Homotopy Theory of Strict ω-Categories

Unofficial notes from the lectures of François Métayer
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Definition. This document is unofficial notes taken from the lectures "Homotopy Theory of Strict ω-Categories" given by François Métayer during the conference "Categories in Homotopy Theory and Rewriting", held at CIRM in September 2017. It can contain many typos and mistakes, and some notations have been changed from the original lecture.

1 Model categories

Throughout this section, we assume the ambient category $\mathscr{C}$ to be complete and cocomplete. We make use of the grammatical composition convention, meaning that the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is written fg.

Lifting properties. Consider the following diagram:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dotted, "k"] & \bullet \end{tikzcd}$$

We say that f has the left lifting property with respect to g if such a morphism k making the two triangles commute exists. We also say that g has the right lifting property with respect to f, and denote it by $f \boxslash g$. If $\mathfrak{I}$ is a class of morphism, let ${}^{\boxslash}\mathfrak{I} = \{f \in \hom \mathscr{C} \mid f \boxslash i, \forall i \in \mathfrak{I}\}$ and similarly for $\mathfrak{I}^{\boxslash}$. The class ${}^{\boxslash}\mathfrak{I}$ of $\mathfrak{I}$-injective morphism is closed under pushouts, sums, retracts, and transfinite composition.

Small objects. We say that an object A is small if the representable $\mathscr{C}(A, -)$ preserves countable directed colimits. If $\mathfrak{I}$ is a class of morphism whose domain are all small, we can apply the following:

Theorem 1.1 (Small object argument)**.** *Under the conditions above, any morphism $f \in \mathfrak{I}$ can be factorized as $f = ip$, with $p \in {}^{\boxslash}\mathfrak{I}$ and $i \in {}^{\boxslash}(\mathfrak{I}^{\boxslash})$.*

Weak factorization systems. A weak factorization system is a pair $(\mathfrak{L}, \mathfrak{R})$ of classes of morphisms, such that: 1. each morphism f of $\mathscr{C}$ factors as $f = rl$ with $r \in \mathfrak{R}$ and $l \in \mathfrak{L}$; 2. $\mathfrak{L} = {}^{\boxslash}\mathfrak{R}$; 3. $\mathfrak{R} = \mathfrak{L}^{\boxslash}$.

Proposition 1.2. *If $\mathfrak{I}$ has small domains, then $({}^{\boxslash}(\mathfrak{I}^{\boxslash}), \mathfrak{I}^{\boxslash})$ is a weak factoriz ation system.*

Model structure. A model structure on $\mathscr{C}$ is the data of three classes of morphisms: $\mathfrak{W}$ (weak equivalences), $\mathfrak{Cof}$ (cofibrations), and $\mathfrak{Fib}$ (fibrations), such that:

1. $\mathfrak{W}$ has the "3 for 2" property (also known as "2 out of 3"): among f, g, fg, if two are in $\mathfrak{W}$ then so is the third;
2. $\mathfrak{W}$, $\mathfrak{Cof}$, and $\mathfrak{Fib}$ are closed under retracts;
3. $\mathfrak{W} \cap \mathfrak{Cof} \subseteq {}^{\boxslash}\mathfrak{Fib}$, and $\mathfrak{W} \cap \mathfrak{Fib} \subseteq \mathfrak{Cof}^{\boxslash}$;
4. we have two weak factorization systems: $(\mathfrak{W} \cap \mathfrak{Cof}, \mathfrak{Fib})$, and $(\mathfrak{Cof}, \mathfrak{W} \cap \mathfrak{Fib})$.

This is the classical definition, that we shall not use. The following one is proved equivalent in [1]: a model structure on $\mathscr{C}$ is the data of two classes of morphisms: $\mathfrak{W}$ (weak equivalences) and $\mathfrak{I}$ (generating cofibrations), such that:

1. $\mathfrak{W}$ has the "3 for 2" property and is stable under retracts;
2. $\mathfrak{I}^{\boxslash} \subseteq \mathfrak{W}$;
3. $\mathfrak{W} \cap {}^{\boxslash}(\mathfrak{I}^{\boxslash})$ is stable under pushouts and transfinite composition;
4. the "solution set condition" holds: for $i \in \mathfrak{I}$, let J_i be the class of mor-

phisms j that factor an outer square as follows:

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[ddd, "i"'] \arrow[dr, dotted] & & \bullet \arrow[ddd, "w \in \mathfrak{W}"] \\
& \bullet \arrow[ur, dotted] \arrow[d, dotted, "j"'] & \\
& \bullet \arrow[dr, dotted] & \\
\bullet \arrow[rr] \arrow[ur, dotted] & & \bullet
\end{tikzcd}$$

we then ask that $J_i \subseteq \mathfrak{W} \cap {}^{\boxslash}(\mathfrak{J}^{\boxslash})$.

Theorem 1.3 (Jeff Smith)**.** *Under the above conditions, letting* $\mathfrak{Cof} = {}^{\boxslash}(\mathfrak{J}^{\boxslash})$ *and* $\mathfrak{Fib} = (\mathfrak{W} \cap \mathfrak{Cof})^{\boxslash}$ *yields a model structure in the classical definition.*

2 ω-categories

Globular sets. Let $\mathbb{G}$ be the category of globes, having objects the natural numbers, and morphisms $\sigma, \tau : n \longrightarrow n+1$, for all $n \in \mathbb{N}$, subject to the following globularity conditions:

$$\begin{cases} \sigma\sigma = \sigma\tau \\ \tau\sigma = \tau\tau. \end{cases}$$

The category of globular sets is defined as $\hat{\mathbb{G}} = [\mathbb{G}^{\mathrm{op}}, \mathscr{S}et]$.

ω-categories. An ω-category $\mathscr{C}$ is a globular set

$$\mathscr{C}_0 \xleftarrow{s,t} \mathscr{C}_1 \xleftarrow{s,t} \cdots \xleftarrow{s,t} \mathscr{C}_n \xleftarrow{s,t} \mathscr{C}_{n+1} \xleftarrow{s,t} \cdots,$$

with:

1. compositions: for $x, y \in \mathscr{C}_n$ with $tx = sy$, we have a composition $x \circ_{n-1} y \in \mathscr{C}_n$;

2. identity cells: for $x \in \mathscr{C}_n$ we have $\mathrm{id}(x) \in \mathscr{C}_{n+1}$ being left and right neutral for $\circ_n$;

such that:

1. composition is associative;

2. the exchange rule holds:

$$(x \circ_i y) \circ_j (z \circ_i w) = (x \circ_j z) \circ_i (y \circ_j w)$$

whenever those expressions are defined.

Let $\mathscr{C}at_\omega$ be the category of strict ω-categories and natural transformations that preserve composition and identities, also called ω-functors.

For $\mathscr{C} \in \mathscr{C}at_\omega$ and $x, y \in \mathscr{C}_0$, the hom $[x, y]$ is also an ω-category. For $u \in \mathscr{C}_{n+1}$ such that $s^0 u = x$ (iterated source at dimension 0) and $t^0 = y$, we have a cell $[u] \in [x, y]_n$.

The forgetful functor $\mathscr{C}at_\omega \longrightarrow \hat{\mathbb{G}}$ has a left adjoint. Let $O[n]$ be the free ω-category over the representable $\mathbb{G}(-, n)$. Let $\partial O[n]$ be the boundary of $O[n]$, i.e. the free ω-category over $\mathbb{G}(-, n)$ with the n-cell removed. We have canonical inclusions $i_n : \partial O[n] \hookrightarrow O[n]$, and let $\mathfrak{I} = \{i_n \mid i \in \mathbb{N}\}$.

Definition 2.1. This definition is made by mutual coinduction.

- For $n \geq 1$, a n-cell $u : x \longrightarrow y$ is reversible if there is a cell $\bar{u} : y \longrightarrow x$ such that $u \circ_{n-1} \bar{u} \sim \mathrm{id}(x)$ and $\bar{u} \circ_{n-1} u \sim \mathrm{id}(y)$;
- For $n \geq 1$, two parallel n-cells f and g are ω-equivalent, denoted by $f \sim g$, if there exists a reversible $(n+1)$-cell $u : f \longrightarrow g$.

For instance, identities are reversible cells. An ω-functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a weak equivalence if

1. $F_0 : \mathscr{C}_0 \longrightarrow \mathscr{D}_0$ is surjective up to $\sim$;
2. for parallel $x, y \in \mathscr{C}_n$, of there is $w : Fx \longrightarrow Fy$, then there exists $v : x \longrightarrow y$ such that $Fv \sim w$.

Let $\mathfrak{W}$ be the class of such weak equivalences.

Theorem 2.2. *The category $\mathscr{C}at_\omega$ together with $\mathfrak{W}$ as the class of weak equivalences and $\mathfrak{I}$ as the class of generating cofibrations is a model category.*

3 Cofibrant objects and polygraphs

Free category over a cellular extension. Let $\mathbb{G}|_n$ be the category $\mathbb{G}$ truncated at dimension n, i.e. the full subcategory generated by objects $0, \dots, n$. We call elements of $\widehat{\mathbb{G}|_n}$ n-globular sets. Consider the following pullback:

$$\begin{tikzcd}
\mathscr{C}at_n^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{\mathbb{G}|_{n+1}} \arrow[d] \\
\mathscr{C}at_n \arrow[r] & \widehat{\mathbb{G}|_n},
\end{tikzcd}$$

where the cospan contains the forgetful functors. An object of $\mathscr{C}at_n^+$ is then a n-category $\mathscr{C}$ endowed with a cellular extension $S_{n+1} \xrightarrow{s,t} \mathscr{C}_n$ satisfying the globularity condition. The outer square

$$\begin{tikzcd}
\mathscr{C}at_{n+1} \arrow[dr, dotted, "\exists! W"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathscr{C}at_n^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{\mathbb{G}|_{n+1}} \arrow[d] \\
& \mathscr{C}at_n \arrow[r] & \widehat{\mathbb{G}|_n},
\end{tikzcd}$$

factors uniquely trough a forgetful functor $W : \mathscr{C}at_{n+1} \longrightarrow \mathscr{C}at_n^+$ that forgets the composition $\circ_n$. It has a left adjoint L, so that from a cellular extension $\mathscr{C}^+ = (S_{n+1} \xrightarrow{s,t} \mathscr{C})$ in $\mathscr{C}at_n^+$ we can build a free $L\mathscr{C}^+ \in \mathscr{C}at_{n+1}$.

Polygraph. A 0-polygraph is a set endowed with its identity:

$$\begin{tikzcd}
P_0 \arrow[d, "\mathrm{id}"'] \\
P_0^*
\end{tikzcd}$$

By induction, a $(n+1)$-polygraph is a n polygraph with a cellular extension $P_{n+1} \xrightarrow{s,t} P_n^*$, which generate a free $(n+1)$-category, whose set of $(n+1)$-cells is denoted by P_{n+1}^*:

$$\begin{tikzcd}
P_0 \arrow[d] & P_1 \arrow[d] \arrow[dl, "s,t"'] & \cdots \arrow[dl, "s,t"'] & P_n \arrow[d] \arrow[dl, "s,t"'] & P_{n+1} \arrow[d] \arrow[dl, "s,t"'] \\
P_0^* & P_1^* \arrow[l, "s,t"] & \cdots \arrow[l, "s,t"] & P_n^* \arrow[l, "s,t"] & P_{n+1}^* \arrow[l, "s,t"]
\end{tikzcd}$$

A polygraph $P = (P_0, P_1, \ldots)$ has an underlying ω-category, denoted by P^*. Morphisms of polygraphs are ω-functors that map generators to generators. We denote by $\mathscr{P}oly_\omega$ the category of polygraphs and such morphisms.

Proposition 3.1. *If $\mathscr{C} = P^*$ is a ω-category generated by a polygraph, then $\mathscr{C}$ is cofibrant. Conversely, any cofibrant ω-category is of this form.*

Sketch. The successive truncations $\mathscr{C}|_n$ of $\mathscr{C}$ can be obtain by the following pushout:

$$\begin{tikzcd}
\coprod_{P_{n+1}} \partial O[n+1] \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner"] & \mathscr{C}|_n \arrow[d, tail, "\eta_n"] \\
\coprod_{P_{n+1}} O[n+1] \arrow[r] & \mathscr{C}|_{n+1}
\end{tikzcd}$$

and since the left arrow is a cofibration, so is the right one. Then the initial inclusion map $\emptyset \longrightarrow \mathscr{C}$ is obtained via the following transfinite composite of cofibrations, and hence is a cofibration:

$$\emptyset \rightarrowtail \mathscr{C}|_0 \overset{\eta}{\rightarrowtail} \mathscr{C}|_1 \rightarrowtail \cdots \longrightarrow \mathscr{C} \cong \operatorname*{colim}_n \mathscr{C}|_n.$$

The converse implication is much more technical and not covered in this lecture. $\square$

Let $P\mathscr{C}$ be the normal resolution of an ω-category $\mathscr{C}$, i.e. the natural polygraph such that $P\mathscr{C}_n = \mathscr{C}_n$.

Proposition 3.2. *The normal resolution gives rise to an adjunction $(-)^* : \mathscr{P}oly_\omega \overset{\dashv}{\longleftrightarrow} \mathscr{C}at_\omega : P$. Moreover, the counit $\varepsilon_{\mathscr{C}} : (P\mathscr{C})^* \longrightarrow \mathscr{C}$ is a trivial fibration, making the comonad $(P-)^*$ a cofibrant replacement.*

Universal property. Assume we have defined a n-functor $f|_n : P|_n \longrightarrow \mathscr{C}|_n$, and that we have a map $f_{n+1} : P_{n+1} \longrightarrow \mathscr{C}_{n+1}$ satisfying the globularity condition. Then there is a unique factorization $f_{n+1} : P^*_{n+1} \longrightarrow \mathscr{C}_{n+1}$ making $f|_{n+1} : P|_{n+1} \longrightarrow \mathscr{C}|_{n+1}$ into a $(n+1)$-functor.

$$\begin{tikzcd}
\cdots & P_n \arrow[d] & P_{n+1} \arrow[dl, "{s,t}"'] \arrow[d] \arrow[dd, bend left, "f_{n+1}"] \\
\cdots & P_n^* \arrow[d, "f_n"'] & P_{n+1}^* \arrow[l, "{s,t}"'] \arrow[d, dotted, "f_{n+1}"'] \\
\cdots & \mathscr{C}_n & \mathscr{C}_{n+1}
\end{tikzcd}$$

Take $P \in \mathscr{P}oly_n$ and consider the following n-category

$$1 \overset{s,t}{\longleftarrow} 1 \overset{s,t}{\longleftarrow} \cdots \overset{s,t}{\longleftarrow} \mathbb{Z}P_n$$

The inclusion $P_n \hookrightarrow \mathbb{Z}P_n$ gives rise to an n-functor whose n-th component $u : P_n^* \longrightarrow \mathbb{Z}P_n$ "counts" generating cells occurrences in elements of P_n^*.

4 Homology in $\mathscr{C}at_\omega$

Abelianization. Define $\mathrm{Ab} : \mathscr{C}at_\omega \longrightarrow \mathscr{C}h_{\mathbb{Z}}$ by:

$$\begin{cases} \mathrm{Ab}(\mathscr{C})_n = \dfrac{\mathbb{Z}\mathscr{C}_n}{\langle x \circ_i y - x - y \mid 0 \leq i < n, t^i x = s^i y \rangle}, \\ \partial[x] = [tx] - [sx]. \end{cases}$$

An ω-functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ naturally induces

$$\begin{aligned} \mathrm{Ab}(f) : \mathrm{Ab}(\mathscr{C}) &\longrightarrow \mathrm{Ab}(\mathscr{D}) \\ [x] &\longmapsto [fx]. \end{aligned}$$

Proposition 4.1. *The abelianization functor* Ab *is 1. left adjoint; 2. takes weak equivalences between cofibrant objects to quasi-isomorphisms; 3. a left Quillen functor.*

Polygraphic homology. We have:

$$\begin{tikzcd} \mathscr{C}at_\omega \arrow[r, "\mathrm{Ab}"] \arrow[d] & \mathscr{C}h_{\mathbb{Z}} \arrow[d] \\ \mathrm{Ho}(\mathscr{C}at_\omega) \arrow[r, "\mathsf{L}\mathrm{Ab}"] & \mathrm{Ho}(\mathscr{C}h_{\mathbb{Z}}) \end{tikzcd}$$

This square doesn't commute! But it does if we restrict $\mathscr{C}at_\omega$ to $\mathscr{P}oly_\omega$. We have a definition for polygraphic homology:

$$H_\bullet^{\mathrm{poly}}(\mathscr{C}) = H_\bullet(\mathrm{Ab}(P)),$$

for $\mathscr{C} \in \mathscr{C}at_\omega$ and $P^* \overset{\sim}{\twoheadrightarrow} \mathscr{C}$ a polygraphic resolution.

Classical homology. The oriental functor $O : \mathbb{\Delta} \longrightarrow \mathscr{C}at_\omega$ gives rise to a simplicial nerve $N : \mathscr{C}at_\omega \longrightarrow \hat{\mathbb{\Delta}}$. We may define the homology of $\mathscr{C} \in \mathscr{C}at_\omega$ by

$$H_\bullet(\mathscr{C}) = H_\bullet(\mathbb{Z}N\mathscr{C}).$$

In general, $H_\bullet(\mathscr{C})$ and $H_\bullet^{\mathrm{poly}}(\mathscr{C})$ don't match.

Proposition 4.2. *If $\mathscr{C}$ is a monoid, then $H_\bullet(\mathscr{C}) \cong H_\bullet^{\mathrm{poly}}(\mathscr{C})$.*

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