Name : Raushan Sharma Roll : 18MA20058

Functional Analysis Test 2

1) Continuous for $p=1$ only.
Discontinuous for other values.

2) $N(A) = \{ x(t) = c \ \forall t \in [a,b] \}$ = set of all constant functions

$\dim N(A) = 1$ and A is an unbounded, thus discontinuous linear operator

3) True.

4) True.

5) $R(A)$ is a closed subspace of Y.

Scanned with CamScanner

2) $X = C'[a,b]$ and $Y = C[a,b]$ w.r.t $\|\cdot\|_\infty$ and $A: X \to Y$ defined as: $Ax = x'$.

Then, $N(A) = \{x \in X \mid Ax = x' = 0\} = \{x \in X \mid x = c\}$
= set of all constant functions.

which is a closed subspace of X.

$\dim(N(A)) = 1$ ~~as~~ $\{1\}$ is a ~~basi~~ basis for $N(A)$
↳ constant function 1.

Now, consider: $x_n(t) = t^n$, ~~$t \in [a,b]$~~ $t \in [0,1]$

Then, $\|x_n\|_\infty = 1$, $\|Ax\|_\infty = n$

$\Rightarrow A$ is not continuous. A is unbounded.

1) let $X = C_{00}$ with $\|\cdot\|_\infty$ and $f: X \to \mathbb{R}$ by: $f(x) = \sum_{j=1}^{\infty} x(j)$ $\forall x \in X$

Let $x_n = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

$\|x_n\|_\infty = \sup\{|x(i)|\} = 1$ and $|f(x_n)| = |\sum_{j=1}^{\infty} x_n(j)|$
i.e. $\{x_n\}$ is a bounded seq.

(in this case equality holds) $= \sum_{j=1}^{\infty} |x_n(j)|$
$= n$

$|f(x_n)| = n \to \infty$ as $n \to \infty$

Thus for a bounded sequence $\{x_n\}$, $\{f(x_n)\}$ is unbounded.

$\therefore f$ is discontinuous. $\to$ [For $p = \infty$ in $\|\cdot\|_p$]

For $p = 1$: $X = C_{00}$ w.r.t $\|\cdot\|_1$

~~consider same seq. $\{x_n\}$~~

$f(x) = \sum_{j=1}^{\infty} x(j)$

Then, $|f(x)| = |\sum_{j=1}^{\infty} x(j)| \leq \sum_{j=1}^{\infty} |x(j)| = \|x\|_1$.

$\therefore |f(x)| \leq \|x\|_1$

$\Rightarrow f$ is bounded w.r.t $\|\cdot\|_1$. $\Rightarrow f$ is continuous w.r.t $\|\cdot\|_1$ on X.

For $p=2$: $X = c_{00}$ w.r.t $\|\cdot\|_2$

Consider ~~...~~ $x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots) \in c_{00}$

Then, $\|x_n\|_2^2 = \sum_{j=1}^{n} \frac{1}{j^2} \leq \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < \infty$

So, $\{x_n\}$ is a bounded seq.

But $f(x_n) = \sum_{j=1}^{n} \frac{1}{j} \to \infty$ as $n \to \infty$

$\therefore |f(x_n)| \to \infty$ as $n \to \infty$

$\therefore f$ is discontinuous w.r.t X on $\|\cdot\|_2$.

3) X is a n.l.s and $f: X \to K$ is a linear functional.

Then, if $N(f)$ is closed in X, then f is continuous

~~However, converse need not be true.~~

Let $x_0 \in X$ such that $f(x_0) \neq 0$. Then for every $x \in X$,

$$x = x - \frac{f(x)}{f(x_0)} x_0 + \frac{f(x)}{f(x_0)} \cdot x_0$$

$$= y + \alpha x_0$$

$\Rightarrow y = x - \alpha x_0$, $\alpha = \frac{f(x)}{f(x_0)}$

Then $f(y) = f(x) - \frac{f(x)}{f(x_0)} f(x_0) = 0$

$\Rightarrow y \in N(f)$

Scanned with CamScanner

$\therefore \ dist(x, N(f)) = dist(y + \alpha x_0, N(f)) = dist(\alpha x_0, N(f))$ $[\because y \in N(f)]$

$= |\alpha| \ dist(x_0, N(f))$ .... (i)

$\because x_0 \notin N(f) \Rightarrow dist(x_0, N(f)) > 0$

From (i), $\left|\frac{f(x)}{f(x_0)}\right| = |\alpha| = \frac{dist(x, N(f))}{dist(x_0, N(f))}$

$\Rightarrow |f(x)| = \frac{|f(x_0)|}{dist(x_0, N(f))} dist(x, N(f))$

$\leq \frac{|f(x_0)|}{dist(x_0, N(f))} \cdot \|x\|$

$\Rightarrow \|f\| \leq \frac{|f(x_0)|}{dist(x_0, N(f))}$ $\Rightarrow f$ is continuous.

Converse: Suppose f is continuous. Then for $x_n \to x$, we have $f(x_n) \to f(x)$. Let $x_n \in N(f)$ then $f(x_n) = 0$.

Using continuity, $f(x) = \lim_{n \to \infty} f(x_n) = 0$

Hence, $x \in N(f)$ and thus $N(f)$ is closed.

So, given statement is true.

4) X is finite dimensional n.l.s and $\{A_n\}$ is $\in L(X,X)$

$\{A_n x\}$ converges for every $x \in X$.

$\Rightarrow \{A_n\}$ is pointwise bounded.

X is finite dimensional. So ~~$\{A_n\}$~~ pointwise bdd. $\Rightarrow$ uniformly bdd.

As $\{A_n\}$ is uniformly bounded, then $\|A\|$ is bounded $\forall n$

So, $\{\|A_n\|\}$ is bounded.

Thus, $\|A_n x - Ax\| \to 0 \Rightarrow \|A_n - A\| \to 0$ as $n \to \infty$.

where $Ax = \lim_{n\to\infty} A_n x$, $x \in X$.

$\therefore$ The given statement is true.

5) X and Y are Banach spaces, $X_0 \subseteq X$

$A : X_0 \subseteq X \to Y$ is closed operator and A is one - one.

By theorem, this implies $A^{-1} : R(A) \to X$ is closed.

Now, A^{-1} is also continuous, i.e. bounded.

$\therefore A^{-1} : R(A) \to X$ is both closed and bounded.

Also X is a Banach space,

$\Rightarrow R(A)$ is a closed subspace of Y.