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Contents

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Homework | **Stat231**

1 General Instructions

Prove the following:

Q1. When $\{A_k\}$ is either expanding or contracting, we say that it is monotone, and for monotone sequence $\{A_k\}$, $\lim\limits_{n\to\infty} A_n$ is defined as follows:

$$\lim_{n\to\infty} A_n = \begin{cases} \bigcup\limits_{k=1}^{\infty} A_k & \text{if } \{A_k\} \text{ is expanding} \\ \bigcap\limits_{k=1}^{\infty} A_k & \text{if } \{A_k\} \text{ is contracting} \end{cases}.$$

Q2. **Monotone Sequential Continuity**. If A_k is monotone, then

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}(A_n).$$

2 Solutions

Q1

Preliminaries: Before jumping into the proof, we will setup some tools needed for proving first.

For an infinite sequence $A_1, A_2, \cdots$ one can define two events from $\lim\limits_{n\to\infty} A_n$, i.e.

$$\lim_{n\to\infty} A_n = \begin{cases} \lim\limits_{n\to\infty} \sup\limits_{k\in[n,\infty)} \{A_k\} \\ \lim\limits_{n\to\infty} \inf\limits_{k\in[n,\infty)} \{A_k\} \end{cases}. \tag{1}$$

Now the $\sup\limits_{k\in[n,\infty)}\{A_k\}$ and $\inf\limits_{k\in[n,\infty)}\{A_k\}$ are defined as follows:

$$\sup_{k\in[n,\infty)}\{A_k\} = \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \inf_{k\in[n,\infty)}\{A_k\} = \bigcap_{k=n}^{\infty} A_k.$$

Hence,

$$\lim_{n\to\infty}\sup_{k\in[n,\infty)}\{A_k\} = \lim_{n\to\infty}\bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \lim_{n\to\infty}\inf_{k\in[n,\infty)}\{A_k\} = \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k.$$

Since $\bigcup\limits_{k=n}^{\infty} A_k$ is an event that *"at least one A_k occurs"*, then $\bigcup\limits_{k=n}^{\infty} A_k$ occurs for all n. This statement simply defines the intersection, and thus we have

$$\lim_{n\to\infty}\sup_{k\in[n,\infty)}\{A_k\} = \lim_{n\to\infty}\bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k. \tag{2}$$

Now for $\bigcap\limits_{k=n}^{\infty} A_k$, it can be observed that, for every n, $\{A_k\}$ must occur for all k, $k\in[n,\infty)$. This statement is sometimes not satisfied on some n, wherein an empty set is obtained if no common values between the sequence $\{A_k\}$ are observed. Therefore the event, $\bigcap\limits_{k=n}^{\infty} A_k$ occurs for some n, that is,

$$\lim_{n\to\infty}\inf_{k\in[n,\infty)}\{A_k\} = \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k. \tag{3}$$

Finally, Equations (2) and (3) can now be equated to Equation (1),

$$\lim_{n\to\infty} A_n = \begin{cases} \lim\limits_{n\to\infty}\sup\limits_{k\in[n,\infty)}\{A_k\} = \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} A_k \\ \lim\limits_{n\to\infty}\inf\limits_{k\in[n,\infty)}\{A_k\} = \bigcup\limits_{n=1}^{\infty}\bigcap\limits_{k=n}^{\infty} A_k \end{cases}.$$

Now that we have our necessary tools, let us proceed to proving.

Proof. If the sequence $\{A_k\}$ is expanding, then the inner union $\left(\bigcup\limits_{k=n}^{\infty} A_k\right)$ in $\lim\limits_{n\to\infty}\sup\limits_{k\in[n,\infty)}\{A_k\}$ remains the same independently of n, so that $\lim\limits_{n\to\infty}\sup\limits_{k\in[n,\infty)}\{A_k\} = \bigcup\limits_{k=1}^{\infty} A_k$. On the other hand, the inner intersection $\left(\bigcap_{k=n}^{\infty} A_k\right)$ in $\lim_{n\to\infty} \inf_{k\in[n,\infty)} \{A_k\}$, is equal to A_n, implying that the limit is $\bigcup_{n=1}^{\infty} A_n$. Since both $\lim_{n\to\infty} \sup_{k\in[n,\infty)} \{A_k\}$ and $\lim_{n\to\infty} \inf_{k\in[n,\infty)} \{A_k\}$ converge to the same event, then that proves the first part of the problem.

Now if the sequence $\{A_k\}$ is contracting. Then the inner union $\left(\bigcup_{k=n}^{\infty} A_k\right)$ in $\lim_{n\to\infty} \sup_{k\in[n,\infty)} \{A_k\}$ is equal to A_n, so that $\lim_{n\to\infty} \sup_{k\in[n,\infty)} \{A_k\} = \bigcap_{n=1}^{\infty} A_n$. On the other hand, the inner intersection $\left(\bigcap_{k=n}^{\infty} A_k\right)$ in $\lim_{n\to\infty} \inf_{k\in[n,\infty)} \{A_k\}$, remains the same independently of n, and thus the limit is $\bigcap_{k=1}^{\infty} A_k$. Since both $\lim_{n\to\infty} \sup_{k\in[n,\infty)} \{A_k\}$ and $\lim_{n\to\infty} \inf_{k\in[n,\infty)} \{A_k\}$ converge to the same event, then that proves the second part of the problem. $\square$

Q2

Proof. If $\{A_k\}$ is monotone, then

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \begin{cases} \mathrm{P}\left(\bigcup_{k=1}^{\infty} A_k\right) & \text{if } \{A_k\} \text{ is expanding} \\ \mathrm{P}\left(\bigcap_{k=1}^{\infty} A_k\right) & \text{if } \{A_k\} \text{ is contracting} \end{cases}.$$

So if A_k is expanding, then we can write $\bigcup_{k=1}^{\infty} A_k$ as disjoint unions,

$$\begin{aligned} \bigcup_{k=1}^{\infty} A_k &= A_1 \cup (A_2 \cap A_1^c) \cup (A_3 \cap A_2^c) \cup \cdots \\ &= A_1 \cup (A_2 \backslash A_1) \cup (A_3 \backslash A_2) \cup \cdots \end{aligned}$$

Then it follows that

$$\begin{aligned} \mathrm{P}\left(\bigcup_{k=1}^{\infty} A_k\right) &= \mathrm{P}(A_1) + \mathrm{P}\left[\sum_{k=2}^{\infty} (A_k - A_{k-1})\right] \\ &= \mathrm{P}(A_1) + \mathrm{P}\left[\lim_{n\to\infty} \sum_{k=2}^{n} (A_k - A_{k-1})\right] \\ &= \mathrm{P}(A_1) + \lim_{n\to\infty} \mathrm{P}\left[\sum_{k=2}^{n} (A_k - A_{k-1})\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{P}(A_1) + \lim_{n\to\infty} [(\mathrm{P}(A_2) - \mathrm{P}(A_1)) + (\mathrm{P}(A_3) - \mathrm{P}(A_2)) + \cdots \\
&\quad + (\mathrm{P}(A_n) - \mathrm{P}(A_{n-1}))] \\
&= \mathrm{P}(A_1) + \lim_{n\to\infty} [\mathrm{P}(A_n) - \mathrm{P}(A_1)] \\
&= \mathrm{P}(A_1) - \mathrm{P}(A_1) + \lim_{n\to\infty} \mathrm{P}(A_n) = \lim_{n\to\infty} \mathrm{P}(A_n)
\end{aligned}
$$

On the other hand, when A_k is contracting then

$$
\begin{aligned}
\mathrm{P}\left(\bigcap_{k=1}^{\infty} A_k\right) &= \mathrm{P}(A_1) \cap \mathrm{P}(A_2) \cap \mathrm{P}(A_3) \cap \cdots \\
&= \lim_{n\to\infty} \bigcap_{k=1}^{n} \mathrm{P}(A_k) \\
&= \lim_{n\to\infty} \mathrm{P}(A_n), \quad \text{since } A_k \text{ is contracting.}
\end{aligned}
$$

And that proves the problem. $\square$