From multiple degree-of-freedom to distributed systems: linear strings and sound synthesis

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Outline

Multiple degree-of-freedom systems

- Equations of motion
- Simple example
- Normal modes and frequencies

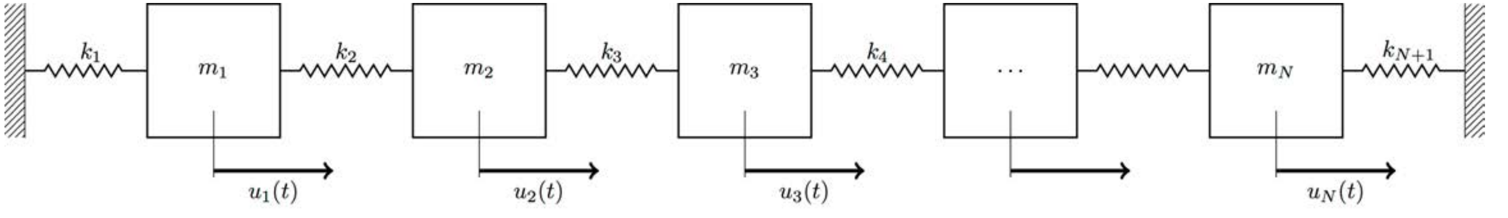
Sound synthesis applications

- Modal approach
- Finite difference approach

Multiple degree-of-freedom systems

Mass-spring system with N DOFs

Consider an array of N masses connected by $N+1$ springs, as in the following figure.



We wish to calculate the equation of motion of the system and do some experiments with it.

Deriving the equations of motion

Elegant approach by **Prof. Gilbert Strang**, available online at this link:

http://www.courses.com/massachusetts-institute-of-technology/computational-science-and-engineering-i/2 (http://www.courses.com/massachusetts-institute-of-technology/computational-science-and-engineering-i/2)

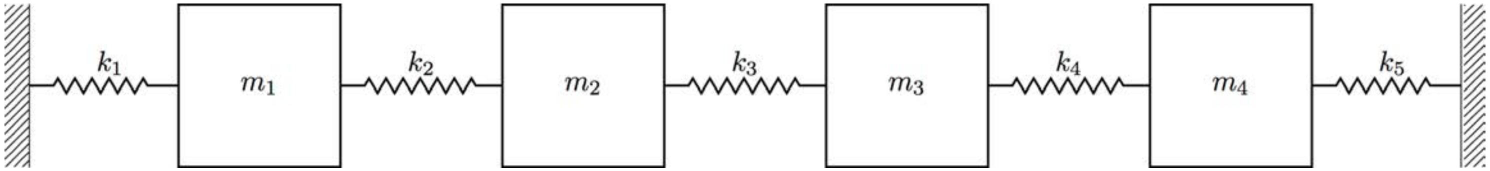
Start from the individual **positions of the masses**, link those to the **elongations of the springs**, which in turn give rise to **forces** determined by Hooke's law. Then, relate these to the **total force** acting on each mass and eventually to their **acceleration**.

Here's a diagram:

$$\vec{u} \rightarrow \vec{e} \rightarrow \vec{f} \rightarrow \vec{F} \rightarrow \ddot{\vec{u}}$$

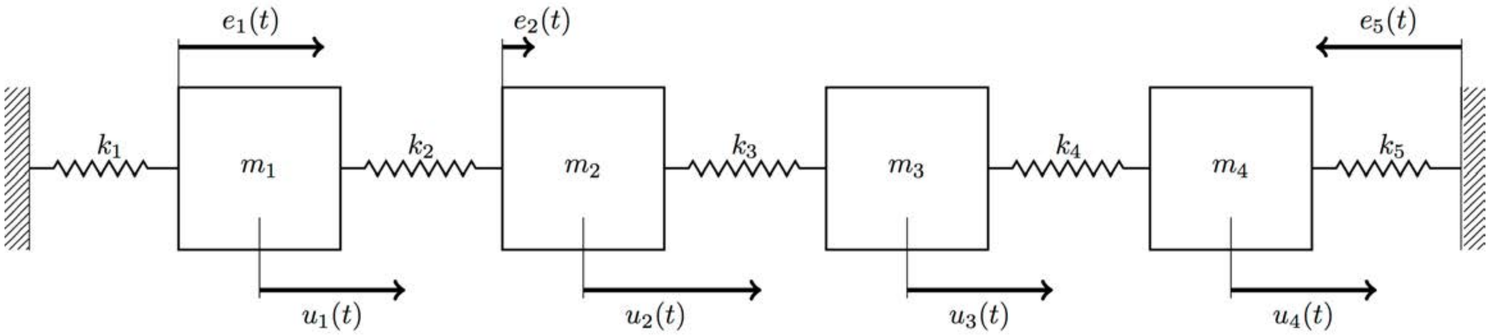
Four steps, where each arrow is actually a matrix...

To start, consider 4 masses and 5 springs, but the argument can easily be generalised.



$$\vec{u} \rightarrow \vec{e} \rightarrow \vec{f} \rightarrow \vec{F} \rightarrow \ddot{\vec{u}}$$

1. Elongations of the springs, $\vec{\mathbf{u}} \rightarrow \vec{\mathbf{e}}$



- $e_1 = u_1$
- $e_2 = u_2 - u_1$
- $\ldots$
- $e_5 = -u_4$

1. Elongations of the springs, $\vec{\mathbf{u}} \to \vec{\mathbf{e}}$

Putting everything in matrix form, one can write:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \implies \vec{e} = A\vec{u},$$

where the matrix A is rectangular (5x4).

2. Hooke's law, $\vec{\mathbf{e}} \to \vec{\mathbf{f}}$

Relate the elongations to the forces acting on the masses using Hooke's law: for each elongation e_j, a recall force f_j proportional to e_j is generated. Mathematically,

$$f_j = -k_j e_j,$$

where k_j is the stiffness constant of the j-th spring.

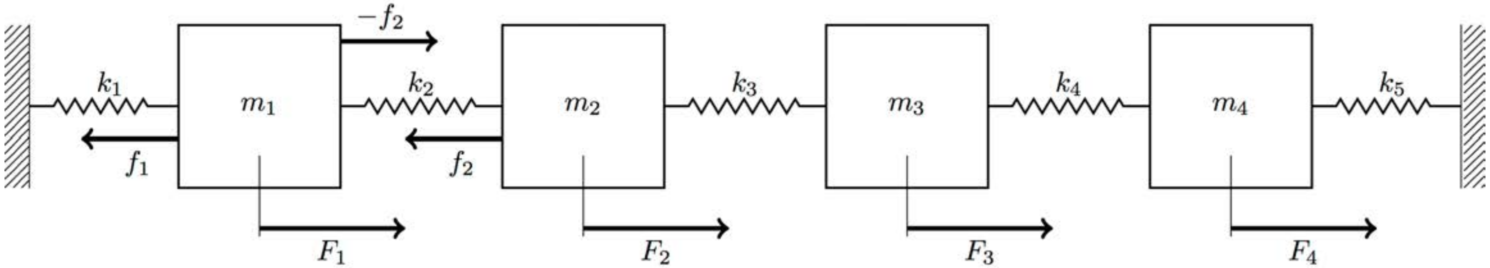
2. Hooke's law, $\vec{\mathbf{e}} \rightarrow \vec{\mathbf{f}}$

In matrix form, this becomes

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = -\begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} \quad \Longrightarrow \quad \vec{f} = -C\,\vec{e}$$

where C is the diagonal square matrix (5x5) of the stiffness coefficients.

3. Resultant force, $\vec{\mathbf{f}} \to \vec{\mathbf{F}}$



Calculate the total forces by summing the vectors of the individual spring forces acting on each mass.

- $F_1 = f_1 - f_2$
- $F_2 = f_2 - f_3$
- $\dots$

3. Resultant force, $\vec{\mathbf{f}} \rightarrow \vec{\mathbf{F}}$

You can easily see where this is going once we move to matrix notation:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \quad \Longrightarrow \quad \vec{F} = A^T \vec{f}.$$

The interesting fact here is that the matrix we obtain is exactly the transpose of the elongation matrix A!

4. Newton's law, $\vec{\mathbf{F}} \rightarrow \ddot{\vec{\mathbf{u}}}$

Each resultant force F_j can be related to the acceleration $\ddot{u}_j$ of the j-th mass via Newton's second law,

$$F_j = m_j \ddot{u}_j .$$

4. Newton's law, $\vec{\mathbf{F}} \rightarrow \ddot{\vec{\mathbf{u}}}$

In matrix form

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 1 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} \quad \Longrightarrow \quad \vec{F} = M\ddot{\vec{u}}.$$

Putting it all together, $\vec{\mathbf{u}} \to \vec{\mathbf{e}} \to \vec{\mathbf{f}} \to \vec{\mathbf{F}} \to \ddot{\vec{\mathbf{u}}}$

Combine all the previous steps backwards to get the equations of motion:

$$M\ddot{\vec{u}} = \vec{F} = A^T\vec{f} = -A^T C\,\vec{e} = -A^T C\,A\,\vec{u}.$$

We can rewrite this in a more familiar form as

$$M\ddot{\vec{u}} = -K\vec{u}, \qquad K \equiv A^T CA.$$

A simple example

Suppose now that all the masses are equal as well as all springs. The above equation simplifies to

$$\ddot{\vec{u}} = -\omega^2 A^T A\vec{u}, \qquad \omega^2 = k/m.$$

So, accelerations and positions are related by the symmetric matrix $A^T A$... but how does this matrix look like?

$\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Let's do the math in Matlab

In [4]:
```
A = [1 0 0 0; -1 1 0 0; 0 -1 1 0; 0 0 -1 1; 0 0 0 -1]
```

Out[4]:
```
A =

     1     0     0     0
    -1     1     0     0
     0    -1     1     0
     0     0    -1     1
     0     0     0    -1
```

In [5]:
```
AT = A'
```

Out[5]:
```
AT =

     1    -1     0     0     0
     0     1    -1     0     0
     0     0     1    -1     0
     0     0     0     1    -1
```

In [6]:
```
D = AT * A
```

Out[6]:
```
D =

   2  -1   0   0
  -1   2  -1   0
   0  -1   2  -1
   0   0  -1   2
```

The product of the two matrices $A^T A$ is:

$$D = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The matrix D, or better its negative version, is particularly important in finite difference applications, as we will see shortly. For the moment, let's investigate the eigenvalues and eigenvectors.

Eigenvalues and eigenvectors

Suppose we know that a solution of $\ddot{\vec{u}} = -\omega^2 D\vec{u}$ can be written in this form:

$$\vec{u}(t) = \vec{U}\sin(\Omega t + \phi),$$

where $\vec{U}$ is some constant vector, Ω a non-negative frequency and ϕ the phase.

Then, the differential equation becomes an eigenvector problem,

$$\Omega^2 \vec{U} = \omega^2 D \vec{U}.$$

Do the calculations in Matlab

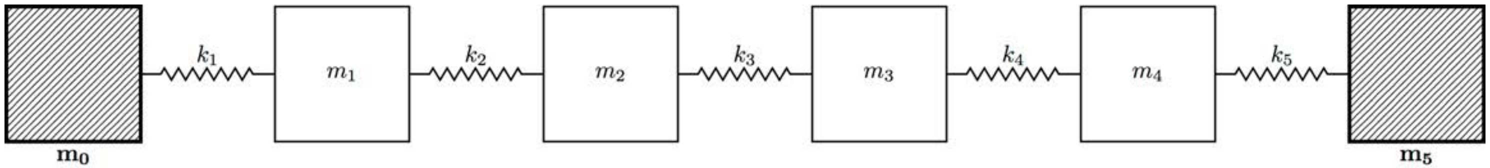
In [7]:
```
[V,E] = eig(D)
```

Out[7]:
```
V =

    0.3717   -0.6015   -0.6015   -0.3717
    0.6015   -0.3717    0.3717    0.6015
    0.6015    0.3717    0.3717   -0.6015
    0.3717    0.6015   -0.6015    0.3717


E =

    0.3820         0         0         0
         0    1.3820         0         0
         0         0    2.6180         0
         0         0         0    3.6180
```

Add extra immobile masses at the ends

Imagine that we have two extra masses, m_0 and m_5, at the two ends of the system, held fixed by some invisible agent.

These correspond to the boundary conditions that we are implicitly applying to the array of masses.

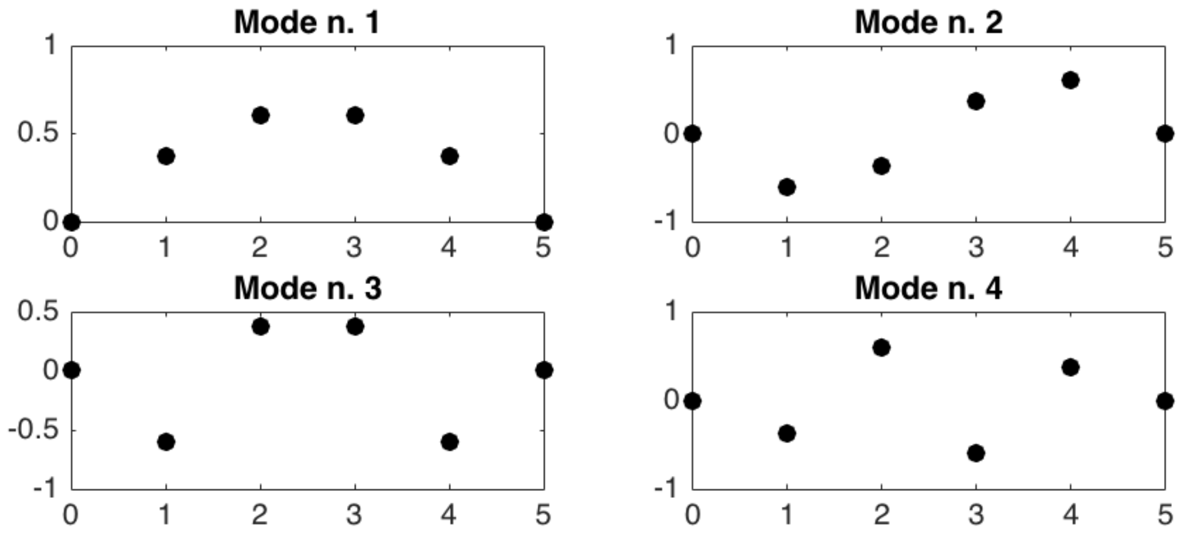


In [8]:
```
V = [zeros(1,4); V; zeros(1,4)]
```

Out[8]:
```
V =

        0         0         0         0
   0.3717   -0.6015   -0.6015   -0.3717
   0.6015   -0.3717    0.3717    0.6015
   0.6015    0.3717    0.3717   -0.6015
   0.3717    0.6015   -0.6015    0.3717
        0         0         0         0
```

Plot the modal shapes

In [9]:
```
for II=1:4
subplot(2,2,II)
plot(0:5, V(:,II), '.k', 'markersize', 22)
title(['Mode n. ', num2str(II, '%d')])
xlim([0, 5])
end
```



The modes of this system look suspiciously similar to those of a linear string... Let's see what happens when we increase the number of masses.

First, however, we need a way to generalise the creation of the matrix D to N masses...

Matlab function that creates $\mathbf{D}$

In [10]:

```
createTridiag = @(N)(full(spdiags(...
        [-ones(N,1), 2*ones(N,1), -ones(N,1)], -1:1, N, N)))
```

Out[10]:

```
createTridiag =

    @(N)(full(spdiags([-ones(N,1),2*ones(N,1),-ones(N,1)],-1:1,N,N)))
```

Test with N=10

In [11]:
```
N = 10;
D10 = createTridiag(N)
```

Out[11]:
```
D10 =

    2   -1    0    0    0    0    0    0    0    0
   -1    2   -1    0    0    0    0    0    0    0
    0   -1    2   -1    0    0    0    0    0    0
    0    0   -1    2   -1    0    0    0    0    0
    0    0    0   -1    2   -1    0    0    0    0
    0    0    0    0   -1    2   -1    0    0    0
    0    0    0    0    0   -1    2   -1    0    0
    0    0    0    0    0    0   -1    2   -1    0
    0    0    0    0    0    0    0   -1    2   -1
    0    0    0    0    0    0    0    0   -1    2
```

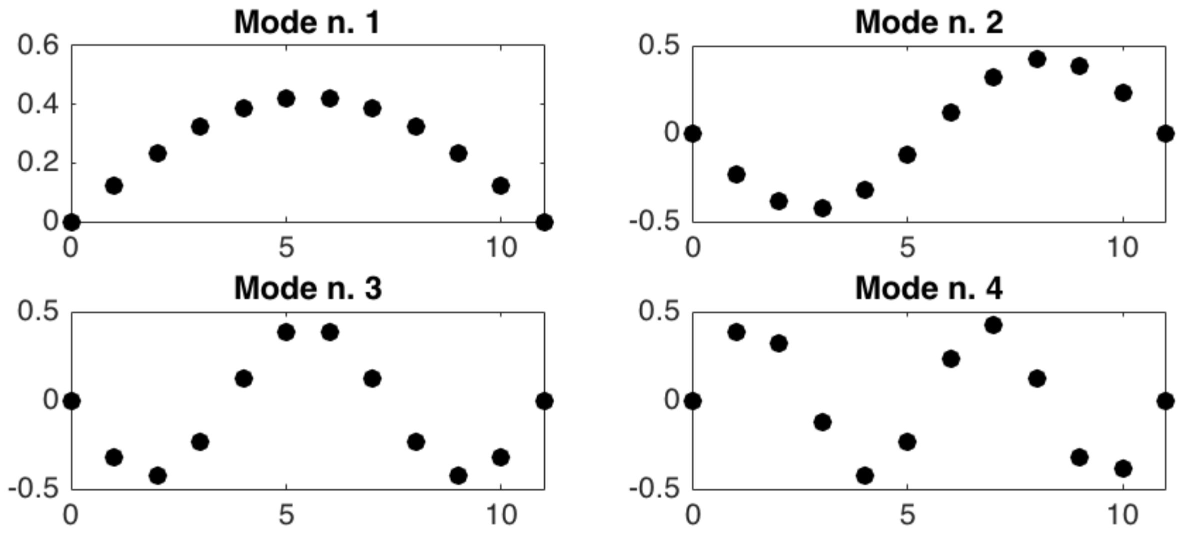
Array with 10 DOFs

Create the eigenvectors and add the fixed masses.

In [12]:
```
[V10,E10] = eig(D10);
V10 = [zeros(1,N); V10; zeros(1,N)];
```

... then plot the modes!

In [13]:
```
for II=1:4
subplot(2,2,II)
plot(0:N+1, V10(:,II), '.k', 'markersize', 22)
title(['Mode n. ', num2str(II, '%d')])
xlim([0, N+1])
end
```



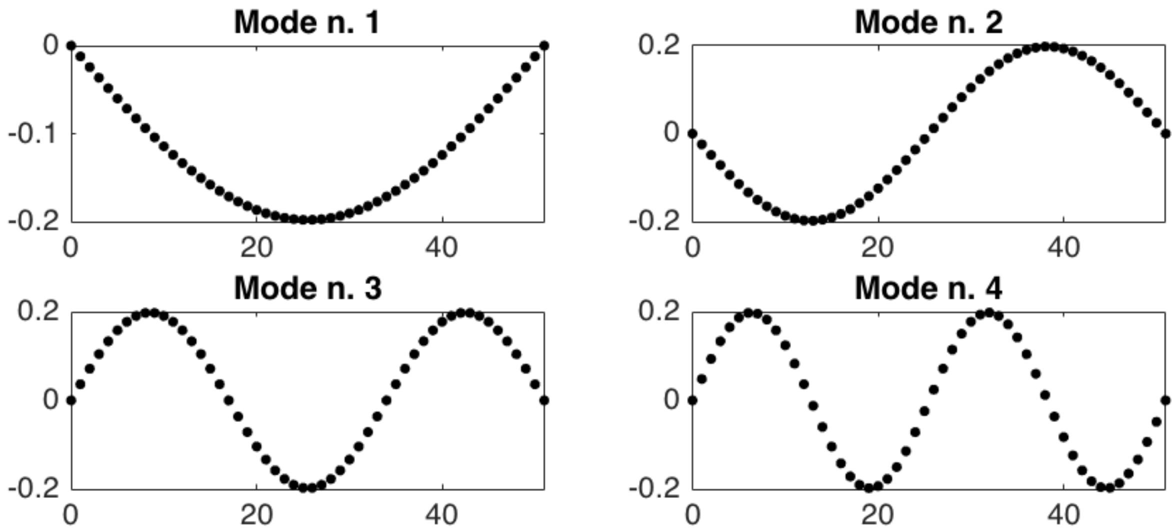
Array with 50 DOFs

Again, create the matrix D, calculate the eigenvectors and add the fixed masses.

In [14]:
```
N = 50;
D50 = createTridiag(N);
[V50,E50] = eig(D50);
V50 = [zeros(1,N); V50; zeros(1,N)];
```

... then plot the modes!

In [15]:
```
for II=1:4
subplot(2,2,II)
plot(0:N+1, V50(:,II), '.k', 'markersize', 20)
title(['Mode n. ', num2str(II, '%d')])
xlim([0, N+1])
end
```



Results

The modal shapes are sinusoids with a period that is an integer multiple of $L/2$, where L is the string length.

These are the modes of a linear string!

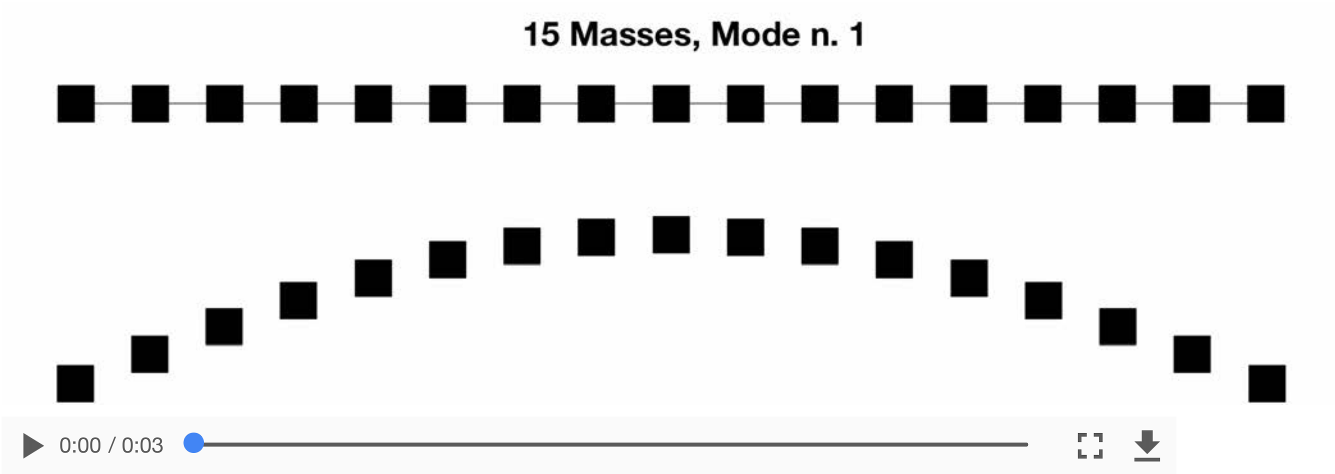
Confirm this by math calculation...

See http://www.physics.usu.edu/riffe/3750/lecture_notes.htm (http://www.physics.usu.edu/riffe/3750/lecture_notes.htm), Lecture 5, for some help!

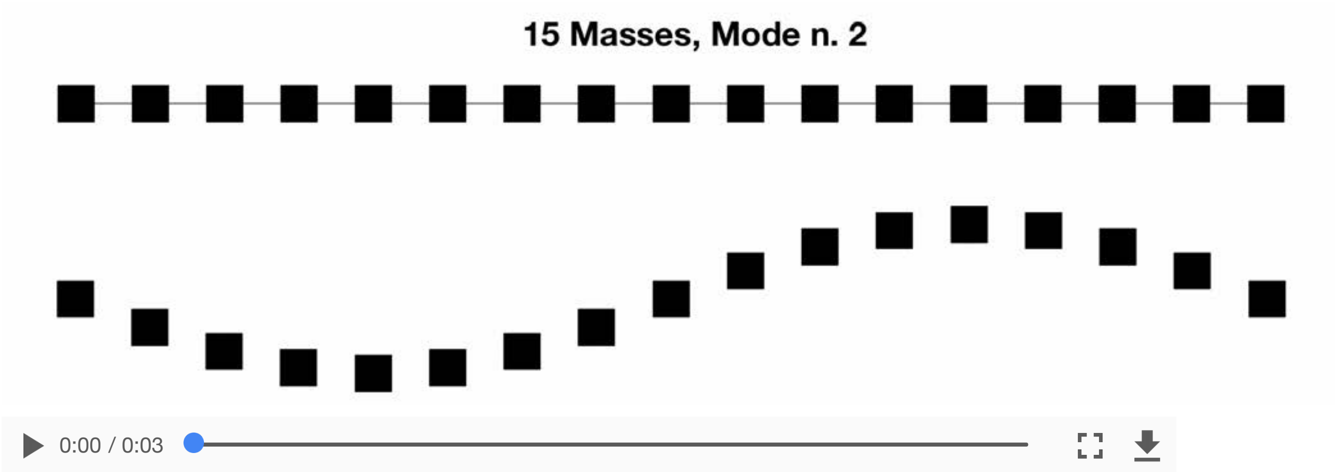
Important comment

For a string, the vibration is transverse to the length of the string. In the case of the mass-spring system, the vibrations are along the chain of oscillators, as the next videos show.

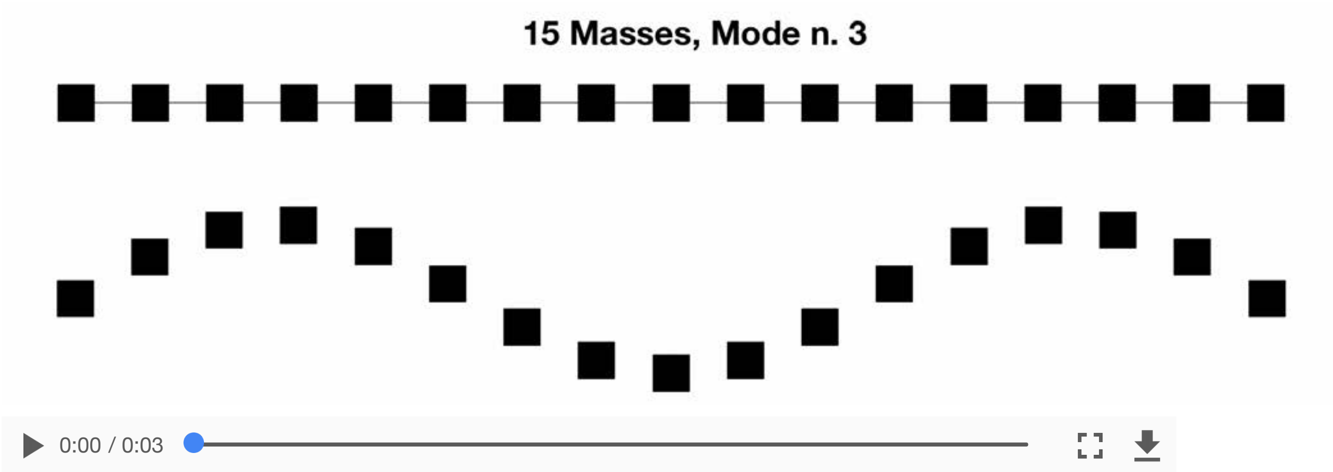
In [16]:
```
%%html
<video style="display:block; margin: 0 auto;" width="1000" height="300" controls l
oop>
  <source src="files/N15_mode1.mp4" type="video/mp4">
</video>
```



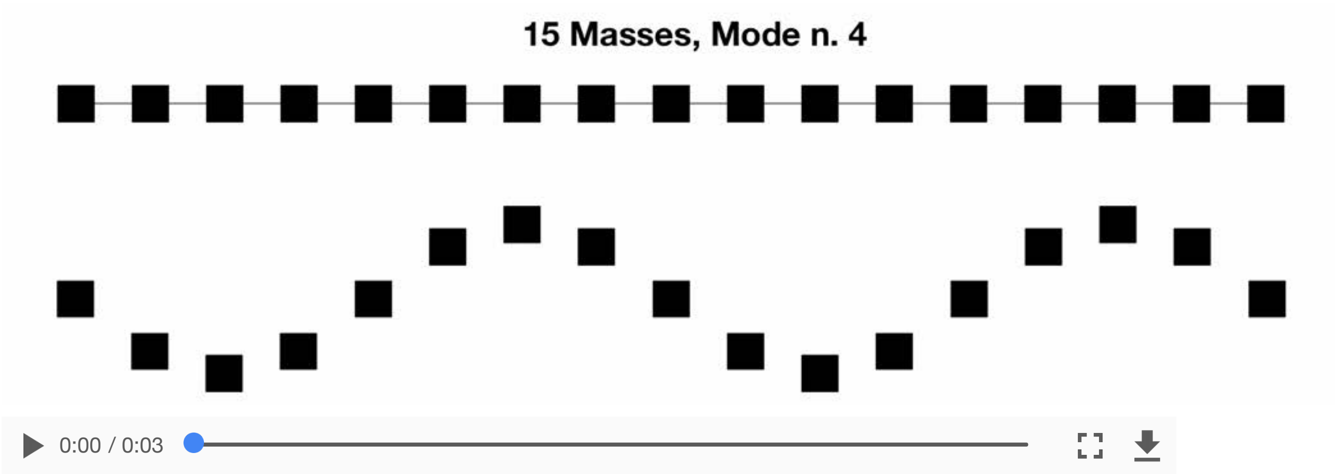
In [17]:
```
%%html
<video style="display:block; margin: 0 auto;" width="1000" height="300" controls l
oop>
  <source src="files/N15_mode2.mp4" type="video/mp4">
</video>
```



In [18]:
```
%%html
<video style="display:block; margin: 0 auto;" width="1000" height="300" controls l
oop>
  <source src="files/N15_mode3.mp4" type="video/mp4">
</video>
```



In [19]:
```
%%html
<video style="display:block; margin: 0 auto;" width="1000" height="300" controls l
oop>
  <source src="files/N15_mode4.mp4" type="video/mp4">
</video>
```



Frequencies of vibration

From the eigenvalue equation $\Omega^2 \vec{U} = \omega^2 D\vec{U},$ the frequencies are given by

$$\frac{\Omega}{2\pi} = \frac{\omega}{2\pi}\sqrt{\mathrm{eig}(D)}.$$

Frequencies of a string

They are integer multiples of the frequency of the first mode... think of a guitar string!

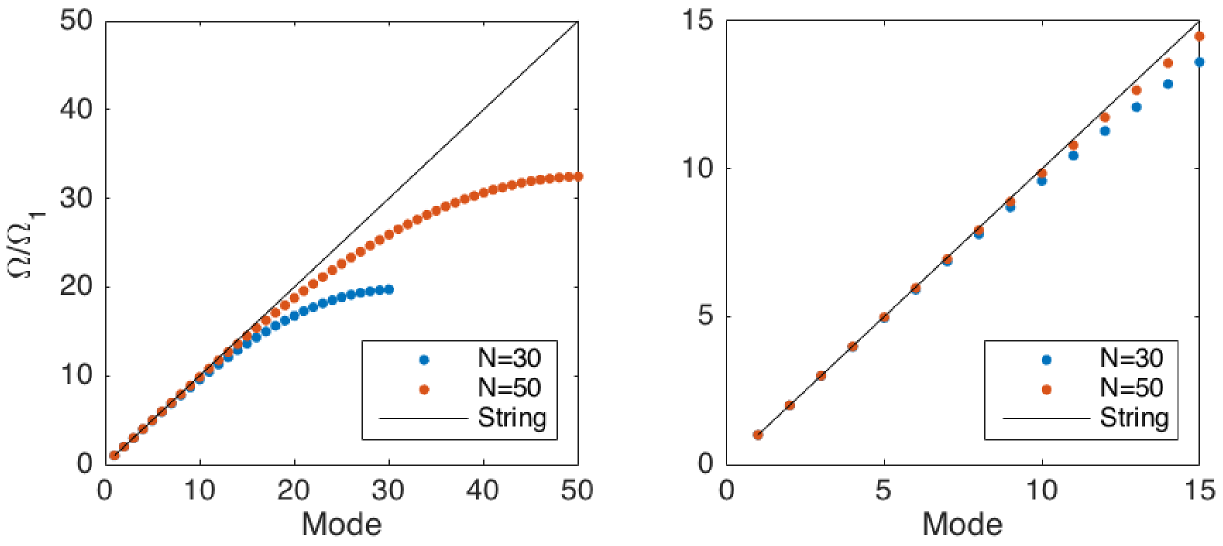
Find eigenvalues for N=30 and N=50

In [20]:
```
[V30, E30] = eig(createTridiag(30));
[V50, E50] = eig(createTridiag(50));
```

… then normalise and sqrt

In [21]:
```
ev30 = diag(E30); ev50 = diag(E50);
y30 = sqrt(ev30/ev30(1)); y50 = sqrt(ev50/ev50(1));
```

In [22]:
```
subplot(1,2,1), plot(y30,'.','markersize',20), hold on
plot(y50,'.','markersize',20), plot(1:50, 'k'), xlim([0, 50]), hold off
ylabel('\Omega/\Omega_1'), xlabel('Mode')
legend('N=30', 'N=50', 'String', 'location', 'best')
subplot(1,2,2), plot(y30,'.','markersize',20), hold on
plot(y50,'.','markersize',20), plot(1:50, 'k'), xlim([0, 15]), hold off
xlabel('Mode'), legend('N=30', 'N=50', 'String', 'location', 'best')
```



$$\Omega = \omega\sqrt{\mathrm{eig}(D)}, \qquad \Omega_1 = \text{fundamental}$$

Key points

- Modal shapes are the same as those of a string (sinusoidal shapes)
- Frequencies of the oscillator system are "detuned" w.r.t. those of a string (difference dispersion relation)

Sound synthesis applications

Sound synthesis

- Diverse range of techniques (especially digital) to create sounds
  - Additive synthesis, Wavetable synthesis, AM and FM synthesis, etc.
  - Physical modelling

Physical modelling

- Physical description of the instrument underlying the numerical algorithm
    - Lumped mass-spring networks (CORDIS-ANIMA by Cadoz et al.)
    - Modal synthesis (MOSAIC-Modalys at IRCAM, Paris)
    - Digital waveguides (J. Smith III, Stanford)
    - Time stepping methods

Modal approach

Back to the mass-spring system

The global solution to the equations of motion can be written as a linear combination of the solutions to the N eigenvalue problems for the system.

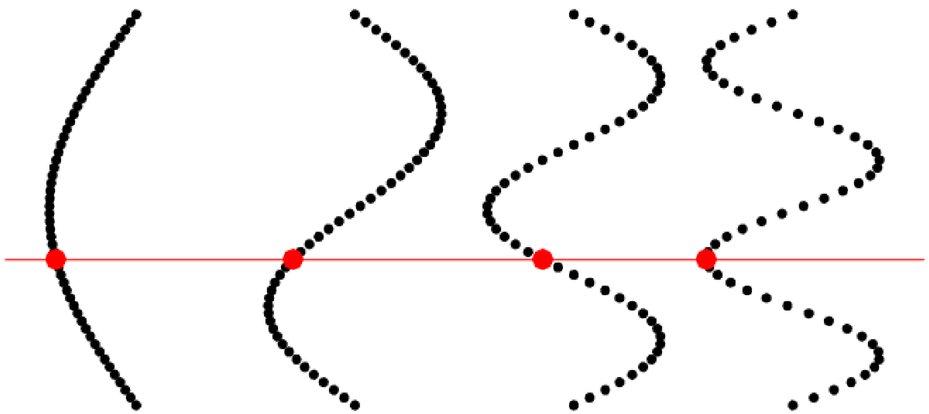
$$\vec{u}(t) = \sum_{1}^{N} A_j \, \vec{U}_j \, sin(\Omega_j t + \phi_j)$$

It requires, therefore, N amplitudes and N phases to be specified, one for each of the N eigenmodes for the system.

Create the eigenmodes

In [23]:
```
[V50, E50] = eig(createTridiag(50));
V50 = [zeros(1,50); V50; zeros(1,50) ];
```

In [24]:
```
plot(V50(:,1), 0:N+1, 'k.', 'markersize', 20),
hold on, plot(V50(20,1), 19, 'r.', 'markersize', 25)
plot(V50(:,2)+0.5, 0:N+1, 'k.', 'markersize', 20), plot(V50(20,2)+0.5, 19, 'r.',
'markersize', 25)
plot(V50(:,3)+1, 0:N+1, 'k.', 'markersize', 20), plot(V50(20,3)+1, 19, 'r.', 'mark
ersize', 25)
plot(V50(:,4)+1.5, 0:N+1, 'k.', 'markersize', 20), plot(V50(20,4)+1.5, 19, 'r.',
'markersize', 25)
plot([-0.3, 1.8], [19, 19], 'r')
axis off
```



Sound in Matlab

In [59]:
```
DT = 2;           % sound duration (in s)
t = (0:DT*44100)/44100;
y = sin(2*pi*t*440) + sin(2*pi*t*660);
soundsc(y , 44100 )
```

How does the mass-spring system sound like?

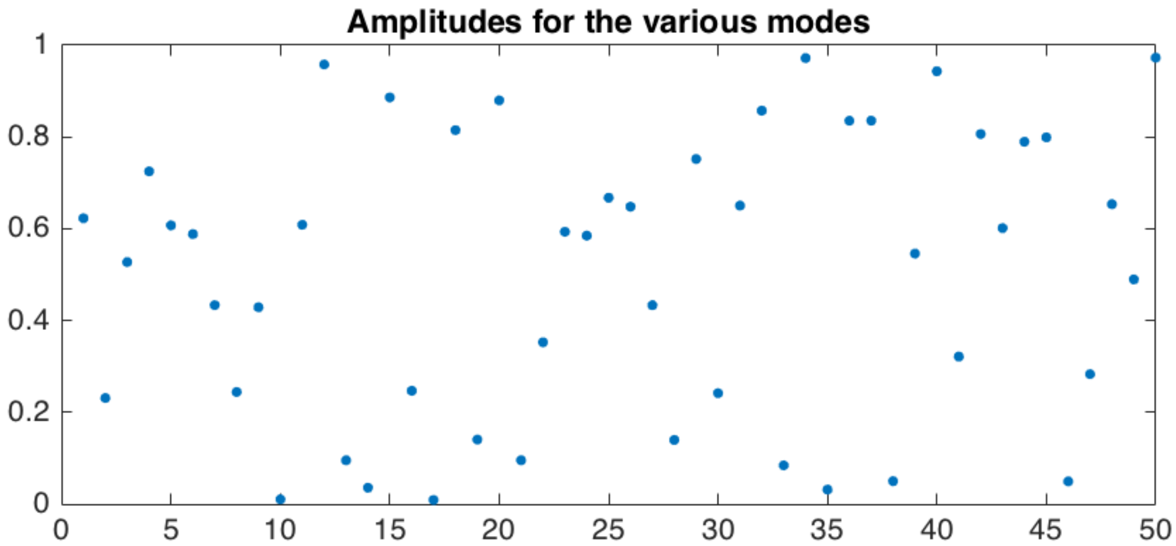
Create system matrix $\mathbf{D}$ and find the e.values

In [54]:
```
[V50,E50] = eig(createTridiag(50));
ev50 = sqrt(diag(E50));
```

Define ω, amplitudes and phases

In [55]:
```
omega = 2*pi*440;
A = rand(50,1);
%A = (50:-1:1)'/50;
%A = (((1:50)-25).^2/25^2)';
phi = zeros(50,1);
```

In [56]:
```
plot(A, '.', 'markersize', 20)
title('Amplitudes for the various modes')
```



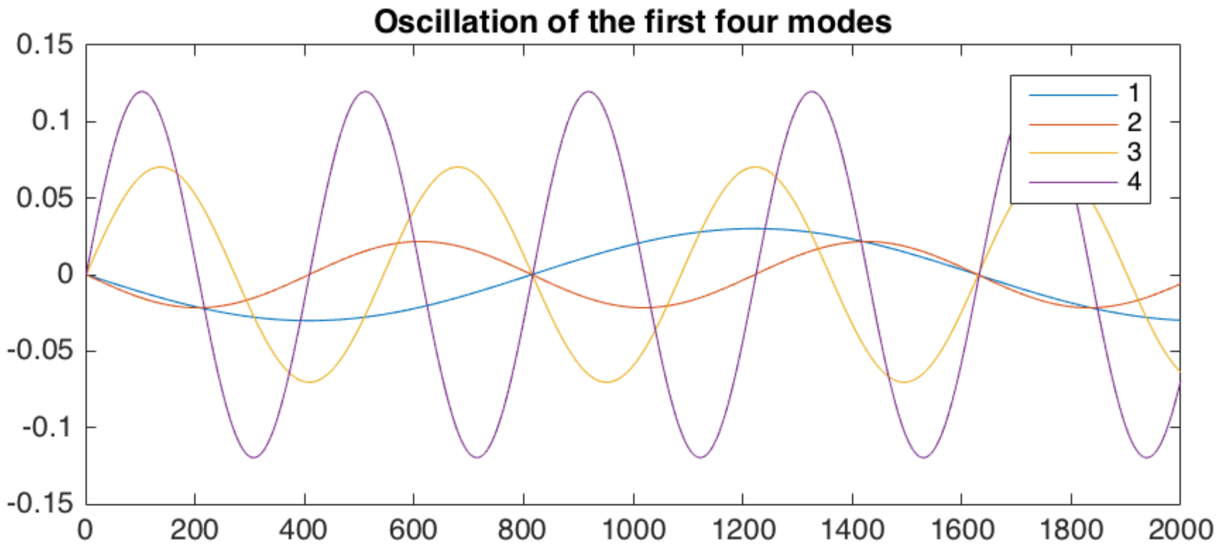
Select a point and create the oscillators

In [29]:
```
pt = 23;
```

Create amplitudes for each point and individual sounds

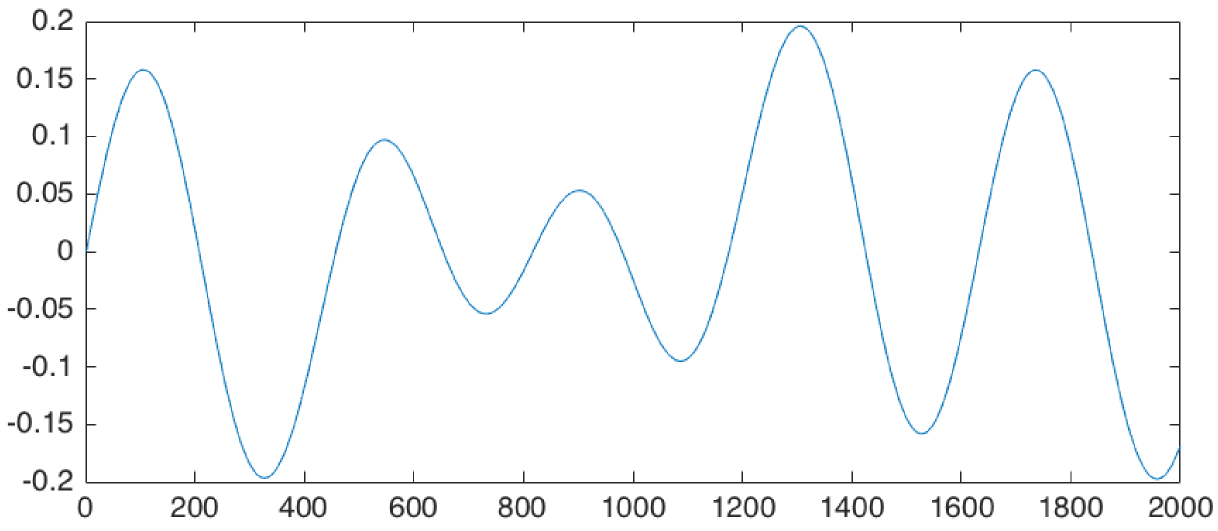
In [60]:
```
% Create the argument of sine function
T = kron(omega*ev50, t) + kron(phi, ones(numel(t),1)');

Amps = V50(pt, :)'.*A;
snd = sparse(diag(Amps)) * sin(T);
```

In [61]:
```
plot(snd(1,1:2000))
hold on, plot(snd(2,1:2000))
plot(snd(3,1:2000)), plot(snd(4,1:2000))
legend('1','2','3','4')
title('Oscillation of the first four modes')
hold off
```



Plot the final signal

In [62]:
```
plot(snd(1,1:2000) + snd(2,1:2000) + snd(3,1:2000) + snd(4,1:2000))
```



Let's play!

In [44]:
```
soundsc(sum(snd,1), 44100)
```

Try the code yourself!

`MDoF_Modal.m` available at: https://github.com/atorin/Dynamics3-Lecture (https://github.com/atorin/Dynamics3-Lecture)

Finite difference approach

Discretise time

$$u(t) \quad \longrightarrow \quad u(nk) = u^n,$$

with n integer and k the time step.

Recursions

Time differentiation becomes recursions: knowing u^n you can calculate u^{n+1}.

Second difference

$$\ddot{u} \quad \longrightarrow \quad \frac{u^{n+1} - 2u^n + u^{n-1}}{k^2}$$

Notice the pattern (1, -2, 1)...

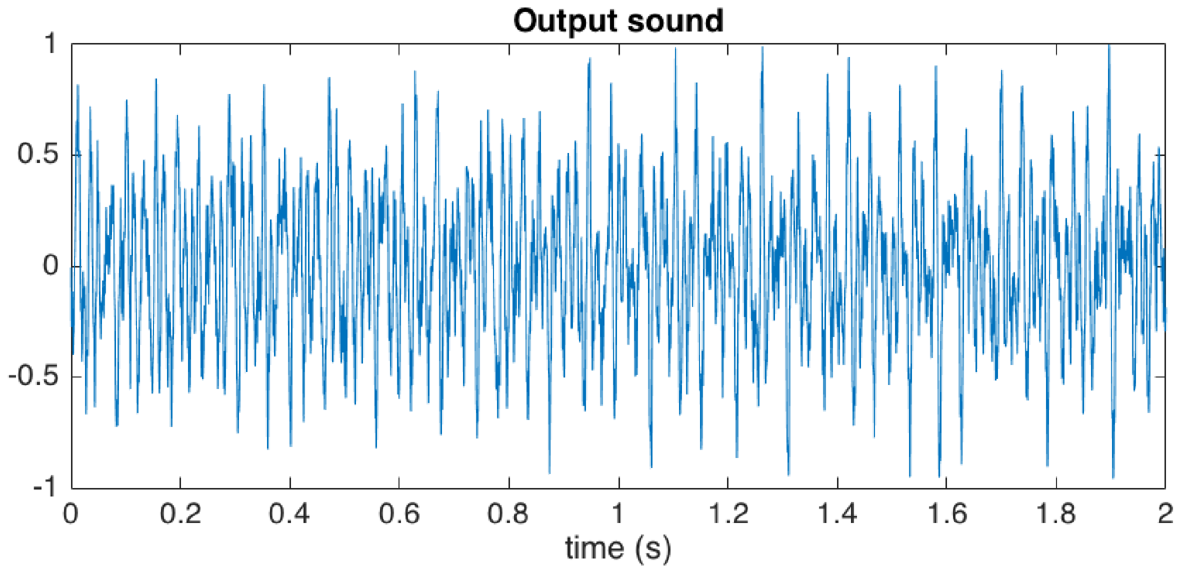
Equation for the mass-spring system

$$M\frac{\vec{u}^{n+1} - 2\vec{u}^n + \vec{u}^{n-1}}{k^2} = -\omega^2 D\vec{u}^n$$

Try the code yourself!

`MDoF_FD.m` available at: https://github.com/atorin/Dynamics3-Lecture (https://github.com/atorin/Dynamics3-Lecture)

In [47]:
```
%for II=1:5
MDoF_FD
%pause(rand)
%end
```



Conclusions

- We discussed a general method to find the equations of motion of a mass-spring system

- We looked at the normal modes and frequencies in the case of large N, and compared them with those of a string

- We had some fun with basic sound synthesis techniques

Thank you for your attention!

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Slides and Codes on Github: http://github.com/atorin/Dynamics3-Lecture (http://github.com/atorin/Dynamics3-Lecture)