Method of Moving Points

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1 Introduction

Inspired by problem 4 on this IMO mock by Evan Chen.

2 Cross Ratios

Definition 2.1 – Cross Ratios Given 4 distinct points A, B, C, D on a line, the cross ratio $(A, B; C, D)$ is defined as

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD}$$

Where the lengths are taken to be directed ($XY = -YX$).

We can actually extend the definition of the cross ratio to not just points on a line, but also four points on a conic γ (the most commonly used conic in Olympiad geometry is a circle), and also a *pencil* of lines through a particular point. In the latter case A, B, C, D will correspond to lines rather than points.

In the case of a pencil, the cross ratio can actually be thought of as the ratio of the sines of the angles between these four lines.

3 Projective Transformations

A *projective transformation* is any transformation that preserves the cross ratio. Specifically:

Definition 3.1 – Projective Transformations A projective map f is defined as a function $f : \mathcal{C}_1 \to \mathcal{C}_2$ (where $\mathcal{C}_1$ and $\mathcal{C}_2$ are both conics, lines or pencils of lines) such that for any 4 points $A, B, C, D \in \mathcal{C}_1$,

$$(A, B; C, D) = (f(A), f(B); f(C), f(D))$$

Notice that a projective map is bijective. Now, here are two results that would come in handy.

Theorem 3.1 – Projective Compositions The composition $f \circ g$ of two projective functions f and g is projective.

Proof.

$$(A,B;C,D) = (g(A),g(B);g(C),g(D)) = (f\circ g(A), f\circ g(B); f\circ g(C), f\circ g(D))$$

□

Theorem 3.2 – Inverse of a Projective Map The inverse f^{-1} of a projective map f is also projective.

Proof.

$$(f(A),f(B);f(C),f(D)) = (A,B;C,D) = (f^{-1}f(A), f^{-1}f(B); f^{-1}f(C), f^{-1}f(D))$$

□

We give a few examples of common projective transformations below. These are taken from this blog post.

3.1 Common Projective Transformations

3.1.1 Projection from a line to a pencil of lines

Given a line l and a point P not on l, we can project every point Q on l to the line PQ. This is a projective map, as

$$(A,B;C,D) = (PA,PB;PC,PD)$$

Where PA, PB, PC, PD are lines going through P.

3.1.2 Projection from a line to another line

To project a line l_1 to another line l_2, we take a point P not on either line. We will first project l_1 onto the pencil of lines going through P, then project this pencil onto l_2. This gives the desired effect.

3.1.3 Rotating a pencil of lines

This is a projective map as the cross ratio of a pencil of lines only depend on the angle between them.

3.1.4 Reflection across a line

This is true by symmetry: imagine four points A, B, C, D. When we reflect them across any arbitrary line l, The distances between them (in the case they are points) or the angle between them (in in case they are lines) stays fixed.

3.1.5 Projection from a conic to a pencil of lines

Consider a conic γ. We take a point $P \in \gamma$, and for every other point $Q \in \gamma, Q \neq P$, we will project Q onto the line PQ.

3.1.6 Projection from a conic to points on that same conic

Consider the conic γ once again. Now we will take a point P not on the conic. For every point Q on the conic, we will project Q to $PQ \cap \gamma$ different from Q (Unless PQ is a tangent, in which case Q gets mapped to itself).

3.1.7 Inversion

It turns out, interestingly, Inversion also preserves the cross ratio. Therefore inversion is actually also a type of projective transformation.

Proof. Consider an inversion about a circle with radius r and center O. By the distance formula:

$$A'B' = \frac{r^2}{OA \cdot OB} \cdot AB$$

Now we have:

$$(A', B'; C', D') = \frac{A'C' \cdot B'D'}{B'C' \cdot A'D'} = \frac{\frac{r^4}{OA \cdot OB \cdot OC \cdot OD} \cdot AC \cdot BD}{\frac{r^4}{OA \cdot OB \cdot OC \cdot OD} \cdot BC \cdot AD} = \frac{AC \cdot BD}{BC \cdot AD} = (A, B; C, D)$$

□

4 The Method

The essence of the method of moving points boils down to one important theorem:

Theorem 4.1 If $f, g : \mathcal{C}_1 \to \mathcal{C}_2$ are projective, then $f \equiv g$ iff $f(A) = g(A)$ for at least 3 different values of A.

Proof. Necessity is simple. Now for sufficiency: consider 3 points A, B, C such that $f = g$ on these three points. Consider another point $X \in \mathcal{C}_1/_{\{A,B,C\}}$ Then we see:

$$(f(A), f(B); f(C), f(X)) = (A, B; C, X) = (g(A), g(B); g(C), g(X)) = (f(A), f(B); f(C), g(X))$$

Which is enough to conclude $f(X) = g(X)$. □

So now to solve a given geometry problem, If we can find two projective maps f and g, such that f is equivalent to the condition given by the problem, and g is equivalent to the result we want to prove, then we only need to check 3 special values of a moving point where $f = g$. If we can do that, then by the above theorem we would've proved that $f \equiv g$ and in fact we would be done.

5 Example Problem

Sourced from this blog.

Problem 5.1 – USA Winter TST for IMO 2019 Problem 1 Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

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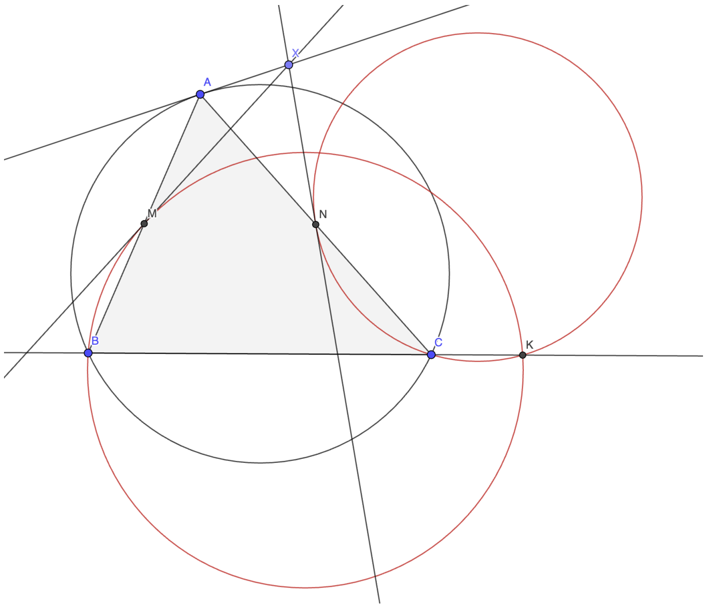

Proof. Call the tangent at A l_1, and call line BC l_2.

We will define a map $f : l_1 \to l_2$ such that $f(X) = K$. Specifically, see that f can be thought of as 3 projections: l_1 will be projected to the pencil of lines at N, then we rotate this pencil at N by $\angle NCK$. We finally project this rotated pencil onto (NKC) which coincides with l_2 at K (This works via the tangent property).

Similarly, lets define another projection g that does the same thing to the other side. Now we have two projective functions f and g, so it suffices to check three particular cases to prove that $f \equiv g$.

Case 1: $X = A$. Then we see $f(X) = P_\infty$, and similarly $g(X) = P_\infty$, so $f(X) = g(X)$ here.

Case 2: Pick X such that $\angle XNC = 180 - \angle ACB$. We see that in this case, $f(X) = C$. Now we aim to show that $g(X) = C$ too. I will now prove $\angle XMC = \angle ABC$, which would imply the result. Notice that $\triangle ANX \sim \triangle MNA$ via an angle chase. Hence, $\frac{XA}{AN} = \frac{AM}{MN} \iff \frac{AM}{XA} = \frac{MN}{AN} = \frac{MN}{NC}$. Also $\angle MNC = \angle MAX$, so $\triangle MAX \sim \triangle MNC$. Now we have $\angle XMC = \angle NMA = \angle ABC$, so we have $g(X) = C$ too.

Case 3: Pick X to be the corresponding point to Case 2, but on the other side. By the similar argument we have $f(X) = g(X) = B$, so we're done.

Now we have found 3 points, we may conclude that $f \equiv g$ and so we're done. $\square$