School of Statistics, University of the Philippines (Diliman)
Linangan ng Estadistika, Unibersidad ng Pilipinas (Diliman)

12 of March 2016

Al-Ahmadgaid B. Asaad
PS 1 | **Stat 234**

1. Assuming $\mathbf{A}$ and $\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}}$ are nonsingular, prove

$$(\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}}{(1+\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})} \tag{1}$$

Proof. Since $\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}}$ is nonsingular, then the inverse of its inverse exists. And using the fact that

$$(\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1}(\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}}) = \mathbf{I}, \tag{2}$$

then we want to show that

$$\left[\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}}{(1+\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})}\right](\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}}) = \mathbf{I}. \tag{3}$$

Simpliying this leads to the following

$$\mathbf{A}^{-1}(\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}}) - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}}{(1+\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})}(\mathbf{A}+\mathbf{u}\mathbf{v}^{\mathrm{T}}) \stackrel{?}{=} \mathbf{I} \tag{4}$$

$$\mathbf{I} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}}{(1+\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})} \stackrel{?}{=} \mathbf{I} \tag{5}$$

For brevity, let the scalar $(1+\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})$ factor be α. Then

$$\mathbf{I} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}}{\alpha} \stackrel{?}{=} \mathbf{I} \tag{6}$$

Note that $\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}$ is a scalar, then

$$\mathbf{I} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} + (\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}}{\alpha} \stackrel{?}{=} \mathbf{I} \tag{7}$$

$$\mathbf{I} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} - \frac{[1+(\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})]\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}}{\alpha} \stackrel{?}{=} \mathbf{I} \tag{8}$$

$$\mathbf{I} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}} - \frac{\alpha\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}}{\alpha} \stackrel{\checkmark}{=} \mathbf{I} \tag{9}$$

$\square$

2. **Rayleigh's quotient**.
   Assume $\mathbf{S} \geq \mathbf{0}$ in $\mathbb{R}_n^n$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and corresponding eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$. Prove:

   (a)
   $$\lambda_n \leq \frac{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \leq \lambda_1, \quad \forall \mathbf{x} \neq \mathbf{0}. \tag{10}$$

   (b) For any fixed $j = 2, \cdots, n$,
   $$\frac{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \leq \lambda_j, \quad \forall \mathbf{x} \neq \mathbf{0} \tag{11}$$
   such that $\langle \mathbf{x}, \mathbf{x}_1\rangle = \cdots = \langle \mathbf{x}, \mathbf{x}_{j-1}\rangle = 0$.

3. Given a bivariate copula d.f. $C(t_1, t_2)$, two measures of association are Spearman's ρ and Kendall's τ,

$$\rho = 12\int_{[0,1]^2} t_1 t_2 \mathrm{d}C(t_1, t_2) - 3, \tag{12}$$

$$\tau = 4\int_{[0,1]^2} C(t_1, t_2)\mathrm{d}C(t_1, t_2) - 1, \tag{13}$$

   $|\rho| \leq 1$ and $|\tau| \leq 1$. Now, let $|\alpha| < 1/3$ in the bivariate Morgenstern copula

$$C(t_1, t_2) = t_1 t_2[1 + 3\alpha(1-t_1)(1-t_2)]. \tag{14}$$

   Verify this copula is parameterized by Spearman's meausre, or

$$\alpha = 12\int_{[0,1]^2} t_1 t_2 \mathrm{d}C(t_1, t_2) - 3. \tag{15}$$

Proof. Consider the following differentiation,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t_1\mathrm{d}t_2}C(t_1,t_2) &= \frac{\mathrm{d}^2}{\mathrm{d}t_1\mathrm{d}t_2}(t_1t_2+3\alpha t_1t_2-3\alpha t_1t_2^2-3\alpha t_1^2t_2+3\alpha t_1^2t_2^2)\\ &= \frac{\mathrm{d}}{\mathrm{d}t_2}(t_2+3\alpha t_2-3\alpha t_2^2-6\alpha t_1t_2+6\alpha t_1t_2^2)\\ &= 1+3\alpha-6\alpha t_2-6\alpha t_1+12\alpha t_1t_2.\end{aligned}$$

We want to show that $\rho=\alpha$. So that

$$\begin{aligned}\rho &= 12\int_{[0,1]^2} t_1t_2\mathrm{d}C(t_1,t_2)-3\\ &= 12\int_{[0,1]}\int_{[0,1]} t_1t_2\mathrm{d}^2C(t_1,t_2)-3\\ &= 12\int_{[0,1]}\int_{[0,1]} t_1t_2\frac{\mathrm{d}^2}{\mathrm{d}t_1\mathrm{d}t_2}C(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2-3\\ &= 12\int_{[0,1]}\int_{[0,1]} t_1t_2(1+3\alpha-6\alpha t_2-6\alpha t_1+12\alpha t_1t_2)\mathrm{d}t_1\mathrm{d}t_2-3\\ &= 12\int_{[0,1]}\int_{[0,1]} (t_1t_2+3\alpha t_1t_2-6\alpha t_1t_2^2-6\alpha t_1^2t_2+12\alpha t_1^2t_2^2)\mathrm{d}t_1\mathrm{d}t_2-3\\ &= 12\int_{[0,1]}\left(\frac{t_1^2}{2}t_2+3\alpha\frac{t_1^2}{2}t_2-3\alpha t_1^2t_2^2-2\alpha t_1^3t_2+4\alpha t_1^3t_2^2\right)\Bigg|_{t_1=0}^{t_1=1}\mathrm{d}t_2-3\\ &= 12\int_{[0,1]}\left(\frac{1}{2}t_2+3\alpha\frac{1}{2}t_2-3\alpha t_2^2-2\alpha t_2+4\alpha t_2^2\right)\mathrm{d}t_2-3\\ &= 12\left(\frac{1}{4}t_2^2+3\alpha\frac{1}{4}t_2^2-\alpha t_2^3-\alpha t_2^2+\frac{4}{3}\alpha t_2^3\right)\Bigg|_0^1-3\\ &= 12\left(\frac{1}{4}+3\alpha\frac{1}{4}-2\alpha+\frac{4}{3}\alpha\right)-3\\ &= 3+9\alpha-24\alpha+16\alpha-3=\alpha\end{aligned}$$

□

4. **Corollary 3.2 (Marginal Dirichlet)** If $\mathbf{x}_1=(x_{i_1},\cdots,x_{i_K})^{\mathrm{T}}$ denotes any subset of the coordinates, then $\mathbf{x}_1\sim\mathcal{D}_K(\mathbf{p}_1,q)$ with $\mathbf{p}_1=(p_{i_1},\cdots,p_{i_K})^{\mathrm{T}}$ and $p=q+\sum_{j=1}^K p_{i_j}$.

Proof. Let $\mathbf{y}\sim\mathcal{D}_{n-1}(\mathbf{p},p_n)$ where $\mathbf{p}=(p_1,\cdots,p_{n-1})^{\mathrm{T}},p_i>0,i=1,\cdots,n$. Suppose $\mathbf{y}=(x_1,\cdots,x_n)^{\mathrm{T}}$ where $x_n=1-\sum_j^{n-1}x_j$, then $\mathbf{y}$ forms the vector coordinates of the probability simplex such that $\sum_{i=1}^n x_i=1$. If $\mathbf{x}_1=(x_{i_1},\cdots,x_{i_K})^{\mathrm{T}}$ is any subset of the coordinate vector $\mathbf{y}$, then it should be understood that $K<n$ (for example, if the probability simplex has $n=3$ dimension, then the plane x_1x_2 forms another simplex on $K=2$ dimensional space). Recall that the Dirichlet distribution is mathematically given by:

$$\mathbb{P}(\mathbf{y}|\boldsymbol{\alpha}) \triangleq \frac{\Gamma(\mathfrak{S})}{\prod_{i=1}^n\Gamma(\alpha_i)}\prod_{i=1}^n x_i^{\alpha_i-1} \tag{16}$$

$$=\frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\times\cdots\times\Gamma(\alpha_n)} \tag{17}$$

$$\times\, x_1^{\alpha_1-1}\times\cdots\times x_{n-1}^{\alpha_{n-1}-1}\left[1-\sum_{j=1}^{n-1}x_j^{\alpha_j-1}\right]^{\alpha_n-1}. \tag{18}$$

The marginal distribution of $\mathbf{x}_1$ is done by integrating out $x_{i_l^\star}$ variables such that $i_l^\star\neq i_k,\forall k=1,\cdots,K$ and $\forall l=1,\cdots,n-1-K$. That is,

$$\mathbb{P}(\mathbf{x}_1)=\int\int\cdots\int\mathbb{P}(x_{i_1},\cdots,x_{i_K},x_{i_1^\star},\cdots,x_{i_{n-1-K}^\star})\mathrm{d}x_{i_1^\star}\mathrm{d}x_{i_2^\star}\cdots\mathrm{d}x_{i_{n-1-K}^\star}$$

Note that for purpose of generalization, the $x_{i_l^\star}$s are not necessarily in a particular order or sequence so long as $\sum_{i=1}^n x_i=\sum_{k=1}^K x_{i_k}+\sum_{l=1}^{n-K}x_{i_l^\star}=1$. So that

$$\begin{aligned}\mathbb{P}(\mathbf{x}_1)=&\frac{\Gamma(\alpha_{i_1}+\cdots+\alpha_{i_{n-K}^\star})}{\Gamma(\alpha_{i_1})\times\cdots\times\Gamma(\alpha_{i_{n-K}^\star})}x_{i_1}^{\alpha_{i_1}-1}\times\cdots\times x_{i_K}^{\alpha_{i_K}-1}\\ &\times\int\cdots\int x_{i_1^\star}^{\alpha_{i_1^\star}-1}\times\cdots\times x_{i_{n-1-K}^\star}^{\alpha_{i_{n-1-K}^\star}-1}\\ &\times\left[1-\sum_{k=1}^K x_{i_k}-\sum_{l=1}^{n-2-K}x_{i_l^\star}-x_{i_{n-1-K}^\star}\right]^{\alpha_{i_{n-K}^\star}-1}\mathrm{d}x_{i_{n-1-K}^\star}\cdots\mathrm{d}x_{i_1^\star}\end{aligned}$$

Now we want to prove by induction that for all l, the marginal distribution of $\mathbf{x}_1$ is a Dirichlet distribution. In particular, for $l=n-1-K$: let $x_{i_{n-1-K}^\star}=\left[1-\sum_{k=1}^K x_{i_k}-\sum_{l=1}^{n-2-K}x_{i_l^\star}\right]u$. Then evaluating the first integration with respect to $x_{i^\star_{n-1-K}}$ gives:

$$\begin{aligned}
&\int \frac{\Gamma(\alpha_{i_1}+\cdots+\alpha_{i^\star_{n-K}})}{\Gamma(\alpha_{i_1})\times\cdots\times\Gamma(\alpha_{i^\star_{n-K}})} x_{i_1}^{\alpha_{i_1}-1}\times\cdots\times x_{i_K}^{\alpha_{i_K}-1} \\
&\times x_{i^\star_1}^{\alpha_{i^\star_1}-1}\times\cdots\times\left\{\left[1-\sum_{k=1}^{K}x_{i_k}-\sum_{l=1}^{n-2-K}x_{i^\star_l}\right]u\right\}^{\alpha_{i^\star_{n-1-K}}-1} \\
&\times\left\{(1-u)\left[1-\sum_{k=1}^{K}x_{i_k}-\sum_{l=1}^{n-2-K}x_{i^\star_l}\right]\right\}^{\alpha_{i^\star_{n-K}}-1}\mathrm{d}x_{i^\star_{n-1-K}} \\
=&\frac{\Gamma(\alpha_{i_1}+\cdots+\alpha_{i^\star_{n-K}})}{\Gamma(\alpha_{i_1})\times\cdots\times\Gamma(\alpha_{i^\star_{n-K}})} x_{i_1}^{\alpha_{i_1}-1}\times\cdots\times x_{i_K}^{\alpha_{i_K}-1} \\
&\times x_{i^\star_1}^{\alpha_{i^\star_1}-1}\cdots\left\{\left[1-\sum_{k=1}^{K}x_{i_k}-\sum_{l=1}^{n-2-K}x_{i^\star_l}\right]\right\}^{\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}}-1} \\
&\times\underbrace{\int u^{\alpha_{i^\star_{n-1-K}}-1}(1-u)^{\alpha_{i^\star_{n-K}}-1}\mathrm{d}x_{i^\star_{n-1-K}}}_{\text{Kernel of Beta}\left(\alpha_{i^\star_{n-1-K}},\alpha_{i^\star_{n-K}}\right)}\cdots\mathrm{d}x_{i^\star_1} \\
=&\frac{\Gamma(\alpha_{i_1}+\cdots+\alpha_{i^\star_{n-K}})}{\Gamma(\alpha_{i_1})\times\cdots\times\Gamma(\alpha_{i^\star_{n-K}})} x_{i_1}^{\alpha_{i_1}-1}\times\cdots\times x_{i_K}^{\alpha_{i_K}-1} \\
&\times x_{i^\star_1}^{\alpha_{i^\star_1}-1}\cdots\left\{\left[1-\sum_{k=1}^{K}x_{i_k}-\sum_{l=1}^{n-2-K}x_{i^\star_l}\right]\right\}^{\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}}-1} \\
&\frac{\Gamma(\alpha_{i^\star_{n-1-K}})\Gamma(\alpha_{i^\star_{n-K}})}{\Gamma(\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}})}\mathrm{d}x_{i^\star_{n-2-K}}\cdots\mathrm{d}x_{i^\star_1}.
\end{aligned}$$

Ignoring for now other remaining integrals returns the marginal distribution for the remaining variables $x_{i_1},\cdots,x_{i_K},x_{i^\star_1},\cdots,x_{i^\star_{n-2-K}}$ and is given by

$$\begin{aligned}
&\frac{\Gamma(\alpha_{i_1}+\cdots+\alpha_{i^\star_{n-K}})}{\Gamma(\alpha_{i_1})\times\cdots\times\Gamma(\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}})} x_{i_1}^{\alpha_{i_1}-1}\times\cdots\times x_{i_K}^{\alpha_{i_K}-1}x_{i^\star_1}^{\alpha_{i^\star_1}-1} \\
&\times\cdots x_{i^\star_{n-2-K}}^{\alpha_{i^\star_{n-2-K}}-1}\left\{\left[1-\sum_{k=1}^{K}x_{i_k}-\sum_{l=1}^{n-2-K}x_{i^\star_l}\right]\right\}^{\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}}-1}.
\end{aligned}\tag{19}$$

Hence the distribution of the subset coordinate $\mathbf{x}_1^{(1)} \triangleq (x_{i_1},\cdots,x_{i_K},x_{i^\star_1},\cdots,x_{i^\star_{n-2-K}})^{\mathrm{T}}$ is Dirichlet distribution with parameters $(\mathbf{p}_1^{(1)},q^{(1)})$ where $\mathbf{p}_1^{(1)} = (\alpha_{i_1},\cdots,\alpha_{i_K},\alpha_{i^\star_1},\cdots,\alpha_{i^\star_{n-2-K}})^{\mathrm{T}}$ and $q^{(1)} = \alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}}$. Equivalently, since $\mathbf{x}_1$ is arbitrary then $\mathbf{x}_1^{(1)}$ is one possible vector of $\mathbf{x}_1$ with parameters $\mathbf{p}_1=\mathbf{p}_1^{(1)}$ and if we let $p\triangleq\sum_{i=1}^n\alpha_i=\sum_{k=1}^K\alpha_{i_k}+\sum_{l=1}^{n-K}\alpha_{i^\star_l}$ then $q^{(1)}=p-\mathbf{1}^{\mathrm{T}}\mathbf{p}_1^{(1)}=p-\sum_{k=1}^K\alpha_{i_k}-\sum_{l=1}^{n-2-K}\alpha_{i^\star_l}$.

If we continue with the remaining integration this time with respect to $x_{i^\star_{n-2-K}}$, the marginal distribution is still Dirichlet distribution but with parameters $(\mathbf{p}_1^{(2)},q^{(2)})$ where $\mathbf{p}_1^{(2)}=(\alpha_{i_1},\cdots,\alpha_{i_K},\alpha_{i^\star_1},\cdots,\alpha_{i^\star_{n-3-K}})^{\mathrm{T}}$ and $q^{(2)}=\alpha_{i^\star_{n-2-K}}+\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}}=p-\mathbf{1}^{\mathrm{T}}\mathbf{p}_1^{(2)}=p-\sum_{k=1}^K\alpha_{i_k}-\sum_{l=1}^{n-3-K}\alpha_{i^\star_l}$. To see this, apply the change variable technique, that is for $l=n-2-K$: let $x_{i^\star_{n-2-K}}=\left[1-\sum_{k=1}^K x_{i_k}-\sum_{l=1}^{n-3-K}x_{i^\star_l}\right]u$. Then with a little algebra aiming for the kernel of Beta distribution, this time with parameters $\alpha_{i^\star_{n-2-K}}$ and $\alpha_{i^\star_{n-1-K}}+\alpha_{i^\star_{n-K}}$, one will obtain the said marginal distribution.

Therefore in general, for any subset coordinate vector $\mathbf{x}_1$, the distribution of $\mathbf{x}_1$ is Dirichlet with parameters stated in the Corollary that is now verified to be true. □

5. **Corollary 3.4** If $S=\mathbf{x}^{\mathrm{T}}\mathbf{1}=\sum_{i=1}^n x_i$ and again $\mathbf{x}_1=(x_{i_1},\cdots,x_{i_K})^{\mathrm{T}}, K<n$ is any subset, let $\mathbf{w}_1\overset{d}{=}\frac{1}{S}\mathbf{x}_1$. We find $\mathbf{w}_1\sim\mathcal{D}_K(\mathbf{p}_1;r)$ with $\mathbf{p}_1=(p_{i_1},\cdots,p_{i_K})^{\mathrm{T}}$ as before but this time, $p-p_{n+1}=r+\sum_{j=1}^K p_{i_j}$.

Proof. From Corollary 3.2, we proved that $\mathbf{x}_1\sim\mathcal{D}_k(\mathbf{p}_1,q)$ and $p=q+\sum_{j=1}^K p_{i_j}$. Then

$$\mathbb{P}(\mathbf{x}_1|\mathbf{p},q)\triangleq\frac{\Gamma(\mathfrak{S})}{\prod_{i=1}^{n-K}\Gamma(\alpha_i)\Gamma(q)}\prod_{i=1}^{n-K-1}x_i^{\alpha_i-1}\left(1-\sum_{j=1}^{n-K-1}x_i^{\alpha_i-1}\right)^{q-1}\tag{20}$$

where $\mathfrak{S}=\sum_{i=1}^{n-K}\alpha_i$. To show that $\mathbf{w}_1\overset{d}{=}\frac{1}{S}\mathbf{x}_1$, we have to consider

Jacobian transformation. That is,

$$\mathbf{x}_1 \stackrel{d}{=} S\mathbf{w}_1 \quad \text{or} \quad \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_K} \end{bmatrix} \stackrel{d}{=} \begin{bmatrix} Sw_1 \\ Sw_2 \\ \vdots \\ S \end{bmatrix}. \tag{21}$$

Now using vector differentiation and noting that $x_{i_K} = S(1 - \sum_{k=1}^{K-1} x_{i_k})$, we have

$$\frac{\partial \mathbf{x}_1}{\partial \mathbf{w}_1} = \begin{vmatrix} \frac{\partial x_{i_1}}{\partial w_1} & \frac{\partial x_{i_1}}{\partial w_2} & \cdots & \frac{\partial x_{i_1}}{\partial w_K} \\ \frac{\partial x_{i_2}}{\partial w_1} & \frac{\partial x_{i_2}}{\partial w_2} & \cdots & \frac{\partial x_{i_2}}{\partial w_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{i_K}}{\partial w_1} & \frac{\partial x_{i_K}}{\partial w_2} & \cdots & \frac{\partial x_{i_K}}{\partial w_K} \end{vmatrix}$$

The rest of the proof is analogous to the proof of Proposition 3.1. Hence, following the proof of the said Proposition, Corollary 3.4 then follows. $\square$