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PS 1 | **Stat 233**

1. $V^\perp$ and $V|W$ are subspace.

Proof.

(a) Let $\mathbf{w} \in V$ where $V \subset \mathbb{R}^n$ is a subspace. And consider $\mathbf{u}, \mathbf{v} \in V^\perp$ the orthogonal complement of V. Then $\langle \mathbf{u}, \mathbf{w}\rangle = 0$ and $\langle \mathbf{v}, \mathbf{w}\rangle = 0$, $\forall\, \mathbf{u}, \mathbf{v} \in V^\perp$. Now it follows that:

$$\langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = 0, \text{ i.e. } \mathbf{u}+\mathbf{v} \in V^\perp.$$

Also,

$$\langle k\mathbf{u}, \mathbf{w}\rangle = k\langle \mathbf{u}, \mathbf{w}\rangle = k \cdot 0 = 0$$
$$\langle k\mathbf{v}, \mathbf{w}\rangle = k\langle \mathbf{v}, \mathbf{w}\rangle = k \cdot 0 = 0.$$

Implying, $V^\perp$ is closed under addition and scalar multiplication. Therefore, $V^\perp$ is a subspace.

(b) By definition of quotient space, $V|W = V \cap W^\perp$ and that $W \subset V$. Consider the following cases:

Case 1. Let $\mathbf{w} \in W$ and $\mathbf{v}_1, \mathbf{v}_2 \in V|W$, then $\langle \mathbf{w}, \mathbf{v}_1\rangle = 0$ and $\langle \mathbf{w}, \mathbf{v}_2\rangle = 0$ since $W \perp V|W$ by definition of quotient space ($\cdot$ mod $\cdot$). So that,

$$\langle \mathbf{w}, \mathbf{v}_1\rangle + \langle \mathbf{w}, \mathbf{v}_2\rangle = \langle \mathbf{w}, \mathbf{v}_1 + \mathbf{v}_2\rangle = 0.$$

And

$$\langle k\mathbf{w}, \mathbf{v}_1\rangle = k\langle \mathbf{w}, \mathbf{v}_1\rangle = 0$$
$$\langle k\mathbf{w}, \mathbf{v}_2\rangle = k\langle \mathbf{w}, \mathbf{v}_2\rangle = 0.$$

Hence $V|W$ is closed under addition and scalar multiplication.

Case 2. Let $\mathbf{u} \in V^\perp$ and $\mathbf{v}_1, \mathbf{v}_2 \in V|W$, then $\langle \mathbf{u}, \mathbf{v}_1\rangle = 0$ and $\langle \mathbf{u}, \mathbf{v}_2\rangle = 0$ since $V^\perp$ is orthogonal to $V|W$. It follows that

$$\langle \mathbf{u}, \mathbf{v}_1\rangle + \langle \mathbf{u}, \mathbf{v}_2\rangle = \langle \mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2\rangle = 0.$$

Further,

$$\langle k\mathbf{u}, \mathbf{v}_1\rangle = k\langle \mathbf{u}, \mathbf{v}_1\rangle = 0$$
$$\langle k\mathbf{u}, \mathbf{v}_2\rangle = k\langle \mathbf{u}, \mathbf{v}_2\rangle = 0.$$

Then again $V|W$ is closed under addition and scalar multiplication.

From two cases above, one can conclude that $V|W$ is a subspace.

□

2. Let $W \subset V \subset \mathbb{R}^n$. Then $\dim(V|W) = \dim V - \dim W$.

To prove this problem, let's formally define what the dimension (dim) of a subspace first.

Definition 1. If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the dimension of S, denoted by $\dim(S)$.

Proof. By definition of quotient space, if $\mathbf{x} \in V|W$ and $\mathbf{y} \in W$, then $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ since $V|W \perp W$. Now consider $\mathbf{u}_1, \cdots, \mathbf{u}_p$ be mutually orthogonal nonzero vectors that span $V|W$. Also, let $\mathbf{u}_{p+1}, \cdots, \mathbf{u}_{p+k}$ be mutually orthogonal nonzero vectors that span W. If $\mathbf{X} = [\mathbf{u}_1, \cdots, \mathbf{u}_p]$ and $\mathbf{Y} = [\mathbf{u}_{p+1}, \cdots, \mathbf{u}_{p+k}]$ be matrices with column vectors $\mathbf{u}_i$ and $\mathbf{u}_j$, respectively, $i = 1, \cdots, p$ and $j = p+1, \cdots, p+k$. Then $\mathbf{X}$ and $\mathbf{Y}$ forms the basis of $V|W$ and W since $\mathbf{u}_i$s and $\mathbf{u}_j$s are linearly independent vectors $\forall\, i, j$ by Lemma 1.6. So that,

$$\dim(V|W) = \#\text{ of columns in } \mathbf{X} = p,$$

$$\dim(W) = \#\text{ of columns in } \mathbf{Y} = k$$

Given that $W \subset V$, then $W = V \cap W$. Thus,

$$\dim(V|W) = \dim(V \cap W^\perp) = \#\text{ of columns in } \mathbf{X} = p,$$

$$\dim(W) = \dim(V \cap W) = \#\text{ of columns in } \mathbf{Y} = k.$$

Because subspace is a set of all vectors that span the said space, then from set theory, it follows that

$$V = \{V \cap W\} \cup \{V \cap W^{\perp}\}, \text{ for } W \subset V.$$

What is left to show now is that,

$$\mathcal{L}(\{V \cap W\} \cup \{V \cap W^{\perp}\}) = V,$$

that is, the span of the vectors in the set $\{V \cap W\} \cup \{V \cap W^{\perp}\}$ is the subspace V. Let $\mathbf{v} \in V$, then by Theorem 1.4(f),

$$\begin{aligned} P_{V|W}\mathbf{v} &= P_V\mathbf{v} - P_W\mathbf{v} \\ P_V\mathbf{v} &= P_{V|W}\mathbf{v} + P_W\mathbf{v} \end{aligned}$$

It follows that $P_W\mathbf{v} \in W$, and because $W = \{\mathbf{w} : \mathbf{Xa} = \mathbf{w}\}$ for $\mathbf{X} = [\mathbf{u}_{p+1}, \cdots, \mathbf{u}_{p+k}]$ the basis matrix of W is a subset of V. We have,

$$P_W\mathbf{v} = \sum_{j=p+1}^{p+k} x_j a_j \text{ spans } W.$$

Also $P_{V|W}\mathbf{v} \in V|W$ since $V|W = \{\mathbf{v} : \mathbf{Ya} = \mathbf{v}\}$ for $\mathbf{Y} = [\mathbf{u}_1, \cdots, \mathbf{u}_p]$ the basis matrix of $V|W$ is a subset of V. So

$$P_{V|W}\mathbf{v} = \sum_{i=1}^{p} x_i a_i \text{ spans } V|W.$$

Implying

$$P_V\mathbf{v} = P_W\mathbf{v} + P_{V|W}\mathbf{v} = \sum_{i=1}^{p} x_i a_i + \sum_{j=p+1}^{p+k} x_j a_j = \sum_{i=1}^{p+k} x_i a_i,$$

that is, $P_V\mathbf{v} \in V$ and that $V = \{\mathbf{v} : [\mathbf{X}, \mathbf{Y}]\,\mathbf{b} = \mathbf{v}\}$. And because $\mathbf{X}$ and $\mathbf{Y}$ are bases of orthogonal subspaces ($W \perp V|W$) then the matrix $\mathbf{Z} = [\mathbf{X}, \mathbf{Y}]$ say, forms a basis matrix for V since the columns of $\mathbf{Z}$ are linearly independent. Therefore from the definition of dimension of a matrix,

$$\begin{aligned} \dim(V) &= \dim(W) + \dim(V|W) \\ \dim(V|W) &= \dim(V) - \dim(W) \end{aligned}$$

□

3. Let $W \subset V$. Then $V|(V|W) = W$.

   Proof. We need to show the following first:

   (a) $W \subset V|(V|W)$; and,
   (b) $\dim(W) = \dim(V|(V|W))$

   So that by result from Linear Algebra, $V|(V|W) = W$.

   (a) Let $\mathbf{x} \in W$, then $\mathbf{x} \in V$. If $\mathbf{y} \in V|W$, then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, since ($V|W = V \cap W^{\perp}$) $\perp$ W. Implies, $V|(V|W) = V \cap (V|W)^{\perp}$. Now since a subspace is a set of all vectors that span the said space, then by set theory, it follows that

   $$\begin{aligned} V \cap (V|W)^{\perp} &= V \cap \{V \cap W^{\perp}\}^{\perp} \\ &= V \cap \{V^{\perp} \cup W\} \\ &= \{V \cap V^{\perp}\} \cup \{V \cap W\} \\ &= V \cap W = W. \end{aligned}$$

   Hence, if $\mathbf{x} \in W$, then $\mathbf{x} \in V|(V|W)$. And therefore, we conclude that $W \subset V|(V|W)$.

   (b) From problem 2, if $W \subset V \subset \mathbb{R}^n$, then $\dim(V|W) = \dim V - \dim W$. Thus,

   $$\begin{aligned} \dim(V|(V|W)) &= \dim V - \dim V|W \\ &= \dim V - (\dim V - \dim W) \\ &= \dim W. \end{aligned}$$

   With results from part (a) $W \subset V|(V|W)$; and part (b) $\dim(W) = \dim(V|(V|W))$. $V|(V|W) = W$. □

4. Show that $W \subset V$ if and only if $W^{\perp} \supset V^{\perp}$

   Proof.

   (a) Assuming $W \subset V$, then $W^{\perp} \supset V^{\perp}$.

   Case 1. Let $\mathbf{u} \in W$, then $\mathbf{u} \in V$. Implies that $\mathbf{u} \notin W^{\perp}$ and $\mathbf{u} \notin V^{\perp}$ since

   $$\langle \mathbf{u}, \mathbf{a} \rangle = 0 \quad \text{and} \quad \langle \mathbf{u}, \mathbf{b} \rangle = 0$$

   $\forall\, \mathbf{a} \in W^{\perp}$ and $\forall\, \mathbf{b} \in V^{\perp}$.

Case 2. Now if $\mathbf{u} \notin W$ but $\mathbf{u} \in V$, then $\mathbf{u} \in W^{\perp}$ and $\mathbf{u} \notin V^{\perp}$. That is,

$$\langle \mathbf{u}, \mathbf{a} \rangle \neq 0 \quad \text{and} \quad \langle \mathbf{u}, \mathbf{b} \rangle = 0$$

$\forall\, \mathbf{a} \in W^{\perp}$ and $\forall\, \mathbf{b} \in V^{\perp}$.

Case 3. Further, if $\mathbf{u} \notin W$ and $\mathbf{u} \notin V$, then $\mathbf{u} \in W^{\perp}$ and $\mathbf{u} \in V^{\perp}$, that is

$$\langle \mathbf{u}, \mathbf{a} \rangle \neq 0 \quad \text{and} \quad \langle \mathbf{u}, \mathbf{b} \rangle \neq 0$$

$\forall\, \mathbf{a} \in W^{\perp}$ and $\forall\, \mathbf{b} \in V^{\perp}$.

To summarize, $\mathbf{u} \in W^{\perp} \cap V^{\perp}$ in Case 3. And in Case 2, $\mathbf{u} \in W^{\perp} \backslash V^{\perp}$. Therefore, $W^{\perp} \supset V^{\perp}$.

(b) Assuming $W^{\perp} \supset V^{\perp}$, then $W \subset V$.
Let's prove this by contradiction, suppose not. That is, suppose $W^{\perp} \supset V^{\perp}$ implies $W \supset V$.

Case 1. Let $\mathbf{x} \in V^{\perp}$, then $\mathbf{x} \in W^{\perp}$. It follows that, $\mathbf{x} \notin V$ and $\mathbf{x} \notin W$ since

$$\langle \mathbf{x}, \mathbf{a} \rangle = 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{b} \rangle = 0$$

$\forall\, \mathbf{a} \in V$ and $\forall\, \mathbf{b} \in W$.

Case 2. Now consider $\mathbf{x} \notin V^{\perp}$ but $\mathbf{x} \in W^{\perp}$. Then $\mathbf{x} \in V$ and $\mathbf{x} \notin W$. That is,

$$\langle \mathbf{x}, \mathbf{a} \rangle \neq 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{b} \rangle = 0$$

$\forall\, \mathbf{a} \in V$ and $\forall\, \mathbf{b} \in W$.

Case 3. Finally, if $\mathbf{x} \notin V^{\perp}$ and $\mathbf{x} \notin W^{\perp}$. Then $\mathbf{x} \in V$ and $\mathbf{x} \in W$. That is,

$$\langle \mathbf{x}, \mathbf{a} \rangle \neq 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{b} \rangle \neq 0$$

$\forall\, \mathbf{a} \in V$ and $\forall\, \mathbf{b} \in W$.

To summarize, $\mathbf{x} \in V \cap W$ in Case 3, and $\mathbf{x} \in V \backslash W$ in Case 2. Therefore $W \subset V$, which is a contradiction $\lightning$.

$\square$

5. Let $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then $(\mathbf{Y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu}) \sim \chi^2_n(\mathbf{0})$.

Proof. If $\mathbf{Z} = \mathbf{Y} - \boldsymbol{\mu}$, then $\mathbf{Z} \sim N_n(\mathbf{0}, \boldsymbol{\Sigma})$. So that from Theorem 2.13(a), $(\mathbf{Y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu}) \sim \chi^2_n((\boldsymbol{\mu} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}))$. Or simply,

$$(\mathbf{Y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu}) \sim \chi^2_n(\mathbf{0}).$$

$\square$