# AMATH 351

## Homework 5

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**Problem 1**

If $a$, $b$, and $c$ are positive constants, show that all solutions of

$$ay'' + by' + cy = 0$$

approach zero as $t \to \infty$.

The characteristic equation is $a\lambda^2 + b\lambda + c = 0$.
By the quadratic formula, $\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.
Set $\lambda_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\lambda_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.
The conditions do not specify whether $b^2 - 4ac$ is greater than, equal to, or less than 0, so we have three cases.

(1) Consider the case where $b^2 - 4ac > 0$.
Then, the roots are real and $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$.
By substitution, $y = C_1 e^{\frac{-b+\sqrt{b^2-4ac}}{2a}t} + C_2 e^{\frac{-b-\sqrt{b^2-4ac}}{2a}t}$.
Rearranging, $y = C_1 e^{\frac{-b}{2a}t} e^{\frac{\sqrt{b^2-4ac}}{2a}t} + C_2 e^{\frac{-b}{2a}t} e^{\frac{\sqrt{-b^2-4ac}}{2a}t}$.
Since $a$, $b > 0$, $\frac{-b}{2a} < 0$.
As $t \to \infty$, $e^{\frac{-b}{2a}t} \to 0$.
By substitution, as $t \to \infty$, $y = C_1 e^{\frac{-b}{2a}t} e^{\frac{\sqrt{b^2-4ac}}{2a}t} + C_2 e^{\frac{-b}{2a}t} e^{\frac{\sqrt{-b^2-4ac}}{2a}t} \to C_1(0)e^{\frac{\sqrt{b^2-4ac}}{2a}t} + C_2(0)e^{\frac{\sqrt{-b^2-4ac}}{2a}t} = 0$.

(2) Consider the case where $b^2 - 4ac = 0$.
Then, the roots are repeated and $y = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$.
By substitution, $y = C_1 e^{\frac{-b}{2a}t} + C_2 t e^{\frac{-b}{2a}t}$.
Since $a$, $b > 0$, $\frac{-b}{2a} < 0$.
As $t \to \infty$, $e^{\frac{-b}{2a}t} \to 0$.
By substitution, as $t \to \infty$, $y = C_1 e^{\frac{-b}{2a}t} + C_2 t e^{\frac{-b}{2a}t} \to C_1(0) + C_2(\infty)(0) = 0$.

(3) Consider the case where $b^2 - 4ac < 0$.
Then, the roots are complex conjugates and $y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$.
By substitution, $y = e^{\frac{-b}{2a}t}(C_1 \cos\left(\frac{\sqrt{b^2-4ac}}{2a}t\right) + C_2 \sin\left(\frac{\sqrt{b^2-4ac}}{2a}t\right))$.
Since $a$, $b > 0$, $\frac{-b}{2a} < 0$.
As $t \to \infty$, $e^{\frac{-b}{2a}t} \to 0$.
By substitution, as $t \to \infty$, $y = e^{\frac{-b}{2a}t}(C_1 \cos\left(\frac{\sqrt{b^2-4ac}}{2a}t\right) + C_2 \sin\left(\frac{\sqrt{b^2-4ac}}{2a}t\right)) \to 0(C_1 \cos(\infty) + C_2 \sin(\infty)) = 0$.

Hence, all solutions of the differential equation approach zero as $t \to 0$.

**Problem 2**

(a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 1 is no longer true, but that all solutions are bounded as $t \to \infty$.

In this case, the differential equation is $ay' + cy = 0$.
Thus, the characteristic equation is $a\lambda^2 + c = 0$.
By the quadratic formula, $\lambda = \pm\frac{\sqrt{-4ac}}{2a}$.
Since $a$, $c > 0$, $-4ac < 0$.
Thus, the roots are complex conjugates and $y = e^0(C_1 \cos\left(\frac{\sqrt{-4ac}}{2a}t\right) + C_2 \sin\left(\frac{\sqrt{-4ac}}{2a}t\right)) = C_1 \cos\left(\frac{\sqrt{-4ac}}{2a}t\right) + C_2 \sin\left(\frac{\sqrt{-4ac}}{2a}t\right)$.
As $t \to \infty$, $y = C_1 \cos\left(\frac{\sqrt{-4ac}}{2a}t\right) + C_2 \sin\left(\frac{\sqrt{-4ac}}{2a}t\right) \to C_1 \cos(\infty) + C_2 \sin(\infty) = \sqrt{C_1 + C_2} \cos(\infty)$.
Since $\cos t$ oscillates between $-1$ and 1, all solutions to the differential equation are bounded s.t. $-\sqrt{C_1^2 + C_2^2} \leq y \leq \sqrt{C_1^2 + C_2^2}$.

(b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 1 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0$, $y'(0) = y_0'$.

In this case, the differential equation is $ay'' + by' = 0$.
Thus, the characteristic equation is $a\lambda^2 + b\lambda = 0$.
By the quadratic formula, $\lambda = \frac{-b \pm \sqrt{b^2}}{2a} = \frac{-b \pm b}{2a}$.
So, the two real roots are $\frac{-b}{a}$ and 0.
Thus the general solution is $y = C_1 e^{\frac{-b}{a}t} + C_2 e^{0t} = C_1 e^{\frac{-b}{a}t} + C_2$.
As $t \to \infty$, $y \to C_1 e^{\frac{-b}{a}\infty} + C_2$.
Since $a$, $b > 0$, $\frac{-b}{a} < 0$.
Thus, $y \to C_2$, a constant that depends on the initial conditions.
Given the initial condition $y(0) = y_0$, $y_0 = C_1 e^{\frac{-b}{a}0} + C_2 = C_1 + C_2 \Rightarrow C_2 = y_0 - C_1$.
Differentiating the general solution, $y'(t) = \frac{-b}{a} C_1 e^{\frac{-b}{a}t}$.
Given the initial condition $y'(0) = y_0'$, $y_0' = \frac{-b}{a} C_1 \Rightarrow C_1 = \frac{-a}{b} y_0' \Rightarrow C_2 = y_0 - C_1 = y_0 + \frac{a}{b} y_0'$.
Thus, the particular solution is $y = \frac{-a}{b} y_0' e^{\frac{-b}{a}t} + y_0 + \frac{a}{b} y_0' = \frac{a}{b} y_0'(-e^{\frac{-b}{a}t} + 1) + y_0$.

**Problem 3**

Solve the following ordinary differential equations *(manually and using a computer algebra system (CAS)):*

(a) $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin 3t$

Solving by the method of undetermined coefficients, $y = y_C + y_P$.
To find the complimentary solution, consider the homogeneous differential equation: $y" + 3y' = 0$.

The characteristic equation is: $\lambda^2 + 3\lambda = 0$.
Thus, the real roots are $\lambda = \frac{-3\pm\sqrt{9}}{2} = \frac{-3\pm3}{2} \Rightarrow \lambda_1 = 0,\ \lambda_2 = -3$.
Therefore, $y_C = C_1e^{-3t} + C_2e^{0t} = C_1e^{-3t} + C_2$.
So, $y = C_1e^{-3t} + C_2 + y_P$.
To find $y_P$, consider $r(t) = 2t^4 + t^2e^{-3t} + \sin 3t$.
Set $r_1(t) = 2t^4$.
Let $y_{P1} = At^5 + Bt^4 + Ct^3 + Dt^2 + Et$.
So, $y'_{P1} = 5At^4 + 4Bt^3 + 3Ct^2 + 2Dt + E$ and $y''_{P1} = 20At^3 + 12Bt^2 + 6Ct + 2D$.
By substitution, $y''_{P1} + 3y'_{P1} = 20At^3 + 12Bt^2 + 6Ct + 2D + 15At^4 + 12Bt^3 + 9Ct^2 + 6Dt + 3E = 15At^4 + (20A + 12B)t^3 + (12B + 9C)t^3 + (6C + 6D)t + 2D + 3E = 2t^4$.
Solving for the coefficients, $y_{P1} = \frac{2}{15}t^5 - \frac{2}{9}t^4 + \frac{8}{27}t^3 - \frac{8}{27}t^2 + \frac{16}{81}t$.
Set $r_2(t) = t^2e^{-3t}$.
Let $y_{P2} = e^{-3t}(Ht^3 + It^2 + Jt)$.
So, $y'_{P2} = -3e^{-3t}(Ht^3 + It^2 + Jt) + e^{-3t}(3Ht^2 + 2It + J)$ and $y''_{P2} = 9e^{-3t}(Ht^3 + It^2 + Jt) - 6e^{-3t}(3Ht^2 + 2It + J) + e^{-3t}(6Ht + 2I)$.
By substitution, $y''_{P2} + 3y'_{P2} = -Ht^2e^{-3t} + te^{-3t}(\frac{-2}{3}I + 6H) + e^{-3t}(\frac{J}{3} + 2I)$.
Solving for the coefficients, $y_{P2} = e^{-3t}(\frac{-1}{9}t^3 - \frac{1}{9}t^2 - \frac{2}{27}t)$.
Set $r_3(t) = \sin 3t$.
Let $y_{P3} = F\cos 3t + G\sin 3t$.
So, $y'_{P3} = -3F\sin 3t + 3G\cos 3t$ and $y''_{P3} = -9F\cos 3t - 9G\sin 3t$.
By substitution, $y''_{P3} + 3y'_{P3} = 9(-F + G)\cos 3t - 9(F + G)\sin 3t$.
Solving for the coefficients, $y_{P3} = \frac{-1}{18}\cos 3t - \frac{1}{18}\sin 3t$.
Thus, the particular solution is $y_P = \frac{2}{15}t^5 - \frac{2}{9}t^4 + \frac{8}{27}t^3 - \frac{8}{27}t^2 + \frac{16}{81}t + e^{-3t}(\frac{-1}{9}t^3 - \frac{1}{9}t^2 - \frac{2}{27}t) + \frac{-1}{18}\cos 3t - \frac{1}{18}\sin 3t$.
Hence, the solution to the differential equation is $y = y_C = y_P = C_1e^{-3t} + C_2 + \frac{2}{15}t^5 - \frac{2}{9}t^4 + \frac{8}{27}t^3 - \frac{8}{27}t^2 + \frac{16}{81}t + e^{-3t}(\frac{-1}{9}t^3 - \frac{1}{9}t^2 - \frac{2}{27}t) + \frac{-1}{18}\cos 3t - \frac{1}{18}\sin 3t$.

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In[1]:= eq = y''[t] + 3 y'[t] == 2 t^4 + t^2 Exp[-3 t] + Sin[3 t];
        sol = DSolve[eq, y[t], t]
        FullSimplify[sol]
        y[t] /. sol // Expand
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Out[2]= $\left\{\left\{y[t] \to c_2 + \frac{1}{810}\left(2 e^{-3t}\left(10\left(-3+8e^{3t}\right)t - 15\left(3+8e^{3t}\right)t^2 + 15\left(-3+8e^{3t}\right)t^3 - 90e^{3t}t^4 + 54e^{3t}t^5 - 5(2+27c_1)\right) - 45\,\mathrm{Cos}[3t] - 45\,\mathrm{Sin}[3t]\right)\right\}\right\}$

Out[3]= $\left\{\left\{y[t] \to \frac{1}{810}\left(4t(40 + 3t(-20 + t(20 + 3t(-5 + 3t)))) - 10e^{-3t}(2 + 3t(2 + 3t(1 + t)) + 27c_1) + 810c_2 - 45\,\mathrm{Cos}[3t] - 45\,\mathrm{Sin}[3t]\right)\right\}\right\}$

Out[4]= $\left\{-\frac{2}{81}e^{-3t} + \frac{16t}{81} - \frac{2}{27}e^{-3t}t - \frac{8t^2}{27} - \frac{1}{9}e^{-3t}t^2 + \frac{8t^3}{27} - \frac{1}{9}e^{-3t}t^3 - \frac{2t^4}{9} + \frac{2t^5}{15} - \frac{1}{3}e^{-3t}c_1 + c_2 - \frac{1}{18}\mathrm{Cos}[3t] - \frac{1}{18}\mathrm{Sin}[3t]\right\}$

(b) $y'' - 5y' + 6y = e^t\cos 2t + e^{2t}(3t + 4)\sin t$

Solving by the method of undetermined coefficients, $y = y_C + y_P$.
To find the complementary solution, consider the homogeneous differential equation: $y'' - 5y' + 6y = 0$.
The characteristic equation is: $\lambda^2 - 5\lambda + 6 = 0 = (\lambda - 2)(\lambda - 3)$.
Thus, the real roots are $\lambda_1 = 2,\ \lambda_2 = 3$.
Therefore, $y_C = C_1e^{2t} + C_2e^{3t}$.
So, $y = C_1e^{2t} + C_2e^{3t} + y_P$.
To find $y_P$, consider $r(t) = e^t\cos 2t + e^{2t}(3t + 4)\sin t$.
Set $r_1(t) = e^t\cos 2t$.
Let $y_{P1} = Ae^t\cos 2t + Be^t\sin 2t$.
So, $y'_{P1} = Ae^t\cos 2t + Be^t\sin 2t$ and $y''_{P1} = -3Ae^t\cos 2t - 4Ae^t\sin 2t - 3Be^t\sin 2t +$

$4Be^t \cos 2t$.
By substitution, $y''_{P1} - 5y'_{P1} + 6y_{P1} = (-2A - 6B)e^t \cos 2t + (6A - 2B)e^t \sin 2t$.
Solving for the coefficients, $y_{P1} = \frac{-1}{20}e^t \cos 2t - \frac{3}{20}e^t \sin 2t$.
Set $r_2(t) = e^{2t}(3t + 4)\sin t$.
Let $y_{P2} = e^{2t}((Ct + D)\cos t + (Et + F)\sin t)$.
So, $y'_{P2} = e^{2t}(((2C+E)t+(2D+C+F))\cos t+((2E-C)t+(2F-D+E))\sin t)$ and $y''_{P2} = e^{2t}(((3C+4E)t+(3D+4C+4F+2E))\cos t+((3E-4C)t+(3F-4D+4C-2C))\sin t)$.
By substitution, $y''_{P2} - 5y'_{P2} + 6y_{P2} = e^{2t}((-(C + E)t + (2E - C - D - F))\cos t + ((C - E)t + (D - 2C - E - F))\sin t)$.
Solving for the coefficients, $y_{P2} = e^{2t}((\frac{3}{2}t + \frac{1}{2})\cos t - (\frac{3}{2}t + 5)\sin t$.
Thus, the particular solution is $y_P = \frac{-1}{20}e^t \cos 2t - \frac{3}{20}e^t \sin 2t + e^{2t}((\frac{3}{2}t + \frac{1}{2})\cos t - (\frac{3}{2}t + 5)\sin t$.
Hence, the solution to the differential equation is $y = y_C + y_P = C_1e^{2t} + C_2e^{3t} - \frac{1}{20}e^t \cos 2t - \frac{3}{20}e^t \sin 2t + e^{2t}((\frac{3}{2}t + \frac{1}{2})\cos t - (\frac{3}{2}t + 5)\sin t$.

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In[5]:= sol = DSolve[y''[t] - 5 y'[t] + 6 y[t] == Exp[t] Cos[2 t] + Exp[2 t] (3 t + 4) Sin[t], y[t], t];
        y[t] /. sol[[1]]
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Out[6]= $e^{2t} c_1 + e^{3t} c_2 - \frac{1}{20} e^t (-10 e^t \text{Cos}[t] - 30 e^t t\, \text{Cos}[t] + \text{Cos}[2t] + 100 e^t \text{Sin}[t] + 30 e^t t\, \text{Sin}[t] - 10 \text{Cos}[t]\, \text{Sin}[t] + 8 \text{Sin}[2t])$

**Problem 4**

Solve the following ordinary differential equations using method of variation of parameters.

(a) $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$

The solution can be split into the homogeneous and particular solutions, $y = y_H + y_P$.
The homogeneous equation is $y'' + 4y' + 4y = 0$.
Thus, the characteristic equation is $\lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)(\lambda + 2)$.
Therefore, the roots are repeated and $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$.
So the homogeneous solution is $y_H = C_1e^{-2t} + C_2te^{-2t}$.
To find the particular solution, solve the equation $y_P = u_1y_1 + u_2y_2$.
Taking the Wronskian, $W(y_1, y_2) = y_1y'_2 - y'_1y_2 = e^{-2t}e^{-2t} - 2te^{-2t}e^{-2t} - 2te^{-2t}e^{-2t} = e^{-4t}$.
$u'_1 = \frac{y_2g(t)}{W(y_1,y_2)} = \frac{te^{-2t}t^{-2}e^{-2t}}{e^{-4t}} = t^{-1}$.
$u'_2 = \frac{y_1g(t)}{W(y_1,y_2)} = \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}} = t^{-2}$.
Integrating, $u_1 = -\int t^{-1}dt = -\ln|t|$ and $u_2 = \int t^{-2}dt = \frac{-1}{t}$.
By substitution, $y_P = u_1y_1 + u_2y_2 = -\ln|t|e^{-2t} - \frac{1}{t}te^{-2t} = -\ln|t|e^{-2t} - e^{-2t}$.
Hence, the solution to the differential equation is $y = C_1e^{-2t} + C_2te^{-2t} - \ln|t|e^{-2t}$.

(b) $y'' - 2y' + y = \frac{e^t}{1+t^2}$

The solution can be split into the homogeneous and particular solutions, $y = y_H + y_P$.
The homogeneous equation is $y'' - 2y' + y = 0$.
Thus, the characteristic equation is $\lambda^2 - 2\lambda + 1 = 0 = (\lambda - 1)(\lambda - 1)$.
Therefore, the roots are repeated and $y_1 = e^t$ and $y_2 = te^t$.
So the homogeneous solution is $y_H = C_1e^t + C_2te^t$.
To find the particular solution, solve the equation $y_P = u_1y_1 + u_2y_2$.
Taking the Wronskian, $W(y_1, y_2) = y_1y'_2 - y'_1y_2 = e^te^t + te^te^t - te^te^t = e^{2t}$.

$u_1' = \frac{y_2 g(t)}{W(y_1,y_2)} = \frac{te^t \frac{e^t}{1+t^2}}{e^{2t}} = \frac{te^{2t}}{(1+t^2)e^{2t}} = \frac{t}{1+t^2}$.
$u_2' = \frac{y_1 g(t)}{W(y_1,y_2)} = \frac{e^t \frac{e^t}{1+t^2}}{e^{2t}} = \frac{e^{2t}}{(1+t^2)e^{2t}} = \frac{1}{1+t^2}$.
Integrating, $-\int \frac{t}{1+t^2}dt = \frac{-1}{2}\int \frac{1}{u}du = \frac{-1}{2}\ln|u| = \frac{-1}{2}\ln(1+t^2)$ and $\int \frac{1}{1+t^2}dt = \arctan t$.
By substitution, $y_P = u_1 y_1 + u_2 y_2 = \frac{-e^t}{2}\ln(1+t^2) + te^t \arctan t$.
Hence, the solution to the differential equation is $y = C_1 e^t + C_2 te^t - \frac{e^t}{2}\ln(1+t^2) + te^t \arctan t$