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TEST-2

ANSWERS

(1) $p = 1$

(2) $N(A)$ = closed subspace of X. , $\dim N(A) = 0$
A is a bounded operator

(3) True

(4) False

(5) $R(A)$ is closed subspace of X.

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Solution

(1) $X = c_{00}$ with norm $\|\cdot\|_p$, $1 \le p \le \infty$

$f: X \to K$ $f(x) = \sum_{i=0}^{\infty} x(i)$,

$x = (x(1), x(2) \ldots) \in X$.

let $x_n = (\underbrace{1,1,1,\ldots,1}_{n \text{ times}}, 0, 0, \ldots)$

$\|x_n\|_\infty = \sup |x_n(i)| = 1$

& $|f(x_n)| = \left|\sum_{j=1}^{\infty} x_n(j)\right| \le \sum_{j=1}^{n} |x_n(j)| = n$

$|f(x_n)| = n \to \infty$ as $n \to \infty$

thus for a bounded sequence $\{x_n\}$, $\{f(x_n)\}$ is unbounded.

$\therefore$ f is dis continuous

So, it is dis continuous for $p = \infty$.

similarly it is dis continuous for $p \geq 1$

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(2) $X = C'[0,b]$, $Y = C[0,b]$. both with norm $\|\cdot\|$

& $A: X \to Y$ be defined as.

$Ax = x'$, $x \in X$.

$\because$ $\bar{X} = X$. for $x \in X - Y$.

$\exists$ a sequence $\{x_n\}$ in X such that $x_n \to x \in Y$

$\therefore$ the sequence $\{(x_n, Ax_n)\}$ is a sequence in the graph of A, & $(x_n, Ax_n) \to (x, x) \notin G(A)$.

$\therefore$ $G(A)$ is not closed subspace of $Y \times Y$.

$\therefore$ A is not a closed operator

but if the domain of a bounded operator is. closed subspace, then, it is a closed operator

Now, we know that if $A \in BL(X, Y)$, then the null space, $N(A)$ is a closed subspace of X.

Now, we have to see. $\dim(N(A))$

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(3) $X \to$ normed linear space &
$f: X \to K$ is a linear functional.
we know that a ~~non linear~~ linear form is continuous iff its kernel is closed.

Now if $N(f)$ is closed in X.
then f is continuous for every $x_0 \in X - N(f)$.

$$\|f\| = \frac{|f(x_0)|}{dist(x_0, N(f))}.$$

let $x_0 \in X$ such that $f(x_0) \neq 0$.
then $\forall\ x \in X$.

$$x = x - \frac{f(x)}{f(x_0)} x_0 + \frac{f(x)}{f(x_0)} x_0$$

$$= y + \alpha x_0$$

where $y = x - \frac{f(x)\, x_0}{f(x_0)}$ $\alpha = \frac{f(x)}{f(x_0)}$,

$$f(y) = f\left(x - \frac{f(x)}{f(x_0)} x_0\right).$$

$$= f(x) - \frac{f(x)}{f(x)} f(x_0)$$

$$= 0 \quad \Rightarrow y \in N(f)$$

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$$\therefore dist(x, N(f)) = dist(y + \alpha x_0, N(f))$$
$$= dist(\alpha x_0, N(f))$$

$$= |\alpha|\ dist(x_0, N(f))$$

$\because x_0 \notin N(f) \Rightarrow dist(x_0, N(f)) \neq 0$.

$$\therefore \left|\frac{f(x)}{f(x_0)}\right| = |\alpha| = \frac{dist(x, N(f))}{dist(x_0, N(f))}$$

$$\Rightarrow \|f\| \leq \frac{|f(x_0)|}{dist(x_0, N(f))}$$

$\Rightarrow f$ is continuous

& also $\forall\ u \in N(f)$

$$|f(x_0)| = |f(x_0) - f(u)|$$

$$= |f(x_0 - u)|$$

$$\leq \|f\|\ \|x_0 - u\|.$$

thus
the null space $N(f)$ is closed space of X if and only if f is continuous.

True

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(4) $X \to$ finite dimensional n.l.s.

if $\{A_n x\}$ converges for every $x \in X$.
then, a function $A: X \to Y$ is defined
$Ax = \lim_{n\to\infty} A_n x$, $x \in X$.

for any $x, y \in X$, $\alpha, \beta \in k$

$A(\alpha x + \beta y) = \lim_{n\to\infty} A_n(\alpha x + \beta y)$

$= \lim_{n\to\infty} [\alpha A_n x + \beta A_n y]$

$= \alpha Ax + \beta Ay$

$\Rightarrow A \in L(X, Y)$

$\therefore$ if $\|A_n - A\| \to 0$ as $n \to \infty$

ie. A_n is a bounded operator

but A is not a bounded operator.

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(5) $X, Y \to$ Banach spaces &
$X_0 \to$ subspace of X.
& $A: X_0 \subset X \to Y$ be a injective

if $A^{-1}: R(A) \to X$ is continuous

Since A is injective & closed, then $R(A)$ is also close. & bounded

& Since X is a Banach Space
$\Rightarrow R(A)$ is a closed subspace of X