HW # handwrite
31465306.

Problem #8.3

① prove. $w'_{n+1}(x_i) = \prod_{j \neq i} (x_i - x_j)$ where w_{n+1} is the nodal poly.

$$w'_{n+1}(x_i) = \lim_{x \to x_i} \frac{w_{n+1}(x) - w_{n+1}(x_i)}{x - x_i} = \lim_{x \to x_i} \frac{\prod_{j=0}^{n} (x - x_j) - \cancel{\prod_{j=0}^{n} (x_i - x_j)}}{x - x_i}$$

Since $(x_i - x_j) = 0$ as $j = i$, the whole term equals to zero.

$$w'_{n+1}(x_i) = \lim_{x \to x_i} \frac{\prod_{j=0}^{n} (x - x_j)}{(x - x_i)} = \prod_{j \neq i}^{n} (x_i - x_j)$$

② check (8.5). $\Pi_n(x) = \sum_{i=0}^{n} \frac{w_{n+1}(x)}{(x - x_i)\, w'_{n+1}(x_i)} \cdot y_i$

from definition

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{\prod_{j=0, j \neq i}^{n} (x - x_j)}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)}$$

with part ①

$$l_i(x) = \frac{\prod_{i=0}^{n} (x - x_j)}{x - x_i} \cdot \frac{1}{w'_{n+1}(x_i)} = \frac{w_{n+1}(x)}{(x - x_i)\, w'_{n+1}(x_i)} \quad \text{for } x \neq x_i$$

$$\Pi_n(x) = \sum_{i=0}^{n} l_i(x)\, y_i = \sum_{i=0}^{n} \frac{w_{n+1}(x)}{(x - x_i)\, w'_{n+1}(x_i)} \quad \text{for } x \notin \{x_1, x_2 \dots x_n\}$$

Problem 2

Show that $1 = \sum_{i=0}^{n} l_i(x)$.

Let P be a polynomial as

$$P(x) = \sum_{i=0}^{n} l_i(x) - 1 = \left(\sum_{i=0}^{n} \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \right) - 1$$

Then, we have $P(x_i) = 0, \quad \forall i \in \{0, 1, \dots n\}$

Since P has $n+1$ distinct root, yet P has the degree at most n,
By Fundamental Theorem of algebra, $P(x)$ is constant and $P(x) = 0$.

Then, $1 = \sum_{i=0}^{n} l_i(x)$