AMATH 351

Homework 4

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Problem 1

Solve the following differential equations.

(a) $y(x-y)dx - x^2dy = 0$.

Rearranging, $y(x-y)dx = x^2dy$.
$\frac{x^2dy}{y(x-y)dx} = 1$
$\frac{dy}{dx} = \frac{y(x-y)}{x^2} = \frac{y}{x} - (\frac{y}{x})^2$.
Thus, f is a homogeneous differential equation.
Set $v = \frac{y}{x}$, so $\frac{dy}{dx} = v - v^2$.
Since $v = \frac{y}{x}$, $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
By substitution, $v + x\frac{dv}{dx} = v - v^2$.
$x\frac{dv}{dx} = -v^2$
$\frac{1}{v^2}dv = \frac{-1}{x}dx$
Integrating, $\int \frac{1}{v^2}dv = \int \frac{-1}{x}dx$.
$\frac{-1}{v} = -\ln|x| + C$
$\frac{1}{v} = \ln|x| + C$
$v = \frac{1}{\ln|x|+C}$
Since $y = vx$, $y(x) = \frac{x}{\ln|x|+C}$.

(b) $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$.

f is a homogeneous differential equation.
Set $v = \frac{y}{x}$, so $\frac{dy}{dx} = v + \sin v$.
Since $v = \frac{y}{x}$, $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
By substitution, $v + x\frac{dv}{dx} = v + \sin v$.
$\frac{1}{\sin v}dv = \frac{1}{x}dx$.
Integrating, $\int \frac{1}{\sin v}dv = \int \frac{1}{x}dx$.
$\int \frac{\sin v}{\sin^2 v}dv = \ln|x| + C$
$= \int \frac{\sin v}{1-\cos^2 v}dv = \ln|x| + C$
Let $u = \cos v$, so $= \int \frac{-1}{1-u^2}du = \ln|x| + C$.
$\frac{1}{2}\ln\frac{|u-1|}{|u+1|} = \ln|x| + C$
$\frac{1}{2}\ln\frac{|\cos v-1|}{|\cos v+1|} = \ln|x| + C$
$\ln\left(\tan\frac{v}{2}\right) = \ln|x| + C$
$\tan\frac{v}{2} = Cx$
$\frac{v}{2} = \arctan Cx$
$v = 2\arctan Cx$

Since $y = vx$, $y(x) = 2x \arctan Cx$

(c) $\frac{dy}{dx} + \frac{2}{x}y = x^2 y^{\frac{1}{2}}$.

This is a Bernoulli's differential equation, since it is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$.
Dividing both sides by $y^{\frac{1}{2}}$, $y^{\frac{-1}{2}}\frac{dy}{dx} + \frac{2y^{\frac{1}{2}}}{x}$.
Let $v = y^{\frac{1}{2}}$, so $\frac{dv}{dx} = \frac{1}{2}y^{\frac{-1}{2}}\frac{dy}{dx}$.
By substitution, $\frac{dv}{dx} + (\frac{1}{2})\frac{2v}{x} + (\frac{1}{2})x^2 = 0$.
Simplifying, $\frac{dv}{dx} + \frac{v}{x} = \frac{x^2}{2}$.
Since this is a linear ODE, it can be solved with an integrating factor.
$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$
$x\frac{dv}{dx} + v = \frac{x^3}{2}$
$\frac{d}{dx}(xv) = \frac{x^3}{2}$
Integrating, $\int xdv = \frac{1}{2}\int x^3 dx$.
$xv = \frac{x^4}{8} + C$
$v = \frac{x^3}{8} + \frac{C}{x}$
By substitution, $y = (\frac{x^3}{8} + \frac{C}{x})^2$.

Problem 2

In each of the following problems, determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

(a) $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$

The characteristic equation is $\lambda^2 - (2\alpha - 1)\lambda + \alpha(\alpha - 1) = 0$
Using the quadratic formula, $\lambda = \frac{(2\alpha-1)\pm\sqrt{4\alpha^2-4\alpha+1-4\alpha(\alpha-1)}}{2}$
Simplifying, $\lambda = \frac{2\alpha-1\pm1}{2}$.
So $\lambda_1 = \alpha$ and $\lambda_2 = \alpha - 1$.
Thus, the general solution is $y = C_1e^{\alpha t} + C_2e^{(\alpha-1)t}$.
Therefore, $y \to 0$ as $t \to \infty$ when α and $\alpha - 1 < 0$.
$\alpha < 0$ is the stricter condition, so $y \to 0$ as $t \to \infty\ \forall\ \alpha < 0$.
From the general solution, $y \to \pm\infty$ as $t \to \infty$ when $\alpha > 0$ (for $C_1 \neq 0$) or $\alpha - 1 > 0$.
Thus, $y \to \pm\infty$ as $t \to \infty\ \forall\ \alpha > 0$ when $C \neq 0$ and $\alpha > 1$ when $C = 0$.

(b) $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$

The characteristic equation is $\lambda^2 + (3 - \alpha)\lambda - 2(\alpha - 1) = 0$.
Using the quadratic formula, $\lambda = \frac{-(3-\alpha)\pm\sqrt{(3-\alpha)^2-4(-2(\alpha-1))}}{2} = \frac{-3+\alpha\pm\sqrt{\alpha^2-6\alpha+9+8\alpha-8}}{2} = \frac{-3+\alpha\pm\sqrt{\alpha^2+2\alpha+1}}{2} = \frac{-3+\alpha\pm\sqrt{(\alpha+1)^2}}{2} = \frac{-3+\alpha\pm(\alpha+1)}{2}$.
Thus, $\lambda_1 = \frac{-3+\alpha+\alpha+1}{2} = \frac{-2+2\alpha}{2} = \alpha - 1$, and $\lambda_2 = \frac{-3+\alpha-\alpha-1}{2} = \frac{-4}{2} = -2$.
Therefore, the general solution is $y = C_1e^{(\alpha-1)t} + C_2e^{-2t}$.
As $t \to \infty$, $C_2e^{-2t} \to \infty$, so the behavior of y only depends on $C_1e^{(\alpha-1)t}$.

Thus, $y \to 0$ as $t \to \infty$ if $\alpha - 1 < 0$.
Hence, $y \to 0 \; \forall \; \alpha < 1$.
Following, $y \to \pm\infty$ if $\alpha - 1 > 0$ and $C_1 \neq 0$.
Hence, $y \to \pm\infty \; \forall \; \alpha > 1$ where $C_1 \neq 0$.

Problem 3

If the Wronskian of f and g is $t\cos t - \sin t$, and if
$u = f + 3g$, $v = f - g$,
find the Wronskian of u and v.

By the definition of the Wronskian, $W(f, g) = fg' - f'g$.
By substitution, $f = t$, $f' = 1$, $g = \sin t$, $g' = \cos t$.
Thus, $u = f + 3g = t + 3\sin t$, and $v = f - g = t - \sin t$.
Therefore, the Wronskian of u and v is $W(u, v) = uv' - u'v$.
By substitution, $W(u, v) = (t + 3\sin t)(1 - \cos t) - (1 + 3\cos t)(t - \sin t)$.
$W(u, v) = (t - t\cos t + 3\sin t - 3\sin t\cos t) - (t - \sin t + 3t\cos t - 3\sin t\cos t)$
Simplifying, $W(u, v) = -t\cos t + 3\sin t + \sin t - 3t\cos t$.
$W(u, v) = 4\sin t - 4t\cos t$
Thus, $W(u, v) = 4(\sin t - t\cos t)$.

Problem 4

Consider the initial value problem
$y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = \beta$,
where $\beta > 0$

(a) Solve the initial value problem.

The characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$.
Factoring, $(\lambda + 2)(\lambda + 3) = 0$.
Thus, $\lambda_1 = -2$, $\lambda_2 = -3$.
Therefore, the general solution is $y(t) = C_1 e^{-2t} + C_2 e^{-3t}$.
Taking the derivative, $y' = -2C_1 e^{-2t} - 3C_2 e^{-3t}$.
Applying the initial condition $y(0) = 2$, $2 = C_1 + C_2 \Rightarrow C_2 = 2 - C_1$ by substitution.
Applying the initial condition $y'(0) = \beta$, $\beta = -2C_1 - 3C_2$ by substitution.
Since $C_2 = 2 - C_1$ and $\beta = -2C_1 - 3C_2$, $\beta = -2C_1 - 3(2 - C_1)$.
$\beta = -2C_1 - 6 + 3C_1$
$\beta = C_1 - 6$.
Thus, $C_1 = \beta + 6$, and $C_2 = 2 - (\beta + 6) = -\beta - 4 = -(\beta + 4)$. Therefore, the particular solution is $y(t) = (\beta + 6)e^{-2t} - (\beta + 4)e^{-3t}$.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.

Differentiating the particular solution, $y'(t) = -2(\beta + 6)e^{-2t} + 3(\beta + 4)e^{-3t}$.
When $y(t)$ attains its maximum, $y'(t) = 0$, so $y'(t_m) = -2(\beta + 6)e^{-2t_m} + 3(\beta + 4)e^{-3t_m} = 0$.

Rearranging, $2(\beta+6)e^{-2t_m} = 3(\beta+4)e^{-3t_m}$.
Dividing by e^{-3t_m}, $2(\beta+6)e^{t_m} = 3(\beta+4)$.
$e^{t_m} = \frac{3(\beta+4)}{2(\beta+6)}$.
$t_m = \ln|\frac{3(\beta+4)}{2(\beta+6)}|$.
Since $\beta > 0$, $t_m = \ln(\frac{3(\beta+4)}{2(\beta+6)})$.
Substituting t_m into the particular solution, $y_m = y(t_m) = (\beta+6)e^{-2t_m} - (\beta+4)e^{-3t_m} = (\beta+6)e^{-2\ln\left(\frac{3(\beta+4)}{2(\beta+6)}\right)} - (\beta+4)e^{-3\ln(\frac{3(\beta+4)}{2(\beta+6)})}$.
Simplifying, $y_m = (\beta+6)(\frac{3(\beta+4)}{2(\beta+6)})^{-2} - (\beta+4)(\frac{3(\beta+4)}{2(\beta+6)})^{-3}$.
$y_m = (\beta+6)(\frac{2(\beta+6)}{3(\beta+4)})^2 - (\beta+4)(\frac{2(\beta+6)}{3(\beta+4)})^3$
$y_m = (\beta+6)\frac{4(\beta+6)^2}{9(\beta+4)^2} - (\beta+4)\frac{8(\beta+6)^3}{27(\beta+4)^3}$
$y_m = \frac{12(\beta+6)^3}{9(\beta+4)^2} - \frac{8(\beta+6)^3}{27(\beta+4)^2}$
$y_m = \frac{12(\beta+6)^3 - 8(\beta+6)^3}{27(\beta+4)^2}$
$y_m = \frac{4(\beta+6)^3}{27(\beta+4)^3}$.
Thus, the coordinates of the maximum $(t_m, y_m) = (\ln(\frac{3(\beta+4)}{2(\beta+6)}), \frac{4(\beta+6)^3}{27(\beta+4)^3})$.

(c) Determine the smallest value of β for which $y_m \geq 4$.

By substitution, $y_m = \frac{4(\beta+6)^3}{27(\beta+4)^2} \geq 4$.
$4(\beta+6)^3 \geq 4(27(\beta+4)^2)$
$(\beta+6)^3 \geq 27(\beta+4)^2$.
Let $u = \beta + 4$.
Thus, $\beta = u - 4$.
By substitution, $(u-4+6)^3 \geq 27(u-4+4)^2$.
$(u+2)^3 \geq 27u$
$u^3 + 6u^2 + 12u + 8 \geq 27u^2$
$u^3 - 21u^2 + 12u + 8 \geq 0$
Factoring, $(u-1)(u^2 - 20u - 8) \geq 0$.
Solving the quadratic, $u = \frac{20 \pm \sqrt{400+32}}{2} = \frac{20 \pm \sqrt{432}}{2} = \frac{20 \pm 12\sqrt{3}}{2} = 10 \pm \sqrt{3}$.
Thus, $u = 1,\ 10 \pm 6\sqrt{3}$.
Since $\beta > 0$, $u = \beta + 4 > 4$.
So, the only feasible solution is $u = 10 + 6\sqrt{3}$.
Therefore, $\beta = u - 4 = 10 + 6\sqrt{3} - 4 = 6 + 6\sqrt{3} = 6(1+\sqrt{3})$.
Hence, the smallest value of β for which $y_m \geq 4$ is $6(1+\sqrt{3})$.

(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

As $\beta \to \infty$, $t_m = \ln\frac{3(\beta+4)}{2(\beta+6)} \to \ln\frac{3(\infty+4)}{2(\infty+6)} \to \ln\frac{3}{2}$.
As $\beta \to \infty$, $y_m = \frac{4(\beta+6)^3}{27(\beta+4)^2} \to \frac{4(\infty+6)^3}{27(\infty+4)^2} \to \frac{\infty^3}{\infty^2} = \infty$.
Thus, as $\beta \to \infty$, $(t_m, y_m) \to (\ln\frac{3}{2}, \infty)$