IMO Shortlist Writeups

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1 Introduction

Here's a compiled list of my typed up solutions to various IMO shortlist problems during my preparation for the 66th IMO.

2 Problems

2.1 ISL 2022

Problem 2.1 – 2022 A1 Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers with the property that

$$(a_{n+1})^2 + a_n a_{n+2} \leq a_n + a_{n+2}$$

for all positive integers n. Show that $a_{2022} \leq 1$.

Solution. Define a sequence $b_i = a_i - 1\ \forall i$. The given condition is equivalent to

$$b_n b_{n+2} + b_{n+1}(b_{n+1} + 2) \leq 0$$

Where $b_i > -1\ \forall i$. Now FTSOC $b_{2022} > 0$. Notice we also have

$$b_{n-1} b_{n+1} + b_n(b_n + 2) \leq 0$$

Summing the two gives

$$b_n(b_{n-1} + b_n + 2) + b_{n+1}(b_{n+1} + b_{n-1} + 2) \leq 0$$

Substituting $n = 2022$ and $n = 2021$ into the above eqeuation, we get that $b_{2021} < 0$ and also $b_{2023} < 0$ respectively. As a result, considering $n = 2021$ in the first equation gives us a contradiction, and we're done. □

2.2 ISL 2019

Problem 2.2 – 2019 G4 Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2

be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

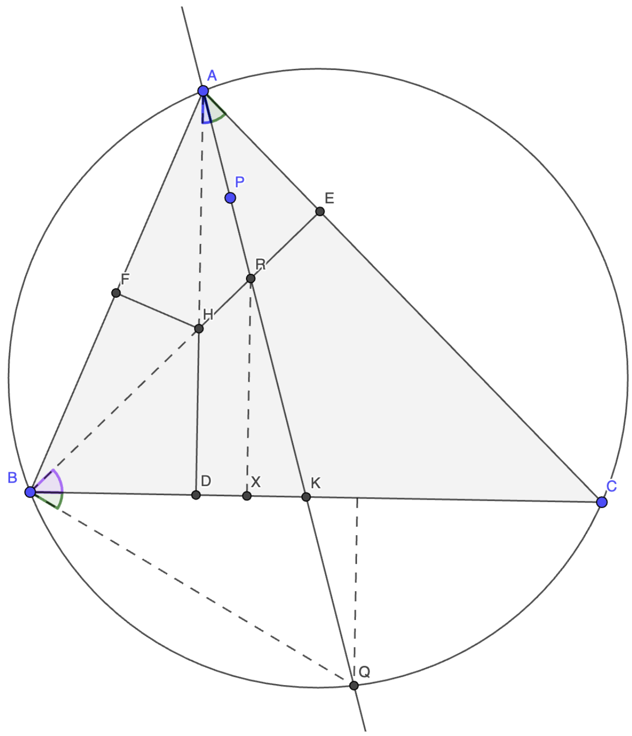

Fig 1: Diagram for 2019 G4

Proof. We will first prove the case where ABC is acute, and then deal with the obtuse case.

Let H denote the orthocenter of the triangle ABC. Drop the perpendiculars from H to each of the sides BC, AC, AB to be D, E, F respectively. Now, WLOG point P lies inside the quadrilateral $AEFH$. We will prove that the reflection of point P across K will be outside of (ABC) (notice I have renamed a few points for convenience). In fact, we just need to prove $KR > KQ$ for this to work.

Now, WLOG $FH < EH$. P can lie on either the same side of AH as F, or the opposite. If P lies on the same side of AH as F, we have $KR > HD > KQ$ so we're done. Henceforth we assume P lies ont he same side of AH as E.

In the above diagram, let

$$\alpha = \angle QBC = \angle QAC \text{ (green)}$$
$$\beta = \angle CBE = \angle DAC \text{ (purple)}$$
$$\gamma = \angle DAK \text{ (blue)}$$

We start with the following manipulation:

$$\begin{aligned} \sin(2\beta)\cos(2\gamma) - \cos(2\beta)\sin(2\gamma) &< \sin(2\beta) \\ \sin(2(\beta-\gamma)) &< \sin(2\beta) \\ \sin(2\alpha) &< \sin(2\beta) \\ \cos(\alpha)\sin(\alpha) &< \cos(\beta)\sin(\beta) \\ \frac{\cos(\alpha)}{\cos(\beta)} &< \frac{\sin(\beta)}{\sin(\alpha)} \end{aligned}$$

Now, we also have by sine rule in $\triangle ARB$ and $\triangle AQB$:

$$\frac{BR}{\sin(\angle BAR)} = \frac{AB}{\sin(\angle ARB)} = \frac{AB}{\sin(\angle ARE)} = \frac{AB}{\sin(90-\alpha)} = \frac{AB}{\cos(\alpha)}$$

$$\frac{BQ}{\sin(\angle BAR)} = \frac{AB}{\sin(\angle C)} = \frac{AB}{\sin(90-\beta)} = \frac{AB}{\cos(\beta)}$$

Hence

$$\frac{BQ}{BR} = \frac{\cos(\alpha)}{\cos(\beta)} < \frac{\sin(\beta)}{\sin(\alpha)}$$

and we have

$$BQ\sin(\alpha) < BR\sin(\beta) \iff YQ < RX$$

But this is enough to deduce $KQ < KR$ as $\triangle RKX \sim \triangle QKY$. So we're done in the ABC acute case. Now we handle obtuse:

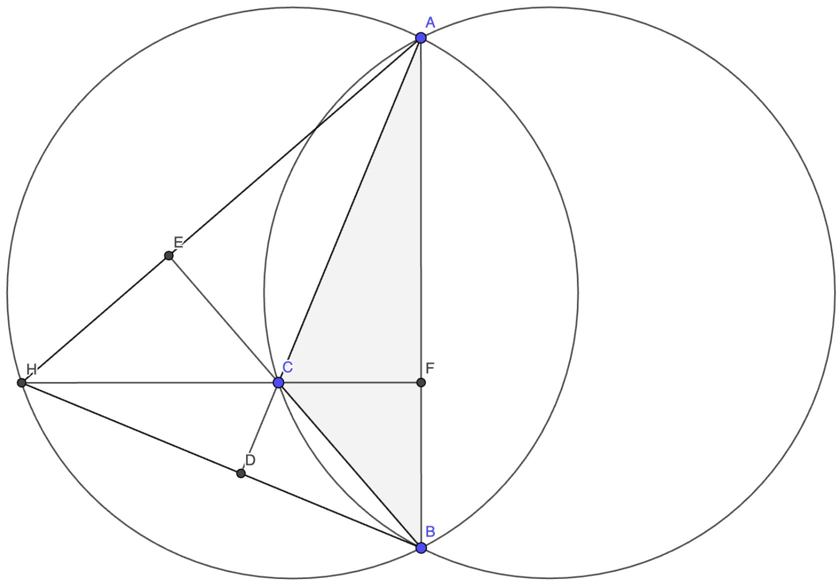

Fig 2: Obtuse case of 2019 G4.

We still construct the orthocenter H. Notice that P must be within the shaded region, and by our work on acute triangles before, the reflection of P across BD or AC will always lie outside of (AHB), as $\triangle AHB$ is acute. Finally, notice that this is enough to finish the problem, as (AHB) completely encloses the arc ACB on the circumcircle of $\triangle ABC$. $\square$

Remark: As I was attempting this problem, I told myself that barycentric coordinates would be an easy way to solve this, but I didn't know how they worked :(It turns out, this problem is much easier with a bary bash.

Problem 2.3 – 2019 N3 We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Proof. I claim that the answer is $S \in \mathbb{Z}$. Clearly this works. Now we prove that if we start with the set $S = \{2^a - 2^b \mid a, b \in \mathbb{Z}^+\}$, we can then get every integer with the appropriate choices of coefficients.

First we see 1 is in S by taking $P(x) = 2x - 2$.

Now, notice that if k is in S, $-k$ must also be in S by taking $P(x) = x + k$. Therefore, we restrict our search to only positive integers, as the negative integers will follow.

We will take the minimal positive integer m such that $m \notin S$ and aim to find a contradiction. Let $m = 2^e p$, where p is odd. Consider the number $M = 2^{e+\varphi(p)+1} - 2^{e+1} = 2^{e+1}(2^{\varphi(p)} - 1)$. Clearly $M \in S$ and $m \mid M$ (by Euler's Theorem).

We write $M = b_1 m + b_2 m^2 + b_3 m^3 + \cdots + b_n m^n$. Then we can consider the polynomial

$$P(x) = -M + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n$$

This works as $b_i \in S \ \forall i$ since $b_i < m$. Finally, notice that m must be a root to the above polynomial, and so $m \in S$, and we're done. $\square$

2.3 ISL 2017

Problem 2.4 – 2017 A4 A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that $|a_n| \leq M$ for all positive integers n.

Proof. Let's denote a_x to be the element with the maximum absolute value in the set $\{a_1, a_2, \cdots a_{2017}\}$. We split the problem into cases:

Case 1: $a_x = 0$. This case is trivial as all values in the sequence is equal to 0.

Case 2: $a_x > 0$. Let $M = a_x$, I will prove that $-2M \leq a_i \leq M \ \forall i$. Proof: Notice that

$$\max_{i+j=2018}(a_i + a_j) \geq a_x + a_{2018-x} \geq 0$$

So $a_{2018} = -\max_{i+j=2018}(a_i + a_j) \le 0$, i.e. It's bounded above by 0. We also know that

$$\max_{i+j=2018}(a_i + a_j) \le M + M = 2M$$

so $a_{2018} = -\max_{i+j=2018}(a_i + a_j) \ge -2M$. So we have

$$-2M \le a_{2018} \le 0$$

Now, if $-M \le a_{2018} \le 0$, we can carry on this process iteratively to get that the next element also has the bound stated above. Otherwise, assume that $-2M \le a_{2018} < -M$. We see that

$$\max_{i+j=2019}(a_i + a_j) \ge a_x + a_{2019-x} \ge M + (-2M) = -M$$

So that means $a_{2019} = -\max_{i+j=2019}(a_i + a_j) \le M$. But also,

$$\max_{i+j=2019}(a_i + a_j) \le M + M = 2M$$

So we have

$$-2M \le a_{2019} \le M$$

Thus we may continue this process iteratively to get that $-2M \le a_i \le M \ \forall i$.

Case 3: $a_x < 0$. Let $-M = a_x$, $M > 0$.

Notice that

$$\max_{i+j=2018}(a_i + a_j) \le 2M$$
$$\max_{i+j=2018}(a_i + a_j) \ge -2M$$

So we achieve the bound that $-2M \le a_{2018} \le 2M$.

Case 3.1: If $M < a_{2018} \le 2M$, we can refer to Case 2 above to see that the sequence is bounded.

Case 3.2: If $-M \le a_{2018} \le M$, We can iterate this process again, as $a_x = -M$ is still the a_i with the largest absolute value.

Case 3.3: $-2M \le a_{2018} < -M$.

Let $a_{2018} = -k$. Therefore there must exist p, q such that $p + q = 2018$ and $a_p + a_q = k$. WLOG let $a_p \ge \frac{k}{2}$.

We see

$$\max_{i+j=2019}(a_i + a_j) \ge a_p + a_{2019-p} \ge \frac{k}{2} + (-k) = \frac{-k}{2} \ge \frac{-2M}{2} = -M$$

and also

$$\max_{i+j=2019}(a_i + a_j) \le M + M = 2M$$

So we actually see that

$$-2M \le a_{2019} \le M$$

But we're done here, by considering the most negative element $a_n = -N$. There must be an a_i with $a_i > \frac{N}{2}$, so the lower bound for $\max_{i+j=n}(a_i + a_j)$ is $\frac{N}{2} + (-N) = \frac{-N}{2} \ge -M$. The upper bound of $2M$ is obvious to see.

So when when calculate the next values of the sequence, the upper and lower bounds for $\max_{i+j=n}(a_i + a_j)$ are fixed at $2M$ and $-M$ respectively, so we're done.

$\square$