School of Statistics, University of the Philippines (Diliman)
Linangan ng Estadistika, Unibersidad ng Pilipinas (Diliman)

14 of April 2015

Al-Ahmadgaid B. Asaad
alstated@gmail.com

Problem Set 3 | **Stat232**

7.22 This exercise will prove the assertions in Example 7.2.16, and more. Let $X_1, \cdots, X_n$ be a random sample from a $\text{n}(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $\text{n}(\mu, \tau^2)$. Here we assume that σ^2, μ and τ^2 are all known.

(a) Find the joint pdf of $\bar{X}$ and θ.

(b) Show that $m(\bar{x}|\sigma^2, \mu, \tau^2)$, the marginal distribution of $\bar{X}$, is $n(\mu, (\sigma^2/n) + \tau^2)$.

(c) Show that $\pi(\theta|\bar{x}, \sigma^2, \mu, \tau^2)$ the posterior distribution of θ, is normal with means and variance given by Equation (7.2.10).

Solution

(a) The distribution of $\bar{X}$, assuming X_i's are iid, is obtain using the mgf technique, that is

$$\begin{aligned}
M_{\frac{1}{n}\sum X_i}(t) &= \text{E}e^{\frac{t}{n}\sum X_i} = \text{E}e^{\frac{t}{n}X_1} \cdots \text{E}e^{\frac{t}{n}X_n} \\
&= \exp\left[\theta\frac{t}{n} + \frac{\sigma^2}{2}\left(\frac{t}{n}\right)^2\right] \cdots \exp\left[\theta\frac{t}{n} + \frac{\sigma^2}{2}\left(\frac{t}{n}\right)^2\right] \\
&= \exp\left[\theta\frac{t}{n} + \sigma^2\frac{t^2}{2n^2}\right]^n = \exp\left[\theta t + \frac{\sigma^2}{n}\frac{t^2}{2}\right] \\
&= \text{mgf of n}(\theta, \sigma^2/n)
\end{aligned}$$

Let $Y = \bar{X}$, then the joint density of Y and θ is,

$$\begin{aligned}
f(y, \theta) &= f(y|\theta)\pi(\theta) \\
&= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left[-n\frac{(y-\theta)^2}{2\sigma^2}\right] \frac{1}{\sqrt{2\pi}\tau} \exp\left[-\frac{(\theta-\mu)^2}{2\tau^2}\right] \\
&= \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\left[\frac{n}{\sigma^2}(y-\theta)^2 + \frac{(\theta-\mu)^2}{\tau^2}\right]\right\}.
\end{aligned}$$

(b) From part (a), the marginal distribution of $Y = \bar{X}$ is,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y,\theta)\,\mathrm{d}\,\theta &= \int_{-\infty}^{\infty} \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\left[\frac{n}{\sigma^2}(y-\theta)^2 + \frac{(\theta-\mu)^2}{\tau^2}\right]\right\}\mathrm{d}\,\theta \\
&= \int_{-\infty}^{\infty} \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\left[\frac{ny^2}{\sigma^2} - \frac{2y\theta n}{\sigma^2} + \frac{n\theta^2}{\sigma^2} + \right.\right. \\
&\qquad\qquad \left.\left. \frac{\theta^2}{\tau^2} - \frac{2\theta\mu}{\tau^2} + \frac{\mu^2}{\tau^2}\right]\right\}\mathrm{d}\,\theta \\
&= \frac{\sqrt{n}}{2\pi\sigma\tau} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\theta^2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right) + \right.\right. \\
&\qquad\qquad \left.\left.\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)\right]\right\}\mathrm{d}\,\theta. \qquad (1)
\end{aligned}$$

To simplify this, consider the integral of the function below over the reals of θ,

$$\int_{-\infty}^{\infty} \exp\left[-a\theta^2 + b\theta + c\right]\mathrm{d}\,\theta = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2}{4a}+c\right].$$

Above identity is true according to Wikipedia (Topic: Gaussian Integral). Hence for our case, we can write the integrand of Equation (1) into the following simple form,

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(a\theta^2 - 2b\theta + c)\right] &= \int_{-\infty}^{\infty} \exp\left[-\frac{a\theta^2}{2} + b\theta - \frac{c}{2}\right] \\
&= \sqrt{\frac{\pi}{\frac{a}{2}}}\exp\left[\frac{b^2}{\frac{4a}{2}} - \frac{c}{2}\right] \qquad (2) \\
&= \sqrt{\frac{2\pi}{a}}\exp\left[-\frac{1}{2}\left(c - \frac{b^2}{a}\right)\right].
\end{aligned}$$

Thus, substituting $a = \left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right), b = \left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right), c = \left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y,\theta)\,\mathrm{d}\,\theta &= \frac{\sqrt{n}}{2\pi\sigma\tau}\sqrt{\frac{2\pi}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}\exp\left\{-\frac{1}{2}\left[\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right) - \frac{\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]\right\} \\
&= \frac{\sqrt{2\pi n}}{2\pi\sigma\tau\sqrt{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}\exp\left\{-\frac{1}{2}\left[\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right) - \frac{\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]\right\}
\end{aligned}$$

(c) Let $Y = \bar{X}$,

$$
\begin{aligned}
\pi(\theta|y,\sigma^2,\mu,\tau^2) &= \frac{f(y,\theta)}{m(y|\sigma^2,\mu,\tau^2)} \\
&= \frac{\frac{\sqrt{n}}{2\pi\sigma\tau}\exp\left\{-\frac{1}{2}\left[\frac{n}{\sigma^2}(y-\theta)^2+\frac{(\theta-\mu)^2}{\tau^2}\right]\right\}}{\frac{\sqrt{2\pi n}}{2\pi\sigma\tau\sqrt{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}\exp\left\{-\frac{1}{2}\left[\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)-\frac{\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]\right\}} \\
&= \frac{\exp\left\{-\frac{1}{2}\left[\theta^2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)-2\theta\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)+\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)\right]\right\}}{\sqrt{\frac{2\pi}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}\exp\left\{-\frac{1}{2}\left[\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)-\frac{\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]\right\}} \\
&= \frac{\exp\left[-\frac{\theta^2}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)+\theta\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)-\frac{1}{2}\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)\right]}{\sqrt{\frac{2\pi}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}\exp\left[-\frac{1}{2}\left(\frac{ny^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)+\frac{1}{2}\frac{\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]} \\
&= \frac{\exp\left[-\frac{\theta^2}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)+\theta\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)-\frac{1}{2}\frac{\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]}{\sqrt{\frac{2\pi}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}} \\
&= \frac{1}{\sqrt{\frac{2\pi}{\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}}\exp\left\{-\frac{\left[\theta\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)-\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)\right]^2}{2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}\right\},
\end{aligned}
$$

which is a normal density with mean, $\left(\frac{yn}{\sigma^2}+\frac{\mu}{\tau^2}\right)$, and variance, $\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)$.

7.24 Let $X_1,\cdots,X_n$ be iid Poisson(λ), and let λ have a gamma(α,β) distribution, the conjugate family for the Poisson.

(a) Find the posterior distribution of λ.

(b) Calculate the posterior mean and variance.

Solution

(a) The posterior distribution of λ is given by,

$$\pi(\lambda|\mathbf{x}) = \frac{f(\mathbf{x}|\lambda)\pi(\lambda)}{m(\mathbf{x})}, \tag{3}$$

where the joint density of the sample is obtain using the mgf technique, that is

$$\begin{aligned}
M_{\sum_{i=1}^n X_i}(t) &= \mathrm{E}e^{t\sum_{i=1}^n X_i} = \mathrm{E}e^{tX_1}\cdots\mathrm{E}e^{tX_n} \\
&= \exp[\lambda(e^t-1)]^n = \exp[n\lambda(e^t-1)] \\
&= \text{mgf of Poisson}(n\lambda).
\end{aligned}$$

So that, if $Y = \sum_{i=1}^n X_i$ then,

$$f(y|\lambda) = \frac{\exp[-n\lambda](n\lambda)^y}{y!},$$

and the joint density of Y and λ is,

$$f(y,\lambda) = f(y|\lambda)\pi(\lambda) = \frac{\exp[-n\lambda](n\lambda)^y}{y!} \times \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}\exp[-\lambda/\beta].$$

Implies that the marginal distribution of λ is,

$$\begin{aligned}
m(\mathbf{x}) = \int_0^\infty f(y,\lambda)\,\mathrm{d}\,\lambda &= \int_0^\infty \frac{\exp[-n\lambda](n\lambda)^y}{y!} \times \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}\exp[-\lambda/\beta]\,\mathrm{d}\,\lambda \\
&= \frac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\int_0^\infty \exp[-n\lambda-\lambda/\beta]\lambda^{y+\alpha-1}\,\mathrm{d}\,\lambda \\
&= \frac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\int_0^\infty \exp\left[\frac{-\lambda(n\beta+1)}{\beta}\right]\lambda^{y+\alpha-1}\,\mathrm{d}\,\lambda \\
&= \frac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\int_0^\infty \lambda^{(y+\alpha)-1}\exp\left[\frac{-\lambda}{\frac{\beta}{(n\beta+1)}}\right]\mathrm{d}\,\lambda.
\end{aligned}$$

Notice the integrand is the kernel of gamma$\left(y+\alpha, \frac{\beta}{n\beta+1}\right)$. Thus,

$$m(\mathbf{x}) = \frac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\Gamma(y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}.$$

Substituting this to Equation (3),

$$
\begin{aligned}
\pi(\lambda|\mathbf{x}) &= \frac{\dfrac{\exp[-n\lambda](n\lambda)^y}{y!} \times \dfrac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}\exp[-\lambda/\beta]}{\dfrac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\Gamma(y+\alpha)\left(\dfrac{\beta}{n\beta+1}\right)^{y+\alpha}} \\
&= \frac{\dfrac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\lambda^{(y+\alpha)-1}\exp\left[\dfrac{-\lambda}{\frac{\beta}{(n\beta+1)}}\right]}{\dfrac{n^y}{\Gamma(\alpha)\beta^\alpha y!}\Gamma(y+\alpha)\left(\dfrac{\beta}{n\beta+1}\right)^{y+\alpha}} \\
&= \frac{\lambda^{(y+\alpha)-1}\exp\left[\dfrac{-\lambda}{\frac{\beta}{(n\beta+1)}}\right]}{\Gamma(y+\alpha)\left(\dfrac{\beta}{n\beta+1}\right)^{y+\alpha}} \\
&= \text{density of gamma}\left(y+\alpha, \frac{\beta}{n\beta+1}\right).
\end{aligned}
$$

(b) From part (a) the posterior mean and variance is,

$$
\begin{aligned}
\mathrm{E}(\lambda|\mathrm{x}) &= \frac{\beta(y+\alpha)}{n\beta+1}, \\
\mathrm{Var}(\lambda|\mathrm{x}) &= (y+\alpha)\left[\frac{\beta}{n\beta+1}\right]^2.
\end{aligned}
$$

7.25 We examine a generalization of the heirarchical (Bayes) model considered in Example 7.2.16 and Exercise 7.22. Suppose that we observe $X_1, \cdots, X_n$, where

$$
\begin{aligned}
X_i|\theta_i &\sim \mathrm{n}(\theta_i, \sigma^2), \quad i = 1, \cdots, n, \quad \text{independent}, \\
\theta_i &\sim \mathrm{n}(\mu, \tau^2), \quad i = 1, \cdots, n, \quad \text{independent}.
\end{aligned}
$$

(a) Show that the marginal distribution of X_i is $\mathrm{n}(\mu, \sigma^2+\tau^2)$ and that, marginally, $X_1, \cdots, X_n$ are iid. (*Empirical Bayes analysis would use the marginal distribution of the* X_is to estimate the prior parameters μ and τ^2. See Miscellanea 7.5.6)

(b) Show, in general, that if

$$
\begin{aligned}
X_i|\theta_i &\sim f(x|\theta_i), \quad i = 1, \cdots, n, \quad \text{independent}, \\
\theta_i &\sim \pi(\theta, \tau), \quad i = 1, \cdots, n, \quad \text{independent},
\end{aligned}
$$

then marginally, $X_1, \cdots, X_n$ are iid.

Solution

(a) For a particular i, the marginal distribution of X_i is,

$$
\begin{aligned}
m(x|\mu,\sigma^2,\tau^2) &= \int_{-\infty}^{\infty} f(x|\theta,\sigma^2)\pi(\theta|\tau^2)\,\mathrm{d}\,\theta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right]\frac{1}{\sqrt{2\pi\tau^2}}\exp\left[-\frac{(\theta-\mu)^2}{2\tau^2}\right]\mathrm{d}\,\theta \\
&= \frac{1}{2\pi\sigma\tau}\int_{-\infty}^{\infty}\exp\left[-\frac{(x^2-2x\theta+\theta^2)}{2\sigma^2}-\frac{(\theta^2-2\theta\mu+\mu^2)}{2\tau^2}\right]\mathrm{d}\,\theta \\
&= \frac{1}{2\pi\sigma\tau}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left[\frac{x^2}{\sigma^2}-\frac{2x\theta}{\sigma^2}+\frac{\theta^2}{\sigma^2}+\frac{\theta^2}{\tau^2}-\frac{2\theta\mu}{\tau^2}+\frac{\mu^2}{\tau^2}\right]\right\}\mathrm{d}\,\theta \\
&= \frac{1}{2\pi\sigma\tau}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left[\theta^2\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)-2\theta\left(\frac{x}{\sigma^2}+\frac{\mu}{\tau^2}\right)+\right.\right. \\
&\qquad\qquad\left.\left.\left(\frac{x^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)\right]\right\}\mathrm{d}\,\theta.
\end{aligned}
$$

To simplify this, we use again Equation (2), where in this case $a = \left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)$, $b = \left(\frac{x}{\sigma^2}+\frac{\mu}{\tau^2}\right)$, and $c = \left(\frac{x^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)$.

$$
\begin{aligned}
m(x|\mu,\sigma^2,\tau^2) &= \sqrt{\frac{2\pi}{a}}\exp\left[-\frac{1}{2}\left(c-\frac{b^2}{a}\right)\right] \\
&= \sqrt{\frac{2\pi}{\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)}}\exp\left\{-\frac{1}{2}\left[\left(\frac{x^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)-\frac{\left(\frac{x}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)}\right]\right\}.
\end{aligned}
$$

(b) For a particular observation of X and θ, the marginal pdf of X is,

$$m(x|\tau) = \int_{-\infty}^{\infty} f(x|\theta)\pi(\theta|\tau)\,\mathrm{d}\,\theta.$$

Thus the marginal pdf of the joint pdf of the random vector $\mathbf{X} = (X_1,\cdots,X_n)$ and $\boldsymbol{\theta} = (\theta_1,\cdots,\theta_n)$ is

$$
\begin{aligned}
m(\mathbf{x}|\tau) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{n} f(x_i|\theta_i)\pi(\theta_i|\tau)\,\mathrm{d}\,\theta_1\cdots\mathrm{d}\,\theta_n. \\
&= \int_{-\infty}^{\infty}\cdots\left\{\int_{-\infty}^{\infty} f(x_1|\theta_1)\pi(\theta_1|\tau)\,\mathrm{d}\,\theta_1\right\}\prod_{i=2}^{n} f(x_i|\theta_i)\pi(\theta_i|\tau)\,\mathrm{d}\,\theta_2\cdots\mathrm{d}\,\theta_n.
\end{aligned}
$$

Observe that we can factor the integrals into a product of the marginal pdf of x_is. That is the first factor that we can evaluate above is the one inside the braces, which is the marginal of x_1. Applying this procedure to the remaining $n-1$ x_is, we can simplify this into the following equation

$$m(\mathbf{x}|\tau) = \prod_{i=1}^{n} m(x_i|\tau),$$

implying that marginally $X_1, \cdots, X_n$ are independent and identically distributed.

7.38 For each of the following distribution, let $X_1, \cdots, X_n$ be a random sample. Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao Lower Bound? If so, find it. If not, show why not.

(a) $f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0$

(b) $f(x|\theta) = \frac{\log(\theta)}{\theta-1}\theta^x, \quad 0 < x < 1, \quad \theta > 1$

Solution

(a) To answer the question, we use the CRLB Attainment Corollary, that is

$$\frac{\partial}{\partial\theta}\ell(\theta|\mathbf{x}) = \frac{\partial}{\partial\theta}\log\left(\prod_{i=1}^n \theta x_i^{\theta-1}\right) = \frac{\partial}{\partial\theta}\left[n\log\theta + (\theta-1)\sum_{i=1}^n \log x_i\right]$$
$$= \frac{n}{\theta} + \sum_{i=1}^n \log x_i = -n\left(-\sum_{i=1}^n \frac{\log x_i}{n} - \frac{1}{\theta}\right)$$

Therefore, the function of θ, $g(\theta) = \frac{1}{\theta}$, with unbiased estimator $-\sum_{i=1}^n \frac{\log x_i}{n}$ attains the CRLB, implies that the estimator is the UMVUE of population parameter $\frac{1}{\theta}$.

(b) Using CRLB Attainment Corollary, we have

$$\begin{aligned}
\frac{\partial}{\partial\theta}\ell(\theta|\mathbf{x}) &= \log\left(\prod_{i=1}^n \frac{\log\theta}{\theta-1}\theta^{x_i}\right) \\
&= \frac{\partial}{\partial\theta}\left[n\log(\log\theta) - n\log(\theta-1) + \log\theta\sum_{i=1}^n x_i\right] \\
&= \frac{n}{\log\theta}\frac{1}{\theta} - \frac{n}{\theta-1} + \frac{1}{\theta}\sum_{i=1}^n x_i \\
&= \frac{n}{\theta}\left[\frac{1}{n}\sum_{i=1}^n x_i - \left(\frac{\theta}{\theta-1} - \frac{n}{n\log\theta}\right)\right]
\end{aligned}$$

Thus we can write $a(\theta) = \frac{n}{\theta}$ and the CRLB of the estimator $\bar{X}$ is attainable, hence the statistic is the UMVUE of the population parameter $\frac{\theta}{\theta-1} - \frac{n}{n\log\theta}$.

7.44 Let $X_1, \cdots, X_n$ be iid $\mathrm{n}(\theta, 1)$. Show that the best unbiased estimator of θ^2 is $\bar{X}^2 - (1/n)$. Calculate its variance(use Stein's Identity from Section 3.6), and show that it is greater than the Cramér-Rao Lower Bound.

Solution

Since $X_i \sim \mathrm{n}(\theta, 1)$, which is an exponential family model. Then by Example 3.4.4, Theorem 6.2.10, and Theorem 6.2.25, it should be clear that $\sum_{i=1}^n X_i$ is a complete sufficient statistic for θ. And because $\bar{X}^2 - 1/n$ is a function of $\sum_{i=1}^n X_i$, then by Theorem 7.3.23, $\bar{X}^2 - 1/n$ is the unique best unbiased estimator of θ^2. Using mgf technique like what we did in Exercise 7.22, $\bar{X} \sim \mathrm{n}(\theta, 1/n)$, so that

$$\mathrm{E}\left[\bar{X}^2 - \frac{1}{n}\right] = \mathrm{E}\bar{X}^2 - \frac{1}{n},$$

where

$$\mathrm{E}\bar{X}^2 = \mathrm{Var}\bar{X} + \left(\mathrm{E}\bar{X}\right)^2 = \frac{1}{n} + \theta^2.$$

So that,

$$\mathrm{E}\left[\bar{X}^2 - \frac{1}{n}\right] = \frac{1}{n} + \theta^2 - \frac{1}{n} = \theta^2,$$

implies that $\bar{X}^2 - 1/n$ is the UMVUE of θ^2. Next we calculate the variance,

$$\mathrm{Var}[\bar{X}^2 - 1/n] = \mathrm{Var}\bar{X}^2 = \mathrm{E}\bar{X}^4 - (\mathrm{E}\bar{X}^2)^2,$$

where,

$$\begin{aligned}
\mathrm{E}\bar{X}^4 &= \mathrm{E}[\bar{X}^3(\bar{X} - \theta + \theta)] = \mathrm{E}[\bar{X}^3(\bar{X} - \theta)] + \mathrm{E}\bar{X}^3\theta \\
&= \frac{3}{n}\mathrm{E}\bar{X}^2 + \mathrm{E}\bar{X}^3\theta = \frac{3}{n^2} + \frac{3\theta^2}{n} + \mathrm{E}\bar{X}^3\theta, \quad \text{by Stein's Lemma} \\
\mathrm{E}\bar{X}^3 &= \mathrm{E}[\bar{X}^2(\bar{X} - \theta + \theta)] = \mathrm{E}[\bar{X}^2(\bar{X} - \theta)] + \mathrm{E}\bar{X}^2\theta \\
&= \frac{2\theta}{n} + \frac{\theta}{n} + \theta^3
\end{aligned}$$

So that,

$$\mathrm{E}\bar{X}^4 = \frac{3}{n^2} + \frac{3\theta^2}{n} + \frac{3\theta^2}{n} + \theta^4 = \frac{3}{n^2} + \frac{6\theta^2}{n} + \theta^4.$$

Substituting this,

$$\begin{aligned}
\mathrm{Var}\bar{X}^2 &= \frac{3}{n^2} + \frac{6\theta^2}{n} + \theta^4 - \left(\frac{1}{n} + \theta^2\right)^2 \\
&= \frac{3}{n^2} + \frac{6\theta^2}{n} + \theta^4 - \frac{1}{n^2} - \frac{2\theta^2}{n} - \theta^4 \\
&= \frac{2}{n^2} + \frac{4\theta^2}{n}
\end{aligned}$$

Now we compare this with the CRLB of the estimator, which is obtain as follows:

$$\begin{aligned}
\text{CRLB} &= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{E}\bar{X}^2\right)^2}{n\mathrm{E}\left\{\left[\frac{\partial}{\partial\theta}\log f(X|\theta)\right]^2\right\}} = \frac{4\theta^2}{n\mathrm{E}\left\{\left[\frac{\partial}{\partial\theta}\log\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-\theta)^2}{2}\right)\right]^2\right\}} \\
&= \frac{4\theta^2}{n\mathrm{E}\left\{\left[\frac{\partial}{\partial\theta}\left(-\log\sqrt{2\pi}-\frac{(x-\theta)^2}{2}\right)\right]^2\right\}} = \frac{4\theta^2}{n\mathrm{E}\left\{\left[-(x-\theta)(-1)\right]^2\right\}} \\
&= \frac{4\theta^2}{n\mathrm{Var}X} = \frac{4\theta^2}{n}.
\end{aligned}$$

Equating this with the computed variance of the estimator, we have

$$\mathrm{Var}\bar{X}^2 = \frac{2}{n^2} + \frac{4\theta^2}{n} > \frac{4\theta^2}{n} = \text{CRLB}$$

Q.E.D.

7.52 Let $X_1, \cdots, X_n$ be iid Poisson(λ), and let $\bar{X}$ and S^2 denote the sample mean and variance, respectively. We now complete Example 7.3.8 in a different way. There we used the Cramér-Rao Bound, now we use completeness.

(a) Prove that $\bar{X}$ is the best unbiased estimator λ without using the Cramér-Rao Theorem.

(b) Prove the rather remarkable identity $\mathrm{E}(S^2|\bar{X}) = \bar{X}$, and use it to explicitly demonstrate that $\mathrm{Var}S^2 > \mathrm{Var}\bar{X}$.

(c) Using completeness, can a general theorem be formulated for which the identity in part (b) is a special case?

Solution

(a)

$$\mathrm{E}\bar{X} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}X_i = \frac{1}{n}\sum_{i=1}^{n}\lambda = \lambda$$

So $\bar{X}$ is an unbiased estimator of λ. Now consider the density of the sample,

$$\frac{\exp(-\lambda)\lambda^x}{x!} = \frac{1}{x!}\exp(-\lambda)\exp(x\log\lambda)$$

where $h(x) = \frac{1}{x!}, c(\lambda) = \exp(-\lambda), w(\lambda) = \log\lambda, t(x) = x$, tells us that this is an exponential family model. Then by Theorem 6.2.10 and Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic, also since $\bar{X}$ is an estimator based only on $T(\mathbf{X})$, then by Theorem 7.3.23, $\bar{X}$ is the unique best unbiased estimator of λ.

7.66 (a) Show that the MLE of θ^2, $\left(\sum_{i=1}^{n} X_i/n\right)^2$, is biased estimator of θ^2.

(b) Derive the one-step jackknife estimator based on the MLE.

(c) Show that the one-step jackknife estimator is an unbiased estimator of θ^2. (In general, jackknifing only reduces bias. In special case, however, it removes it entirely.)

(d) Is this jackknife estimator the best unbiased estimator of θ^2? If so, prove it. If not, find the best unbased estimator.

Solution

(a) For $\hat{\theta}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$, if we let $Y = \sum_{i=1}^{n} X_i$, then $Y \sim \text{Binomial}(n, \theta)$ since $X_i \sim \text{Bernoulli}(\theta)$. Then

$$\begin{aligned}
\mathrm{E}\hat{\theta}^2 &= \frac{1}{n^2}\mathrm{E}Y^2 = \frac{1}{n^2}\left[\mathrm{Var}Y + (\mathrm{E}Y)^2\right] = \frac{1}{n^2}\left[n\theta(1-\theta) + n^2\theta^2\right] \\
&= \frac{\theta(1-\theta)}{n} + \theta^2 = \frac{\theta}{n} - \frac{\theta^2}{n} + \theta^2,
\end{aligned}$$

which is what we wanted to show.

(b) The MLE is defined as $\hat{\theta} = \frac{1}{n^2}Y^2$, where $Y = \sum_{i=1}^{n} X_i$. And since $X_i \sim \text{Bernoulli}(\theta)$, then the support of X_i is either 0 or 1. Thus, if $T_n^{(j)} = \frac{1}{n-1}\sum_{i\neq j}^{n} X_i$, implying that we compute the statistic using the $n-1$ observations by dropping out the jth observation, whose value is either 0 or 1, then for each j there will be two factors:

Factor 1 For $Y = \sum_{i=1}^{n} X_i$ values,

$$T_n^{(j)} = \frac{(Y-1)^2}{(n-1)^2},$$

since the dropped out jth observation is 1; and,

Factor 2 For the remaining, $n - Y = n - \sum_{i}^{n} X_i$ values,

$$T_n^{(j)} = \frac{(Y)^2}{(n-1)^2},$$

since the dropped out jth observation is 0.

So that the jackknife estimator of θ is given by,

$$\mathrm{JK}(T_n) = n\frac{Y^2}{n^2} - \frac{n-1}{n}\left(Y\frac{(Y-1)^2}{(n-1)^2} + (n-Y)\frac{Y^2}{(n-1)^2}\right) = \frac{Y^2 - Y}{n(n-1)}$$

(c) The expected value of jackknife estimator of θ is,

$$\begin{aligned}
\mathrm{EJK}(T_n) &= \frac{1}{n(n-1)}\left(\mathrm{E}Y^2 - \mathrm{E}Y\right) \\
&= \frac{1}{n(n-1)}\left(n\theta(1-\theta) + n^2\theta^2 - n\theta\right) = \theta^2
\end{aligned}$$