

Homework #08

Convective Boundary Conditions

Problem 1: Derive the one-dimensional equation

Handwritten derivation of the one-dimensional equation for convective boundary conditions. The derivation starts with the energy balance for a control volume of thickness Δx and cross-sectional area A . The heat fluxes are defined as $F_0 = \frac{\alpha \Delta T}{\Delta x^2}$ and $B_1 = \frac{h \Delta T}{k}$, where $\alpha = \frac{k}{\rho c_p}$. The energy balance equation is then written as $\rho c_p V \frac{dT}{dt} = kA \frac{dT}{dx} + kA \frac{dT}{dy}$. This is simplified to $\frac{dT}{dt} = \frac{kA}{\rho c_p V} \frac{dT}{dx} + \frac{kA}{\rho c_p V} \frac{dT}{dy}$. The next step is to write the energy balance for a control volume of thickness Δx and cross-sectional area A , leading to $\frac{dT}{dt} = \frac{k \Delta T}{\rho c_p \Delta x^2} \frac{dT}{dx} + \frac{k \Delta T}{\rho c_p \Delta x^2} \frac{dT}{dy}$. This is then simplified to $\frac{dT}{dt} = F_0 \frac{dT}{dx} + F_0 \frac{dT}{dy}$. The final equation is $T = (1 - 4F_0) T_{i-1} + F_0 (T_{i-1,j-1} + 2T_{i-1} + T_{i,j+1}) + F_0 (T_{i-1,k-1} - 2T_{i-1} + T_{i-1,k+1})$.

$$\rho c_p V \frac{dT}{dt} = kA \frac{dT}{dx} + kA \frac{dT}{dy}$$

$$\frac{dT}{dt} = \frac{kA}{\rho c_p V} \frac{dT}{dx} + \frac{kA}{\rho c_p V} \frac{dT}{dy}$$

$$\frac{dT}{dt} = \frac{k}{\rho c_p \Delta x^2} \frac{dT}{dx} + \frac{k}{\rho c_p \Delta y^2} \frac{dT}{dy}$$

$$\frac{dT}{dt} = \frac{k \Delta T}{\rho c_p \Delta x^2} \frac{dT}{dx} + \frac{k \Delta T}{\rho c_p \Delta y^2} \frac{dT}{dy}$$

$$\frac{dT}{dt} = F_0 \frac{dT}{dx} + F_0 \frac{dT}{dy}$$

$$T - T_{i-1} = F_0 (T_{i-1,j-1} + 2T_{i-1} + T_{i,j+1}) + F_0 (T_{i-1,k-1} - 2T_{i-1} + T_{i-1,k+1})$$

$$T = \underbrace{T_{i-1} + F_0 T_{i-1,j-1} - 2F_0 T_{i-1} + F_0 T_{i,j+1}}_{T_{i-1}} + \underbrace{F_0 T_{i-1,k-1} - 2F_0 T_{i-1} + F_0 T_{i-1,k+1}}_{F_0 T_{i-1,k+1}}$$

$$T = (1 - 4F_0) T_{i-1} + F_0 (T_{i-1,j-1} + T_{i,j+1} + T_{i-1,k-1} + T_{i-1,k+1})$$

Problem 2: Derive the equation for a 2-D plane surface with convection

$$\begin{aligned}
 \rho c_p V \frac{dT}{dt} &= kA \frac{dT}{dx} + kA \frac{dT}{dy} + hA(T_a - T) & \text{sides} \\
 V &= \frac{\Delta x \Delta y \Delta z}{2} & A = \frac{\Delta x \Delta z}{2} & F_0 = \frac{\alpha \Delta t}{\rho c_p} & \alpha = \frac{k}{\rho c_p} \\
 & & A = \frac{\Delta y \Delta z}{2} & B_1 = \frac{h \Delta x}{k} \\
 \frac{dT}{dt} &= \frac{2k \Delta x \Delta z}{\rho c_p \Delta x \Delta y \Delta z} \frac{dT}{dx} + \frac{k \Delta x \Delta z}{\rho c_p \Delta x \Delta y \Delta z} \frac{dT}{dy} + hA(T_a - T) \\
 \frac{dT}{dt} &= \frac{2k}{\rho c_p \Delta x^2} dT + \frac{k}{\rho c_p \Delta y^2} dT + 2B_1 F_0 (T_a - T) \\
 dT &= \frac{2k \Delta t}{\rho c_p \Delta x^2} dT + \frac{k \Delta t}{\rho c_p \Delta y^2} dT + 2B_1 F_0 (T_a - T) \\
 dT &= 2F_0 dT + F_0 dT + 2B_1 F_0 (T_a - T) \\
 T - T_{i-1} &= 2F_0(T_{i-1,j-1} - T_{i-1,j}) + F_0(T_{i-1,k-1} - 2T_{i-1} + T_{i-1,k+1}) + 2B_1 F_0 T_a - 2B_1 F_0 T_{i-1} \\
 T &= \overline{T_{i-1}} + 2F_0 \overline{T_{i-1,j-1}} - 2F_0 \overline{T_{i-1}} + F_0 \overline{T_{i-1,k-1}} - 2F_0 \overline{T_{i-1}} + F_0 \overline{T_{i-1,k+1}} + 2B_1 F_0 T_a - 2B_1 F_0 \overline{T_{i-1}} \\
 T &= (1 - 4F_0 - 2B_1 F_0) T_{i-1} + 2F_0(T_{i-1,j-1} + \frac{1}{2}(T_{i-1,k-1} + T_{i-1,k+1}) + B_1 T_a)
 \end{aligned}$$

Problem 3: derive the equation for the exterior corner

Corner

$$\rho C_p V \frac{dT}{dt} = kA \frac{dT}{dx} + kA \frac{dT}{dy} + hA(T_a - T) + hA(T_b - T)$$

$$V = \frac{\Delta x \Delta y \Delta z}{4} \quad A = \frac{\Delta y \Delta z}{2} \quad A = \frac{\Delta x \Delta z}{2} \quad F_0 = \frac{hA \Delta T}{k} \quad B_1 = \frac{hA \Delta T}{k}$$

$$\frac{dT}{dt} = \frac{2k}{\rho C_p \Delta x^2} dT + \frac{2k}{\rho C_p \Delta y^2} dT + \frac{2h}{\rho C_p \Delta x} (T_a - T) + \frac{2h}{\rho C_p \Delta y} (T_b - T)$$

$$dT = 2F_0 dT + 2F_0 dT + 2B_1 F_0 (T_a - T) + 2B_1 F_0 (T_b - T)$$

$$T - T_{i-1} = 2F_0 (T_{i-1,j-1} - T_{i-1}) + 2F_0 (T_{i-1,k-1} - T_{i-1}) + 2B_1 F_0 (T_a - T_{i-1}) + 2B_1 F_0 (T_b - T_{i-1})$$

$$T = \overline{T}_{i-1} + 2F_0 \overline{T}_{i-1,j-1} - 2F_0 \overline{T}_{i-1} + 2F_0 \overline{T}_{i-1,k-1} - 2F_0 \overline{T}_{i-1} + 2B_1 F_0 T_a - 2B_1 F_0 \overline{T}_{i-1} + 2B_1 F_0 T_b - 2B_1 F_0 \overline{T}_{i-1}$$

$$T = (1 - 4F_0 - 4B_1 F_0) \overline{T}_{i-1} + 2F_0 (T_{i-1,j-1} + T_{i-1,k-1} + 2B_1 T_a)$$

Problem 4: Snickers Bar

Equations Used:

eqTSide(snickBlock,fo,bi,j,k,ambientTemp)

eqBSide(snickBlock,fo,bi,j,k,ambientTemp)

eqLSide(snickBlock,fo,bi,j,k,ambientTemp)

eqLCorner(snickBlock,fo,bi,j,k,ambientTemp)

eqRCorner(snickBlock,fo,bi,j,k,ambientTemp)

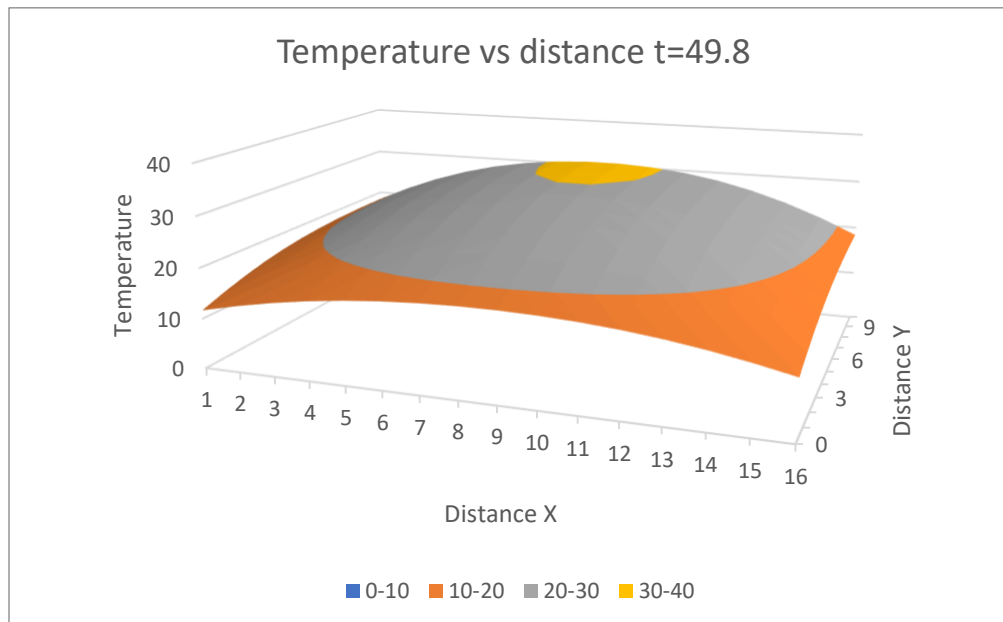
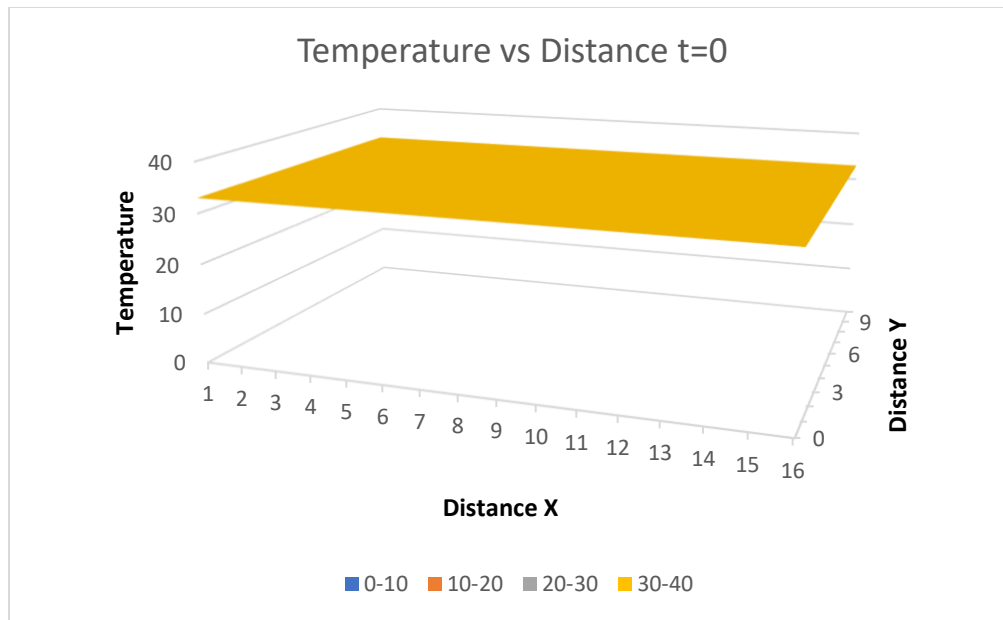
eqRSide(snickBlock,fo,bi,j,k,ambientTemp)

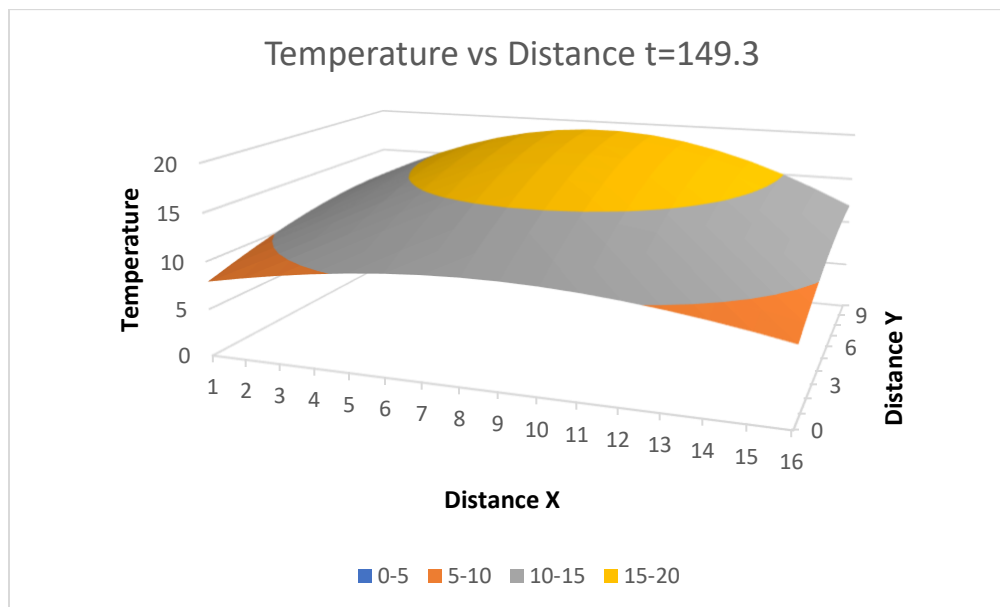
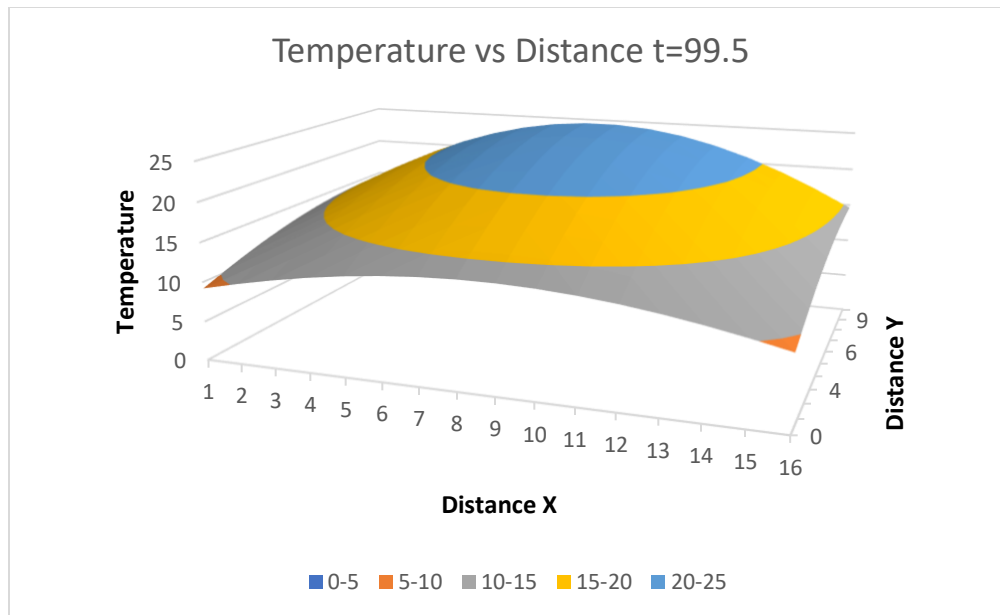
eqLBCorner(snickBlock,fo,bi,j,k,ambientTemp)

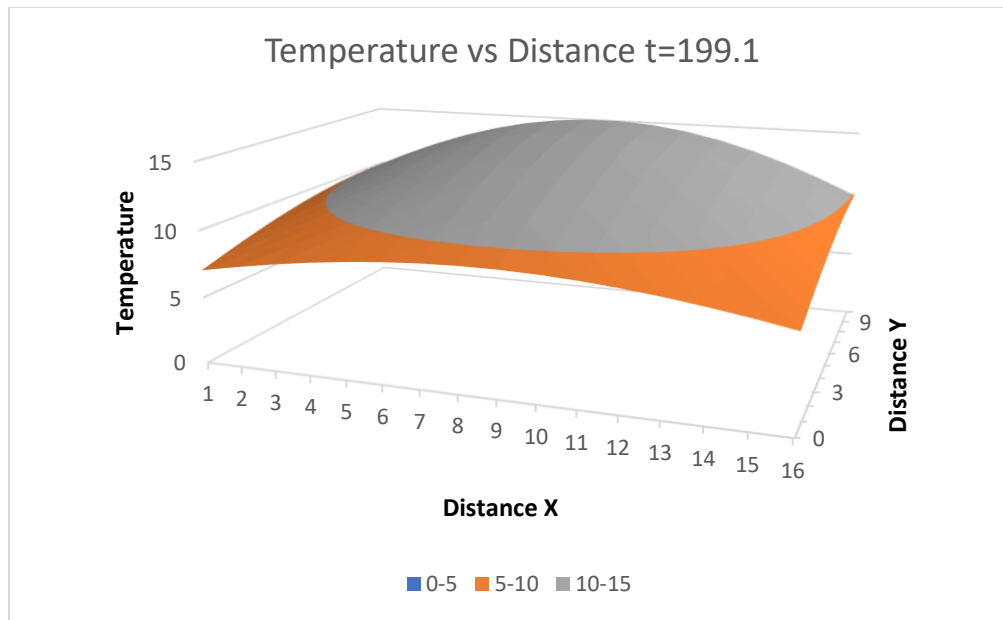
eqRBCorner(snickBlock,fo,bi,j,k,ambientTemp)

eqCenter(snickBlock,fo,j,k,ambientTemp)

Temperature Profile Plots:







How long does the tunnel need to be?

$$\text{Time} = 199.1\text{s} \cdot (1\text{min}/60\text{s}) = 3.318\text{min}$$

$$\text{Length} = 1\text{m/min} \cdot (3.318\text{min}) = \underline{\underline{3.318\text{meters}}}$$