# Impulse Response of $\mathrm{sinc}^N$ FIR Filters

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***Abstract*—A simple recurrence formula is presented for computing the impulse response coefficients of the $\mathrm{sinc}^N$ FIR filter, consisting of a cascade of $N$ sinc filters, each of length $M$. A closed form expression is also given for the first $M$ coefficients.**

***Index Terms*—Impulse response, $\mathrm{sinc}^N$ filters, digital filters.**

## I. INTRODUCTION

A $\mathrm{sinc}^N$ filter has the magnitude response

$$|H(e^{j\omega})| = \left|\frac{\sin(M\omega/2)}{M\sin(\omega/2)}\right|^N \tag{1}$$

and has the transfer function

$$H(z) = \left[\frac{(1-z^{-M})}{M(1-z^{-1})}\right]^N \tag{2}$$

$$= M^{-N}\left[1+z^{-1}+z^{-2}+\cdots+z^{-(M-1)}\right]^N. \tag{3}$$

As can be seen from (2) and (3), the $\mathrm{sinc}^N$ filter is a cascade of $N$ sinc or moving average filters, each of length $M$. Such filters are useful in decimation and interpolation applications [1]–[11]. Several architectures are available for realizing these filters, of which the cascaded-integrator-comb (CIC) [1] and the nonrecursive FIR architectures [3], [4] are well known. The latter approach requires off-line calculation of all the impulse response coefficients. In a recent paper, Shiraishi [8] presented a generalization of the simultaneous coefficient calculation method for the second approach, which was earlier investigated for $N = 3$. For this purpose, he developed a multiple summation formula for the coefficients and also its recurrence form. In this paper, we present a simple recurrence formula for the coefficients, and also give a closed form expression for the first $M$ coefficients.

## II. RECURRENCE RELATION

For simplicity, we consider the scaled transfer function

$$G(z) = M^N H(z) = \left[\frac{(1-z^{-M})}{(1-z^{-1})}\right]^N. \tag{4}$$

Obviously, $G(z)$ is a polynomial of the form

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots + g_{N(M-1)} z^{-N(M-1)}. \tag{5}$$

Manuscript received March 3, 2005; revised June 20, 2005. This paper was recommended by Associate Editor C.-T. Len.

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Digital Object Identifier 10.1109/TCSII.2005.858319

Our problem is to find the $g_n$'s, $n = 0$ to $N(M-1)$. However, from (3), it is obvious that $G(z)$ is a symmetrical coefficient linear phase polynomial i.e.,

$$g_n = g_{N(M-1)-n}. \tag{6}$$

Thus, we need to find $g_n$'s for $n = 0$ to

$$n_0 = [N(M-1) - q]/2 \tag{7}$$

where $q = 1$ or 0 depending on whether $N(M-1)$ is odd or even.

Now from (4), (5) and the binomial theorem, we get

$$\begin{aligned}
&1 - \binom{N}{1} z^{-M} + \binom{N}{2} z^{-2M} - \binom{N}{3} z^{-3M} \\
&\quad + \cdots + (-1)^N \binom{N}{N} z^{-MN} \\
&\qquad = \left[1 - \binom{N}{1} z^{-1} + \binom{N}{2} z^{-2} - \binom{N}{3} z^{-3}\right. \\
&\qquad\quad \left. + \cdots + (-1)^N \binom{N}{N} z^{-N}\right] \\
&\qquad\quad \times \left[g_0 + g_1 z^{-1} + g_2 z^{-2}\right. \\
&\qquad\quad \left. + \cdots g_{N(M-1)} z^{-N(M-1)}\right].
\end{aligned} \tag{8}$$

By performing the multiplication on the right-hand side of (8), we get (9) as shown at the bottom of the next page, where, it can be observed, the same order terms on the right-hand side are located on a line slanting to the left. In equating the coefficients of $z^{-n}$ on both sides, we can easily see that there are two different cases. For $n = rM$, where $r$ is an integer and $1 \le r \le N$, the sum of the coefficients of $z^{-n}$ on the right-hand side must be $(-1)^r \binom{N}{r}$. Thus

$$\sum_{i=0}^{p} (-1)^i \binom{N}{i} g_{n-i} = (-1)^r \binom{N}{r} \tag{10}$$

where

$$p = \min(N, n) \tag{11}$$

i.e., the last term in the left-hand side summation in (10) contains either $\binom{N}{N}$ or $g_0$. On the other hand, for $n \ne rM$, the sum of the coefficients of $z^{-n}$ on the right-hand side must be zero, i.e.,

$$\sum_{i=0}^{p} (-1)^i \binom{N}{i} g_{n-i} = 0. \tag{12}$$

Solving (10) and (12) for $g_n$, we get

$$g_n = 1 \quad \text{for } n = r = 0 \tag{13}$$

$$g_n = (-1)^r \binom{N}{r} + \sum_{i=1}^{p} (-1)^{i-1} \binom{N}{i} g_{n-i} \quad \text{for } n = rM, \quad 1 \le r \le N \tag{14}$$

$$g_n = \sum_{i=1}^{p} (-1)^{i-1} \binom{N}{i} g_{n-i} \quad \text{for } n \ne rM, \quad 1 \le r \le N. \tag{15}$$

In general, therefore, the required recurrence relation can be written as

$$g_n = (-1)^r \binom{N}{r} \delta(n - rM) - \sum_{i=1}^{p} (-1)^i \binom{N}{i} g_{n-i} \tag{16}$$

where

$$\begin{aligned} g_0 &= 1 \\ \delta(n) &= \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases} \\ p &= \min(N, n), \qquad 1 \le r \le N. \end{aligned} \tag{17}$$

## III. Explicit Expression for $g_n, n < M$

It should be clear from (8) that for finding $g_n, n < M$, all terms on the left hand side, except 1, can be ignored. That is, $g_n, n < M$, is the same as that which occurs in the expansion of $(1 - z^{-1})^{-N}$. By Taylor's series expansion, we get after putting $z^{-1} = y$, for simplicity

$$(1-y)^{-N} = \sum_{n=0}^{\infty} \left. \frac{d^n (1-y)^{-N}}{dy^n} \right|_{y=0} \times \frac{y^n}{n!} = \sum_{n=0}^{\infty} \frac{N(N+1)\dots(N+n-1)}{n!} y^n. \tag{18}$$

Thus

$$g_n = \frac{N(N+1)\dots(N+n-1)}{n!} = \binom{N+n-1}{n} \tag{19}$$

which is the required explicit formula. This agrees with the formula obtained by putting $j = 1$ and $\ell = 0$ in equation (A.10) of [1].

## IV. Examples

As examples, the computed results for $N = 2, 3$ and $4$ and $M = 2$ through $8$ are shown in Table I. It can be seen that $g_n$'s for $M = M_1$ and $M = M_2$ are the same if $n$ is less than both $M_1$ and $M_2$. Also, $g_0$ is always unity and $g_1$ is always equal to $N$.

## V. Conclusion

In this paper, a simple explicit formula has been presented for the first $M$ impulse response coefficients of the $\mathrm{sinc}^N$ filter having the transfer function given by (2). A recurrence formula has also been given for computing the rest of the $n_0 - M$ coefficients, where $n_0$ is given by (7).

A comparison of these formulas with those given in [8] appears to be appropriate at this point. The multiple summation formula in [8], viz.,

$$g_n = \sum_{k_{N-1}=0}^{n} \sum_{k_{N-2}=0}^{k_{N-1}} \cdots \sum_{k_0=0}^{k_1} \times \left[ \sum_{r=0}^{N} (-1)^r \binom{N}{r} \delta(k_0 - Mr) \right] \tag{20}$$

indeed looks formidable, and keeping track of the indexes $k_0, k_1, \dots k_{N-1}$ proves to be difficult. However, when translated to the recurrence form, viz.,

$$\begin{aligned} g_n &= f_N(n) \\ f_m(n) &= \sum_{k=0}^{n} f_{m-1}(k), \qquad \text{for } 1 \le m \le N \\ f_0(n) &= \sum_{r=0}^{n} (-1)^r \binom{N}{r} \delta(n - Mr) \end{aligned} \tag{21}$$

$$\begin{array}{llllll}
1 - & -\binom{N}{1} z^{-M} & +\binom{N}{2} z^{-2M} & -\binom{N}{3} z^{-3M} & + \dots\dots + (-1)^N \binom{N}{N} z^{-MN} \\
= g_0 & -\binom{N}{1} g_0 z^{-1} & +\binom{N}{2} g_0 z^{-2} & -\binom{N}{3} g_0 z^{-3} & + \dots\dots + (-1)^N \binom{N}{N} g_0 z^{-N} \\
+ g_1 z^{-1} & -\binom{N}{1} g_1 z^{-2} & +\binom{N}{2} g_1 z^{-3} & -\binom{N}{3} g_1 z^{-4} & + \dots\dots + (-1)^N \binom{N}{N} g_1 z^{-(N+1)} \\
+ g_2 z^{-2} & -\binom{N}{1} g_2 z^{-3} & +\binom{N}{2} g_2 z^{-4} & -\binom{N}{3} g_2 z^{-5} & + \dots\dots + (-1)^N \binom{N}{N} g_2 z^{-(N+2)} \\
+ g_3 z^{-3} & -\binom{N}{1} g_3 z^{-4} & +\binom{N}{2} g_3 z^{-5} & -\binom{N}{3} g_3 z^{-6} & + \dots\dots + (-1)^N \binom{N}{N} g_3 z^{-(N+3)} \\
+ \quad \dots & \dots & \dots & \dots & \dots \\
+ g_{N(M-1)} z^{-N(M-1)} & -\binom{N}{1} g_{N(M-1)} z^{-[N(M-1)+1]} & +\binom{N}{2} g_{N(M-1)} z^{-[N(M-1)+2]} & -\binom{N}{3} g_{N(M-1)} z^{-[N(M-1)+3]} & + \dots\dots + (-1)^N \binom{N}{N} g_{N(M-1)} z^{-MN}.
\end{array} \tag{9}$$

TABLE I
VALUES OF $g(n)$ FOR $N = 2, 3, 4$ AND $M = 2$ TO $8$

| N=2 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| M | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ | $g_{12}$ | $g_{13}$ | $g_{14}$ |
| 2 | 1 | 2 | | | | | | | | | | | | | |
| 3 | 1 | 2 | 3 | | | | | | | | | | | | |
| 4 | 1 | 2 | 3 | 4 | | | | | | | | | | | |
| 5 | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | | | | | | |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | | | | | |
| N=3 | | | | | | | | | | | | | | | |
| 2 | 1 | 3 | | | | | | | | | | | | | |
| 3 | 1 | 3 | 6 | 7 | | | | | | | | | | | |
| 4 | 1 | 3 | 6 | 10 | 12 | | | | | | | | | | |
| 5 | 1 | 3 | 6 | 10 | 15 | 18 | 19 | | | | | | | | |
| 6 | 1 | 3 | 6 | 10 | 15 | 21 | 25 | 27 | | | | | | | |
| 7 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 33 | 37 | | | | | | |
| 8 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 46 | 48 | | | | | |
| N=4 | | | | | | | | | | | | | | | |
| 2 | 1 | 4 | | | | | | | | | | | | | |
| 3 | 1 | 4 | 10 | 16 | 19 | | | | | | | | | | |
| 4 | 1 | 4 | 10 | 20 | 31 | 40 | 44 | | | | | | | | |
| 5 | 1 | 4 | 10 | 20 | 35 | 52 | 68 | 80 | 85 | | | | | | |
| 6 | 1 | 4 | 10 | 20 | 35 | 56 | 80 | 104 | 125 | 140 | 146 | | | | |
| 7 | 1 | 4 | 10 | 20 | 35 | 56 | 84 | 116 | 149 | 180 | 206 | 224 | 231 | | |
| 8 | 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 | 161 | 204 | 246 | 284 | 315 | 336 | 344 |

also given in [8], the computational effort is considerably reduced. In fact, the computational complexity then becomes somewhat comparable to that of the method given in this paper. However, a quantitative comparison of the computational effort is difficult to formulate because of the difference in the nature of the computations. Our method has the advantages of simplicity, compactness and explicitness, not only for the first $M$ impulse response coefficients, but in the computation of the rest of the coefficients also. Both methods can be easily implemented in computer software or hardware. Also, in either method, there are no potential numerical problems.

## ACKNOWLEDGMENT

The author thanks Prof. S. D. Joshi, and Prof. R. K. Patney for helpful discussions and Prof. Jayadeva for his help in the preparation of this paper. He is also thankful to the reviewers for their insightful queries, constructive criticisms and helpful suggestions, which improved the presentation of the paper considerably.

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