AMATH 351

Homework 6

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Problem 1

Find the inverse Laplace transform of the given functions.

(a) $F(s) = \frac{8s^2-4s+12}{s(s^2+4)}$

By partial fractions, $\frac{8s^2-4s+12}{s(s^2+4)} = \frac{As+B}{s} + \frac{Cs+D}{s^2+4}$.
Solving for the coefficients, $\frac{8s^2-4s+12}{s(s^2+4)} = \frac{3}{s} + \frac{5s+4}{s^2+4}$.
By substitution, $F(s) = \frac{3}{s} + \frac{5s+4}{s^2+4} = \frac{3}{s} + \frac{5s}{s^2+4} + \frac{4}{s^2+4}$.
Thus, $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{3}{s}\} + \mathcal{L}^{-1}\{\frac{5s}{s^2+4}\} - \mathcal{L}^{-1}\{\frac{4}{s^2+4}\}$.
Solving, $f(t) = 3 + 5\cos 2t - 4\sin 2t$.

(b) $F(s) = \frac{2s+2}{s^2+2s+5}$

Factoring, $F(s) = \frac{2(s+1)}{(s+1)^2+4}$.
Thus, $\mathcal{L}^{-1}\{F(s)\} = 2\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+4}\}$.
By the frequency-shift operation, $f(t) = 2e^{-t}\cos 2t$.

(c) $F(s) = \frac{2s-3}{s^2-4}$

Separating the numerator, $F(s) = \frac{2s}{s^2-4} - \frac{3}{s^2-4}$.
Thus, $\mathcal{L}^{-1}\{F(s)\} = 2\mathcal{L}^{-1}\{\frac{s}{s^2-4}\} - 3\mathcal{L}^{-1}\{\frac{1}{s^2-4}\}$.
By the hyperbolic Laplace transforms, $f(t) = s\cosh 2t - 3\sinh 2t$.

(d) $F(s) = \frac{1-2s}{s^2+4s+5}$

Factoring, $F(s) = \frac{1-2s}{(s+1)(s+5)}$.
By partial fractions, $F(s) = \frac{A}{s+1} + \frac{B}{s+5}$.
Solving for coefficients, $F(s) = \frac{3}{4}(\frac{1}{s+1}) - \frac{11}{4}(\frac{1}{s+5})$.
Thus, $\mathcal{L}^{-1}\{F(s)\} = \frac{3}{4}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{11}{4}\mathcal{L}^{-1}\{\frac{1}{s+5}\}$.
Solving, $f(t) = \frac{3}{4}e^{-t} - \frac{11}{4}e^{-5t}$.

Problem 2

Use the Laplace transform to solve the given initial value problems.

(a) $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 1$.

Taking the Laplace transform, $\mathcal{L}\{y^{(4)}\} - 4\mathcal{L}\{y'''\} + 6\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$.
Solving, $s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4s^3Y + 4s^2y(0) + 4sy'(0) + 4y''(0) + 6s^2Y - 6sy(0) - 6y'(0) - 4sY + 4y(0) + Y = 0$.
Substituting in the initial values, $s^4Y - s^2 - 1 - 4s^3Y + 4s + 6s^2Y - 6 - 4sY + Y = 0$.
Simplifying, $s^4Y - 4s^3Y + 6s^2Y - 4sY + Y = s^2 - 4s + 7$.
Factoring, $Y(s^4 - 4s^3 + 6s^2 - 4s + 1) = s^2 - 4s + 7$.
Thus, $Y = \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1}$.
Factoring, $Y = \frac{s^2-4s+7}{(s-1)^4} = \frac{((s-1)+1)^2-4((s-1)+1)+7}{(s-1)^4}$.
Let $u = s - 1$.
By substitution, $Y = \frac{(u+1)^2-4(u+1)+7}{u^4}$.
Rearranging, $Y = \frac{u^2-2u+4}{u^4} = \frac{u^2}{u^4} - \frac{2u}{u^4} + \frac{4}{u^4}$.
Back-substituting, $Y = \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}$.
Using the inverse Laplace transform, $\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{1}{(s-1)^2}\} - 2\mathcal{L}^{-1}\{\frac{1}{(s-1)^3}\} + 4\mathcal{L}^{-1}\{\frac{1}{(s-1)^4}\}$.
Rearranging, $\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{1!}{(s-1)^2}\} - \mathcal{L}^{-1}\{\frac{2!}{(s-1)^3}\} + \frac{2}{3}\mathcal{L}^{-1}\{\frac{3!}{(s-1)^4}\}$.
Solving, $y = te^t = t^2e^t + \frac{2}{3}t^3e^t$.
Thus, $y = e^t(t - t^2 + \frac{2}{3}t^3)$.

(b)

$$y'' + 4y = \begin{cases} 1 & 0 \le t \le \pi, \\ 0 & \pi \le t \le \infty; \end{cases}$$

y(0)=1, y'(0)=0.

Using the Laplace transform on the LHS, $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = s^2Y - sy(0) - y'(0) + 4Y$.
Substituting in the initial values, $\mathcal{L}\{y'' + 4y\} = s^2Y - s + 4Y$.
Since the RHS is a piecewise function, the Laplace transform is $\mathcal{L}\{f(t)\} = F(s) = \int_0^\pi e^{-st}dt + \int_\pi^\infty 0dt = \int_0^\pi e^{-st}dt$.
Integrating, $[\frac{-e^{-st}}{s}]_0^\pi = \frac{-1}{s}(e^{-\pi s} - e^{-0s}) = \frac{-1}{s}(e^{-\pi s} - 1) = \frac{1-e^{-\pi s}}{s}$.
Combining, $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{f(t)\}$ becomes $s^2Y - s + 4Y = \frac{1-e^{-\pi s}}{s}$.
Rearranging, $s^2Y = 4Y = \frac{1-e^{-\pi s}}{s} + s$.
Factoring, $Y(s^2 + 4) = \frac{1-e^{-\pi s}}{s} + s$.
Thus, $Y = \frac{1}{s(s^2+4)} - e^{-\pi s}\frac{1}{s(s^2-4)} + \frac{s}{s^2-4}$.
By partial fractions, $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$.
Solving for the coefficients, $\frac{1}{s(s^2+4)} = \frac{1}{4}(\frac{1}{s}) - \frac{1}{4}(\frac{1}{s^2+4})$.
By substitution, $Y = \frac{s}{s^2-4} + \frac{1}{4}(\frac{1}{s}) - \frac{1}{4}(\frac{s}{s^2+4}) - \frac{1}{4}e^{-\pi s}(\frac{1}{s}) + \frac{1}{4}e^{-\pi s}(\frac{s}{s^2+4})$.
Using the inverse Laplace transform, $\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{s}{s^2-4}\} + \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{4}\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} -$

$\frac{1}{4}\mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s}\} + \frac{1}{4}\mathcal{L}^{-1}\{\frac{se^{-\pi s}}{s^2+4}\}$.
Solving using the shift operator, $y = \cos 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4}(1-\pi) + \frac{1}{4}\cos 2(t-\pi)$.
As a piecewise function,

$$y = \begin{cases} \frac{1}{4} + \frac{3}{4}\cos 2t & 0 \le t \le \pi, \\ \cos 2t & \pi \le t \le \infty. \end{cases}$$

Problem 3

Use the property $\frac{d}{ds}F(s) = -\mathcal{L}\{tf(t)\}$ to find $\mathcal{L}^{-1}\{\frac{d}{ds}(\frac{1}{s^2+a^2})\}$.

By the definition of Laplace transforms, $\mathcal{L}^{-1}\{\frac{d}{ds}F(s)\} = \mathcal{L}^{-1}(-\mathcal{L}\{tf(t)\}) = -tf(t)$.
Thus, $\mathcal{L}^{-1}\{\frac{d}{ds}(\frac{1}{s^2+a^2})\} = -\frac{t}{a}\sin at$.