AMATH 351

Homework 1

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Problem 1

Differentiate the following functions with respect to x:

(a) $f(x) = x^2e^{3x}\sin x$
By the product rule, $f'(x) = (x^2e^{3x})'\sin x + (x^2e^{3x})(\sin x)'$.
Again, by the product rule, $f'(x) = (x^2)'(e^{3x})\sin x + (x^2)(e^{3x})'\sin x + (x^2e^{3x})(\sin x)'$.
Differentiating, $f'(x) = 2xe^{3x}\sin x + 3x^3e^{3x}\sin x + x^2e^{3x}\cos x$.
Simplifying, $f'(x) = e^{3x}(2x\sin x + 3x^3\sin x + x^2\cos x)$.

(b) $g(x) = \ln(x^2 + \sqrt{x+1})$
By natural logarithm properties, $g'(x) = \frac{(x^2+\sqrt{x+1})'}{x^2+\sqrt{x+1}} = \frac{(x^2)'}{x^2+\sqrt{x+1}} + \frac{(\sqrt{x+1})'}{x^2+\sqrt{x+1}}$.
By the chain rule, $g'(x) = \frac{2x}{x^2+\sqrt{x+1}} + \frac{1}{2\sqrt{x+1}(x^2+\sqrt{x+1})} = \frac{2x}{x^2+\sqrt{x+1}} + \frac{1}{2(x^2\sqrt{x+1}+x+1)}$.

(c) $h(x) = \arctan(\frac{x}{1+x^2})$
By the chain rule, $h'(x) = (\arctan'(\frac{1}{1+x^2}))(\frac{1}{1+x^2})' = (\frac{1}{1+(\frac{x}{1+x^2})^2})(\frac{1}{1+x^2})'$.
By the quotient rule, $h'(x) = (\frac{1}{1+(\frac{x}{1+x^2})^2})(\frac{1-x^2}{(1+x^2)^2})$.
Simplifying, $h'(x) = (\frac{1}{1+\frac{x^2}{(1+x^2)^2}})(\frac{1-x^2}{(1+x^2)^2}) = (\frac{(1+x^2)^2}{(1+x^2)^2+x^2})(\frac{1-x^2}{(1+x^2)^2}) = \frac{1-x^2}{(1+x^2)^2+x^2}$

(d) $p(x) = x^x$
By properties of logarithms and exponents, $p(x) = x^x = e^{\ln(x^x)}$.
By the chain rule, $p'(x) = e^{x\ln x}(x\ln x)' = e^{x\ln x}(\ln x + 1)$.
Substituting back, $p'(x) = x^x(\ln x + 1)$.

(e) $q(x) = \frac{e^x\sin(x)}{1+x^2}$
By the quotient rule, $q'(x) = \frac{(e^x\sin x)'(1+x^2)-(e^x\sin x)(1+x^2)'}{(1+x^2)^2}$.
By the product rule, $\frac{((e^x)'(\sin x)+(e^x)(\sin x)')(1+x^2)-(e^x\sin x)(1+x^2)'}{(1+x^2)^2}$.
Simplifying, $q'(x) = \frac{e^x(\sin x+\cos x)(1+x^2)-(e^x\sin x)(2x)}{(1+x^2)^2} = \frac{e^x((\sin x+\cos x)(1+x^2)-2x\sin x)}{(1+x^2)^2}$

Problem 2

Evaluate the following integrals:

(a) $\int x^2e^{2x}dx$
Using integration by parts, $\int x^2e^{2x}dx = (x^2)(\frac{1}{2}e^{2x}) - \int(\frac{1}{2}e^{2x})(2xdx) = \frac{x^2e^{2x}}{2} - \int xe^{2x}dx$.
Again, using integration by parts, $\frac{x^2e^{2x}}{2} - \int xe^{2x}dx = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \int\frac{e^{2x}}{2}$.
Integrating, $\frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \int\frac{e^{2x}}{2} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C = \frac{e^{2x}}{2}(x^2 - x + \frac{1}{2}) + C$

(b) $\int \frac{\ln x}{x^2} dx$
Using integration by parts, $\int \frac{\ln x}{x^2} dx = (\ln x)(\frac{-1}{x}) - \int (\frac{-1}{x})(\frac{1}{x} dx)$.
Simplifying, $(\ln x)(\frac{-1}{x}) - \int (\frac{-1}{x})(\frac{1}{x} dx) = -\frac{\ln x}{x} - \frac{1}{x} + C$.

(c) $\int \frac{2x+3}{x^2+3x+2} dx$
Factoring the denominator, $\int \frac{2x+3}{x^2+3x+2} dx = \int \frac{2x+3}{(x+1)(x+2)} dx$.
By partial fractions, $\frac{2x+3}{(x+1)(x+2)} dx = (\frac{A}{x+1} + \frac{B}{x+2}) dx$.
Multiplying out, $2x + 3 = A(x + 2) + B(x + 1)$.
Expanding, $2x + 3 = Ax + 2A + Bx + B = (A + B)x + (2A + B)$.
By substitution, $A + B = 2$ and $2A + B = 3$.
Solving the system of equations, $A = 1$ and $B = 1$.
Thus, by substitution, $\int \frac{2x+3}{(x+1)(x+2)} dx = \int (\frac{1}{x+1} + \frac{1}{x+2}) dx = \int \frac{1}{x+1} dx + \int \frac{1}{x+2} dx$.
Integrating, $\int \frac{1}{x+1} dx + \int \frac{1}{x+2} dx = \ln|x + 1| + \ln|x + 2| + C$
By properties of natural logarithms, $\ln|x + 1| + \ln|x + 2| + C = \ln(x + 1)(x + 2) + C$.

(d) $\int x \sin x \cos x dx$
By trigonometric identities, $\sin x \cos x = \frac{1}{2} sin 2x$.
Thus, by substitution, $\int x \sin x \cos x dx = \frac{1}{2} \int x \sin 2x$.
Using integration by parts, $\frac{1}{2} \int x \sin 2x dx = \frac{-x}{4} \cos 2x + \frac{1}{4} \int \cos 2x dx$.
Integrating, $\frac{-x}{4} \cos 2x + \frac{1}{4} \int \cos 2x dx = \frac{-x}{4} \cos 2x + \frac{1}{8} \sin 2x$.

(e) $\int \frac{dx}{x\sqrt{\ln(x)}}$
Let $u = \ln x$.
Thus, $du = \frac{1}{x} dx$ and $\int \frac{dx}{x\sqrt{\ln(x)}} = \int \frac{1}{\sqrt{u}} du = \int u^{\frac{-1}{2}} du$.
Integrating, $\int u^{\frac{-1}{2}} du = 2u^{\frac{1}{2}} + C = 2\sqrt{u} + C$,
Substituting, $2\sqrt{u} + C = 2\sqrt{\ln x} + C$.

Problem 3

A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 grams of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.
$Q(t)$ = grams of undesirable chemical at time t
$\frac{dQ}{dt}$ = inflow - outflow
Inflow = $(\frac{0.01g}{1gal})(\frac{300gal}{1hr}) = 3$ grams/hr
Outflow = $(\frac{300gal}{1hr})(\frac{Q(t)g}{1{,}000{,}000gal}) = 0.0003Q(t)$ grams/hr
$\frac{dQ}{dt} = 3 - 0.0003Q(t), Q(0) = Q_0$.

(b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?
At equilibrium, $\frac{dQ}{dt} = 0$.
Thus, $0 = 3 - 0.0003Q(t) => 0.0003Q(t) = 3 => Q(t) = 10,000$. After a very long

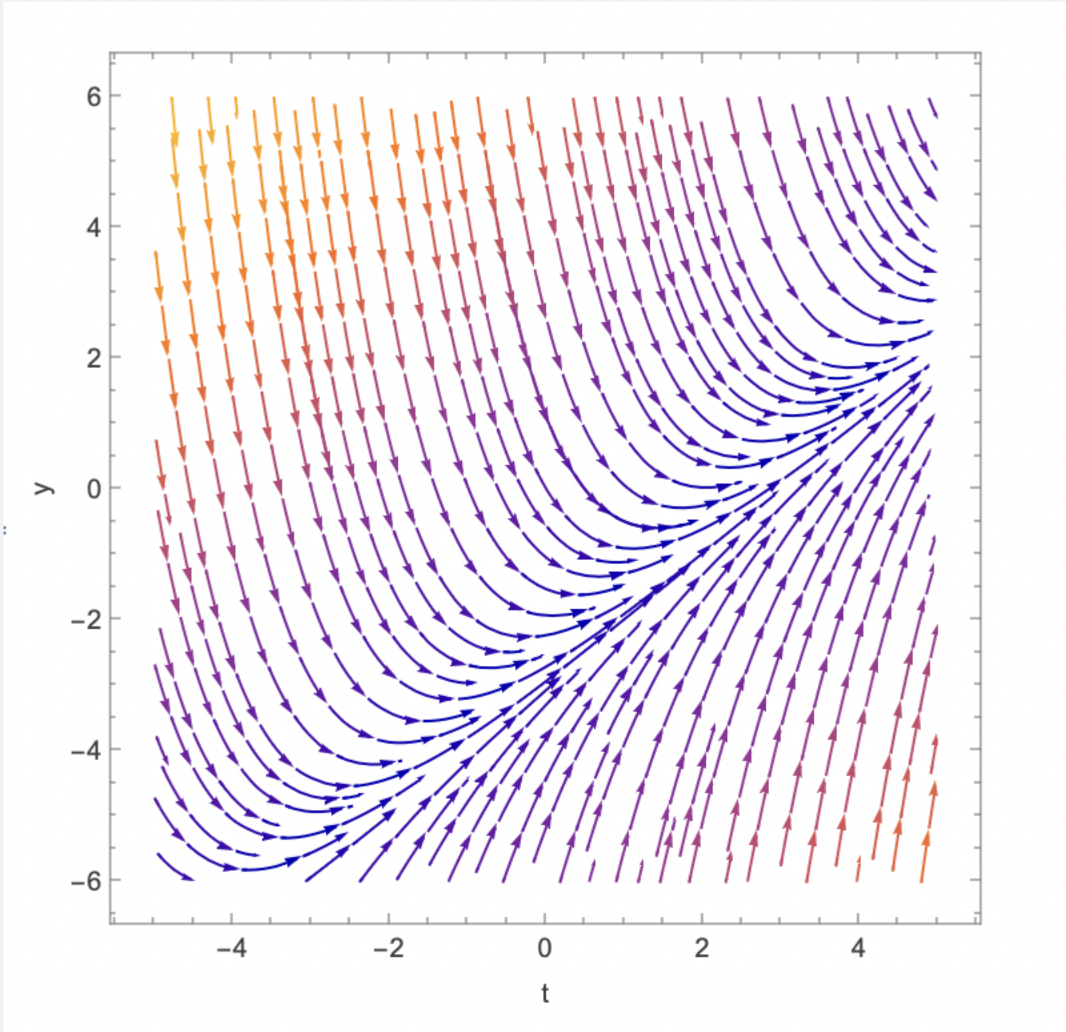
time, there will be 10,000 grams of the chemical in the pond.
The differential equation doesn't include Q_0, so the initial amount does not influence this limiting amount. Instead, the limiting amount is only dependent on concentration, flow rate, and volume.

(c) Write a differential equation for the concentration of the chemical in the pond at time t.
Hint: The concentration is $c = \frac{Q}{v} = \frac{Q(t)}{10^6}$.
Rearranging, $Q = vc$.
Differentiating, $\frac{dQ}{dt} = v\frac{dc}{dt}$.
Substituting into $\frac{dQ}{dt} = 3 - 0.0003Q(t)$, $v\frac{dc}{dt} = 3 - 0.0003vc(t) = 1,000,000\frac{dc}{dt} = 3 - 300c(t)$.
Thus, $\frac{dc}{dt} = \frac{3-300c(t)}{10^6}$.

Problem 4

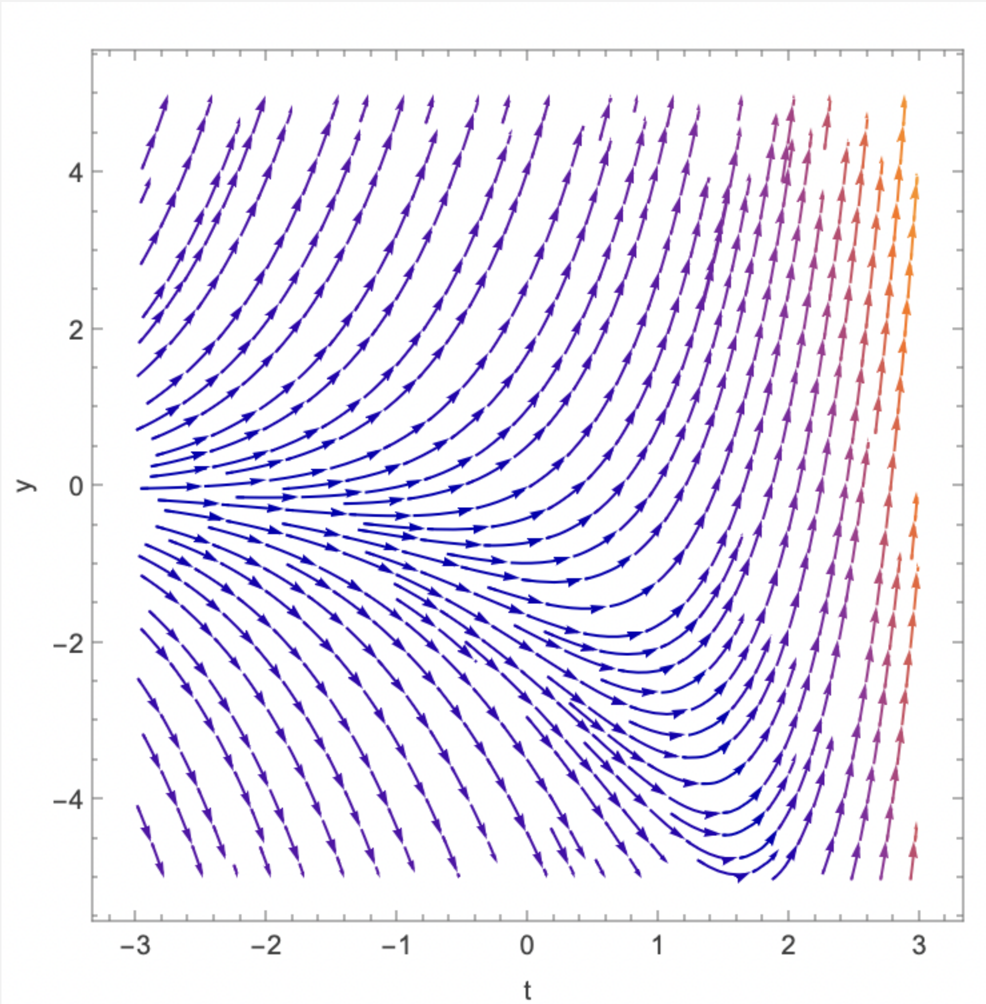
Draw a direction field for the given differential equation (using Mathematica). Based on the direction field, determine the behavior of y as $t \to \infty$.

(a) $y' = -2 + t - y$



As $t \to \infty$, $y \to t - 3$.

(b) $y' = e^t + y$



As $t \to \infty$, $y \to \infty$.