#5. (12.23) $\|\tau_h\|_h^2 \leq 3\left(\|f\|_h^2 + \|f'\|_{L^2(0,1)}^2\right)$.

$$R_4(x_j+h) = \int_{x_j}^{x_j+h}\left(u'''(t) - u'''(x_j)\right)\frac{(x_j+h-t)^2}{2}\,dt.$$

$$R_4(x_j-h) = -\int_{x_j-h}^{x_j}\left(u'''(t) - u'''(x_j)\right)\frac{(x_j-h-t)^2}{2}\,dt$$

$$T_h(x_j) = \frac{1}{h^2}\left(R_4(x_j+h) + R_4(x_j-h)\right) \quad (12.21)$$

$$\longrightarrow \quad R_4(x_j+h) = \int_{x_j}^{x_j+h}u'''(t)\frac{(x_j+h-t)^2}{2}\,dt - u'''(x_j)\int_{x_j}^{x_j+h}\frac{(x_j+h-t)^2}{2}\,dt.$$

$$\underbrace{\phantom{\int_{x_j}^{x_j+h}u'''(t)\frac{(x_j+h-t)^2}{2}\,dt}}_{①} \qquad \underbrace{\phantom{u'''(x_j)\int_{x_j}^{x_j+h}\frac{(x_j+h-t)^2}{2}\,dt}}_{②}$$

① Let. $du^* = u'''(t) \implies a^* = u''(t)$.

$v = \dfrac{(x_j+h-t)^2}{2} \implies dv = x_j+h-t$

$\implies ① = \dfrac{h^2}{2}u''\Big|_{x_j}^{x_j+h} - \int_{x_j}^{x_j+h}u''(x_j+h-t)$

$② = -\dfrac{h^2}{2}u'''(x_j)$

$$\implies R_4(x_j+h) = -\frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \int_{x_j}^{x_j+h}u''(x_j+h-t)\,dt.$$

Similarly, $R_4(x_j-h) = \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \int_{x_j-h}^{x_j}u''(x_j-h-t)\,dt.$

$\implies T_h(x_j) = -u''(x_j) + \frac{1}{h^2}\left(\int_{x_j}^{x_j+h}u''(t)(x_j+h-t)\,dt - \int_{x_j-h}^{x_j}u''(t)(x_j-h-t)\,dt\right)$

On noted that.

$(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we have.

$$\frac{1}{3}\, \tau_h^2(x_j) = u''(x_j)^2 + \frac{1}{h^4}\left(\int_{x_j}^{x_j+h} u'''(t)(x_j+h-t)\,dt\right)^2 + \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} u'''(t)(x_j-h-t)\,dt\right)^2$$

$$\underbrace{\phantom{\frac{1}{h^4}\left(\int_{x_j}^{x_j+h} u'''(t)(x_j+h-t)\,dt\right)^2}}_{\text{(A)}} \quad \underbrace{\phantom{\frac{1}{h^4}\left(\int_{x_j-h}^{x_j} u'''(t)(x_j-h-t)\,dt\right)^2}}_{\text{(B)}}$$

(A) by Cauchy-Schwarz.

$$\text{(A)} = \frac{h}{h^4}\int_{x_j}^{x_j+h} u'''(t)^2\,dt \int_{x_j}^{x_j+h}(x_j-h-t)^2\,dt$$

$$\leq \frac{1}{3h}\cdot\int_{x_j}^{x_j+h} u'''^2(t)\,dt \qquad \text{since } |x_j-h-t| \leq h \text{ for } h \in [x_j-h, x_j].$$

Similarly.

$$\text{(B)} \leq \frac{1}{3h}\int_{x_j-h}^{x_j} u'''(t)\,dt.$$

$$\Rightarrow. \quad \sum_{j=1}^{n-1} \tau_h(x_j)^2 \leq 3\sum_{j=1}^{n-1} u''(x_j)^2 + \frac{1}{h}\sum_{j=1}^{n-1}\int_{x_j-h}^{x_j} u'''(t)^2\,dt + \frac{1}{h}\sum_{j=1}^{n-1}\int_{x_j}^{x_j+h} u'''(t)^2\,dt.$$

$$\Rightarrow \quad \|\tau_h\|_h^2 \leq h\sum_{j=1}^{n-1}\tau_h(x_j)^2 \leq 3h\sum_{j=1}^{n-1} u''(x_j)^2 + \sum_{j=1}^{n-1}\int_{x_j-h}^{x_j} u'''(t)\,dt + \sum_{j=1}^{n-1}\int_{x_j}^{x_j+h} u'''(t)\,dt.$$

$$\leq 3\|u''\|_h^2 + 3\int_{x_0}^{x_n} u'''(t)^2\,dt.$$

$$= 3\|u''\|_h^2 + 3\|u'''\|_{L^2(0,1)}^2.$$

$$= 3\|f\|_h^2 + 3\|f'\|_{L^2(0,1)}^2 \qquad (-u'' = f) \qquad \square$$

#7. (12.25) $w_n = T_h g$, $w_n = \sum_{k=1}^{n-1} g(x_k) G^k$

$$w_h(x_j) = \sum_{k=1}^{n-1} g(x_k) G^k(x_j) \text{, given } g(x_k) = 1, \text{ and } G^k(x_j) = h\, G(x_j, x_k$$

$$T_h g(x_j) = h\left(\sum_{k=1}^{n-1} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right)$$

$$= h\left((1-x_j)\sum_{k=1}^{j} x_k + x_j\sum_{k=j+1}^{n-1}(1-x_k)\right) \quad\text{---} (*).$$

On uniform grid, $x_k = kh$,

$$(*) = h\left((1-jh)\frac{(1+j)j}{2}h + jh\left((n-j-1) - h\frac{(n-j-1)(n+j)}{2}\right)\right)$$

$(1-jh)(h+jh)$ [marked x_2]

$$= jh\cdot\frac{1}{2}\left(h - j h^2 + jh - j^2 h^2 + 2nh - 2jh - 2h - n^2 h^2 + j^2 h^2 + nh^2 + jh^2\right)$$

$$= \frac{x_j}{2}\left(2nh - jh - h - n^2 h^2 + nh^2\right) = \frac{x_j}{2}\cdot h\cdot(n-j) = \frac{x_j}{2}(1-x_j)\left(\text{take } h = \frac{1}{n}\right)$$

#8 Prove Young's inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, $\forall a, b \in \mathbb{R}$, $\forall \varepsilon > 0$. (12.40)

Consider $0 \le \left(\sqrt{\varepsilon}\, a - \frac{b}{2\sqrt{\varepsilon}}\right)^2 = \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2$.

$\Rightarrow ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$.

#9 Show that $\|v_n\|_h \le \|v_n\|_{h,\infty}$ $\forall v_h \in V_h$.

$$\|v_n\|_h^2 = h\sum_{j=1}^{n-1} v_h^2(x_j), \quad \|v_n\|_{h,\infty}^2 = \max_{1\le j\le n-1} v_n^2$$

By def, $(v_n(x_j))^2 \le (\|v_n\|_{h,\infty})^2$ for all j.

$\Rightarrow h\sum_{j=1}^{n-1}(v_h(x_j))^2 \le h\sum_{j=1}^{n-1}\|v_n\|_{h,\infty}^2 = h\cdot(n-1)\|v_n\|_{h,\infty}^2 \le \|v_n\|_{h,\infty}^2$

$\Rightarrow \|v_n\|_h^2 \le \|v_n\|_{h,\infty}^2 \Rightarrow \|v_n\|_h^2 \le \|v_n\|_{h,\infty}^2$ □

11   the fourth-order $L_h(x) = -u^{(iv)}(x)$ from centered finite difference.

$$L_h w(x_j) = - \frac{w_{j+1} - 2w_j + w_{j-1}}{h^2}$$

$$x_{j+1} = - \frac{w_{j+2} - 2w_{j+1} + w_j}{h^2}$$

$$x_{j-1} = - \frac{w_j - 2w_{j-1} + w_{j-2}}{h^2}$$

$$(L_h w)^2(x_j) = - \frac{L_h w_{j+1} - 2 L_h w_j + L_h w_{j-1}}{h^2}$$

$$= -\frac{1}{h^2} \cdot \left(-\frac{1}{h^2}\right) \left(\underbrace{w_{j+2} - 2w_{j+1} + w_j}_{\Delta} \underbrace{-2 w_{j+1} + 4 w_j - 2 w_{j-1}}_{\Delta} + \underbrace{w_j - 2w_{j-1} + w_{j-2}}_{\Delta} \right)$$

$$= \frac{1}{h^4} \left(w_{j+2} - 4 w_{j+1} + 6 w_j - 4 w_{j-1} + w_{j-2} \right)$$

$$\Rightarrow L_h(x) = -u^{(iv)}(x) = -L_h^2 w(x_j) = \frac{-1}{h^4} \left(w_{j+2} - 4w_{j+1} + 6 w_j - 4 w_{j-1} + w_{j-2} \right)$$