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Homework | **Stat230**

1 General Instructions

Prove the following:

Q1. If $\{A_n\}_{n=1}^{k}$ is any finite collection of sets, disjoint or not, then

$$\mu^*\left(\bigcup_{n=1}^{k} A_n\right) \leq \sum_{n=1}^{k} \mu^*(A_n) \tag{1}$$

2 Solutions

Q1. *Proof.* This finite subadditivity property follows from countable subadditivity by taking $A_n = \emptyset$ for $n > k$. Then there exists a finite collection $\{I_{n,i}\}$ of open intervals such that,

$$\{A_n\}_{n=1}^{k} \subset \bigcup_{i=1}^{k} \{I_{n,i}\}_{n=1}^{k}.$$

It follows that, $\displaystyle\sum_{i=1}^{k} l(I_{n,i}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}$. Note that $\displaystyle\bigcup_{n=1}^{k} A_n$ is covered by $\{I_{n,i}\}$. Therefore, from the definition of outer measure, we have

$$\begin{aligned}
\mu^*\left(\bigcup_{n=1}^{k} A_n\right) &\leq \sum_{n=1}^{k}\left[\sum_{i=1}^{k} l(I_{n,i})\right] \\
&< \sum_{n=1}^{k}\left[\mu^*(A_n) + \frac{\varepsilon}{2^n}\right] \\
&< \sum_{n=1}^{k} \mu^*(A_n) + \sum_{n=1}^{k} \frac{\varepsilon}{2^n} \\
&< \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon\left[\sum_{n=1}^{\infty} \frac{1}{2^n}\right], \text{ since } A_n = \emptyset \text{ for } k < n \to \infty
\end{aligned}$$

$$= \sum_{n=1}^{k} \mu^*(A_n) + \varepsilon, \text{ from previous argument, } \sum_{n=1}^{k} \mu^*(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$$

And because $\varepsilon > 0$ is very small, then $\varepsilon = 0$ holds. $\square$