251107, hw
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P1

(12.1) $-u''(x) = f(x), \quad 0 < x < 1$

(12.2) $u(0) = u(1) = 0.$

(12.3) $u(x) = x\int_0^1 (1-s)f(s)\,ds - \int_0^x (x-s)f(s)\,ds.$

with $f(x) = \frac{1}{x}$; prove that $u(x) = -x\log(x)$

Green function is :

$$G(x,s) = \begin{cases} (1-x)s & \text{if } 0<s<x<1 \\ x(1-s) & \text{if } 0<x<s<1 \end{cases}$$

$$\Rightarrow u(x) = \int_0^1 G(x,s)\frac{1}{s}\,ds = \int_0^x (1-x)\,ds + \int_x^1 \frac{x(1-s)}{s}\,ds$$

$$= (1-x)\underbrace{\int_0^x ds}_{=x} + x\int_x^1 \frac{1-s}{s}\,ds \quad \text{——} (*)$$

Notice that

$$\int_x^1 \frac{1-s}{s}\,ds = \int_x^1 \frac{1}{s}\,ds - \int_x^1 ds = -\log x - 1 + x$$

$$\Rightarrow (*) \to u(x) = \cancel{x} - \cancel{x^2} - x\log x + \cancel{x} + \cancel{x^2} = -x\log x.$$

$u(x) = -x\log x$, not define at $x=0$

$u' = -\log x - 1$, not define at $x=0$

$u'' = -\frac{1}{x}$, not define at $x=0$.

Thus, $u \notin C^2(0,1)$, $u(0), u'(0), u''(0)$ not exist.

$\boxed{P2}$

(i) Verify the Sumation by Part Formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = \sum_{j=0}^{n-1} w_{j+1}v_j - \sum_{j=0}^{n-1} w_j v_j$$

$$= \sum_{j=0}^{n} w_{j+1}v_{j+1} + w_0v_0 - w_nv_n.$$

$$= \sum_{j=0}^{n-1}(v_j - v_{j+1})w_{j+1} + w_nv_n - w_0v_0.$$

$$= w_nv_n - w_0v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_j)w_{j+1}$$

(ii) show, for $v_n \in V_n^0$, $(L_nv_n, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$

$$(L_hv_n, v_n)_h = h\sum_{j=1} -L_n v_j \cdot v_j = h\sum_{j=1}^{n-1}\frac{v_{j+1}-2v_j+v_{j-1}}{h^2}v_j$$

$$= -\frac{1}{h}\left(v_2v_1 - 2v_1^2 + v_0v_1 + v_3v_2 - 2v_2^2 + v_2v_1 + \dots v_nv_{n-1} - 2v_{n-1}^2 + v_{n-1}v_{n-2}\right)$$

$$= -\frac{1}{h}\left(-\cancel{v_0}v_1 + \cancel{v_0^2} + \cancel{v_n^2} - \cancel{v_n}v_{n-1} - \sum_{j=0}^{n-1}(v_{j+1}-v_j)^2\right)$$

$$= +h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$$

$\boxed{P5}$ at node x_j: $L_hG^k(x_j) = \frac{1}{h^2}\left(G^k(x_{j+1}) - 2G^k(x_j) + G^k(x_{j-1})\right)$ (*)

Since G is piecewise linear except at $x = x_k$

① if $j \neq k$ $\quad L_hG^k(x_j) = 0.$

② if $j = k$. (*) gives $L_hG_k = -\frac{x_k(1-x_k-h) - 2x_k(1-x_k) + (x_k-h)(1-x_k)}{h^2}$

$$= -\frac{\cancel{x_k} - \cancel{x_k^2} - hx_k - \cancel{2x_k} + \cancel{2x_k^2} - \cancel{x_k^2} + \cancel{x_k} - h + \cancel{hx_k}}{h^2} = \frac{1}{h}$$

Take e^k as k-th standard basis vector,

Then it shows $L_hG^k = e^k \cdot \frac{1}{h}$

$$\Rightarrow G^k(x_j) = hG(x_j, x_k).$$