Lessons in Maths Olympiads

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1 Introduction

This document aims to summarise all of the lessons that I've learnt and the mistakes I've made during my preparation for the 2025 IMO.

2 Lessons

2.1 Hands Dirty

2.2 Wishful Thinking

"Ohhhh... wouldn't it be SOOOOOO nice if this was true?"

Often times it is beneficial to employ this technique to look "into the future", to try and find conjectures that we wish to proof that would further our investigation.

This comes in many forms: Hoping that four points are cyclic, this sequence is bounded, the function is injective ...

Let's look at an example.

Example 2.1 – 2021 ISL N1 Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and

$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

My first thought was to write $a^2+b+3 \mid ab+3b+8$, and start cancelling terms. This was certainly the right idea, however without the following motivation, I would be running into deadends left right and center not knowing what I should cancel.

"Huh.. the degree of b on both parts of the fraction is 1, so wouldn't it be nice if I had a fraction in just a?"

With this motivation we do the following:

$$a^2 + b + 3 \mid b(a+3) + 8$$
$$a^2 + b + 3 \mid b(a+3) + 8 - (a+3)(a^2 + b + 3)$$
$$a^2 + b + 3 \mid b(a+3) + 8 - a^2(a+3) - b(a+3) - 3(a+3)$$
$$a^2 + b + 3 \mid -(a+1)^3$$

Which is actually really nice, as we know a^2+b+3 is cube free. So hence we have $a^2+b+3 \mid (a+1)^2$. But $a^2 + 2b + 5 > 2a$ for any positive integers a, which gives $2(a^2 + b + 3) > (a+1)^2$. therefore we must have $(a+1)^2 = a^2 + b + 3 \iff a = 2b - 2$. We can now sub back into the original equation to get $n = 2$ is the only solution, and we're done.

2.3 Visualisation

This will come pretty naturally in most combinatorics questions, however we are not restricted to just that. As a learn-by-picture guy myself I often times find turning things such as sequences into something that I can see in my head helps me solve problems much faster.