

# Method of Moving Points

Alston Yam

## 1 Introduction

Inspired by [problem 4](#) on this IMO mock by Evan Chen.

## 2 Cross Ratios

**Definition 2.1 – Cross Ratios** Given 4 distinct points  $A, B, C, D$  on a line, the cross ratio  $(A, B; C, D)$  is defined as

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD}$$

Where the lengths are taken to be directed ( $XY = -YX$ ).

We can actually extend the definition of the cross ratio to not just points on a line, but also four points on a conic  $\gamma$  (the most commonly used conic in Olympiad geometry is a circle), and also a *pencil* of lines through a particular point. In the latter case  $A, B, C, D$  will correspond to lines rather than points.

In the case of a pencil, the cross ratio can actually be thought of as the ratio of the sines of the angles between these four lines.

## 3 Projective Transformations

A *projective transformation* is any transformation that preserves the cross ratio. Specifically:

**Definition 3.1 – Projective Transformations** A projective map  $f$  is defined as a function  $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  (where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are both conics, lines or pencils of lines) such that for any 4 points  $A, B, C, D \in \mathcal{C}_1$ ,

$$(A, B; C, D) = (f(A), f(B); f(C), f(D))$$

Notice that a projective map is bijective. Now, here are two results that would come in handy.

**Theorem 3.1 – Projective Compositions** The composition  $f \circ g$  of two projective functions  $f$  and  $g$  is projective.

*Proof.*

$$(A, B; C, D) = (g(A), g(B); g(C), g(D)) = (f \circ g(A), f \circ g(B); f \circ g(C), f \circ g(D))$$

□

**Theorem 3.2 – Inverse of a Projective Map** The inverse  $f^{-1}$  of a projective map  $f$  is also projective.

*Proof.*

$$(f(A), f(B); f(C), f(D)) = (A, B; C, D) = (f^{-1}f(A), f^{-1}f(B); f^{-1}f(C), f^{-1}f(D))$$

□

We give a few examples of common projective transformations below. These are taken from [this blog post](#).

## 3.1 Common Projective Transformations

### 3.1.1 Projection from a line to a pencil of lines

Given a line  $l$  and a point  $P$  not on  $l$ , we can project every point  $Q$  on  $l$  to the line  $PQ$ . This is a projective map, as

$$(A, B; C, D) = (PA, PB; PC, PD)$$

Where  $PA, PB, PC, PD$  are lines going through  $P$ .

### 3.1.2 Projection from a line to another line

To project a line  $l_1$  to another line  $l_2$ , we take a point  $P$  not on either line. We will first project  $l_1$  onto the pencil of lines going through  $P$ , then project this pencil onto  $l_2$ . This gives the desired effect.

### 3.1.3 Rotating a pencil of lines

This is a projective map as the cross ratio of a pencil of lines only depend on the angle between them.

### 3.1.4 Reflection across a line

This is true by symmetry: imagine four points  $A, B, C, D$ . When we reflect them across any arbitrary line  $l$ , The distances between them (in the case they are points) or the angle between them (in in case they are lines) stays fixed.

### 3.1.5 Projection from a conic to a pencil of lines

Consider a conic  $\gamma$ . We take a point  $P \in \gamma$ , and for every other point  $Q \in \gamma, Q \neq P$ , we will project  $Q$  onto the line  $PQ$ .

### 3.1.6 Projection from a conic to points on that same conic

Consider the conic  $\gamma$  once again. Now we will take a point  $P$  not on the conic. For every point  $Q$  on the conic, we will project  $Q$  to  $PQ \cap \gamma$  different from  $Q$  (Unless  $PQ$  is a tangent, in which case  $Q$  gets mapped to itself).

### 3.1.7 Inversion

It turns out, interestingly, Inversion also preserves the cross ratio. Therefore inversion is actually also a type of projective transformation.

*Proof.* Consider an inversion about a circle with radius  $r$  and center  $O$ . By the distance formula:

$$A'B' = \frac{r^2}{OA \cdot OB} \cdot AB$$

Now we have:

$$(A', B'; C', D') = \frac{A'C' \cdot B'D'}{B'C' \cdot A'D'} = \frac{\frac{r^4}{OA \cdot OB \cdot OC \cdot OD} \cdot AC \cdot BD}{\frac{r^4}{OA \cdot OB \cdot OC \cdot OD} \cdot BC \cdot AD} = \frac{AC \cdot BD}{BC \cdot AD} = (A, B; C, D)$$

□

## 4 The Method

The essence of the method of moving points boils down to one important theorem:

**Theorem 4.1** If  $f, g : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  are projective, then  $f \equiv g$  iff  $f(A) = g(A)$  for at least 3 different values of  $A$ .

*Proof.* Necessity is simple. Now for sufficiency: consider 3 points  $A, B, C$  such that  $f = g$  on these three points. Consider another point  $X \in \mathcal{C}_1 / \{A, B, C\}$  Then we see:

$$(f(A), f(b); f(C), f(X)) = (A, B; C, X) = (g(A), g(B); g(C), g(X)) = (f(A), f(b); f(C), g(X))$$

Which is enough to conclude  $f(X) = g(X)$ . □

So now to solve a given geometry problem, If we can find two projective maps  $f$  and  $g$ , such that  $f$  is equivalent to the condition given by the problem, and  $g$  is equivalent to the result we want to prove, then we only need to check 3 special values of a moving point where  $f = g$ . If we can do that, then by the above theorem we would've proved that  $f \equiv g$  and in fact we would be done.

## 5 Example Problems

These problems are sourced from [here](#) and [here](#).

**Problem 5.1 – USA Winter TST for IMO 2019 Problem 1** Let  $ABC$  be a triangle and let  $M$  and  $N$  denote the midpoints of  $\overline{AB}$  and  $\overline{AC}$ , respectively. Let  $X$  be a point such that  $\overline{AX}$  is tangent to the circumcircle of triangle  $ABC$ . Denote by  $\omega_B$  the circle through  $M$  and  $B$  tangent to  $\overline{MX}$ , and by  $\omega_C$  the circle through  $N$  and  $C$  tangent to  $\overline{NX}$ . Show that  $\omega_B$  and  $\omega_C$  intersect on line  $BC$ .

*Merlijn Staps*