AMATH 351

Homework 8

Althea Lutz
2263506
Bhavna

Problem 1

Find the general solution near $x = 0$ of the following differential equations.

(a) $8x^2y'' + 10xy' + (x-1)y = 0.$

Let $P(x) = 8x^2$, $Q(x) = 10x$, and $R(x) = x - 1$.
$P(x)$ vanishes when $8x^2 = 0 \Rightarrow x = 0$, so $x = 0$ is a singular point.
$\lim_{x\to\infty}(x-0)\frac{10x}{8x^2} = \lim_{x\to\infty}\frac{10x^2}{8x^2} = \lim_{x\to\infty}\frac{5}{4} \to \frac{5}{4}$, which is finite.
$\lim_{x\to\infty}(x-0)^2\frac{x-1}{8x^2} = \lim_{x\to\infty}\frac{x^2(x-1)}{8x^2} = \lim_{x\to\infty}\frac{x-1}{8} \to \frac{-1}{8}$, which is finite.
Thus $x = 0$ is a regular singular point and we solve the differential equation using the Frobenius Method.
The solution is of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_n \neq 0$.
Then, $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.
Substituting into the differential equation, $8x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 10x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (x-1)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$.
Rearranging, $8\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + 10\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+1} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$.
Making all powers equal, $8\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + 10\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=1}^{\infty}a_{n-1}x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$.
Factoring, $\sum_{n=0}^{\infty}[8(n+r)(n+r-1) + 10(n+r) - 1]a_n x^{n+r} + \sum_{n=0}^{\infty}a_{n-1}x^{n+r} = 0$.
When $n = 0$, $(8r(r-1) + 10r - 1)a_0 = 0$.
Since $a_0 \neq 0$, the indicial equation is $8r(r-1) + 10r - 1 = 0 \Rightarrow 8r^2 + 2r - 1 = 0$.
By the quadratic formula, $r = \frac{-2\pm\sqrt{4+32}}{16} = \frac{-2\pm 6}{16} \Rightarrow r_1 = \frac{1}{4}$, $r_2 = -\frac{1}{2}$.
$r_1 - r_2 = \frac{3}{4}$, which is not an integer.
Thus there are two independent solutions and the general solution is of the form $y = C_1y_1 + C_2y_2$.
For $n \geq 1$, $[8(n+r)(n+r-1) + 10(n+r) - 1]a_n + a_{n-1} = 0$.
Thus, the recurrence relation is $a_n = \frac{-a_{n-1}}{8(n+r)(n+r-1)+10(n+r)-1}$.
For $r_1 = \frac{1}{4}$, $a_n = \frac{-a_{n-1}}{2n(4n+3)}$ when $n \geq 1$.
When $n = 1$, $a_1 = \frac{-a_0}{14}$.
When $n = 2$, $a_2 = \frac{a_0}{616}$.
When $n = 3$, $a_3 = \frac{-a_0}{55440}$.
Thus, $y_1(x) = x^{\frac{1}{4}}(1 - \frac{x}{14} + \frac{x^2}{616} - \frac{x^3}{55440} + ...)$.
For $r_2 = -\frac{1}{2}$, $a_n = \frac{-a_{n-1}}{2n(4n-3)}$ when $n \geq 1$.
When $n = 1$, $a_1 = -\frac{a_0}{2}$.
When $n = 2$, $a_2 = \frac{a_0}{4}$.
When $n = 3$, $a_3 = -\frac{a_0}{2160}$.

Thus, $y_2(x) = x^{-\frac{1}{2}}(1 - \frac{x}{2} + \frac{x^2}{40} - \frac{x^3}{2160} + ...)$.
Hence, the general solution to the differential equation is $y(x) = C_1 x^{\frac{1}{4}}(1 - \frac{x}{14} + \frac{x^2}{616} - \frac{x^3}{55440} + ...) + C_2^{-\frac{1}{2}}(1 - \frac{x}{2} + \frac{x^2}{40} - \frac{x^3}{2160} + ...)$.

(b) $x^2y'' + (x^2 - 2x)y' + 2y = 0$.

Let $P(x) = x^2$, $Q(x) = x^2 - 2x$, and $R(x) = 2$.
$P(x)$ vanishes when $x^2 = 0 \Rightarrow x = 0$, so $x = 0$ is a singular point.
$\lim_{x\to\infty}(x-0)\frac{x^2-2x}{x^2} = \lim_{x\to\infty}\frac{x^3}{x^2} - \lim_{x\to\infty}\frac{2x^2}{x^2} = \lim_{x\to\infty} x - \lim_{x\to\infty} 2 \to -2$, which is finite.
$\lim_{x\to\infty}(x-0)^2\frac{2}{x^2} = \lim_{x\to\infty}\frac{2x^2}{x^2} = \lim_{x\to\infty} 2 \to 2$, which is finite.
Thus $x = 0$ is a regular singular point and we solve the differential equation using the Frobenius Method.
The solution is of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_n \neq 0$.
Then, $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.
Substituting into the differential equation, $x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 2x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$.
Rearranging, $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} - 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$.
Factoring, $\sum_{n=0}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}$.
When $n = 0$, $r(r-1) - 2r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r_1 = 2$, $r_2 = 1$.
$r_1 - r_2 = 1$, which is a positive integer.
Thus, r_1 gives a normal Frobenius solution and r_2 leads to a breakdown in recursion and the general solution is of the form $y = C_1y_1 + C_2y_2$ where $y_1 = x^{r_1}\sum_{n=0}^{\infty}a_n x^n$ and $y_2 = Cy_1 \ln x + x^{r_2}\sum_{n=0}^{\infty}c_n x^n$.
For $n \geq 1$, $[(n+r)(n+r-1) - 2(n+r) + 2]a_n + (n+r-1)a_{n-1} = 0$.
Thus, the recurrence relation is $a_n = \frac{-a_{n-1}}{n+r-2}$.
For $r_1 = 2$, $a_n = \frac{-a_{n-1}}{n}$.
When $n = 1$, $a_1 = -a_0$.
When $n = 2$, $a_2 = \frac{a_0}{2}$.
When $n = 3$, $a_3 = \frac{-a_0}{6}$.
Thus, $y_1(x) = x^2e^{-x}$.
By the Wronskian property, $y_2 = y_1\int\frac{e^{-\int(1-\frac{2}{x})dx}}{y_1^2}dx = x^2e^{-x}\int\frac{e^{-(x-2\ln x)}}{(x^2e^{-x})^2}dx = x^2e^{-x}\int\frac{x^2e^{-x}}{x^4e^{-2x}}dx = x^2e^{-x}\int^x\frac{e^t}{t^2}dt$.
Hence, the general solution to the differential equation is $y(x) = C_1x^2e^{-x} + C_2x^2e^{-x}\int^x\frac{e^t}{t^2}dt$.

(c) $x^2y'' + xy' + x^2y = 0$.

Let $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2$.
$P(x)$ vanishes when $x^2 = 0 \Rightarrow x = 0$, so $x = 0$ is a singular point.
$\lim_{x\to\infty}(x-0)\frac{x}{x^2} = \lim_{x\to\infty}\frac{x^2}{x^2} = \lim_{x\to\infty} 1 \to 1$, which is finite.
$\lim_{x\to\infty}(x-0)^2\frac{x^2}{x^2} = \lim_{x\to\infty}\frac{x^4}{x^2} = \lim_{x\to\infty} x^2 \to 0$, which is finite.

Thus $x = 0$ is a regular singular point and we solve the differential equation using the Frobenius Method.
The solution is of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_n \neq 0$.
Then, $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.
Substituting into the differential equation, $x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$.
Rearranging, $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$.
Factoring, $\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)]a_n x^{n+r} + \sum_{n=0}^{\infty} a_{n-2}x^{n+r} = 0$.
When $n = 0$, $r(r-1)+r = 0 \Rightarrow r^2 = 0$.
$r_1 = r_2 = r = 0$, a repeated root.
Since the indicial equation has a repeated root, the solutions are $y_1 = x^r \sum_{n=0}^{\infty} a_n x^n$ and $y_2 = y_1 \ln x + x^r \sum_{n=0}^{\infty} c_n x^n$.
By substitution, $y_1 = \sum_{n=0}^{\infty} a_n x^n$ and $y_2 = y_1 \ln x + \sum_{n=0}^{\infty} c_n x^n$.

(d) $3x^2y'' - xy' + y = 0$.

Let $P(x) = 3x^2$, $Q(x) = -x$, and $R(x) = 1$.
$P(x)$ vanishes when $3x^2 = 0 \Rightarrow x = 0$, so $x = 0$ is a siingular point.
$\lim_{x\to\infty}(x-0)\frac{-x}{3x^2} = \lim_{x\to\infty}\frac{-x^2}{3x^2} = \lim_{x\to\infty}\frac{-1}{3} \to -\frac{1}{3}$, which is finite.
$\lim_{x\to\infty}(x-0)^2\frac{1}{3x^2} = \lim_{x\to\infty}\frac{x^2}{3x^2} = \lim_{x\to\infty}\frac{1}{3} \to \frac{1}{3}$, which is finite.
Thus $x = 0$ is a regular singular point and we solve the differential equation using the Frobenius Method.
The solution is of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_n \neq 0$.
Then, $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.
Substituting into the differential equation, $3x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$.
Rearranging, $3\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$.
Factoring, $\sum_{n=0}^{\infty}[3(n+r)(n+r-1)-(n+r)+1]a_n x^{n+r} = 0$.
When $n = 0$, $3r(r-1) - r + 1 = 0 \Rightarrow 3r^2 - 4r + 1 = 0$.
By the quadratic formula, $r = \frac{4\pm\sqrt{16-12}}{6} = \frac{4\pm 2}{6} \Rightarrow r_1 = 1$, $r_2 = \frac{1}{3}$.
$r_1 - r_2 = \frac{2}{3}$, which is not an integer, so there are two independent Frobenius solutions.
By substitution, $y_1 = x\sum_{n=0}^{\infty} a_n x^n$ and $y_2 = x^{\frac{1}{3}}\sum_{n=0}^{\infty} b_n x^n$.

(e) $2x^2y'' + 7x(x+1)y' - 3y = 0$.

Let $P(x) = 2x^2$, $Q(x) = 7x(x+1)$, and $R(x) = -3$.
$P(x)$ vanishes when $2x^2 = 0 \Rightarrow x = 0.$, so $x = 0$ is a singular point.
$\lim_{x\to\infty}(x-0)\frac{7x^2+7x}{2x^2} = \lim_{x\to\infty}\frac{7x^3+7x^2}{2x^2} = \lim_{x\to\infty}\frac{7x+7}{2} \to \frac{7}{2}$, which is finite.
$\lim_{x\to\infty}(x-0)^2\frac{7x^2+7x}{2x^2} = \lim_{x\to\infty}\frac{7x^4+7x^3}{2x^2} = \lim_{x\to\infty}\frac{7x^2+7x}{2} \to 0$, which is finite.
Thus $x = 0$ is a regular singular point and we solve the differential equation using the Frobenius Method.
The solution is of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_n \neq 0$.
Then, $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.
Substituting into the differential equation, $2x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 7x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 7x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 3\sum_{n=0}^{\infty} a_n x^{n+r} = 0$.

Rearranging, $2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + 7\sum_{n=0}^{\infty}(n+r)a_nx^{n+r+1} + 7\sum_{n=0}^{\infty}(n+r)a_nx^{n+r} - 3\sum_{n=0}^{\infty} a_nx^{n+r} = 0$.
Factoring, $\sum_{n=0}^{\infty}[2(n+r)(n+r-1)+7(n+r)-3]a_nx^{n+r} + 7\sum_{n=0}^{\infty}(n+r-1)a_{n-1}x^{n+r}$.
When $n = 0$, $2r(r-1) + 7r - 3 = 0 \Rightarrow 2r^2 + 5r - 3 = 0$.
By the quadratic formula, $r = \frac{-5\pm\sqrt{5^2-4(2)(-3)}}{2(2)} = \frac{-5\pm 7}{4} \Rightarrow r_1 = \frac{1}{2}$, $r_2 = -3$.
$r_1 - r_2 = \frac{7}{2}$, which is not an integer, so there are two independent Frobenius solutions.
By substitution, $y_1 = x^{\frac{1}{2}}\sum_{n=0}^{\infty} a_nx^n$ and $y_2 = x^{-3}\sum_{n=0}^{\infty} b_nx^n$.