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Problem Set 4 | **Stat232**

8.6 Suppose that we have two independent random samples $X_1, \cdots, X_n$ are exponential(θ), and $Y_1, \cdots, Y_m$ are exponential(μ).

(a) Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}.$$

(c) Find the distribution of T when H_0 is true.

Solution

(a) The Likelihood Ratio Test is given by

$$\lambda(\mathbf{x}, \mathbf{y}) = \frac{\sup\limits_{\theta=\mu} \mathrm{P}(\mathbf{x}, \mathbf{y}|\theta, \mu)}{\sup\limits_{\theta>0,\mu>0} \mathrm{P}(\mathbf{x}, \mathbf{y}|\theta, \mu)},$$

where the denominator is evaluated as follows:

$$\sup_{\theta>0,\mu>0} \mathrm{P}(\mathbf{x}, \mathbf{y}|\theta, \mu) = \sup_{\theta>0} \mathrm{P}(\mathbf{x}|\theta) \sup_{\mu>0} \mathrm{P}(\mathbf{y}|\mu), \quad \text{by independence.}$$

So that,

$$\sup_{\theta>0} \mathrm{P}(\mathbf{x}|\theta) = \sup_{\theta>0} \prod_{i=1}^{n} \frac{1}{\theta} \exp\left[-\frac{x_i}{\theta}\right] = \sup_{\theta>0} \frac{1}{\theta^n} \exp\left[-\frac{\sum_{i=1}^n x_i}{\theta}\right]$$
$$= \frac{1}{\bar{x}^n} \exp\left[-\frac{\sum_{i=1}^n x_i}{\bar{x}}\right] = \frac{1}{\bar{x}^n} \exp[-n],$$

since $\bar{x}$, or the sample mean is the MLE of θ. Also,

$$\sup_{\mu>0} \mathrm{P}(\mathbf{y}|\mu) = \sup_{\mu>0} \prod_{j=1}^{m} \frac{1}{\mu} \exp\left[-\frac{y_j}{\mu}\right] = \sup_{\mu>0} \frac{1}{\mu^m} \exp\left[-\frac{\sum_{j=1}^m y_j}{\mu}\right]$$
$$= \frac{1}{\bar{y}^m} \exp\left[-\frac{\sum_{j=1}^m y_j}{\bar{y}}\right] = \frac{1}{\bar{y}^m} \exp[-m].$$

Now the numerator is evaluated as follows,

$$\sup_{\theta=\mu} \mathrm{P}(\mathbf{x}, \mathbf{y}|\theta, \mu) = \sup_{\theta=\mu} \mathrm{P}(\mathbf{x}|\theta)\mathrm{P}(\mathbf{y}|\mu), \quad \text{by independence.}$$
$$= \sup_{\theta=\mu} \prod_{i=1}^{n} \frac{1}{\theta} \exp\left[-\frac{x_i}{\theta}\right] \prod_{j=1}^{m} \frac{1}{\mu} \exp\left[-\frac{y_j}{\mu}\right]$$
$$= \sup_{\theta=\mu} \frac{1}{\theta^n} \exp\left[-\frac{\sum_{i=1}^n x_i}{\theta}\right] \frac{1}{\mu^m} \exp\left[-\frac{\sum_{j=1}^m y_j}{\mu}\right]$$
$$= \frac{1}{\mu^n} \exp\left[-\frac{\sum_{i=1}^n x_i}{\mu}\right] \frac{1}{\mu^m} \exp\left[-\frac{\sum_{j=1}^m y_j}{\mu}\right]$$
$$= \frac{1}{\mu^{n+m}} \exp\left\{-\frac{1}{\mu}\left[\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right]\right\}$$

Note that μ is a nuisance parameter, and so we will also maximize this over its domain. And to do that we take the log-likeihood function first,

$$\ell(\mu|\mathbf{x}, \mathbf{y}) = -\log(\mu^{n+m}) - \frac{1}{\mu}\left[\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right]$$
$$= -(n+m)\log(\mu) - \frac{1}{\mu}\left[\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right].$$

Taking the derivative with respect to μ, gives us

$$\frac{\mathrm{d}}{\mathrm{d}\,\mu}\ell(\mu|\mathbf{x}, \mathbf{y}) = -(n+m)\frac{1}{\mu} + \frac{1}{\mu^2}\left[\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right],$$

equate this to zero to obtain the stationary point,

$$\begin{aligned}
-(n+m)\frac{1}{\mu}+\frac{1}{\mu^2}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]&=0\\
-(n+m)\mu+\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]&=0\\
\mu=\frac{1}{n+m}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right].
\end{aligned}$$

To verify if this is the MLE, we take the second derivative test for the log-likelihood function,

$$\frac{\mathrm{d}^2}{\mathrm{d}\,\mu^2}\ell(\mu|\mathbf{x},\mathbf{y})=(n+m)\frac{1}{\mu^2}-\frac{2}{\mu^3}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]<0,$$

since $\frac{1}{\mu^2}<\frac{2}{\mu^3}$, implying $\hat{\mu}=\frac{1}{n+m}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]$ is the MLE of μ. Thus the LRT, $\lambda(\mathbf{x},\mathbf{y})$ would be,

$$\begin{aligned}
\lambda(\mathbf{x},\mathbf{y})&=\frac{\frac{1}{\mu^{n+m}}\exp\left\{-\frac{1}{\mu}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]\right\}}{\frac{1}{\bar{x}^n}\frac{1}{\bar{y}^m}\exp[-(n+m)]}\\
&=\left(\frac{1}{\frac{1}{(n+m)^{n+m}}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]^{n+m}}\times\right.\\
&\qquad\left.\exp\left\{-\frac{1}{\frac{1}{n+m}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]\right\}\right)\Bigg/\\
&\qquad\frac{1}{\bar{x}^n}\frac{1}{\bar{y}^m}\exp[-(n+m)]\\
&=\frac{\dfrac{1}{\frac{1}{(n+m)^{n+m}}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]^{n+m}}\times\exp[-(n+m)]}{\frac{1}{\bar{x}^n}\frac{1}{\bar{y}^m}\exp[-(n+m)]}\\
&=\frac{\bar{x}^n\bar{y}^m}{\frac{1}{(n+m)^{n+m}}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]^{n+m}}.
\end{aligned}$$

And we say that H_0 is rejected if $\lambda(\mathbf{x},\mathbf{y})\le c$.

(b) If we do some algebra on the LRT in part (a), we obtain the following:

$$\begin{aligned}
\lambda(\mathbf{x},\mathbf{y})&=\frac{\bar{x}^n\bar{y}^m}{\frac{1}{(n+m)^{n+m}}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]^{n+m}}\\
&=\frac{\frac{1}{n^n}\left(\sum_{i=1}^n x_i\right)^n\frac{1}{m^m}\left(\sum_{j=1}^m y_j\right)^m}{\frac{1}{(n+m)^{n+m}}\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]^{n+m}}\\
&=\frac{(n+m)^{n+m}\left(\sum_{i=1}^n x_i\right)^n\left(\sum_{j=1}^m y_j\right)^m}{n^n m^m\left[\sum_{i=1}^n x_i+\sum_{j=1}^m y_j\right]^{n+m}}\\
&=\frac{(n+m)^{n+m}}{n^n m^m}\left[\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i+\sum_{j=1}^m y_j}\right]^m\left[\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i+\sum_{j=1}^m y_j}\right]^n
\end{aligned}$$

$$= \frac{(n+m)^{n+m}}{n^n m^m}\left[1-\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i+\sum_{j=1}^{m} y_j}\right]^m\left[\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i+\sum_{j=1}^{m} y_j}\right]^n$$

$$= \frac{(n+m)^{n+m}}{n^n m^m}\left[1-T\right]^m\left[T\right]^n.$$

Hence, the LRT can be based on the statistic T.

(c) The distribution of $\sum X_i$ is obtain using the MGF technique, that is

$$\begin{aligned}
\mathrm{M}_{\Sigma X_i}(t) &= \mathrm{E}\exp[t\Sigma X_i] = \mathrm{E}\exp[tX_1+\cdots+tX_n]\\
&= \mathrm{E}\exp[tX_1]\times\cdots\times\mathrm{E}\exp[tX_n], \quad \text{by independence.}\\
&= \frac{1}{1-\theta t}\times\cdots\times\frac{1}{1-\theta t}\\
&= \left(\frac{1}{1-\theta t}\right)^n = \text{MGF of gamma}(n,\theta).
\end{aligned}$$

Now, when H_0 is true then $\sum X_i$ is gamma(m,θ). For brevity, let $X=\sum_{i=1}^n X_i$ and $Y=\sum_{j=1}^m Y_j$. The joint distribution of X and Y is given below,

$$f_{XY}(x,y) = \frac{1}{\Gamma(n)\theta^n}x^{n-1}\exp[-x/\theta]\times\frac{1}{\Gamma(m)\theta^m}y^{m-1}\exp[-y/\theta].$$

Let $U=\frac{X}{X+Y}$ and $V=X+Y$, then the support of (X,Y) is $\mathcal{A}=\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+\}$. Since the transformations U and V is one-to-one and onto, then $\mathcal{B}=\{(u,v)\in\mathbb{R}^+\times\mathbb{R}^+\}$. Consider the following transformations

$$u=g_1(x,y)=\frac{x}{x+y} \quad \text{and} \quad v=g_2(x,y)=x+y.$$

Then,

$$u=\frac{x}{x+y}\Rightarrow x=\frac{uy}{1-u} \tag{1}$$

and

$$v=x+y\Rightarrow y=v-x. \tag{2}$$

Substitute Equation (2) to Equation (1), then

$$x=\frac{u(v-x)}{1-u}\Rightarrow x(1-u)=u(v-x)$$
$$x-ux=uv-ux\Rightarrow x=uv=h_1(u,v).$$

Substitute x above to Equation (2) to obtain,

$$y=v(1-u)=h_2(u,v).$$

Then the Jacobian matrix is,

$$\mathbf{J}=\begin{vmatrix} v & u\\ -v & 1-u\end{vmatrix}=v(1-u)+uv=v.$$

So that,

$$\begin{aligned}
f_{UV}(u,v) &= f_{XY}(h_1(u,v),h_2(u,v))|\mathbf{J}| = f_{XY}(uv,v(1-u))|v|\\
&= \frac{1}{\Gamma(n)\theta^n}(uv)^{n-1}\exp[-uv/\theta]\times\\
&\quad\frac{1}{\Gamma(m)\theta^m}(v(1-u))^{m-1}\exp[-v(1-u)/\theta]v\\
&= \frac{1}{\Gamma(n)\theta^n}(uv)^{n-1}\exp[-uv/\theta]\times\\
&\quad\frac{1}{\Gamma(m)\theta^m}(v(1-u))^{m-1}\exp[-v/\theta]\exp[uv/\theta]v\\
&= \frac{1}{\Gamma(n)\theta^n}u^{n-1}v^{n-1}\times\frac{1}{\Gamma(m)\theta^m}v^{m-1}(1-u)^{m-1}\exp[-v/\theta]v\\
&= \frac{1}{\Gamma(n)}\underbrace{u^{n-1}(1-u)^{m-1}}_{\text{Beta}(n,m)\text{ kernel}}\frac{1}{\Gamma(m)\theta^{m+n}}v^{m-1}v^{n-1}\exp[-v/\theta]v\\
&= \frac{\Gamma(m)\Gamma(m+n)}{\Gamma(m)\Gamma(m+n)}\frac{u^{n-1}(1-u)^{m-1}}{\Gamma(n)}\times\\
&\quad\frac{1}{\Gamma(m)\theta^{m+n}}v^{m-1}v^n\exp[-v/\theta]\\
&= \underbrace{\frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)}u^{n-1}(1-u)^{m-1}}_{\text{Beta}(n,m)}\times\\
&\quad\underbrace{\frac{1}{\Gamma(m+n)\theta^{m+n}}v^{m+n-1}\exp[-v/\theta]}_{\text{Gamma}(m+n,\theta)}.
\end{aligned}$$

Therefore, the marginal density of $U=\frac{\sum X_i}{\sum X_i+\sum Y_i}$ is Beta(n,m).

8.7 We have already seen the usefulness of the LRT in dealing with problems with nuisance parameters. We now look at some other nuisance parameter problems.

(a) Find the LRT of

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0$$

base on a sample $X_1, \cdots, X_n$ from a population with probability density function $f(x|\theta, \lambda) = \frac{1}{\lambda} \exp\left[-\frac{(x-\theta)}{\lambda}\right] \mathrm{I}_{[\theta,\infty)}(x)$, where both θ and λ are unknown.

(b) We have previously seen that the exponential pdf can be considered as a special case of the Weibull(γ,β). The Weibull pdf, which reduces to the exponential if $\gamma = 1$, is very important in modeling reliability of systems. Suppose that $X_1, \cdots, X_n$ is a random sample from a Weibull population with both γ and β unknown. Find the LRT of $H_0 : \gamma = 1$ versus $H_1 : \gamma \neq 1$.

Solution

(a) We are only insterested in θ, so λ is a nuisance parameter. The parameter space of the null hypothesis is $\Theta_0 = [\theta, 0]$, and that $\Theta = [\theta, \infty)$, so that the LRT is obtained as follows:

$$\lambda(\mathbf{x}) = \frac{\sup\limits_{\theta \leq 0} \mathcal{L}(\theta, \lambda|\mathbf{x})}{\sup\limits_{\theta < \infty} \mathcal{L}(\theta, \lambda|\mathbf{x})}. \tag{3}$$

where the denominator is

$$\begin{aligned}
\sup_{\theta<\infty} \mathcal{L}(\theta, \lambda|\mathbf{x}) &= \sup_{\theta<\infty} \prod_{i=1}^{n} \frac{1}{\lambda} \exp\left[-\frac{(x_i - \theta)}{\lambda}\right] \mathrm{I}_{[\theta,\infty)}(x_i) \\
&= \sup_{\theta<\infty} \frac{1}{\lambda^n} \exp\left[\frac{-\sum_{i=1}^n x_i + n\theta}{\lambda}\right] \prod_{i=1}^{n} \mathrm{I}_{[\theta,\infty)}(x_i) \\
&= \sup_{\theta<\infty} \frac{1}{\lambda^n} \exp\left[\frac{-\sum_{i=1}^n x_i + n\theta}{\lambda}\right] \mathrm{I}_{\{x_{(1)} \geq \theta\}} \\
&= \sup_{\theta<\infty} \frac{1}{\lambda^n} \exp\left[-\frac{n\bar{x}}{\lambda}\right] \exp\left[\frac{n\theta}{\lambda}\right] \mathrm{I}_{\{x_{(1)} \geq \theta\}}.
\end{aligned}$$

Notice that the right hand side is increasing as a function of θ, regardless of what λ is. And because $\theta \leq x_{(1)}$, then $\hat{\theta} = x_{(1)}$ is the MLE of θ. Next we solve for the MLE of λ,

$$\begin{aligned}
\ell(\theta, \lambda|\mathbf{x}) &= -n \log \lambda - \frac{n\bar{x}}{\lambda} + \frac{n\theta}{\lambda}, \quad x_{(1)} \geq \theta \\
\frac{\partial\, \ell(\theta, \lambda|\mathbf{x})}{\partial\, \lambda} &= -\frac{n}{\lambda} + \frac{n\bar{x}}{\lambda^2} - \frac{n\theta}{\lambda^2}.
\end{aligned}$$

Equating this to zero,

$$\begin{aligned}
-\frac{n}{\lambda} + \frac{n\bar{x} - n\theta}{\lambda^2} &= 0 \\
-n\lambda &= n\theta - n\bar{x} \\
\lambda &= \bar{x} - \theta = \bar{x} - x_{(1)}, \quad \text{since } \hat{\theta} = x_{(1)}.
\end{aligned}$$

To see if this is the maximum, we use the second-derivative test. That is,

$$\begin{aligned}
\frac{\partial^2\, \ell(\theta, \lambda|\mathbf{x})}{\partial\, \lambda^2} &= \frac{n}{\lambda^2} - \frac{2n(\bar{x} - \theta)}{\lambda^3}\bigg|_{\lambda = \bar{x} - x_{(1)}} \\
&= \frac{n}{(\bar{x} - x_{(1)})^2} - \frac{2n(\bar{x} - \theta)}{(\bar{x} - x_{(1)})^3} \\
&= \frac{n(\bar{x} - x_{(1)}) - 2n\bar{x} + 2n\theta}{(\bar{x} - x_{(1)})^3} \\
&= \frac{n\bar{x} - nx_{(1)} - 2n\bar{x} + 2nx_{(1)}}{(\bar{x} - x_{(1)})^3}
\end{aligned}$$

$$\begin{aligned}
&= \frac{nx_{(1)} - n\bar{x}}{(\bar{x} - x_{(1)})^3} = \frac{-n(\bar{x} - x_{(1)})}{(\bar{x} - x_{(1)})^3} \\
&= \frac{-n}{(\bar{x} - x_{(1)})^2} < 0, \quad \forall \lambda = \bar{x} - x_{(1)}.
\end{aligned}$$

So $\hat{\theta} = x_{(1)}$ and $\hat{\lambda} = \bar{x} - x_{(1)}$ are the MLEs for the unrestricted domain. For restricted domain, the MLEs are the same but the supremum under Θ_0 is

$$\hat{\theta}_0 = \begin{cases} 0 & \text{if } x_{(1)} > 0 \\ x_{(1)} & \text{if } x_{(1)} \leq 0, \end{cases}$$

and $\hat{\lambda}_0 = \bar{x} - \hat{\theta}_0$. So that the ratio in Equation (3) would be

$$\lambda(\mathbf{x}) = \begin{cases} \frac{\mathcal{L}(\hat{\theta},\hat{\lambda}|\mathbf{x})}{\mathcal{L}(\hat{\theta},\hat{\lambda}|\mathbf{x})}, & \text{if } x_{(1)} \leq 0 \\ \frac{\mathcal{L}(0,\bar{x}|\mathbf{x})}{\mathcal{L}(\hat{\theta},\hat{\lambda}|\mathbf{x})}, & \text{if } x_{(1)} > 0 \end{cases} = \begin{cases} 1, & \text{if } x_{(1)} \leq 0 \\ \frac{\mathcal{L}(0,\bar{x}|\mathbf{x})}{\mathcal{L}(\hat{\theta},\hat{\lambda}|\mathbf{x})}, & \text{if } x_{(1)} > 0, \end{cases}$$

where

$$\begin{aligned}
\frac{\mathcal{L}(0,\bar{x}|\mathbf{x})}{\mathcal{L}(\hat{\theta},\hat{\lambda}|\mathbf{x})} &= \frac{\frac{1}{\bar{x}^n}\exp\left[-\frac{n\bar{x}}{\bar{x}}\right]}{\frac{1}{(\bar{x}-x_{(1)})^n}\exp\left[-\frac{n\bar{x}}{\bar{x}-x_{(1)}}\right]\exp\left[\frac{nx_{(1)}}{\bar{x}-x_{(1)}}\right]} \\
&= \frac{\frac{1}{\bar{x}^n}\exp\left[-n\right]}{\frac{1}{(\bar{x}-x_{(1)})^n}\exp\left[\frac{-n(\bar{x}-x_{(1)})}{(\bar{x}-x_{(1)})}\right]} \\
&= \left[1 - \frac{x_{(1)}}{\bar{x}}\right]^n.
\end{aligned}$$

And thus we reject the likelihood ratio test if $\lambda(\mathbf{x}) \leq c$, implying

$$\begin{aligned}
\left[1 - \frac{x_{(1)}}{\bar{x}}\right]^n &\leq c \\
-\frac{x_{(1)}}{\bar{x}} &\leq c^{1/n} - 1 \\
\frac{x_{(1)}}{\bar{x}} &\geq 1 - c^{1/n} = c^*.
\end{aligned}$$

Therefore rejecting H_0 if $\lambda(\mathbf{x}) \leq c$, is equivalent to rejecting H_0 if $\frac{x_{(1)}}{\bar{x}} \geq c^*$.

(b) The Weibull density is defined as follows:

$$f(x|\gamma,\beta) = \frac{\gamma}{\beta}x^{\gamma-1}\exp\left[-\frac{x^\gamma}{\beta}\right], \quad 0 \leq x < \infty, \quad \gamma > 0, \quad \beta > 0.$$

The parameter space of the null hypothesis is $\Gamma_0 = \{1\}$, and that the domain of the parameter γ is $\Gamma = (0,\infty)$. So that the LRT is

$$\lambda(\mathbf{x}) = \frac{\sup\limits_{\gamma=1}\mathcal{L}(\gamma,\beta|\mathbf{x})}{\sup\limits_{\gamma>0}\mathcal{L}(\gamma,\beta|\mathbf{x})},$$

where the denominator is

$$\begin{aligned}
\sup_{\gamma>0}\mathcal{L}(\gamma,\beta|\mathbf{x}) &= \sup_{\gamma>0}\prod_{i=1}^n \frac{\gamma}{\beta}x_i^{\gamma-1}\exp\left[-\frac{x_i^\gamma}{\beta}\right] \\
&= \sup_{\gamma>0}\left(\frac{\gamma}{\beta}\right)^n \exp\left[-\frac{1}{\beta}\sum_{i=1}^n x_i^\gamma\right]\prod_{i=1}^n x_i^{\gamma-1}.
\end{aligned}$$

Now taking the log-likelihood we have,

$$\ell(\gamma,\beta|\mathbf{x}) = n\log\left(\frac{\gamma}{\beta}\right) - \frac{1}{\beta}\sum_{i=1}^n x_i^\gamma + (\gamma-1)\sum_{i=1}^n \log x_i.$$

So that differentiating this with respect to γ,

$$\begin{aligned}
\frac{\partial\,\ell(\gamma,\beta|\mathbf{x})}{\partial\,\gamma} &= n\frac{\partial}{\partial\,\gamma}(\log\gamma - \log\beta) - \frac{1}{\beta}\sum_{i=1}^n \frac{\partial}{\partial\,\gamma}x_i^\gamma + 1 \\
&= n\frac{1}{\gamma} - \frac{\gamma}{\beta}\sum_{i=1}^n x_i^{\gamma-1} + 1.
\end{aligned}$$

Equating this to zero and solving for γ gives us,

$$\begin{aligned}
n\frac{1}{\gamma} - \frac{\gamma}{\beta}\sum_{i=1}^n x_i^{\gamma-1} &= -1 \\
\frac{\gamma^2}{\beta}\sum_{i=1}^n x_i^{\gamma-1} - \gamma &= n.
\end{aligned}$$

Above equation is an implicit form of γ, and the only way to maximize this is to use numerical maximization, like the popular Newthon-Raphson method. On the other hand, the MLE of β is,

$$\begin{aligned}
\frac{\partial\,\ell(\gamma,\beta|\mathbf{x})}{\partial\,\beta} &= n\frac{\partial}{\partial\,\beta}(\log\gamma - \log\beta) + \frac{1}{\beta^2}\sum_{i=1}^n x_i^\gamma \\
&= -\frac{n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^n x_i^\gamma.
\end{aligned}$$

Equating this to zero and solving for β gives us,

$$\begin{aligned}
-\frac{n}{\beta}+\frac{1}{\beta^2}\sum_{i=1}^n x_i^\gamma &= 0\\
n\beta &= \sum_{i=1}^n x_i^\gamma\\
\beta &= \frac{1}{n}\sum_{i=1}^n x_i^\gamma.
\end{aligned}$$

Taking the second derivative, we have

$$\begin{aligned}
\frac{\partial^2\,\ell(\gamma,\beta|\mathbf{x})}{\partial\,\beta^2} &= \frac{n}{\beta^2}-\frac{2}{\beta^3}\sum_{i=1}^n x_i^\gamma\bigg|_{\beta=\frac{1}{n}\sum_{i=1}^n x_i^\gamma}\\
&= \frac{n}{\left(\frac{1}{n}\sum_{i=1}^n x_i^\gamma\right)^2}-\frac{2}{\left(\frac{1}{n}\sum_{i=1}^n x_i^\gamma\right)^3}\sum_{i=1}^n x_i^\gamma\\
&= \frac{n^3-2n^3}{\left(\sum_{i=1}^n x_i^\gamma\right)^2}<0\Rightarrow \hat{\beta}_{\text{MLE}}=\frac{1}{n}\sum_{i=1}^n x_i^\gamma.
\end{aligned}$$

On the other hand, the numerator has nuisance parameter β, which from above derivation is maximized at $\bar{x}$. So that the supremum for $\gamma=1$ would be

$$\begin{aligned}
\sup_{\gamma=1}\mathcal{L}(\gamma,\bar{x}|\mathbf{x}) &= \mathcal{L}(1,\bar{x}|\mathbf{x})\\
&= \left(\frac{1}{\bar{x}}\right)^n\exp\left[-\frac{1}{\bar{x}}n\bar{x}\right]=\left(\frac{1}{\bar{x}}\right)^n\exp[-n].
\end{aligned}$$

Therefore, the LRT is given by

$$\lambda(\mathbf{x})=\frac{\left(\frac{1}{\bar{x}}\right)^n\exp[-n]}{\sup\limits_{\gamma>0}\left(\frac{\gamma}{\beta}\right)^n\exp\left[-\frac{1}{\beta}\sum_{i=1}^n x_i^\gamma\right]\prod_{i=1}^n x_i^{\gamma-1}}.$$

The denominator cannot be simplified unless we do numerical maximization on γ as stated before.

8.12 For samples of size $n=1,4,16,64,100$ from a normal population with mean μ and known variance σ^2, plot the power function of the following LRTs. Take $\alpha=.05$.

(a) $H_0:\mu\leq 0$ versus $H_1:\mu>0$

(b) $H_0:\mu=0$ versus $H_1:\mu\neq 0$

Solution

(a) The LRT statistic is given by

$$\lambda(\mathbf{x})=\frac{\sup\limits_{\mu\leq 0}\mathcal{L}(\mu|\mathbf{x})}{\sup\limits_{-\infty<\mu<\infty}\mathcal{L}(\mu|\mathbf{x})},\text{ since }\sigma^2\text{ is known.}$$

The denominator can be expanded as follows:

$$\begin{aligned}
\sup_{-\infty<\mu<\infty}\mathcal{L}(\mu|\mathbf{x}) &= \sup_{-\infty<\mu<\infty}\prod_{i=1}^n\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]\\
&= \sup_{-\infty<\mu<\infty}\frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\sum_{i=1}^n\frac{(x_i-\mu)^2}{2\sigma^2}\right]\\
&= \frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\sum_{i=1}^n\frac{(x_i-\bar{x})^2}{2\sigma^2}\right],\\
&\quad\text{since }\bar{x}\text{ is the MLE of }\mu.\\
&= \frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\frac{n-1}{n-1}\sum_{i=1}^n\frac{(x_i-\bar{x})^2}{2\sigma^2}\right]\\
&= \frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\frac{(n-1)s^2}{2\sigma^2}\right],
\end{aligned}$$

while the numerator is evaluated as follows:

$$\begin{aligned}
\sup_{\mu\leq 0}\mathcal{L}(\mu|\mathbf{x}) &= \sup_{\mu\leq 0}\prod_{i=1}^n\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]\\
&= \sup_{\mu\leq 0}\frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\sum_{i=1}^n\frac{(x_i-\mu)^2}{2\sigma^2}\right].
\end{aligned}$$

Above expression will attain its maximum if the value inside the exponential function is small. And for negative values of $\mu \in (-\infty, 0)$ the quantity $(x_i - \mu)^2$ would be large, implies that the exponential term would become small. Therefore, the only value that will give us the supremum likelihood is $\mu = \mu_0 = 0$. Hence,

$$
\begin{aligned}
\sup_{\mu \leq 0} \mathcal{L}(\mu|\mathbf{x}) &= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{2\sigma^2}\right] \\
&= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\sum_{i=1}^{n} \frac{(x_i-\bar{x}+\bar{x}-\mu_0)^2}{2\sigma^2}\right] \\
&= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left\{-\sum_{i=1}^{n} \left[\frac{(x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu_0) + (\bar{x}-\mu_0)^2}{2\sigma^2}\right]\right\} \\
&= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\frac{(n-1)s^2 + n(\bar{x}-\mu_0)^2}{2\sigma^2}\right], \\
&\qquad \text{since the middle term is 0.} \\
&= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\frac{(n-1)s^2 + n\bar{x}^2}{2\sigma^2}\right], \text{since } \mu_0 = 0.
\end{aligned}
$$

So that

$$
\begin{aligned}
\lambda(\mathbf{x}) &= \frac{\frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\frac{(n-1)s^2+n\bar{x}^2}{2\sigma^2}\right]}{\frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\frac{(n-1)s^2}{2\sigma^2}\right]} \\
&= \exp\left[-\frac{n\bar{x}^2}{2\sigma^2}\right].
\end{aligned} \tag{4}
$$

And we reject the null hypothesis if $\lambda(\mathbf{x}) \leq c$, that is

$$
\begin{aligned}
\exp\left[-\frac{n\bar{x}^2}{2\sigma^2}\right] &\leq c \\
-\frac{n\bar{x}^2}{2\sigma^2} &\leq \log c \\
\frac{|\bar{x}|}{\sigma/\sqrt{n}} &\geq \sqrt{-2\log c} = c'.
\end{aligned}
$$

Hence, rejecting the null hypothesis if $\lambda(\mathbf{x}) \leq c$, is equivalent to rejecting H_0 if $\frac{\bar{x}}{\sigma/\sqrt{n}} \geq c' \in [0, \infty)$. Figure 1 depicts the plot of the

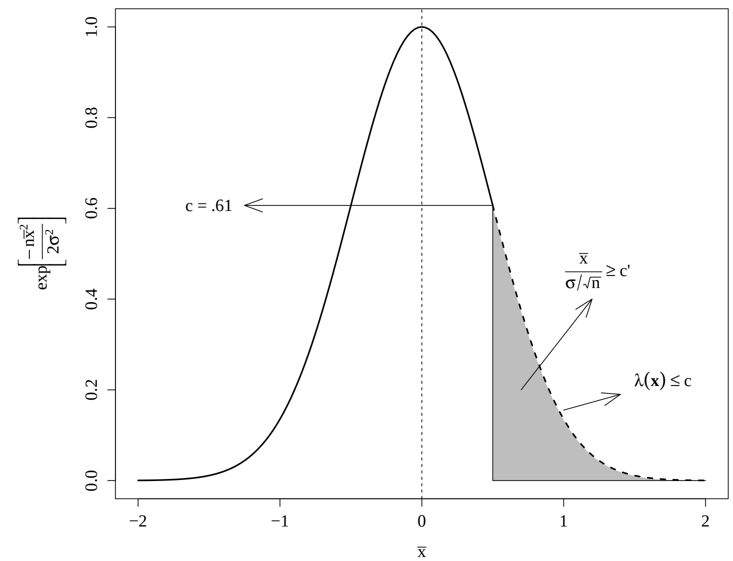

Figure 1: Likelihood Ratio Test Plot for $n = 4, \sigma = 1$.

LRT, the shaded region is on the positive side because that's where the alternative region is, $H_1 : \mu > 0$, in a sense that if the LRT is small enough to reject H_0, then it simply tells us that the plausibility of the parameter in the alternative in explaining the sample is higher compared to the null hypothesis. And if that's the case, we expect the sample to come from the parameter proposed by the H_1, so it should fall on the side (shaded region) of the alternative.

So that the power function, that is the probability of rejecting the null hypothesis given that it is true (the probability of Type I error) is,

$$
\begin{aligned}
\beta(\mu) &= \mathrm{P}\left[\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq c'\right], \quad \mu_0 = 0 \\
&= 1 - \mathrm{P}\left[\frac{\bar{x}+\mu-\mu-\mu_0}{\sigma/\sqrt{n}} < c'\right]
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \mathrm{P}\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}} < c'\right] \\
&= 1 - \mathrm{P}\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < c' + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right] \\
&= 1 - \Phi\left[c' + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right].
\end{aligned}
$$

Values taken by Φ are negative and so it decreases, but since we subtracted it to 1, then $\beta(\mu)$ is an increasing function. So that for $\alpha = .05$,

$$
\begin{aligned}
\alpha &= \sup_{\mu\leq\mu_0} \beta(\mu) \\
.05 &= \beta(\mu_0) \Rightarrow \beta(\mu_0) = 1 - \Phi(c') \\
.95 &= \Phi(c') \Rightarrow c' = 1.645.
\end{aligned}
$$

Since,

$$
\Phi(1.645) = \int_{-\infty}^{1.645} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}\, x = .9500151.
$$

Therefore for $c' = 1.645, \mu_0 = 0, \sigma = 1$, the plot of the power function as a function of μ for different sample size, n, is shown in Figure 2. For example, for $n = 1$ we compute for the function

$$
\begin{aligned}
\beta(\mu) &= 1 - \Phi\left[c' + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right] \\
&= 1 - \Phi\left[1.645 + \frac{0-\mu}{1/\sqrt{1}}\right] \\
&= 1 - \int_{-\infty}^{\left(1.645+\frac{0-\mu}{1/\sqrt{1}}\right)} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}\, x.
\end{aligned} \tag{5}
$$

The obtained values would be the y. For $n = 64$

$$
\begin{aligned}
\beta(\mu) &= 1 - \Phi\left[c' + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right] \\
&= 1 - \Phi\left[1.645 + \frac{0-\mu}{1/\sqrt{64}}\right] \\
&= 1 - \int_{-\infty}^{\left(1.645+\frac{0-\mu}{1/\sqrt{64}}\right)} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}\, x,
\end{aligned}
$$

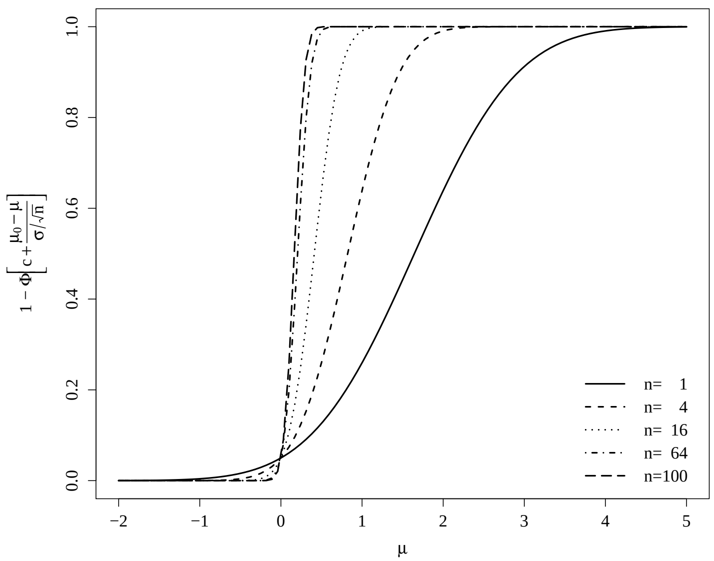

Figure 2: Plots of the Power Function for Different Values of n.

and so on.

(b) The LRT statistic is given by

$$
\lambda(\mathbf{x}) = \frac{\sup_{\mu=0} \mathcal{L}(\mu|\mathbf{x})}{\sup_{-\infty<\mu<\infty} \mathcal{L}(\mu|\mathbf{x})}, \text{ since } \sigma^2 \text{ is known.}
$$

The denominator can be expanded as follows:

$$
\begin{aligned}
\sup_{-\infty<\mu<\infty} \mathcal{L}(\mu|\mathbf{x}) &= \sup_{-\infty<\mu<\infty} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \\
&= \sup_{-\infty<\mu<\infty} \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{2\sigma^2}\right],
\end{aligned}
$$

since $\bar{x}$ is the MLE of μ.

$$= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\frac{n-1}{n-1}\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2\sigma^2}\right]$$
$$= \frac{1}{(2\pi\sigma^2)^{1/n}} \exp\left[-\frac{(n-1)s^2}{2\sigma^2}\right],$$

and the numerator is evaluated as follows:

$$\begin{aligned}
\sup_{\mu=0}\mathcal{L}(\mu|\mathbf{x}) &= \sup_{\mu=0}\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]\\
&= \sup_{\mu=0}\frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right]\\
&= \frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\sum_{i=1}^{n}\frac{(x_i-0)^2}{2\sigma^2}\right]\\
&= \frac{1}{(2\pi\sigma^2)^{1/n}}\exp\left[-\frac{(n-1)s^2+n\bar{x}^2}{2\sigma^2}\right],
\end{aligned}$$

we skip some lines in the above simplification since we've done this already in part (a). And by Equation (4), $\lambda(\mathbf{x}) = \exp\left[-\frac{n\bar{x}^2}{2\sigma^2}\right]$. So that $\lambda(\mathbf{x}) \leq c$ would be

$$\begin{aligned}
\exp\left[-\frac{n\bar{x}^2}{2\sigma^2}\right] &\leq c\\
-\frac{n\bar{x}^2}{2\sigma^2} &\leq \log c\\
\frac{|\bar{x}-\mu_0|}{\sigma/\sqrt{n}} &\geq \sqrt{-2\log c} = c', \quad \mu_0 = 0.
\end{aligned}$$

So rejecting the null hypothesis if $\lambda(\mathbf{x}) \leq c'$ is equivalent to rejecting H_0 if $\frac{|\bar{x}|}{\sigma/\sqrt{n}} \geq c'$. And since H_1 is two-sided, then we reject H_0 if $\frac{\bar{x}}{\sigma/\sqrt{n}} \geq c'$ or $\frac{\bar{x}}{\sigma/\sqrt{n}} \leq -c'$. To illustrate this, consider Figure 3 where the two shaded regions are the lower and upper rejection regions.

So that the power function is,

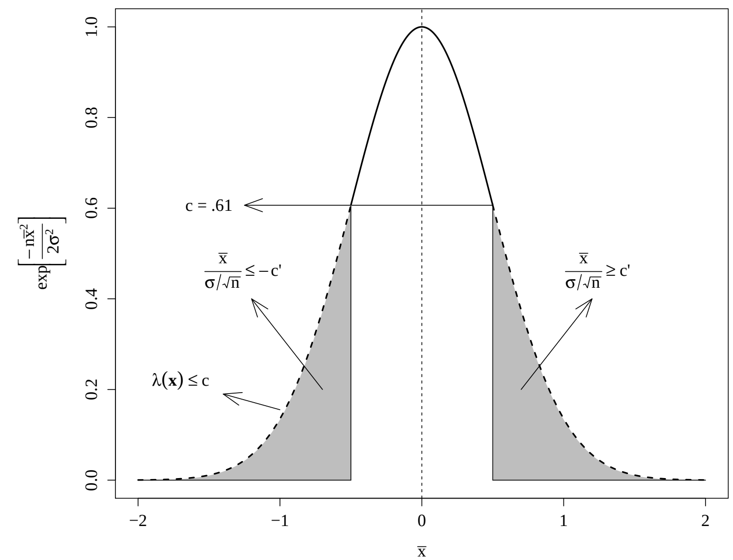

Figure 3: Likelihood Ratio Test Plot for $n = 4, \sigma = 1$.

$$\begin{aligned}
\beta(\mu) &= \mathrm{P}\left[\frac{|\bar{x}|}{\sigma/\sqrt{n}} \geq c'\right]\\
&= 1 - \mathrm{P}\left[\frac{|\bar{x}|}{\sigma/\sqrt{n}} < c'\right]\\
&= 1 - \mathrm{P}\left[-c' < \frac{\bar{x}}{\sigma/\sqrt{n}} < c'\right]\\
&= 1 - \left\{\mathrm{P}\left[\frac{\bar{x}}{\sigma/\sqrt{n}} < c'\right] - \mathrm{P}\left[\frac{\bar{x}}{\sigma/\sqrt{n}} < -c'\right]\right\}\\
&= 1 - \left\{\mathrm{P}\left[\frac{\bar{x}+\mu-\mu}{\sigma/\sqrt{n}} < c'\right] - \mathrm{P}\left[\frac{\bar{x}+\mu-\mu}{\sigma/\sqrt{n}} < -c'\right]\right\}\\
&= 1 - \mathrm{P}\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < c' - \frac{\mu}{\sigma/\sqrt{n}}\right] + \mathrm{P}\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < -c' - \frac{\mu}{\sigma/\sqrt{n}}\right]\\
&= \underbrace{1 - \Phi\left[c' - \frac{\mu}{\sigma/\sqrt{n}}\right]}_{\Phi_1} + \underbrace{\Phi\left[-c' - \frac{\mu}{\sigma/\sqrt{n}}\right]}_{\Phi_2}.
\end{aligned}$$

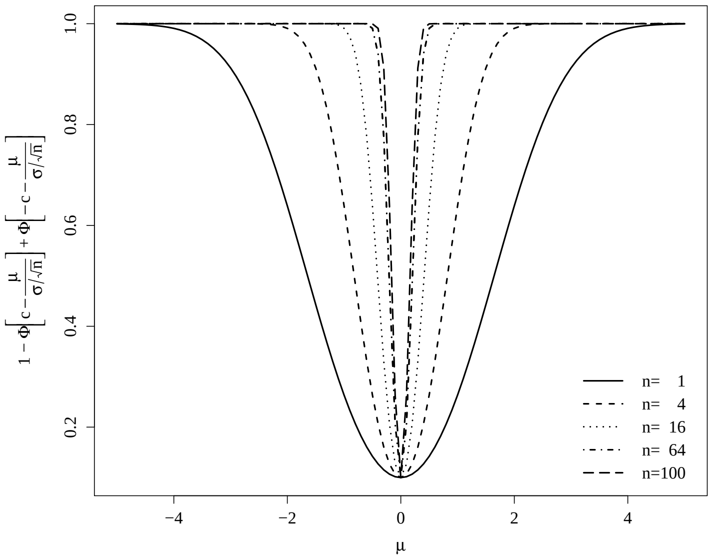

Figure 4: Two-Sided Power Function for Different n.

Notice that Φ_1 is an increasing function, while Φ_2 is decreasing as a function of μ. We expect this since the alternative hypothesis is a two-sided one, so does the power. To see this, consider Figure 4 for different values of n.

The points in the plot are computed by substituting values of $\mu = 0, \sigma = 1$ and n to the power function just like we did in Equation (5).

8.14 For a random sample $X_1, \cdots, X_n$ of Bernoulli(p) variables, it is desired to test

$$H_0 : p = .49 \quad \text{versus} \quad H_1 : p = .51.$$

Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about .01. Use a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.

Solution

We reject H_0 if $\sum_{i=1}^{n} X_i > c$, for any constant c. Because $Y_n = \sum_{i=1}^{n} X_i$ is binomial(n, p), then the probability of Type I Error is,

$$\beta(p) = \mathrm{P}_{p_0}\left[\sum_{i=1}^{n} X_i > c\right].$$

Since $\mathrm{E}Y_n = np$ and $\mathrm{Var}Y_n = np(1-p)$, then the random variable $Z_n = \frac{Y_n - np}{\sqrt{np(1-p)}}$ is asymptotically normally distributbed with mean 0 and variance 1. Therefore,

$$\begin{aligned}
\beta(p) &= \mathrm{P}_{p_0}\left[\sum_{i=1}^{n} X_i > c\right] \\
&= 1 - \mathrm{P}_{p_0}\left[\sum_{i=1}^{n} X_i \leq c\right] \\
&= 1 - \mathrm{P}_{p_0}\left[\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}} \leq \frac{c - np}{\sqrt{np(1-p)}}\right] \\
&= 1 - \Phi_{p_0}\left[\frac{c - np}{\sqrt{np(1-p)}}\right] \\
.01 &= 1 - \Phi\left[\frac{c - n(.49)}{\sqrt{n(.49)(.51)}}\right], \quad \text{since } p_0 = .49 \\
.99 &= \Phi\left[\frac{c - n(.49)}{\sqrt{n(.49)(.51)}}\right].
\end{aligned}$$

Implying,

$$2.326 = \frac{c - n(.49)}{\sqrt{n(.49)(.51)}}, \tag{6}$$

since

$$\Phi(2.326) = \int_{-\infty}^{2.326} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}\,x = .99.$$

Now the probability of the Type II Error which is the probability of rejecting the alternative hypothesis given that it is true, is

$$\begin{aligned}1-\beta(p) &= \mathrm{P}_{p_1}\left[\sum_{i=1}^{n} X_i \leq c\right]\\ .01 &= \Phi\left[\frac{c-n(.51)}{\sqrt{n(.51)(.49)}}\right], \quad \text{since } p_1 = .51.\end{aligned}$$

Implying,

$$-2.326 = \frac{c-n(.51)}{\sqrt{n(.51)(.49)}}. \tag{7}$$

Using Equations (6) and (7), we solve for c and n. From (6),

$$c = 2.326\sqrt{n(.49)(.51)} + .49n. \tag{8}$$

Substitute (8) to (7),

$$\begin{aligned}-2.326 &= \frac{2.326\sqrt{n(.49)(.51)} + .49n - .51n}{\sqrt{n(.51)(.49)}}\\ -2.326\sqrt{n(.51)(.49)} &= 2.326\sqrt{n(.49)(.51)} - .02n\\ 2(2.326\sqrt{n(.51)(.49)}) &= .02n\\ 2^2(2.326)^2(.51)(.49)n &= .02^2n^2\\ n &= \frac{2^2(2.326)^2(.51)(.49)}{.02^2}\\ &= 13520.2797, \quad \text{take } n = 13521.\end{aligned}$$

So that $c = 6760.1399$ for non-rounded value of n.

8.15 Show that for a random $X_1, \cdots, X_n$ from a $\mathrm{n}(0, \sigma^2)$ population the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi(\Sigma X_i^2) = \begin{cases}1 & \text{if } \Sigma X_i^2 > c\\ 0 & \text{if } \Sigma X_i^2 \leq c\end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

Solution

Proof. Since the hypotheses are simple, then we use the Neyman-Pearson Lemma to show the most powerful test. Accordingly, we reject H_0 if the ratio

$$\frac{f(\mathbf{x}|\sigma_1)}{f(\mathbf{x}|\sigma_0)} > k.$$

So that by independence,

$$\begin{aligned}\frac{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left[-\frac{x_i^2}{2\sigma_1^2}\right]}{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left[-\frac{x_i^2}{2\sigma_0^2}\right]} &= \frac{\frac{1}{(2\pi)^{n/2}\sigma_1^n}\exp\left[-\frac{\sum_{i=1}^n x_i^2}{2\sigma_1^2}\right]}{\frac{1}{(2\pi)^{n/2}\sigma_0^n}\exp\left[-\frac{\sum_{i=1}^n x_i^2}{2\sigma_0^2}\right]}\\ &= \frac{\sigma_0^n}{\sigma_1^n}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}x_i^2\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\right].\end{aligned}$$

Implying that,

$$\begin{aligned}\frac{\sigma_0^n}{\sigma_1^n}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}x_i^2\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\right] &> k\\ -\frac{1}{2}\sum_{i=1}^{n}x_i^2\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right) &> \log\left[k\frac{\sigma_1^n}{\sigma_0^n}\right]\\ \frac{1}{2}\sum_{i=1}^{n}x_i^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right) &> \log\left[k\frac{\sigma_1^n}{\sigma_0^n}\right]\\ \sum_{i=1}^{n}x_i^2 &> 2\log\left[k\frac{\sigma_1^n}{\sigma_0^n}\right]\left(\frac{\sigma_1^2\sigma_0^2}{\sigma_0^2-\sigma_1^2}\right)\\ \sum_{i=1}^{n}x_i^2 &> c.\end{aligned}$$

Hence we reject the null hypothesis if $\sum_{i=1}^n x_i^2 > c$. That is, if we write this as an indicator function, say ϕ. Then

$$\phi(\Sigma X_i^2) = \begin{cases}1 & \text{if } \Sigma X_i^2 > c\\ 0 & \text{if } \Sigma X_i^2 \leq c.\end{cases}$$

To determine the value of c, impose the level requirement,

$$\alpha = \mathrm{P}_{\sigma_0}\left[\sum_{i=1}^{n} x_i^2 > c\right].$$

Evaluating the probability above at $\sigma = \sigma_0$, then $X_i \sim \mathrm{n}(0, \sigma_0^2)$, so that if $Y_i = \frac{X_i - 0}{\sigma_0}$, then $Y_i \sim \mathrm{n}(0,1)$. Implying $Y_i^2 \sim \chi^2_{(1)}$, and thus $\sum_{i=1}^{n} Y_i^2 \sim \chi^2_{(n)}$. To continue,

$$\begin{aligned}
\alpha = \sup_{\sigma=\sigma_0} \beta(\sigma) = \beta(\sigma_0) &= \mathrm{P}_{\sigma_0}\left[\sum_{i=1}^{n} \frac{x_i^2}{\sigma_0^2} > \frac{c}{\sigma_0^2}\right] \\
&= 1 - \mathrm{P}_{\sigma_0}\left[\sum_{i=1}^{n} Y_i^2 \leq \frac{c}{\sigma_0^2}\right] \\
\mathrm{P}_{\sigma_0}\left[\sum_{i=1}^{n} Y_i^2 \leq \frac{c}{\sigma_0^2}\right] &= 1 - \alpha.
\end{aligned}$$

Hence the quantile of $1-\alpha$ probability, which is $\chi^2_{(n),\alpha}$, is equal to $\frac{c}{\sigma_0^2}$. That is $\frac{c}{\sigma_0^2} = \chi_{(n),\alpha} \Rightarrow c = \chi_{(n),\alpha}\sigma_0^2$. To conclude, we reject the null hypothesis if $\sum_{i=1}^{n} x_i^2 > \chi_{(n),\alpha}\sigma_0^2$. □

8.19 The random variable X has pdf $f(x) = \exp[-x], x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ needs to be constructed. Find the UMP level $\alpha = .1$ test and compute the Type II Error probability.

Solution

From Neyman-Pearson Lemma, the most powerful rejection region is,

$$\frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)} > k. \tag{9}$$

To obtain the distribution of Y, we need to note that the transformation, $g(X)$, is an increasing function. To see this, consider the gradient of the function

$$\frac{\mathrm{d}}{\mathrm{d}\,x} g(x) = \frac{\mathrm{d}}{\mathrm{d}\,x} X^\theta = \theta x^{\theta-1},$$

which is positive $\forall x, \theta \geq 0$, implying that the slopes of the curve is positive and hence increasing. And by Theorem 2.1.5,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}\,x} g^{-1}(y)\right|,$$

where $y = g(x) = x^\theta \Rightarrow x = y^{1/\theta} = g^{-1}(y)$. Substituting this,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}\,x} g^{-1}(y)\right| \\
&= \exp\left[-y^{1/\theta}\right]\frac{1}{\theta} y^{\frac{1}{\theta}-1}, \quad y > 0.
\end{aligned}$$

Since only one observation is involve, then by Equation (9),

$$\begin{aligned}
\frac{f(y|\theta_1)}{f(y|\theta_0)} &= \frac{\exp\left[-y^{1/\theta_1}\right]\frac{1}{\theta_1} y^{\frac{1}{\theta_1}-1}}{\exp\left[-y^{1/\theta_0}\right]\frac{1}{\theta_0} y^{\frac{1}{\theta_0}-1}} \\
&= \exp\left[y^{1/\theta_0} - y^{1/\theta_1}\right]\frac{\theta_0}{\theta_1} y^{\frac{1}{\theta_1}-\frac{1}{\theta_0}} \\
&= \exp\left[y - y^{1/2}\right]\frac{1}{2} y^{-\frac{1}{2}} > k.
\end{aligned}$$

Figure 5 depicts the plot of the above function along with its assumed critical value.

From the plot, we observe that the likelihood ratio test function is decreasing at $y < 1$ and increasing at $y > 1$. To confirm this stationary point, we'll use the first derivative test. That is,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\frac{\exp\left[y - y^{1/2}\right]}{2y^{1/2}} &= \frac{2y^{1/2}\exp\left[y - y^{1/2}\right]\left(1 - \frac{1}{2y^{1/2}}\right) - \exp\left[y - y^{1/2}\right]\frac{1}{y^{1/2}}}{4y} \\
&= \frac{\exp\left[y - y^{1/2}\right]\left\{\left(1 - \frac{1}{2y^{1/2}}\right) - \frac{1}{2y}\right\}}{2y^{1/2}} \\
&= \frac{\exp\left[y - y^{1/2}\right]\left(\frac{2y - y^{1/2} - 1}{2y}\right)}{2y^{1/2}} \\
&= \exp\left[y - y^{1/2}\right]\left(\frac{2y - y^{1/2} - 1}{4y^{3/2}}\right).
\end{aligned}$$

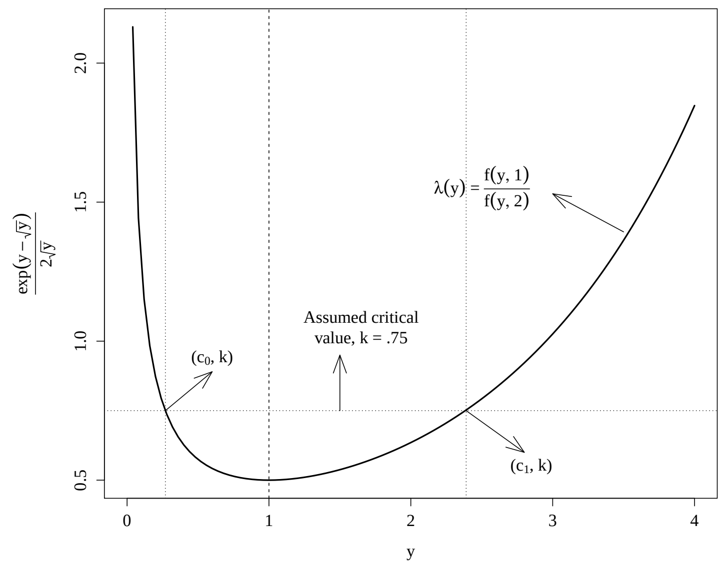

Figure 5: Likelihood Ratio Test base on Neyman-Pearson Lemma.

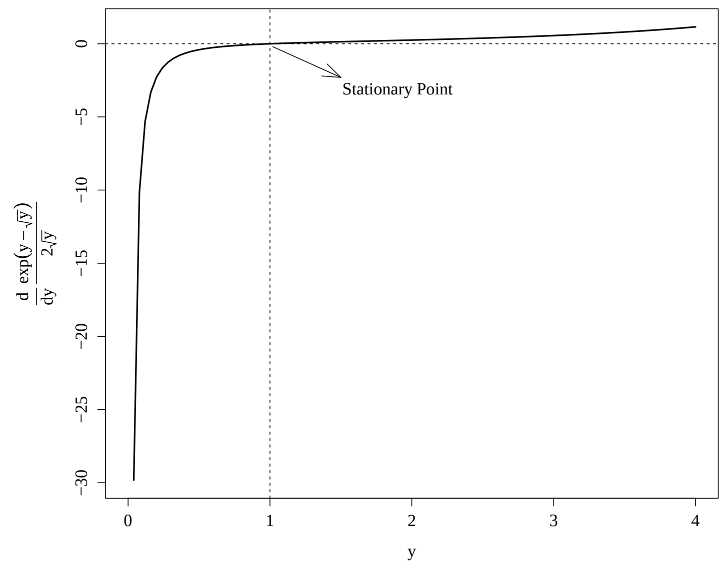

Figure 6: Derivative of the Likelihood Ratio Test in Figure 5.

The critical point above is $y = 1$. Figure 6 is the curve of the derivative of the above equation. In it, the function is negative at $y < 1$, implying that the curve of the LRT is decreasing at $y < 1$, and increasing at $y > 1$.

Thus, rejecting H_0 if $\lambda(x) = \frac{f(y|1)}{f(y|2)} > k$ is equivalent to rejecting H_0 if $y \leq c_0$ or $y \geq c_1$. So that the uniform most powerful level $\alpha = .1$ test, is a test with critical values c_0 and c_1 that solves the following equations

$$
\begin{aligned}
\alpha &= \mathrm{P}_{\theta_0}(y \leq c_0) + \mathrm{P}_{\theta_0}(y \geq c_1) = \mathrm{P}_{\theta_0}(y \leq c_0) + 1 - \mathrm{P}_{\theta_0}(y < c_1)\\
&= \int_0^{c_0} \exp\left[-y^{1/\theta_0}\right] \frac{1}{\theta_0} y^{\frac{1}{\theta_0}-1}\,\mathrm{d}\,x + 1 - \int_0^{c_1} \exp\left[-y^{1/\theta_0}\right] \frac{1}{\theta_0} y^{\frac{1}{\theta_0}-1}\,\mathrm{d}\,x\\
&= \int_0^{c_0} \exp[-y]\,\mathrm{d}\,x + 1 - \int_0^{c_1} \exp[-y]\,\mathrm{d}\,x\\
&= -\exp[-c_0] - 1 + 1 - (-\exp[-c_1] - 1)\\
.1 &= 1 + \exp[-c_1] - \exp[-c_0].
\end{aligned}
$$

To obtain the values of c_0 and c_1, consider Figure 5, notice that the ratios below are equal, since the two intersections of the dot lines are the points (c_0, k) and (c_1, k), which corresponds to the same rejection line k in the y axis.

$$\frac{f(c_0|2)}{f(c_0|1)} = \frac{f(c_1|2)}{f(c_1|1)}.$$

So that we have the following equations for solving c_0 and c_1:

$$.1 = 1 + \exp[-c_1] - \exp[-c_0] \tag{10}$$

and

$$\exp\left[c_0 - c_0^{1/2}\right] \frac{1}{2} c_0^{-\frac{1}{2}} = \exp\left[c_1 - c_1^{1/2}\right] \frac{1}{2} c_1^{-\frac{1}{2}}. \tag{11}$$

From Equation (10), $c_0 = \log(.1) + c_1$, subsituting this to Equation (11), we then have

$$\begin{aligned}
&\exp\left\{\log(.1) + c_1 - [\log(.1)+c_1]^{1/2}\right\}\frac{1}{2}[\log(.1)+c_1]^{-1/2}\\
&\qquad = \exp\left[c_1 - c_1^{1/2}\right]\frac{1}{2}c_1^{-1/2}\\
\Rightarrow\ &\log(.1) + c_1 - [\log(.1)+c_1]^{1/2} - \frac{1}{2}\log[\log(.1)+c_1] - \log(2)\\
&\qquad = c_1 - c_1^{1/2} - \frac{1}{2}\log(c_1) - \log(2)\\
\Rightarrow\ &\log(.1) = [\log(.1)+c_1]^{1/2} + \frac{1}{2}\log[\log(.1)+c_1] - c_1^{1/2} - \frac{1}{2}\log(c_1).
\end{aligned}$$

I've tried it already, but this function cannot be solved symbolically. The only way to obtain values of c_0 and c_1, is to solve this numerically. In doing so, we need to minimize the following equation,

$$\begin{aligned}
0 = \Big\{&[\log(.1)+c_1]^{1/2} + \frac{1}{2}\log[\log(.1)+c_1]\\
&-c_1^{1/2} - \frac{1}{2}\log(c_1) - \log(.1)\Big\}^2
\end{aligned} \tag{12}$$

The square of the above equation will help us omit the possibility of negative values. The plot of the RHS of the above equation is shown in Figure 7.

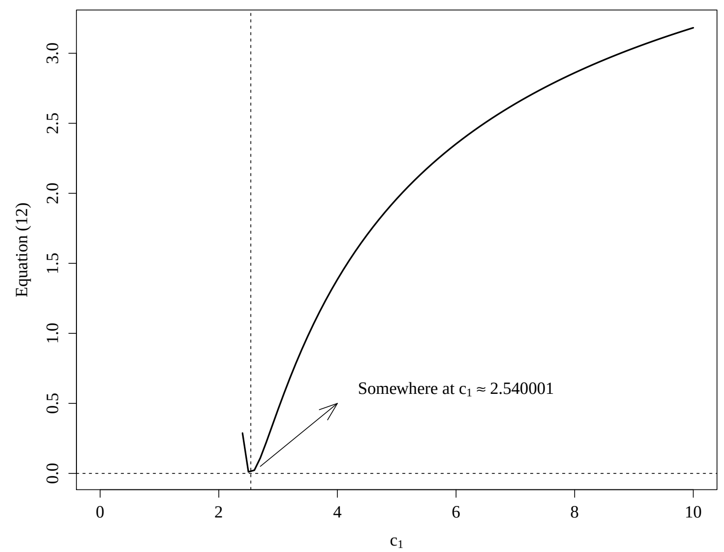

Figure 7: Plot of the RHS of Equation (12) as a function of c_1.

Base on the plot, the value of c_1, obtained by trial and error, subjectively falls somewhere at 2.540001. Therefore, the corresponding value of $c_0 = 0.2374154$.

Now to compute for the Type II Error, which is

$$\begin{aligned}
1-\beta(\theta_1) &= 1 - [\mathrm{P}_{\theta_1}(y \le c_0) + \mathrm{P}_{\theta_1}(y \ge c_1)]\\
&= \mathrm{P}_{\theta_1}(y < c_1) - \mathrm{P}_{\theta_1}(y \le c_0)\\
&= \mathrm{P}_{\theta_1}(c_0 < y < c_1), \text{ since continuous, we can replace } \le \text{ by } < .\\
&= \int_{c_0}^{c_1} \exp\left[-y^{1/\theta_1}\right]\frac{1}{\theta_1}y^{\frac{1}{\theta_1}-1}\,\mathrm{d}\,x\\
&= \int_{c_0}^{c_1} \exp\left[-y^{1/2}\right]\frac{1}{2y^{1/2}}\,\mathrm{d}\,x.
\end{aligned}$$

To evaluate this, let $u = y^{1/2}$, then $\mathrm{d}\,u = \frac{1}{2y^{1/2}}\,\mathrm{d}\,y$. And if the lower limit, $y = c_0 \Rightarrow u = c_0^{1/2}$, so that the upper limit is $u = c_1^{1/2}$. Hence,

$$\begin{aligned}
1-\beta(\theta_1) &= \int_{c_0}^{c_1}\exp[-u]\,\mathrm{d}\,u\\
&= -\exp[-u]\Big|_{u=\sqrt{c_0}}^{u=\sqrt{c_1}}\\
&= \exp[-\sqrt{c_0}] - \exp[-\sqrt{c_1}]\\
&= \exp[-\sqrt{0.2374154}] - \exp[-\sqrt{2.540001}] = 0.4111469.
\end{aligned}$$

8.20 Let X be a random variable whose pmf under H_0 and H_1 is given in Table 1. Use the Neyman-Pearson Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = .04$. Compute the probability of Type II Error for this test.

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_1)$	.06	.05	.04	.03	.02	.01	.79

Table 1: Probability Mass Function under H_0 and H_1.

Solution

The likelihood ratio test base on Neyman-Pearson Lemma is

$$\lambda(x) = \frac{f(x|H_1)}{f(x|H_0)} > k.$$

So that the ratio is

x	1	2	3	4	5	6	7
$\frac{f(x\|H_1)}{f(x\|H_0)}$	6	5	4	3	2	1	0.84

The plot of this is in Figure 8, where the assumed $k = 4.3$. The mass points (x's) with LRT greater than k are colored in white. Hence rejecting H_0 if $\lambda(x) > 4.3$, is equivalent to rejecting H_0 if $x \leq 2$. Or in general, rejecting H_0 for higher values of the ratio, corresponds to rejecting it at small values of x.

The most powerful test with size α is obtain as follows:

$$\alpha = \mathrm{P}_{H_0}(x \leq c) = F_X(c|H_0) = \sum_{i=1}^{c} f(x_i|H_0).$$

If $\alpha = .04$, then $c = 4$, since

$$\sum_{i=1}^{4} f(x_i|H_0) = .01 + \cdots + .01 = .04.$$

Using $c = 4$, we now compute the probability of the Type II Error, or the probability of rejecting the alternative hypothesis given that it is true, is

$$\begin{aligned} \mathrm{P}_{H_1}(x > 4) &= 1 - F_X(4|H_1) \\ &= 1 - \sum_{i=1}^{4} f(x_i|H_1) = 1 - (.06 + \cdots + .03) = .82. \end{aligned}$$

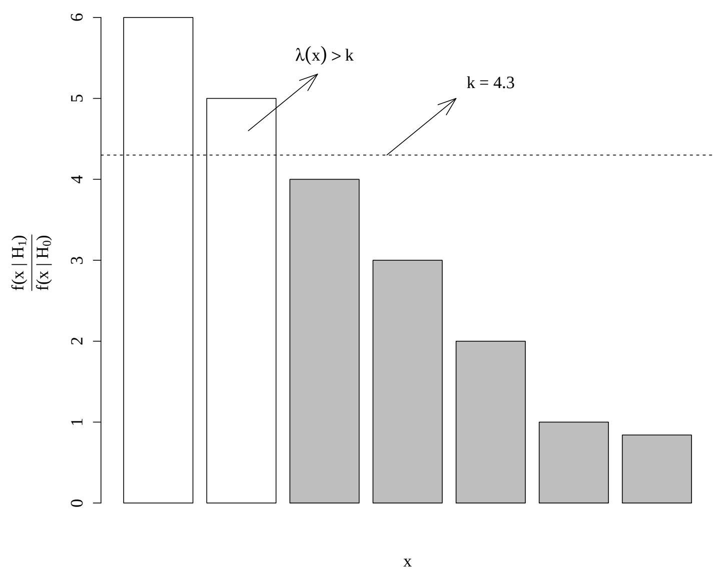

Figure 8: Likelihood Ratio Test base on Neyman-Pearson Lemma.