NAME: ALTAF AHMAD
ROLL: 18MA20005

FUNCTIONAL ANALYSIS

PAGE NO:
DATE: 16/11/21

TEST - III

ANSWERS

1. closed map

2. ~~False~~ TRUE

3. TRUE
R(A) is Y.

4. A is a surjective Linear map

5. TRUE
FALE

6.

7. Yes

8.

9. isomorphic to l^2.

10.

PAGE NO :
DATE :

SOLUTIONS

1. $X, Y \Rightarrow$ Banach Spaces and $A: X \to Y$ is an injective bounded linear map such that $R(A)$ is closed in Y.

Now according to the Closed Graph Theorem, we can say that a function $f: X \to Y$ is closed if for every closed set F in X, the image of ~~$f(F)$~~ $F \to F^{-1}$ is closed in Y.

Now, since $R(A)$ is closed $\Rightarrow F^{-1}: R(A) \to$ is a closed map.

2. X is a Banach Space wrt. two norms $\|\cdot\|_1$ & $\|\cdot\|_2$ such that

$\|x\|_1 \leq \alpha \|x\|_2 \quad \forall x \in X$ (1)

for some $\alpha > 0$.

If we consider the identity map f on X, as a linear map from $(X, \|\cdot\|_2)$ to $(X, \|\cdot\|_1)$, then according to (1), f is continuous.

Thus f is open and thus a homeomorphism.

$\therefore$ its inverse is also continuous & the two norms are equivalent

PAGE NO :
DATE :

3. We know that $F: X \to Y$ is continuous iff for every open set E in Y its preimage $F^{-1}(E)$ is also open in X.

This is because, X & Y $\to$ N.L.S.

$A: X \to Y$ is a bounded linear map.

& $\exists \; \gamma > 0$ st. $\gamma \|x\| \leq \|Ax\|. \quad \forall x \in X$.

Thus A is an open map

thus A^{-1} is also continuous

Also $R(A) = Y$

4. X & Y are N.L.S. & $A: X \to Y$ is an open linear map.

Since A is open and linear therefore $A(X) \subset Y$ must be an open linear subspace.

The only such subspace is Y.

let $y_0 \notin A(X) \Rightarrow y_0 \neq 0$.

consider $a_n \in \mathbb{R}$ with $a_n \to 0$.

$\because A(X)$ is a subspace & $y_0 \notin A(X)$.

$\Rightarrow a_n y_0 \notin A(X)$. for all n.

But $a_n y_0 \to 0$ thus $Y / A(X)$ is ~~not~~ closed

this is a contradiction

thus $A(X)$ is open

& A is a surjective linear map.

PAGE NO :
DATE :

(5) $X = C[a,b]$ with $\|\cdot\|_\infty$.
A $Y = C[a,b]$ with $\|x\|_1 = \int_a^b |x(t)| \, dt$

For A, we can say that a funct is continues on X for every open set the the inverse of f is open Y.

thus A is continuous linear map
But $A^{-1} : Y \to X$ is not continu

(6)

PAGE NO :
DATE :

(7) $X \to$ complex inner product spce.
thus $\forall$ $x, y \in X$.
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{—(1)}$$
we know that
$$\|x+y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + 2\langle x,y\rangle + \|y^2\|$$
Now if (1) holds,
$\Rightarrow$ $2\langle x,y\rangle = 0$
$\Rightarrow$ $x \perp y$

(8)

PAGE NO :

DATE :

(9) According to the fundamental theorem of infinite dimensional Vector Spaces; if H is an infinite dimensional separable Hilbert space, then it is isomorphic to l^2. This is because all Hilbert Spaces with a countable infinite orthonormal basis are isomorphic, and all of them are isomorphic to l^2.