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TEST

FUNCTIONAL ANALYSIS

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1. Closure of c_{00} in (i) $(\ell^1, \|\cdot\|_1) \to c_0$
   (ii) $(\ell^2, \|\cdot\|_2) \to c_0$
   (iii) $(\ell^\infty, \|\cdot\|_\infty) \to c_0$

2. $\|\cdot\|'$ & $\|\cdot\|_{1,\infty}$ are not equivalent
   $\|\cdot\|_{1,\infty}$ & $\|\cdot\|_\infty$ are not equivalent on X.

3.

4. Yes, X is finite dimensional.

5. Yes, c, the space of all convergent sequences ; is a closed subspace of the space ℓ^∞.

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Solution

1. $X = c_{00}$ with $\|\cdot\|_p$ $1 \le p \le \infty$ is not a Banach space.

Let $x \in c_0$ & $n \in \mathbb{N}$, let

$x_n = (x(1), x(2), \ldots, x(n), 0, 0, \ldots) \in c_{00}$

$x = (x(1), x(2), x(3) \ldots) \in c_0$.

Then $\|x_n - x\|_\infty = \sup \{|x(j)| \mid j \ge n\} \to 0$ as $n \to 0$

$\Rightarrow \overline{c_{00}} = c_0$ wrt $\|\cdot\|_\infty$.

(iii) Thus it is the closure of c_{00}

(iii) Let $\{x_n\}$ be a sequence in c such that $x_n \to x$ in ℓ^∞; let $\epsilon > 0$ $\exists$ $m \in \mathbb{N}$ such that $\|x_m - x\|_\infty < \epsilon/3$.

Since x_m is a cauchy sequence in K, then $\exists$ $P \in \mathbb{N}$ such that

$|x_m(i) - x_m(j)| < \epsilon/3$ $\forall$ $i, j \ge P$.

$\Rightarrow |x(i) - x(j)| \le |(x - x_m)(i) - (x - x_m)(j)| + |x_m(i) - x_m(j)|$

$< \frac{2\epsilon}{3} + \epsilon/3 = \epsilon$ $\forall$ $i, j \ge P$.

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Hence $x(i)$ is a cauchy sequence in $\mathbb{K}$ so $x \in C$.
C_{00} is dense in C_0, $\Rightarrow \overline{C_{00}} = C_0$
Since C_0 is complete, and given a sequence in C_0 which is convergent in ℓ^∞, $\therefore$ it is converges in C_0,
thus, C_0 is the closure of C_{00} in ℓ^∞.

(2) $X := C'[a,b]$, for $x \in X$, $\|x\|' := \max\{|x(a)|, \|x'\|_\infty\}$
& $\|x\|_{1,\infty} := \max\{\|x\|_\infty, \|x'\|_\infty\}$.
we know that two norms are said to be equivalent if there exists $c_1 > 0$ & $c_2 > 0$ such that
$$c_1\|x\|' \le \|x\|_{1,\infty} \le c_2\|x\|' \quad \forall x \in X.$$
consider $\|x\|' = \max\{x(a), \|x'\|_\infty\}$
~~let's assume that they are~~ equivalent $= \|x\|_\infty$
Now, we know that if $\|\cdot\|_\infty$ & $\|\cdot\|_{1,\infty}$ are equivalent norms, then X is a Banach space w.r.t $\|\cdot\|_\infty$ iff X is a Banach space w.r.t $\|\cdot\|_{1,\infty}$.

Now, we know that $C'[a,b]$ is not a Banach space w.r.t $\|\cdot\|_\infty$
thus our assumption was wrong & it is not equivalent.

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Given:
(Q3) Y is a finite dimensional proper subspace of a normed linear space $(X, \|\cdot\|)$.

To Prove : $\exists$ some $x \in X$ with $\|x\| = 1$ & $\text{dist}(x, Y) = 1$

Proof: Let us assume $z \in X \setminus Y$

Now, $\because$ Y is closed, $\therefore t = \text{dist}(z, Y) > 0$.
Now, $\because$ $t^{-1}Y = Y$
$\Rightarrow \text{dist}(t^{-1}z, Y) = 1$.
let $\{y_n\}$ be a sequence in Y with $\|y_n - t^{-1}z\| \to 1$
thus, $\{y_n\}$ is bounded ($\because Y$ is finite dim.)

$\exists$ a subsequence of $\{y_n\}$ which converges to some $y \in Y$.
let $x = y - t^{-1}z$.

& $\|y_n - t^{-1}z\| \to 1$
$\Rightarrow \text{dist}(x, Y) = 1$.
$\|x\| = \|y_n - t^{-1}\| \to 1$

thus we have proved that $\exists$ some $x \in X$ with $\|x\| = 1$ & $\text{dist}(x, Y) = 1$

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Q4) **Given**: The subset $\{x \in X, \|x\| \leq 1\}$ of a normed linear space X is compact

To Prove: X is finite dimension

Proof: We know that $\{x \in X \mid \|x\| \leq 1\}$ is closed & bounded.

Let if possible $\{y_1, y_2, y_3, \ldots\}$ be infinite linearly independent subsets of X and consider $Z_n = [y_1, y_2, \ldots, y_n]$ $n = 1, 2, \ldots$

then, Z_n is finite dimensional.

thus it is a closed subspace of Z_{n+1}.

Also $Z_n \neq Z_{n+1}$

$\therefore$ by Rietz lemma, $\exists$ $x_n \in Z_{n+1}$ such that,

$\|x_n\| = 1$ & $dist(x_n, Z_n) \geq 1/2$

(Consider $r = 1/2$ in Rietz lemma)

$\Rightarrow$ $\|x_n - x_m\| \geq 1/2$ $\forall$ $n \neq m$

So $\{x_n\}$ is a sequence in $\{x \in X \mid \|x\| \leq 1\}$ having no convergent subsequence

$\Rightarrow$ $\{x \in X \mid \|x\| \leq 1\}$ is not compact which is a contradiction.

thus our assumption was wrong

& X has to be finite dimensional

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Q5) **Given**: c is the space of all convergent sequences.

To Prove: c is a closed subspace of l^∞

Proof: Consider any convergent sequence $\{C_n\}_n$ of elements of C.

Since l^∞ is complete $\Rightarrow$ $\{C_n\}$ converges to $x \in l^\infty$.

thus $\|C_n - x\|_\infty \to 0$ as $n \to \infty$.

Let's take K with $\|x - C_K\|_\infty < \epsilon$ for n_0 such that $|C_K(m) - C_K(n)| < \epsilon$ $\forall$ $m, n > n_0$.

$|x(m) - x(n)| \leq |x(m) - C_K(m)| + |C_K(m) - C_K(n)| + |C_K(n) - x(n)|$

$\leq 2\|x - C_K\|_\infty + |C_K(m) - C_K(n)|$

$\leq 3\epsilon$

thus x is Cauchy

Now $\because$ $C_n \in C$

thus $x \in C$

thus we have proved the C is a closed subspace of l^∞