hw5, 314653006.

P1 Prove that Heun's method has order 2 w.r.t h.

By hinted, $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h)$, By Heun's method,

$$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - \frac{h}{2}\Big(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\Big)$$

$$= \underbrace{\int_{t_n}^{t_{n+1}} f(s, y)\,ds - \frac{h}{2}\Big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\Big)}_{(E1)} +$$

$$+ \frac{h}{2}\Big(\cancel{f(t_n, y_n)} + f(t_{n+1}, y_{n+1})\Big) \underbrace{- \frac{h}{2}\Big(\cancel{f(t_n, y_n)} + f(t_{n+1}, y_n + hf(t_n, y_n)\Big)}_{(E2)}$$

(E1) is error of numerical integer of trapezoid rule. (9.12)

$a = t_n$, $b = t_{n+1}$, then $E_1 = \frac{-h^3}{12} f''(\xi)$, where $\xi \in [a, b]$.

$$\Rightarrow \boxed{E_1 = O(h^3)}$$

(E2) ① expand the exact solution at t_n,

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + O(h^2)$$

$$= y_n + hf(t_n, y_n) + O(h^2)$$

② let $\tilde{y} = y_n + hf(t_n, y_n)$, with ①

$$\Rightarrow y(t_{n+1}) - \tilde{y} = O(h^2)$$

③ expand $f(t_{n+1}, y_{n+1})$ around $(t_{n+1}, \tilde{y})$

$$f(t_{n+1}, y_{n+1}) = f(t_{n+1}, \tilde{y}) + f_y(t_{n+1}, \tilde{y})(y_{n+1} - \tilde{y}) - O((y_{n+1} - \tilde{y})^2)$$

$$\Rightarrow E_2 = \frac{h}{2}\Big(f(t_{n+1}, y_{n+1}) - f(t_{n+1}, \tilde{y})\Big) = \frac{1}{2}\Big(O(1)\,O(h^2) - O(h^4)\Big)$$

$$\boxed{E_2 = O(h^3)}$$

Therefore, we have $E_1 + E_2 = O(h^3)$.

$$h\tau_{n+1} = O(h^3) \Rightarrow \boxed{\tau_{n+1} = O(h^2)} \Rightarrow \text{order} = 2$$

P2 Prove that Crank-Nicoloson method has order 2 w.r.t h

$$h\tau_{n+1} = y_{n+1} - y_n - \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

$$= \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds - \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

that is, the quadreture error of trapezoid rule (9.12)

with $a = t_n$, $b = t_{n+1}$

$\Rightarrow h\tau_{n+1} = \frac{-h^3}{12} f''(\xi)$ where $\xi \in (a, b)$,

$\Rightarrow \tau_{n+1} = O(h^2)$ $\therefore$ Crank-Nicolson method has order = 2