Functional Analysis Topics Distribution

Lecture 1 -> Linear Space, Norm, normed Space, linear space, Some other properties of norm and named linear spaces. inequalities

Lecture 2 -> proving it and finding the euclidean norm, holding inequality, Vector space.
Let Ohm be any non empty set, let's denote B(ohm) as the set of k-valued bounded functions ont Ohm, is a vector space w.rt. addition and scalar multiplication
Holder's inequality

Lecture 3 -> subspace of F(ohm) . Jensen's inequality

Lecture 4 -> Linear Map. Banach Spaces . Complete metric space, cauchy sequence
If every cauchy sequence is also a convergent sequence then we say that R is complete.
* since a subset Y of a complete metric space X is complete iff it is closed in X, it follows that " A subspace Y of a Banach Space X is a Banach space iff Y is a closed subspace of X. "
open set , closed subspace of a banach space

Lecture 5 -> l^p is a banach space , The space C = { x = (x(1) , x(2) , ...) is a closed subspace
C_0, C both are banach space
C_00 is nota closed subspace of l^(infinity)
(C[a,b] , || . ||_(infinity)) is a banach space

Lecture 6 -> P [a,b] = C [a, b]
x_n ----> x
C_00 is not closed
c_00 is nota banach space
Baire Category Theorem - If X is a complete metric space and {x_n} is a sequence of subsets of X such that X = U x_n , then there exists sone i in N such that interior of X_j is non empty
Lemma - The interior of a proper subspace of a normed linear space X is empty.
Theorem - A banach space cannot have denumerable basis

Lecture 7 -> Since a basis of a NLS depends on the linear structure of the space and not on the norm, from the left theorem, we can say that if a linear space is Banach space wrt same norm on it, then it cannot have a denumerable basis.
(ii) if a linear space X has a denumerable basis, then no norm on X masks it banach space
That is a banach space X has either finite basis or uncountable basis.
Ex : 1. X = P , the linear space of all polynomials with coefficients in the field K.
Since { u_j(t) = t^j, j = 0,1,2,3,4,.. } 7is a denumerable basis for p, so X is not abanach space wrt to any name on it.

2 . X = C_oo is not a banach space wrt any norm on it. Since {e_1, e_2, .. } with e_i (j) = del_ij is a denumerable basis for C_00
Schauder Basis : - a countable subset ofa banach space is called a schauder basis if ||x_n|| = 1, for all n in N and if every x in X, there are unique scalars in K such that x = summation k_j x_j

Equivalent norms , comparable norms , stronger norm

Lecture 8 -> coset of an element x wrt Y is defined as
x + Y = {x + y | y in Y}
any 2 cosets are either disjoint or identical and distinct cosets form the partition of X.
additive identity, additive inverse.
X / Y is a normed linear space.
A series summation x_n in X is said to be absolutely summable if summation ||X_n|| < inf
a series is said to be summable if s_n = summation x_j --> x in X
Theorem : A normed linear space X is a banach space iff every absolutely summable series is summable in X
converse is also true

Thoerem : Let X be a normed linear space and Y be a closed subspace of X. Then X is a banach space iff Y and X/Y are banach spaces in the induced norms respectively.

Lecture 9 ->Riesz lemma , Riesz lemma says that if Y is a proper closed subspace of rls X, then there exists a poin on the unit sphere of X whose distance from Y is very small.

Lecture 10 -> Let X be a nls and Y be a subspace of X
(a) for x in X, y in Y and k in K, we have || kx + y || >= |k| dist(x, y)
(b) let Y be a finite dimensional . Then Y is a complete. In particular it is closed in X
* an infinite dimensional subspace of a nls X need not be closed in X
Let X be a nls . Then the following are equivalent.
(i) every closed and bounded subset of X is compact
(ii) the subset of X is compact
(iii) X is finite dimentional

Lecture 11 - > Continuity of a linear Map .
Theorem : Let X and Y be nls and F : X --> Y be a linear map. If F is bounded on U (o,r) r > 0 then there exists alpha > 0 st, ||F(x) || <= alpha ||x|| for all x in X
Theorem : Let F : X --> Y be a linear map. Then F is continuous on X iff there exists alpha > 0 st, ||F(x) || <= alpha ||x|| for all x in X

Definition : A linear Map F : X -> Y is said to be bounded on X if there exists some alpha > 0 such that, ||F(x) || <= alpha ||x|| for all x in X

Lecture 12 -> : A linear Map F : X -> Y is said to be bounded if there exists some m > 0 such that, ||F(x) ||_Y <= m ||x||_X for all x in X

Let X and Y be nls. If : A linear Map F : X -> Y is continuous linear map, then it is uniformly continuous
Theorem :Let X and Y be nls. If : A linear Map F : X -> Y , then the following are equivalent:
(i) F is continuous at the origin
(ii) F is continuous at every x in X
(ii) F is unifromly continuous on x
(iv) There exists alpha > 0 such that ||F(x) || <= alpha ||x|| for all x in X
(v) {F(x) | ||x|| = 1, x in X} is a bounded set in Y
(vi) for every bounded set E belongs to X, teh est
F(E) = {F(x) | x in E} is bounded in Y
Theorem : Let X and Y be nls. If : A linear Map F : X -> Y , Let Z(F) be a null space of X.
Then F is continous iff Z(F) is closed in X and F tilde : X/ Z(F) -> Y defined by F_tilde(x + Z(F)) = F(x) is continuous

Lecture 13 -> Produce a bounded sequence {x_n} in X such that {A x_n} is unbounded in Y.
X = C'[0,1] with ||.||_infinity
Define f : X -> K by f(x) = x'(1) for all x in X
clearly f is a linear map but it isnt continuous
(2) Let X = C'[0,1] with ||.||_infinity
and Y = C[0,1] with ||.||_infinity
define A : X --> Y by
Ax(t) = x'(t) for all t in [0,1]
since x_n(t) = t_n => ||x_n||_infinity = 1

A linear map on a linear space X may be continuous wrt the same norm on X, but may be discontinuous wrt some other norm on X.

Linear map f_1 : C_oo => K
infinite matrix

Lecture 14 ->Ax(s) = \integral from a to b k(s,t) x(t) dt
Show that BL(X,Y) is a linear space under the pointwise operations

Lecture 15 ->since F is linear , we have ||F(x) || = || F (rx/ ||x||) || ||x||/ r
<= sup{||F(z) || / z in X, || z || = r}
supremum over all x in X
sup _{a <= s <= b} \integral {a to b} |k(s,t) | dt = \integral _{a to b} |k(s_o, t)| dt

Theorem : Let X be a nls and f : X -> K be a nonzero linear functional on X such that null space N(f) is closed. Then f is continuous and for every x_o in X - N(f)
||f || = |f(x_o) | / dist (x_o, N (f))

Lecture 16 -> Let X be a NLS and f : X -> K be a nonzero linear functional on X such that N(f) is closed in X. Then f is continuous, and for every x_0 in X - N(f),
||f|| = |f(x_0)| / dist(x_0 , N(f))
Ex : X = C' [0,1] with || . ||_infinity and f : X -> K be defined by f(x) = x'(1), for all x in X, we know that f is discontinous linear functional on X => N(f) is not closed by above theorem

Suppose Y and X be nls and {A_n} be a sequence of operatores from X to Y {A_n} belongs to L(X,Y). If {A_n x} converges for every x in X, then the function, A: X -> Y defined by
Ax = lim n-> inf A_n x , x in X is also a linear operator
boundedness of A

Lecture 17 ->let a = t1 < t2 < ... < tn = b be the nodes with weights, w1, w2, ..., wn. Define Qn : C[a,b] -> K by Qn x = \sum j = 1 to n   Coj x (tj) \approx equal to \integral of x(t) dt from a to b
therefore Qn is a bounded / continuous linear functional
Thorem : Qnx convergs to Qx for every x in E , which is a subset of C[a,b]
Inner product space, legendre polynomial
Gauss quadrature formula
Completeness of BL(X,Y).
A is pointwise bounded on X if for each x in X , there exists M_x > 0 such that
|| Ax || <= M_x || x ||
we say A is uniformly bounded

Lecture 18 -> uniformly bounded implies pointwise bounded. but converse need not be true
If X is finite dimentional, then pointwise bounded implies uniformly bounded.
family of operators, A , is pointwse bounded
Uniform boundedness principle.
let X be a banach space and Y be a nls and A is a subset of BL(X,Y), if A is pointwise bounded, then A is uniformly bounded.
Baire Category theorem
Corollary (Banach-Steinhaus theorem)

Lecture 19 -> Let X be a Banach space and Y be a nls and {A_n} be a sequence in BL(X,Y) such that {A_n x} converges for every x in X. Let A : X -> Y be defined by Ax = lim n to inf A_n x
Then for every totally bounded subset S in sup || A_n x - A x|| -> 0 as n -> infinity

Closed operator

Test 1

- Q1 -> lecture 4, pg 1-3
- Q2 -> yes, lecture 7, pg 14-18
- Q3 -> https://people.math.gatech.edu/~heil/6338/summer08/section1d_details.pdf
- Q4 -> not a banach space. Lecture 5, pg 7
- Q5 -> Lecture 2 : Holder's inequality.