{\mathrm{i}b + \omega^* c - \omega a}{\omega^2(\mathrm{i}b + \omega^* c - \omega a)} \\
\mathrm{ei}\theta &= \frac{1}{\omega^2} \\
\mathrm{e}^{\mathrm{i}\theta} &= \mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}
\end{aligned}
$$

Therefore we have $\theta = -\dfrac{\pi}{3}$ and so the line $g_{bc} - g_{ab}$ is a rotation of $g_{ca} - g_{ab}$ of an angle $\dfrac{\pi}{3}$ *clockwise* about the point g_{ab}. Hence T_2 is equilateral.

Question 4

4 The plane Π has equation $\mathbf{r}\,.\,\mathbf{n} = 0$ where $\mathbf{n}$ is a unit vector. Let P be a point with position vector $\mathbf{x}$ which does not lie on the plane Π. Show that the point Q with position vector $\mathbf{x} - (\mathbf{x}\,.\,\mathbf{n})\mathbf{n}$ lies on Π and that PQ is perpendicular to Π.

(i) Let transformation T be a reflection in the plane $ax + by + cz = 0$, where $a^2 + b^2 + c^2 = 1$.

Show that the image of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ under T is $\begin{pmatrix} b^2 + c^2 - a^2 \\ -2ab \\ -2ac \end{pmatrix}$, and find the images of $\mathbf{j}$ and $\mathbf{k}$ under T.

Write down the matrix $\mathbf{M}$ which represents transformation T.

(ii) The matrix

$$\begin{pmatrix} 0.64 & 0.48 & 0.6 \\ 0.48 & 0.36 & -0.8 \\ 0.6 & -0.8 & 0 \end{pmatrix}$$

represents a reflection in a plane. Find the cartesian equation of the plane.

(iii) The matrix $\mathbf{N}$ represents a rotation through angle π about the line through the origin parallel to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where $a^2 + b^2 + c^2 = 1$. Find the matrix $\mathbf{N}$.

(iv) Identify the single transformation which is represented by the matrix $\mathbf{NM}$.

Examiner's report

This was not a popular question but it received a respectable number of attempts with about one sixth trying it. The average score was a little under half marks, but on each part of the question, if the part was attempted it was generally fully correct. Most candidates had no problem demonstrating the desired properties, and if they used this in part **(i)** they had little problem obtaining full marks. Even if they could not apply the stem in **(i)**, they nearly all found the images of $\mathbf{j}$ and $\mathbf{k}$ correctly using symmetry and hence the matrix $\mathbf{M}$. In part **(ii)**, almost all the candidates could solve the equations, though some lost marks by working inaccurately. The few that attempted part **(iii)** either got it completely correct or scored nothing: those getting it correct generally drew a parallel with the technique used in **(i)**. As a consequence, only a small number attempted part **(iv)**, and few scored both marks, either losing a mark for insufficient justification, or for describing the transformation as a rotation about the origin.

Solution

In the stem we are asked to show that Q lies on Π and that PQ is perpendicular to Π. We are told that $\mathbf{n}$ is a unit vector and so we have $\mathbf{n} \,.\, \mathbf{n} = 1$. Note also that in the plane equation, $\mathbf{n}$ is a vector which is perpendicular to the plane.

If Q lies on Π if and only if the vector representing Q satisfies $\mathbf{r} \,.\, \mathbf{n} = 0$. Using the given position vector of Q we have:

$$\begin{aligned}\Big(\mathbf{x} - (\mathbf{x} \,.\, \mathbf{n})\mathbf{n}\Big) \,.\, \mathbf{n} &= \mathbf{x} \,.\, \mathbf{n} - (\mathbf{x} \,.\, \mathbf{n})\mathbf{n} \,.\, \mathbf{n} \\ &= \mathbf{x} \,.\, \mathbf{n} - (\mathbf{x} \,.\, \mathbf{n}) \times 1 \\ &= 0\end{aligned}$$

Therefore Q lies on Π.

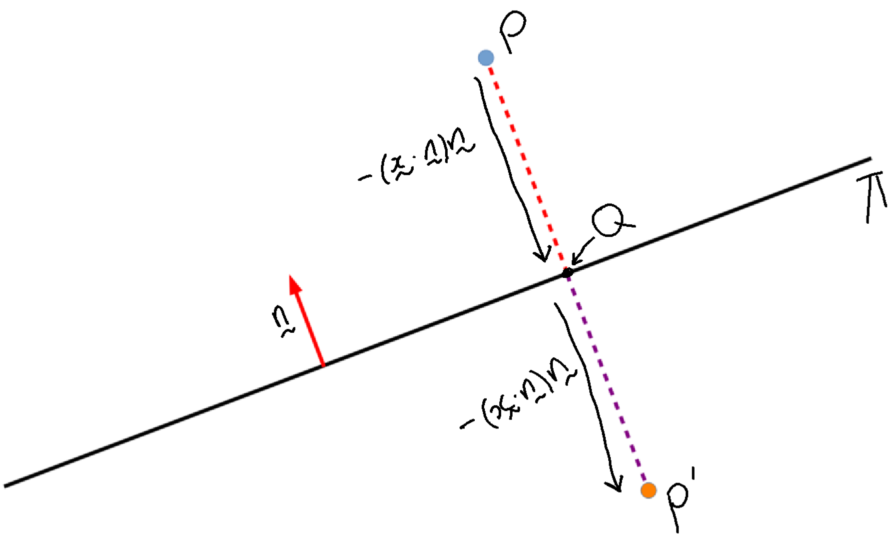
To show that PQ is perpendicular to Π consider the vector $\overrightarrow{PQ}$. We have:

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \Big(\mathbf{x} - (\mathbf{x} \,.\, \mathbf{n})\mathbf{n}\Big) - \mathbf{x} \\ &= -(\mathbf{x} \,.\, \mathbf{n})\mathbf{n}\end{aligned}$$

Hence PQ is parallel to $\mathbf{n}$, and so it is perpendicular to the plane Π.

It is quite helpful to draw a sketch of what we now know. We know that Q lies in the plane, and that PQ is perpendicular to the plane, so PQ is the shortest distance from point P to the plane.

The next part of the question is about reflections, so I have included the reflection of P (labelled P') in the diagram below.



(i) From the work in the stem, and the diagram, we can see that the reflection of P in plane Π is P' which has position vector $\mathbf{x} - 2(\mathbf{x} \,.\, \mathbf{n})\mathbf{n}$.

The given equation of the plane means that we have $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and since $a^2 + b^2 + c^2 = 1$, $\mathbf{n}$ is a unit vector.

This means that the image of $\mathbf{i}$ is:

$$\begin{aligned}
\mathbf{i} - 2(\mathbf{i} \,.\, \mathbf{n})\mathbf{n} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2a \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= \begin{pmatrix} 1 - 2a^2 \\ -2ab \\ -2ac \end{pmatrix} \\
&= \begin{pmatrix} b^2 + c^2 - a^2 \\ -2ab \\ -2ac \end{pmatrix}
\end{aligned}$$

where the last line uses $a^2 + b^2 + c^2 = 1$.

Using the same method, the image of $\mathbf{j}$ is $\begin{pmatrix} -2ab \\ a^2 + c^2 - b^2 \\ -2bc \end{pmatrix}$ and the image of $\mathbf{k}$ is $\begin{pmatrix} -2ac \\ -2bc \\ a^2 + b^2 - c^2 \end{pmatrix}$.

Using the fact that the images of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ form the columns of the transformation matrix we have:

$$\mathbf{M} = \begin{pmatrix} b^2 + c^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 + c^2 - b^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{pmatrix}.$$

(ii) Using the general form of $\mathbf{M}$ above we have:

$$b^2 + c^2 - a^2 = 0.64 \qquad (1)$$
$$a^2 + c^2 - b^2 = 0.36 \qquad (2)$$
$$a^2 + b^2 - c^2 = 0 \qquad (3)$$

Summing equation (1) and (2) gives $2c^2 = 1$, and so $c = \pm\dfrac{1}{\sqrt{2}}$. WLOG take $c = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ (we could multiply throughout by -1 to change the sign of c if we wished).

Summing (1) and (3) gives $2b^2 = 0.64$. We also know that $-2bc = -0.8$, so if c is positive, b must be positive. This gives $b = \sqrt{0.32} = \sqrt{\dfrac{32}{100}} = \dfrac{4\sqrt{2}}{10}$.

Summing (2) and (3) gives $2a^2 = 0.18$, and since $-2ac = 0.6$ this means that a is negative. We have $a = -\sqrt{0.18} = -\dfrac{3\sqrt{2}}{10}$.

The cartesian equation of the plane is therefore:

$$\begin{aligned}
&-\frac{3\sqrt{2}}{10}x + \frac{4\sqrt{2}}{10}y + \frac{\sqrt{2}}{2}z = 0 \\
\implies \quad & 3\sqrt{2}x - 4\sqrt{2}y - 5\sqrt{2}z = 0 \\
& \implies 3x - 4y - 5z = 0
\end{aligned}$$

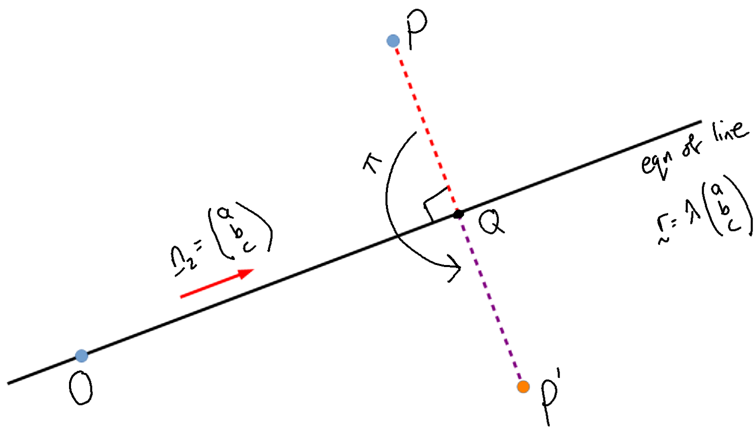
If an equation is easy to rationalise/simplify then it is good practice to do so!

(iii) I spent an embarrassingly long time trying to work out how this was different to the reflection. The key point that I was missing is that the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is *parallel* to the line we are rotating around, but it was *perpendicular* to the plane.

The line passes through the origin and is parallel to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, so the equation of the line is:

$$\mathbf{r} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

To work out where the image of P is we can imagine the line segment PQ (where PQ is perpendicular to the line) rotating by π. The picture below shows what is happening (and also explains why I was finding it difficult to distinguish between the two situations, and my drawing for the previous parts looks almost identical). Trying to represent a 3D situation on a 2D piece of paper can be challenging! With hindsight, it would have been better to draw this digram with the line perpendicular to the position of the plane in the previous diagram.



Let the vector representing Q be $\mathbf{q}$, and let the vector representing P be $\mathbf{x}$ (as before). As LaTeX-ing up column vectors is a bit of a nuisance, I am going to use $\mathbf{n}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Q is at the point where PQ is perpendicular to the line, so we have:

$$(\mathbf{q} - \mathbf{x}).\mathbf{n}_2 = 0$$

We also know that $\mathbf{q} = \lambda \mathbf{n}_2$, for some value of λ, so we have:

$$\begin{aligned} (\lambda \mathbf{n}_2 - \mathbf{x}).\mathbf{n}_2 &= 0 \\ \lambda \mathbf{n}_2^2 - \mathbf{x}.\mathbf{n}_2 &= 0 \\ \lambda &= \mathbf{x}.\mathbf{n}_2 \end{aligned}$$

Noting that $\mathbf{n}_2$ is a unit vector in this case as well.

This means that we have $\mathbf{q} = (\mathbf{x}.\mathbf{n}_2)\mathbf{n}_2$, and so $\overrightarrow{PQ} = (\mathbf{x}.\mathbf{n}_2)\mathbf{n}_2 - \mathbf{x}$.

The image of P has position vector:

$$\begin{aligned}\overrightarrow{OP'} &= \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QP'} \\ &= \mathbf{x} + \Big[(\mathbf{x}.\mathbf{n}_2)\mathbf{n}_2 - \mathbf{x}\Big] + \Big[(\mathbf{x}.\mathbf{n}_2)\mathbf{n}_2 - \mathbf{x}\Big] \\ &= 2(\mathbf{x}.\mathbf{n}_2)\mathbf{n}_2 - \mathbf{x}\end{aligned}$$

Using this, the image of $\mathbf{i}$ is given by:

$$\begin{aligned}(2a)\mathbf{n}_2 - \mathbf{i} &= \begin{pmatrix} 2a^2-1 \\ 2ab \\ 2ac \end{pmatrix} \\ &= \begin{pmatrix} a^2-b^2-c^2 \\ 2ab \\ 2ac \end{pmatrix}\end{aligned}$$

Similarly the image of $\mathbf{j}$ is $\begin{pmatrix} 2ab \\ b^2-c^2-a^2 \\ 2bc \end{pmatrix}$ and the image of $\mathbf{k}$ is $\begin{pmatrix} 2ac \\ 2bc \\ c^2-a^2-b^2 \end{pmatrix}$.

This means we have

$$\mathbf{N} = \begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-c^2-a^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{pmatrix} = -\mathbf{M}\,.$$

(iv) Using the fact that $\mathbf{N} = -\mathbf{M}$ we have:

$$\begin{aligned}\mathbf{NM} &= -\mathbf{MM} \\ &= -\mathbf{I}\end{aligned}$$

Where the last line occurs because $\mathbf{M}$ is a reflection, so is self-inverse. This means that we have $\mathbf{i} \to -\mathbf{i}$ etc and the transformation is an enlargement, scale factor -1, centre the origin.

Note that the matrix for an enlargement, scale factor k, with centre of enlargement at the origin is:

$$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

Question 5

5 Show that for positive integer n, $x^n - y^n = (x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1}$.

(i) Let F be defined by

$$\mathrm{F}(x) = \frac{1}{x^n(x-k)} \quad \text{for } x \neq 0, k$$

where n is a positive integer and $k \neq 0$.

(a) Given that

$$\mathrm{F}(x) = \frac{A}{x-k} + \frac{\mathrm{f}(x)}{x^n},$$

where A is a constant and $\mathrm{f}(x)$ is a polynomial, show that

$$\mathrm{f}(x) = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right).$$

Deduce that

$$\mathrm{F}(x) = \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r}.$$

(b) By differentiating $x^n\,\mathrm{F}(x)$, prove that

$$\frac{1}{x^n(x-k)^2} = \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}}.$$

(ii) Hence evaluate the limit of

$$\int_2^N \frac{1}{x^3(x-1)^2}\,\mathrm{d}x$$

as $N \to \infty$, justifying your answer.

Examiner's report

A popular question, which was well attempted with a fair degree of success: it was marginally less popular than question **2**, but marginally more successful. Most submitted quite a large amount of work, and were able to attempt later parts even if earlier parts were not successful as key results (requiring proof) were quoted in each part. The stem was mostly well completed, by a variety of methods, namely, re-summing indices, induction, or geometric series, though there were some candidates who seemed to think it was obvious and produced no working. Part **(i)(a)** was also well completed though few received full marks. The main problems were finding A and that $\mathrm{F}(x)$

is not defined for $x = k$. The second result in this part was better done, though some candidates struggled with re-summing when changing indices. For **(i)(b)**, many did not realise that they needed to differentiate both sides. Differentiation errors and confusion thwarted many that did differentiate. Part **(ii)** was well done by candidates that attempted it with most realising that they could use the result of **(i)(b)**. Though many lost marks for failing to show how to take the limit of the logarithm, most realised that they need to use partial fractions to complete the integral. Some candidates sadly left their expressions in terms of k.

Solution

Note that this question has lots of opportunities to get back into it and gain marks for later parts even if you could not complete an earlier part.

For the stem, start on the RHS. We have:

$$
\begin{aligned}
(x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1} &= x\sum_{r=1}^{n} x^{n-r}y^{r-1} - y\sum_{r=1}^{n} x^{n-r}y^{r-1} \\
&= x^n + \cancel{x^{n-1}y} + \cancel{x^{n-2}y^2} + \cancel{x^{n-3}y^3} + \cdots + \cancel{xy^{n-1}} \\
&\quad - \cancel{x^{n-1}y} - \cancel{x^{n-2}y^2} + \cancel{x^{n-3}y^3} - \cdots - \cancel{xy^{n-1}} - y^n \\
&= x^n - y^n
\end{aligned}
$$

(i)(a) Equating the expressions for $\mathrm{F}(x)$ gives:

$$
\begin{aligned}
\frac{1}{x^n(x-k)} &= \frac{A}{x-k} + \frac{\mathrm{f}(x)}{x^n} \\
1 &= Ax^n + \mathrm{f}(x)(x-k)
\end{aligned}
$$

At this point we can substitute $x = k$, which gives $1 = Ak^n \implies A = \dfrac{1}{k^n}$. This gives:

$$
\begin{aligned}
\mathrm{f}(x) &= \frac{1}{x-k}\left(1 - Ax^n\right) \\
&= \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right)
\end{aligned}
$$

Using this we have:

$$
\begin{aligned}
\mathrm{F}(x) &= \frac{A}{x-k} + \frac{\mathrm{f}(x)}{x^n} \\
&= \frac{1}{k^n(x-k)} + \frac{1}{x^n} \times \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right) \\
&= \frac{1}{k^n(x-k)} + \frac{1}{x-k}\left[\left(\frac{1}{x}\right)^n - \left(\frac{1}{k}\right)^n\right] \\
&= \frac{1}{k^n(x-k)} - \frac{1}{x-k}\left[\left(\frac{1}{k}\right)^n - \left(\frac{1}{x}\right)^n\right] \\
&= \frac{1}{k^n(x-k)} - \frac{1}{x^n k^n(x-k)}\left[x^n - k^n\right]
\end{aligned}
$$

The penultimate step is done so that we have a negative sign in the middle of the expression to help us find the required form of $\mathrm{F}(x)$.

Using the result from the stem this becomes:

$$\begin{aligned}
\mathrm{F}(x) &= \frac{1}{k^n(x-k)} - \frac{1}{x^n k^n (x-k)}\left[(x-k)\sum_{r=1}^{n} x^{n-r}k^{r-1}\right] \\
&= \frac{1}{k^n(x-k)} - \left[\sum_{r=1}^{n} \frac{x^{n-r}k^{r-1}}{x^n k^n}\right] \\
&= \frac{1}{k^n(x-k)} - \left[\sum_{r=1}^{n} \frac{1}{x^{n-(n-r)}k^{n-(r-1)}}\right] \\
&= \frac{1}{k^n(x-k)} - \sum_{r=1}^{n} \frac{1}{x^r k^{n-r+1}} \\
&= \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n} \frac{1}{x^r k^{n-r}}
\end{aligned}$$

(i)(b) Considering $x^n\mathrm{F}(x)$ we have:

$$\begin{aligned}
\frac{1}{x-k} &= \frac{x^n}{k^n(x-k)} - \frac{x^n}{k}\sum_{r=1}^{n} \frac{1}{x^r k^{n-r}} \\
&= \frac{x^n}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n} \frac{x^{n-r}}{k^{n-r}}
\end{aligned}$$

Differentiating gives:

$$\frac{-1}{(x-k)^2} = \frac{1}{k^n}\left[\frac{n(x-k)x^{n-1} - x^n}{(x-k)^2}\right] - \frac{1}{k}\sum_{r=1}^{n} \frac{(n-r)x^{n-r-1}}{k^{n-r}}$$

and then multiplying throughout by $\frac{-1}{x^n}$ gives:

$$\begin{aligned}
\frac{1}{x^n(x-k)^2} &= -\frac{1}{x^n k^n}\frac{n(x-k)x^{n-1}}{(x-k)^2} + \frac{1}{x^n k^n}\frac{x^n}{(x-k)^2} + \frac{1}{kx^n}\sum_{r=1}^{n} \frac{(n-r)x^{n-r-1}}{k^{n-r}} \\
&= -\frac{1}{xk^n}\frac{n}{(x-k)} + \frac{1}{k^n}\frac{1}{(x-k)^2} + \frac{1}{k}\sum_{r=1}^{n} \frac{(n-r)x^{n-r-1}}{k^{n-r}x^n} \\
&= -\frac{n}{xk^n(x-k)} + \frac{1}{k^n(x-k)^2} + \frac{1}{k}\sum_{r=1}^{n} \frac{(n-r)}{k^{n-r}x^{r+1}} \\
&= \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n} \frac{(n-r)}{k^{n+1-r}x^{r+1}}
\end{aligned}$$

as required.

(ii) Using the result shown in part **(i)(b)** with $n = 3$ and $k = 1$ we have[1]:

$$\int_2^N \frac{1}{x^3(x-1)^2}\,\mathrm{d}x = \int_2^N \left[\frac{1}{(x-1)^2} - \frac{3}{x(x-1)} + \sum_{r=1}^{3} \frac{(3-r)}{x^{r+1}}\right] \mathrm{d}x$$

The first term and the terms in the sum can be integrated directly. The second term can be split up using partial fractions:

$$\frac{3}{x(x-1)} = \frac{3}{x-1} - \frac{3}{x}$$

The integral becomes:

$$\begin{aligned}
&\int_2^N \left[\frac{1}{(x-1)^2} - \left(\frac{3}{x-1} - \frac{3}{x}\right) + \frac{2}{x^2} + \frac{1}{x^3}\right] \mathrm{d}x \\
&= \left[-\frac{1}{x-1} - 3\ln(x-1) + 3\ln x - \frac{2}{x} - \frac{1}{2x^2}\right]_2^N \\
&= \left(-\frac{1}{N-1} + 3\ln\left(\frac{N}{N-1}\right) - \frac{2}{N} - \frac{1}{2N^2}\right) \\
&\qquad\qquad - \left(-\frac{1}{2-1} + 3\ln\left(\frac{2}{2-1}\right) - \frac{2}{2} - \frac{1}{2\times 2^2}\right)
\end{aligned}$$

As $N \to \infty$, $\left(\frac{N}{N-1}\right) \to 1$ and so $\ln\left(\frac{N}{N-1}\right) \to 0$. All of the other terms in the first bracket also tend to 0 as $N \to \infty$, so we have:

$$\begin{aligned}
\int_2^\infty \frac{1}{x^3(x-1)^2}\,\mathrm{d}x &= -\left(-1 + 3\ln 2 - 1 - \frac{1}{8}\right) \\
&= 2 + \frac{1}{8} - 3\ln 2 \\
&= \frac{17}{8} - 3\ln 2
\end{aligned}$$

[1]Note that you could have attempted this part even if you had not done any of the previous parts. This question might have been a good choice for a sixth question, in the hope that you could pick up a few extra marks.

Question 6

6 **(i)** Sketch the curve $y = \cos x + \sqrt{\cos 2x}$ for $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$.

(ii) The equation of curve C_1 in polar co-ordinates is

$$r = \cos\theta + \sqrt{\cos 2\theta} \qquad -\tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi.$$

Sketch the curve C_1.

(iii) The equation of curve C_2 in polar co-ordinates is

$$r^2 - 2r\cos\theta + \sin^2\theta = 0 \qquad -\tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi.$$

Find the value of r when $\theta = \pm\frac{1}{4}\pi$.

Show that, when r is small, $r \approx \frac{1}{2}\theta^2$.

Sketch the curve C_2, indicating clearly the behaviour of the curve near $r = 0$ and near $\theta = \pm\frac{1}{4}\pi$.

Show that the area enclosed by curve C_2 and above the line $\theta = 0$ is $\dfrac{\pi}{2\sqrt{2}}$.

Examiner's report

This was quite a popular question, being attempted by about three fifths of the candidates, but on average scoring only a bit better than one third marks. Most candidates were broadly successful at sketching the first graph in part **(i)**, but though they had differentiated, many did not consider the gradients at the endpoints. Attempts to draw the sketch for part **(ii)** were usually less successful, and few dealt well with the behaviour near the endpoints. Few candidates gave a completely accurate justification of the small r approximation in **(iii)**. Many candidates did not solve the equation of curve C_2 for r and thus did not realise that C_1 was one branch of C_2. Most only drew one or other branch, and very few considered how to join the branches. Most candidates did not know how to compute areas in polar coordinates: successful ones realised the area was a difference of two polar integrals and used trigonometric substitutions to perform the integral.

Solution

(i) **Symmetry**
Since we have $\cos(-x) = \cos x$ we know that the expression for y is even and hence we have reflection symmetry in the y axis.

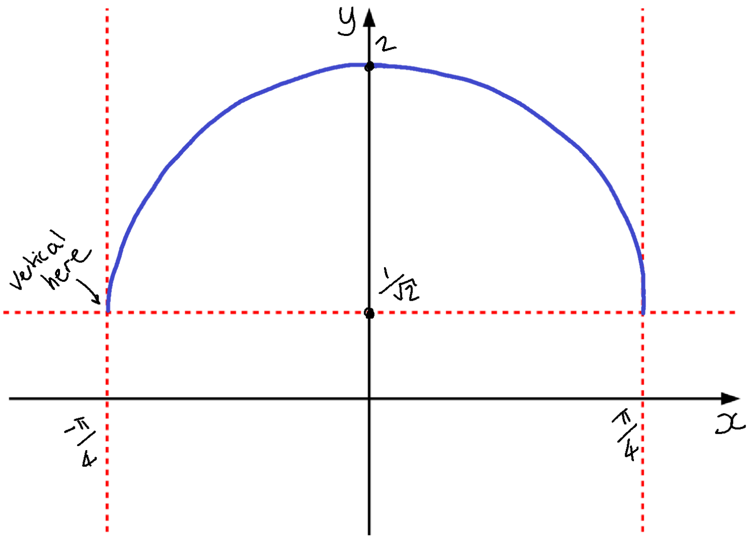
Axes and ranges
When $x = 0$, $y = 2$. We also know that in the range $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$, $\frac{1}{\sqrt{2}} \leqslant \cos x \leqslant 1$, and $0 \leqslant \sqrt{\cos 2x} \leqslant 1$. This means that we have $\frac{1}{\sqrt{2}} \leqslant y \leqslant 2$, and $y = \frac{1}{\sqrt{2}}$ when $x = \pm\frac{1}{4}\pi$.

Gradient
We have:

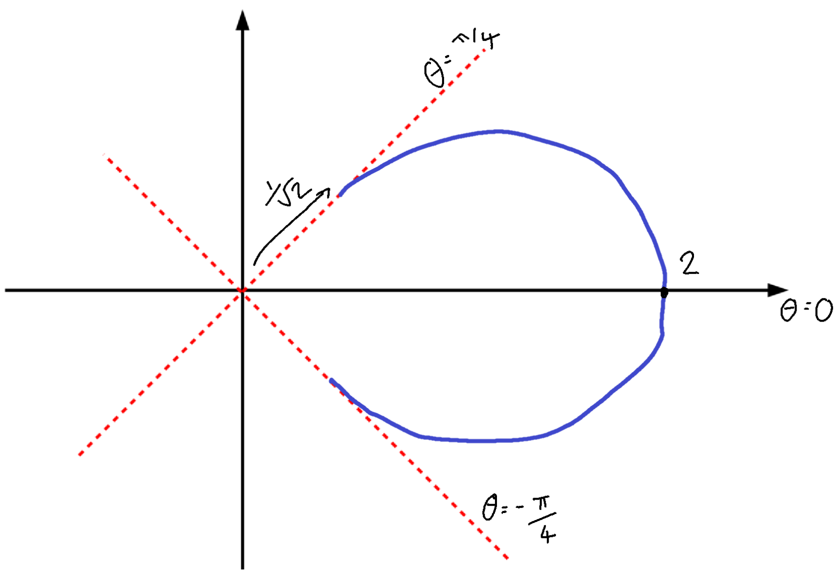
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin x - \tfrac{1}{2} \times 2\sin 2x \times (\cos 2x)^{-\frac{1}{2}} \\ &= -\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}}\end{aligned}$$

This means that when $x = 0$ we have a stationary point, when x is positive the gradient is negative and when x is negative the gradient is positive. As x tends towards $\pm\frac{1}{4}\pi$ we have $\cos 2x \to 0$, and so as $x \to \pm\frac{1}{4}\pi$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} \to \mp\infty$.

This gives enough information to sketch the graph.



(ii) Part **(i)** shows what happens to r as θ varies. We know that r had a maximum value of 2 when $\theta = 0$, and that the minimum value of r (which is $\frac{1}{\sqrt{2}}$) occurs when $\theta = \pm\frac{1}{4}\pi$.



Note that when $\theta = 0$, $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 0$, which means that the radius is staying the same nearby - this is why the curve is vertical where it crosses the $\theta = 0$ line. We also know that the curve is symmetrical in θ (as both $\cos\theta$ and $\cos 2\theta$ are symmetrical in θ), which also implies that the curve must be vertical when it crosses the $\theta = 0$ line (there is reflection symmetry across this line!).

As $\theta \to \pm\frac{1}{4}\pi$, $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} \to \infty$, which means that near here the radius is tangential to the $\theta = \pm\frac{1}{4}\pi$ lines. This is basically because near these values of θ, r is changing a lot for a very small change in θ.

(iii) When $\theta = \pm\frac{1}{4}\pi$ we have:

$$\begin{aligned}
r^2 - 2r\cos\theta + \sin^2\theta &= 0 \\
r^2 - \sqrt{2}r + \frac{1}{2} &= 0 \\
\left(r - \frac{\sqrt{2}}{2}\right)^2 - \frac{2}{4} + \frac{1}{2} &= 0 \\
\left(r - \frac{\sqrt{2}}{2}\right)^2 &= 0
\end{aligned}$$

So $r = \dfrac{\sqrt{2}}{2}\left(= \dfrac{1}{\sqrt{2}}\right)$.

Completing the square on the given equation gives:

$$\begin{aligned}
r^2 - 2r\cos\theta + \sin^2\theta &= 0 \\
(r - \cos\theta)^2 - \cos^2\theta + \sin^2\theta &= 0 \\
(r - \cos\theta)^2 &= \cos^2\theta - \sin^2\theta \\
(r - \cos\theta)^2 &= \cos 2\theta \\
r - \cos\theta &= \pm\sqrt{\cos 2\theta}
\end{aligned}$$

One of these, $r = \cos\theta + \sqrt{\cos 2\theta}$, is the situation described in the previous two parts. Here we know that when $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$ we have $\frac{1}{\sqrt{2}} \leqslant r \leqslant 2$, and so we cannot have r being close to 0 here (*small* usually means that you are looking at what happens as you tend to 0).

Therefore when r is small we have $r = \cos\theta - \sqrt{\cos 2\theta}$. When $r \approx 0$ we have:

$$\begin{aligned}
\cos\theta &\approx \sqrt{\cos 2\theta} \\
\cos^2\theta &\approx \cos 2\theta \\
\cos^2\theta &\approx \cos^2\theta - \sin^2\theta \\
\sin^2\theta &\approx 0
\end{aligned}$$

and since we are in the range $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$, $\sin^2\theta \approx 0 \implies \theta \approx 0$.

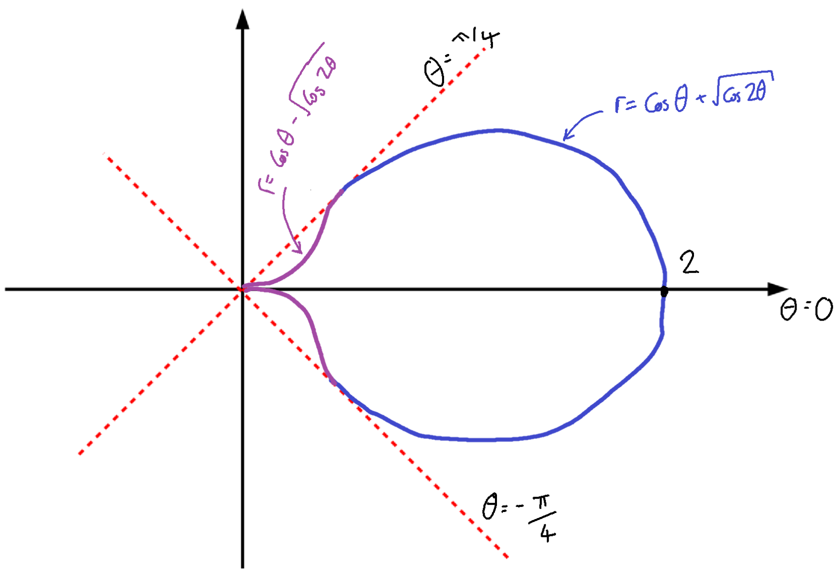
When $\theta \approx 0$ we have $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Substituting this into r gives:

$$\begin{aligned} r &= \cos\theta - \sqrt{\cos 2\theta} \\ r &\approx 1 - \tfrac{1}{2}\theta^2 - \sqrt{\left[1 - \tfrac{1}{2}(2\theta)^2\right]} \\ r &\approx 1 - \tfrac{1}{2}\theta^2 - (1 - 2\theta^2)^{\frac{1}{2}} \\ r &\approx 1 - \tfrac{1}{2}\theta^2 - \left[1 - \tfrac{1}{2} \times 2\theta^2 + O(\theta^4)\right] \\ r &\approx 1 - \tfrac{1}{2}\theta^2 - 1 + \theta^2 \\ r &\approx \tfrac{1}{2}\theta^2 \end{aligned}$$

And so when we get close to the origin, we have $r \approx \frac{1}{2}\theta^2$.

The notation $O(\theta^4)$ means terms in θ^4 and higher powers of θ. Since θ is small the terms in θ^4 and so on are much smaller than the term in θ^2, and so can be ignored.

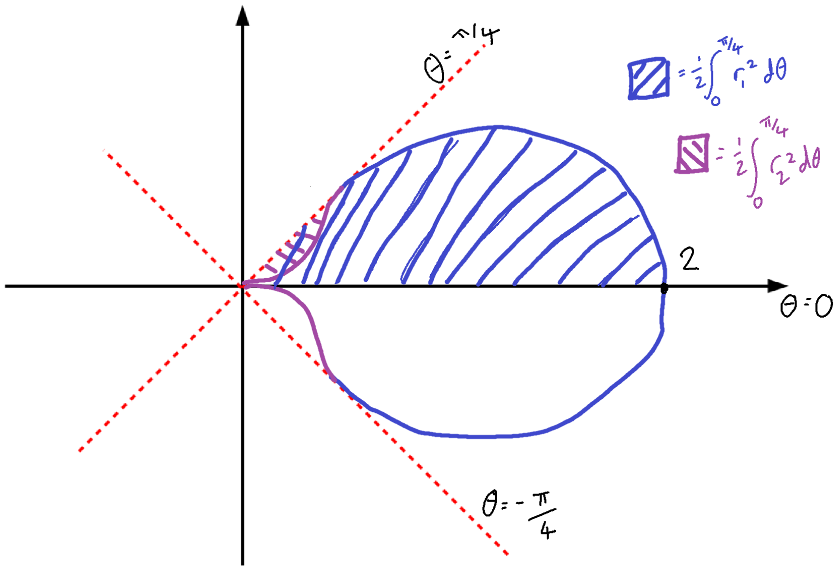
This is my sketch of the curve - the behaviour near $r = 0$ is rather exaggerated! Here is a neater version.



The area contained between the lines $\theta = \alpha$ and $\theta = \beta$ is given by:

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta$$

The area we are being asked to find is actually a difference between two areas, as I have tried to indicate below:



The required area is:

$$\frac{1}{2}\int_0^{\frac{1}{4}\pi}\left(\cos\theta+\sqrt{\cos 2\theta}\right)^2 \mathrm{d}\theta - \frac{1}{2}\int_0^{\frac{1}{4}\pi}\left(\cos\theta-\sqrt{\cos 2\theta}\right)^2 \mathrm{d}\theta$$

$$=\frac{1}{2}\int_0^{\frac{1}{4}\pi}\left[\left(\cos\theta+\sqrt{\cos 2\theta}\right)^2-\left(\cos\theta-\sqrt{\cos 2\theta}\right)^2\right]\mathrm{d}\theta$$

$$=\frac{1}{2}\int_0^{\frac{1}{4}\pi}\left[\cancel{\cos^2\theta}+2\cos\theta\sqrt{\cos 2\theta}+\cancel{\cos 2\theta}-\cancel{\cos^2\theta}+2\cos\theta\sqrt{\cos 2\theta}-\cancel{\cos 2\theta}\right]\mathrm{d}\theta$$

$$=2\int_0^{\frac{1}{4}\pi}\cos\theta\sqrt{\cos 2\theta}\,\mathrm{d}\theta$$

$$=2\int_0^{\frac{1}{4}\pi}\cos\theta\sqrt{1-2\sin^2\theta}\,\mathrm{d}\theta$$

Let $\sqrt{2}\sin\theta = u$, so we have $\theta = 0 \implies u = 0$ and $\theta = \frac{1}{4}\pi \implies u = 1$. We also have $\frac{\mathrm{d}u}{\mathrm{d}\theta} = \sqrt{2}\cos\theta$. This transforms our integral into:

$$2\int_0^1 \cos\theta\sqrt{1-u^2}\times\frac{1}{\sqrt{2}\cos\theta}\,\mathrm{d}u = \sqrt{2}\int_0^1\sqrt{1-u^2}\,\mathrm{d}u$$

Now using a substitution of $u = \sin t$, so that $u = 0 \implies t = 0$, $u = 1 \implies t = \frac{1}{2}\pi$ and $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \cos t$ we have:

$$\begin{aligned}
\sqrt{2}\int_0^1 \sqrt{1-u^2}\,\mathrm{d}u &= \sqrt{2}\int_0^{\frac{1}{2}\pi} \sqrt{1-\sin^2 t} \times \cos t\,\mathrm{d}t \\
&= \sqrt{2}\int_0^{\frac{1}{2}\pi} \cos^2 t\,\mathrm{d}t \\
&= \sqrt{2}\int_0^{\frac{1}{2}\pi} \frac{\cos 2t + 1}{2}\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{2}\left[\frac{\sin 2t}{2} + t\right]_0^{\frac{1}{2}\pi} \\
&= \frac{\sqrt{2}}{2}\left(\cancel{\frac{\sin \pi}{2}} + \frac{1}{2}\pi\right) \\
&= \frac{\sqrt{2}\pi}{4} \\
&= \frac{\pi}{2\sqrt{2}}
\end{aligned}$$

Question 7

7 **(i)** Given that the variables x, y and u are connected by the differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{f}(x)u = \mathrm{h}(x) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y = u,$$

show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (\mathrm{g}(x) + \mathrm{f}(x))\frac{\mathrm{d}y}{\mathrm{d}x} + (\mathrm{g}'(x) + \mathrm{f}(x)\mathrm{g}(x))y = \mathrm{h}(x). \qquad (1)$$

(ii) Given that the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(1 + \frac{4}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2}{x} + \frac{2}{x^2}\right)y = 4x + 12 \qquad (2)$$

can be written in the same form as (1), find a first order differential equation which is satisfied by $\mathrm{g}(x)$.

If $\mathrm{g}(x) = kx^n$, find a possible value of n and the corresponding value of k.

Hence find a solution of (2) with $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3$ at $x = 1$.

Examiner's report

This was the second most popular question, but the most successful with a mean score of nearly two thirds marks. All but the weakest candidates managed to do part **(i)** perfectly well. Similarly, finding the first order differential equation for $\mathrm{g}(x)$ in part **(ii)** caused very few problems. Most candidates that attempted to substitute the given expression for $\mathrm{g}(x)$ in the first order differential equation obtained the correct polynomial equation, and a few gave up having done this. Most guessed the value $n = -1$ and then found that $k = 2$ works, whilst some just wrote the values of k and n, without any explanation. It wasn't uncommon for candidates to get stuck finding k or n, usually due to arithmetic errors. Most candidates attempting to find $u(x)$ were able to find the integrating factor and perform the integration, although a significant proportion got the integral wrong. Regardless of accuracy, everyone attempted inserting the initial conditions. Some candidates also tried using a particular and complimentary solution method to integrate, but only a few who attempted that got the complimentary part correct. If candidates solved for $u(x)$ correctly, they usually did so for y as well.

Solution

(i) Differentiating the second equation gives:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{g}'(x)y + \mathrm{g}(x)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}$$

Substituting this into the first equation, and substituting for u as well, gives:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x} \qquad\qquad + \qquad \mathrm{f}(x)u \qquad\qquad &= \mathrm{h}(x)\\ \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{g}'(x)y + \mathrm{g}(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \mathrm{f}(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y\right) &= \mathrm{h}(x)\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \Big(\mathrm{f}(x) + \mathrm{g}(x)\Big)\frac{\mathrm{d}y}{\mathrm{d}x} + \Big(\mathrm{g}'(x) + \mathrm{f}(x)\mathrm{g}(x)\Big)y &= \mathrm{h}(x)\end{aligned}$$

(ii) We are told that this differential equation can be written in the same form as (1). Looking at the coefficients of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and y we have:

$$\begin{aligned}\mathrm{f}(x) + \mathrm{g}(x) &= 1 + \frac{4}{x}\\ \mathrm{g}'(x) + \mathrm{f}(x)\mathrm{g}(x) &= \frac{2}{x} + \frac{2}{x^2}\end{aligned}$$

This means that:

$$\mathrm{f}(x) = 1 + \frac{4}{x} - \mathrm{g}(x)$$

and then substituting this into the y coefficient equation gives:

$$\mathrm{g}'(x) + \left(1 + \frac{4}{x} - \mathrm{g}(x)\right)\mathrm{g}(x) = \frac{2}{x} + \frac{2}{x^2}$$

which is a first order differential equation in $\mathrm{g}(x)$.

If $\mathrm{g}(x) = kx^n$ then $\mathrm{g}'(x) = knx^{n-1}$. Substituting this into the differential equation for $\mathrm{g}(x)$ we have:

$$\begin{aligned}knx^{n-1} + \left(1 + \frac{4}{x} - kx^n\right)kx^n &= \frac{2}{x} + \frac{2}{x^2}\\ knx^{n-1} + kx^n + 4kx^{n-1} - k^2x^{2n} &= \frac{2}{x} + \frac{2}{x^2}\\ k(n+4)x^{n-1} + kx^n - k^2x^{2n} &= \frac{2}{x} + \frac{2}{x^2}\\ k(n+4)x^{n+1} + kx^{n+2} - k^2x^{2n+2} &= 2x + 2\\ k^2x^{2n+2} - kx^{n+2} - k(n+4)x^{n+1} + 2x + 2 = 0\end{aligned}$$

Really this is an identity rather than an equation, we want this to be true for all x!

If we take $n > 0$, then k^2x^{2n+2} term cannot be cancelled out by any of the other terms (for example, if $n = 1$ then we have $k^2x^4 - kx^3 - 5kx^2 + 2x + 2$ which can not be identically 0).

If $n = 0$ then we have:

$$k^2x^2 - kx^2 - 4kx + 2x + 2 = 0$$

There is nothing here to cancel out the 2.

If $n = -1$ we have:

$$k^2x^0 - kx^1 - k(-1+4)x^0 + 2x + 2 = 0$$

Equating the coefficients of x gives $k = 2$. Substituting this gives:

$$\begin{aligned} &k^2x^0 - kx^1 - k(-1+4)x^0 + 2x + 2 \\ =&4 - 2x - 6 + 2x + 2 \\ =&0 \end{aligned}$$

and so $n = -1, k = 2$ is a possible pair of values.

You are only asked to find a pair of values, so you can stop here! If you wanted to show that this was the only possible value of n then you could consider $n \leqslant -2$. If $n \leqslant -2$, then $2n + 2 \leqslant -2$, $n + 2 \leqslant 0$ and $n + 1 \leqslant -1$, so there is nothing to cancel with the $2x$ term.

If $\mathrm{g}(x) = \dfrac{2}{x}$, then $\mathrm{f}(x) = 1 + \dfrac{4}{x} - \dfrac{2}{x} = 1 + \dfrac{2}{x}$. A quick check verifies that this gives $\mathrm{g}'(x) + \mathrm{f}(x)\mathrm{g}(x) = \dfrac{2}{x} + \dfrac{2}{x^2}$ as required. Substituting these, and $\mathrm{h}(x) = 4x + 12$, into the first pair of differential equations gives:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x} + \left(1 + \frac{2}{x}\right)u &= 4x + 12 \\ \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y &= u \end{aligned}$$

The first of these is a linear first order differential equation in u, and can be solved with an integrating factor. The integrating factor is:

$$\begin{aligned} \mathrm{e}^{\int\left(1+\frac{2}{x}\right)\,\mathrm{d}x} &= \mathrm{e}^{x+2\ln x} \\ &= \mathrm{e}^x \times \mathrm{e}^{\ln(x^2)} \\ &= x^2\mathrm{e}^x \end{aligned}$$

Multiplying throughout by the integrating factor gives:

$$\begin{aligned} x^2\mathrm{e}^x\frac{\mathrm{d}u}{\mathrm{d}x} + x^2\mathrm{e}^x\left(1 + \frac{2}{x}\right)u &= x^2\mathrm{e}^x(4x + 12) \\ x^2\mathrm{e}^x\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{e}^x\left(x^2 + 2x\right)u &= \mathrm{e}^x(4x^3 + 12x^2) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x u\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^3\mathrm{e}^x\right) \\ x^2\mathrm{e}^x u &= 4x^3\mathrm{e}^x + c \end{aligned}$$

We know that $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3$ when $x = 1$, and since $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2}{x}y = u$ this means that when $x = 1$ we have $u = -3 + \frac{2}{1} \times 5 = 7$. Hence we have $7\mathrm{e}^1 = 4\mathrm{e}^1 + c \implies c = 3\mathrm{e}$.

Hence $u = \dfrac{4x^3\mathrm{e}^x + 3\mathrm{e}}{x^2\mathrm{e}^x} = 4x + 3\mathrm{e}x^{-2}\mathrm{e}^{-x}$. Substituting this into $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2}{x}y = u$ gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 4x + 3\mathrm{e}x^{-2}\mathrm{e}^{-x}$$

The integrating factor this time is:

$$\mathrm{e}^{\int \frac{2}{x}\,\mathrm{d}x} = \mathrm{e}^{2\ln x} = x^2$$

and so we have:

$$\begin{aligned} x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy &= 4x^3 + 3\mathrm{e}\times\mathrm{e}^{-x} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right) &= 4x^3 + 3\mathrm{e}\times\mathrm{e}^{-x} \\ x^2y &= x^4 - 3\mathrm{e}\times\mathrm{e}^{-x} + c' \end{aligned}$$

I have avoided using k as the constant of integration as this was used earlier.

When $x = 1$, $y = 5$ and so $5 = 1 - 2\mathrm{e}\times\mathrm{e}^{-1} + c' \implies c' = 7$. Hence we have:

$$\begin{aligned} x^2y &= x^4 - 3\mathrm{e}\times\mathrm{e}^{-x} + 7 \\ y &= x^2 - \frac{3\mathrm{e}^{-x+1}}{x^2} + \frac{7}{x^2} \end{aligned}$$

Question 8

8 A sequence u_k, for integer $k \geqslant 1$, is defined as follows.

$$u_1 = 1$$
$$u_{2k} = u_k \text{ for } k \geqslant 1$$
$$u_{2k+1} = u_k + u_{k+1} \text{ for } k \geqslant 1$$

(i) Show that, for every pair of consecutive terms of this sequence, except the first pair, the term with odd subscript is larger than the term with even subscript.

(ii) Suppose that two consecutive terms in this sequence have a common factor greater than one. Show that there are then two consecutive terms earlier in the sequence which have the same common factor. Deduce that any two consecutive terms in this sequence are co-prime (do not have a common factor greater than one).

(iii) Prove that it is not possible for two positive integers to appear consecutively in the same order in two different places in the sequence.

(iv) Suppose that a and b are two co-prime positive integers which do not occur consecutively in the sequence with b following a. If $a > b$, show that $a - b$ and b are two co-prime positive integers which do not occur consecutively in the sequence with b following $a - b$, and whose sum is smaller than $a + b$. Find a similar result for $a < b$.

(v) For each integer $n \geqslant 1$, define the function f from the positive integers to the positive rational numbers by $\text{f}(n) = \dfrac{u_n}{u_{n+1}}$. Show that the range of f is all the positive rational numbers, and that f has an inverse.

Examiner's report

About 60% of the candidates attempted this question, but it was the second least successful question on the paper with a mean score of about one third marks. There were some very good solutions to this question, but most candidates only provided fragmentary answers, and stopped after the first couple of parts.

A common mistake was to use the condition $u_{2k} = u_k$ along with $u_1 = 1$ to erroneously conclude that all the terms with an even subscript are equal to 1. This might have been avoided if the candidates had written out the first 10 (or so) terms of the sequence to help them get a "feel" for what was happening, which could have also help stopped some other misconceptions along the way.

Part **(i)** was generally well done, but some candidates did not consider both cases $u_{2k} > u_{2k}$ and $u_{2k-1} > u_{2k}$. Other candidates concluded that $u_{k+1} + u_k > u_k$ without justifying this inequality by stating that all the terms are positive.

Part **(ii)** was attempted well by many candidates but was less successful than part **(i)**. Some candidates who correctly considered both the (u_{2k-1}, u_{2k}) and (u_{2k}, u_{2k+1}) cases in part **(i)** then failed to consider both in this part. Some candidates erroneously assumed that if two terms p, q share a common factor and $p < q$ then it must be the case that $q = kp$.

Part **(iii)** was only answered well by a few of the candidates. Some did not appreciate that "consecutively" means appears one directly after another, instead taking it to mean that the second one occurs at some position after the first one. Only a small minority of attempts considered both the (u_{2k-1}, u_{2k}) and (u_{2k}, u_{2k+1}) cases. A lot of candidates erroneously stated that "if $u_k = a$ then if a is going to reappear then the next index must be $2^n k$ for some integer n". A look at the first few terms of the sequence shows that $u_5 = u_7 = 3$ which contradicts that statement.

Part **(iv)** was not well attempted. Some candidates did not process the wording (which was designed to help with the next part), and some tried to show instead that if a and b were two co-prime integers which do occur consecutively in the sequence etc. The most successful candidates used contradiction here to show that if $a - b$ and b do occur consecutively then this means that a and b must occur consecutively. Some candidates correctly showed the first result, but when trying to find the similar result for $a < b$ ended up with a following b and so essentially proved the same result again.

Part **(v)** was answered by only a few of the candidates attempting this question. There were some very well-reasoned arguments, including some candidates who used a construction method to justify that all possible rational numbers are in the range of $\mathrm{f}(n)$. Only a very small number connected part **(iv)** to this part of the question.

Solution

There are a lot of words in the solution here, probably more than you would need in an exam script. This question attracted quite a lot of "last question" attempts, that is it was often attempted by candidate as a sixth question in the hope of picking up a few marks, which is not necessarily a bad strategy and might explain the high attempt rate and low score.

With these sorts of questions, it can be a good idea to find a few values of the sequence first. This helps you to understand how the sequence is generated, and might also stop you making some false claims later in the question. We have:

$$\begin{aligned}
u_1 &= 1 \\
u_2 &= u_1 = 1 \\
u_3 &= u_1 + u_2 = 2 \\
u_4 &= u_2 = 1 \\
u_5 &= u_2 + u_3 = 3 \\
u_6 &= u_3 = 2 \\
u_7 &= u_3 + u_4 = 3 \\
u_8 &= u_4 = 1
\end{aligned}$$

and so the sequence is $1, 1, 2, 1, 3, 2, 3, 1$. From this you can see that all the terms of the form $u_{(2^k)}$ (i.e. $u_1, u_2, u_4, u_8, u_{16}$ etc.) are equal to 1, but not all the terms of the form $u_{(2k)}$.

Note that because of the way the sequence is constructed, all of the terms must be positive.

(i) To show this part we need to show that each term with an odd subscript is larger than both the term before and the term after (so, for example, we need both $u_7 > u_6$ and $u_7 > u_8$). This means that we need to show that $u_{2k+1} > u_{2k}$ and $u_{2k+1} > u_{2k+2}$.

With inequalities it is often easiest to show that they are greater than or less than 0. Start by considering $u_{2k+1} - u_{2k}$, where $k \geqslant 1$. We have:

$$\begin{aligned}
u_{2k+1} - u_{2k} &= (u_k + u_{k+1}) - u_k && \text{(using the definitions in the stem)}\\
u_{2k+1} - u_{2k} &= u_{k+1} && \text{(cancelling } u_k\text{)}\\
u_{2k+1} - u_{2k} &> 0 && \text{(as all terms are positive)}\\
\therefore u_{2k+1} &> u_{2k}
\end{aligned}$$

Considering $u_{2k+1} - u_{2k+2}$, where $k \geqslant 1$, we have

$$\begin{aligned}
u_{2k+1} - u_{2k+2} &= (u_k + u_{k+1}) - u_{k+1} && \text{(using the definitions in the stem)}\\
u_{2k+1} - u_{2k+2} &= u_k && \text{(cancelling } u_{k+1}\text{)}\\
u_{2k+1} - u_{2k+2} &> 0 && \text{(as all terms are positive)}\\
\therefore u_{2k+1} &> u_{2k+2}
\end{aligned}$$

Therefore in any pair of consecutive terms the term (apart from the first pair which are both equal to 1) the term with the odd subscript is greater than the term with the even subscript.

(ii) Assume that u_{2k} and u_{2k+1} share a common factor q, where $q > 1$. Let $u_{2k} = qx$ and let $u_{2k+1} = qy$ (and we know that $y > x$ as the odd terms are larger than the even ones).

Then we have:

$$\begin{aligned}
u_{2k} &= u_k\\
\implies u_k &= qx\\
u_{2k+1} &= u_k + u_{k+1}\\
qy &= qx + u_{k+1}\\
\implies u_{k+1} &= q(y-x)
\end{aligned}$$

and so both u_k and u_{k+1} have a common factor of q.

Note that we cannot use this as part of a *descent* argument yet as we started with a pair where the first term was an even-subscript term, and we do not know if k is even.

Now assume that $u_{2k+1} = qy$ and $u_{2k+2} = qx$ (you could use different letters to x and y this time, but I felt I was running out of alphabet - quite a lot of letters turn up later in the question).

We have:

$$\begin{aligned}
u_{2k+2} &= u_{k+1}\\
\implies u_{k+1} &= qx\\
u_{2k+1} &= u_k + u_{k+1}\\
qy &= qx + u_k\\
\implies u_k &= q(y-x)
\end{aligned}$$

and so u_k and u_{k+1} share a common factor.

Hence if a consecutive pair of terms share a common factor $q > 1$, then a previous pair of terms contains the same common factor. You can keep repeating this until you end up at the first two terms. These are both equal to 1 so this is a contradiction (as the only factor common to 1 and 1 is 1). Hence all consecutive terms are co-prime.

(iii) Let $u_{2k} = x$ and $u_{2k+1} = y$, and let $u_{2k+m} = x$ and $u_{2k+m+1} = y$ where $m \geqslant 1$. We know that $x < y$ as this is an even subscript — odd subscript pair, hence m must also be even. Let $m = 2n$, so we have $2k + m = 2(k + n)$ (where $n \geqslant 1$).

Since $u_{2k} = x$ and $u_{2k+1} = y$ we have:

$$\begin{aligned} u_k &= x \\ u_{k+1} &= u_{2k+1} - u_k \\ u_{k+1} &= y - x \end{aligned}$$

Since $u_{2(k+n)} = x$ and $u_{2(k+n)+1} = y$ we have:

$$\begin{aligned} u_{k+n} &= x \\ u_{k+n+1} &= u_{2(k+n)+1} - u_{k+n} \\ u_{k+n+1} &= y - x \end{aligned}$$

So we have $(u_k = x, u_{k+1} = y - x)$ and $(u_{k+n} = x, u_{k+n+1} = y - x)$. Hence if we have a pair of consecutive terms, where the first is an even subscript term, which occurs twice in the sequence, then there is another pair of consecutive terms which appears twice. Since $k < 2k$ and $k + n < 2k + 2n$ for $k \geqslant 1, n \geqslant 1$ then this pair of terms occurs earlier in the sequence.

Now we need to consider the other case, i.e. where the first term of the pair is an odd-subscript term. If $u_{2k+1} = y$ and $u_{2k+2} = x$, where $x < y$ and also $u_{2k+m+1} = y$ and $u_{2k+m+2} = x$ then, as before, m must be even so let $m = 2n$ (and $n \geqslant 1$).

Since $u_{2k+2} = x$ and $u_{2k+1} = y$ we have:

$$\begin{aligned} u_{k+1} &= x \\ u_k &= u_{2k+1} - u_{k+1} \\ u_k &= y - x \end{aligned}$$

Since $u_{2(k+n)+2} = x$ and $u_{2(k+n)+1} = y$ we have:

$$\begin{aligned} u_{k+n+1} &= x \\ u_{k+n} &= u_{2(k+n)+1} - u_{k+n+1} \\ u_{k+n} &= y - x \end{aligned}$$

So we have $(u_k = y - x, u_{k+1} = x)$ and $(u_{k+n} = y - x, u_{k+n+1} = x)$. Hence if we have a pair of consecutive terms, where the first is an odd subscript term, which occurs twice in the sequence, then there is another pair of consecutive terms which appears twice. Since $k < 2k + 1$ and $k + n < 2k + 2n + 1$ for $k \geqslant 1$, $n \geqslant 1$ then this pair of terms occurs earlier in the sequence.

We have now considered both cases (odd subscript followed by even and even subscript followed by odd). Hence wherever there is a repeated pair of consecutive terms, then there is a repeated pair earlier in the sequence. This argument can be applied until you get to the first pair, which equal $(1,1)$. This then implies that there is another, later pair, which is equal to $(1,1)$ which is a contradiction of part **(i)**.

In the exam, you could have written in places "and hence the same argument follows and..."

(iv) This part is can be a little hard to understand at first reading. It is saying that if (a,b) do not occur consecutively (with b after a) at any point in the sequence, and if $a > b$, then we cannot have $(a-b,b)$ occurring anywhere consecutively in the sequence.

An example can be helpful to pin down the ideas here. Suppose that 12 and 5 (which are co-prime) do not appear anywhere in the sequence as a consecutive pair in that order, so nowhere in the sequence do we see $\cdots, 12, 5, \cdots$. This case has $a > b$, and so we are asked to show that if $\cdots, 12, 5, \cdots$ never occurs in the sequence then $\cdots, (12-5), 5, \cdots$ can never occur in the sequence. The bit about the sum seems a little odd, but this actually helps with part **(v)**. The fact that a and b are co-prime is not needed to show that the smaller pair cannot appear consecutively, but is also useful in part **(v)**.

Let's try proof by contradiction. Assume that there exists a pair a, b (which appear in that order) with $a > b$ which does not appear consecutively, but that **there does exist** a consecutive pair $(a-b,b)$ (in that order!).

Assume that we have $u_k = a-b$ and $u_{k+1} = b$, i.e. the terms $a-b$ and b appear consecutively with b after $a-b$.

We then have:

$$\begin{aligned}
u_{2k+1} &= u_k + u_{k+1} \\
u_{2k+1} &= (a-b) + b \\
u_{2k+1} &= a \\
\text{and } u_{2k+2} &= u_{k+1} \\
u_{2k+2} &= b
\end{aligned}$$

Hence we have a consecutive pair $u_{2k+1} = a$, $u_{2k+2} = b$ which contradicts our first statement. Hence if the pair a, b does not appear consecutively then the pair $(a-b), b$ cannot appear consecutively. The sum of this second pair is $(a-b)+b = a < a+b$ since $b > 0$. Since a, b have no common factors greater than 1, $a-b, b$ have no common factors greater than 1 so they are also co-prime.

For the other part you need to be careful not to be in a "double" negative situation! We have considered the case when $\cdots, 12, 5, \cdots$ does not appear, but we have not considered the case when $\cdots, 5, 12, \cdots$ does not appear. This is the case when b follows a (as before), but we have $a < b$.

Using proof by contradiction again, assume that there exists a pair a, b (which appear in that order) with $a < b$ which does not appear consecutively in the sequence, but **there does exist** a consecutive pair $(a, b-a)$ (in that order).

To work out what the smaller pair would be I started with $u_{2k} = a, u_{2k+1} = b$ but the proof needs to work in the other direction.

Let $u_k = a$ and $u_{k+1} = b - a$. We then have:

$$\begin{aligned} u_{2k+1} &= u_k + u_{k+1} \\ u_{2k+1} &= a + (b - a) \\ u_{2k+1} &= b \\ \text{and } u_{2k} &= u_k \\ u_{2k} &= a \end{aligned}$$

This is a contradiction as we now have a consecutive pair $u_{2k} = a, u_{2k+1} = b$. If a and b are co-prime then $a, (b-a)$ are co-prime, and the sum of the second pair is $a+(b-a) = b < a+b$.

What we have shown this is there is a pair of numbers a, b which do not appear consecutively in that order anywhere in the sequence then either the pair $a, (b - a)$ or $(a - b), b$ (depending on whether $a > b$ or $a < b$) cannot appear consecutively anywhere in the sequence.

(v) This part is the "punchline" to the question, where all the other parts come together!

$f(n)$ here is the fractions formed by considering the ratio of consecutive terms of the sequence. Using the first few values of u_n found earlier we have $f(1) = 1; f(2) = \frac{1}{2}; f(3) = 2;$ $f(4) = \frac{1}{3}; f(5) = \frac{3}{2}; f(6) = \frac{2}{3}; f(7) = 3$. It looks as if this might be generating the positive rational numbers (including integers) in their simplest form.

In this part we need to show that **(a)** all of the rational numbers are generated in this way **(b)** each rational number is generated only once (so that $f(n)$ has an inverse).

We have already shown that any two consecutive terms must be co-prime, so all of the rationals generated will be in their simplest form. For **(a)** start by assuming that there is a co-prime ordered[2] pair (a, b) which does not appear consecutively in the sequence. By part **(iv)** this means that there is another pair of co-prime integers with a smaller sum which do not appear in the sequence. You can keep repeating this, and the sum of the two integers keeps reducing, until you can conclude that the pair of integers with a sum of 2 cannot exist as a consecutive pair of terms. This is a contradiction as the first two terms are $1, 1$, and so every possible ordered pair of co-prime integers must be in the list. Hence the range of f includes all of the rational numbers.

By part **(iii)** we have show that two integers cannot appear as a consecutive pair in the same order twice in the sequence. This means that we will not have the same rational number occurring twice, and so $f(n)$ is one-to-one and has an inverse.

Another way of expressing this is to say that for every rational number q, there exists an n, and only one n, so that $q = \dfrac{u_n}{u_{n+1}}$. Or you could say for every q there exists a **unique** n such that $q = \dfrac{u_n}{u_{n+1}}$.

[2]This means that the order matters, so we are considering $(5, 12)$ and not $(12, 5)$.

Question 9

9 Two inclined planes Π_1 and Π_2 meet in a horizontal line at the lowest points of both planes and lie on either side of this line. Π_1 and Π_2 make angles of α and β, respectively, to the horizontal, where $0 < \beta < \alpha < \frac{1}{2}\pi$.

A uniform rigid rod PQ of mass m rests with P lying on Π_1 and Q lying on Π_2 so that the rod lies in a vertical plane perpendicular to Π_1 and Π_2 with P higher than Q.

(i) It is given that both planes are smooth and that the rod makes an angle θ with the horizontal. Show that $2\tan\theta = \cot\beta - \cot\alpha$.

(ii) It is given instead that Π_1 is smooth, that Π_2 is rough with coefficient of friction μ and that the rod makes an angle ϕ with the horizontal. Given that the rod is in limiting equilibrium, with P about to slip down the plane Π_1, show that

$$\tan\theta - \tan\phi = \frac{\mu}{(\mu + \tan\beta)\sin 2\beta}$$

where θ is the angle satisfying $2\tan\theta = \cot\beta - \cot\alpha$.

Examiner's report

Just over 10% of the candidates attempted this question, with most understanding the setup and writing down some resolve and moment equations. A few candidates misunderstood the setup, specifically the "(planes) meet in a horizontal line at the lowest points of both planes and lie on either side of this line" and sketched a 'tilted wedge'. There was a moderate degree of success with the mean score being just short of half marks.

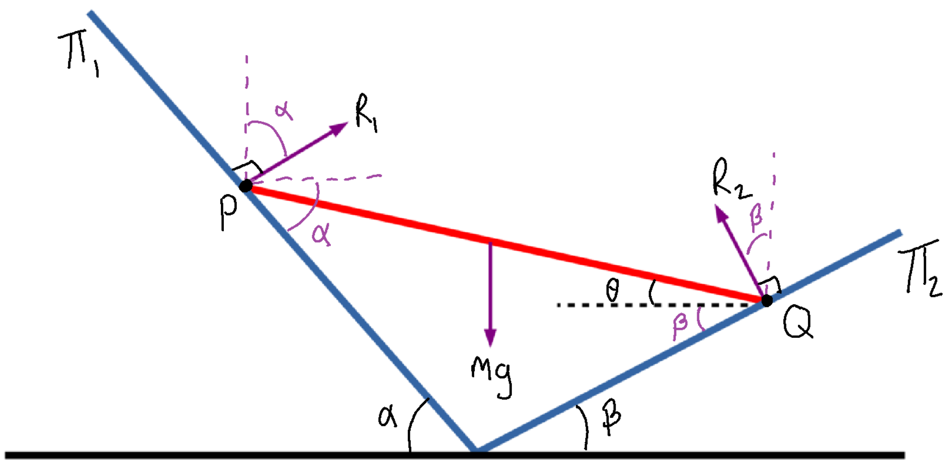
Part **(i)** was generally done well. Candidates who were unable to progress usually forgot about the moment equation. The resolve equations were done in various ways, with the most popular being horizontal-vertical and perp-parallel to the rod. All kinds of combinations of resolve and moment equations were used. With the horizontal resolve equation and moments about the centre of mass one only needs two equations to do this part. Most candidates were able to do the required algebra, and the failure to reach the answer usually stemmed from incorrect trigonometry in the equilibrium equations.

Most candidates who succeeded in part **(i)** then proceeded to do part **(ii)**. Most candidates were successful at incorporating the friction and writing down the new equations. At this point trig errors were common, and people who were resolving perp-parallel to the rod made more errors. Many candidates were put off by the difficult algebra that was about to follow. Of those who persisted, a good number arrived at the final answer, with some submitting many pages of attempts to do the algebra. The most common mistake was failing to eliminate α systematically.

Solution

I had to read the stem a couple of times to make sure I had understood all the information, and that I had the correct angle attached to the correct plane.

The diagram below shows the information in the stem, and the angle between the rod and the horizontal (given in part **(i)**) The planes are smooth so the only forces acting on the rod are the reaction forces from the two planes and the weight of the rod, which can be modelled as acting at the centre of the rod.



Note that since $0 < \beta < \alpha < \frac{1}{2}\pi$, we have $0 < \sin\alpha, \sin\beta, \cos\alpha, \cos\beta < 1$, so we can divide by these without any problems.

(i) The options are to resolve parallel and perpendicular to the rod (or even one of the planes), or to resolve horizontally and vertically.

Resolving horizontally:

$$R_1 \sin\alpha = R_2 \sin\beta \tag{1}$$

You might instead use $R_1 \cos\left(\frac{\pi}{2} - \alpha\right) = R_2 \cos\left(\frac{\pi}{2} - \beta\right)$, and then use $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$.

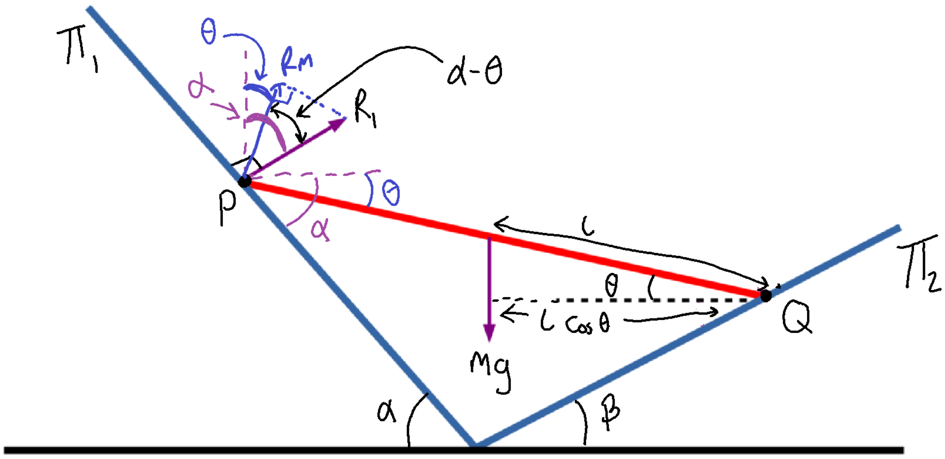
Resolving vertically:

$$mg = R_1 \cos\alpha + R_2 \cos\beta \tag{2}$$

We need some other information here. Let the length of the rod be $2l$. Taking moments about Q we have:

$$lmg\cos\theta = 2lR_1\cos(\alpha - \theta) \tag{3}$$

The diagram below shows how I found the moments:



For some reason I found it easiest to find the perpendicular distance of mg from Q, but when considering the moment from R_1 I used the component of R_1 which was perpendicular to the rod.

Another way of thinking about the clockwise moment is to split the reaction force R_1 into vertical and horizontal components and looks at these separately. This would give the clockwise moment as $R_1\cos\alpha \times 2l\cos\theta + R_1\sin\alpha \times 2l\sin\theta$, which is the same as $2lR_1\cos(\alpha-\theta)$.

Dividing equation (3) throughout by l and then dividing (3) by (1) gives:

$$\cos\theta = \frac{2R_1\cos(\alpha-\theta)}{R_1\cos\alpha + R_2\cos\beta} \tag{4}$$

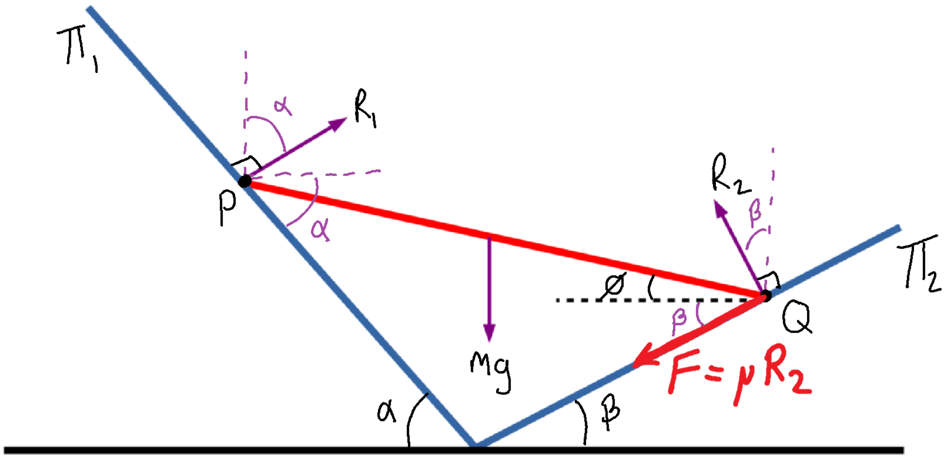
Using (2) to get $R_2 = R_1\frac{\sin\alpha}{\sin\beta}$ and substituting this into (4) gives:

$$\begin{aligned}
\cos\theta &= \frac{2R_1\cos(\alpha-\theta)}{R_1\cos\alpha + R_1\frac{\sin\alpha\cos\beta}{\sin\beta}} \\
\cos\theta &= \frac{2\sin\beta\cos(\alpha-\theta)}{\sin\beta\cos\alpha + \sin\alpha\cos\beta}
\end{aligned}$$

Multiplying up:

$$\begin{aligned}
\cos\theta(\sin\beta\cos\alpha + \sin\alpha\cos\beta) &= 2\sin\beta\cos(\alpha-\theta) \\
\cos\theta\sin\beta\cos\alpha + \cos\theta\sin\alpha\cos\beta &= 2\sin\beta\cos\alpha\cos\theta + 2\sin\beta\sin\alpha\sin\theta \\
\sin\theta\Big[2\sin\alpha\sin\beta\Big] &= \cos\theta\Big[\sin\alpha\cos\beta - \sin\beta\cos\alpha\Big] \\
2\tan\theta &= \frac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\sin\alpha\sin\beta} \\
2\tan\theta &= \frac{\cancel{\sin\alpha}\cos\beta}{\cancel{\sin\alpha}\sin\beta} - \frac{\cancel{\sin\beta}\cos\alpha}{\sin\alpha\cancel{\sin\beta}} \\
2\tan\theta &= \cot\beta - \cot\alpha
\end{aligned}$$

(ii) If the rod is on the point of slipping so that P moves downwards, then Q is on the point of moving upwards, so the friction on end Q acts down the slope.



Resolving horizontally:

$$R_1 \sin\alpha = R_2 \sin\beta + \mu R_2 \cos\beta \tag{5}$$

Resolving vertically:

$$mg + \mu R_2 \sin\beta = R_1 \cos\alpha + R_2 \cos\beta \tag{6}$$

Taking moments about Q we have:

$$lmg\cos\phi = 2lR_1\cos(\alpha - \phi) \tag{7}$$

Note that since the friction force acts through point Q, this equation is unchanged!

Dividing (7) throughout by l and then using (6) to eliminate mg gives:

$$\Big[R_1\cos\alpha + R_2\cos\beta - \mu R_2\sin\beta\Big]\cos\phi = 2R_1\cos(\alpha-\phi)$$

$$\Big[R_1\sin\alpha\cos\alpha + R_2\sin\alpha\cos\beta - \mu R_2\sin\alpha\sin\beta\Big]\cos\phi = 2R_1\sin\alpha\cos(\alpha-\phi)$$

Where the second line is obtained by multiplying throughout by $\sin\alpha$. Using (5) to eliminate $R_1\sin\alpha$ on the LHS of this gives:

$$\begin{aligned}
&\Big[R_1\sin\alpha\cos\alpha + R_2\sin\alpha\cos\beta - \mu R_2\sin\alpha\sin\beta\Big]\cos\phi \\
=&\Big[(R_2\sin\beta + \mu R_2\cos\beta)\cos\alpha + R_2\sin\alpha\cos\beta - \mu R_2\sin\alpha\sin\beta\Big]\cos\phi \\
=&R_2\cos\phi\Big[\sin\beta\cos\alpha + \sin\alpha\cos\beta + \mu(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\Big] \\
=&R_2\cos\phi\Big[\sin(\alpha+\beta) + \mu\cos(\alpha+\beta)\Big]
\end{aligned}$$

And on the RHS we have:

$$\begin{aligned}
&2(R_2\sin\beta + \mu R_2\cos\beta)\cos(\alpha-\phi) \\
=&2R_2\sin\beta(\cos\alpha\cos\phi + \sin\alpha\sin\phi) + 2\mu R_2\cos\beta(\cos\alpha\cos\phi + \sin\alpha\sin\phi)
\end{aligned}$$

Putting these two sides back together and dividing throughout by $R_2 \cos\phi$ gives:

$$\sin(\alpha+\beta)+\mu\cos(\alpha+\beta) = 2\sin\beta\cos\alpha + 2\sin\beta\sin\alpha\tan\phi + 2\mu\cos\beta\cos\alpha + 2\mu\cos\beta\sin\alpha\tan\phi$$

Rearranging:

$$\begin{aligned} 2\tan\phi\big[\sin\beta\sin\alpha+\mu\cos\beta\sin\alpha\big] &= \sin(\alpha+\beta)-2\sin\beta\cos\alpha+\mu\big[\cos(\alpha+\beta)-2\cos\beta\cos\alpha\big] \\ &= \sin\alpha\cos\beta-\sin\beta\cos\alpha+\mu\big[-\cos\alpha\cos\beta-\sin\alpha\sin\beta\big] \end{aligned}$$

Dividing by $\sin\alpha$ gives:

$$2\tan\phi\big[\sin\beta+\mu\cos\beta\big] = \cos\beta-\sin\beta\cot\alpha-\mu(\cot\alpha\cos\beta+\sin\beta)$$

and then dividing by $\cos\beta$ gives:

$$2\tan\phi\big[\tan\beta+\mu\big] = 1-\tan\beta\cot\alpha-\mu(\cot\alpha+\tan\beta)$$

Looking at the required equation, this has no α terms, but there is a θ term, and from part **(i)** we know that $2\tan\theta = \cot\beta-\cot\alpha$. Substituting $\cot\alpha = \cot\beta - 2\tan\theta$ gives:

$$\begin{aligned} 2\tan\phi\big[\tan\beta+\mu\big] &= 1-\tan\beta(\cot\beta-2\tan\theta)-\mu[(\cot\beta-2\tan\theta)+\tan\beta] \\ 2\tan\phi\big[\tan\beta+\mu\big] &= \tan\theta\big[2\tan\beta+2\mu\big]+\cancel{1}-\cancel{\tan\beta\cot\beta}-\mu\big[\cot\beta+\tan\beta\big] \\ 2(\tan\theta-\tan\phi)(\tan\beta+\mu) &= \mu(\cot\beta+\tan\beta) \\ \tan\theta-\tan\phi &= \frac{\mu(\cot\beta+\tan\beta)}{2(\tan\beta+\mu)} \end{aligned}$$

This looks almost there! Having another quick look at the required result, we need a $\sin 2\beta$ term in the denominator, so try multiplying top and bottom by $\sin\beta\cos\beta$.

$$\begin{aligned} \tan\theta-\tan\phi &= \frac{\mu(\cot\beta+\tan\beta)\times\sin\beta\cos\beta}{2\sin\beta\cos\beta(\tan\beta+\mu)} \\ &= \frac{\mu(\cos^2\beta+\sin^2\beta)}{\sin 2\beta(\mu+\tan\beta)} \\ &= \frac{\mu}{(\mu+\tan\beta)\sin 2\beta} \end{aligned}$$

as required (phew!).

Question 10

10 A light elastic spring AB, of natural length a and modulus of elasticity kmg, hangs vertically with one end A attached to a fixed point. A particle of mass m is attached to the other end B. The particle is held at rest so that $AB > a$ and is released.

Find the equation of motion of the particle and deduce that the particle oscillates vertically.

If the period of oscillation is $\dfrac{2\pi}{\Omega}$, show that $kg = a\Omega^2$.

Suppose instead that the particle, still attached to B, lies on a horizontal platform which performs simple harmonic motion vertically with amplitude b and period $\dfrac{2\pi}{\omega}$.

At the lowest point of its oscillation, the platform is a distance h below A.

Let x be the distance of the particle above the lowest point of the oscillation of the platform. When the particle is in contact with the platform, show that the upward force on the particle from the platform is

$$mg + m\Omega^2(a + x - h) + m\omega^2(b - x).$$

Given that $\omega < \Omega$, show that, if the particle remains in contact with the platform throughout its motion,

$$h \leqslant a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}.$$

Find the corresponding inequality if $\omega > \Omega$.

Hence show that, if the particle remains in contact with the platform throughout its motion, it is necessary that

$$h \leqslant a\left(1 + \frac{1}{k}\right) + b,$$

whatever the value of ω.

Examiner's report

This was the least popular question on the paper being attempted by slightly less than 8% of the candidates. It was also the least successful scoring, on average, just short of one quarter marks. Four of the five results are given in the question, and many candidates tried to work backwards, albeit in disguised manners. The first results of the question related to SHM. In many cases, candidates did not clearly choose axis or positive directions, and ended with a second order differential equation without a negative sign.

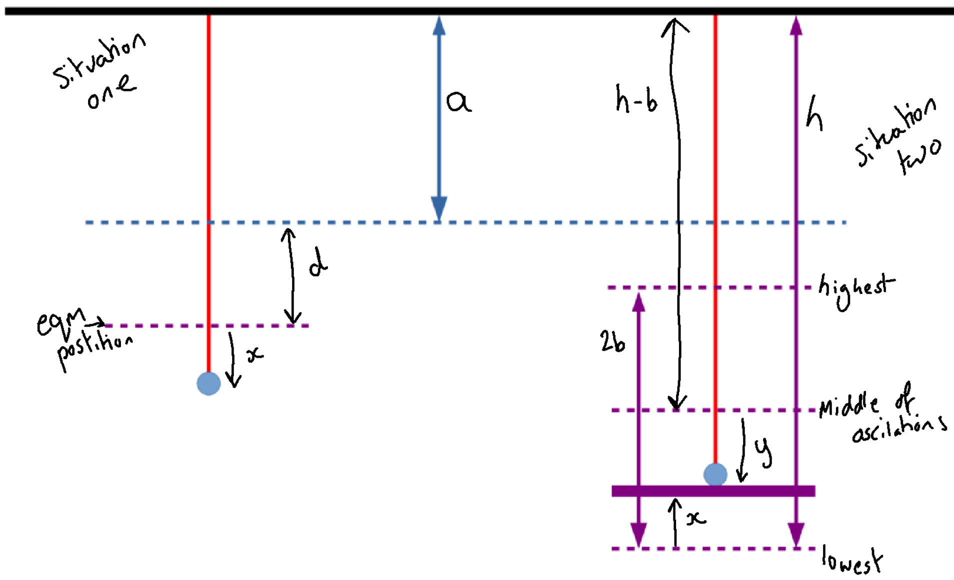
It was clear that, in the next part, some did not understand that the particle, being on the platform the whole time, would have the same acceleration as the platform; when writing the equation of motion for the particle, they often included an extra force "from the platform on the particle" equal to $m\omega^2(b - x)$, using the given result. Many also just wrote down the standard equation of motion for SHM, either without having or obtaining a $b - x$ term on the RHS.

A few attempted the next section but scored no points. They understood that $R \geqslant 0$ for the platform to remain in contact with the particle, but at no point did they mention the range for x. The last two sections were rarely attempted.

Solution

There is a lot of text in this question, and it needs to be carefully read to make sure that you have not missed anything.

In the diagram below I have tried to show both situations. I am taking the positive direction as being downwards, so in situation one the bigger (more positive) x is the more stretched the string is. Note that the definition of x changes between the two situations.



First thing to do is to find the point where the particle would be if it was at equilibrium. At this point the force due to gravity would equal the tension force in the spring. If we assume at this point that the extension in length of the spring is d then we have:

$$mg = \frac{(kmg)d}{a} \qquad (*)$$

using Hooke's Law $F = \dfrac{\lambda x}{l}$. Cancelling mg and rearranging gives $d = \dfrac{a}{k}$.

If the extension when the particle is released is x away from the equilibrium point (so the total extension is $d + x$) then we have:

$$\begin{aligned}
m\ddot{x} &= mg - \frac{kmg(d+x)}{a} \\
&= \cancel{mg} - \cancel{\frac{kmgd}{a}} - \frac{kmgx}{a} \qquad \text{using } (*) \\
&= -\frac{kmgx}{a} \\
\implies \ddot{x} &= -\left(\frac{kg}{a}\right)x
\end{aligned}$$

This is SHM[3] as it has the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{\dfrac{kg}{a}}$ and so the particle is oscillating around the equilibrium position. The period of oscillation of the general SHM equation is given by $\dfrac{2\pi}{\omega}$, and so for this case we have:

$$\begin{aligned} \Omega &= \sqrt{\frac{kg}{a}} \\ \implies a\Omega^2 &= kg \end{aligned} \tag{†}$$

In the second situation, the particle is resting on a platform which is being driven so that it is moving with SHM (I imagined it to be a platform on top of a piston which is going up and down). The forces acting on the particle now are the reaction force from the platform, and the tension in the spring and weight of the particle similarly to before.

x has now been defined to be the distance above the lowest point of the platform. Let y be the distance below the centre point between the highest and lowest positions of the platform (see diagram on previous page), so we have $y + x = b \implies y = b - x$. We are told that the platform moves under SHM with time period $\dfrac{2\pi}{\omega}$, so we have $\ddot{y} = -\omega^2 y$. The particle is at a distance $h - b + y$ below A, which means an extension of $h - b + y - a$.

When the particle is on the platform it is moving with acceleration $\ddot{y}$ i.e. the same as the platform. Let R be the reaction force between the particle and the platform (which is changing throughout the motion!). Resolving vertically (with down positive) we have:

$$m\ddot{y} = mg - \frac{kmg(h - b + y - a)}{a} - R$$

Substituting $\ddot{y} = -\omega^2 y$ and $y = b - x$ gives:

$$\begin{aligned} -m\omega(b-x)^2 &= mg - \frac{kmg(h-a-x)}{a} - R \\ \implies R &= mg - \frac{kmg(h-a-x)}{a} + m\omega(b-x)^2 \\ &= mg - \frac{kg}{a} \times m(h-a-x) + m\omega(b-x)^2 \\ &= mg + \Omega^2 \times m(a+x-h) + m\omega(b-x)^2 \end{aligned}$$

If the particle is to stay in contact throughout the motion, then we have $R \geqslant 0$ throughout the motion, i.e. for all x in the range $0 \leqslant x \leqslant 2b$. We have:

$$R = m\Big[g + \Omega^2(a-h) + \omega^2 b + x(\Omega^2 - \omega^2)\Big]$$

Since $\Omega^2 - \omega^2 > 0$, the minimum of this will be when $x = 0$, and so we have:

$$R_{\min} = m\Big[g + \Omega^2(a-h) + \omega^2 b\Big] \geqslant 0$$

[3] Simple Harmonic Motion

Hence we have:

$$\begin{aligned} g + \Omega^2(a-h) + \omega^2 b &\geqslant 0 \\ \frac{g + \Omega^2 a + \omega^2 b}{\Omega^2} &\geqslant h \\ \frac{g}{\Omega^2} + a + \frac{\omega^2 b}{\Omega^2} &\geqslant h \\ \frac{a}{k} + a + \frac{\omega^2 b}{\Omega^2} &\geqslant h \qquad \text{using } (\dagger) \\ a\left(\frac{1}{k}+1\right) + \frac{\omega^2 b}{\Omega^2} &\geqslant h \end{aligned}$$

If instead we have $\omega > \Omega$, then the minimum value of R is when $x = 2b$, as the term that varies in x is $+x(\Omega^2 - \omega^2)$ and so is negative. In this case we have:

$$R_{\text{min}} = m\Big[g + \Omega^2(a-h) + \omega^2 b + 2b(\Omega^2 - \omega^2)\Big] \geqslant 0$$

Rearranging for h gives:

$$\begin{aligned} g + \Omega^2(a-h) + \omega^2 b + 2b(\Omega^2 - \omega^2) &\geqslant 0 \\ \frac{g}{\Omega^2} + a + \frac{\omega^2 b}{\Omega^2} + 2b - 2\frac{\omega^2 b}{\Omega^2} &\geqslant h \\ a\left(\frac{1}{k}+1\right) + 2b - \frac{\omega^2 b}{\Omega^2} &\geqslant h \end{aligned}$$

If we have $\omega < \Omega$, then the condition on h can be written as:

$$\begin{aligned} h &\leqslant a\left(1+\frac{1}{k}\right) + \left(\frac{\omega^2}{\Omega^2}\right) b \\ &\leqslant a\left(1+\frac{1}{k}\right) + b \qquad \text{since } \frac{\omega^2}{\Omega^2} \leqslant 1 \end{aligned}$$

If instead we have $\omega > \Omega$ then we have:

$$\begin{aligned} h &\leqslant a\left(\frac{1}{k}+1\right) + 2b - \frac{\omega^2 b}{\Omega^2} \\ &\leqslant a\left(\frac{1}{k}+1\right) + 2b - b \qquad \text{since } \frac{\omega^2}{\Omega^2} \geqslant 1 \\ \implies h &\leqslant a\left(\frac{1}{k}+1\right) + b \end{aligned}$$

So in both the cases $\omega < \Omega$ and $\omega > \Omega$ we have:

$$h \leqslant a\left(\frac{1}{k}+1\right) + b$$

We also need to consider the case $\omega = \Omega$. Going back to our reaction equation we have:

$$R = m\Big[g + \Omega^2(a-h) + \omega^2 b + x(\Omega^2 - \omega^2)\Big]$$

If $\omega = \Omega$ then the value of R does not change throughout the motion and we have:

$$\begin{aligned} g + \Omega^2(a-h) + \omega^2 b &\geqslant 0 \\ \frac{g}{\Omega^2} + a + \frac{\omega^2 b}{\Omega^2} &\geqslant h \\ a\left(\frac{1}{k}+1\right) + b &\geqslant h \qquad \text{since } \omega = \Omega \end{aligned}$$

Hence the condition on h is the same for all values of ω.

Question 11

11 The continuous random variable X is uniformly distributed on $[a, b]$ where $0 < a < b$.

(i) Let f be a function defined for all $x \in [a, b]$

- with $\mathrm{f}(a) = b$ and $\mathrm{f}(b) = a$,
- which is strictly decreasing on $[a, b]$,
- for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all $x \in [a, b]$.

The random variable Y is defined by $Y = \mathrm{f}(X)$. Show that

$$\mathrm{P}(Y \leqslant y) = \frac{b - \mathrm{f}(y)}{b - a} \quad \text{for } y \in [a, b].$$

Find the probability density function for Y and hence show that

$$\mathrm{E}(Y^2) = -ab + \int_a^b \frac{2x\mathrm{f}(x)}{b - a}\,\mathrm{d}x.$$

(ii) The random variable Z is defined by $\dfrac{1}{Z} + \dfrac{1}{X} = \dfrac{1}{c}$ where $\dfrac{1}{c} = \dfrac{1}{a} + \dfrac{1}{b}$. By finding the variance of Z, show that

$$\ln\left(\frac{b - c}{a - c}\right) < \frac{b - a}{c}.$$

Examiner's report

Just one candidate more attempted this question than question **12**, and with 20% attempting it, it was the most popular of the applied questions. Overall, there was only moderate success with the mean score just slightly better than 40%. However, there was a wide range of attempts, and although only a few obtained full marks, there were a number of strong attempts that just dropped a few marks in passing.

The first part of the question was generally well attempted, with many candidates gaining full marks. However, some struggled with the initial justification, often by failing to properly use and justify the decreasing property of the function, whilst others were led astray by attempting to find an explicit form for the function, by attempting to sketch a graph instead of providing a proof, or by failing to notice the reversal of the inequality at all.

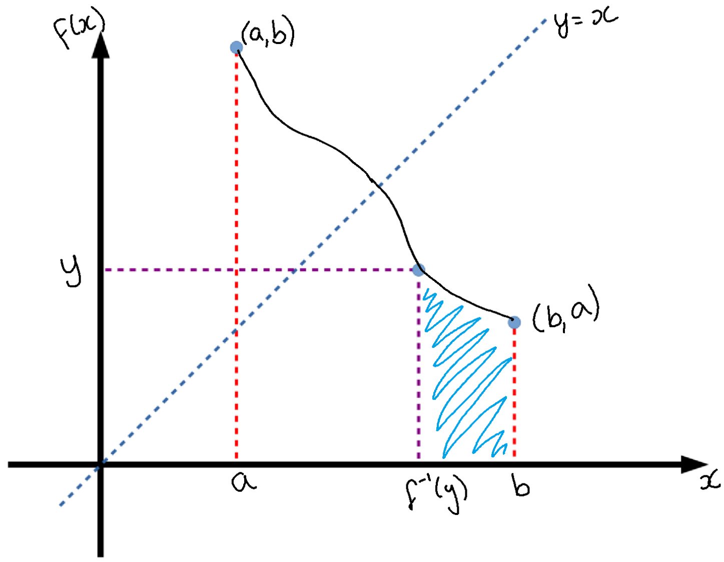
Candidates had more difficulty with the second part of the question. Some failed to justify the use of the previous part, whilst others confused $\mathrm{f}(x)$ with the pdf of Z or Y. Many candidates correctly realised that they would need to use the strict positivity of the variance, but due to algebraic errors or other issues were unable to simplify to the required result. Finally, to receive full marks, candidates needed to ensure that relevant terms were positive in order to rearrange the inequality, which many failed to do.

Solution

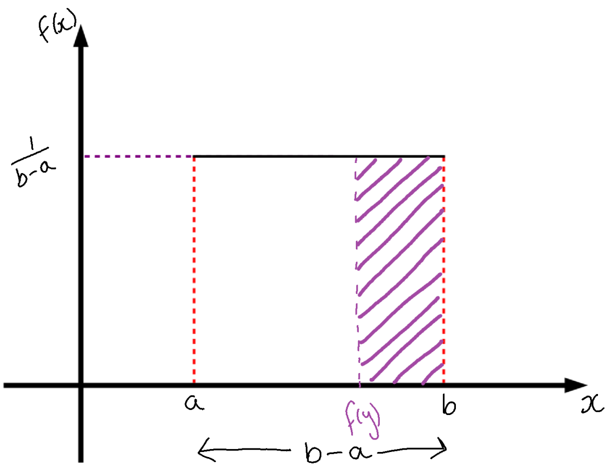
(i) We have:

$$\begin{aligned} \mathrm{P}\big(Y \leqslant y\big) &= \mathrm{P}\big(\mathrm{f}(X) \leqslant y\big) \\ &= \mathrm{P}\big(X \geqslant \mathrm{f}^{-1}(y)\big) && \text{as } \mathrm{f}(x) \text{ is strictly decreasing} \\ &= \mathrm{P}\big(X \geqslant \mathrm{f}(y)\big) && \text{as } \mathrm{f}(x) = \mathrm{f}^{-1}(x) \end{aligned}$$

This picture below shows a possible $\mathrm{f}(x)$, and if $\mathrm{f}(x) \leqslant y$ then $x \geqslant \mathrm{f}^{-1}(y)$ (the blue shaded bit).



Since X is uniformly distributed on the region $[a, b]$ then the p.d.f.[4] of X looks like this:



In order for the whole rectangle to have area 1 the height of the p.d.f. of X has to be equal to $\dfrac{1}{b-a}$. The probability that $X \geqslant \mathrm{f}(y)$ is equal to the shaded purple area so we have:

$$\mathrm{P}\big(Y \leqslant y\big) = \mathrm{P}\big(X \geqslant \mathrm{f}(y)\big) = \frac{b - \mathrm{f}(y)}{b-a}$$

[4]Probability Density Function

$\mathrm{P}\big(Y \leqslant y\big)$ is the c.d.f.[5] of Y, and we can differentiate this to find the p.d.f. of y (let this be $\mathrm{g}(y)$). We have:

$$\mathrm{g}(y) = \frac{-\mathrm{f}'(y)}{b-a}$$

We now have:

$$\begin{aligned}
\mathrm{E}(Y^2) &= \int_a^b y^2 \times \frac{-\mathrm{f}'(y)}{b-a}\,\mathrm{d}y \\
&= \left[\frac{-y^2\mathrm{f}(y)}{b-a}\right]_a^b + \int_a^b \frac{2y\mathrm{f}(y)}{b-a}\,\mathrm{d}y \\
&= -\frac{b^2\mathrm{f}(b)}{b-a} + \frac{a^2\mathrm{f}(a)}{b-a} + \int_a^b \frac{2y\mathrm{f}(y)}{b-a}\,\mathrm{d}y \\
&= -\frac{b^2 a}{b-a} + \frac{a^2 b}{b-a} + \int_a^b \frac{2y\mathrm{f}(y)}{b-a}\,\mathrm{d}y \qquad \text{as } \mathrm{f}(a)=b, \mathrm{f}(b)=a \\
&= -\frac{(b^2a - a^2b)}{b-a} + \int_a^b \frac{2y\mathrm{f}(y)}{b-a}\,\mathrm{d}y \\
&= -\frac{ab\cancel{(b-a)}}{\cancel{b-a}} + \int_a^b \frac{2y\mathrm{f}(y)}{b-a}\,\mathrm{d}y \\
&= -ab + \int_a^b \frac{2x\mathrm{f}(x)}{b-a}\,\mathrm{d}x \qquad \text{using the substitution } y = x
\end{aligned}$$

(ii) The first step is to try to work out how this part relates to the previous part. Rearranging the given relationship between Z and X gives:

$$\begin{aligned}
\frac{1}{Z} &= \frac{1}{c} - \frac{1}{X} \\
\frac{1}{Z} &= \frac{X-c}{cX} \\
Z &= \frac{cX}{X-c}
\end{aligned}$$

So we have $Z = \mathrm{f}(X)$ where $\mathrm{f}(X) = \dfrac{cX}{X-c}$. Note that $\dfrac{1}{c} = \dfrac{a+b}{ab} \implies c = \dfrac{ab}{a+b}$ which will be useful later.

If instead we rearranged to make X the subject we would get $X = \dfrac{cZ}{Z-c}$, and so we know that the function is self-inverse, i.e. $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$.

To show that $\mathrm{f}(x)$ is decreasing, we can re-write the function as:

$$\begin{aligned}
\mathrm{f}(X) &= \frac{c(X-c)+c^2}{X-c} \\
&= c + \frac{c^2}{X-c}
\end{aligned}$$

so as X increases, Z decreases.

Alternatively you could differentiate $\mathrm{f}(X)$ and show that it is always negative.

[5] Cumulative distribution function

Looking back at the original definition of Z, and substituting for c gives:

$$\frac{1}{Z}+\frac{1}{X}=\frac{1}{a}+\frac{1}{b}$$

and so when $X=a$, $Z=b$ and when $X=b$, $Z=a$.

Therefore all of the conditions for part **(i)** hold, and so we can use the results found in part **(i)** in this part.

The variance of Z is given by $\text{Var}(Z)=\text{E}(Z^2)-\big[\text{E}(Z)\big]^2$. From part **(i)** we have:

$$\begin{aligned}
\text{E}(Z^2) &= -ab+\int_a^b \frac{2x}{b-a}\times\frac{cx}{x-c}\,\mathrm{d}x\\
&= -ab+\frac{2c}{b-a}\int_a^b \frac{x^2}{x-c}\,\mathrm{d}x\\
&= -ab+\frac{2c}{b-a}\int_a^b \frac{x(x-c)+cx}{x-c}\,\mathrm{d}x\\
&= -ab+\frac{2c}{b-a}\int_a^b \frac{x(x-c)+c(x-c)+c^2}{x-c}\,\mathrm{d}x\\
&= -ab+\frac{2c}{b-a}\int_a^b x+c+\frac{c^2}{x-c}\,\mathrm{d}x\\
&= -ab+\frac{2c}{b-a}\left[\frac{x^2}{2}+cx+c^2\ln(x-c)\right]_a^b\\
&= -ab+\frac{2c}{b-a}\left[\frac{b^2-a^2}{2}+c(b-a)+c^2\big[\ln(b-c)-\ln(a-c)\big]\right]\\
&= -ab+c(b+a)+2c^2+\frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right)
\end{aligned}$$

Simplifying $c(b+a)$ gives:

$$c(b+a)=\frac{ab}{b+a}\times(b+a)=ab$$

and so we have:

$$\text{E}(Z^2)=2c^2+\frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

Using the p.d.f. from before, $g(y) = \dfrac{-f'(y)}{b-a}$, and also $f(x) = \dfrac{cx}{x-c} \implies f'(x) = \dfrac{-c^2}{(x-c)^2}$, therefore we have:

$$\begin{aligned}
E(Z) &= \int_a^b \frac{-xf'(x)}{b-a}\,dx \\
&= \frac{c^2}{b-a}\int_a^b \frac{x}{(x-c)^2}\,dx \\
&= \frac{c^2}{b-a}\int_a^b \frac{(x-c)+c}{(x-c)^2}\,dx \\
&= \frac{c^2}{b-a}\int_a^b \frac{1}{x-c} + \frac{c}{(x-c)^2}\,dx \\
&= \frac{c^2}{b-a}\left[\ln(x-c) - \frac{c}{x-c}\right]_a^b \\
&= \frac{c^2}{b-a}\left[\ln(b-c) - \ln(a-c) - \frac{c}{b-c} + \frac{c}{a-c}\right] \\
&= \frac{c^2}{b-a}\left[\ln\left(\frac{b-c}{a-c}\right) + \frac{c\big[(b-c)-(a-c)\big]}{(a-c)(b-c)}\right] \\
&= \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right) + \frac{c^2}{\cancel{(b-a)}} \times \frac{c\cancel{(b-a)}}{(a-c)(b-c)} \\
&= \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right) + \frac{c^3}{(a-c)(b-c)}
\end{aligned}$$

Since $c = \dfrac{ab}{a+b}$ we have $a - c = a - \dfrac{ab}{a+b} = \dfrac{a^2+ab-ab}{a+b} = \dfrac{a^2}{a+b}$ and similarly for $b-c$. Hence:

$$\begin{aligned}
\frac{c^3}{(a-c)(b-c)} &= c^3 \times \frac{a+b}{a^2} \times \frac{a+b}{b^2} \\
&= c^3 \times \frac{(a+b)^2}{(ab)^2} \\
&= c
\end{aligned}$$

and so we have $E(Z) = \dfrac{c^2}{b-a}\ln\left(\dfrac{b-c}{a-c}\right) + c$.

Hence:

$$\begin{aligned}
\operatorname{Var}(Z) &= E(Z^2) - \big[E(Z)\big]^2 \\
&= 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) - \left[\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right) + c\right]^2 \\
&= 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) - \left[\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right]^2 - \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) - c^2 \\
&= c^2 - \left[\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right]^2
\end{aligned}$$

We know that $\text{Var}(Z) > 0$ (as Z is not constant and variance is non-negative). Hence we have:

$$\begin{aligned}
c^2 &> \left[\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right]^2 \\
\implies c &> \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right) && \text{as both sides positive} \\
\implies c \times \frac{b-a}{c^2} &> \ln\left(\frac{b-c}{a-c}\right) && \text{as } c^2 > 0,\ b-a > 0 \\
\implies \frac{b-a}{c} &> \ln\left(\frac{b-c}{a-c}\right)
\end{aligned}$$

and so we have $\ln\left(\dfrac{b-c}{a-c}\right) < \dfrac{b-a}{c}$ as required.

Note that in this question, X is defined in the stem (the bit before parts **(i)** and **(ii)**) and so this definition holds for both parts.

Question 12

12 A and B both toss the same biased coin. The probability that the coin shows heads is p, where $0 < p < 1$, and the probability that it shows tails is $q = 1 - p$.

Let X be the number of times A tosses the coin until it shows heads. Let Y be the number of times B tosses the coin until it shows heads.

(i) The random variable S is defined by $S = X + Y$ and the random variable T is the maximum of X and Y. Find an expression for $\mathrm{P}(S = s)$ and show that

$$\mathrm{P}(T = t) = pq^{t-1}(2 - q^{t-1} - q^t).$$

(ii) The random variable U is defined by $U = |X - Y|$, and the random variable W is the minimum of X and Y. Find expressions for $\mathrm{P}(U = u)$ and $\mathrm{P}(W = w)$.

(iii) Show that $\mathrm{P}(S = 2 \text{ and } T = 3) \neq \mathrm{P}(S = 2) \times \mathrm{P}(T = 3)$.

(iv) Show that U and W are independent, and show that no other pair of the four variables S, T, U and W are independent.

Examiner's report

As well as the popularity of this question being similar to that of question **11**, the success was very similar too. It was just below question **11** with its mean score. Very few scored full marks, partly because very few recognised the need to consider the case $U = 0$ separately in parts **(ii)** and **(iv)**, and of those who did, many made mistakes in other places or forgot to also consider it in **(iv)** after correctly considering it in **(ii)**. However, the question was also rather forgiving, in the sense that it was possible to make substantial progress on the question even with errors in the earlier parts.

A common error in parts **(i)** and **(ii)** was to "double count" the case $X = Y$, when finding the distribution of T and U. It was also rather common for candidates to think that Y was the number of tosses until B got a tail (rather than a head). Many candidates identified correct counter-examples for the last part of **(iv)**, but a significant proportion failed to justify that their joint probabilities were equal to zero. There were also a number of candidates who made their lives significantly harder by injudicious choice of counterexamples; e.g. candidates who chose $S = 2$, and then $U = 0$ who then had to do much more work to prove the probabilities were not equal, than if they had made any other choice of U would give a contradiction simply and immediately.

Solution

X and Y here follow a *geometric distribution*. Explicit knowledge of this type of distribution is not necessary, and STEP questions will not use the term. You will be expected to be able to apply your knowledge of geometric series to these questions! The probability that $X = r$ is given by $\mathrm{P}(X = r) = q^{r-1}p$, as if $X = r$ then that means that the first $r - 1$ tosses were tails and the r^{th} toss was a head.

We have $\mathrm{P}(X = 1), \mathrm{P}(X = 2), \mathrm{P}(X = 3) = p, qp, q^2p$ etc, which is a geometric series (hence the name geometric distribution). There are lots of cases in this question where a finite or infinite geometric sum is found.

Throughout this question, the events X and Y are independent, so we have
$\mathrm{P}(X = x \text{ and } Y = y) = \mathrm{P}(X = x) \times \mathrm{P}(Y = y)$.

(i) The smallest value that either X or Y can take is 1, which is when either A or B gets a head on the first toss.

If $S = s$, then we can have $X = 1$ and $Y = s - 1$, or $X = 2$ and $Y = s - 2$, etc. which gives:

$$\begin{aligned}\mathrm{P}(S = s) &= \mathrm{P}(X = 1 \cap Y = s-1) + \mathrm{P}(X = 2 \cap Y = s-2) + \cdots + \mathrm{P}(X = s-1 \cap Y = 1)\\ &= \left(p \times q^{s-2}p\right) + \left(qp \times q^{s-3}p\right) + \left(q^2p \times q^{s-4}p\right) + \cdots + \left(q^{s-2}p \times p\right)\\ &= (s-1)q^{s-2}p^2\end{aligned}$$

Here I have used the "cap" notation ($\cap$) instead of writing out "and", which was a decision I made purely because the first line wouldn't fit nicely as one line otherwise. It is fine to use the words "and" and "or" when answering probability questions!

If $T = t$ then either $X = t$ and $Y < t$, or $Y = t$ and $X < t$, or both X and Y are equal to t. We have:

$$\begin{aligned}\mathrm{P}(T = t) &= P(X = t \cap Y < t) + P(X < t \cap Y = t) + P(X = t \cap Y = t)\\ &= q^{t-1}p \times \left(p + qp + q^2p + \cdots + q^{t-2}p\right)\\ &\qquad + \left(p + qp + q^2p + \cdots + q^{t-2}p\right) \times q^{t-1}p\\ &\qquad\qquad + q^{t-1}p \times q^{t-1}p\\ &= q^{t-1}p \times \frac{\cancel{p}(1 - q^{n-1})}{\cancel{1-q}} + \frac{\cancel{p}(1 - q^{t-1})}{\cancel{1-q}} \times q^{t-1}p + q^{2(t-1)}p^2\\ &= q^{t-1}p - q^{2(t-1)}p + q^{t-1}p - q^{2(t-1)}p + q^{2(t-1)}p^2\\ &= pq^{t-1}\left(1 - q^{t-1} + 1 - q^{t-1} + q^{t-1}p\right)\\ &= pq^{t-1}\left(2 - 2q^{t-1} + q^{t-1}(1-q)\right)\\ &= pq^{t-1}\left(2 - 2q^{t-1} + q^{t-1} - q^t\right)\\ &= pq^{t-1}\left(2 - q^{t-1} - q^t\right)\end{aligned}$$

You do need to be careful of double counting. If you had considered the events ($X = t$ and $Y \leqslant t$) and ($Y = t$ and $X \leqslant t$), then you would have included $X = t$ and $Y = t$ twice.

(ii) If $U = u$, then either X is u more than Y, or the other way around. If X is u more than Y then we could have $X = u + 1$ and $Y = 1$ (remember that the smallest possible value of Y is 1), or $X = u + 2$ and $Y = 2$ etc.

We have, for $u \geqslant 1$:

$$\begin{aligned}
\mathrm{P}(U = u) &= \sum_{i=1}^{\infty} \mathrm{P}(X = u + i \cap Y = i) + \sum_{i=1}^{\infty} \mathrm{P}(X = i \cap Y = u + i) \\
&= \sum_{i=1}^{\infty} \left(q^{u+i-1}p \times q^{i-1}p\right) + \sum_{i=1}^{\infty} \left(q^{i-1}p \times q^{u+i-1}p\right) \\
&= 2p^2q^u \sum_{i=1}^{\infty} q^{2i-2} \\
&= 2p^2q^u\left(1 + q^2 + q^4 + \cdots\right) \\
&= 2p^2q^u \times \frac{1}{1-q^2} \\
&= 2p^2q^u \times \frac{1}{(1-q)(1+q)} \\
&= \frac{2pq^u}{(1+q)}
\end{aligned}$$

When $u = 0$, then we have $X = Y$, and so:

$$\begin{aligned}
\mathrm{P}(U = 0) &= \sum_{i=1}^{\infty} \mathrm{P}(X = i \cap Y = i) \\
&= \sum_{i=1}^{\infty} \left(q^{i-1}p \times q^{i-1}p\right) \\
&= p^2 \sum_{i=1}^{\infty} q^{2i-2} \\
&= p^2\left(1 + q^2 + q^4 + \cdots\right) \\
&= p^2 \times \frac{1}{1-q^2} \\
&= \frac{p^2}{(1+q)(1-q)} \\
&= \frac{p}{(1+q)}
\end{aligned}$$

For W, the process is very similar to T apart from we will have infinite sums to consider. We have:

$$\begin{aligned}
\mathrm{P}(W=w) &= P(X=w\cap Y>w)+P(X>w\cap Y=w)+P(X=w\cap Y=w)\\
&= \sum_{i=w+1}^{\infty} P(X=w\cap Y=i)+\sum_{i=w+1}^{\infty} P(X=i\cap Y=w)+P(X=w\cap Y=w)\\
&= q^{w-1}p\Big(q^{w}p+q^{w+1}p+\cdots\Big)+\Big(q^{w}p+q^{w+1}p+\cdots\Big)q^{w-1}p+q^{2(w-1)}p^2\\
&= q^{w-1}p\times q^{w}p\Big(1+q+q^2+\cdots\Big)+\Big(1+q+q^2+\cdots\Big)q^{w}p\times q^{w-1}p+q^{2(w-1)}p^2\\
&= \frac{q^{2w-1}p^2}{1-q}+\frac{q^{2w-1}p^2}{1-q}+q^{2(w-1)}p^2\\
&= 2pq^{2w-1}+p^2q^{2(w-1)}\\
&= pq^{2(w-1)}\big(2q+p\big)\\
&= pq^{2(w-1)}\big(2q+(1-q)\big)\\
&= pq^{2(w-1)}\big(1+q\big)
\end{aligned}$$

(iii) The idea behind this part is that you are trying to show that S and T are not independent.

If $S=2$ and $T=3$ then this means the sum of X and Y is 2, which means that we must have $X=Y=1$, and also the maximum of X and Y is 3. It is not possible to satisfy both of these together, and so $\mathrm{P}(S=2 \text{ and } T=3)=0$.

We also have $\mathrm{P}(S=2)=(2-1)q^{2-2}p^2=p^2$ (which makes sense as $S=2 \implies X=Y=1$), and $\mathrm{P}(T=t)=pq^{3-1}\big(2-q^{3-1}-q^3\big)$. Neither of these are equal to 0, and so we have $\mathrm{P}(S=2 \text{ and } T=3)\neq \mathrm{P}(S=2)\times\mathrm{P}(T=3)$

(iv) To show that U and W are independent we need to show that $\mathrm{P}(U=u \text{ and } W=w)=\mathrm{P}(U=u)\times\mathrm{P}(W=w)$ for all possible values of u and w.

If $U=u$ and $W=w$ then that means that the minimum value of X and Y is w, and the difference between the two values is u. This means that, in the case when $u\geqslant 1$, either $X=w, Y=w+u$ or $X=w+u, Y=w$. Therefore:

$$\begin{aligned}
\mathrm{P}(U=u \text{ and } W=w) &= \mathrm{P}(X=w \text{ and } Y=w+u)+\mathrm{P}(X=w+u \text{ and } Y=w)\\
&= q^{w-1}p\times q^{w+u-1}p+q^{w+u-1}p\times q^{w-1}p\\
&= 2p^2q^{2w+u-2}
\end{aligned}$$

Using the results found in part **(ii)** we have (for $u\geqslant 1$):

$$\begin{aligned}
\mathrm{P}(U=u)\times\mathrm{P}(W=w) &= \frac{2pq^u}{1+q}\times pq^{2(w-1)}(1+q)\\
&= 2p^2q^{u+2(w-1)}
\end{aligned}$$

which is the same as $\mathrm{P}(U=u \text{ and } W=w)$.

When $u=0$ we have $X=Y=w$, and so:

$$\begin{aligned}
\mathrm{P}(U=0 \text{ and } W=w) &= \mathrm{P}(X=Y=w)\\
&= q^{w-1}p\times q^{w-1}p\\
&= p^2q^{2(w-1)}
\end{aligned}$$

We also have:

$$\begin{aligned}\mathrm{P}(U=0)\times\mathrm{P}(W=w) &= \frac{p}{1+q}\times pq^{2(w-1)}(1+q)\\ &= p^2q^{2(w-1)}\end{aligned}$$

and hence for all values of u and w we have $\mathrm{P}(U=u \text{ and } W=w)=\mathrm{P}(U=u)\times\mathrm{P}(W=w)$, and so U and W are independent.

We have shown in part **(iii)** that S and T are not independent, note that is is only necessary to find one counterexample to show this! The pairs left are $(S,U);(S,W);(T,U);(T,W)$. It can help to think about the situations in order to find a counterexample, and also remember that U has a different probability expression when $U=0$.

* (S,U): S is the sum, and U is the difference. If we consider the case $S=2, U=1$ we know that $S=2 \implies X=Y=1$, and so if $S=2$ we must have $U=0$. Hence $\mathrm{P}(S=2 \text{ and } U=1)=0$.

  Using the result found in previous parts we have $\mathrm{P}(S=2)\times\mathrm{P}(U=1)=p^2\times\dfrac{2pq}{1+q}\neq 0$, and so S and U are not independent.
* (S,W): Here S is the sum, and W is the minimum. Taking $S=2$ again, consider $W=3$. The event "$S=2$ and $W=3$" means that we want a pair of numbers so that the sum is 2, and the minimum is 3. This is not possible, so we have $\mathrm{P}(S=2 \text{ and } W=3)=0$.

  We also have $\mathrm{P}(S=2)\times\mathrm{P}(W=3)=p^2\times pq^4(1+q)\neq 0$, and so S and W are not independent.

  Notice how helpful $S=2$ has been — the choice of counterexample in part **(iii)** was supposed to help you in this part.
* (T,U): Now we are considering the maximum and difference between the values of X and Y. In a similar way to before, if we take $T=1$ then this means that we must have both $X=Y=1$. This means that $\mathrm{P}(T=1 \text{ and } U=1)=0$.

  Using the previous results, we have $\mathrm{P}(T=1)\times\mathrm{P}(U=1)=p(2-1-q)\times\dfrac{2pq}{1+q}\neq 0$. Hence T and U are not independent.
* (T,W): T is the maximum, and W is the minimum. You cannot have $T=2$ and $W=3$ (as the minimum has to be less than or equal to the maximum), so $\mathrm{P}(T=2 \text{ and } W+3)=0$. We also have $\mathrm{P}(T=2)\times\mathrm{P}(W=3)=pq(2-q-q^2)\times pq^4(1+q)\neq 0$, so T and W are not independent.

There are lots of other counterexamples you could use, but a very useful technique (hinted at by part **(ii)**) is to start with the "and" situation being impossible.