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PS 1 | **Stat231**

Contents

1 General Instructions

Answer the following:

1.12 It was noted in Section 1.2.1 that statisticians who follow the deFinetti school do not accept the Axiom of Countable Additivity, instead adhering to the Axiom of Finite Additivity.

(a) Show that the Axiom of Countable Additivity implies Finite Additivity.

(b) Although, by itself, the Axiom of Finite Additivity does not imply Countable Additivity, suppose we supplement it with the following. Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ be an infinite sequence of nested sets whose limit is the empty set, which we denote by $A_n \downarrow \emptyset$. Consider the following:

Axiom of Continuity: If $A_n \downarrow \emptyset$, then $P(A_n) \to 0$

Prove that the Axiom of Continuity and the Axiom of Finite Additivity imply Countable Additivity.

1.28 A way of approximating large factorials is through the use of *Stirling's Formula:*

$$n! \approx \sqrt{2\pi} n^{n+(1/2)} e^{-n},$$

a complete derivation of which is difficult. Instead, prove the easier fact,

$$\lim_{n\to\infty} \frac{n!}{n^{n+(1/2)}} e^{-n} = \text{a constant}.$$

(*Hint:* Feller 1968 proceeds by using the monotonicity of the logarithm to establish that

$$\int_{k-1}^{k} \log x \, dx < \log k < \int_{k}^{k+1} \log x \, dx, \quad k = 1, \cdots, n,$$

and hence

$$\int_{0}^{n} \log x \, dx < \log n! < \int_{1}^{n+1} \log x \, dx.$$

Now compare $\log n!$ to the average of the two integrals. See Exercise 5.35 for another derivation.)

1.38 Prove each of the following statements. (Assume that any conditioning event has positive probability.)

(a) If $P(B) = 1$, then $P(A|B) = P(A)$ for any A.

(b) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

(c) If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}. \tag{1}$$

(d) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

1.47 Prove that the following functions are cdfs.

(a) $\frac{1}{2} + \frac{1}{\pi} \arctan(x), x \in (-\infty, \infty)$

(b) $(1 + e^{-x})^{-1}, x \in (-\infty, \infty)$

(c) $e^{-e^{-x}}, x \in (-\infty, \infty)$

(d) $1 - e^{-x}, x \in (0, \infty)$

(e) the function defined in (1.5.6)

1.49 A cdf F_X is *stochastically* greater than a cdf F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then

$$P(X > t) \geq P(Y > t) \text{ for every } t$$

and

$$P(X > t) > P(Y > t), \; textforsome \; t$$

that is, X tends to be bigger than Y.

1.52 Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function

$$g(x) = \begin{cases} f(x)/[1 - F(x_0)] & x \geq x_0 \\ 0 & x < x_0. \end{cases} \tag{2}$$

Prove that $g(x)$ is a pdf. (Assume that $F(x_0) < 1$.)

1.54 For each of the following, determine the value of c that makes $f(x)$ a pdf.

(a) $f(x) = \mathrm{c} \sin x, 0 < x < \pi/2$

(b) $f(x) = \mathrm{c}e^{-|x|}, -\infty < x < \infty$

2 Solutions

1.12 (a) *Proof.* Consider $A_1, A_2, \cdots \in \mathscr{B}$ are pairwise disjoint, then by countable additivity

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Now

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigcup_{i=1}^{n} A_i \cup \bigcup_{i=n+1}^{\infty} A_i\right) \\ &= P\left(\bigcup_{i=1}^{n} A_i\right) + \left(\bigcup_{i=n+1}^{\infty} A_i\right), \text{ (since } A_i's \text{ are disjoints)} \\ &= P(A_1) + \cdots + P(A_n) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right), \text{ (by finite additivity)} \\ &= \sum_{i=1}^{n} P(A_i) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right) \end{aligned}$$

Now for any n, we can consider $P(A_i),\ i > n$ to be empty. Implying

$$P\left(\bigcup_{i=n+1}^{\infty} A_i\right) = \sum_{i=n+1}^{\infty} P(A_i) = P(\emptyset) + P(\emptyset) + \cdots,$$

that is,

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i) \\ &= \sum_{i=1}^{n} P(A_i) + P(\emptyset) + P(\emptyset) + \cdots \end{aligned}$$

$\therefore$ countable additivity implies finite additivity. $\square$

(b) *Proof.* From (a), we have shown that countable additivity implies finite additivity, i.e.,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} P(A_i) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right)$$

Now if we supplement this with the following condition, that $A_1 \supset A_2 \supset A_3 \supset \cdots$. By Axiom of Continuity, $\lim_{n\to\infty} A_n = \emptyset$, and by Monotone Sequential Continuity, $P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = 0$.

Notice that we can write $A_1 \supset A_2 \supset A_3 \supset \cdots$ as

$$B_k = \bigcup_{i=k}^{\infty} A_i, \text{ such that } B_{k+1} \supset B_k, \text{implying } \lim_{k\to\infty} B_k = \emptyset \tag{3}$$

Thus, finite additivity plus axiom of continuity, we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} P(A_i) + P(B_{n+1})\right) \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} P(A_i)\right) + \lim_{n\to\infty} P(B_{n+1}) \\ &= \sum_{i=1}^{\infty} P(A_i) + 0, \text{ (by axiom of continuity).} \end{aligned}$$

Implying countable additivity. □

1.38 .

(a) *Proof.* If $P(B) = 1$, then $P(S) = P(B) = 1$. Because $A \subseteq S$, implies $A \subseteq B$. Thus, $A \cap B = A$, and therefore

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = P(A) \tag{4}$$

□

(b) *Proof.* If $A \subseteq B$ then

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

and,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

□

(c) *Proof.* If A and B are mutually exclusive, then

$$\begin{aligned} P(A|A\cup B) &= \frac{P(A\cap(A\cup B))}{P(A\cup B)} \\ &= \frac{P(A)\cup[P(A)\cap P(B)]}{P(A)+P(B)} \\ &= \frac{P(A)}{P(A)+P(B)} \end{aligned}$$

□

(d) *Proof.* Consider,

$$P(A|B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

Hence,

$$P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C)$$

Now $P(B \cap C) = P(B|C)P(C)$, therefore

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$$

$\square$

1.47 $F(x)$ is a cdf if it satisfies the following conditions:

i $\lim_{n\to-\infty} F(x) = 0$ and $\lim_{n\to\infty} F(x) = 1$

ii $F(x)$ is increasing.

iii $F(x)$ is right-continuous.

(a) *Proof.* $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x), x \in (-\infty, \infty)$

i

$$\begin{aligned}
\lim_{n\to-\infty} F(x) &= \lim_{n\to-\infty}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(x)\right) \\
&= \frac{1}{2} + \frac{1}{\pi}\lim_{n\to-\infty}(\arctan(x)) \\
&= \frac{1}{2} + \frac{1}{\pi}\left(\frac{-\pi}{2}\right), \text{ since } \lim_{n\to-\frac{\pi}{2}}\frac{\sin(x)}{\cos(x)} = -\infty \\
&= 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
\lim_{n\to\infty} F(x) &= \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(x)\right) \\
&= \frac{1}{2} + \frac{1}{\pi}\lim_{n\to\infty}(\arctan(x)) \\
&= \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right), \text{ since } \lim_{n\to\frac{\pi}{2}}\frac{\sin(x)}{\cos(x)} = \infty \\
&= 1
\end{aligned}$$

ii To test if $F(x)$ is nondecreasing, recall in Calculus that, first differentiation of the function helps us decide if a function is decreasing or increasing. In particular, $\frac{dF(x)}{dx} > 0$ tells us that the function is increasing in a given interval of x. Thus,

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(x)\right) = \frac{1}{\pi(1+x^2)}$$

Since x^2 is always positive for all x, thus $\frac{dF(x)}{dx} > 0$, implying $F(x)$ is increasing.

iii $F(x)$ is continuous, implies that $F(x)$ is right-continuous.

$\square$

(b) *Proof.*

$$F(x) = \frac{1}{1+e^{-x}}, x \in (-\infty, \infty)$$

i

$$\begin{aligned}\lim_{n\to-\infty} F(x) &= \lim_{n\to-\infty}\left(\frac{1}{1+e^{-x}}\right)\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} F(x) &= \lim_{n\to\infty}\left(\frac{1}{1+e^{-x}}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{e^x}}\right)\\ &= 1\end{aligned}$$

ii Using the same method we did in (a), we have

$$\begin{aligned}\frac{dF(x)}{dx} &= \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right)\\ &= \frac{e^{-x}}{(1+e^{-x})^2}\end{aligned}$$

$\frac{dF(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} > 0,\ \forall\, x \in (-\infty, \infty)$. Thus the function is increasing in the interval of x.

iii $F(x)$ is continuous, implies the function is right-continuous.

$\square$

(c) *Proof.* $F(x) = e^{-e^{-x}}, x \in (-\infty, \infty)$

i

$$\begin{aligned}\lim_{n\to-\infty} F(x) &= \lim_{n\to-\infty}\left(e^{-e^{-x}}\right)\\ &= \lim_{n\to-\infty}\left(\frac{1}{e^{\frac{1}{e^x}}}\right)\\ &= 0\end{aligned} \tag{6}$$

$$\begin{aligned}\lim_{n\to\infty} F(x) &= \lim_{n\to\infty}\left(e^{-e^{-x}}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{e^{\frac{1}{e^x}}}\right)\\ &= 1\end{aligned}$$

ii Like what we did in (a), $\frac{dF(x)}{dx}$ is,

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left(e^{-e^{-x}}\right) = e^{-x}e^{-e^{-x}} > 0$$

Because $e^{-x}e^{-e^{-x}} > 0,\ \forall\ x \in (-\infty, \infty)$. Then we say $F(x)$ is an increasing function in the interval of x.

iii $F(x)$ is continuous, implies that $F(x)$ is right-continuous.

□

(d) *Proof.*

$$F(x) = 1 - \frac{1}{e^{-x}}, x \in (0, \infty) \tag{7}$$

i

$$\begin{aligned}\lim_{x\to-\infty} F(x) = \lim_{x\to 0^+} F(x) &= 1 - \lim_{x\to 0^+}\left(\frac{1}{e^x}\right)\\ &= 0\end{aligned} \tag{8}$$

$$\lim_{x\to-\infty} F(x) = 1 - \lim_{x\to\infty}\left(\frac{1}{e^x}\right) = 1$$

ii

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left(1 - \frac{1}{e^{-x}}\right) = 0 - (-e^{-x}) = \frac{1}{e^x} \tag{9}$$

$F(x)$ is an increasing function since $\frac{1}{e^{-x}} > 0,\ \forall\ x \in (0, \infty)$.

iii $F(x)$ is right-continuous, since it is continuous.

□

(e) *Proof.* The function in Equation (1.5.6) is given by,

$$F_Y(y) = \begin{cases} \dfrac{1-\varepsilon}{1+e^{-y}} & \text{if } y < 0,\ \text{for some } \varepsilon, 1 > \varepsilon > 0 \\ \varepsilon + \dfrac{1-\varepsilon}{1+e^{-y}} & \text{if } y \geq 0,\ \text{for some } \varepsilon, 1 > \varepsilon > 0 \end{cases}$$

i

$$\lim_{n\to-\infty} F_Y(y) = \lim_{n\to-\infty}\left(\frac{1-\varepsilon}{1+e^{-y}}\right) = \lim_{n\to-\infty}\left(\frac{1-\varepsilon}{1+\frac{1}{e^y}}\right) = 0$$

$$\lim_{n\to\infty} F(y) = \lim_{n\to\infty}\left(\varepsilon + \frac{1-\varepsilon}{1+e^{-y}}\right) = \varepsilon + \lim_{n\to\infty}\left(\frac{1-\varepsilon}{1+\frac{1}{e^y}}\right) = 1$$

ii For $y < 0$, we have

$$\begin{aligned}\frac{d}{dx}\left(\frac{1-\varepsilon}{1+e^{-y}}\right) &= (1-\varepsilon)\frac{(d)}{dx}\left(\frac{1}{1+e^{-y}}\right)\\ &= (1-\varepsilon)\frac{(1+\varepsilon^{-y})\cdot 0 - 1\cdot e^{-y}(-1)}{(1+e^{-y})^2}\\ &= \frac{(1-\varepsilon)e^{-y}}{(1+e^{-y})^2}\end{aligned}$$

$(1-\varepsilon) > 0$ since $0 < \varepsilon < 1$. Thus for all $y < 0$, $\frac{(1-\varepsilon)e^{-y}}{(1+e^{-y})^2} > 0$, implying that the function is increasing.

For $y \geq 0$,

$$\frac{d}{dx}\left(\varepsilon + \frac{1-\varepsilon}{1+e^{-y}}\right) = \varepsilon + \frac{(1-\varepsilon)e^{-y}}{(1+e^{-y})^2}$$

The function is increasing since $\varepsilon + \frac{(1-\varepsilon)e^{-y}}{(1+e^{-y})^2} > 0$ for all $y \geq 0$.

iii Since the function is continuous, then the function is right-continuous.

□

1.49 *Proof.* We know that,

$$P(X > t) = 1 - P(X \leq t) = 1 - F_X(t) \tag{10}$$

and

$$P(Y > t) = 1 - P(Y \leq t) = 1 - F_Y(t) \tag{11}$$

Hence we have,

$$P(X > t) = 1 - F_X(t) \overset{?}{\geq} 1 - F_Y(t) = P(Y > t)$$

Since $F_X(t) \leq F_Y(t)$, then the difference $1 - F_X(t)$ tends to get larger, contrary to $1 - F_Y(t)$. Thus for all t, $P(X > t) \geq P(X > t)$.

Now if $F_X(t) < F_Y(t)$ for some t, then using the same argument above, $P(X > t) \geq P(X > t)$ for some t. □

1.52 For a function to be a pdf, it has to satisfy the following:

A. $g(x) \geq 0$ for all x;

B. $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = 1$.

Proof. For any arbitrary x_0, $F(x_0) < 1$. Thus, $g(x)$ is always positive. Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{x_0} g(x)\,dx + \int_{x_0}^{\infty} g(x)\,dx \\
&= \int_{x_0}^{\infty} g(x)\,dx \\
&= \int_{x_0}^{\infty} \frac{f(x)}{(1-F(x_0))}\,dx \\
&= \frac{1}{1-F(x_0)}\int_{x_0}^{\infty} f(x)\,dx \\
&= \frac{1}{1-F(x_0)}[F(\infty) - F(x_0)] \\
&= \frac{1}{1-F(x_0)}[1 - F(x_0)] = 1, \text{ since } \lim_{x\to\infty} F(x) = 1
\end{aligned} \tag{12}$$

□

1.54 In order for $f(x)$ to be a pdf, it has to integrate to 1.

(a)

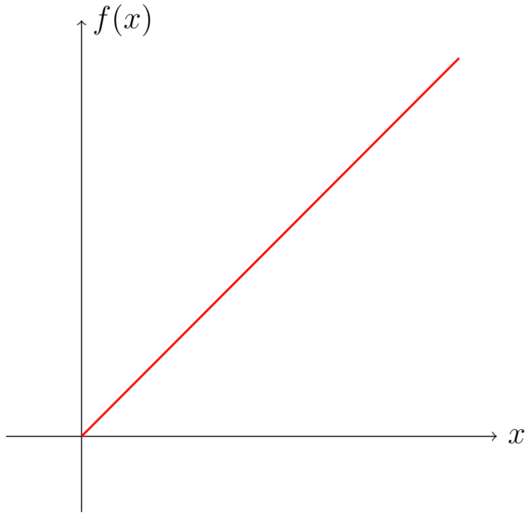
$$\begin{aligned}\int_{-\infty}^{\infty} f(x) = \int_{0}^{\frac{pi}{2}} \mathrm{c}\sin x &= -(\mathrm{c})\cos x|_{0}^{\frac{\pi}{2}} \\ &= -\mathrm{c}\left(\cos\left(\frac{\pi}{2}\right) - \cos(1)\right) \\ &= -\mathrm{c}(0-1) = 1\mathrm{c}\end{aligned} \tag{13}$$

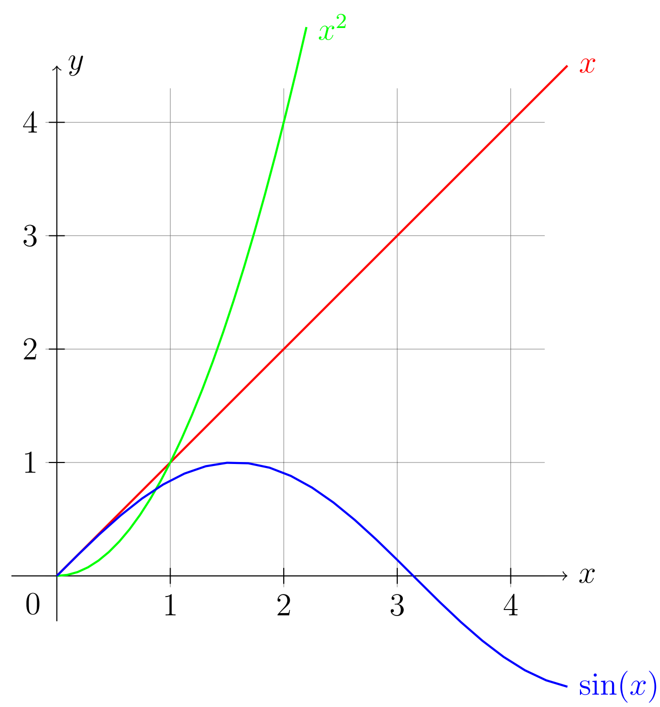
Hence, c is 1.

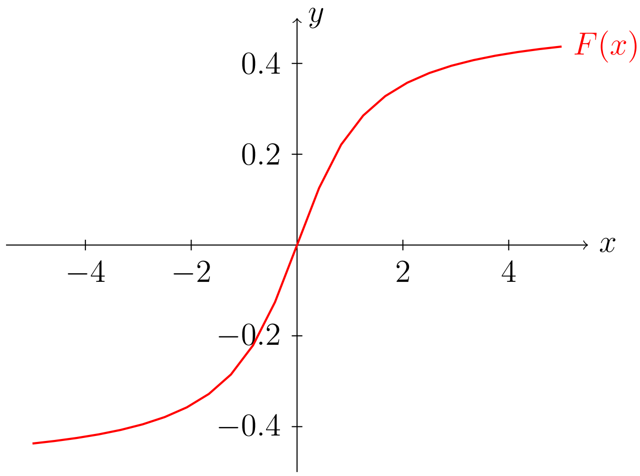
(b)

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) &= \int_{-\infty}^{\infty} \mathrm{c}\, e^{-|x|} \\ &= \mathrm{c}\left(\int_{-\infty}^{0} e^{x}\, dx + \int_{0}^{\infty} e^{-x}\, dx\right) \\ &= (e^{0} - e^{-\infty}) - (e^{-\infty} - e^{0}) \\ &= 1 + 1 = 2\end{aligned} \tag{14}$$

Hence, c is $\frac{1}{2}$.



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0.2
F(x)
x
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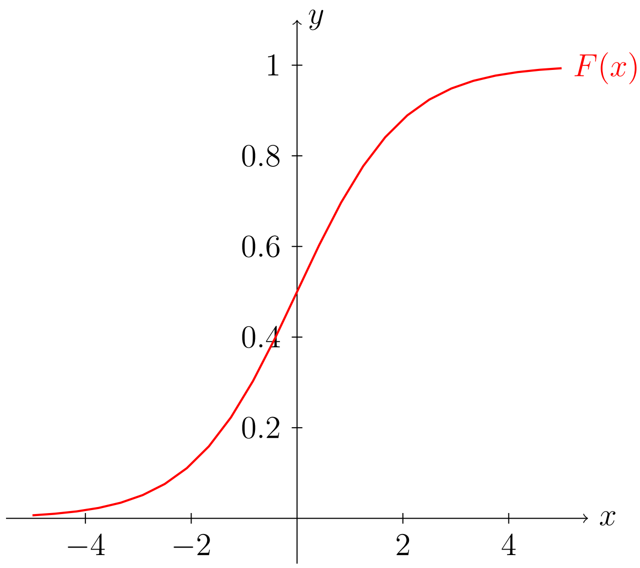
2
4
−0.2
−0.4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F(x)
x
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2
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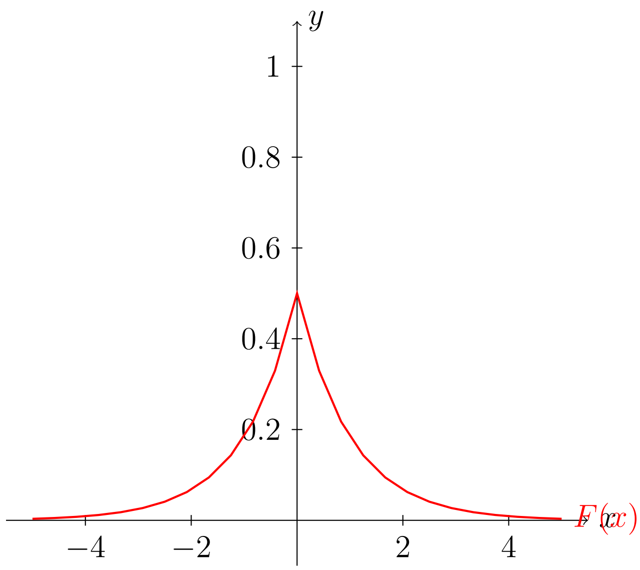
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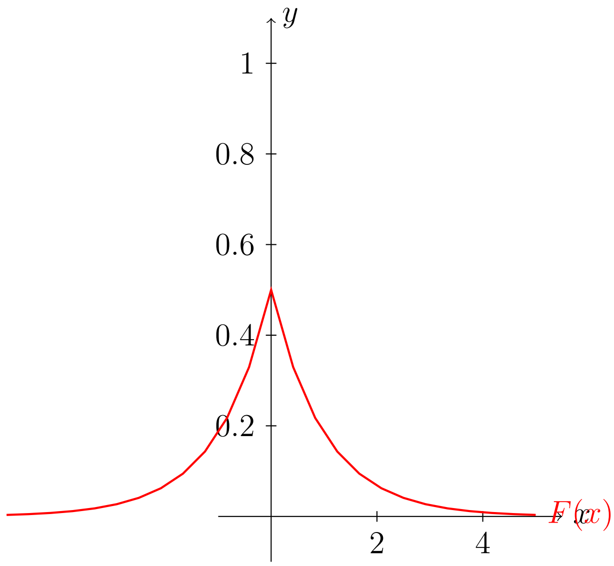

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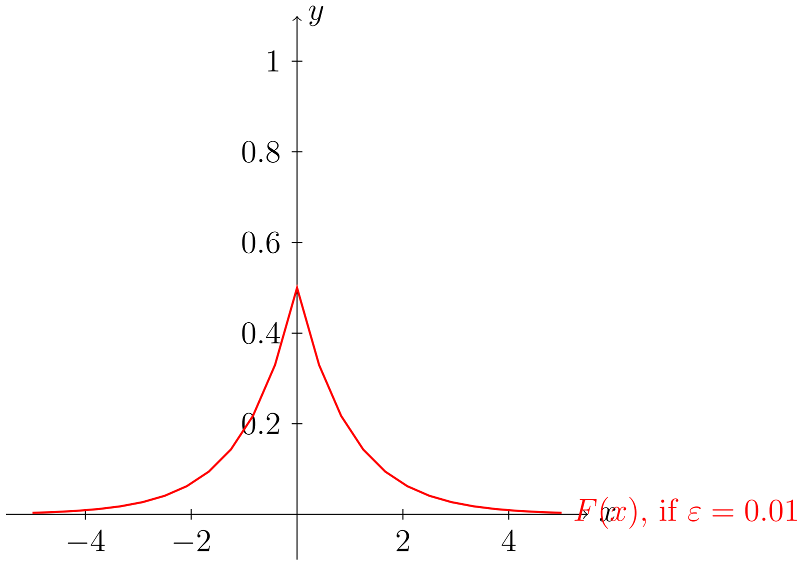
x
$F(x)$



y
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2
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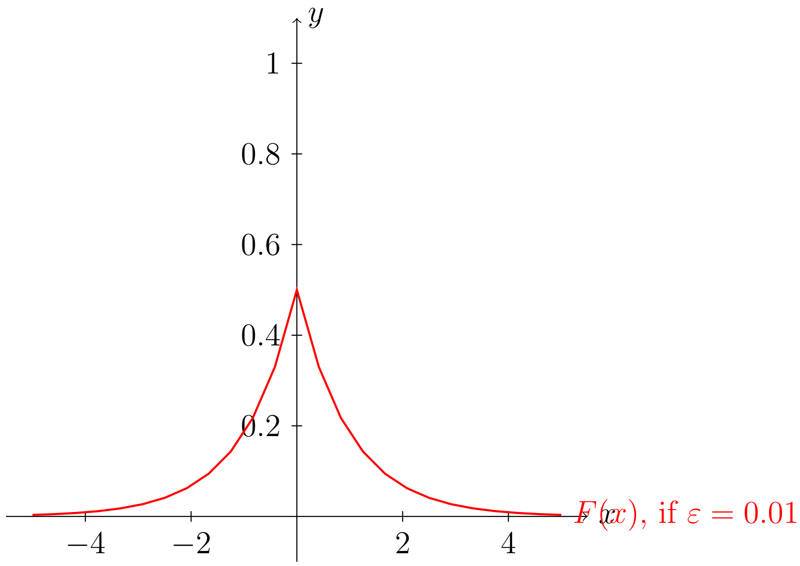
x
$F(x)$, if $\varepsilon = 0.01$



y
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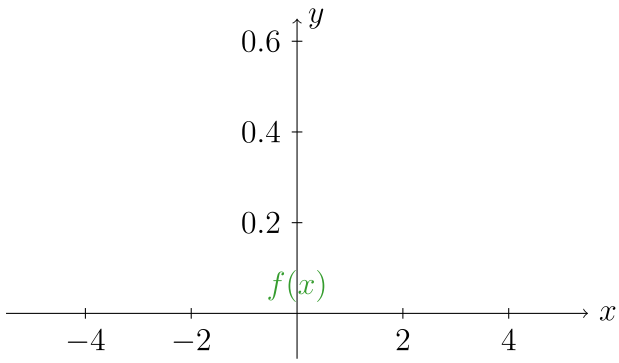
0.2
-4
-2
2
4
x
$F(x)$, if $\varepsilon = 0.01$



y
0.6
0.4
0.2
$f(x)$
-4
-2
2
4
x