HW3
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P9.1 Let $E_0(f)$ and $E_1(f)$ be quadrature errors in (9.6) and (9.12).
Prove $|E_1(f)| \simeq 2|E_0(f)|$

(9.6). $E_0(f) = \frac{h^3}{3} f''(\xi)$, $h = \frac{b-a}{2}$ $\Rightarrow E_0(f) = \frac{h^3}{24} f''(\xi_0)$

(9.12) $E_1(f) = -\frac{h^3}{12} f''(\xi)$, $h = b-a$. $\Rightarrow E_1(f) = \frac{h^3}{-12} f''(\xi_1)$

compare the value, write $2|E_0(f)| \simeq |E_1(f)|$ due to difference of ξ_0 and ξ_1

P9.3 Compare $I_2(f)$ and $\int_{-1}^{1} f(x)\,dx$

(a) $f(x) = x^0$, $I_2(f) = \frac{2}{3} \cdot 3 = \boxed{2}$, $\int_{-1}^{1} f(x)\,dx = \boxed{2}$ (equal)

$f(x) = x^1$, $I_2(f) = \frac{2}{3} \cdot (-1 - 0 + 1) = \boxed{0}$, $\int_{-1}^{1} f(x)\,dx = \boxed{0}$ (equal)

$f(x) = x^2$, $I_2(f) = \frac{2}{3}\left(\frac{1}{2} - 0 + \frac{1}{2}\right) = \boxed{\frac{2}{3}}$, $\int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \boxed{\frac{2}{3}}$ (equal)

$f(x) = x^3$, $I_2(f) = \frac{2}{3}\left(-\frac{1}{4} - 0 + \frac{1}{4}\right) = \boxed{0}$, $\int_{-1}^{1} x^3\,dx = \boxed{0}$ (equal)

$f(x) = x^4$, $I_2(f) = \frac{2}{3}\left(\frac{1}{8} - 0 + \frac{1}{8}\right) = \boxed{\frac{1}{6}}$, $\int_{-1}^{1} x^4\,dx = \frac{1}{5}x^5\Big|_{-1}^{1} = \boxed{\frac{2}{5}}$ (not equal)

So, the degree of exactness $r = 3$

(b) $f(x) = x^0$, $I_4(f) = \frac{1}{4} \cdot 8 = \boxed{2}$, $\int_{-1}^{1} f(x)\,dx = \boxed{2}$ (equal)

$f(x) = x^1$, $I_4(f) = \frac{1}{4} \cdot 0 = \boxed{0}$, $\int_{-1}^{1} f(x)\,dx = \boxed{0}$ (equal)

$f(x) = x^2$, $I_4(f) = \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{3} + 1\right) = \boxed{\frac{2}{3}}$, $\int_{-1}^{1} f(x)\,dx = \boxed{\frac{2}{3}}$ (equal)

$f(x) = x^3$, $I_4(f) = \boxed{0}$, $\int_{-1}^{1} f(x)\,dx = \boxed{0}$ (equal)

$f(x) = x^4$, $I_4(f) = \frac{1}{4}\left(1 + \frac{1}{27} + \frac{1}{27} + 1\right) = \boxed{\frac{14}{27}}$, $\int_{-1}^{1} f(x)\,dx = \boxed{\frac{2}{5}}$ (not equal)

degree of exactness $r = 3$

$\rightarrow$ Order of infinitesemal, I_2 and I_4 are both close Newton-Cotes formulae, (yet I_4 is actually with 4 nodes that $n=3$)

By theorem 9.2, the order of infinitesemal of I_2 and I_4 are $p = 5$.

P9.5 $I_w(f) = \int_0^1 w(x) f(x)\, dx$ with $w(x) = \sqrt{x}$, consider $Q(f) = a f(x_1)$

Find a and x_1 s.t. Q has max degree of exactness r.

$r=0$ $\quad I_w(f) = \int_0^1 x^{\frac{1}{2}} dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \boxed{\frac{2}{3}}$, $Q(f) = \boxed{a} \Rightarrow a = \frac{2}{3}$

$r=1$ $\quad I_w(f) = \int_0^1 x^{\frac{3}{2}} dx = \frac{2}{5} x^{\frac{5}{2}} \Big|_0^1 = \boxed{\frac{2}{5}}$ $\quad Q(f) = \boxed{\frac{2}{3} \cdot x_1} \Rightarrow x_1 = \frac{3}{5}$

$r=2$ $\quad I_w(f) = \int_0^1 x^{\frac{5}{2}} dx = \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}$, $Q(f) = \frac{2}{3} \cdot \left(\frac{3}{5}\right)^2 = \frac{6}{25}$ (not match)

Thus, $r=1$, $a = \frac{2}{3}$, $x_1 = \frac{3}{5}$

P9.6 Consider $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for

$$I(f) = \int_0^1 f(x)\, dx, \quad f \in C^1([0,1])$$

find $\alpha_1, \alpha_2, \alpha_3$ that Q has degree of exactness $r=2$.

$f(x) = 1$, $\quad I(f) = 1$, $Q(f) = \alpha_1 + \alpha_2$.

$f(x) = x$, $\quad I(f) = \frac{1}{2}$, $Q(f) = \alpha_2 + \alpha_3$.

$f(x) = x^2$, $\quad I(f) = \frac{1}{3}$, $Q(f) = \alpha_2$

By def of degree of exactness.

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_2 + \alpha_3 = \frac{1}{2} \\ \alpha_2 = \frac{1}{3} \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{2}{3} \\ \alpha_2 = \frac{1}{3} \\ \alpha_3 = \frac{1}{6} \end{cases}$$