#7

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad z \in \mathbb{C}$$

$$\Rightarrow \Gamma(z+1) = \int_0^\infty e^{-t} t^{z}\, dt, \quad \text{since } \int e^{-t} = -e^{-t}$$

$$= -e^{-t} t^{z} \Big|_0^\infty - z \int_0^\infty -e^{-t} t^{z-1}\, dt$$

$$= (0+0) + z\,\Gamma(z) = z\,\Gamma(z).$$

#9 $\boxed{\text{test}}$ take $\alpha = 1$, then $u_{n+1} = u_n + h\left(\frac{1}{2} f(x_n, u_n) + \frac{1}{2} f(x_{n+1}, u_{n+1})\right)$

$$y'(x) = -10\, y(x) \Rightarrow f(x_n, u_n) = -10 u_n$$

$$f(x_{n+1}, u_{n+1}) = -10 u_{n+1}$$

$$\Rightarrow u_{n+1} - u_n = \frac{h}{2}\cdot\left(-10\left(u_n + u_{n+1}\right)\right) \Rightarrow (1+5h)\, u_{n+1} = (1-5h)\, u_n$$

$$\Rightarrow u_{n+1} = \frac{1-5h}{1+5h} u_n = \left(\frac{1-5h}{1+5h}\right)^n \cdot y_0$$

Absolute stability requires $\left|\frac{1-5h}{1+5h}\right| < 1$

$\Rightarrow$ It is absolutely stable for all $h > 0$.

#9. $\boxed{\text{study}}$ expand $y(x_{n+1})$ and $f(x_{n+1}, y(x_{n+1}))$ about x_n.

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y_n' f_y + O(h)^3 \quad \text{—①}$$

$$f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + h \frac{\partial f}{\partial x}(x_n, y_n) + h \frac{\partial f}{\partial y}(x_n, y_n)\, y_n' + O(h^2) \quad \text{—②}$$

Subsititute $\Rightarrow$ $T_{n+1} = \underbrace{y_{n+1}}_{①} - y_n - h\left(\left(1-\frac{\alpha}{2}\right) f(x_n, y_n) + \frac{\alpha}{2} \underbrace{f(x_{n+1}, y_{n+1})}_{②}\right)$

$$= \cancel{h y_n'} + \frac{h^2}{2} y_n'' - \cancel{h\left(1-\frac{\alpha}{2}\right) y_n'} - h\frac{\alpha}{2}\left(\cancel{y_n'} + h\left(f_x + f_y y_n'\right)\right) + O(h^3)$$

$$= \frac{h^2}{2}\left(f_x + f_y f\right) - \frac{h^2}{2}\alpha\left(f_x + f_y f\right) + O(h^3) = \frac{h^2}{2}(1-\alpha)\left(f_x + f_y f\right) + O(h^3)$$

So the $T_{n+1} = \begin{cases} O(h^2) & \text{if } \alpha \neq 1 \\ O(h^3) & \text{if } \alpha = 1 \end{cases}$, The method is consistent for any α.