Running Non-Local Games on IBM's Cloud Quantum Computers: An Implementation of a mod-n Generalization of CHSH using Qiskit

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Outline

- Motivation
- A Generalization of CHSH
- Implementation Using Qiskit
- Results
- Next Steps

Motivation: Entanglement

- A composite state $|\psi\rangle$ is entangled iff $|\psi\rangle \neq |a\rangle \otimes |b\rangle$ (Nielsen & Chuang, 2010)

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

- Some states are more entangled than others (entanglement entropy)

No Entanglement | Some Entanglement | Maximum Entanglement

$$|00\rangle \qquad \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \qquad \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

- How can we systematically measure the entanglement entropy?

Motivation: Entanglement Entropy (1)

Schmidt Decomposition

Given $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ with both dimension n, there exists orthonormal bases $\{|i_A\rangle\}_{i=0}^{n-1}$ for $\mathcal{H}_A$ and $\{|i_B\rangle\}_{i=0}^{n-1}$ for $\mathcal{H}_B$ and unique non-negative real numbers $\{\lambda_i\}_{i=0}^{n-1}$ such that

$$|\psi\rangle = \sum_{i=0}^{n-1} \lambda_i \, |i_A\rangle |i_B\rangle$$

Motivation: Entanglement Entropy (2)

- Using the Schmidt decomposition, we define the entanglement entropy S_ψ of a state as

$$S_\psi = \sum_{i=0}^{n-1} \lambda_i^2 \log(\lambda_i^2)$$

- If $|a\rangle \otimes |b\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ the maximum entanglement entropy is log(n)

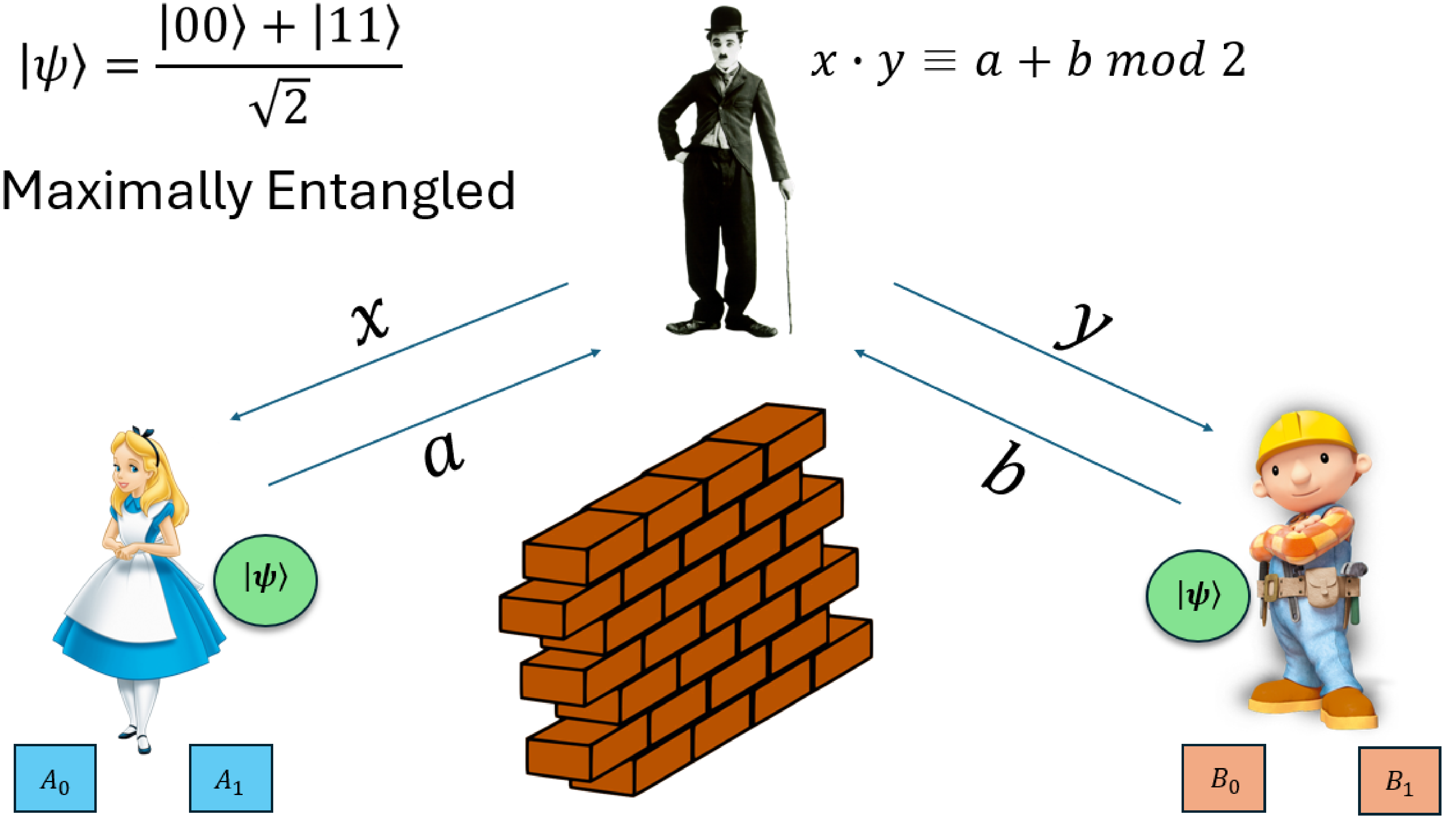
- **Can we make use of states that are not maximally entangled?**

CHSH: The Canonical Game

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Maximally Entangled

$$x \cdot y \equiv a + b \bmod 2$$



Classical Value: 75%

Quantum Value: ~85%

CHSH in the form of an LCS Game

Eq 0: $x_0 x_1 = 1$

$$\iff \quad x \cdot y \equiv a + b \bmod 2$$

Eq 1: $x_0 x_1 = -1$

- Ref sends Alice 0 and Bob 0
  - They win iff they output the **same** values
- Ref sends Alice 0 and Bob 1
  - They win iff they output **different** values
  - Vice-Versa
- Ref sends Alice 1 and Bob 1
  - They win iff they output **same** values

Win conditions are the same as CHSH

A Generalization of CHSH: LCS Game

Let $\omega_n = e^{\frac{2\pi i}{n}}$ and $x_0, x_1 \epsilon \mathbb{Z}_n$.

Eq 0: $x_0 x_1 = 1$

Eq 1: $x_0 x_1 = \omega_n$

- Ref sends Alice an equation (0 or 1)
  - Alice responds with a variable assignment to each variable in her eqn
- Ref sends Bob a variable in Alices eqn (0 or 1)
  - Bob assigns his variable
- Ref verifies Alice's and Bob's responses are consistent

Classical Value for any n: 75%

Win Conditions for mod-n CHSH

Let $\omega_n = e^{\frac{2\pi i}{n}}$ and $a, b \in \mathbb{Z}_n$. The win conditions for each question pair are:

$$V(0,0,a,b) = 1 \iff a = b$$

$$V(0,1,a,b) = 1 \iff ab = 1$$

$$V(1,0,a,b) = 1 \iff a = b$$

$$V(1,1,a,b) = 1 \iff ab = \omega_n$$

When n=2, we have CHSH

A Strategy for mod-n CHSH: The State

- Alice and Bob share the entangled state

$$|\psi_n\rangle = \frac{1}{\gamma_n}\sum_{i=0}^{n-1}(1 - z^{n+2i+1})|\sigma^i(0), \sigma^{-i}(0)\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B,$$

- We only know the state is optimal for $n = 2$ and $n = 3$

$$|\psi_3\rangle = \frac{\sqrt{10}}{10}\left[(1 - e^{\frac{2i\pi}{3}})|00\rangle + 2|12\rangle + (1 - e^{\frac{-2i\pi}{3}})|21\rangle\right]$$

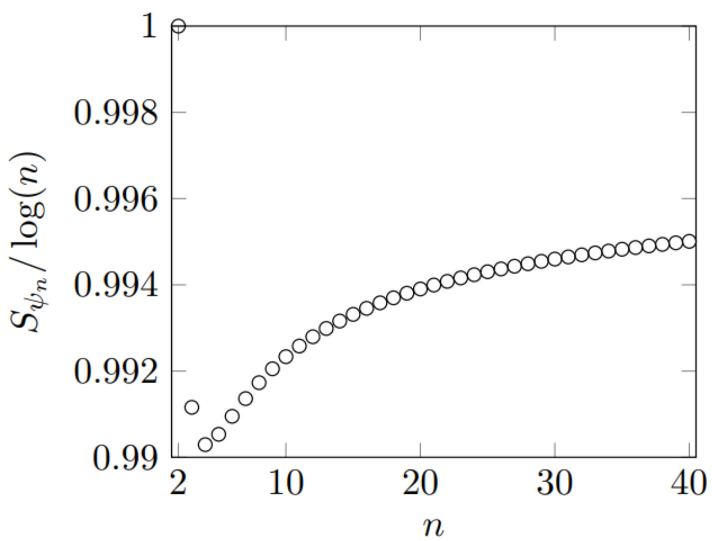


Fig1: Closeness to maximum entanglement of the strategy's state

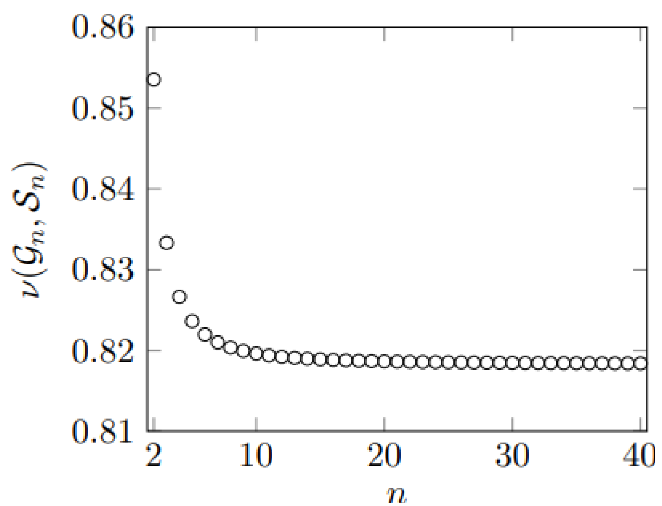


Fig2: Winning probability of the strategy

Optimal Strategy for mod-n CHSH: Observables

- Each observable arises from a PVM $\{E_0, \dots, E_{n-1}\}$

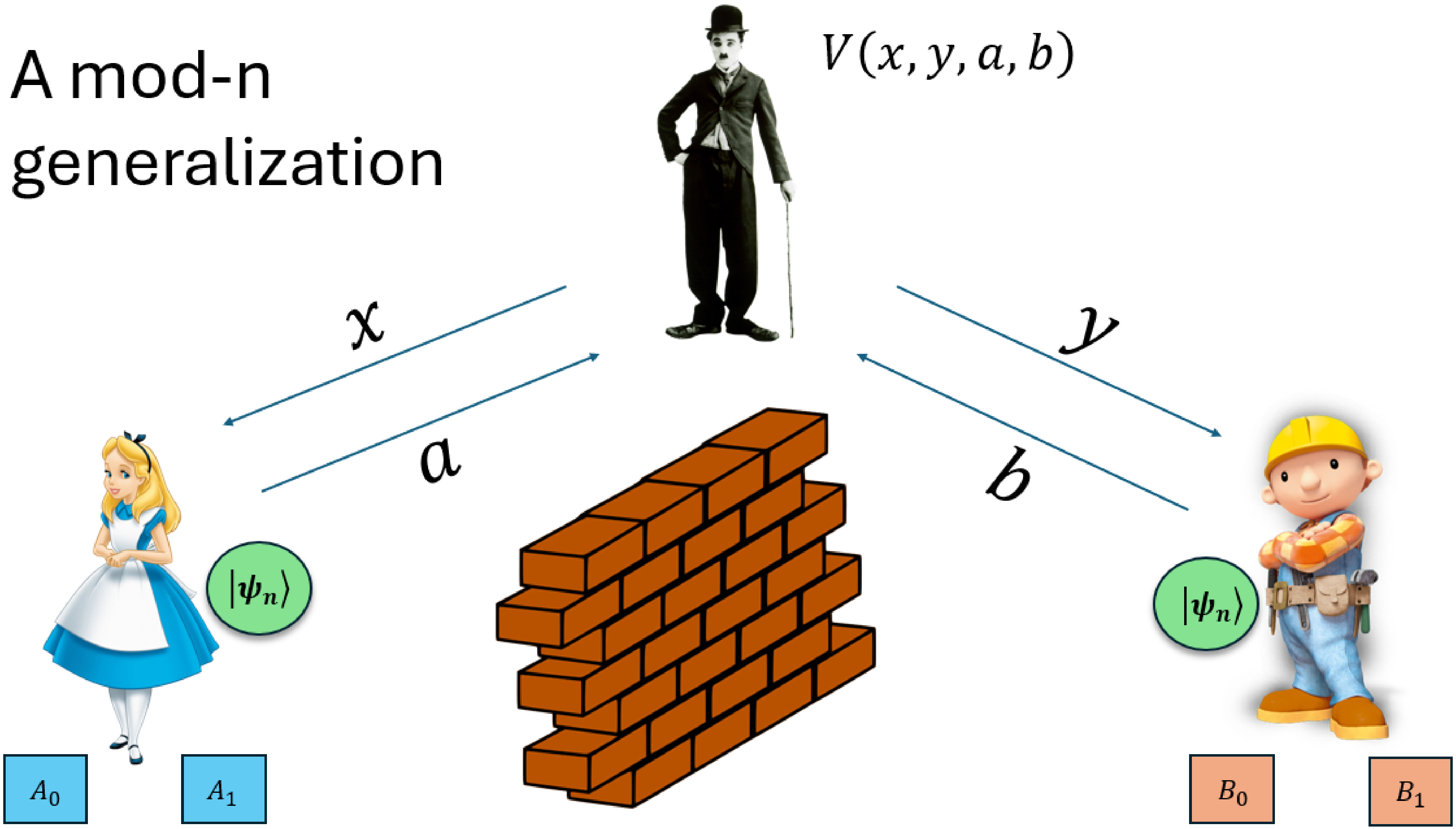
$$O = \sum_{i=0}^{n-1} \omega_n^i \, E_i$$

- Alice and Bob have order-n observables that have eigenvalues from elements of $\mathbb{Z}_n$

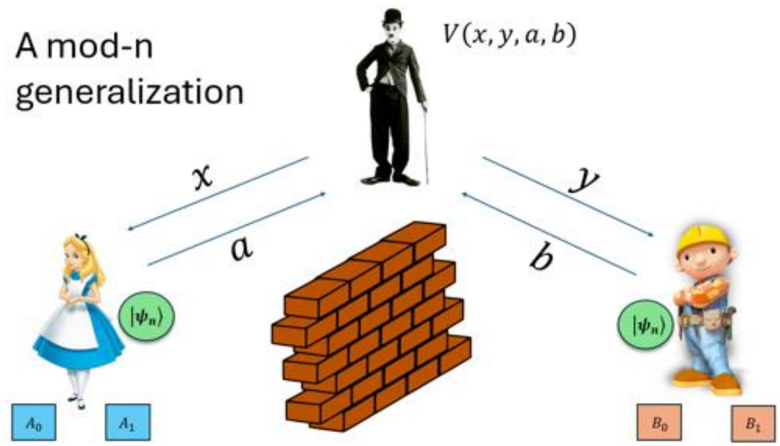
- **The eigenvectors for each eigenvalue form a non-standard measurement basis**

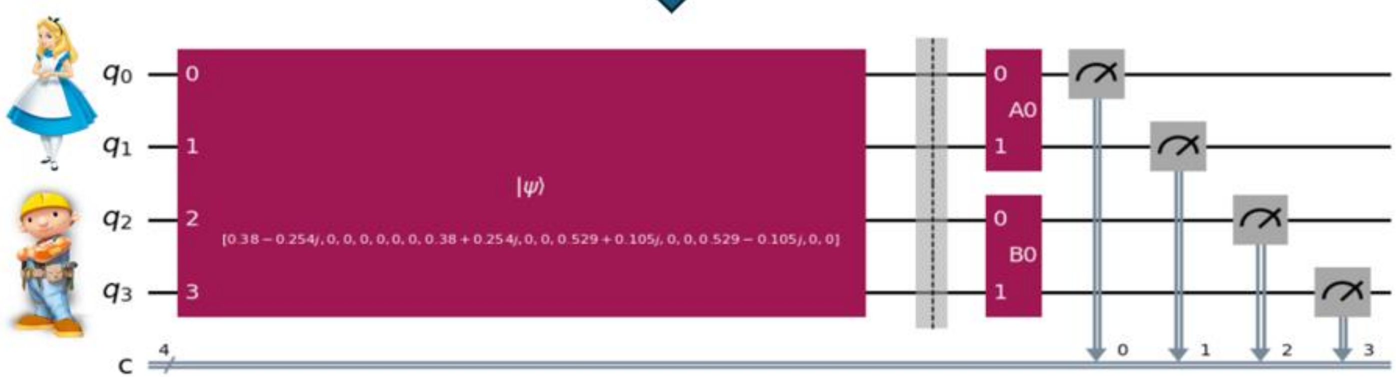
$$A_0 = B_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 0 & -e^{i\pi/3} \\ e^{i\pi/3} & 0 & 0 \\ 0 & e^{i\pi/3} & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 & -e^{i\pi/3} & 0 \\ 0 & 0 & e^{i\pi/3} \\ e^{i\pi/3} & 0 & 0 \end{pmatrix}$$

A mod-n generalization



A mod-n generalization





Implementation Using Qiskit

1. Simulating Qudit states on a Qubit framework

2. Observables into correct Qiskit Gates

3. Converting Alice and Bob measurement results to $\mathbb{Z}_n$

4. Simulating on IBM's Cloud Computers

Simulating Qudit States

- A single Qiskit register only stores qubits
- What if we want 3 or 4 dimensional qudit registers?

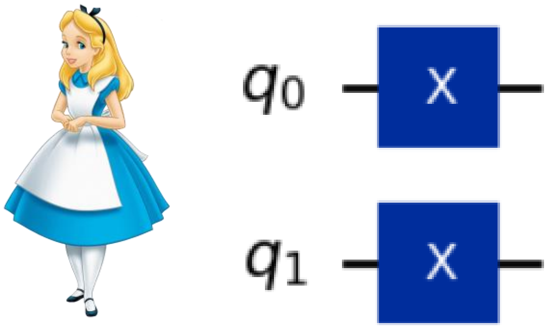
To do this, we use the isomorphism $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$

$$|0\rangle \rightarrow |00\rangle$$

$$|1\rangle \rightarrow |01\rangle$$

$$|2\rangle \rightarrow |10\rangle$$

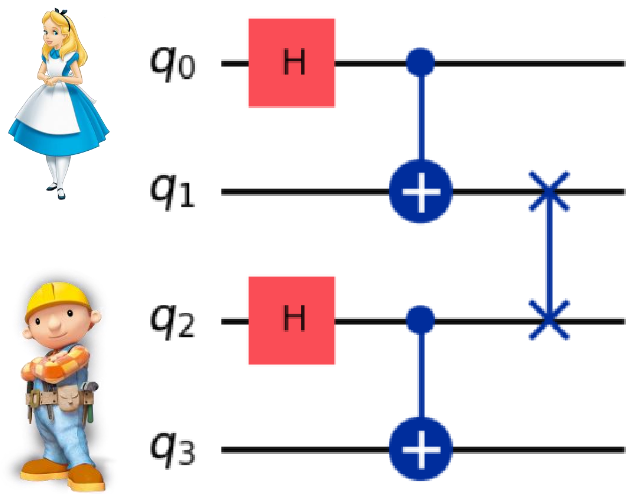
$$|3\rangle \rightarrow |11\rangle$$



$$X \otimes X\,(|00\rangle) = |11\rangle \cong |3\rangle$$

How about Bell State in $\mathbb{C}^4 \otimes \mathbb{C}^4$?

$$\frac{1}{2}\left(|00\rangle + |11\rangle + |22\rangle + |33\rangle\right) \cong \frac{1}{2}\left(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle\right)$$



Observables into correct Qiskit Gates

- Qiskit measures in the standard basis but our observables have non-standard bases

- Let $\{|e_i\rangle\}_{i=0}^{n-1}$ be the standard basis for $\mathbb{C}^n$. Let $\{|d_i\rangle\}_{i=0}^{n-1}$ be an ordered orthonormal set of eigenvectors of an observable. The gate for an observable is

$$U = \sum_{i=0}^{n-1} |e_i\rangle\langle d_i|$$

```
In [8]: A0
```

Out[8]:
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

```
In [9]: get_unitary_from_observable(A0)
```

Out[9]:
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & -\frac{1}{2} & \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

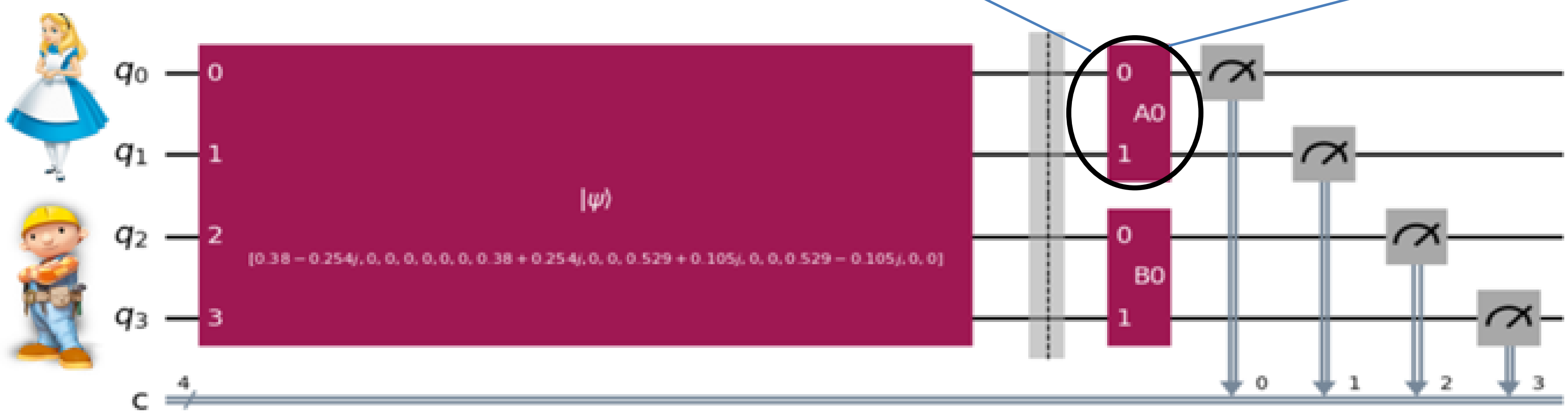
What about gates for qutrit states?

- As of now, the setup only supports registers for qubits with dimensions 2^n
- What if we want to support any dimensional qubit?
  - We add a rows/columns of $(0,0,\ldots0,1)$ to our unitary
  - Add the required zeroes in our state prep

Ex; The n=3 game

$$\begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{6}+\frac{i}{2} & -\frac{\sqrt{3}}{6}-\frac{i}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{6}-\frac{i}{2} & -\frac{\sqrt{3}}{6}+\frac{i}{2} & \frac{\sqrt{3}}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{6}+\frac{i}{2} & -\frac{\sqrt{3}}{6}-\frac{i}{2} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{6}-\frac{i}{2} & -\frac{\sqrt{3}}{6}+\frac{i}{2} & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Converting Alice and Bob measurement results to $\mathbb{Z}_n$

- Qiskit measurement returns a python dictionary of measurement results. We need to convert to $\mathbb{Z}_n$

```
{'1111': 211,
 '0101': 213,
 '1100': 28,
 '0000': 219,
 '0001': 35,
 '1010': 179,
 '1011': 38,
 '0111': 6,
 '1110': 4,
 '0100': 4,
 '0110': 29,
 '0010': 6,
 '0011': 6,
 '1101': 6,
 '1000': 9,
 '1001': 7}
```

Outcome Occurrences

$$\underbrace{1\,0}_{\text{Bob Output}}\;\underbrace{1\,1}_{\text{Alice Output}}$$

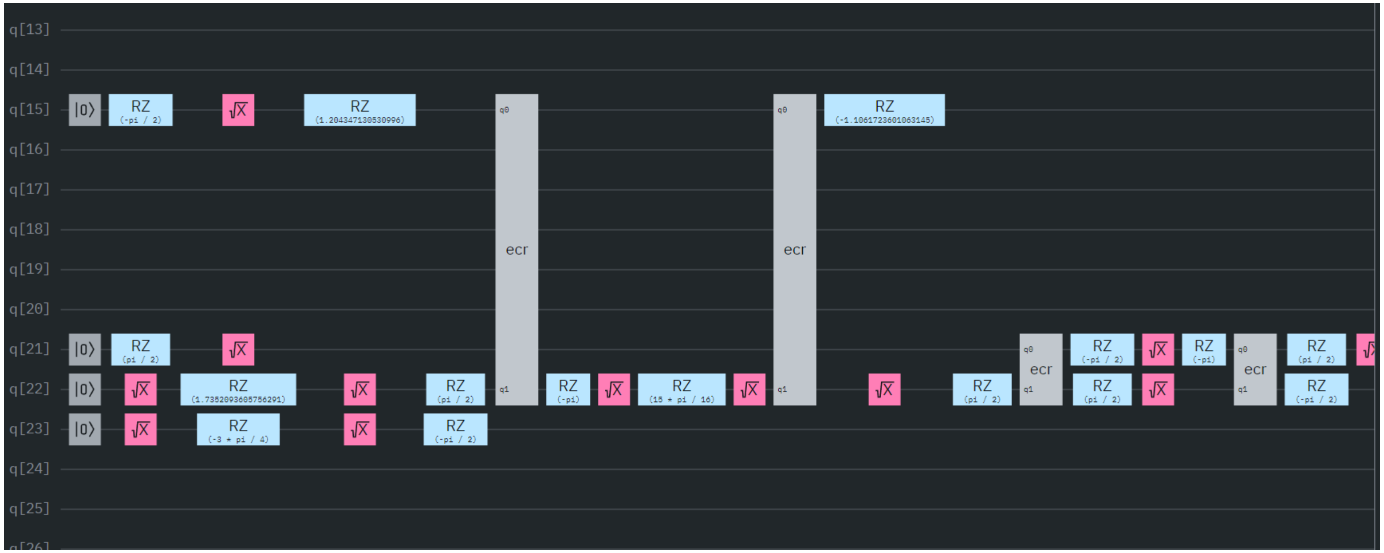
$$-1 \quad -i$$

$$V(x, y, -i, -1)$$

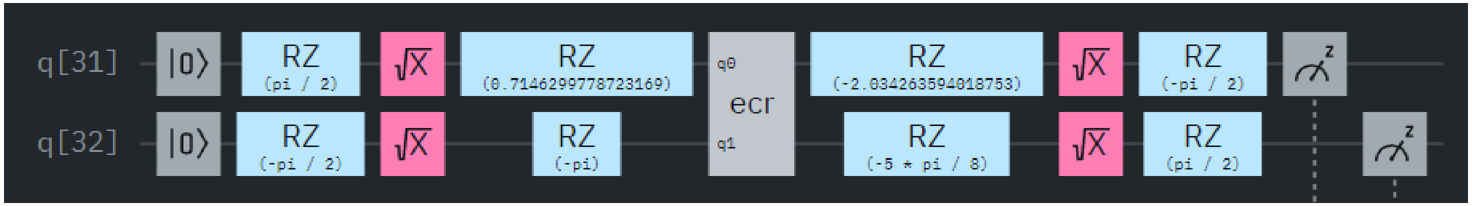
Simulating on IBM's Cloud Computers

- The theoretical simulations yield the correct win percentage for n=2,3,4
- The IBM simulations do not do well on the n=3 and n=4 games

Transpilation of n=4 game (190 gates)



Transpilation of n=2 game (13 gates)



Next Steps

- CMC partnership with them to unspool the state prep
- Make improvements on the math side different state and observables that might be more tolerant of noise
- Generalize the code to play the game for any n
- Search for the observables that give the highest win percentage with the typical Bell State