

# Method of Moving Points

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## 1 Introduction

Inspired by [problem 4](#) on this IMO mock by Evan Chen.

## 2 Cross Ratios

**Definition 2.1 – Cross Ratios** Given 4 distinct points  $A, B, C, D$  on a line, the cross ratio  $(A, B; C, D)$  is defined as

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD}$$

Where the lengths are taken to be directed ( $XY = -YX$ ).

We can actually extend the definition of the cross ratio to not just points on a line, but also four points on a conic  $\gamma$  (the most commonly used conic in Olympiad geometry is a circle), and also a *pencil* of lines through a particular point. In the latter case  $A, B, C, D$  will correspond to lines rather than points.

In the case of a pencil, the cross ratio can actually be thought of as the ratio of the sines of the angles between these four lines.

## 3 Projective Transformations

A *projective transformation* is any transformation that preserves the cross ratio. Specifically:

**Definition 3.1 – Projective Transformations** A projective map  $f$  is defined as a function  $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  (where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are both conics, lines or pencils of lines) such that for any 4 points  $A, B, C, D \in \mathcal{C}_1$ ,

$$(A, B; C, D) = (f(A), f(B); f(C), f(D))$$

There are two useful results that would come in handy.

**Theorem 3.1 – Projective Compositions** The composition  $f \circ g$  of two projective functions  $f$  and  $g$  is projective.

**Theorem 3.2 – Inverse of a Projective Map** The inverse  $f^{-1}$  of a projective map  $f$  is also projective.

We give a few examples of common projective transformations below. These are taken from [this blog post](#).

### 3.1 Common Projective Transformations

3.1.1 Projection from a line to a pencil of lines

3.1.2 Projection from a line to another line

3.1.3 Reflection across a line

3.1.4 Projection from a conic to a pencil of lines

3.1.5 Projection from a conic to points on that same conic

3.1.6 Inversion

## 4 The Method

The essence of the method of moving points boils down to one important theorem:

**Theorem 4.1** hello