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PS 2 | **Stat 234**

6.4.1 Obtain the pdf of $\mathbf{X} \sim \mathcal{N}_q^p(\mathbf{M}, \mathbf{A} \otimes \mathbf{B})$, where $\mathbf{A} > \mathbf{0}$ is in $\mathbb{R}_p^p$ and $\mathbf{B} > \mathbf{0}$ is in $\mathbb{R}_q^q$:

$$f(\mathbf{X}) = (2\pi)^{-\frac{pq}{2}} |\mathbf{A}|^{-\frac{q}{2}} |\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-1}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\right] \quad (1)$$

where $\operatorname{etr} = \exp(\operatorname{tr}(\cdot))$.
Hint: Let $\mathbf{X} = \mathbf{A}^{\frac{1}{2}}\mathbf{Z}\mathbf{B}^{\frac{1}{2}} + \mathbf{M}$, where $\mathbf{Z} \sim \mathcal{N}_q^p(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_q)$, and use Corollary 6.1.

Proof. From the hint, let $\mathbf{Z} \sim \mathcal{N}_q^p(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_q)$. Now recall that the Multivariate Gaussian distribution for $\mathbf{Z} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ we have

$$\begin{aligned} f(\mathbf{X}) &= \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right] &(2)\\ &= \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \operatorname{etr}\left[-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]. &(3)\end{aligned}$$

Since the factors in the exponential function returns a constant, then we can apply the trace function. So that if $\mathbf{Z} \sim \mathcal{N}_q^p(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_q)$ then this is equivalent to $\operatorname{vec}(\mathbf{Z}^{\mathrm{T}}) \sim \mathcal{N}_{pq}(\operatorname{vec}(\mathbf{0}^{\mathrm{T}}), \mathbf{I}_p \otimes \mathbf{I}_q)$ or mathematically equivalent to:

$$\begin{aligned} f(\mathbf{Z}) &= \frac{1}{(2\pi)^{\frac{pq}{2}}|\mathbf{I}_p \otimes \mathbf{I}_q|^{\frac{1}{2}}} \operatorname{etr}\left[-\frac{1}{2}\mathbf{Z}^{\mathrm{T}}(\mathbf{I}_p \otimes \mathbf{I}_q)^{-1}\mathbf{Z}\right] &(4)\\ &= \frac{1}{(2\pi)^{\frac{pq}{2}}} \operatorname{etr}\left[-\frac{1}{2}\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\right] &(5)\end{aligned}$$

Now consider the following transformations $\mathbf{X} = \mathbf{A}^{\frac{1}{2}}\mathbf{Z}\mathbf{B}^{\frac{1}{2}} + \mathbf{M}$, then $\mathbf{X} - \mathbf{M} = \mathbf{A}^{\frac{1}{2}}\mathbf{Z}\mathbf{B}^{\frac{1}{2}}$, and since $\mathbf{A} \in \mathcal{P}_p$ and $\mathbf{B} \in \mathcal{P}_q$. Then both $\mathbf{A}$ and $\mathbf{B}$ are nonsingular. So that, $\mathbf{A}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-\frac{1}{2}} = \mathbf{Z}$. Using Corollary 6.1, $J(\mathbf{Z} \to \mathbf{X}) = |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}}$. So that substituting the terms into Equation (5), we have

$$\begin{aligned} f(\mathbf{X}) &= \frac{1}{(2\pi)^{\frac{pq}{2}}} \operatorname{etr}\left[-\frac{1}{2}(\mathbf{A}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-\frac{1}{2}})^{\mathrm{T}}(\mathbf{A}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-\frac{1}{2}})\right] |J(\mathbf{Z} \to \mathbf{X})|\\ &= \frac{1}{(2\pi)^{\frac{pq}{2}}} |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}(\mathbf{B}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\mathbf{A}^{-\frac{1}{2}})(\mathbf{A}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-\frac{1}{2}})\right]\\ &= \frac{1}{(2\pi)^{\frac{pq}{2}}} |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}(\mathbf{B}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-\frac{1}{2}})\right]. \end{aligned}$$

And since trace is invariant under circular permutation, then

$$\begin{aligned} f(\mathbf{X}) &= \frac{1}{(2\pi)^{\frac{pq}{2}}} |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}\mathbf{B}^{-\frac{1}{2}}\mathbf{B}^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{M})\right]\\ &= \frac{1}{(2\pi)^{\frac{pq}{2}}} |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}\mathbf{B}^{-1}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{M})\right]\\ &= \frac{1}{(2\pi)^{\frac{pq}{2}}} |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-1}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\mathbf{A}^{-1}\right]\\ &= (2\pi)^{-\frac{pq}{2}} |\mathbf{A}|^{-\frac{q}{2}}|\mathbf{B}|^{-\frac{p}{2}} \operatorname{etr}\left[-\frac{1}{2}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{B}^{-1}(\mathbf{X}-\mathbf{M})^{\mathrm{T}}\right]. \end{aligned}$$

□

6.4.8 Assume $(x_i, y_i)^{\mathrm{T}}, i = 1, \cdots, n$ are i.i.d.

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \quad (6)$$

and let r be the sample correlation coefficient. Prove the asymptotic result $\sqrt{n}(r-\rho) \xrightarrow{d} \mathcal{N}(0, (1-\rho^2)^2)$.

Proof. Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then recall that the MGF of $M_{\mathbf{v}}(t) = \exp\left(\frac{1}{2}\mathbf{t}^{\mathrm{T}}\Sigma\mathbf{t} + \mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}\right) = \exp\left(\frac{1}{2}\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}\right)$. Now let $\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \boldsymbol{\Sigma} =$

$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Then,

$$\mathbf{t}^{\mathrm{T}}\Sigma\mathbf{t} = [t_1 \quad t_2]\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = t_1^2 + \rho t_1 t_2 + \rho t_1 t_2 + t_2^2$$

So that $\exp\left(\frac{1}{2}\mathbf{t}^{\mathrm{T}}\Sigma\mathbf{t}\right) = \exp\left(\frac{1}{2}t_1^2 + \rho t_1 t_2 + \frac{1}{2}t_2^2\right)$. Now from Example 6.4, note that

$$\sqrt{n}\left[\begin{pmatrix} s_1^2 \\ s_{12} \\ s_2^2 \end{pmatrix} - \begin{pmatrix} \sigma_1^2 \\ \sigma_{12} \\ \sigma_2^2 \end{pmatrix}\right] \xrightarrow{d} \mathcal{N}_3(0, \Omega) \tag{7}$$

where

$$\Omega = \begin{pmatrix} \mu_4^1 - (\mu_2^1)^2 & \mu_{31}^{12} - \mu_{11}^{12}\mu_2^1 & \mu_{22}^{12} - \mu_2^1\mu_2^2 \\ \cdot & \mu_{22}^{12} - (\mu_{11}^{12})^2 & \mu_{13}^{12} - \mu_{11}^{12}\mu_2^2 \\ \cdot & \cdot & \mu_4^2 - (\mu_2^2)^2 \end{pmatrix} \tag{8}$$

Next is to compute for the values of the components of Ω, and from the problem $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$. Thus

$$\begin{aligned}
\mu_4^1 &= \mathbb{E}(x - \mu_1)^4 = \mathbb{E}x^4 \\
\mu_2^1 &= \mathbb{E}(x - \mu_1)^2 = \sigma_1^2 = 1 \\
\mu_{31}^{12} &= \mathbb{E}(x - \mu_1)^3(y - \mu_2) = \mathbb{E}x_1^3 x_2 \\
\mu_{11}^{12} &= \mathbb{E}(x - \mu_1)(y - \mu_2) = \sigma_{12} \\
\mu_2^2 &= \mathbb{E}(y - \mu_2)^2 = \sigma_2^2 = 1 \\
\mu_{22}^{12} &= \mathbb{E}(x - \mu_1)^2(y - \mu_2)^2 = \mathbb{E}x^2y^2 \\
\mu_{13}^{12} &= \mathbb{E}(x - \mu_1)(y - \mu_2)^3 = \mathbb{E}xy^3 \\
\mu_4^2 &= \mathbb{E}(y - \mu_2)^4 = \mathbb{E}y^4
\end{aligned}$$

To avoid confusion, we stressed that t_1 is for x and t_2 is for y. So that

(a) $\mathbb{E}x^4 = \frac{\partial^4}{\partial t_1 \partial t_1} M_{\mathbf{v}}(t_1, t_2)\Big|_{t_1=t_2=0}$, and let $u = \frac{1}{2}t_1^2 + \rho t_1 t_2 + \frac{1}{2}t_2^2$, then $\frac{\partial}{\partial t_1}u = t_1 + \rho t_2$, and $f = \exp(u)$. Thus,

$$\begin{aligned}
f' &= (t_1 + \rho t_2)\exp(u) \\
f'' &= \exp(u) + (t_1 + \rho t_2)^2\exp(u) \\
f''' &= (t_1 + \rho t_2)\exp(u) + 2(t_1 + \rho t_2)(1)\exp(u) + (t_1 + \rho t_2)^3\exp(u) \\
f'''' &= \exp(u) + (t_1 + \rho t_2)^2\exp(u) + 2(1)\exp(u) + 2(t_1 + \rho t_2)^2\exp(u) \\
&\quad + 3(t_1 + \rho t_2)^2\exp(u) + (t_1 + \rho t_2)^4\exp(u)
\end{aligned}$$

If $t_1 = t_2 = 0$, then $u|_{t_1=t_2=0} = \frac{1}{2}(0)^2 + \rho(0) + \frac{1}{2}(0)^2 = 0$. Implies that,

$$f''''|_{t_1=t_2=0} = 1 + 0 + 2 + 0 + 0 = 3. \tag{9}$$

(b) $\mu_{31}^{12} = \mathbb{E}x^3y = \frac{\partial^4}{(\partial t_1)^3\partial t_2} M_{[x \quad y]^{\mathrm{T}}}(t_1, t_2)\Big|_{t_1=t_2=0}$. Now let $u = \frac{1}{2}t_1^2 + \rho t_1 t_2 + \frac{1}{2}t_2^2$, and $a = \frac{\partial u}{\partial t_1} = t_1 + \rho t_2$; $b = \frac{\partial u}{\partial t_2} = \rho t_1 + t_2$.

$$\begin{aligned}
\frac{\partial}{(\partial t_1)^3\partial t_2}\exp(u) =& \frac{\partial}{(\partial t_1)^3}\exp(u)(\rho t_1 + t_2) \\
=& \frac{\partial}{(\partial t_1)^2}\left[(t_1 + \rho t_2)\exp(u)(\rho t_1 + t_2) + \exp(u)\rho\right]
\end{aligned} \tag{10}$$

$$\begin{aligned}
=& \frac{\partial}{(\partial t_1)^2}\left[(\rho t_1^2 + t_1 t_2 + \rho^2 t_1 t_2 + \rho t_2^2)\exp(u)\right. \\
& \left. + \exp(u)\rho\right] \\
=& \rho\exp(u)(t_1 + \rho t_2)^2 + \rho\exp(u) \\
& + (\rho t_1^3 + 2\rho^2 t_1 t_2 + \rho^3 t_1 t_2^2 + t_1^2 t_2 + 2\rho t_1 t_2^2 + \rho^2 t_2^3) \\
& \times \exp(u)(t_1 + \rho t_2) + \exp(u) \\
& \times (3\rho t^2 + 4\rho^2 t_1 t_2 + \rho^3 t_2^2 + 2t_1 t_2 + \rho t_2^2) \\
& + (2\rho t_1 + \rho^2 t_2 + t_2)\exp(u)(t_1 + \rho t_2) + \exp(u)(2\rho)
\end{aligned}$$

Substituting $t_1 = t_2 = 0$ to above expression returns:

$$\frac{\partial}{(\partial t_1)^3\partial t_2}\exp(u)|_{t_1=t_2=0} = 3\rho. \tag{11}$$

(c) $\mu_{11}^{12} = \mathbb{E}(x - \mu_1)(y - \mu_2) = \sigma_{12}$. It follows from Equation (10)

$$\frac{\partial}{\partial t_1\partial t_2}\exp(u) = (t_1 + \rho t_2)\exp(u)(\rho t_1 + t_2) + \exp(u)\rho \tag{12}$$

So that,

$$\frac{\partial}{\partial t_1\partial t_2}\exp(u)|_{t_1=t_2=0} = \rho \tag{13}$$

(d) $\mu_{22}^{12} = \mathbb{E}x^2y^2 = \frac{\partial}{(\partial t_1)^2(\partial t_2)^2} M_{[x\ \ y]}(t_1,t_2)\Big|_{t_1=t_2=0}$. Thus,

$$\begin{aligned}\mu_{22}^{12} =& \frac{\partial^2}{(\partial t_1)^2}\frac{\partial}{\partial t_2}\left[\exp(u)(\rho t_1+t_2)\right] \\ =& \frac{\partial^2}{(\partial t_1)^2}\left[\exp(u)+(\rho t_1+t_2)^2\exp(u)\right] \\ =& \frac{\partial}{\partial t_1}\left[\exp(u)(t_1+\rho t_2)+(\rho t_1+t_2)^2\exp(u)\right] \\ =& \exp(u)(1)+(t_1+\rho t_2)\exp(u)(t_1+\rho t_2) \\ &+(\rho t_1+t_2)^2\left[\exp(u)(1)+(t_1+\rho t_2)\exp(u)(t_1+\rho t_2)\right] \\ &+2\rho(\rho t_1+t_2)\exp(u)(t_1+\rho t_2) \\ &+\exp(u)(t_1+\rho t_2)2(\rho t_1+t_2)\rho+\exp(u)2\rho^2\end{aligned}$$

Therefore,

$$\frac{\partial}{(\partial t_1)^2(\partial t_2)^2} M_{[x\ \ y]}(t_1,t_2)\Big|_{t_1=t_2=0} = 1+2\rho^2. \tag{14}$$

Applying the same process we obtain $\mu_4^1 = 3 = \mu_4^2$ and $\mu_{31}^{12} = 3\rho = \mu_{13}^{12}$. Thus,

$$\begin{aligned}\mu_4^1 = 3;&\quad \mu_2^1 = 1 \\ \mu_{31}^{12} = 3\rho;&\quad \mu_{11}^{12} = \rho \\ \mu_2^2 = 1;&\quad \mu_{22}^{12} = 1+2\rho^2 \\ \mu_{13}^{12} = 3\rho;&\quad \mu_4^2 = 3.\end{aligned}$$

Therefore,

$$\begin{aligned}\Omega &= \begin{pmatrix} \mu_4^1-(\mu_2^1)^2 & \mu_{31}^{12}-\mu_{11}^{12}\mu_2^1 & \mu_{22}^{12}-\mu_2^1\mu_2^2 \\ \cdot & \mu_{22}^{12}-(\mu_{11}^{12})^2 & \mu_{13}^{12}-\mu_{11}^{12}\mu_2^2 \\ \cdot & \cdot & \mu_4^2-(\mu_2^2)^2 \end{pmatrix} \\ &= \begin{pmatrix} 3-1 & 3\rho-\rho(1) & 1+2\rho^2-1 \\ 3\rho-\rho(1) & 1+2\rho^2-\rho^2 & 3\rho-\rho \\ 1+2\rho^2-1 & 3\rho-\rho & 3-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2\rho & 2\rho^2 \\ 2\rho & 1+\rho^2 & 2\rho \\ 2\rho^2 & 2\rho & 2 \end{pmatrix}\end{aligned}$$

By Delta Method, define $g : \mathbb{R}^3 \to \mathbb{R}^1$,

$$g(\cdot) = \frac{s_{12}}{\sqrt{s_1^2 s_2^2}} = r. \tag{15}$$

Thus,

$$\sqrt{n}(x_n - c) = \sqrt{n}\left(\begin{bmatrix} s_1^2 \\ s_{12} \\ s_2^2 \end{bmatrix} - \begin{bmatrix} \sigma_1^2 \\ \sigma_{12} \\ \sigma_2^2 \end{bmatrix}\right) \xrightarrow{d} \mathcal{N}_3(0,\Omega) \tag{16}$$

So that,

$$\sqrt{n}(g(x_n)-g(c)) = \sqrt{n}(r-\rho) \xrightarrow{d} \mathbf{Dg}(c)\mathbf{Z}, \tag{17}$$

and

$$\mathbf{Dg}(c)\mathcal{N}_3(0,\Omega) \sim \mathcal{N}_3(\mathbf{0}, \mathbf{Dg}(c)^{\mathrm{T}}\Omega\mathbf{Dg}(c)) \tag{18}$$

$$\mathbf{Dg}(c)^{\mathrm{T}} = \mathbf{D}\left(\frac{\sigma_{12}}{\sqrt{\sigma_1^2\sigma_2^2}}\right)^{\mathrm{T}} = \begin{bmatrix} \frac{\partial \mathbf{g}(c)}{\partial \sigma_1^2} & \frac{\partial \mathbf{g}(c)}{\partial \sigma_{12}} & \frac{\partial \mathbf{g}(c)}{\partial \sigma_2^2} \end{bmatrix} \tag{19}$$

But $\frac{\sigma_{12}}{\sqrt{\sigma_1^2\sigma_2^2}} = \sigma_{12}(\sigma_1^2)^{-\frac{1}{2}}(\sigma_2^2)^{-\frac{1}{2}}$, then

$$\begin{aligned}\frac{\partial \mathbf{g}(c)}{\partial \sigma_1^2} &= -\frac{1}{2}\sigma_{12}(\sigma_1^2)^{-\frac{3}{2}}(\sigma_2^2)^{-\frac{1}{2}} \\ \frac{\partial \mathbf{g}(c)}{\partial \sigma_{12}} &= (\sigma_1^2)^{-\frac{1}{2}}(\sigma_2^2)^{-\frac{1}{2}} \\ \frac{\partial \mathbf{g}(c)}{\partial \sigma_2^2} &= -\frac{1}{2}\sigma_{12}(\sigma_1^2)^{-\frac{1}{2}}(\sigma_2^2)^{-\frac{3}{2}}\end{aligned}$$

And using the fact that $\sigma_{12} = \rho, \sigma_1^2 = 1$, and $\sigma_2^2 = 1$. Then

$$\begin{aligned}\frac{\partial \mathbf{g}(c)}{\partial \sigma_1^2} &= -\frac{\rho}{2} \\ \frac{\partial \mathbf{g}(c)}{\partial \sigma_{12}} &= 1 \\ \frac{\partial \mathbf{g}(c)}{\partial \sigma_2^2} &= -\frac{\rho}{2}\end{aligned}$$

Thus $\mathbf{Dg}(c)^{\mathrm{T}} = \left[-\frac{\rho}{2} \quad 1 \quad -\frac{\rho}{2}\right]$, so that

$$\begin{aligned}
&\left[-\frac{\rho}{2} \quad 1 \quad -\frac{\rho}{2}\right] \begin{pmatrix} 2 & 2\rho & 2\rho^2 \\ 2\rho & 1+\rho^2 & 2\rho \\ 2\rho^2 & 2\rho & 2 \end{pmatrix} \begin{bmatrix} -\frac{\rho}{2} \\ 1 \\ -\frac{\rho}{2} \end{bmatrix} \\
&= \left[\rho-\rho^3 \quad 1-\rho^2 \quad \rho-\rho^3\right] \begin{bmatrix} -\frac{\rho}{2} \\ 1 \\ -\frac{\rho}{2} \end{bmatrix} \\
&= \left(-\frac{\rho(\rho-\rho^3)}{2} + 1 - \rho^2 - \frac{\rho(\rho-\rho^3)}{2}\right) = (-\rho(\rho-\rho^3) + 1 - \rho^2) \\
&= (\rho^4 - \rho^2 - \rho^2 + 1) \\
&= 1 - 2\rho^2 + \rho^4 = (1-\rho^2)^2.
\end{aligned}$$

Therefore,

$$\sqrt{n}(r-\rho) \xrightarrow{d} \mathcal{N}(0, (1-\rho^2)^2). \tag{20}$$

□

7.5.5 Assume $\mathbf{W} \sim W_p(m), m \geq p$ and $\mathbf{A} > \mathbf{0}$. Prove:

(a) $\mathbb{E}\mathbf{W} = m\mathbf{I}$

(b) $\mathbb{E}\mathbf{W}^{-1} = \mathbf{I}/(m-p-1)$.

(a) *Proof.* Recall that $\mathbf{W} \sim W_p(m)$ iff $\sum_{i=1}^m \mathbf{z}_i\mathbf{z}_i^{\mathrm{T}} \overset{d}{=} \mathbf{W}, \mathbf{z}_i$ i.i.d. $\mathcal{N}(\mathbf{0}, \mathbf{I})$. Then using the fact that $\mathbb{C}\mathrm{ov}(\mathbf{z}_i, \mathbf{z}_i) = \mathbb{E}(\mathbf{z}_i\mathbf{z}_i^{\mathrm{T}}) + \mathbb{E}\mathbf{z}_i\mathbb{E}\mathbf{z}_i^{\mathrm{T}} = \mathbb{C}\mathrm{ov}(\mathbf{z}_i)$. So that $\mathbb{E}(\mathbf{z}_i\mathbf{z}_i^{\mathrm{T}}) = \mathbb{C}\mathrm{ov}(\mathbf{z}_i) - \mathbb{E}\mathbf{z}_i\mathbb{E}\mathbf{z}_i^{\mathrm{T}} = \mathbf{I} + \mathbf{0} = \mathbf{I}$. Therefore, $\sum_{i=1}^m \mathbf{z}_i\mathbf{z}_i^{\mathrm{T}} = m\mathbf{I}$. □

(b) *Proof.* Consider the following Theorem,

Theorem 0.1. *If* $\mathbf{M} \sim W_p(n, \Sigma)$ *and* $\mathbf{a} \in \mathbb{R}^p$ *and* $n > p-1$, *then*

$$\frac{\mathbf{a}^{\mathrm{T}}\Sigma^{-1}\mathbf{a}}{\mathbf{a}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{a}} \sim \chi^2_{n-p+1}. \tag{21}$$

Using the above Theorem suppose $\mathbf{W} \sim W_p(m, \mathbf{I})$ and $\mathbf{b} \in \mathbb{R}^\rho$, then

$$\begin{aligned}
&\frac{\mathbf{b}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{b}} \sim \chi^2_{m-p+1} \\
\Rightarrow &\left(\frac{\mathbf{b}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{b}}\right)^{-1} \sim \text{inverse } \chi_{(m-p+1)-2} \\
&= \frac{\mathbf{b}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{b}} \sim \text{inverse } \chi_{(m-p+1)-2}.
\end{aligned}$$

So that,

$$\mathbb{E}\left[\frac{\mathbf{b}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{b}}\right] = \frac{1}{(m-p+1)-2}$$

Now $\mathbb{E}\left[\frac{\mathbf{b}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{b}}\right] = \frac{\mathbf{b}^{\mathrm{T}}\mathbb{E}(\mathbf{W}^{-1})\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{b}} \Rightarrow \mathbf{b}^{\mathrm{T}}\mathbb{E}(\mathbf{W}^{-1})\mathbf{b} = \frac{\mathbf{b}^{\mathrm{T}}\mathbf{I}\mathbf{b}}{(m-p+1)-2} = \frac{\mathbf{b}^{\mathrm{T}}\mathbf{I}\mathbf{b}}{m-p-1}$. And because $\mathbf{b}$ is arbitrary, then it follows that $\mathbb{E}\mathbf{W}^{-1} = \frac{\mathbf{I}}{m-p-1}$. □

7.5.7 **Wishart Density**
Obtain the p.d.f. of $\mathbf{V} \sim W_p(m, \mathbf{\Sigma}), m \geq p, \mathbf{\Sigma} > \mathbf{0}$:

$$f_{\mathbf{V}}(\mathbf{V}) = \frac{1}{2^{mp/2}\Gamma_p\left(\frac{1}{2}m\right)|\mathbf{\Sigma}|^{m/2}}|\mathbf{V}|^{(m-p-1)/2}\mathrm{etr}\left(-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{V}\right), \tag{22}$$

$\mathbf{V} > \mathbf{0}$.

Proof. From the definition of the Wishart distribution, $\mathbf{V} \sim W_p(m, \mathbf{\Sigma})$ iff $\mathbf{V} \overset{d}{=} \mathbf{AWA}^{\mathrm{T}}$, $\mathbf{W} \sim W_p(m), \mathbf{\Sigma} = \mathbf{AA}^{\mathrm{T}}$. So for $p = 1$,

$$f_V(V) = \frac{1}{2^{m/2}\Gamma\left(\frac{m}{2}\right)\Sigma^{m/2}}V^{(m-2)/2}\mathrm{etr}\left(-\frac{1}{2}\frac{V}{\Sigma}\right) \tag{23}$$

Note that $\Sigma = A^2$. Also for $p = 1$, $W \sim \chi^2_m$, so that

$$f_W(w) = \frac{1}{2^{m/2}\Gamma\left(\frac{m}{2}\right)}w^{(m-2)/2}\exp\left(-\frac{1}{2}w\right), w > 0. \tag{24}$$

Now consider the Jacobian transformation $y = a^2w = awa$, then $\frac{y}{a^2} = w$.

Then the Jacobian transformation J is $J(y \to w) = \frac{1}{a^2}$. Hence,

$$f_W(w) = \frac{1}{2^{m/2}\Gamma\left(\frac{m}{2}\right)}\left(\frac{y}{a^2}\right)^{(m-2)/2}\exp\left(-\frac{y}{a^2}\right)\left|\frac{1}{a^2}\right| \tag{25}$$

$$= \frac{1}{2^{m/2}\Gamma\left(\frac{m}{2}\right)}\frac{y^{(m-2)/2}}{(a^2)^{(m-2)/2}}\exp\left(-\frac{y}{a^2}\right)\left(\frac{1}{a^2}\right) \tag{26}$$

$$= \frac{1}{2^{m/2}\Gamma\left(\frac{m}{2}\right)}\frac{y^{(m-2)/2}}{(a^2)^{(m)/2}}\exp\left(-\frac{y}{a^2}\right) \tag{27}$$

$$= \frac{1}{2^{m/2}\Gamma\left(\frac{m}{2}\right)}\frac{y^{(m-2)/2}}{(a^2)^{(m)/2}}\mathrm{etr}\left(-\frac{y}{a^2}\right). \tag{28}$$

Hence Equations (23) and (28) are equivalent if we let $V = y$ and $\Sigma = a^2$. Therefore Equation (22) is true for $p = 1$. Now let $r = 1$ and $s = p - 1$ in Proposition 7.9, then

$$V_{11.2} \sim W(m-p+1, \Sigma_{11.2})$$
$$\mathbf{V}_{21}|\mathbf{V}_{22} \sim \mathcal{N}_{p-1}(\mathbf{V}_{22}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}, \mathbf{V}_{22}\Sigma_{11.2})$$
$$\mathbf{V}_{22} \sim W_{p-1}(m, \mathbf{\Sigma}_{22})$$

where $V_{11.2} \perp (\mathbf{V}_{21}, \mathbf{V}_{22})$. Thus the joint density is given by,

$$\frac{V_{11.2}^{(m-p+1)/2-1}\mathrm{etr}\left(-\frac{1}{2}\frac{V_{11.2}}{\Sigma_{11.2}}\right)}{2^{(m-p+1)/2}\Gamma\left(\frac{m-p+1}{2}\right)\Sigma_{11.2}^{(m-p+1)/2}}$$
$$\times\frac{\exp\left\{-\frac{1}{2}\left(\mathbf{V}_{21} - \mathbf{V}_{22}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)^{\mathrm{T}}(\Sigma_{11.2}\mathbf{V}_{22})^{-1}\left(\mathbf{V}_{21} - \mathbf{V}_{22}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)\right\}}{(2\pi)^{(p-1)/2}|\Sigma_{11.2}\mathbf{V}_{22}|^{1/2}}$$
$$\times\frac{|\mathbf{V}_{22}|^{(m-(p-1)-1)/2}\mathrm{etr}(-\frac{1}{2}\mathbf{\Sigma}_{22}^{-1}\mathbf{V}_{22})}{2^{(m(p-1))/2}\Gamma_{p-1}\left(\frac{m}{2}\right)|\mathbf{\Sigma}_{22}|^{m/2}} \tag{29}$$

We make change of variables $(V_{11.2}, \mathbf{V}_{21}, \mathbf{V}_{22}) \to (V_{11}, \mathbf{V}_{21}, \mathbf{V}_{22})$, and since $V_{11.2} = V_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}$. Then, $J[(V_{11.2}, \mathbf{V}_{21}, \mathbf{V}_{22}) \to (V_{11}, \mathbf{V}_{21}, \mathbf{V}_{22})] = |1|_+$. So that combining the factors in Equation (29) should lead to the Wishart density in Equation (22). In particular, the following enumeration attempts to summarize the process of simplifying Equation (29):

(a) Exponents of $\frac{1}{2}$:

$$\frac{(m-p+1)+p-1+m(p-1)}{2} = \frac{mp}{2}$$

(b) Combining the $\Gamma\left(\frac{m-p+1}{2}\right), \Gamma_{p-1}\left(\frac{m}{2}\right)$, and $\pi^{(p-1)/2}$: Note that

$$\Gamma_{p-1}\left(\frac{m}{2}\right) = \pi^{(p-1)(p-2)/4}\prod_{i=1}^{p-1}\Gamma_{p-1}\left[\frac{m}{2} - \frac{1}{2}(i-1)\right] \tag{30}$$

So that if $i = p$, then

$$\Gamma_{p-1}\left[\frac{m}{2} - \frac{1}{2}(p-1)\right] = \Gamma_{p-1}\left(\frac{m-p+1}{2}\right). \tag{31}$$

Thus,

$$\pi^{(p-1)/2}\pi^{(p-1)(p-2)/4}\Gamma_{p-1}\left(\frac{m-p+1}{2}\right)\prod_{i=1}^{p-1}\Gamma_{p-1}\left[\frac{m}{2} - \frac{1}{2}(i-1)\right]$$
$$= \pi^{2(p-1)/4}\pi^{(p-1)(p-2)/4}\prod_{i=1}^{p}\Gamma_{p-1}\left[\frac{m}{2} - \frac{1}{2}(i-1)\right]$$
$$= \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma_{p-1}\left[\frac{m}{2} - \frac{1}{2}(i-1)\right]$$
$$= \Gamma_{p-1}\left(\frac{m}{2}\right).$$

(c) Combining $V_{11.2}^{(m-p+1)/2-1}, |\mathbf{V}_{22}|^{1/2}, |\mathbf{V}_{22}^{(m-p)/2}$:

$$V_{11.2}^{(m-p+1)/2-1}|\mathbf{V}_{22}|^{-1/2}|\mathbf{V}_{22}|^{(m-p)/2} = V_{11.2}^{(m-p+1)/2-1}|\mathbf{V}_{22}|^{(m-p-1)/2}$$
$$= |\mathbf{V}_{22}|^{(m-p-1)/2}$$

(d) Combining $\Sigma_{11.2}^{(m-p+1)/2}, \Sigma_{11.2}^{-1/2}, |\mathbf{\Sigma}_{22}|^{m/2}$:

$$\Sigma_{11.2}^{(m-p+1)/2}\Sigma_{11.2}^{-1/2}|\mathbf{\Sigma}_{22}|^{m/2} = \Sigma_{11.2}^{(m-p)/2}|\mathbf{\Sigma}_{22}|^{m/2}$$
$$= \Sigma_{11.2}^{m/2}\Sigma_{11.2}^{-p/2}|\mathbf{\Sigma}_{22}|^{m/2}$$
$$= \Sigma_{11.2}^{-p/2}(\Sigma_{11.2}|\mathbf{\Sigma}_{22}|)^{m/2}$$
$$= \Sigma_{11.2}^{-p/2}|\mathbf{\Sigma}_{22}|^{m/2}.$$

(e) Combining the exponential functions:

$$\mathrm{etr}\left(-\frac{1}{2}\frac{V_{11.2}}{\Sigma_{11.2}}\right)\mathrm{etr}\left(-\frac{1}{2}\mathbf{\Sigma}_{22}^{-1}\mathbf{V}_{22}\right)$$
$$\times\exp\left\{-\frac{1}{2}\left(\mathbf{V}_{21} - \mathbf{V}_{22}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)^{\mathrm{T}}(\Sigma_{11.2}\mathbf{V}_{22})^{-1}\left(\mathbf{V}_{21} - \mathbf{V}_{22}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)\right\}.$$

Simplifying this and combining with the previous items should generate the Wishart distribution.

$\square$