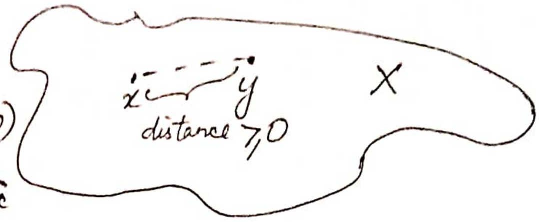
Functional Analysis

Metric spaces

X set

$d : X \times X \to [0, \infty)$



is called a metric if

1) $d(x,y) = 0 \Leftrightarrow x = y$
2) $d(x,y) = d(y,x)$

~~3) $d(x,y) + d(y,z) \geq d(x,z)$~~

3) $d(x,y) \leq d(x,z) + d(z,y)$

metric space (X, d)

eg: a) $X = \mathbb{C}$, $d(x,y) = |x-y|$

b) $X = \mathbb{R}^n$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2}$

Euclidean metric

c) X any set $(\neq \phi)$

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$ discrete metric

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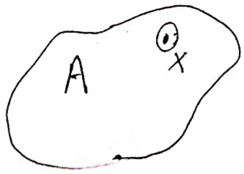
$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$

(open ball of radius $\varepsilon > 0$ centred at x)

Open sets :



$A \subseteq X$ is called open if for each $x \in A$ there is an open ball with $B_\varepsilon(x) \subseteq A$.

Boundary pts : $A \subseteq X$, $x \in X$ is called a boundary pt for A if $\forall\ \varepsilon > 0$,

$B_\varepsilon(x) \cap A \neq \phi$ and

$B_\varepsilon(x) \cap A^c \neq \phi$ $\quad [A^c = X \setminus A]$

$\partial A = \{x \in X \mid x$ is boundary pt for $A\}$

(*) A open $\Leftrightarrow A \cap \partial A = \phi$

(*) A closed $\Leftrightarrow A \cup \partial A = A$

closed sets : $A \subseteq X$ is called closed if $A^c = X \setminus A$ is open.

(all boundary pts belong to A & not A^c)

Closure : $A \cup \partial A = \bar{A}$ (closure of A) (always closed, smallest closed set containing A)

eg $X = (1, 3] \cup (4, \infty)$

$d(x, y) = |x - y|$

$A = (1, 3] \subseteq X$ an open?

a) $A = (1, 3]$

for $x \in A$, $x \neq 3$, define $\varepsilon = \frac{1}{2}\min(|1-x|, |3-x|)$

Then $B_\varepsilon(x) \subseteq A$

for $x = 3$, $B_1(x) = \{y \in X \mid d(x, y) < 1\}$
$= (2, 3] \subseteq A$

$[\because (3, 4] \notin X]$

$\therefore A$ is an open set.

b) A is also closed!

Sequence - ordered set of pts. inside metric space.

Sequence in X : $(x_1, x_2, x_3, \ldots)$ or $(x_n)_{n \in \mathbb{N}}$

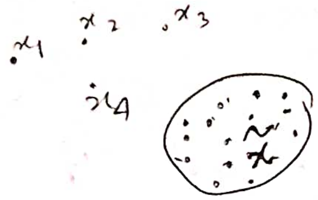
or $x : \mathbb{N} \to X$ map

$n \to x_n$

<u>Convergence</u> : A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is called convergent if there is $\tilde{x} \in X$ (limit pt.) with $\forall \varepsilon > 0 \ \exists N \in \mathbb{N},$
$\forall n \geq N : d(x_n, \tilde{x}) < \varepsilon$

$x_n \to \tilde{x}$ as $n \to \infty$

$\lim_{n\to\infty} x_n = \tilde{x}$



ε - ball centred at $\tilde{x}$

<u>Proposition</u> :

$A \subseteq X$ is closed

$\Leftrightarrow$ For every convergent sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, one has

$$\lim_{n\to\infty} a_n \in A$$

<u>Pf</u> :

$(\Leftarrow)$ Show by contraposition.

Assume A is not closed

$A^c = X \setminus A$ is not open

$\Rightarrow \ \exists \ \tilde{x} \in A^c$ with $B_\varepsilon(\tilde{x}) \cap A \neq \phi \ \forall \varepsilon > 0$

$\Rightarrow \ \exists$ a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in B_{\frac{1}{n}}(\tilde{x}) \cap A$

$\Rightarrow \lim_{n\to\infty} a_n = \tilde{x} \notin A$

$(\Rightarrow)$: Assume $\exists (a_n)_{n\in\mathbb{N}} \subseteq A$ with $\tilde{x} = \lim_{n\to\infty} a_n \notin A$

$\Rightarrow B_\varepsilon(\tilde{x}) \cap A \neq \phi \ \forall \varepsilon > 0$

$\Rightarrow A^c$ is not open

$\Rightarrow A$ is not closed

Def Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called Cauchy sequence if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \geq N : d(x_n, x_m) < \varepsilon$

~~Every Cauchy sequence is a convergent sequence.~~

Def (X, d) is called complete if all Cauchy sequences converge.

Def : $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let X be a $\mathbb{F}$-vector space. A map $\|\cdot\| : X \to [0, \infty)$ is called norm if

(a) $\|x\| = 0 \Leftrightarrow x = 0$ (positive definite)

b) $\|\lambda . x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{F}, x \in X$ (absolutely homogeneous),

~~c) $\|x+y\| \leq \|x+z\| + \|z+y\|$~~

c) $\|x+y\| \leq \|x\| + \|y\|$ $\forall x, y \in X$ (Δ-inequality)

$(X, \|\cdot\|)$ is a normed space.

(*) If $\|\cdot\|$ is a norm for the $\mathbb{F}$-vector space X, then $d_{\|\cdot\|}(x, y) := \|x - y\|$ defines a metric for set X.

A normed space is a special case of metric space.

If $(X, d_{\|\cdot\|})$ is a complete metric space, then the normed space $(X, \|\cdot\|)$ is called a Banach space.

eg Let $l^p(\mathbb{N}, \mathbb{F})$ (where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, p \in [1, \infty)$) be defined as all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ st

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \quad \text{(converges!)}$$

$$\|\cdot\|_p : l^p \to [0,\infty)$$
with $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$

Inner product

" $\cancel{\langle x, y\rangle = \|x\| \|y\| . \cos(\alpha)}$

$\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let X be a $\mathbb{F}$-vector space.
A map $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{F}$ is called an inner product on X if

1) $\langle x, x\rangle \geq 0 \quad \forall\, x \in X$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$
2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (conjugate symmetry)
3) $\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$
$\langle x, \lambda y\rangle = \lambda \langle x, y\rangle, \quad \forall\, x, y \in X, \lambda \in F$

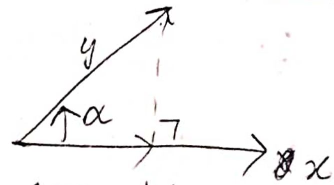
If $\langle\cdot,\cdot\rangle$ is an inner product, then
$\|x\|_{\langle\cdot,\cdot\rangle} = \sqrt{\langle x, x\rangle}$ defines norm.

Hilbert space : $(X, \langle\cdot,\cdot\rangle)$ is called a Hilbert space if $(X, \|\cdot\|_{\langle\cdot,\cdot\rangle})$ is a Banach space.

eg a) $\mathbb{R}^n$, $\mathbb{C}^n$ with $\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$
b) $l^2(\mathbb{N}, \mathbb{F})$ with $\langle x, y\rangle = \sum_{i=1}^{\infty} \overline{x_i}\, y_i$
c) $C([0,1], \mathbb{F})$ with $\langle f, g\rangle = \int_0^1 \overline{f(t)}\, g(t)\, dt$

Cauchy - Schwarz Inequality : Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|x\| := \sqrt{\langle x, x\rangle}$.
Then $\forall\, x, y \in X$:

$|\langle x, y\rangle| \leq \|x\| . \|y\|$



& $|\langle x, y\rangle| = \|x\|.\|y\| \Leftrightarrow x, y$ are LD.

<u>Orthogonality</u> : Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space

a) $x, y \in X$ are called <u>orthogonal</u> if $\langle x, y\rangle = 0$

write $x \perp y$.

b) For $U, V \subseteq X$, write $U \perp V$ if $x \perp y$ $\forall x \in U, y \in V$

c) For $U \subseteq X$, the <u>orthogonal complement</u> of U is $\{x \in X \mid \langle x, u\rangle = 0 \ \forall u \in U\} = U^{\perp}$ (always a subspace in X)

<u>Remark</u> : 1) $\{0\}^{\perp} = X$, $X^{\perp} = \{0\}$.

2) $U \subseteq V \Rightarrow U^{\perp} \supseteq V^{\perp}$

3) If $x \perp y$, then $\|x+y\|^2_{\langle\cdot,\cdot\rangle} = \|x\|^2_{\langle\cdot,\cdot\rangle} + \|y\|^2_{\langle\cdot,\cdot\rangle}$

<u>Continuity for metric spaces</u> : (X, d_X), (Y, d_Y) two metric spaces.

A map $f: X \to Y$ is called

• <u>continuous</u> if $f^{-1}[B]$ is open (in X) $\forall$ open sets $B \subseteq Y$.

• <u>sequentially continuous</u> if $\forall \tilde{x} \in X$ and $(x_n)_{n \in \mathbb{N}} \subseteq X$ with $x_n \to \tilde{x}$ as $n \to \infty$ holds $f(x_n) \to f(\tilde{x})$ as $n \to \infty$

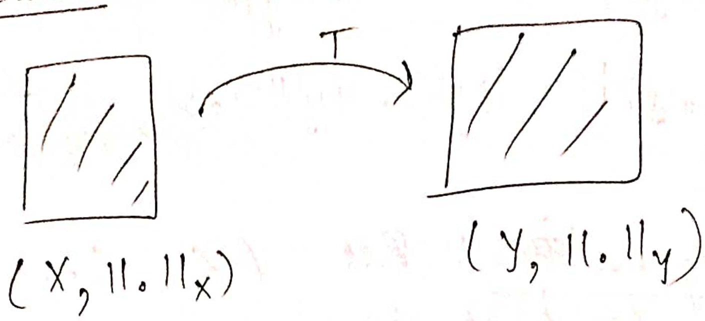
(*) For metric spaces, continuous & sequentially continuous are equivalent.

(*) $(X, \langle\cdot,\cdot\rangle)$ is an inner product space. $U \subseteq X$.

Then $U^{\perp}$ is closed.

Bounded operators

Operator :



$T : X \to Y$: • linear
• continuous (bounded)

<u>Defn</u> : Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces: $T : X \to Y$ is linear map

$$T(x + \tilde{x}) = Tx + T\tilde{x}$$

$$T(\lambda x) = \lambda Tx$$

$$\forall\, x, \tilde{x} \in X, \lambda \in F$$

$\|T\| = \|T\|_{X \to Y}$ is called the operator norm of T.

$$\|T\| = \|T\|_{X \to Y} = \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X, x \neq 0 \right\}$$

If $\|T\| < \infty$, T is called <u>bounded</u>.

(T can be unbounded only if X is infinite dimensional).

<u>Proposition</u> :

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces, $T : X \to Y$ linear

Then following claims are equivalent:

a) T is continuous

b) T is continuous at $x = 0$

c) T is bounded

Riesz Theorem:

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then for each continuous linear map $l : X \to \mathbb{F}$ (a continuous linear functional), there is exactly one $x_l \in X$ s.t.

$$l(x) = \langle x_l, x \rangle \; \forall \; x \in X \text{ and } \|l\|_{X \to \mathbb{F}} = \|x_l\|_X$$

Compactness: $\mathbb{R}^n \supseteq A$

$\left.\begin{array}{l} \cdot A \text{ is closed} \\ \cdot A \text{ is bounded} \end{array}\right\} \Leftrightarrow A$ is compact. (only in $\mathbb{R}^n$ or $\mathbb{C}^n$)

Defn: Let (X, d) be a metric space. $A \subseteq X$ is called (sequentially) compact if for each sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$, one finds a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $\tilde{x} = \lim_{k \to \infty} x_{n_k} \in A$

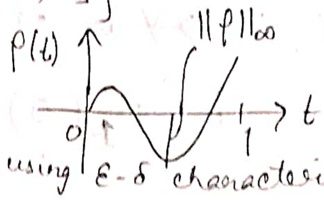
Prop: Let (X, d) be a metric space and $A \subseteq X$ compact. Then A is closed and bounded.

There is an $x \in X$ & $\varepsilon > 0$ st $B_\varepsilon(x) \supseteq A$

Arzelà - Ascoli theorem

Continuous functions: $(C([0,1]), \|\cdot\|_\infty)$, $\to$ Banach space

$\|f\|_\infty = \sup \{ |f(t)| \mid t \in [0,1] \}$



f is called uniformly continuous (using ε-δ characterisation)

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall t_1, t_2 \in [0,1] : |t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon$

$A \subseteq C([0,1])$ is called <u>uniformly equicontinuous</u>

$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{t_1,t_2\in[0,1]}\ \forall_{f\in A} : |t_1-t_2|<\delta \Rightarrow |f(t_1)-f(t_2)|<\varepsilon$

or equivalently $\sup_{f\in A} |f(t_1)-f(t_2)| \xrightarrow{|t_1-t_2|\to 0} 0$

<u>Arzelá - Ascoli Theorem</u> : For $(C([0,1]), \|\,\|_\infty)$ could be any compact metric space holds:

$A \subseteq C([0,1])$ compact $\Leftrightarrow A$ is $\begin{cases} \text{closed} + \\ \text{bounded} + \\ \text{uniformly equicontinuous} \end{cases}$

<u>Compact Operators</u>

$T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ linear (standard norm)

$\Rightarrow T$ is continuous / bounded

$\Rightarrow T[B_1(0)] \subseteq \mathbb{F}^m$ bounded

$\Rightarrow \overline{T[B_1(0)]} \subseteq \mathbb{F}^m$ compact

However : $I : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $p \in [1,\infty)$

$x \longmapsto x$

$\Rightarrow \overline{I[B_1(0)]} = \overline{B_1(0)}$ not <u>compact</u>. (closed unit ball in $\ell^p(\mathbb{N})$)

<u>Defn</u> : Let $(X, \|.\|_X)$, $(Y, \|.\|_Y)$ be 2 normed spaces. A bounded linear operator $T : X \to Y$ is called <u>compact</u> if

$\overline{T[B_1(0)]} \subseteq Y$ is a compact set.

Holder's inequality : (For $\mathbb{F}^n$ & $p \in (1, \infty)$)

$p' \in (1, \infty)$ Holder conjugate

$$\frac{1}{p} + \frac{1}{p'} = 1$$

For $x \in \mathbb{F}^n$:

$$\|x\|_q = \left(\sum_{j=1}^{n} |x_j|^q \right)^{1/q} \quad , q \in [1, \infty)$$

For $x, y \in \mathbb{F}^n$ write: $xy := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{pmatrix}$.

$$\therefore \|xy\|_1 \leq \|x\|_p \|y\|_{p'} \quad \forall \; x, y \in \mathbb{F}^n$$

Young's inequality :

$$a, b > 0 \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

Minkowski's inequality: Δ-inequality for $\|.\|_p$ in $\ell^p(\mathbb{N})$.

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \forall \; x, y \in \ell^p(\mathbb{N})$$

~~Homo~~

Homomorphism : map that preserves structure.

eg. a) Let X, Y be VS & $f: X \to Y$ be a map.

$$\left.\begin{aligned} f(\lambda x) &= \lambda f(x) \\ f(x+x') &= f(x) + f(x') \end{aligned}\right\} \text{linear map}$$

homomorphism = linear map

$$\boxed{\text{isomorphism} = \text{homomorphism} + \text{bijective} + \text{inverse map is also homomorphism}}$$

Isomorphism for Banach spaces X, Y:

$f : X \to Y$ with linear + bijective + $\|f(x)\|_Y = \|x\|$
(often called isometric isomorphism)

Dual spaces : X normed space

$\rightsquigarrow X'$ normed space.

$$X' := \{ l : X \to \mathbb{F} \mid l \text{ linear + bounded} \}$$

Prop : Let X be a normed space. Then $(X', \|\cdot\|_{X \to \mathbb{F}})$ is a Banach space.

(*) Cauchy sequences are always bounded sequences.

eg: Dual space of l^p is isometric isomorphic to $l^{p'}$

Uniform boundedness principle (Banach-Steinhaus theorem)

X, Y normed spaces, X banach space.

$$B(X,Y) = \{ T : X \to Y \mid T \text{ linear + bounded} \}$$

bounded linear operators

Theorem : For every subset $M \subseteq B(X,Y)$ holds:
M is bounded pointwise on X $\Leftrightarrow$ M is uniformly bounded

$$\forall_{x \in X} \exists_{C_x \geq 0} \forall_{T \in M} \|Tx\|_Y \leq C_x \Leftrightarrow \exists_{C \geq 0} \forall_{T \in M} \|T\|_{X \to Y} \leq C$$

Prop: X, Y normed spaces, X Banach space.
Let $T_n \in B(X,Y)$ $\forall\, n \in \mathbb{N}$ with
$\lim_{n\to\infty} T_n x$ exists $\forall\, x \in X$.
Then $T: X \to Y$ defined by $Tx = \lim_{n\to\infty} T_n x$
is linear & bounded.

Hahn-Banach Theorem:

$(X, \|\cdot\|_X)$ is a normed space $\rightsquigarrow (X', \|\cdot\|_{X'})$

$U \subseteq X$ subspace, $u' : U \to \mathbb{F}$ is a continuous linear functional. Then there exists $x' : X \to \mathbb{F}$ is a continuous linear functional with $x'(u) = u'(u)$ $\forall\, u \in U$, $\|x'\|_{X'} = \|u'\|_{U'}$

Applications: $(X, \|\cdot\|_X)$ normed space

a) $\forall\, x \in X$, $x \neq 0$, there is a $x' \in X'$ with $\|x'\|_{X'} = 1$ & $x'(x) = \|x\|_X$

b) X' separates points of X
For $x_1, x_2 \in X$, $x_1 \neq x_2$, there is an $x' \in X'$ with $x'(x_1) \neq x'(x_2)$

c) $\forall\, x \in X$, $\|x\|_X = \sup\{ |x'(x)| \mid x' \in X', \|x'\| = 1\}$

d) Let $U \subseteq X$ be a closed subspace, $x \in X$ with $x \notin U$
Then $\exists\, x' \in X'$ with $x'|_U = 0$ and $x'(x) \neq 0$

Open mapping theorem (Banach-Schauder theorem)

Let (X, d_X), (Y, d_Y) be a metric spaces.

$f : X \to Y$ is called open if

$A \subseteq X$ open in $X \Rightarrow f[A] \subseteq Y$ open in Y.

eg: $f : X \to Y$ is bijective and $f^{-1} : Y \to X$ is continuous, then

$f : X \to Y$ is an open map.

Continuity of f^{-1} : $A \subseteq X$ open in $X \Rightarrow$

$\underbrace{(f^{-1})^{-1}[A]}_{f[A]} \subseteq Y$ open in Y.

eg: a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ open

Open mapping theorem: Let X, Y be Banach spaces. For $T \in B(X, Y)$ holds

T surjective $\Leftrightarrow$ T open map.

Bounded inverse theorem: X, Y Banach spaces. $T \in B(X, Y)$.

Then T bijective $\Rightarrow T^{-1} \in B(Y, X)$ (its continuous)