314653006, hw#2.

P8.5

prove that

$$(n-1)!\,h^{n-1}|(x-x_{n-1})(x-x_n)| \le |w_{n+1}(x)| \le n!\,h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1=x_0<x_1<\cdots<x_n=1$, $x\in(x_{n-1},x_n)$, $h=2/n$.

$w_{n+1}(x)=\prod_{i=0}^{n}(x-x_i)$, Let $N=n/2$, then $w_{n+1}(x)=\prod_{i=0}^{n}(x-(i-N)h)$

Since $x\in(x_{n-1},x_n)$, that is, $x\in((N-1)h, Nh)$.

$i=0$, $\quad (n-1)h \le |x-Nh| \le nh$.

$i=1$. $\quad (n-2)h \le |x-(1-N)h| \le (n-1)h$.

$\vdots$

$i=n-2$. $\quad h \le |x-(N-2)h| \le 2h$.

product of above inequation make

$$(n-1)!\,h^{n-1} \le \frac{|w_{n+1}(x)|}{|x-x_{n-1}||x-x_n|} \le n!\,h^{n-1}$$

Thus, $(n-1)!\,h^{n-1}|x-x_{n-1}||x-x_n| \le |w_{n+1}(x)| \le n!\,h^{n-1}|x-x_{n-1}||x-x_n|$

P8.6

Show that $|w_{n+1}|$ is maximum if $x\in(x_{n-1},x_n)$,

Consider a sequence $S_i=\{|w_{n+1}(x+ih)|,\ i\in\mathbb{Z}, i\in[0,N-1]\}$ that is $|w_{n+1}(x)|, |w_{n+1}(x+h)| \cdots |w_{n+1}(x+(N-1)h)|$, $\forall x\in(0,h)$

$$\left|\frac{S_{i+1}}{S_i}\right| = \left|\frac{w_{n+1}(x+(i+1)h)}{w_{n+1}(x+ih)}\right| = \left|\frac{(x+(N+i+1)h)(x+\cancel{(N+i)h})\cdots(x-\cancel{(N-i-1)h})}{(x+\cancel{(N+i)}h)(x+(\cancel{N+i}-1)h)\cdots(x-(N-i)h)}\right|$$

$$= \left|\frac{x+(N+i+1)h}{x-(N-i)h}\right| > 1,$$ which shows $\{S_i\}$ is strictly increasing.

Thus, maximum of $\{S_i\}$ is $|w_{n+1}(x+(N-1)h)|$, $\forall x\in(0,h)$, or saying $|w_{n+1}(x^*)|$ $\forall x^*\in(x_{n-1},x_n)$

And since $|w_{n+1}|$ is even, maximum of $|w_{n+1}(x)|$ for $x\in(x_0,x_n)$ happen when $x\in(x_{n-1},x_n)$, or $x\in(x_0,x_1)$

$\boxed{P\,8.8}$ Determine an interpolating poly $Hf \in \mathbb{P}_n$ s.t.

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, \dots, n$$, and check

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j .$$

From (8.32), the form of Hermite Interpolation is

$$H_{N-1}(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i} y_i^{(k)} L_{ik}(x),$$

since there is only one node x_0, Determine

$$Hf(x) = \sum_{j=0}^{n} f^{(j)}(x_0) L_j(x) \quad \text{where } L_j = \frac{(x - x_0)^j}{j!}$$

check the derivative conditions

$$(Hf)^{k}(x_0) = \sum_{j=k}^{n} f^{(j)}(x_0) \frac{\frac{j!}{(j-k)!}(x - x_0)^{j-k}}{j!}$$

And easily find that for $x = x_0$,

$$(x - x_0)^{j-k} = \begin{cases} 0, & \text{if } j > k \\ 1, & \text{if } j = k \end{cases}$$

$$\Rightarrow (Hf)^{k}(x_0) = \frac{f^{(k)}(x_0)}{0!} \cdot 1 = f^{(k)}(x_0). \quad \text{Done.}$$

(assignment #2).

Show that, for $n+1$ Chebyshev points of second kind, the barycentric weights are

$$\begin{cases} w_0 = 1/2, \\ w_i = (-1)^i, \quad i = 1, \dots, n-1 \\ w_n = (-1)^n/2. \end{cases}$$

From assignment in week 1, we have the weight

$$w_i = \frac{1}{\omega'_{n+1}(x_i)}$$, where $\{x_i\}$ is set of Chebyshev 2nd kind point.

From lecture note, we have

$$W_{n+1}(x) = (x-1)(x+1)\frac{U_{n-1}(x)}{2^{n-1}}$$

$$= \frac{-1}{2^{n-1}} \sin(n\cos^{-1}(x))\sin(\cos^{-1}(x))$$, let $x=\cos\theta$, $\theta\in[0,\pi]$.

① for $x \in \mathbb{Z} \cup [1, n-1]$

$$\frac{d}{dx}W_{n+1}(x) = \frac{d}{d\theta}\left(\frac{-1}{2^{n-1}}\sin(n\theta)\sin\theta\right)\frac{d\theta}{dx}$$

$$= \frac{-1}{2^{n-1}}\left(n\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta\right)\cdot\frac{1}{-\sin\theta}$$

$$= \frac{1}{2^{n-1}}\left(n\cos(n\theta) + \sin(n\theta)\cdot\cot\theta\right)$$

take $\theta = \frac{i}{n}\pi$, the second term with $\sin(i\pi)$ always equal to 0.

then, $W'_{n+1}(x) = \frac{n}{2^{n-1}}(-1)^i \Rightarrow w_i = \frac{2^{n-1}}{n}(-1)^i$ for $i=1,2\dots n-1$

② for $i=0$ $\Rightarrow$ $\theta=0$, $x=1$, since $\sin\theta=0$ at the point,

use definition of derivative

$$W'_{n+1}(x) = \lim_{\theta\to0}\frac{\frac{-1}{2^{n-1}}\sin(n\theta)\sin(\theta) - 0}{\cos\theta - 1} \overset{(L'H)}{=} \lim_{\theta\to0}\frac{\frac{-1}{2^{n-1}}\left(n\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta\right)}{-\sin\theta}$$

$$= \frac{1}{2^{n-1}}\left(n\cdot\cos(0) + \cos(0)\cdot\lim_{\theta\to0}\frac{\sin(n\theta)}{\sin\theta}\right) = \frac{2n}{2^{n-1}} = \frac{n}{2^{n-2}}$$

$$\Rightarrow w_0 = \frac{2^{n-2}}{n}$$

③ for $i=n$, similar with ②

$$W'_{n+1}(n) = \lim_{\theta\to\pi}\frac{\frac{-1}{2^{n-1}}\left(n\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta\right)}{-\sin\theta}$$

$$= \frac{1}{2^{n-1}}\left(n\cdot\cos(n\pi) + \cos(\pi)\lim_{\theta\to0}(-1)^n\cdot\frac{\sin(n\theta)}{\sin\theta}\right) = \frac{2n}{2^{n-1}}(-1)^n = \frac{n}{2^{n-2}}(-1)^n$$

$$\Rightarrow w_n = \frac{2^{n-2}}{n}(-1)^n$$

Rescaling ①, ②, ③ by multiply with $\frac{n}{2^{n-1}}$, then we have

$$\begin{cases} w_0 = \frac{1}{2} \\ w_i = (-1)^i, \text{ for } i=1,2\dots n-1 \\ w_n = \frac{1}{2}(-1)^n \end{cases}$$