#5. (12.23) $\|\tau_h\|_h^2 \le 3\left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2\right)$.

$$R_4(x_j+h) = \int_{x_j}^{x_j+h} \left(u'''(t) - u'''(x_j)\right) \frac{(x_j+h-t)^2}{2}\, dt.$$

$$R_4(x_j-h) = -\int_{x_j-h}^{x_j} \left(u'''(t) - u'''(x_j)\right) \frac{(x_j-h-t)^2}{2}\, dt$$

$$\tau_h(x_j) = \frac{1}{h^2}\left(R_4(x_j+h) + R_4(x_j-h)\right) \quad (12.21).$$

$$\rightarrow \quad R_4(x_j+h) = \underbrace{\int_{x_j}^{x_j+h} u'''(t) \frac{(x_j+h-t)^2}{2}\, dt}_{①} - \underbrace{u'''(x_j)\int_{x_j}^{x_j+h} \frac{(x_j+h-t)^2}{2}\, dt}_{②}.$$

① Let. $du^* = u'''(t)$ $\Rightarrow$ $u^* = u''(t)$.

$$V = \frac{(x_j+h-t)^2}{2} \Rightarrow dv = x_j+h-t$$

$$\Rightarrow ① = \frac{h^2}{2} u'' \Big|_{x_j}^{x_j+h} - \int_{x_j}^{x_j+h} u''(x_j+h-t).$$

$$② = -\frac{h^2}{2} u'''(x_j)$$

$$\Rightarrow R_4(x_j+h) = -\frac{h^2}{2} u''(x_j) - \frac{h^3}{6} u'''(x_j) + \int_{x_j}^{x_j+h} u''(x_j+h-t)\, dt.$$

Similarly, $R_4(x_j-h) = \frac{h^2}{2} u''(x_j) - \frac{h^3}{6} u'''(x_j) + \int_{x_j-h}^{x_j} u''(x_j-h-t)\, dt.$

$$\Rightarrow \tau_h(x_j) = -u''(x_j) + \frac{1}{h^2}\left(\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\, dt - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt\right)$$

On noted that.

$(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we have.

$$\frac{1}{3}\tau_h^2(x_j) = u''(x_j)^2 + \underbrace{\frac{1}{h^4}\left(\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt\right)^2}_{A} + \underbrace{\frac{1}{h^4}\left(\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt\right)^2}_{B}$$

A by Cauchy-Schwarz.

$$A \le \frac{h}{h^4}\int_{x_j}^{x_j+h} u''(t)^2\,dt \int_{x_j}^{x_j+h} (x_j-h-t)^2\,dt$$

$$\le \frac{1}{3h}\int_{x_j}^{x_j+h} u''^2(t)\,dt \quad \text{since } |x_j-h-t| \le h \text{ for } h\in[x_j-h, x_j].$$

Similarly.

$$B \le \frac{1}{3h}\int_{x_j-h}^{x_j} u''(t)\,dt.$$

$$\Rightarrow \sum_{j=1}^{n-1} \tau_h(x_j)^2 \le 3\sum_{j=1}^{n-1} u''(x_j)^2 + \frac{1}{h}\sum_{j=1}^{n-1}\int_{x_j-h}^{x_j} u''(t)^2\,dt + \frac{1}{h}\sum_{j=1}^{n-1}\int_{x_j}^{x_j+h} u''(t)^2\,dt.$$

$$\Rightarrow \|\tau_h\|_h^2 \le h\sum_{j=1}^{n-1}\tau_h(x_j)^2 \le 3h\sum_{j=1}^{n-1} u''(x_j)^2 + \sum_{j=1}^{n-1}\int_{x_j-h}^{x_j} u''(t)\,dt + \sum_{j=1}^{n-1}\int_{x_j}^{x_j+h} u''(t)\,dt.$$

$$\le 3\|u''\|_h^2 + 3\int_{x_0}^{x_n} u''(t)^2\,dt.$$

$$= 3\|u''\|_h^2 + 3\|u''\|_{L^2(0,1)}^2.$$

$$= 3\|f\|_h^2 + 3\|f\|_{L^2(0,1)}^2 \quad (-u''=f) \qquad \square$$

\# 7. (12.25) $w_h = T_h g$, $w_h = \sum_{k=1}^{n-1} g(x_k) G^k$

\# 6.

$w_k(x_j) = \sum_{k=1}^{n-1} g(x_k) G^k(x_j)$, given $g(x_k) = 1$, and $G^k(x_j) = hG(x_j, x_k)$

$$T_h g(x_j) = h\left(\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right)$$

$$= h\left((1-x_j)\sum_{k=1}^{j} x_k + x_j \sum_{k=j+1}^{n-1}(1-x_k)\right) \quad \text{—— } (*)$$

On uniform grid, $x_k = kh$,

$$(*) = \cancel{h}\left(\underbrace{(1-jh)\frac{(1+j)j}{2}h}_{(1-jh)(h+jh)} + jh\left((n-j-1) - h\,\frac{\overset{n^2-j^2-n-j}{(n-j-1)(n+j)}}{2}\right)\right)$$

$$= jh \cdot \frac{1}{2}\left(h - \cancel{jh^2} + jh - \cancel{j^2h^2} + 2nh - 2jh - 2h - n^2h^2 + \cancel{j^2h^2} + nh^2 + \cancel{jh^2}\right)$$

$$= \frac{x_j}{2}\left(2nh - jh - \cancel{h} - n^2h^{\cancel{2}} + \cancel{nh^2}\right) = \frac{x_j}{2} \cdot h(n-j) = \frac{x_j}{2}(1-x_j) \quad \left(\text{take } h = \frac{1}{n}\right)$$

\# 8 Prove Young's inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, $\forall a, b \in \mathbb{R}$, $\forall \varepsilon > 0$ (12.40)

Consider $0 \le \left(\sqrt{\varepsilon}\, a - \frac{b}{2\sqrt{\varepsilon}}\right)^2 = \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2$

$$\Rightarrow \quad ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$$

\# 9 Show that $\|v_h\|_h \le \|v_h\|_{h,\infty}$ $\forall v_h \in V_h$.

$\|v_h\|_h^2 = h\sum_{j=1}^{n-1} v_h^2(x_j)$, $\|v_h\|_{h,\infty}^2 = \max_{1\le j\le n-1} v_h^2$

By def, $|v_h(x_j)|^2 \le (\|v_h\|_{h,\infty})^2$ for all j.

$$\Rightarrow \quad h\sum_{j=1}^{n-1}(v_h(x_j))^2 \le h\sum_{j=1}^{n-1} \|v_h\|_{h,\infty}^2 = h\cdot(n-1)\,\|v_h\|_{h,\infty}^2 \le \|v_h\|_{h,\infty}^2$$

$$\Rightarrow \quad \|v_h\|_h^2 \le \|v_h\|_{h,\infty}^2 \quad \Rightarrow \quad \|v_h\|_h \le \|v_h\|_{h,\infty} \quad \square$$

#11 the fourth-order $L_h(x) = -u^{(iv)}(x)$ from centered finite difference.

$$L_h w(x_j) = -\frac{w_{j+1} - 2w_j + w_{j-1}}{h^2}$$

$$x_{j+1} = -\frac{w_{j+2} - 2w_{j+1} + w_j}{h^2}$$

$$x_{j-1} = -\frac{w_j - 2w_{j-1} + w_{j-2}}{h^2}$$

$$(L_h w)^2(x_j) = -\frac{L_h w_{j+1} - 2L_h w_j + L_h w_{j-1}}{h^2}$$

$$= -\frac{1}{h^2}\cdot\left(-\frac{1}{h^2}\right)\left(w_{j+2} - 2w_{j+1} + w_j - 2w_{j+1} + 4w_j - 2w_{j-1} + w_j - 2w_{j-1} + w_{j-2}\right)$$

$$= \frac{1}{h^4}\left(w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}\right)$$

$$\Rightarrow L_u(x) = -u^{(iv)}(x) = -L_h^2 w(x_j) = \frac{-1}{h^4}\left(w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}\right)$$