Notes on the Tensor Product of Axiomatized Algebraic Theories and their Stability

Cédric HT

Contents

1 The category of an algebraic theory

1.1 Construction

Let $\mathfrak{L}$ be a first order langage, and denote by $c\mathfrak{L} = f^{(0)}\mathfrak{L}$ the set of its constant symbols, $f^{(n)}\mathfrak{L}$ the set of its function symbols of arity n, $f\mathfrak{L} = \bigcup_{n\in\mathbb{N}} f^{(n)}\mathfrak{L}$, and define $t^{(n)}\mathfrak{L}$, $t\mathfrak{L}$ similarity for terms.

An *identity* is a formula ϕ of $\mathfrak{L}$ of the following form

$$\phi = \quad [\forall \overrightarrow{x_i} \;\; t(\overrightarrow{x_i}) = u(\overrightarrow{x_i})],$$

which we'll more compactly denote $\phi = [t = u]$, where $t, u \in t\mathfrak{L}$. An *algebraic theory* is a pair $\mathtt{T} = (\mathfrak{L}, T)$, where T is a theory only containing identities. We will use the notations $f^{(n)}\mathtt{T}$ instead of $f^{(n)}\mathfrak{L}$, and similarity for terms.

We now create a category that describe $\mathtt{T}$. The first step is to describe its langage $\mathfrak{L}$. Consider $\mathbb{N}$ the category of finite cardinals and set maps. Then the usual addition $+$ is a product in the opposite category $\mathbb{N}^{\mathrm{op}}$. Endow it with an additional morphism $f : n \longrightarrow 1$ for each function symbol $f \in f^{(n)}\mathtt{T}$ and complete it so as it still has finite products. Denote by $\mathscr{L}$ the resulting category.

Each term $t \in t^{(n)}\mathtt{T}$ defines a morphism $t : n \longrightarrow 1$. Define $\sim$ to be the smallest congruence relation on $\mathfrak{L}$ such that $t = u$ in $\mathscr{L}/\sim$ whenever there is an axiom $\phi \in T$ of the form $\phi = [t = u]$. We abuse notations and denote by $\mathtt{T}$ the quotient category.

Let $\mathscr{A}$ be the category of algebraic theories and finite product preserving functors that are identity on objects, i.e. a morphism $F : \mathtt{T} \longrightarrow \mathtt{U}$ is a product preserving functor such that the follwing diagram commutes :

$$\begin{tikzcd} & \mathbb{N}^{\mathrm{op}} \arrow[dl, hook'] \arrow[dr, hook] & \\ \mathtt{T} \arrow[rr, "F"] & & \mathtt{U}. \end{tikzcd}$$

1.2 Structure of $\mathtt{T}(\mathscr{C})$

Take $\mathscr{C}$ a category with finite product, and define a $\mathtt{T}$-*model* in $\mathscr{C}$ to be a product preserving functor $X : \mathtt{T} \xrightarrow{\times} \mathscr{C}$. We abuse notations further and denote $X = X1 \in \mathrm{ob}\,\mathscr{C}$, $f = Xf : X^n \longrightarrow X$ for all $f \in f^{(n)}\mathtt{T}$. Take Y another model, and $\alpha : X \longrightarrow Y$ is a natural transformation. Remark that $\alpha_n = \alpha_1^n : X^n \longrightarrow Y^n$. Hence, we identify α with α_1. Denote by $\mathtt{T}(\mathscr{C})$ to be the category of such models and natural transformations. If $\mathscr{C} = \mathscr{S}et$, denote $\underline{\mathtt{T}} = \mathtt{T}(\mathscr{S}et)$.

Take $X, Y : \mathscr{L} \xrightarrow{\times} \mathscr{C}$, and define

$$\begin{aligned} X \times Y : \mathscr{L} &\longrightarrow \mathscr{C} \\ n &\longmapsto (X \times Y)^n \\ f &\longmapsto (X \times Y)f, \qquad\qquad \forall f \in f^{(n)}\mathtt{T} \end{aligned}$$

where $(X \times Y)f$ is the following composite :

$$(X \times Y)^n \xrightarrow{\tau_{2,n}} X^n \times Y^n \xrightarrow{f \times f} X \times Y.$$

Proposition 1.1. 1. *This operation defines a product on the category of finite product preserving functors $\mathscr{L} \xrightarrow{\times} \mathscr{C}$.*

2. *If $X, Y \in \operatorname{ob} \mathtt{T}(\mathscr{C})$, then so does $X \times Y$. Hence, $\mathtt{T}(\mathscr{C})$ is a category with finite products.*

Proof. Define $(\mathrm{proj}_X)_n$ as being the composite

$$(X \times Y)^n \xrightarrow{\tau_{2,n}} X^n \times Y^n \xrightarrow{\mathrm{proj}} X^n.$$

If $f \in f^{(n)}\mathtt{T}$, then the following diagram commutes by definition :

$$\begin{tikzcd}
(X \times Y)^n \arrow[r, hook, "\tau_{2,n}"] \arrow[d, "(X \times Y)f"'] & X^n \times Y^n \arrow[d, "f \times f"] \\
X \times Y \arrow[r, "\tau_{1,1} = \mathrm{id}"] & X \times Y.
\end{tikzcd}$$

Take $w : p \longrightarrow q$ be a morphism in $\mathscr{L}$, and consider

$$\begin{tikzcd}
(X \times Y)^p \arrow[r, "\tau_{2,p}"] \arrow[d, "(X \times Y)w"'] & X^p \times Y^p \arrow[r, "\mathrm{proj}"] \arrow[d, "w \times w"'] & X^p \arrow[d, "w"] \\
(X \times Y)^q \arrow[r, "\tau_{2,p}"'] & X^q \times Y^q \arrow[r, "\mathrm{proj}"'] & X^q.
\end{tikzcd}$$

The left square commutes by the previous remark whereas the right square commutes by the definition of the product in $\mathscr{C}$. Therefore, the outer square commutes, and we can define a natural transformation $\mathrm{proj}_X = (\mathrm{proj}_X)_\bullet : X \times Y \longrightarrow X$, and similarily for Y.

Take $Z : \mathscr{L} \xrightarrow{\times} \mathscr{C}$ and two natural transformations $X \xleftarrow{\alpha} Z \xrightarrow{\beta} Y$. Using the same argument as before, we obtain a well defined natural transformation $\gamma : Z \longrightarrow X \times Y$ having components $\tau_{n,2}\langle \alpha^n, \beta^n \rangle : Z \longrightarrow (X \times Y)^n$, where $\langle \alpha^n, \beta^n \rangle$ is the morphism induced by the universal property of the product. Clearly, the flowing diagram commutes

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "\alpha"] \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & X \times Y \arrow[u, "\mathrm{proj}_X"'] \arrow[d, "\mathrm{proj}_Y"] \\
 & Y.
\end{tikzcd}$$

Showing that γ is unique with this property is routine verification. From the definition of $(X \times Y)f$, for f a function symbol, one can show that if X and

Y factor through $\mathtt{T}$, then so does $X \times Y$. The terminal object of $\mathtt{T}(\mathscr{C})$ is given by the composite $\mathscr{L} \xrightarrow{!} \star \xrightarrow{!} \mathscr{C}$, where $\star$ is the terminal category endowed with the trivial (and only possible) product. □

1.3 Functoriality

Take $\mathtt{T} \in \operatorname{ob} \mathscr{A}$, and a finite product preserving functor $F : \mathscr{C} \longrightarrow \mathscr{D}$, where $\mathscr{C}$ and $\mathscr{D}$ have finite products. Define

$$\begin{aligned} \mathtt{T}(F) : \mathtt{T}(\mathscr{C}) &\longrightarrow \mathtt{T}(\mathscr{D}) \\ X &\longmapsto FX \\ (X \xrightarrow{m} Y) &\longmapsto (FX \xrightarrow{\mathrm{id}_F * m} FY), \end{aligned}$$

where $*$ stands for the Godment product. It is routine verification to show that $\mathtt{T}(F)$ preserve products, and hence $\mathtt{T}$ induces a functor

$$\mathtt{T}(-) : \mathscr{C}at_\times \longrightarrow \mathscr{C}at_\times,$$

where $\mathscr{C}at_\times$ is the (meta)category of categories with finite product. Let $\alpha : F \longrightarrow G$ be a natural transformation, and define $\mathtt{T}(\alpha)_X = \alpha * \mathrm{id}_X : FX \longrightarrow GX$. If $m : X \longrightarrow Y$ is a morphism, then the following diagram commutes :

$$\begin{tikzcd} FX \arrow[r, hook, "\mathtt{T}(\alpha)_X = \alpha * \mathrm{id}_X"] \arrow[d, hook, "\mathtt{T}(F)m = \mathrm{id}_F * m"'] & GX \arrow[d, "\mathtt{T}(G)m = \mathrm{id}_G * m"] \\ FY \arrow[r, "\mathtt{T}(\alpha)_Y = \alpha * \mathrm{id}_Y"] & GY, \end{tikzcd}$$

making $\mathtt{T}(\alpha) : \mathtt{T}(F) \longrightarrow \mathtt{T}(G)$ a natural transformation. Moreover, composition of natural transformations is preserved under this operation. Finally, we obtain that $\mathtt{T}(-)$ is a 2-functor.

2 Tensor product

2.1 Categorical motivation

Take $n, m \in \mathbb{N}$ two integers, and define the *shuffle operation* $\tau_{n,m} \in \mathfrak{S}_{nm}$ to be such that $\tau_{n,m}((i-1)m + j) = (j-1)n + i$, for $1 \le i \le n$ and $1 \le j \le m$. In other word, it "rearranges m tuples of n elements each into n tuples of m elements each".

Take $\mathtt{T} \in \operatorname{ob} \mathscr{A}$, and $f \in f^{(n)}\mathtt{T}$, $g \in f^{(m)}\mathtt{T}$ two function symbols. We say that f *distributes over* g if

$$f \boxtimes g := \quad [fg^n = gf^m \tau_{m,n}]$$

is true in $T_{\mathtt{T}}$.

Take another $\mathtt{U} \in \operatorname{ob} \mathscr{A}$ and define the *tensor product* of $\mathtt{T}$ and $\mathtt{U}$, denoted by $\mathtt{T} \otimes \mathtt{U}$, to be

- $f^{(n)}(\mathtt{T} \otimes \mathtt{U}) = f^{(n)}\mathtt{T} \cup f^{(n)}\mathtt{U}$, where we implicitely distinguish symbols from $\mathtt{T}$ and $\mathtt{U}$;
- $T_{\mathtt{T}\otimes\mathtt{U}} = T_{\mathtt{T}} \cup T_{\mathtt{U}} \cup \{f \boxtimes g \mid f \in f\mathtt{T}, g \in f\mathtt{U}\}$.

Remark that $f \boxtimes g \iff g \boxtimes f$. Hence $\otimes$ is an associative and commutative operation. It can moreover be seen as functorial in each variable.

Theorem 2.1. *There is a finite product preserving isomorphism of categories* $\mathtt{U}(\mathtt{T}(\mathscr{C})) \cong (\mathtt{U} \otimes \mathtt{T})(\mathscr{C})$.

Proof. Take $X \in \operatorname{ob} \mathtt{U}(\mathtt{T}(\mathscr{C}))$ and define

$$\begin{aligned} \check{X} : \mathtt{U} \otimes \mathtt{T} &\longrightarrow \mathscr{C} \\ n &\longmapsto (X1)n \\ f &\longmapsto (Xf)_1 && \forall f \in f\mathtt{U} \\ g &\longmapsto (X1)g && \forall g \in f\mathtt{T}. \end{aligned}$$

Define the so called *deflation functor* :

$$\begin{aligned} (\check{-}) : \mathtt{U}(\mathtt{T}(\mathscr{C})) &\longrightarrow \mathtt{U} \otimes \mathtt{T}(\mathscr{C}) \\ X &\longmapsto \check{X} \\ (X \xrightarrow{\alpha} X') &\longmapsto (\check{X} \xrightarrow{\alpha_1} \check{X}'). \end{aligned}$$

Conversely, take $Y \in \operatorname{ob} \mathtt{U} \otimes \mathtt{T}(\mathscr{C})$. Define

$$\begin{aligned} \hat{Y} : \mathtt{U} &\longrightarrow \mathtt{T}(\mathscr{C}) \\ n &\longmapsto Y^n \iota \\ f &\longmapsto (Yf)^\bullet && \forall f \in f\mathtt{U}, \end{aligned}$$

Where ι is the canonical functor $\mathtt{T} \longrightarrow \mathtt{U} \otimes \mathtt{T}$, and $(Yf)^\bullet$ is the natural transformation with components $(Yf)^n$. Define the so called *inflation functor* :

$$\begin{aligned} (\hat{-}) : \mathtt{U} \otimes \mathtt{T}(\mathscr{C}) &\longrightarrow \mathtt{U}(\mathtt{T}(\mathscr{C})) \\ Y &\longmapsto \hat{Y} \\ (Y \xrightarrow{\beta} Y') &\longmapsto (\hat{Y} \xrightarrow{\beta^\bullet} \hat{Y}'). \end{aligned}$$

Then one can check that $\check{(-)}$ and $\hat{(-)}$ are indeed finite product preserving and mutually inverse. $\square$

Corollary 2.2. *There is a natural isomorphism* $\mathtt{U}(\mathtt{T}(-)) \cong \mathtt{T}(\mathtt{U}(-))$.

2.2 Stability

Take $\mathtt{T} \in \mathrm{ob}\,\mathscr{A}$, denote by $\eta_k : \mathtt{T}^{\otimes k} \longrightarrow \mathtt{T}^{\otimes k+1}$ the canonical morphism, and $U_k = \eta_k^* : \underline{\mathtt{T}^{\otimes k}} \longrightarrow \underline{\mathtt{T}^{\otimes k+1}}$ the forgetful functor.

The theory $\mathtt{T}$ is said *syntactically stable* at $k \in \mathbb{N}$ if η_k is an isomorphism. It is said *semantically stable* at k if U_k is an equivalence of categories.

Proposition 2.3. *If* $\mathtt{T}$ *is syntactically (resp. semantically) stable at* k*, then it remains so at* $k+1$*.*

Proof. Remark that the following diagram commute :

$$\begin{tikzcd} \mathtt{T}^{\otimes k+1} \arrow[r, "\eta_{k+1}"] \arrow[d, equal] & \mathtt{T}^{\otimes k+2} \arrow[d, equal] \\ \mathtt{T} \otimes \mathtt{T}^{\otimes k} \arrow[r, "\mathtt{T} \otimes \eta_k"] & \mathtt{T} \otimes \mathtt{T}^{\otimes k+1}, \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathtt{T}^{\otimes k+2}} \arrow[r, "U_{k+1}"] \arrow[d, equal] & \underline{\mathtt{T}^{\otimes k+1}} \arrow[d, equal] \\ \underline{\mathtt{T} \otimes \mathtt{T}^{\otimes k+1}} \arrow[r, "(\mathtt{T} \otimes \eta_k)^*"] & \underline{\mathtt{T} \otimes \mathtt{T}^{\otimes k}} \\ \mathtt{T}(\underline{\mathtt{T}^{\otimes k+1}}) \arrow[r, "\mathtt{T}(U_k)"] \arrow[u, "\check{(-)}"] & \mathtt{T}(\underline{\mathtt{T}^{\otimes k}}). \arrow[u, "\check{(-)}"'] \end{tikzcd}$$

Hence, η_{k+1} (resp. U_{k+1}) is an isomorphism (resp. equivalence of categories) whenever η_k (resp. U_k) is. $\square$

Define $\|\mathtt{T}\|$ to be the smallest k such that $\mathtt{T}$ is syntactically stable at k, or ∞ if stability never occurs. Define $|\mathtt{T}|$ similarly for semantics. Clearly, if it is syntactically stable, then it is semantically stable at the same k, and hence $|\mathtt{T}| \geq \|\mathtt{T}\|$.

3 The ε_1 ring

Take $\mathtt{T}, \mathtt{U} \in \mathrm{ob}\,\mathscr{A}$, and consider $\mathtt{T} \otimes \mathtt{U}$. Then every function symbol of $\mathtt{T}$ distributes over every function symbol of $\mathtt{U}$, which we shall conveniently denote by $f\mathtt{T} \boxtimes f\mathtt{U}$.

Proposition 3.1. $t\mathtt{T} \boxtimes t\mathtt{U}$.

Proof. • We first show that $f\mathtt{T} \boxtimes t\mathtt{U}$. Take $f \in f^{(n)}\mathtt{T}$ and $u \in t\mathtt{U}$. If u is a function symbol, then $f \boxtimes u$ by hypothesis. If $u : \operatorname{dom} u \longrightarrow 1$ is a projection, then a short computation shows that the result also hold. Proceed now by induction on the height of u, and write $u = g(\overrightarrow{u_i})$, where $g \in f^{(m)}\mathtt{U}$, $u_i \in t\mathtt{U}$. Then by induction hypothesis, $f \boxtimes u_i$, and

$$\begin{aligned} fu^n &= fg^n \left(\prod_i u_i\right)^n = gf^m \tau_{m,n} \left(\prod_i u_i\right)^n = gf^m \left(\prod_i u_i^n\right) \tau_{m,n} \\ &= g\left(\prod_i fu_i^n\right) \tau_{m,n} = g\left(\prod_i u_i f\right) \tau_{m,n} = uf^m \tau_{m,n}. \end{aligned}$$

- Take $t \in t\mathtt{T}$ and $u \in t^{(m)}\mathtt{U}$. If t is a function symbol or a projection, then $t \boxtimes u$. Proceed now by induction on the height of t and write $t = f(\overrightarrow{t_i})$, where $f \in f^{(n)}\mathtt{T}$, $t_i \in t\mathtt{T}$. Then by induction hypothesis, $t_i \boxtimes u$, and

$$\begin{aligned} tu^n &= f \prod_i t_i u = f \prod_i u t_i^m = uf^m \tau_{m,n} \prod_i t_i^m \\ &= uf^m \left(\prod_i t_i\right)^m \tau_{m,n} = ut^m \tau_{m,n}. \end{aligned}$$

□

Consider $\mathtt{CMon}$ the algebraic theory of commutative monoids, with constant symbol 0 and multiplication $\lambda = \lambda_2$. We shall yet again abuse notation and allow 0 to also denote the composite $1 \xrightarrow{!} 0 \xrightarrow{0} 1$. Denote by λ_m the m-fold multiplication term, for $m \geq 2$.

Take $\mathtt{T}, \mathtt{V} \in \operatorname{ob} \mathscr{A}$ such that $\mathtt{V}$ extends $\mathtt{CMon}$, and consider $\mathtt{V} \otimes \mathtt{T}$. Take $t \in t^{(n)}\mathtt{T}$, $n \geq 1$, and define its *i-th axis*, for $1 \leq i \leq n$, by

$$t^{[i]} = t(0^{i-1} \times \operatorname{id} \times 0^{n-i}).$$

In other words, $t^{[i]}(x) = t(\underbrace{0, \cdots, 0}_{i-1}, x, \underbrace{0, \cdots, 0}_{n-i})$.

Theorem 3.2 (Boardman–Vogt decomposition)**.** *Let $t \in t^{(n)}\mathtt{T}$. Then*

$$t = \lambda_n \prod_{i=1}^{n} t^{[i]}.$$

Moreover this decomposition is unique in the following sense: if $t = \lambda_n \prod_{i=1}^n t_i$ for $t_i \in t^{(1)}(\mathtt{V} \otimes \mathtt{T})$, then $t_i = t^{[i]}$.

Proof. We have

$$t = t\prod_{i=1}^{n}\lambda_n(0^{i-1}\times \mathrm{id} \times 0^{n-i}) = t\lambda_n^n\tau_{n,n}\prod_{i=1}^{n}(0^{i-1}\times \mathrm{id} \times 0^{n-i})$$
$$= \lambda_n^n t^n \prod_{i=1}^{n}(0^{i-1}\times \mathrm{id} \times 0^{n-i}) = \lambda_n \prod_{i} t^{[i]}.$$

To prove uniqueness, remark that every term $u \in t^{(n)}(\mathtt{V}\otimes\mathtt{T})$ distributes over 0, i.e. $u0^n = 0$. Then $\forall 1 \leq k \leq n$ we have :

$$\lambda_n \prod_{i=1}^{n} t^{[i]} = \lambda_n \prod_{i=1}^{n} t_i$$
$$\Longrightarrow \underbrace{\lambda_n\left(\prod_{i=1}^{n} t^{[i]}\right)(0^{k-1}\times \mathrm{id} \times 0^{n-k})}_{=t^{[i]}} = \underbrace{\left(\lambda_n\prod_{i=1}^{n} t_i\right)(0^{k-1}\times \mathrm{id} \times 0^{n-k})}_{=t_i}.$$

□

Corollary 3.3. *If* $\mathtt{U} \in \mathrm{ob}\,\mathscr{A}$ *is another theory, then* $\mathtt{V}\otimes\mathtt{T} = \mathtt{V}\otimes\mathtt{U}$ *if and only if* $\mathrm{end}_{\mathtt{V}\otimes\mathtt{T}}\,1 = \mathrm{end}_{\mathtt{V}\otimes\mathtt{U}}\,1$ *as monoids with respect to composition.*

Consider now the case $\mathtt{V} = \mathtt{Ab}$, the theory of abelian groups, and denote by ι the unary function symbol of inversion. Define $\varepsilon_1\mathtt{T} = \mathrm{end}_{\mathtt{Ab}\otimes\mathtt{T}}\,1$. Take $x, y \in \varepsilon_1\mathtt{T}$, and define

$$x + y = \lambda_2(x\times y)\Delta_2$$
$$-x = \iota x.$$

Then $\varepsilon_1\mathtt{T}$, together with those operations and 0 form an abelian group. Endow it further with the composition and id_1, and it becomes a (unitary) ring. Moreover, if $F : \mathtt{T} \longrightarrow \mathtt{U}$ is a morphism of theories, then $\mathtt{Ab}\otimes F : \mathtt{Ab}\otimes\mathtt{T} \longrightarrow \mathtt{Ab}\otimes\mathtt{U}$ induces a ring homomorphism $\varepsilon_1 F : \varepsilon_1\mathtt{T} \longrightarrow \varepsilon_1\mathtt{U}$. This gives rise to a functor

$$\varepsilon_1 : \mathscr{A} \longrightarrow \mathscr{R}ing.$$

Take R a ring, and consider $\mathtt{Mod}_R$ the theory of left R-modules, which extends $\mathtt{Ab}$ with a unary function symbol r, for each element $r \in R$, and with the appropriate axioms. It is routine verifications to show that $\mathtt{Mod}_R\otimes\mathtt{Mod}_S \cong \mathtt{Mod}_{R\otimes S}$, where R and S are tensored as $\mathbb{Z}$-algebras. We will abuse notation and allow R to represent its module theory $\mathtt{Mod}_R$. Remark that the notation $R\otimes S$ leaves no ambiguity then.

Returning to ε_1, remark that $\mathtt{Ab}\otimes\mathtt{T} \cong \varepsilon_1\mathtt{T}$, and so surprisingly enough, tensoring by $\mathtt{Ab}$ result in module theories.

Proposition 3.4. *The functor ε_1 is monoidal.*

Proof. Take $x \in \varepsilon_1\mathtt{T}$ and $y \in \varepsilon_1\mathtt{U}$. Then by distributivity, $xy = yx$ in $\varepsilon_1(\mathtt{T}\otimes\mathtt{U})$. Hence, every element $t \in \varepsilon_1(\mathtt{T}\otimes\mathtt{U})$ decomposes uniquely as $t = \sum_i x_i y_i$, where $x_i \in \varepsilon_1\mathtt{T}$ and $y_i \in \varepsilon_1\mathtt{U}$. One can show that the following map is an isomorphism :

$$\begin{aligned} \varepsilon_1(\mathtt{T}\otimes\mathtt{U}) &\longrightarrow \varepsilon_1\mathtt{T}\otimes\varepsilon_1\mathtt{U} \\ \sum_i x_i y_i &\longmapsto \sum_i x_i \otimes y_i. \end{aligned}$$

□

Theorem 3.5. *If* $\mathtt{T}$ *is syntactically stable at k, then the canonical ring inclusion* $\tilde{\eta}_k : \varepsilon_1\mathtt{T}^{\otimes k} \hookrightarrow \varepsilon_1\mathtt{T}^{\otimes k+1}$ *is an isomorphism. In particular,* $|\varepsilon_1\mathtt{T}| \leq |\mathtt{T}|$.

4 First results about the (un)stability of classical algebraic theories

$\mathtt{T}$	$\lVert\mathtt{T}\rVert_{\mathscr{S}et}$	$\lvert\mathtt{T}\rvert$	$\varepsilon_1\mathtt{T}$
$\mathtt{Mag}_0$	1	1	$\mathbb{Z}$
$\mathtt{Mag}_1$	∞	∞	$\mathbb{Z}[X]$
$\mathtt{Mag}_n, n \geq 1$	?	∞	$\mathbb{Z}\langle X_1, \dots, X_n\rangle$
$\mathtt{CMag}_n, n \geq 1$	?	∞	$\mathbb{Z}[X]$
$\mathtt{SGrp}$	?	∞	$\frac{\mathbb{Z}[X,Y]}{\langle X(X-1), Y(Y-1)\rangle}$
$\mathtt{CSGrp}$	?	∞	$\frac{\mathbb{Z}[X]}{\langle X(X-1)\rangle}$
$\mathtt{Mon}$	2	2	$\mathbb{Z}$
$\mathtt{CMon}$	1	1	$\mathbb{Z}$
$\mathtt{Grp}$	2	2	$\mathbb{Z}$
$\mathtt{Ab}$	1	1	$\mathbb{Z}$
$\mathtt{Mod}_R$	?	(*)	R
$\mathtt{Alg}_R$	2	2	R

where (*) means the number depends on the ring R.

For the rest of this section, let λ, ι and 0 be respectively the (binary) multiplication, inversion and unit of $\mathtt{Ab}$.

$\mathtt{Mon}$, $\mathtt{CMon}$, $\mathtt{Grp}$, and $\mathtt{Ab}$. Take $\mathtt{T}$ to be one of those theories. By the Heckmann–Hilton argument, $\mathtt{Ab}\otimes\mathtt{T} = \mathtt{Ab}$. We then have an isomorphism $\varepsilon_1\mathtt{T} \xrightarrow{\cong} \mathbb{Z}$ mapping id to 1.

$\mathtt{Mag}_0$. Let c be the unique constant symbol of $\mathtt{Mag}_0$. In $\mathtt{Ab} \otimes \mathtt{Mag}_0$ we have $c = 0$ since $c \boxtimes 0$. Hence $\mathtt{Ab} \otimes \mathtt{Mag}_0 = \mathtt{Ab}$ and $\varepsilon_1 \mathtt{Mag}_0 = \mathbb{Z}$.

Then, in $\mathtt{Mag}_0^{\otimes k}$ with $k \geq 1$, all constant symbols are equal, so $\mathtt{Mag}_0^{\otimes k} = \mathtt{Mag}_0$ and $|\mathtt{Mag}_0| = \|\mathtt{Mag}_0\| = 1$.

$\mathtt{Mag}_n$, with $n \geq 1$. For f the unique n-ary function symbol of $\mathtt{Mag}_1$ we have a ring isomorphism

$$\begin{aligned} \varepsilon_1 \mathtt{Mag}_n &\longrightarrow \mathbb{Z}\langle X_1, \ldots, X_n \rangle \\ \mathrm{id} &\longmapsto 1 \\ f^{[i]} &\longmapsto X_i \qquad\qquad 1 \leq i \leq n. \end{aligned}$$

$\mathtt{CMag}_n$, with $n \geq 1$. For $\sigma \in \mathfrak{S}_n$ we have $f = f\sigma$, so in term of axes

$$\lambda_n \prod_i f^{[i]} = \lambda_n \prod_i f^{[\sigma(i)]}, \quad \forall \sigma \in \mathfrak{S}_n.$$

By uniqueness of axes, $f^{[i]} = f^{[j]}$, $\forall i, j$, and so $\varepsilon_1 \mathtt{CMag}_n \cong \mathbb{Z}[X]$.

$\mathtt{SGrp}$. Let m be the binary multiplication symbol of $\mathtt{SGrp}$. Since $m \boxtimes 0$ we have $m(0 \times 0) = 0$. This theory extends $\mathtt{Mag}_2$ only by the associativity axiom $m(m \times \mathrm{id}) = m(\mathrm{id} \times m)$. In term of axes we have

Axis	$m(m \times \mathrm{id})$	$m(\mathrm{id} \times m)$
1	$m^{[1]} m^{[1]}$	$m^{[1]}$
2	$m^{[1]} m^{[2]}$	$m^{[2]} m^{[1]}$
3	$m^{[2]}$	$m^{[2]} m^{[2]}$

Hence,

$$\begin{aligned} \varepsilon_1 \mathtt{SGrp} &= \frac{\varepsilon_1 \mathtt{Mag}_n}{\langle m^{[1]}(m^{[1]} - 1), m^{[2]}(m^{[2]} - 1), m^{[1]} m^{[2]} - m^{[2]} m^{[1]} \rangle} \\ &= \frac{\mathbb{Z}[X, Y]}{\langle X(X-1), Y(Y-1) \rangle}. \end{aligned}$$

$\mathtt{CSGrp}$. This theory extends $\mathtt{SGrp}$ only by the symmetry axiom, so

$$\varepsilon_1 \mathtt{CSGrp} = \frac{\mathbb{Z}[X]}{\langle X(X-1) \rangle}.$$

$\mathtt{Mod}_R$. Let a, i and o be respectively the addition, inversion and zero of $\mathtt{Mod}_R$. By the Heckmann-Hilton argument, $\lambda = a$, $\iota = i$ and $0 = o$ in $\mathtt{Ab} \otimes \mathtt{Mod}_R$. Let f_r be the r-action unary function symbol, for $r \in R$. In $\mathtt{Mod}_R$, f_r already distributes over a, i and o, so tensoring with $\mathtt{Ab}$ doesn't change the underlying $\mathtt{Mod}_R$, and $\varepsilon_1 \mathtt{Mod}_R = R$.

For the syntactical rank, remark that $\mathtt{Mod}_R^{\otimes k+1} = \mathtt{Mod}_R^{\otimes k} \iff R^{\otimes k+1} = R^{\otimes k}$.

$\mathtt{Alg}_R$. This theory extends $\mathtt{Mod}_R$ only by the multiplication m, the one l and the related axioms. Then in $\mathtt{Ab} \otimes \mathtt{Alg}_R$, $o = 0 = l$ by distributivity. Then,

$$m^{[1]} = m(\mathrm{id} \times 0) = m(\mathrm{id} \times o) = o,$$

and similarly for $m^{[2]}$. Hence $\mathtt{Ab} \otimes \mathtt{Alg}_R = \mathtt{Mod}_R$ and $\varepsilon_1 \mathtt{Alg}_R = R$.

Consider now $\mathtt{Alg}_R^{\otimes k}$ with $k \geq 2$. There, all constants are equal, all additions are equal (by Heckmann-Hilton), and all multiplications are identically o, so equal as well. Hence $|\mathtt{Alg}_R| = 2$. Furthermore, for $k \geq 2$, models of $\mathtt{Alg}_R^{\otimes k}$ are such that the additive and multiplicative units are equal, which only allows for the trivial model, and $\|\mathtt{Alg}_R\| = 2$.

References

[Buc08] M. Buckley. Lawvere theories. http://maths.mq.edu.au/s̃treet/MitchB.pdf, 2008.

[BV79] J. M. Boardman and R. M. Vogt. Tensor products of theories, application to infinite loop spaces. *J. Pure Appl. Algebra*, 14(2):117–129, 1979.

[HP07] Martin Hyland and John Power. The category theoretic understanding of universal algebra: Lawvere theories and monads. In *Computation, meaning, and logic: articles dedicated to Gordon Plotkin*, volume 172 of *Electron. Notes Theor. Comput. Sci.*, pages 437–458. Elsevier, Amsterdam, 2007.

[MT08] P.A. Melliès and N Tabareau. Free models of t-algebraic theories computed as kan extensions. *Hal*, 14(hal-00339331), 2008.