Homotopy Theory of Strict ω-Categories

Notes from the course of François Métayer
LaTeXed by Cédric HT

25 – 28 September 2017

Definition. This document is unofficial notes taken from the course "Homotopy Theory of Strict ω-Categories" given by François Métayer during the conference "Categories in Homotopy Theory and Rewriting", held at CIRM in September 2017. It can contain many typos and mistakes, and some notations have been changed from the orifinal lecture.

1 Preliminaries

Throughout this lecture, we assume the ambient category $\mathscr{C}$ to be complete and cocomplete. We make use of the grammatical composition convention, meaning that the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is written fg.

Lifting properties. Consider the following diagram:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed, "k"] & \bullet \end{tikzcd}$$

We say that f has the left lifting property with respect to g if such a morphism k making the two triangles commute exists. We also say that g has the right lifting property with respect to f, and denote it by $f \boxslash g$. If $\mathfrak{I}$ is a class of morphism, let ${}^{\boxslash}\mathfrak{I} = \{f \in \hom\mathscr{C} \mid f \boxslash i, \forall i \in \mathfrak{I}\}$ and similarily for $\mathfrak{I}^{\boxslash}$. The class ${}^{\boxslash}\mathfrak{I}$ if $\mathfrak{I}$-injective morphism is closed under pushouts, sums, retracts, and transfinite composition.

Small objects. We say that an object A is small if the representable $\mathscr{C}(A, -)$ preserves countable directed colimits. If $\mathfrak{I}$ is a class of morphism whose domain are all small, we can apply the following:

Theorem 1.1 (Small object argument)**.** *Under the conditions above, any morphism $f \in \mathfrak{I}$ can be factorized as $f = ip$, with $p \in {}^{\boxslash}\mathfrak{I}$ and $i \in {}^{\boxslash}(\mathfrak{I}^{\boxslash})$.*

Weak factorization systems. A weak factorization system is a pair $(\mathfrak{L}, \mathfrak{R})$ of classes of morphisms, such that: 1. each morphism f of $\mathscr{C}$ factors as $f = rl$ with $r \in \mathfrak{R}$ and $l \in \mathfrak{L}$; 2. $\mathfrak{L} = {}^{\boxslash}\mathfrak{R}$; 3. $\mathfrak{R} = \mathfrak{L}^{\boxslash}$.

Proposition 1.2. *If $\mathfrak{I}$ has small domains, then $({}^{\boxslash}(\mathfrak{I}^{\boxslash}), \mathfrak{I}^{\boxslash})$ is a weak factoriz ation system.*

Model structure. A model structure on $\mathscr{C}$ is the data of three classes of morphisms: $\mathfrak{W}$ (weak equivalences), $\mathfrak{Cof}$ (cofibrations), and $\mathfrak{Fib}$ (fibrations), such that:

1. $\mathfrak{W}$ has the "3 for 2" property (also known as "2 out of 3"): among f, g, fg, if two are in $\mathfrak{W}$ then so is the third;

2. $\mathfrak{W}$, $\mathfrak{Cof}$, and $\mathfrak{Fib}$ are closed under retracts;

3. $\mathfrak{W} \cap \mathfrak{Cof} \subseteq {}^{\boxslash}\mathfrak{Fib}$, and $\mathfrak{W} \cap \mathfrak{Fib} \subseteq \mathfrak{Cof}^{\boxslash}$;

4. we have two weak factorization systems: $(\mathfrak{W} \cap \mathfrak{Cof}, \mathfrak{Fib})$, and $(\mathfrak{Cof}, \mathfrak{W} \cap \mathfrak{Fib})$.

This is the classical definition, that we shall not use. The following one is proved equivalent in [1]: a model structure on $\mathscr{C}$ is the data of two classes of morphisms: $\mathfrak{W}$ (weak equivalences) and $\mathfrak{I}$ (generating cofibrations), such that:

1. $\mathfrak{W}$ has the "3 for 2" property and is stable under retracts;

2. $\mathfrak{I}^{\boxslash} \subseteq \mathfrak{W}$;

3. $\mathfrak{W} \cap {}^{\boxslash}(\mathfrak{I}^{\boxslash})$ is stable under pushouts and transfinite composition;

4. the "solution set condition" holds: for $i \in \mathfrak{I}$, let J_i be the class of morphisms j that factor an outer square as follows:

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dddd, "i"'] \arrow[dr, dashed] & & \bullet \arrow[dddd, "w \in \mathfrak{W}"] \\
& \bullet \arrow[ur, dashed] \arrow[dd, dashed, "j"'] & \\
& & \\
& \bullet \arrow[dr, dashed] & \\
\bullet \arrow[rr] \arrow[ur, dashed] & & \bullet
\end{tikzcd}$$

we then ask that $J_i \subseteq \mathfrak{W} \cap {}^{\boxslash}(\mathfrak{I}^{\boxslash})$.

Theorem 1.3 (Jeff Smith)**.** *Under the above conditions, letting $\mathfrak{Cof} = {}^{\boxslash}(\mathfrak{I}^{\boxslash})$ and $\mathfrak{Fib} = (\mathfrak{W} \cap \mathfrak{Cof})^{\boxslash}$ yields a model structure in the classical definition.*

2 ω-categories

Globular sets. Let $\mathbb{G}$ be the category of globes, having objects the natural numbers, and morphisms $\sigma, \tau : n \longrightarrow n+1$, for all $n \in \mathbb{N}$, subject to the following globularity conditions:

$$\begin{cases} \sigma\sigma = \sigma\tau \\ \tau\sigma = \tau\tau. \end{cases}$$

The category of globular sets is defined as $\hat{\mathbb{G}} = [\mathbb{G}^{\mathrm{op}}, \mathscr{S}et]$.

ω-categories. An ω-category $\mathscr{C}$ is a globular set

$$\mathscr{C}_0 \xleftarrow{s,t} \mathscr{C}_1 \xleftarrow{s,t} \cdots \xleftarrow{s,t} \mathscr{C}_n \xleftarrow{s,t} \mathscr{C}_{n+1} \xleftarrow{s,t} \cdots,$$

with:

1. compositions: for $x, y \in \mathscr{C}_n$ with $tx = sy$, we have a composition $x \circ_{n-1} y \in \mathscr{C}_n$;
2. identity cells: for $x \in \mathscr{C}_n$ we have $\mathrm{id}(x) \in \mathscr{C}_{n+1}$ being left and right neutral for $\circ_n$;

such that:

1. composition is associative;
2. the exchange rule holds:

$$(x \circ_i y) \circ_j (z \circ_i w) = (x \circ_j z) \circ_i (y \circ_j w)$$

whenever those expressions are defined.

Let $\mathscr{C}at_\omega$ be the category of strict ω-categories and natural transformations that preserve composition and identities, also called ω-functors.

For $\mathscr{C} \in \mathscr{C}at_\omega$ and $x, y \in \mathscr{C}_0$, the hom $[x, y]$ is also an ω-category. For $u \in \mathscr{C}_{n+1}$ such that $s^0 u = x$ (iterated source at dimension 0) and $t^0 = y$, we have a cell $[u] \in [x, y]_n$.

The forgetful functor $\mathscr{C}at_\omega \longrightarrow \hat{\mathbb{G}}$ has a left adjoint. Let $O[n]$ be the free ω-category over the representable $\mathbb{G}(-, n)$. Let $\partial O[n]$ be the boundary of $O[n]$, i.e. the free ω-category over $\mathbb{G}(-, n)$ with the n-cell removed. We have canonical inclusions $i_n : \partial O[n] \hookrightarrow O[n]$, and let $\mathfrak{I} = \{i_n \mid i \in \mathbb{N}\}$.

Definition 2.1. This definition is made by mutual coinduction.

- For $n \geq 1$, a n-cell $u : x \longrightarrow y$ is reversible if there is a cell $\bar{u} : y \longrightarrow x$ such that $u \circ_{n-1} \bar{u} \sim_\omega \mathrm{id}(x)$ and $\bar{u} \circ_{n-1} u \sim_\omega \mathrm{id}(y)$;
- For $n \geq 1$, two parallel n-cells f and g are ω-equivalent, denoted by $f \sim_\omega g$, if there exists a reversible $(n+1)$-cell $u : f \longrightarrow g$.

For instance, identities are reversible cells. An ω-functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a weak equivalence if

1. $F_0 : \mathscr{C}_0 \longrightarrow \mathscr{D}_0$ is surjective up to $\sim_\omega$;
2. for parallel $x, y \in \mathscr{C}_n$, of there is $w : Fx \longrightarrow Fy$, then there exists $v : x \longrightarrow y$ such that $Fv \sim_\omega w$.

Let $\mathfrak{W}$ be the class of such weak equivalences.

Proposition 2.2. *1. We have $\mathfrak{I}^{\boxslash} \subseteq \mathfrak{W}$;*

2. $\mathfrak{W}$ has the "3 for 2".

References

[1] Tibor Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.

[2] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[3] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[4] Ross Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.