AMATH 351

Homework 2

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Problem 1

Show that if a and λ are positive constants, and b is any real number, then every solution of the equation $y' + ay = be^{-e}$ has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Solving the linear ODE with an integrating factor, let $P(x) = a$ and $Q(x) = be^{-\lambda t}$.
Thus, $u(x) = e^{\int adt} = e^{at}$.
Multiplying by $u(x)$, $e^{at}y' + e^{at}ay = e^{at}be^{-\lambda t}$

(1) Consider the case where $a = \lambda$.
By substitution, $e^{at}y' + e^{at}ay = e^{at}be^{-at} = b$.
Thus, $(u(t)y)' = b$.
Integrating, $\int (u(t)y)'dt = \int bdt$.
$u(t)y = bt + C$.
$y = \frac{bt+C}{u(t)}$.
Substituting, $y = (bt + C)e^{-at}$.
As $t \to \infty$, $bt + C \to \pm\infty$ and $e^{-at} \to 0$.
Thus $y = (bt + C)e^{-at} \to 0$.

(2) Consider the case where $a \neq \lambda$.
Simplifying, $(u(t))' = be^{t(a-\lambda)}$.
Integrating, $\int (u(t))'dt = b \int e^{t(a-\lambda)}dt$.
$u(t)y = \frac{b}{a-\lambda}e^{t(a-\lambda)} + C$.
Substituting, $y = \frac{b}{a-\lambda}e^{t(a-\lambda)}e^{-at} + Ce^{-at} = \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at}$.
As $t \to \infty$, $\frac{b}{a-\lambda}e^{-\lambda t} \to 0$ and $Ce^{-at} \to 0$ since λ and a are positive.
Thus $y = \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at} \to 0$.

Therefore, $y \to 0$ as $t \to \infty$.

Problem 2

Consider the initial value problem
$y' = \frac{3}{2}y = 3t + 2e^t, y(0) = y_0$.
Find the value y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

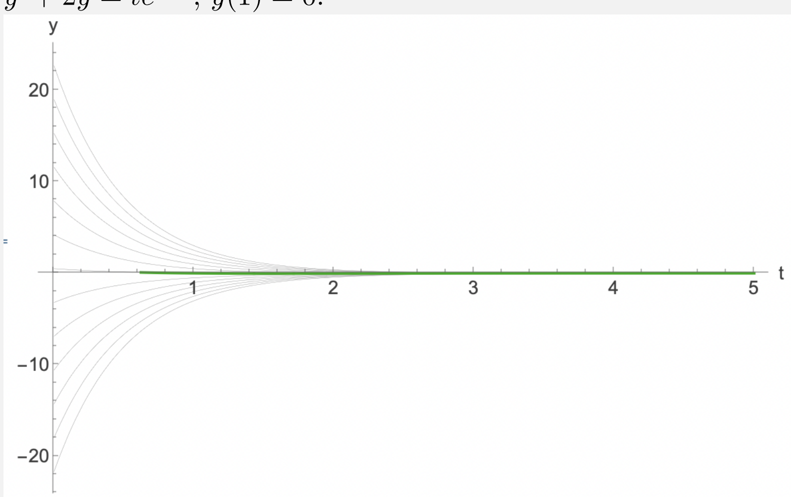
Let $P(t) = -\frac{3}{2}$ and $Q(t) = 3t + 2e^t$.
Then, $u(t) = e^{\int -\frac{3}{2}dt} = e^{-\frac{3t}{2}}$.

Multiplying by $u(t)$, $y'e^{\frac{-3t}{2}} - \frac{3}{2}ye^{\frac{-3t}{2}} = 3te^{\frac{-3t}{2}} + 2e^te^{\frac{-3t}{2}}$.
$y'e^{\frac{-3t}{2}} - \frac{3}{2}ye^{\frac{-3t}{2}} = 3te^{\frac{-3t}{2}} + 2e^{\frac{-t}{2}}$.
Substituting, $(u(t)y)' = 3te^{\frac{-3t}{2}} + 2e^{\frac{-t}{2}}$.
Integrating, $\int (u(t)y)'dt + \int 3te^{\frac{-3t}{2}}dt + \int 2e^{\frac{-t}{2}}dt$.
$u(t)y = -2te^{\frac{-3t}{2}} - \frac{4}{3}e^{\frac{-3t}{2}} - 4e^{\frac{-t}{2}} + C$.
Substituting, $y = e^{\frac{3t}{2}}(-2te^{\frac{-3t}{2}} - \frac{4}{3}e^{\frac{-3t}{2}} - 4e^{\frac{-t}{2}} + C)$.
Thus, $y = -2t - \frac{4}{3} - 4e^{\frac{t}{2}} + Ce^{\frac{3t}{2}}$.
Now, considering the initial condition: $y(0) = y_0$, $y_0 = -2(0) - \frac{4}{3} - 4e^{\frac{0}{2}} + Ce^0$.
$y_0 = -\frac{4}{3} - 4 + C = C - \frac{16}{3}$.
$C = y_0 + \frac{16}{3}$.
The critical value is $y_0 = \frac{-16}{3}$.
When $y_0 < \frac{-16}{3}$, $y(t) \to -\infty$ as $t \to \infty$.
When $y_0 = \frac{-16}{3}$, $y(t) \to 0$ as $t \to \infty$.
When $y_0 > \frac{-16}{3}$, $y(t) \to \infty$ as $t \to \infty$.

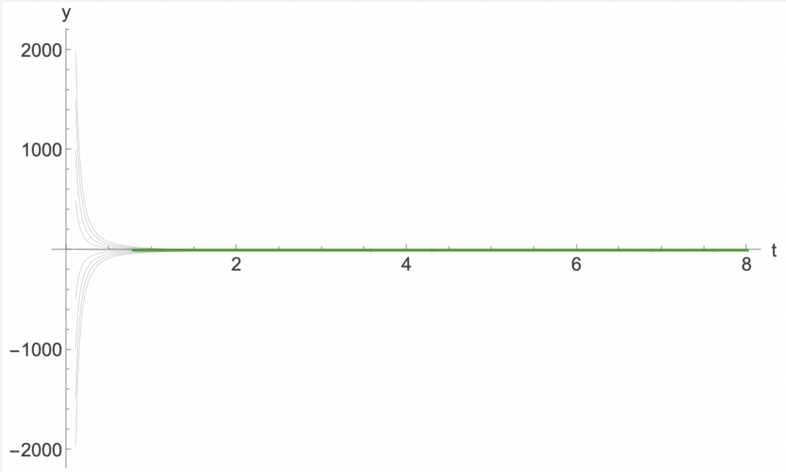
Problem 3

Draw the integral curves (using Mathematica or any other software) of the following first-order linear differential equations subject to the given initial conditions:

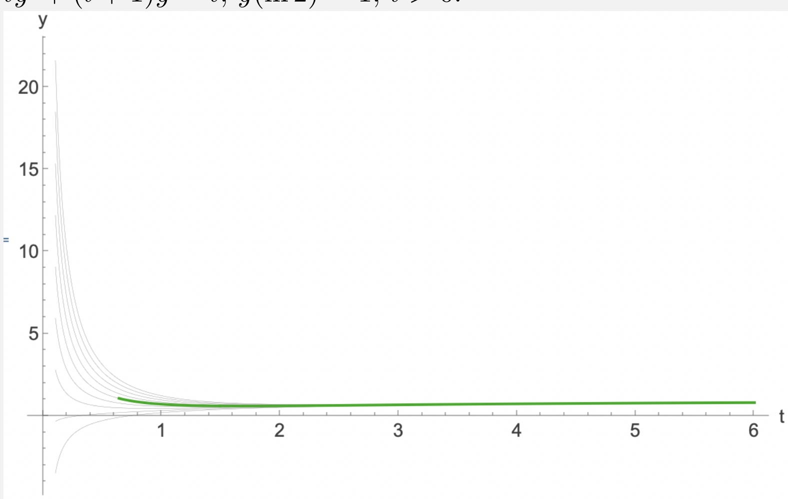
(1) $y' + 2y = te^{-2t}$, $y(1) = 0$.



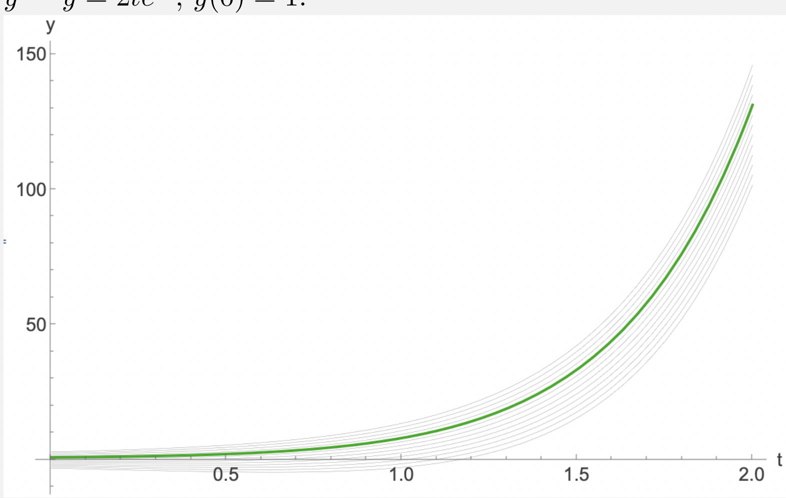
(2) $y' + \frac{2}{t}y = \frac{cost}{t^2}$, $y(\pi) = 0$, $t > 0$.



(3) $ty' + (t+1)y = t$, $y(\ln 2) = 1$, $t > 0$.



(4) $y' - y = 2te^{2t}$, $y(0) = 1$.



Problem 4

Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 minute later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

Newton's law of cooling states $\frac{dT}{dt} = -k(T - T_a)$, where T is the object's temperature, T_a is the ambient temperature, t is time elapsed in minutes, and k is some constant.
Since this is an autonomous differential equation, it can be solved for T by separation.
So, $\frac{1}{T-T_a}dT = -kdt$.
Integrating, $\int \frac{1}{T-T_a}dT = \int -kdt$.
$\ln{(T - T_a)} = kt + C_1$.
$e^{\ln{(T-T_a)}} = e^{(kt+C_1)}$
$T - T_a = e^{C_1}e^{kt}$
$T = T_a + C_2e^{kt}$.
Using the initial condition, $T(0) = 200$ and the fact that $T_a = 70$, $200 = 70+C_2e^{k(0)} = 70+C_2$.
Thus, $C_2 = 130$.
The condition $T(1) = 190$ is given, so by substitution, $190 = 70 + 130e^{k(0)} = 70 + 130e^k$.
$120 = 130e^k$.
$\frac{120}{130} = e^k$.
Thus, $k = \ln \frac{12}{13}$.
To determine when the coffee reaches a temperature of 150°F, substitute in the given and constant values.
$150 = 70 + 130e^{\ln{(\frac{12}{13})}t} = 70 + 130(\frac{12}{13})^t$.
$80 = 130(\frac{12}{13})^t$.
$\frac{80}{130} = (\frac{12}{13})^t$.
$\ln{(\frac{8}{13})} = t\ln{(\frac{12}{13})}$
$t = \frac{\ln\frac{8}{13}}{\ln\frac{12}{13}} \approx 6.066$
It would take just over 6 minutes for the coffee to reach 150°F.