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Convention

All cited Theorems, Lemma and Corollary are all based on the reference book by Arnold.

1. Let $\mathbf{BY}$ be a linear unbiased estimator of $\mathbf{A}\boldsymbol{\mu}$. Then

$$\mathbb{C}\text{ov}\mathbf{BY} - \mathbb{C}\text{ov}\mathbf{A}\hat{\boldsymbol{\mu}} \succeq 0.$$

[That is, $\mathbb{C}\text{ov}\mathbf{BY} - \mathbb{C}\text{ov}\mathbf{A}\hat{\boldsymbol{\mu}}$ is nonnegative definite.]

Setup. Let $\mathbb{E}$ and $\mathbb{V}$ be notations for expected value and variance operators, respectively. Let $\mathbf{Y}$ be an $n \times 1$ vector such that $\mathbb{E}\mathbf{Y} = \boldsymbol{\mu}$ and $\mathbb{C}\text{ov}\mathbf{Y} = \sigma^2\mathbf{I}_n$. Further let $\mathbf{A}$ and $\mathbf{B}$ be $k \times n$ matrices. Consider the following definitions:

Definition 1. $\mathbf{T}(\mathbf{Y})$ is the *best linear unbiased estimator* of $\mathbf{A}\boldsymbol{\mu}$ if the components of $\mathbf{T}$ are best linear unbiased estimator of the components of $\mathbf{A}\boldsymbol{\mu}$.

So that the following should formally define linear unbiased estimator for $\mathbf{A}\boldsymbol{\mu}$:

Definition 2. Let $\mathbf{T}(\mathbf{Y}) = [(\gamma_i)]_{i=1}^k$ and $\mathbf{A}\boldsymbol{\mu} = [(\alpha_i)]_{i=1}^k$. We say that $\mathbf{T}(\mathbf{Y})$ is a *linear unbiased estimator* of $\mathbf{A}\boldsymbol{\mu}$ if it satisfies the following:

(a) $\mathbf{T}(\mathbf{Y}) = [(\gamma_i)]_{i=1}^k$, where γ_i is a linear function of $\mathbf{Y}$; and,

(b) $\mathbb{E}\mathbf{T}(\mathbf{Y}) = \mathbf{A}\boldsymbol{\mu}$ if $[(\mathbb{E}\gamma_i)]_{i=1}^k = [(\alpha_i)]_{i=1}^k$.

Proof. If $\mathbf{T}(\mathbf{Y}) = \mathbf{BY} = [(\gamma_{ib})]_{i=1}^k$ is a linear unbiased estimator of $\mathbf{A}\boldsymbol{\mu} = [(\alpha_i)]_{i=1}^k$, then by Definition 2, $[(\mathbb{E}\gamma_{ib})]_{i=1}^k = [(\alpha_i)]_{i=1}^k$ i.e. $\mathbb{E}\mathbf{T}(\mathbf{Y}) = \mathbb{E}\mathbf{BY} = \mathbf{A}\boldsymbol{\mu}$. From Corollary 1 of Theorem 6.7 (Gauss-Markov, *see* Arnold book), $\mathbf{A}\hat{\boldsymbol{\mu}}$, say with components $[(\gamma_{ia})]_{i=1}^k$, is the BLUE of $\mathbf{A}\boldsymbol{\mu}$. So if $\mathbf{BY}$ is any other linear unbiased estimator of $\mathbf{A}\boldsymbol{\mu}$, then by Definition 1, $\mathbb{V}\gamma_{ib} \geq \mathbb{V}\gamma_{ia}, \forall i$. That is, $\mathbb{V}\gamma_{ib} - \mathbb{V}\gamma_{ia} \geq 0, \forall i$. In terms of covariance, since $[(\gamma_{ib})]_{i=1}^k = [(\mathbf{b}_i\mathbf{Y})]_{i=1}^k$ say $\mathbf{B} = [(\mathbf{b}_i)]_{i=1}^k$ where $\mathbf{b}_i$ is a $1 \times n$ row vector, then $0 \leq \mathbb{V}\gamma_{ib} = \mathbb{V}\mathbf{b}_i\mathbf{Y} = \mathbf{b}_i\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{b}_i^{\mathrm{T}}, \forall i$. Also if $\mathbf{A} = [(\mathbf{a}_i)]_{i=1}^k$ such that $\mathbf{a}_i$ is a $1 \times n$ row vector, then $0 \leq \mathbb{V}\gamma_{ia} = \mathbb{V}(\mathbf{a}_i\mathbf{P}_V\mathbf{Y}) = \mathbf{a}_i\mathbf{P}_V\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{P}_V^{\mathrm{T}}\mathbf{a}_i^{\mathrm{T}}$. Hence $\mathbb{V}\gamma_{ib} \geq \mathbb{V}\gamma_{ia}, \forall i$, implies that $\mathbf{b}_i\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{b}_i^{\mathrm{T}} \geq \mathbf{a}_i\mathbf{P}_V\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{P}_V^{\mathrm{T}}\mathbf{a}_i^{\mathrm{T}}, \forall i$. So that,

$$\mathbf{b}_i\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{b}_i^{\mathrm{T}} - \mathbf{a}_i\mathbf{P}_V\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{P}_V^{\mathrm{T}}\mathbf{a}_i^{\mathrm{T}} \geq 0, \forall i.$$

Generalizing the problem into a variance-covariance matrix, we have $\mathbb{C}\text{ov}\mathbf{BY} = [(\mathbb{C}\text{ov}(\gamma_{ib}, \gamma_{jb}))]_{\forall i,j}$ and $\mathbb{C}\text{ov}\mathbf{A}\hat{\boldsymbol{\mu}} = [(\mathbb{C}\text{ov}(\gamma_{ia}, \gamma_{ja}))]_{\forall i,j}, i = 1, \cdots, k; j = 1, \cdots, k$. Recall that the variance-covariance matrix is a positive semidefinite matrix, so that if $\mathbf{z} \in \mathbb{R}^k$ is any arbitrary vector, then $\mathbf{z}^{\mathrm{T}}\mathbb{C}\text{ov}(\mathbf{BY})\mathbf{z} \geq 0$ and $\mathbf{z}^{\mathrm{T}}\mathbb{C}\text{ov}(\mathbf{A}\hat{\boldsymbol{\mu}})\mathbf{z} \geq 0$, to see if the difference of the two is positive semidefinite, consider the following

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}}[\mathbb{C}\text{ov}(\mathbf{BY}) - \mathbb{C}\text{ov}(\mathbf{A}\hat{\boldsymbol{\mu}})]\mathbf{z} &\overset{?}{\geq} 0 \\
\mathbf{z}^{\mathrm{T}}\mathbb{C}\text{ov}(\mathbf{BY})\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbb{C}\text{ov}(\mathbf{A}\hat{\boldsymbol{\mu}})\mathbf{z} &\overset{?}{\geq} 0 \\
\mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{B}^{\mathrm{T}}\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbb{C}\text{ov}(\hat{\boldsymbol{\mu}})\mathbf{A}^{\mathrm{T}}\mathbf{z} &\overset{?}{\geq} 0 \\
\mathbf{z}^{\mathrm{T}}\mathbf{B}\sigma^2\mathbf{I}_n\mathbf{B}^{\mathrm{T}}\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbb{C}\text{ov}(\mathbf{P}_V\mathbf{Y})\mathbf{A}^{\mathrm{T}}\mathbf{z} &\overset{?}{\geq} 0 \\
\sigma^2\mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbf{P}_V\mathbb{C}\text{ov}(\mathbf{Y})\mathbf{P}_V^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{z} &\overset{?}{\geq} 0 \\
\sigma^2\mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbf{P}_V\sigma^2\mathbf{I}_n\mathbf{P}_V^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{z} &\overset{?}{\geq} 0 \\
\sigma^2\mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{z} - \sigma^2\mathbf{z}^{\mathrm{T}}\mathbf{A}\mathbf{P}_V\mathbf{P}_V^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{z} &\overset{?}{\geq} 0 \\
\sigma^2(\mathbf{B}^{\mathrm{T}}\mathbf{z})^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{z}) - \sigma^2(\mathbf{P}_V^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{z})^{\mathrm{T}}(\mathbf{P}_V^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{z}) &\overset{?}{\geq} 0 \\
\sigma^2\|\mathbf{B}^{\mathrm{T}}\mathbf{z}\|^2 - \sigma^2\|\mathbf{P}_V^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{z}\|^2 &\overset{?}{\geq} 0
\end{aligned}$$

Still working here.....

□

2. Lemma 7.2
$$F = \frac{\|\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}}\|^2}{(p-k)\hat{\sigma}^2}$$
$$\|\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}}\|^2 = \|\hat{\boldsymbol{\mu}} - \mathbf{P}_W\hat{\boldsymbol{\mu}}\|^2 = \|\hat{\boldsymbol{\mu}}\|^2 - \|\mathbf{P}_W\hat{\boldsymbol{\mu}}\|^2.$$

Proof.
$$F = \frac{\|\mathbf{P}_{V|W}\mathbf{Y}\|^2(n-p)}{\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2(p-k)},$$
recall that $\hat{\sigma}^2 = \frac{\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2}{(n-p)}$, so $\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2 = (n-p)\hat{\sigma}^2$, then
$$F = \frac{\|\mathbf{P}_{V|W}\mathbf{Y}\|^2}{\hat{\sigma}^2(p-k)},$$
and because
$$\begin{aligned}\|\mathbf{P}_{V|W}\mathbf{Y}\|^2 &= \|\mathbf{P}_V\mathbf{Y}\|^2 - \|\mathbf{P}_W\mathbf{Y}\|^2 \quad \text{by Theorem 2.5 (f)}\\ &= \|\hat{\boldsymbol{\mu}}\|^2 - \|\mathbf{P}_W\mathbf{Y}\|^2\end{aligned}$$
and further since $W \subset V$, then $\mathbf{P}_W\mathbf{Y} = \mathbf{P}_W\mathbf{P}_V\mathbf{Y} = \mathbf{P}_W\hat{\boldsymbol{\mu}}$ by Theorem 2.5 (e), so that
$$\|\mathbf{P}_{V|W}\mathbf{Y}\|^2 = \|\hat{\boldsymbol{\mu}}\|^2 - \|\mathbf{P}_W\hat{\boldsymbol{\mu}}\|^2 = \|\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}}\|^2,$$
then that proves the problem. □

3. Lemma 7.5. $\mathbf{C} = \tilde{\mathbf{T}}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}$ is a basis matrix for $V|W$.

Setup. Let $\bar{\mathbf{T}} = \frac{1}{n}\sum_{i=1}^n \mathbf{T}_i$ and $\tilde{\mathbf{T}}_i = \mathbf{T}_i - \bar{\mathbf{T}}$, where $\mathbf{T}_i^{\mathrm{T}}$ are known $p-1$ dimensional vector. So that $\tilde{\mathbf{T}} = [\tilde{\mathbf{T}}_1, \cdots, \tilde{\mathbf{T}}_n]^{\mathrm{T}}$ or
$$\tilde{\mathbf{T}} = \begin{bmatrix}\tilde{\mathbf{T}}_1\\ \vdots \\ \tilde{\mathbf{T}}_n\end{bmatrix} = \begin{bmatrix}1 & \mathbf{T}_1\\ \vdots & \vdots\\ 1 & \mathbf{T}_n\end{bmatrix}\begin{bmatrix}-\bar{\mathbf{T}}\\ \mathbf{I}\end{bmatrix} = \mathbf{X}\begin{bmatrix}-\bar{\mathbf{T}}\\ \mathbf{I}\end{bmatrix} \tag{1}$$

Proof. From above setup $\tilde{\mathbf{T}}$ is $n \times (p-1)$ matrix. If $\mathbf{A}$ is a $(p-k)\times(p-1)$ matrix of rank $p-k$, then $\mathbf{C}$ is $n\times(p-k)$ matrix. Let $\mathbf{C} = [\mathbf{C}_1, \cdots, \mathbf{C}_{p-k}]$ and suppose $\mathbf{0} = \sum_{i=1}^{p-k} b_i\mathbf{C}_i = \mathbf{C}\mathbf{b}$. Then
$$\begin{aligned}\mathbf{0} &= (\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\tilde{\mathbf{T}}^{\mathrm{T}}\underbrace{\tilde{\mathbf{T}}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}}_{\mathbf{C}}\mathbf{b}\\ &= (\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\tilde{\mathbf{T}}^{\mathrm{T}}\mathbf{C}\mathbf{b} = \mathbf{b}\end{aligned}$$

$\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is invertible since $\mathbf{A}$ has $p-k$ rank. So the columns of $\mathbf{C}$ are linearly independent. Now let U be the subspace spanned by the columns of $\mathbf{C}$, so that $\mathbf{C}$ is a basis matrix for U. Let $\mathbf{u} \in U$, then
$$\mathbf{u} = \tilde{\mathbf{T}}\underbrace{(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}}_{\text{say } \mathbf{d}} = \tilde{\mathbf{T}}\mathbf{d}$$
then $\mathbf{u} \in V$ follows from the fact that $\tilde{\mathbf{T}}$ is a transformation of the basis of V which is $\mathbf{X}$, refer to Equation (1) to verify. Now the subspace W is the subspace for which $\mathbf{A}\boldsymbol{\gamma} = \mathbf{0} = \mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\tilde{\mathbf{T}}^{\mathrm{T}}\boldsymbol{\mu}$. But $\left[\tilde{\mathbf{T}}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}\right]^{\mathrm{T}}\boldsymbol{\mu} = \mathbf{C}^{\mathrm{T}}\boldsymbol{\mu} = \mathbf{0}$. So $\mathbf{C}_i^{\mathrm{T}}\boldsymbol{\mu} = \mathbf{0}, \forall i$. Therefore, $U \perp \mathbf{u}$, since $\mathbf{C}$ is the basis of U. Then $\forall \boldsymbol{\mu} \in W, U \perp W$. Therefore, $U \subset V$ and $U \perp W$. Since $W \subset V$, U is the orthogonal complement of W relative to V. That is, $U = V|W$. Thus $\mathbf{C}$ is the basis matrix for $V|W$. Since $\mathbf{C}$ has linear independent columns, then $rk(\mathbf{C}) = p-k$. So that $\dim(V|W) = p-k$, implying $\dim W = k$. □

4. For testing that $\mathbf{A}\boldsymbol{\gamma} = \mathbf{0}$ in the regression model with an intercept,
$$F = \frac{(\mathbf{A}\hat{\boldsymbol{\gamma}})^{\mathrm{T}}(\mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{A}\hat{\boldsymbol{\gamma}})}{(p-k)\hat{\sigma}^2},$$
give the noncentrality parameter as well.

Proof. From Lemma 7.2 in problem (2) above, the test statistic for testing $H_0 : \mathbf{A}\boldsymbol{\gamma} = \mathbf{0}$ is,
$$F = \frac{\|\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}}\|^2}{(p-k)\hat{\sigma}^2}$$
$\mathbf{C}$ is the basis matrix for $V|W$, as proven in Lemma 7.5 problem (3) above. Then $\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}} = \mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\hat{\boldsymbol{\mu}}$, and $\|\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}}\|^2 = \hat{\boldsymbol{\mu}}^{\mathrm{T}}\mathbf{P}_{V|W}\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}^{\mathrm{T}}\mathbf{C}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\hat{\boldsymbol{\mu}}$. Now $\mathbf{C}^{\mathrm{T}}\hat{\boldsymbol{\mu}} = [\tilde{\mathbf{T}}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}]^{\mathrm{T}}\hat{\boldsymbol{\mu}} = \mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\tilde{\mathbf{T}}^{\mathrm{T}}\hat{\boldsymbol{\mu}} = \mathbf{A}\hat{\boldsymbol{\gamma}}$, so that
$$\mathbf{C}^{\mathrm{T}}\mathbf{C} = \mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}} = \mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}$$
Therefore,
$$F = \frac{(\mathbf{C}^{\mathrm{T}}\hat{\boldsymbol{\mu}})^{\mathrm{T}}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\hat{\boldsymbol{\mu}}}{(p-k)\hat{\sigma}^2} = \frac{(\mathbf{A}\hat{\boldsymbol{\gamma}})^{\mathrm{T}}[\mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}]^{-1}\mathbf{A}\hat{\boldsymbol{\gamma}}}{(p-k)\hat{\sigma}^2}.$$

So that,

$$F \sim F_{p-k,n-p}\left(\frac{\|\mathbf{P}_{V|W}\boldsymbol{\mu}\|^2}{\sigma^2}\right),$$

but $\|\mathbf{P}_{V|W}\boldsymbol{\mu}\|^2 = (\mathbf{C}^{\mathrm{T}}\boldsymbol{\mu})^{\mathrm{T}}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\boldsymbol{\mu} = (\mathbf{A}\boldsymbol{\gamma})^{\mathrm{T}}[\mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}]^{-1}\mathbf{A}\boldsymbol{\gamma}$. Therefore the noncentrality parameter is

$$\delta = \frac{(\mathbf{A}\boldsymbol{\gamma})^{\mathrm{T}}[\mathbf{A}(\tilde{\mathbf{T}}^{\mathrm{T}}\tilde{\mathbf{T}})^{-1}\mathbf{A}^{\mathrm{T}}]^{-1}\mathbf{A}\boldsymbol{\gamma}}{\sigma^2}.$$

□

5. Show that for testing H_0: all the $\gamma_{ij} = 0$ is

$$F = \frac{m\sum_{i=1}^{r}\sum_{j=1}^{c}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2}{\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{m}(Y_{ijk} - \bar{Y}_{ij.})}$$

Setup. Let $\theta = \bar{\mu}_{..}; \alpha_i = \bar{\mu}_{i.} - \bar{\mu}_{..}; \beta_j = \bar{\mu}_{.j} - \bar{\mu}_{..}; \gamma_{ij} = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$. Thus,

$$\begin{aligned}\mu_{ij} &= \gamma_{ij} + \bar{\mu}_{i.} + \bar{\mu}_{.j} - \bar{\mu}_{..}\\ &= \gamma_{ij} + \alpha_i + \bar{\mu}_{..} + \beta_j + \bar{\mu}_{..} - \bar{\mu}_{..}\\ &= \theta + \alpha_i + \beta_j + \gamma_{ij};\end{aligned}$$

where $\sum_{i=1}^{r}\alpha_i = 0; \sum_{j=1}^{c}\beta_j = 0; \sum_{i=1}^{r}\gamma_{ij} = 0; \sum_{j=1}^{c}\gamma_{ij} = 0.$

Proof. Notice that we can have an equivalent version of the model given by $Y_{ijk} \sim \mathcal{N}(\theta + \alpha_i + \beta_j + \gamma_{ij}, \sigma^2), \exists\, \theta, \alpha_i, \beta_j$ where $\sum_{i=1}^{r}\alpha_i = 0, \sum_{j=1}^{c}\beta_j = 0, \sum_{i=1}^{r}\gamma_{ij} = 0, \sum_{j=1}^{c}\gamma_{ij} = 0, \forall i = 1, \cdots, r; \forall j = 1, \cdots, c$ and $\forall k = 1, \cdots, m$. In matrix linear model form, this can be written as

$$\mathbf{Y} \sim \mathcal{N}_{rcm}(\boldsymbol{\mu}, \sigma^2\mathbf{I})$$

where $\boldsymbol{\mu} = \mu_{ijk}$. Now let

$$V = \Big\{\mu_{ijk}^{(v)} : \mu_{ijk}^{(v)} = \theta + \alpha_i + \beta_j + \gamma_{ij}, i = 1, \cdots, r; j = 1, \cdots, c;$$
$$k = 1, \cdots, m; \sum_{i=1}^{r}\alpha_i = 0; \sum_{j=1}^{c}\beta_j = 0; \sum_{i=1}^{r}\gamma_{ij} = 0; \sum_{j=1}^{c}\gamma_{ij} = 0\Big\}$$

Then V is a subspace since it is closed under formation of linear combinations. Also let

$$V = \Big\{\mu_{ijk}^{(w)} : \mu_{ijk}^{(w)} = \theta + \alpha_i + \beta_j, i = 1, \cdots, r; j = 1, \cdots, c;$$
$$k = 1, \cdots, m; \sum_{i=1}^{r}\alpha_i = 0; \sum_{j=1}^{c}\beta_j = 0\Big\}$$

Hence, testing $\gamma_{ij} = 0$ is similar to testing $H_0 : \boldsymbol{\mu} \in W$ versus $H_1 : \boldsymbol{\mu} \in V$ where

$$\dim V = \underbrace{\cancel{1}}_{\theta} + \underbrace{\cancel{r}}_{\alpha_i} + \underbrace{\cancel{c}}_{\beta_j} + \underbrace{rc}_{\gamma_{ij}} - (\underbrace{\cancel{1}}_{\sum_i\alpha_i} + \underbrace{\cancel{1}}_{\sum_j\beta_j}) - (\underbrace{\cancel{c}}_{\sum_i\gamma_{ij}} + \underbrace{\cancel{r}}_{\sum_j\gamma_{ij}} - \underbrace{\cancel{1}}_{red.}) = rc$$

and

$$\dim W = \underbrace{\cancel{1}}_{\theta} + \underbrace{r}_{\alpha_i} + \underbrace{c}_{\beta_j} - (\underbrace{\cancel{1}}_{\sum_i\alpha_i} + \underbrace{1}_{\sum_j\beta_j}) = r + c - 1$$

Now $\hat{\boldsymbol{\mu}} = \mathbf{P}_V\mathbf{Y}$ is obtain by minimizing

$$\begin{aligned}\|\mathbf{Y} - \boldsymbol{\mu}^{(v)}\|^2 &= \sum_i\sum_j\sum_k\left(Y_{ijk} - \mu_{ijk}^{(v)}\right)^2\\ &= \sum_i\sum_j\sum_k(Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij})^2\end{aligned}$$

For brevity let $S_1 = \|\mathbf{Y} - \boldsymbol{\mu}^{(v)}\|^2$, then

$$\frac{\partial S_1}{\partial\theta} = -2\sum_i\sum_j\sum_k(Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij})^2$$

Obtain the stationary point $\frac{\partial S_1}{\partial\theta} \overset{\text{set}}{=} 0$,

$$\frac{\partial S_1}{\partial\theta} = -2\sum_i\sum_j\sum_k(Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij}) \overset{\text{set}}{=} 0$$

$$Y_{...} - mrc\theta - mc\cancelto{0}{\sum_i\alpha_i} - mr\cancelto{0}{\sum_j\beta_j} - m\sum_i\cancelto{0}{\sum_j\gamma_{ij}} = 0$$

$$\theta = \bar{Y}_{...}$$

where $\frac{\partial^2 S_1}{\partial \theta^2} = 2mrc > 0$, then $\hat{\theta}$ minimizes S_1. Next we write S_1 as follows:

$$S_1 = \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij})^2 + \sum_{i^\star \neq i} \sum_j \sum_k (Y_{i^\star jk} - \theta - \alpha_{i^\star} - \beta_j - \gamma_{i^\star j})^2$$

So that,

$$\frac{\partial S_1}{\partial \alpha_i} = -2 \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij}) \overset{\text{set}}{=} 0$$

$$Y_{i..} - mc\theta - mc\alpha_i - m\cancelto{0}{\sum_j \beta_j} - m\cancelto{0}{\sum_j \gamma_{ij}} = 0$$

$$\hat{\alpha}_i = \bar{Y}_{i..} - \hat{\theta} = \bar{Y}_{i..} - \bar{Y}_{...}$$

And since $\frac{\partial^2 S_1}{\partial \alpha_i^2} = 2mc > 0$, then $\hat{\alpha}_i$ minimizes S_1. Next we write again S_1 as follows:

$$S_1 = \sum_i \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij})^2 + \sum_i \sum_{j^\star \neq j} \sum_k (Y_{ij^\star k} - \theta - \alpha_{i^\star} - \beta_{j^\star} - \gamma_{ij^\star})^2$$

So that,

$$\frac{\partial S_1}{\partial \beta_j} = -2 \sum_i \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij}) \overset{\text{set}}{=} 0$$

$$Y_{.j.} - mr\theta - m\cancelto{0}{\sum_i \alpha_i} - mr\beta_j - m\cancelto{0}{\sum_j \gamma_{ij}} = 0$$

$$\hat{\beta}_j = \bar{Y}_{.j.} - \hat{\theta} = \bar{Y}_{.j.} - \bar{Y}_{...}$$

and since $\frac{\partial^2 S_1}{\partial \beta_j^2} = 2mr > 0$, then $\hat{\beta}_j$ minimizes S_1. And now we can write S_1 as follows:

$$S_1 = \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij})^2 + \sum_{i^\star \neq i} \sum_{j^\star \neq j} \sum_k (Y_{i^\star j^\star k} - \theta - \alpha_i - \beta_j - \gamma_{i^\star j^\star})^2$$

So that

$$\frac{\partial S_1}{\partial \gamma_{ij}} = -2 \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j - \gamma_{ij}) \overset{\text{set}}{=} 0$$

$$Y_{ij.} - m\theta - m\alpha_i - m\beta_j - m\gamma_{ij} = 0$$

$$\gamma_{ij} = \bar{Y}_{ij.} - \theta - \alpha_i - \beta_j,$$

and since $\frac{\partial S_1}{\partial \beta_j} = 2 > 0$, then

$$\begin{aligned} \hat{\gamma}_{ij} &= \bar{Y}_{ij.} - \hat{\theta} - \hat{\alpha}_i - \hat{\beta}_j \\ &= \bar{Y}_{ij.} - \bar{Y}_{...} - \bar{Y}_{i..} + \bar{Y}_{...} - \bar{Y}_{.j.} + \bar{Y}_{...} \\ &= \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...} \end{aligned}$$

So that,

$$\begin{aligned} \hat{\mu}_{ijk} &= \hat{\theta} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij} \\ &= \cancel{\bar{Y}_{...}} + \cancel{\bar{Y}_{i..}} - \cancel{\bar{Y}_{...}} + \cancel{\bar{Y}_{.j.}} - \cancel{\bar{Y}_{...}} + \bar{Y}_{ij.} - \cancel{\bar{Y}_{i..}} - \cancel{\bar{Y}_{.j.}} + \cancel{\bar{Y}_{...}} \\ &= \bar{Y}_{ij.} \end{aligned}$$

Now to obtain $\hat{\hat{\mu}} = \mathbf{P}_W \mathbf{Y}$, we minimize

$$\|\mathbf{Y} - \boldsymbol{\mu}^{(W)}\|^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j)^2$$

For brevity let $S_2 = \|\mathbf{Y} - \boldsymbol{\mu}^{(W)}\|^2$ so that

$$\frac{\partial S_2}{\partial \theta} = -2 \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j)^2$$

Obtain the stationary point $\frac{\partial S_2}{\partial \theta} \overset{\text{set}}{=} 0$,

$$\frac{\partial S_2}{\partial \theta} = -2 \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j) \overset{\text{set}}{=} 0$$

$$Y_{...} - mrc\hat{\hat{\theta}} - mc\cancelto{0}{\sum_i \alpha_i} - mr\cancelto{0}{\sum_j \beta_j} = 0$$

$$\hat{\hat{\theta}} = \bar{Y}_{...}$$

where $\frac{\partial^2 S_2}{\partial \theta^2} = 2mrc > 0$, then $\hat{\theta}$ minimizes S_2. Next we write S_2 as follows:

$$S_2 = \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j)^2 + \sum_{i^\star \neq i} \sum_j \sum_k (Y_{i^\star jk} - \theta - \alpha_{i^\star} - \beta_j)^2$$

So that,

$$\frac{\partial S_2}{\partial \alpha_i} = -2\sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j) \overset{\text{set}}{=} 0$$

$$Y_{i..} - mc\hat{\hat{\theta}} - mc\hat{\hat{\alpha}}_i - m\cancelto{0}{\sum_j \beta_j} = 0$$

$$\hat{\hat{\alpha}}_i = \bar{Y}_{i..} - \hat{\hat{\theta}} = \bar{Y}_{i..} - \bar{Y}_{...}$$

And since $\frac{\partial^2 S_2}{\partial \alpha_i^2} = 2mc > 0$, then $\hat{\hat{\alpha}}_i$ minimizes S_2. Next we write again S_2 as follows:

$$S_2 = \sum_i \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j)^2 + \sum_i \sum_{j^\star \neq j} \sum_k (Y_{ij^\star k} - \theta - \alpha_{i^\star} - \beta_{j^\star})^2$$

So that,

$$\frac{\partial S_2}{\partial \beta_j} = -2\sum_i \sum_k (Y_{ijk} - \theta - \alpha_i - \beta_j) \overset{\text{set}}{=} 0$$

$$Y_{.j.} - mr\hat{\hat{\theta}} - m\cancelto{0}{\sum_i \alpha_i} - mr\hat{\hat{\beta}}_j = 0$$

$$\hat{\hat{\beta}}_j = \bar{Y}_{.j.} - \hat{\hat{\theta}} = \bar{Y}_{.j.} - \bar{Y}_{...}$$

and since $\frac{\partial^2 S_2}{\partial \beta_j^2} = 2mr > 0$, then $\hat{\hat{\beta}}_j$ minimizes S_2. Thus,

$$\begin{aligned}
\hat{\hat{\mu}}_{ijk} &= \hat{\hat{\theta}} + \hat{\hat{\alpha}}_i + \hat{\hat{\beta}}_j \\
&= \bar{Y}_{...} + \bar{Y}_{i..} - \bar{Y}_{...} + \bar{Y}_{.j.} - \bar{Y}_{...} \\
&= \bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}
\end{aligned}$$

Therefore,

$$\begin{aligned}
F &= \frac{\|\mathbf{P}_{V|W}\mathbf{Y}\|^2 (n - \dim V)}{\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2 (\dim V - \dim W)} = \frac{\|\mathbf{P}_V\mathbf{Y} - \mathbf{P}_W\mathbf{Y}\|^2 (mrc - rc)}{\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2 (rc - r - c + 1)} \\
&= \frac{\|\hat{\boldsymbol{\mu}} - \hat{\hat{\boldsymbol{\mu}}}\|^2}{\|\mathbf{Y} - \mathbf{P}_V\mathbf{Y}\|^2} \frac{rc(m-1)}{(r-1)(c-1)} \\
&= \frac{\sum_i \sum_j \sum_k (\hat{\mu}_{ijk} - \hat{\hat{\mu}}_{ijk})}{\sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2} \frac{rc(m-1)}{(r-1)(c-1)} \\
&= \frac{m\sum_i \sum_j (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})}{\sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2} \frac{rc(m-1)}{(r-1)(c-1)}
\end{aligned}$$

And the noncentrality parameter is,

$$\begin{aligned}
\delta &= \frac{\|\mathbf{P}_{V|W}\|^2}{\sigma^2} = \frac{\|\mathbf{P}_V\boldsymbol{\mu} - \mathbf{P}_W\boldsymbol{\mu}\|^2}{\sigma^2} = \frac{\|\boldsymbol{\mu}^{(V)} - \boldsymbol{\mu}^{(W)}\|^2}{\sigma^2} \\
&= \frac{\sum_i \sum_j \sum_k (\mu_{ijk}^{(V)} - \mu_{ijk}^{(W)})^2}{\sigma^2} \\
&= \frac{\sum_i \sum_j \sum_k (\theta + \alpha_i + \beta_j + \gamma_{ij} - \theta - \alpha_i - \beta_j)^2}{\sigma^2} \\
&= \frac{m\sum_i \sum_j (\gamma_{ij})^2}{\sigma^2}.
\end{aligned}$$

Therefore,

$$F \sim F_{(r-1)(c-1), rc(m-1)}\left(\frac{m\sum_i \sum_j (\gamma_{ij})^2}{\sigma^2}\right)$$

□

6. Show that the test statistic for testing $\delta_{ij} = 0, i = 1, \cdots, r; j = 1, \cdots, c$, that is

$$F = \frac{rc(m-1)}{r(c-1)} \frac{m\sum_i \sum_j (\bar{Y}_{ij.} - \bar{Y}_{i..})^2}{\sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{.j.})^2}$$

$$F \sim F_{r(c-1), rc(m-1)}\left(\frac{m\sum_i \sum_j \delta_{ij}^2}{\sigma^2}\right)$$

Setup. Let $\theta = \bar{\mu}_{...}, \alpha_i = \bar{\mu}_{i..} - \bar{\mu}_{...}, \delta_{ij} = \mu_{ijk} - \bar{\mu}_{i..}$. Then

$$\mu_{ijk} = \delta_{ij} + \bar{\mu}_{i..} = \delta_{ij} + \alpha_i + \bar{\mu}_{...} = \theta + \alpha_i + \delta_{ij};$$
$$\sum_i \alpha_i = 0; \sum_j \delta_{ij} = 0, \forall i; \sum_i \delta_{ij} \neq 0, \forall j;$$

Proof. Observe that $Y_{ijk} \sim \mathcal{N}(\theta+\alpha_i+\delta_{ij}, \sigma^2), \exists\, \theta, \alpha_i, \delta_{ij}$ such that $\sum_i \alpha_i = 0, \sum_j \delta_{ij} = 0, \forall i; \sum_i \delta_{ij} \neq 0, \forall j$. We can write $\mathbf{Y} = [(Y_{ijk})]_{(rcm\times 1)}$ and $\boldsymbol{\mu} = [(\theta + \alpha_i + \delta_{ij})_k]_{rcm\times 1}$. So that

$$\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_{rcm}).$$

Now let $V = \{\mu_{ijk} : \mu_{ijk} = \theta + \alpha_i + \delta_{ij}, \exists\, \theta, \alpha_i, \delta_{ij}; \sum_i \alpha_i = 0, \sum_j \delta_{ij} = 0, \forall i; \sum_i \delta_{ij} \neq 0, \forall j\}$ by definition of subspace, V is closed under linear combination of α_i and δ_{ij}. So V is a subspace. Also, let $W = \{\mu_{ijk} : \mu_{ijk} = \theta + \alpha_i, \exists\, \alpha_i \text{ s.t. } \sum_i \alpha_i = 0\}$. Then similar to testing H_0 :All the $\delta_{ij} = 0$ is similar to testing $H_0 : \boldsymbol{\mu} \in W$ versus $H_1 : \boldsymbol{\mu} \in V$. Also

$$\dim V = \underbrace{\cancel{1}}_{\theta} + \underbrace{\cancel{r}}_{\alpha_i} + \underbrace{rc}_{\delta_{ij}} - (\underbrace{\cancel{1}}_{\sum_i \alpha_i} + \underbrace{\cancel{r}}_{\sum_j \delta ij}) = rc$$

and

$$\dim W = \underbrace{\cancel{1}}_{\theta} + \underbrace{r}_{\alpha_i} - \underbrace{\cancel{1}}_{\sum_i \alpha_i} = r$$

Thus, to find $\hat{\boldsymbol{\mu}} = \mathbf{P}_V \mathbf{Y}$ we must minimize

$$\|\mathbf{Y} - \boldsymbol{\mu}^{(V)}\|^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \mu_{ijk}^{(V)})^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij})^2$$

For brevity, suppose $S_1 = \|\mathbf{Y} - \boldsymbol{\mu}\|^2$, then $\frac{\partial S_1}{\partial \theta} = -2 \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij})$. Obtain the stationary point by setting $\frac{\partial S_1}{\partial \theta}$ to 0. i.e.

$$\frac{\partial S_1}{\partial \theta} \overset{\text{set}}{=} 0 \Rightarrow -2 \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij}) = 0$$
$$Y_{...} - rcm\hat{\theta} - cm \cancelto{0}{\sum_i \alpha_i} - m \sum_i \cancelto{0}{\sum_j \delta_{ij}} = 0$$
$$\hat{\theta} = \bar{Y}_{...},$$

and since $\frac{\partial^2 S_1}{\partial S_1^2} = 2rcm > 0$, then $\hat{\theta}$ minimizes S_1. Next we write S_1 as follows

$$S_1 = \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij})^2 + \sum_{i^\star \neq i} \sum_j \sum_k (Y_{i^\star jk} - \theta - \alpha_{i^\star} - \delta_{i^\star j})^2,$$

then

$$\frac{\partial S_1}{\partial \alpha_i} = -2 \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij}) \overset{\text{set}}{=} 0$$
$$Y_{i..} - mc\hat{\alpha}_i - m \cancelto{0}{\sum_j \delta_{ij}} = 0$$
$$\hat{\alpha}_i = \bar{Y}_{i..}$$

and since $\frac{\partial S_1}{\partial \alpha_i} = 2mc > 0$, then $\hat{\alpha}_i$ minimizes S_1. Next we write S_1 as follows

$$S_1 = \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij})^2 + \sum_{i^\star \neq i} \sum_{j^\star \neq j} \sum_k (Y_{i^\star j^\star k} - \theta - \alpha_{i^\star} - \delta_{i^\star j^\star})^2,$$

So that

$$\frac{\partial S_1}{\partial \delta_{ij}} = -2 \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij}) \overset{\text{set}}{=} 0$$
$$Y_{ij.} - m\hat{\theta} - m\hat{\alpha}_i - m\hat{\delta}_{ij} = 0$$
$$\hat{\delta}_{ij} = \bar{Y}_{ij.} - \hat{\theta} - \hat{\alpha}$$
$$= \bar{Y}_{ij.} - \bar{Y}_{...} - \bar{Y}_{i..}$$

Thus,

$$\hat{\mu}_{ijk} = \hat{\theta} + \hat{\alpha}_i + \hat{\delta}_{ij} = \bar{Y}_{...} + \bar{Y}_{i..} + \bar{Y}_{ij.} - \bar{Y}_{...} - \bar{Y}_{i..} = \bar{Y}_{ij.}.$$

Now for $\hat{\hat{\boldsymbol{\mu}}} = \mathbf{P}_W \mathbf{Y}$, we have to minimize

$$\|\mathbf{Y} - \boldsymbol{\mu}^{(W)}\|^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \mu_{ijk}^{(W)})^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i)^2.$$

For brevity, we let $S_2 = \|\mathbf{Y} - \boldsymbol{\mu}\|^2$ then

$$\frac{\partial S_2}{\partial \theta} \overset{\text{set}}{=} 0 \Rightarrow -2\sum_i \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i) = 0$$

$$Y_{...} - rcm\hat{\hat{\theta}} - cm\cancelto{0}{\sum_i \alpha_i} = 0$$

$$\hat{\theta} = \bar{Y}_{...},$$

and since $\frac{\partial^2 S_2}{\partial S_2^2} = 2rcm > 0$, then $\hat{\hat{\theta}}$ minimizes S_2. Next we write S_2 as follows

$$S_2 = \sum_j \sum_k (Y_{ijk} - \theta - \alpha_i)^2 + \sum_{i^\star \neq i} \sum_j \sum_k (Y_{i^\star jk} - \theta - \alpha_{i^\star})^2,$$

then

$$\frac{\partial S_2}{\partial \alpha_i} = -2\sum_j \sum_k (Y_{ijk} - \theta - \alpha_i - \delta_{ij}) \overset{\text{set}}{=} 0$$

$$Y_{i..} - mc\hat{\hat{\theta}} - mc\hat{\hat{\alpha}}_i = 0$$

$$\hat{\hat{\alpha}}_i = \bar{Y}_{i..} - \bar{Y}_{...}$$

and since $\frac{\partial^2 S_2}{\partial \alpha_i^2} = 2mc > 0$, then $\hat{\hat{\alpha}}_i$ minimizes S_2. Therefore,

$$\hat{\hat{\mu}}_{ijk} = \hat{\theta} + \hat{\alpha}_i = \bar{Y}_{...} + \bar{Y}_{i...} - \bar{Y}_{...} = \bar{Y}_{i..}$$

Therefore,

$$\begin{aligned}
F &= \frac{\|\mathbf{P}_{V|W}\mathbf{Y}\|^2(n - \dim V)}{\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2(\dim V - \dim W)} = \frac{\|\mathbf{P}_V\mathbf{Y} - \mathbf{P}_W\mathbf{Y}\|^2(mrc - rc)}{\|\mathbf{P}_{V^\perp}\mathbf{Y}\|^2(rc - r)} \\
&= \frac{\|\hat{\boldsymbol{\mu}} - \hat{\hat{\boldsymbol{\mu}}}\|^2}{\|\mathbf{Y} - \mathbf{P}_V\mathbf{Y}\|^2} \frac{rc(m-1)}{r(c-1)} \\
&= \frac{\sum_i \sum_j \sum_k (\hat{\mu}_{ijk} - \hat{\hat{\mu}}_{ijk})}{\sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{i..})^2} \frac{rc(m-1)}{r(c-1)} \\
&= \frac{m\sum_i \sum_j (\bar{Y}_{ij.} - \bar{Y}_{i..})}{\sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{i..})^2} \frac{rc(m-1)}{r(c-1)}
\end{aligned}$$

And the noncentrality parameter is,

$$\begin{aligned}
\delta &= \frac{\|\mathbf{P}_{V|W}\|^2}{\sigma^2} = \frac{\|\mathbf{P}_V\boldsymbol{\mu} - \mathbf{P}_W\boldsymbol{\mu}\|^2}{\sigma^2} = \frac{\|\boldsymbol{\mu}^{(V)} - \boldsymbol{\mu}^{(W)}\|^2}{\sigma^2} \\
&= \frac{\sum_i \sum_j \sum_k (\mu_{ijk}^{(V)} - \mu_{ijk}^{(W)})^2}{\sigma^2} \\
&= \frac{\sum_i \sum_j \sum_k (\theta + \alpha_i + \delta_{ij} - \theta - \alpha_i)^2}{\sigma^2} \\
&= \frac{m\sum_i \sum_j (\delta_{ij})^2}{\sigma^2}.
\end{aligned}$$

Therefore,

$$F \sim F_{r(c-1), rc(m-1)}\left(\frac{m\sum_i \sum_j (\delta_{ij})^2}{\sigma^2}\right)$$

□

7. Lemma 7.8. Let V^* be the subspace of $\boldsymbol{\mu} = [(\mu_{ijk})]$ such that μ_{ijk} does not depend on k. Then $V = V^*$, and hence does not depend on the weights w_i and v_j.

Proof. Consider the following cases:

Case 1 $V \subset V^*$:
Let $\boldsymbol{\mu} \in V$, then $\mu_{ijk} = \theta + \alpha_i + \beta_j + \gamma_{ij}$. Thus μ_{ijk} does not depend on k, implying $\mu_{ijk} \in V^*, \forall \mu_{ijk} \in V$. Therefore, $V \subset V^*$

Case 2 $V^* \subset V$:
Let $\boldsymbol{\mu} \in V^*$. If V^* does not depend on k, then $V^* = \{\boldsymbol{\mu} : \mu_{ijk} = s_i + t_j + z_{ij}, \exists$ numbers $s_i, t_j, z_{ij}\}$ satisfies the given definition. Now define

$$\alpha_i = s_i - \frac{\sum_i w_i s_i}{\sum_i w_i}; \quad \beta_j = t_j - \frac{\sum_j v_j t_j}{\sum_j v_j}; \quad \gamma_{ij} = z_{ij} - \frac{\sum_i w_i z_{ij}}{\sum_i w_i};$$
$$\theta = \frac{\sum_i w_i s_i}{\sum_i w_i} + \frac{\sum_j v_j t_j}{\sum_j v_j} + \frac{\sum_i w_i z_{ij}}{\sum_i w_i}$$

Then,

$$\mu_{ijk} = \theta + \alpha_i + \beta_j + \gamma_{ij}.$$

and

$$\sum_i w_i \alpha_i = \sum_i w_i \left(s_i - \frac{\sum_i w_i s_i}{\sum_i w_i} \right) = 0$$

$$\sum_j v_i \beta_j = \sum_j v_j \left(t_j - \frac{\sum_j v_j t_j}{\sum_j v_j} \right) = 0$$

$$\sum_i w_i \gamma_{ij} = \sum_i w_i \left(z_{ij} - \frac{\sum_i w_i z_{ij}}{\sum_i w_i} \right) = 0$$

$$\sum_j v_j \gamma_{ij} = \sum_j v_j \left(z_{ij} - \frac{\sum_i w_i z_{ij}}{\sum_i w_i} \right) = 0$$

This suggests that μ_{ijk} satisfies the condition of $V, \forall \boldsymbol{\mu} \in V^*$. Thus $V^* \subset V$.

From Case 1 and Case 2, we conclude $V = V^*$. $\square$

8. Derive the $100(1-\alpha)\%$ simultaneous condfidence intervals for the contrasts associated with testing H_0 : the $\gamma_{ij} = 0$. The contrasts are given by

$$\sum_i \sum_j b_{ij} \gamma_{ij}, \sum_i b_{ij} = 0, j = 1, \cdots, c; \sum_j b_{ij}, i = 1, \cdots, r.$$

The $100(1-\alpha)\%$ simulataneous confidence intervals are given by

$$\sum_i \sum_j b_{ij} \gamma_{ij} \in \sum_i \sum_j b_{ij} \bar{Y}_{ij.} \pm \hat{\sigma} [(r-1)(c-1) F^{\alpha}_{(r-1)(c-1), N-rc} \sum_i \sum_j \frac{b_{ij}^2}{n_{ij}}]^{1/2}$$

where

$$\hat{\sigma}^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2 / (N - rc).$$