Method of Moving Points

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1 Introduction

Inspired by problem 4 on this IMO mock by Evan Chen.

2 Cross Ratios

Definition 2.1 – Cross Ratios Given 4 distinct points A, B, C, D on a line, the cross ratio $(A, B; C, D)$ is defined as

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD}$$

Where the lengths are taken to be directed ($XY = -YX$).

We can actually extend the definition of the cross ratio to not just points on a line, but also four points on a conic γ (the most commonly used conic in Olympiad geometry is a circle), and also a *pencil* of lines through a particular point. In the latter case A, B, C, D will correspond to lines rather than points.

In the case of a pencil, the cross ratio can actually be thought of as the ratio of the sines of the angles between these four lines.

3 Projective Transformations

A *projective transformation* is any transformation that preserves the cross ratio. Specifically:

Definition 3.1 – Projective Transformations A projective map f is defined as a function $f : \mathcal{C}_1 \to \mathcal{C}_2$ (where $\mathcal{C}_1$ and $\mathcal{C}_2$ are both conics, lines or pencils of lines) such that for any 4 points $A, B, C, D \in \mathcal{C}_1$,

$$(A, B; C, D) = (f(A), f(B); f(C), f(D))$$

There are two useful results that would come in handy.

Theorem 3.1 – Projective Compositions The composition $f \circ g$ of two projective functions f and g is projective.

Theorem 3.2 – Inverse of a Projective Map The inverse f^{-1} of a projective map f is also projective.

We give a few examples of common projective transformations below. These are taken from this blog post.

3.1 Common Projective Transformations

3.1.1 Projection from a line to a pencil of lines

3.1.2 Projection from a line to another line

3.1.3 Reflection across a line

3.1.4 Projection from a conic to a pencil of lines

3.1.5 Projection from a conic to points on that same conic

3.1.6 Inversion

4 The Method

The essence of the method of moving points boils down to one important theorem:

Theorem 4.1 hello

5 Example Problems

These problems are sourced from here and here.

Problem 5.1 – USA Winter TST for IMO 2019 Problem 1 Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

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