9231 CAIE Further Math — Complex Numbers

Alston

1 Summations

We use De Moivre's Theorem to get $z^n = cis(n\theta)$ and also $\frac{1}{z^n} = z^{-n} = cis(-n\theta) = \cos(n\theta) - i\sin(n\theta)$. Then, we can sum or subtract these two statements to get:

Theorem 1.1 – Formulae

$$z^n + \frac{1}{z^n} = 2\cos(n\theta)$$

$$z^n - \frac{1}{z^n} = 2i\sin(n\theta)$$

Now, for evaluating things such as $\int \cos^4(\theta)$, just use the formula for $n = 1$ first, then raise it to the 4th power. Expand the bracket in z, and then group the z^n terms.

$$\cos^4(\theta) = (z + \frac{1}{z})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} = 2\cos(4\theta) + 8\cos(2\theta) + 6$$

1.1 Summation of Trig Functions

If it's a summation in cos, realise that you can turn it into the REAL parts of the summation of a sequence in z, which is very easy to calculate as a geometric series. Then turn it back to $z = cis(\theta)$ or $z = e^{i\theta}$ and get the real part. Similarly for summation in sin.

Problem 1.1 Determine the value of $\sum_{n=0}^{\infty} 2^{-n} \sin\left(\frac{n\pi}{2}\right)$

Consider the series $\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$. We actually just want the real part of this series, evaluated at $\theta = \frac{\pi}{2}$.

$$\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = 1 + \frac{z}{2} + \frac{z^2}{4} \cdots = \frac{1}{1 - \frac{z}{2}} = \frac{2}{2 - z}$$

Now, let $z = e^{i\theta}$, $\frac{2}{2-z} = \frac{2}{2-e^{i\theta}}$. Multiply top and bottom by $2 - e^{-i\theta}$ and we have $\frac{2}{2-z} = \frac{4-2e^{-i\theta}}{5-4\cos\theta}$. $\mathrm{Im}\left(\frac{4-2e^{-i\theta}}{5-4\cos\theta}\right) = \frac{2\sin\theta}{5-4\cos\theta}$, so just substitute $\theta = \frac{\pi}{2}$ to get the answer.