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Problem Set 2 | **Stat 245**

3.8 The following data consists of the times to relapse and death of 10 bone marrow transplant patients. In the sample patients 4 and 6 were still alive at the end of the study and patients 7-10 were alive, free of relapse at the end of the study. Suppose the time to relapse had an exponential distribution with hazard rate λ and the time to death had a Weibull distribution with parameters θ and α.

Patient	Relapse Time (months)	Death Time (months)
1	5	11
2	8	12
3	12	15
4	24	35^+
5	32	45
6	17	28^+
7	16^+	16^+
8	17^+	17^+
9	19^+	19^+
10	30^+	30^+

(a) Construct the likelihood for the relapse rate λ.

(b) Construct a likelihood for the parameters θ and α.

(c) Suppose we were only allowed to observe a patients death time if the patient relapsed. Construct the likelihood for θ and α based on this truncated sample, and compare it to the results in (b).

Solution

(a) Let X be the time to relapse, then $X \sim$ exponential(λ). And since the data involve here are the true lifetimes and the right censored observations, thus we have the following likelihood functions:

$$\mathcal{L} \propto \prod_{i\in D} f(x_i) \prod_{i\in R} S(C_r).$$

From Table 2.2 (refer to the book), $f(x) = \lambda\exp(-\lambda x)$, $S(x) = \exp(-\lambda x)$. So that,

$$\begin{aligned}\mathcal{L} &\propto \prod_{i\in D} f(x_i) \prod_{i\in R} S(C_r)\\ &= \prod_{i\in D} \lambda\exp(-\lambda x_i) \prod_{i\in R} \exp(-\lambda x_i)\\ &= \lambda^{\#\{i\in D\}} \exp\left[-\lambda \sum_{i\in D} x_i\right] \times \exp\left[-\lambda\sum_{i\in R} x_i\right],\end{aligned} \tag{1}$$

where $\#\{i \in D\}$ is the cardinality of the set $\{i \in D\}$. Now we take the log-likelihood function,

$$\ell = \#\{i\in D\}\log\lambda - \lambda\sum_{i\in D} x_i - \lambda\sum_{i\in R} x_i.$$

To maximize this log-likelihood function with respect to the parameter λ, we take the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\ell = \#\{i\in D\}\frac{1}{\lambda} - \sum_{i\in D} x_i - \sum_{i\in R} x_i.$$

Now we set it to zero obtain the stationary point,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\ell = \#\{i\in D\}\frac{1}{\lambda} - \sum_{i\in D} x_i - \sum_{i\in R} x_i &\overset{set}{=} 0\\ \Rightarrow \frac{1}{\lambda} &= \frac{1}{\#\{i\in D\}}\left[\sum_{i\in D} x_i + \sum_{i\in R} x_i\right]\\ \Rightarrow \hat{\lambda} &= \frac{\#\{i\in D\}}{\left[\sum_{i\in D} x_i + \sum_{i\in R} x_i\right]}.\end{aligned}$$

To verify if this is the maximum likelihood estimator, we take the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}\,\lambda^2}\ell = -\#\{i\in D\}\frac{1}{\lambda^2},$$

which is negative for all λ, implying $\hat{\lambda}$ is the MLE. Now plugging-in the data, the plausible estimate of the parameter that likely explains the observations is,

$$\lambda = \frac{\#\{i \in D\}}{\sum_{i\in D} x_i + \sum_{i\in R} x_i}.$$

The $\#\{i \in D\} = 6$ since there are 6 patients with uncensored lifetime. And

$$\sum_{i\in D} x_i = 5 + 8 + 12 + 24 + 32 + 17 = 98,$$

also

$$\sum_{i\in R} x_i = 16 + 17 + 19 + 30 = 82.$$

And therefore,

$$\lambda = \frac{6}{98+82} = \frac{1}{30}.$$

Figure 1 is the plot of the likelihood function of X.

(b) Let Y be the time to death, then $Y \sim \text{Weibull}(\theta, \alpha)$, and hence we have the following likelihood function:

$$\mathcal{L} \propto \prod_{i\in D} f(x_i) \prod_{i\in R} S(C_r).$$

The density and the survival function of Y, is given by

$$f(x) = \alpha\theta x^{\alpha-1}\exp(-\theta x^{\alpha}), \quad \text{and} \quad S(x) = \exp(-\theta x^{\alpha}),$$

respectively. So that,

$$\begin{aligned}
\mathcal{L} &\propto \prod_{i\in D} f(x_i) \prod_{i\in R} S(C_r) \\
&= \prod_{i\in D} \alpha\theta x_i^{\alpha-1}\exp(-\theta x_i^{\alpha}) \prod_{i\in R} \exp(-\theta x_i^{\alpha}) \\
&= (\alpha\theta)^{\#\{i\in D\}} \exp\left[-\theta \sum_{i\in D} x_i^{\alpha}\right] \exp\left[-\theta \sum_{i\in R} x_i^{\alpha}\right] \prod_{i\in D} x_i^{\alpha-1}
\end{aligned}$$

where $\#\{i \in D\}$ is the cardinality of the set $\{i \in D\}$. Now we take the log-likelihood function

$$\ell = \#\{i \in D\}(\log\alpha + \log\theta) - \theta\sum_{i\in D} x_i^{\alpha} - \theta\sum_{i\in R} x_i^{\alpha} + (\alpha - 1)\sum_{i\in D}\log x_i.$$

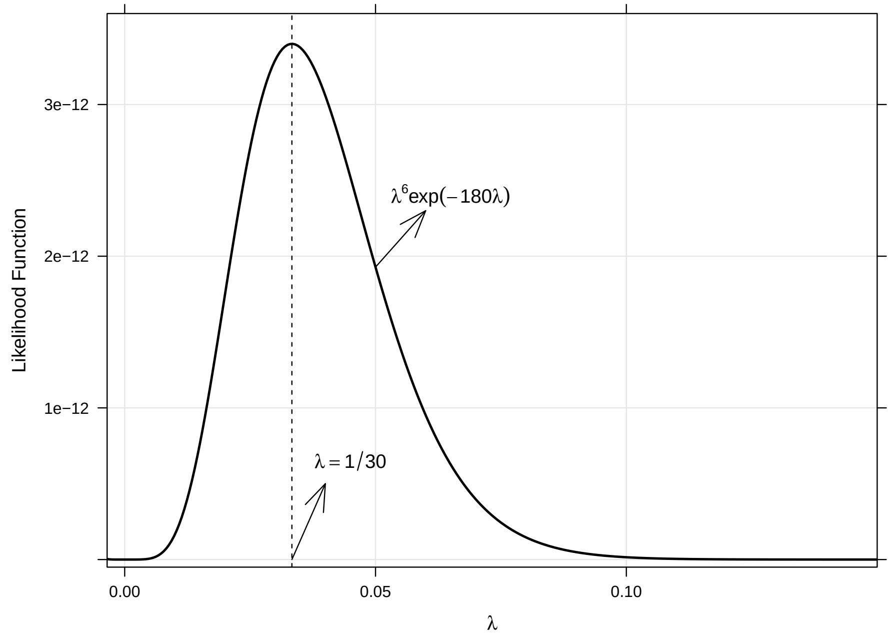

Figure 1: The likelihood function of X.

Now we take the partial derivatives with respect to both parameters,

$$\frac{\partial\,\ell}{\partial\,\theta} = \frac{\#\{i \in D\}}{\theta} - \sum_{i\in D} x_i^{\alpha} - \sum_{i\in R} x_i^{\alpha}$$

Setting this to zero to obtain the critical points, then

$$\begin{aligned}
\frac{\partial\,\ell}{\partial\,\theta} &= \frac{\#\{i \in D\}}{\theta} - \sum_{i\in D} x_i^{\alpha} - \sum_{i\in R} x_i^{\alpha} \overset{set}{=} 0 \\
\Rightarrow \hat{\theta} &= \frac{\#\{i \in D\}}{\sum_{i\in D} x_i^{\alpha} + \sum_{i\in R} x_i^{\alpha}}
\end{aligned}$$

Clearly this is the MLE for θ. Therefore, plugging the data, we have $\#\{i \in D\} = 4$ since there are four patients with exact lifetime.

$$\sum_{i\in D} x_i^{\alpha} = 11^{\alpha} + 12^{\alpha} + 15^{\alpha} + 45^{\alpha}$$

and

$$\sum_{i \in R} x_i^\alpha = 33^\alpha + 28^\alpha + 16^\alpha + 17^\alpha + 19^\alpha + 30^\alpha$$

So that if $\alpha = 1$, then

$$\hat{\theta} = \frac{4}{83 + 143} = \frac{2}{113} = .0177.$$

Speaking about α, the partial derivative with respect to it is,

$$\frac{\partial \ell}{\partial \alpha} = \frac{\#\{i \in D\}}{\alpha} - \theta\alpha \sum_{i \in D} x_i^{\alpha-1} - \theta\alpha \sum_{i \in R} x_i^{\alpha-1} + \sum_{i \in D} \log x_i \overset{set}{=} 0$$

In this case, there is no way to solve α. So we need to use iterative search procedure. The following are steps for Newton-Raphson iterative search:

1. Begin with a first guess $\hat{\alpha}^{(0)}$ for a root of the function $\frac{\partial \ell}{\partial \alpha}$.

   $$\frac{\partial \ell}{\partial \alpha} = \frac{\#\{i \in D\}}{\alpha} - \theta\alpha \sum_{i \in D} x_i^{\alpha-1} - \theta\alpha \sum_{i \in R} x_i^{\alpha-1} + \sum_{i \in D} \log x_i \overset{set}{=} 0,$$

   where θ of course is estimated by its MLE, $\hat{\theta} = \frac{\#\{i \in D\}}{\sum_{i \in D} x_i^\alpha + \sum_{i \in R} x_i^\alpha}$.
2. Solve for $\Delta\hat{\alpha}$, evaluating partial derivatives at the initial value $\hat{\alpha}^{(0)}$.

   $$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \left(-\frac{\partial^2 \ell}{\partial \alpha^2}\right)(\hat{\alpha}^{(1)} - \hat{\alpha}^{(0)}) \\ &= \left(-\frac{\partial^2 \ell}{\partial \alpha^2}\right)\Delta\hat{\alpha} \end{aligned}$$

3. So that $\Delta\hat{\alpha} = \hat{\alpha}^{(1)} - \hat{\alpha}^{(0)} \Rightarrow \hat{\alpha}^{(1)} = \Delta\hat{\alpha} + \hat{\alpha}^{(0)}$.
4. Repeat Step 2, but this time evaluating partial derivatives at $\hat{\alpha}^{(1)}$, and then proceed to Step 3 to obtain the new initial value, $\hat{\alpha}^{(2)}$.

Repeat the process above, until convergence to the value of $\hat{\alpha}$. That is, if the difference between $\hat{\alpha}^{(i)}$ and $\hat{\alpha}^{(i-1)}$ is close to zero at the ith iteration.

The likelihood function of α and θ is plotted in Figure 2, that is plot was generated from the following equation:

$$\begin{aligned} \mathcal{L}(\alpha, \theta|\mathbf{x}) = (\alpha\theta)^4 &\exp[-\theta(11^\alpha + \cdots + 45^\alpha + 33^\alpha + \cdots + 30^\alpha)] \\ &\times (11^{\alpha-1} \times \cdots \times 45^{\alpha-1}) \end{aligned}$$

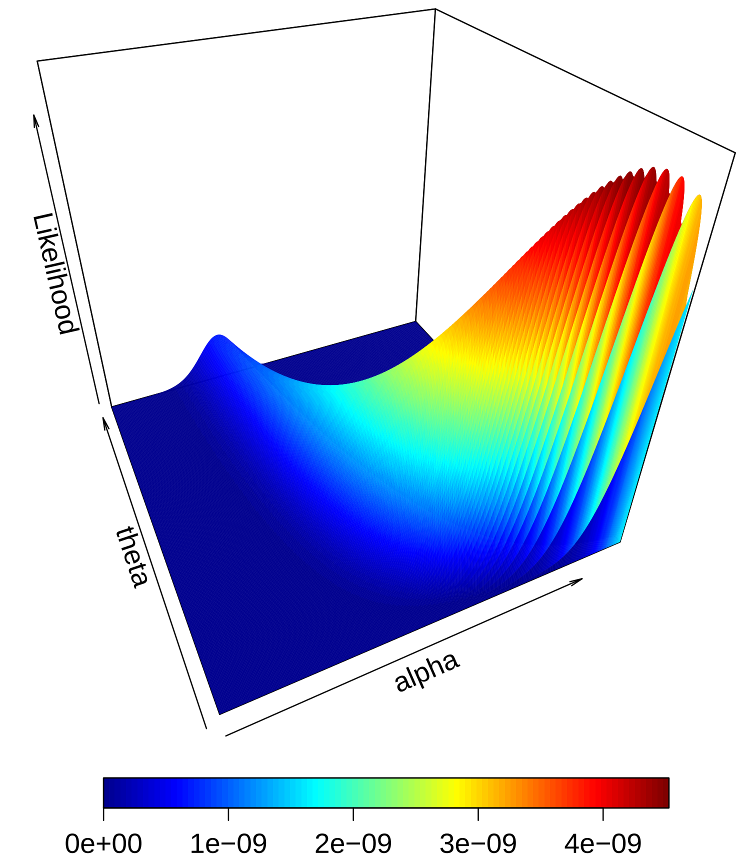

Figure 2: Likelihood Function of Y.

(c) Let Z be the time to death if the patient relapsed. From part (b), the MLE of θ is,

$$\hat{\theta} = \frac{\#\{i \in D\}}{\sum_{i \in D} x_i^\alpha + \sum_{i \in R} x_i^\alpha},$$

with the data, we have $\#\{i \in D\} = 4$, and

$$\sum_{i \in D} x_i^\alpha = 11^\alpha + 12^\alpha + 15^\alpha + 45^\alpha$$

and

$$\sum_{i\in R} x_i^\alpha = 33^\alpha + 28^\alpha.$$

Suppose $\alpha = 1$, then

$$\hat{\theta} = \frac{4}{83+61} = \frac{1}{36} = .0278.$$

As expected of course, the new MLE of θ is greater than the one in part (b) since the sum of the terms in the denominator is lesser with the exclusion of the patients 7 to 10, resulting to an increase in $\hat{\theta}$. The MLE of α, on the other hand, is obtain by Newton-Raphson iterative search

$$\frac{\#\{i \in D\}}{\alpha} - \theta\alpha \sum_{i\in D} x_i^{\alpha-1} - \theta\alpha \sum_{i\in R} x_i^{\alpha-1} + \sum_{i\in D} \log x_i \overset{set}{=} 0.$$

The same procedure with that in part (b) is done to obtain the value of α. Of course, the plausible value of α in explaining this sample is different from the previous one. Finally, in comparison to Figure 2 in part (b), Figure 3 is the likelihood function of the parameters given the truncated sample. Both figures were generated from the same domain, and viewed on the same perspective.

Appendix: R Codes

1. The following R codes generates Figure 1:

```
library(lattice)

LF <- function (x) {
  (x ** 6) * exp(- 180 * x)
}

x <- 1:400; y <- LF(x)
xyplot(0 ~ 0, type = c('l', 'g'), xlim = c(-0.0035, .15),
       ylim = c(-0.05e-12, 3.6e-12),
       lwd = 2, xlab = expression(lambda),
       ylab = "Likelihood Function",
       panel = function (x, y, ...) {
         panel.xyplot(x, y, ...)
         LF <- function (x) {
           (x ** 6) * exp(- 180 * x)
```

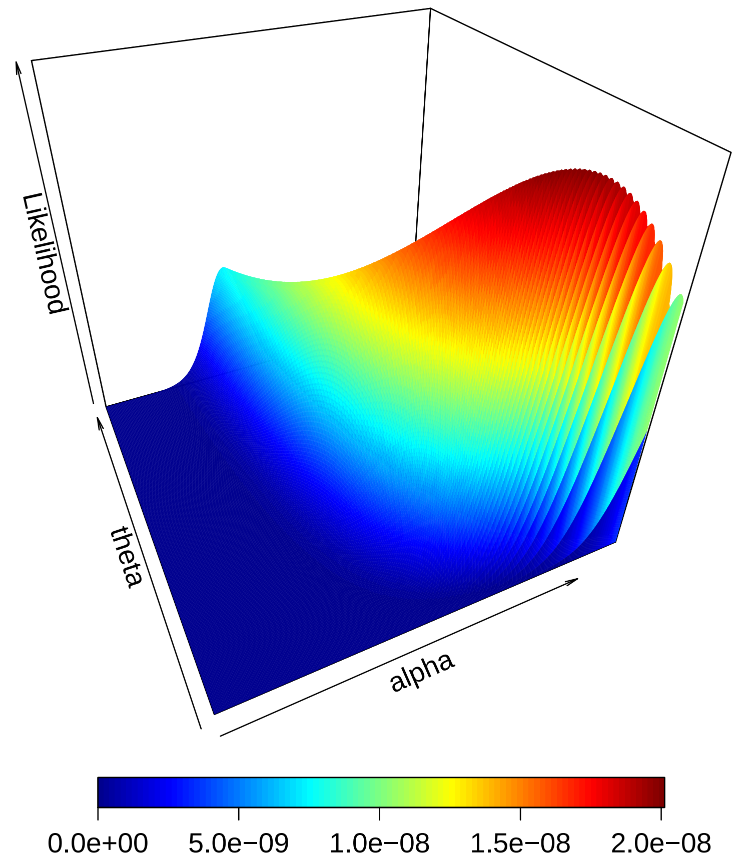

Figure 3: Likelihood Function of Z.

```
}
panel.curve(LF(x), n = 300, ...)
panel.abline(v = 1/30, lty = 'dashed')
panel.arrows(1/30, 0, .04, .5e-12, angle = 20)
panel.text(.045, .65e-12,
           label = expression(lambda==1/30))
panel.arrows(.05, LF(.05), .06, 2.3e-12, angle = 20)
```

```
        panel.text(
          .065, 2.4e-12,
          label = expression(lambda ^ 6 * exp(-180*lambda)))
      })
```

2. The following R codes generates Figure 2:

```
install.packages("plot3D")

x <- seq(0, 2.3, length = 300)
y <- seq(0, .05, length = 300)
f1 <- function(x, y) {
  (x * y) ** 4 * exp(
    - y * (11 ** x + 12 ** x + 15 ** x +
           45 ** x + 33 ** x + 28 ** x +
           16 ** x + 17 ** x + 19 ** x + 30 ** x)) *
    (11 ** (x - 1) *
     12 ** (x - 1) *
     15 ** (x - 1) *
     45 ** (x - 1))
  }
z <- outer(x, y, f1)

par(mai = c(.3,.3,.1,.3))
plot3D::persp3D(x, y, z, theta = -25, phi = 35,
                xlab = "alpha", ylab = "theta",
                zlab = "Likelihood",
                colkey = list(
                  side = 1,
                  length = .75,
                  dist = -0.1))
```

3. The following R codes generates Figure ??:

```
install.packages("plot3D")

x <- seq(0, 2.3, length = 300)
y <- seq(0, .05, length = 300)
f2 <- function(x, y) {
  (x * y) ** 4 * exp(
    - y * (11 ** x + 12 ** x + 15 ** x +
           45 ** x + 33 ** x + 28 ** x)) *
    (11 ** (x - 1) *
     12 ** (x - 1) *
     15 ** (x - 1) *
     45 ** (x - 1)) }
z1 <- outer(x, y, f2)

par(mai = c(.3,.3,.1,.3))
plot3D::persp3D(x, y, z1, theta = -25, phi = 35,
                xlab = "alpha", ylab = "theta",
                zlab = 'Likelihood',
                colkey = list(
                  side = 1,
                  length = .75,
                  dist = -0.1))
```