Contradiction Answers

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1 Solutions

Problem 1.1 Prove that $\sqrt{2}$ is irrational.

Proof. FTSOC, Suppose $\sqrt{2}$ is rational. Then, $\sqrt{2} = \frac{a}{b}$ for some integers a and b with no common factors. Squaring both sides, we get $2 = \frac{a^2}{b^2}$. Thus, $2b^2 = a^2$. Since a^2 is even, a must be even. Let $a = 2k$ for some integer k. Then, $2b^2 = (2k)^2 = 4k^2$. Thus, $b^2 = 2k^2$. Since b^2 is even, b must be even. However, this contradicts the fact that a and b have no common factors. Thus, $\sqrt{2}$ is irrational. $\square$

Problem 1.2 Prove that $\sqrt[m]{n}$ is either irrational or an integer $\forall n, m \in \mathbb{N}$.

Proof. FTSOC, Suppose $\sqrt[m]{n}$ is neither irrational nor an integer. Then, $\sqrt[m]{n} = \frac{a}{b}$ for some integers a and b with no common factors. Raising both sides to the mth power, we get $n = \frac{a^m}{b^m}$. Thus, $nb^m = a^m$. However, notice that since a is an integer, if the RHS is a perfect mth power, then the LHS must also be an mth power. This implies that n is an mth power, i.e. $n = k^m$ for some integer k. This is a contradiction as $\sqrt[m]{n} = k$ which is an integer, Contradiction! $\square$

Problem 1.3 Show that if $2^x = 3$, then x is irrational.

Proof. FTSOC, Suppose x is rational. Then, $x = \frac{a}{b}$ for some integers a and b with no common factors. Raising both sides to the bth power, we get $2^a = 3^b$. However, this is a contradiction as 2^a is even and 3^b is odd. Thus, x is irrational. $\square$

Problem 1.4 Prove that there are no triangular numbers which is two less than a multiple of 11. (Triangular numbers are of the form $n = \frac{k(k+1)}{2}$, where k is a positive integer.)

Proof. FTSOC, Suppose there exists a triangular number $n = \frac{k(k+1)}{2}$ such that $n \equiv 9 \pmod{11}$. Then, $k(k+1) \equiv 18 \equiv 7 \pmod{11}$. Now, we may consider all the different possible residues of k under mod 11. We have the following table: (see next page)

$k \pmod{11}$	$k(k+1)$	$k(k+1) \pmod{11}$
0	0	0
1	2	2
2	6	6
3	12	1
4	20	9
5	30	8
6	42	9
7	56	1
8	72	6
9	90	2
10	110	0

From the table, we see that there are no possible values of k such that $k(k+1) \equiv 7 \pmod{11}$. Contradiction! ☐

Problem 1.5 Prove that the sum of a rational number and an irrational number is irrational.

Proof. *FTSOC,* Suppose the sum of a rational number $\frac{a}{b}$ and an irrational number r is rational. Then, $\frac{a}{b} + r = q$ for some rational number $q = \frac{c}{d}$. Rearranging, we get

$$\frac{a}{b} + r = \frac{c}{d}$$

$$r = \frac{c}{d} - \frac{a}{b}$$

$$r = \frac{bc - ad}{bd}$$

However, since we assumed that r was irrational, it should not have been possible to express it as a fraction. This is a contradiction, so we're done. ☐

Problem 1.6 In a party, friendship forms and breaks all the time. However, you discover that no matter what happens, there are always either 3 people who are friends with each other, or 3 people who are not friends with each other. Prove that there are at least 6 people at the party.

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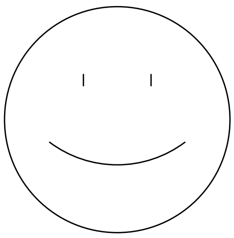
Proof. *FTSOC,* Suppose there are less than $n \leq 5$ people at the party. If we are able to find a construction for 5 people at the party, such that the problem statement is not satisfied, then we would be done as we could choose a subset of size n of the five people to see that this subset also doesn't satisfy the problem. So we go about finding such a construction:



$\square$