

MEAM 620 Advanced Robotics: Homework 3

Due: Wednesday, March 4, 2015, 11:59am

1. (20pts) We have defined an edge point as a point where the gradient magnitude of the image $\|\nabla I\|$ reaches a local maximum along the gradient direction. This means that the derivative of $\|\nabla I\|$ along the gradient direction $\frac{\nabla I}{\|\nabla I\|}$ has a zero crossing. Compute

$$\nabla_{\eta} \|\nabla I\| \quad \text{where} \quad \eta = \frac{\nabla I}{\|\nabla I\|}.$$

You have to know (look up) how to differentiate the magnitude of a vector $\|v\|$ with respect to the vector v .

First we compute the gradient of the function $\|\nabla I\|$. We here use the notation that $\frac{\partial I}{\partial x} = I_x$ and similarly for I_y . Using this we get:

$$\begin{aligned} \nabla \|\nabla I\| &= \left(\frac{\partial \sqrt{I_x^2 + I_y^2}}{\partial x}, \frac{\partial \sqrt{I_x^2 + I_y^2}}{\partial y} \right)^{\top} \\ &= \left(\frac{2I_x I_{xx} + 2I_y I_{xy}}{2\sqrt{I_x^2 + I_y^2}}, \frac{2I_x I_{xy} + 2I_y I_{yy}}{2\sqrt{I_x^2 + I_y^2}} \right)^{\top} \\ &= \frac{1}{\sqrt{I_x^2 + I_y^2}} (I_x I_{xx} + I_y I_{xy}, I_x I_{xy} + I_y I_{yy})^{\top} \\ &= \frac{1}{\|\nabla I\|} \begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix} \nabla I \\ &= \frac{1}{\|\nabla I\|} (\nabla^2 I) \nabla I \end{aligned}$$

So taking the directional derivative of $\frac{\nabla I}{\|\nabla I\|}$ we get:

$$\nabla \|\nabla I\| \cdot \frac{\nabla I}{\|\nabla I\|} = \frac{1}{\|\nabla I\|^2} \nabla I^{\top} (\nabla^2 I) \nabla I$$

2. (80pts) Let M be the autocorrelation matrix of a corner detector

$$M = \sum_{(x,y) \in \mathcal{N}(x_0,y_0)} \begin{pmatrix} I_x(x,y)^2 & I_x(x,y)I_y(x,y) \\ I_x(x,y)I_y(x,y) & I_y(x,y)^2 \end{pmatrix}.$$

I will refer to this as the following:

$$M = \sum_{\vec{x}} \nabla I(\vec{x}) \nabla I(\vec{x})^{\top}$$

- a. What will happen to the trace of the matrix if the image will be dilated $I'(x, y) = I(x/2, y/2)$. Assume that I_x, I_y are the image derivatives directly (without any Gaussian convolution) and that the neighborhood of summation is double the original size.
- b. What will happen to the trace of the matrix if the image will be rotated by 45deg ? We look at rotated image as $I'(\vec{x}) = I(R\vec{x})$. Therefore by the chain rule $\nabla I'(\vec{x}) = R\nabla I(R\vec{x})$. So now with this, we can recompute M :

$$\begin{aligned}
M' &= \sum_{\vec{x}} \nabla I'(\vec{x}) \nabla I'(\vec{x})^\top \\
&= \sum_{\vec{x}} R \nabla I(R\vec{x}) \nabla I(R\vec{x})^\top R^\top \\
&= R \left(\sum_{\vec{x}} \nabla I(R\vec{x}) \nabla I(R\vec{x})^\top \right) R^\top \\
&= R M R^\top
\end{aligned}$$

So now our task is to compute $\text{tr}(R M R^\top)$. It can be shown that $\text{tr}(AB) = \text{tr}(BA)$:

$$\text{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij} B_{ji} = \sum_j \sum_i B_{ji} A_{ij} = \sum_j (BA)_{jj} = \text{tr}(BA)$$

Therefore:

$$\text{tr}(R M R^\top) = \text{tr}(R (M R^\top)) = \text{tr}((M R^\top) R) = \text{tr}(M)$$

So the trace is the same as the original, $\text{tr}(M)$

- c. Compute the eigenvalues of the matrix if the neighborhood contains only one straight edge at 45 degrees orientation:

$$I(x, y) = \begin{cases} 1 & \text{if } x + y \geq 0 \\ 0 & \text{if } x + y < 0 \end{cases}$$

To compute the eigenvalues of M , we need to know $\partial I / \partial x$ and $\partial I / \partial y$. We define in class for discrete variables the derivative as $\frac{1}{2}I(x-1, y) - \frac{1}{2}I(x+1, y)$, so we have:

$$\begin{aligned}
\frac{\partial I}{\partial x} &= \begin{cases} -1 & \text{if } x + y = 0 \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial I}{\partial y} &= \begin{cases} -1 & \text{if } x + y = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

This means that only the (x, y) pairs of the form (k, k) for some k have nonzero derivative.

Let's say there are n of them. We compute the M matrix:

$$\begin{aligned}
 M &= \sum_{\vec{x}} \nabla I(\vec{x}) \nabla I(\vec{x})^\top \\
 &= \sum_k [-1 \ -1]^\top [-1 \ -1] \\
 &= \sum_k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
 &= n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
 \end{aligned}$$

This clearly has eigen values $2n$ and 0 .

$$n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This represents the aperture problem: We can only see change in one direction. If we move the window across the $[1 \ -1]^\top$ direction, then we will see no change in the trace of M .

d. In this last question we want to see whether the big red rectangle is a better Harris corner than the small one.

$$I(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq r \\ 0 & \text{if otherwise} \end{cases}$$

yielding a gradient in the direction of the radius $\nabla I = (\cos \theta, \sin \theta)$. The large rectangle extends for $\theta = 0.. \frac{\pi}{4}$ while the small rectangle extends for $\theta = 0.. \frac{\pi}{8}$. Compute the autocorrelation matrix in both cases by replacing the sum with an integral, i.e., compute $\int \int \frac{\partial I}{\partial x} dx dy$, etc. Compute in both cases the trace and the determinant. Which of the rectangle interiors has more ‘‘corneriness’’ ?

Since the gradient is zero everywhere except the circle, the M matrix becomes:

$$\begin{aligned}
M &= \int_{\theta} [\cos \theta \ \sin \theta]^{\top} [\cos \theta \ \sin \theta] d\theta \\
&= \int_{\theta} \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} d\theta \\
&= \int_{\theta} \begin{bmatrix} \frac{1}{2}(1 + \cos 2\theta) & \frac{1}{2} \sin 2\theta \\ \frac{1}{2} \sin 2\theta & \frac{1}{2}(1 - \cos 2\theta) \end{bmatrix} d\theta \\
&= \frac{1}{2} \int_{\theta} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \right) d\theta \\
&= \frac{1}{2} \int_{\theta=0}^{\theta_{max}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d\theta + \frac{1}{2} \int_{\theta} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} d\theta \\
&= \frac{1}{2} \int_{\theta=0}^{\theta_{max}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d\theta + \frac{1}{2} \int_{\theta=0}^{\theta_{max}} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} d\theta \\
&= \frac{\theta_{max}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \sin 2\theta_{max} - \sin 2 \cdot 0 & -\cos 2\theta_{max} + \cos 2 \cdot 0 \\ -\cos 2\theta_{max} + \cos 2 \cdot 0 & -\sin 2\theta_{max} + \sin 2 \cdot 0 \end{bmatrix} \\
&= \frac{\theta_{max}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \sin 2\theta_{max} & -\cos 2\theta_{max} \\ -\cos 2\theta_{max} & -\sin 2\theta_{max} \end{bmatrix}
\end{aligned}$$

Now we just plug in the values for θ_{max} . As the analytic numbers get confusing, we will use numerical values rather than analytic values. We have for the larger of the two, $\theta_{max} = \pi/2$.

$$M_1 = \frac{\pi}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \sin(\pi) & -\cos(\pi) \\ -\cos(\pi) & -\sin(\pi) \end{bmatrix} = \begin{bmatrix} 0.7854 & 0.5000 \\ 0.5000 & 0.7854 \end{bmatrix}$$

For the smaller one, we have $\theta_{max} = \pi/4$.

$$M_2 = \frac{\pi}{16} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \sin(\pi/2) & -\cos(\pi/2) \\ -\cos(\pi/2) & -\sin(\pi/2) \end{bmatrix} = \begin{bmatrix} 0.6427 & 0.2500 \\ 0.2500 & 0.6427 \end{bmatrix}$$

The 'cornerness' of these matrices is determined by $c(A) = \det(A) - 0.06 \cdot \text{tr}(A)^2$. Computing it on each of these two matrices we get

$$c(M_1) = 0.2188$$

$$c(M_2) = -0.0078$$

Clearly M_1 , the one corresponding to $\theta_{max} = \pi/2$, is the better corner.