MEAM 620 Advanced Robotics: Homework 3
Due: Wednesday, March 4, 2015, 11:59am

1. (20pts) We have defined an edge point as a point where the gradient magnitude of the image $\|\nabla I\|$ reaches a local maximum along the gradient direction. This means that the derivative of $\|\nabla I\|$ along the gradient direction $\frac{\nabla I}{\|\nabla I\|}$ has a zero crossing. Compute

$$\nabla_\eta \|\nabla I\| \qquad \text{where} \qquad \eta = \frac{\nabla I}{\|\nabla I\|}.$$

You have to know (look up) how to differentiate the magnitude of a vector $\|v\|$ with respect to the vector v.

First we compute the gradient of the function $\|\nabla I\|$. We here use the notation that $\frac{\partial I}{\partial x} = I_x$ and similarly for I_y. Using this we get:

$$\begin{aligned}
\nabla||\nabla I|| &= \left(\frac{\partial\sqrt{I_x^2+I_y^2}}{\partial x}, \frac{\partial\sqrt{I_x^2+I_y^2}}{\partial y} \right)^\top \\
&= \left(\frac{2I_xI_{xx}+2I_yI_{xy}}{2\sqrt{I_x^2+I_y^2}}, \frac{2I_xI_{xy}+2I_yI_{yy}}{2\sqrt{I_x^2+I_y^2}} \right)^\top \\
&= \frac{1}{\sqrt{I_x^2+I_y^2}} \left(I_xI_{xx}+I_yI_{xy}, I_xI_{xy}+I_yI_{yy} \right)^\top \\
&= \frac{1}{\|\nabla I\|} \begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix} \nabla I \\
&= \frac{1}{\|\nabla I\|} \left(\nabla^2 I\right) \nabla I
\end{aligned}$$

So taking the directional derivative of $\frac{\nabla I}{\|\nabla I\|}$ we get:

$$\nabla||\nabla I|| \cdot \frac{\nabla I}{\|\nabla I\|} = \frac{1}{\|\nabla I\|^2} \nabla I^\top \left(\nabla^2 I\right) \nabla I$$

2. (80pts) Let M be the autocorrelation matrix of a corner detector

$$M = \sum_{(x,y)\in\mathcal{N}(x_0,y_0)} \begin{pmatrix} I_x(x,y)^2 & I_x(x,y)I_y(x,y) \\ I_x(x,y)I_y(x,y) & I_y(x,y)^2 \end{pmatrix}.$$

I will refer to this as the following:

$$M = \sum_{\vec{x}} \nabla I(\vec{x}) \nabla I(\vec{x})^\top$$

a. What will happen to the trace of the matrix if the image will be dilated $I'(x,y) = I(x/2, y/2)$. Assume that I_x, I_y are the image derivatives directly (without any Gaussian convolution) and that the neighborhood of summation is double the original size.
We will assume a continuous formulation of this. So we have $I'(x,y) = I(x/2, y/2)$, this means $\nabla I'(x/2, y/2) = (1/2)\nabla I(x/2, y/2)$. Let us say the window size is (a, a). The M matrix becomes:

$$M = \int_{-a}^{a}\int_{-a}^{a} \nabla I(x,y)\nabla I(x,y)^\top dxdy$$

So our new matrix M' is:

$$\begin{aligned} M' &= \int_{-2a}^{2a}\int_{-2a}^{2a} \nabla I'(x,y)\nabla I'(x,y)^\top dxdy \\ &= \frac{1}{4}\int_{-2a}^{2a}\int_{-2a}^{2a} \nabla I(x/2,y/2)\nabla I(x/2,y/2)^\top dxdy \end{aligned}$$

We will show by a change of coordinates $x' = x/2$ and $y' = y/2$ that $M' = M$

$$\begin{aligned} M' &= \frac{1}{4}\int_{-2a}^{2a}\int_{-2a}^{2a} \nabla I(x/2,y/2)\nabla I(x/2,y/2)^\top dxdy \\ &= \frac{1}{4}\int_{(-2a)/2}^{(2a)/2}\int_{(-2a)/2}^{(2a)/2} \nabla I(x',y')\nabla I(x',y')^\top (2dx')(2dy') \\ &= \frac{1}{4}\int_{-a}^{a}\int_{a}^{a} \nabla I(x',y')\nabla I(x',y')^\top 4dx'dy' \\ &= \int_{-a}^{a}\int_{a}^{a} \nabla I(x',y')\nabla I(x',y')^\top dx'dy' = M \end{aligned}$$

Therefore since $M' = M$, $\mathbf{tr}(M') = \mathbf{tr}(M)$. In a discrete formulation, this might be an approximation but the results would be similar.

b. What will happen to the trace of the matrix if the image will be rotated by 45deg ?
We look at rotated image as $I'(\vec{x}) = I(R\vec{x})$ (we assume the window is rotation invariant). Therefore by the chain rule $\nabla I'(\vec{x}) = R\nabla I(R\vec{x})$. So now with this, we can recompute M:

$$\begin{aligned} M' &= \sum_{\vec{x}} \nabla I'(\vec{x})\nabla I'(\vec{x})^\top \\ &= \sum_{\vec{x}} R\nabla I(R\vec{x})\nabla I(R\vec{x})^\top R^\top \\ &= R\left(\sum_{\vec{x}} \nabla I(R\vec{x})\nabla I(R\vec{x})^\top\right) R^\top \\ &= RMR^\top \end{aligned}$$

The last equality comes since we are summing over all $\vec{x}$. So now our task is to compute $\mathbf{tr}(RMR^\top)$. It can be shown that $\mathbf{tr}(AB) = \mathbf{tr}(BA)$:

$$\mathbf{tr}(AB) = \sum_i (AB)_{ii} = \sum_i\sum_j A_{ij}B_{ji} = \sum_j\sum_i B_{ji}A_{ij} = \sum_j (BA)_{jj} = \mathbf{tr}(BA)$$

Therefore:

$$\mathbf{tr}(RMR^\top) = \mathbf{tr}(R(MR^\top)) = \mathbf{tr}((MR^\top)R) = \mathbf{tr}(M)$$

So the trace is the same as the original, $\mathbf{tr}(M)$

c. Compute the eigenvalues of the matrix if the neighborhood contains only one straight edge at 45 degrees orientation:

$$I(x,y) = \begin{cases} 1 & \text{if } x+y \geq 0 \\ 0 & \text{if } x+y < 0 \end{cases}$$

To compute the eigenvalues of M, we need to know $\partial I/\partial x$ and $\partial I/\partial y$. We defined in class for images the derivative in the x direction as $\frac{1}{2}I(x-1,y) - \frac{1}{2}I(x+1,y)$, which is the same as a discrete way convolution with $[-1/2, 0, 1/2]$. Similarly for the y direction. So we have:

$$\frac{\partial I}{\partial x} = \begin{cases} -1/2 & \text{if } -1 \leq x+y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial I}{\partial y} = \begin{cases} -1/2 & \text{if } -1 \leq x+y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

This means that only the (x,y) pairs of the form $(k,-k)$ or $(k,-k-1)$ for some k have nonzero derivative. Let's say there are n non-zero entires in the matrix. We compute the M matrix:

$$\begin{aligned} M &= \sum_{\vec{x}} \nabla I(\vec{x}) \nabla I(\vec{x})^\top \\ &= \sum_k [-1/2 \ -1/2]^\top [-1/2 \ -1/2] \\ &= \frac{1}{4} \sum_k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{n}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

This clearly has eigen values $n/2$ and 0.

$$\frac{n}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{n}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{n}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This represents the apature problem: We can only see change in one direction. If we move the window accross the $[1 \ -1]^\top$ direction, then we will see no change in the trace of M.

d. In this last question we want to see whether the big red rectangle is a better Harris corner than the small one.

$$I(x,y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq r \\ 0 & \text{if otherwise} \end{cases}$$

yielding a gradient in the direction of the radius $\nabla i = (\cos\theta, \sin\theta)$. The large rectangle extends for $\theta = 0..\frac{\pi}{4}$ while the small rectangle extends for $\theta = 0..\frac{\pi}{8}$. Compute the autocorrelation matrix in both cases by replacing the sum with an integral, i.e., compute $\int\int \frac{\partial I}{\partial x}\, dxdy$, etc. Compute in both cases the trace and the determinant. Which of the rectangle interiors has more "cornerness" ?

Since the gradient is zero everywhere except the circle, the gradient is in polar coordinates $\delta(r'-r)\nabla i$, where δ is the dirac delta function. The delta function is to model the sharp edge at the boundary of the circle (like the derivative of the step function).

So we get the M computed in polor coordinates is:

$$\begin{aligned}
M &= \int_0^R \int_0^{\theta_{max}} \delta(r'-r)\nabla i(\theta)\nabla i(\theta)^\top r' dr' d\theta \\
&= \int_0^R \delta(r'-r) r' dr' \int_0^{\theta_{max}} \nabla i(\theta)\nabla i(\theta)^\top d\theta \\
&= r \int_0^{\theta_{max}} \nabla i(\theta)\nabla i(\theta)^\top d\theta
\end{aligned}$$

As we don't know what r is, we will ignore it and compute the other side part of the integral to determine the "cornerness" of the two images.

$$\begin{aligned}
M &= \int_\theta [\cos\theta \ \sin\theta]^\top [\cos\theta \ \sin\theta] d\theta \\
&= \int_\theta \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} d\theta \\
&= \int_\theta \begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1-\cos 2\theta) \end{bmatrix} d\theta \\
&= \frac{1}{2}\int_\theta \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \right) d\theta \\
&= \frac{1}{2}\int_{\theta=0}^{\theta_{max}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d\theta + \frac{1}{2}\int_\theta \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} d\theta \\
&= \frac{1}{2}\int_{\theta=0}^{\theta_{max}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d\theta + \frac{1}{2}\int_{\theta=0}^{\theta_{max}} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} d\theta \\
&= \frac{\theta_{max}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \sin 2\theta_{max} - \sin 2\cdot 0 & -\cos 2\theta_{max} + \cos 2\cdot 0 \\ -\cos 2\theta_{max} + \cos 2\cdot 0 & -\sin 2\theta_{max} + \sin 2\cdot 0 \end{bmatrix} \\
&= \frac{\theta_{max}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \sin 2\theta_{max} & -\cos 2\theta_{max} \\ -\cos 2\theta_{max} & -\sin 2\theta_{max} \end{bmatrix}
\end{aligned}$$

Now we just plug in the values for θ_{max}. As the analytic numbers get confusing, we will use numerical values rather than analytic values. We have for the larger of the two, $\theta_{max} = \pi/2$.

$$M_1 = \frac{\pi}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \sin(\pi) & -\cos(\pi) \\ -\cos(\pi) & -\sin(\pi) \end{bmatrix} = \begin{bmatrix} 0.7854 & 0.5000 \\ 0.5000 & 0.7854 \end{bmatrix}$$

For the smaller one, we have $\theta_{max} = \pi/4$.

$$M_2 = \frac{\pi}{16}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}\sin(\pi/2) & -\cos(\pi/2)\\-\cos(\pi/2) & -\sin(\pi/2)\end{bmatrix} = \begin{bmatrix}0.6427 & 0.2500\\0.2500 & 0.6427\end{bmatrix}$$

The 'cornerness' of these matrices is determined by $c(A) = \det(A) - 0.06 \cdot \mathbf{tr}(A)^2$. Computing it on each of these two matrices we get

$$c(M_1) = 0.2188$$
$$c(M_2) = -0.0078$$

Clearly M_1, the one corresponding to $\theta_{max} = \pi/2$, is the better corner.