AMATH 351

Homework 3

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Problem 1

Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y be the proportion of infectious individuals; then $x + y = 1$.
Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread $\frac{dy}{dt}$ is proportional to the number of such contacts. Further, assume that members of both groups move freely among each other, so the number of contacts is proportional to the product of x and y.
Since $x = 1 - y$, we obtain the initial value problem

$$\frac{dy}{dt} = \alpha y(1 - y), y(0) = y_0 \tag{1}$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

(a) Find the equilibrium points for the differential equation (1) and determine whether each is asymptotically stable, semi-stable, or unstable.

The equation achieves equilibrium when $\frac{dy}{dt} = 0$, so the equilibrium points are $y = 0$ and $y = 1$.
When $y < 0$, $\frac{dy}{dt} = y(1 - y) < 0$, so y decreases.
When $0 < y < 1$, $\frac{dy}{dt} = y(1 - y) > 0$, so y increases.
When $y > 1$, $\frac{dy}{dt} = y(1 - y) < 0$, so y decreases.
When $y = 0$, solutions move away on either side.
Thus, $y = 0$ is asymptotically unstable.
When $y = 1$, solutions move closer on either side.
Thus, $y = 1$ is asymptotically stable.

(b) Solve the initial value problem (1) and verify that the conclusions you reached in part (a) are correct. Show that

$$y(t) \to 1 \text{ as } t \to \infty,$$

which means that ultimately the disease spreads through the entire population.

Separating the variables, $\frac{1}{y(1-y)}dy = \alpha dy$.
By partial fractions, $\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$.
$1 = A(1 - y) + B(y)$.
Thus, $A = 1$, $B = 1$.
The equation becomes $\frac{1}{y}dy + \frac{1}{1-y}dy = \alpha dt$.

Integrating, $\int \frac{1}{y}dy + \int \frac{1}{1-y}dy = \int \alpha dt$.
$\ln(y) - \ln(1-y) = \alpha t + c$
$e^{\ln(\frac{1}{1-y})} = e^{(\alpha t + c)}$
$|\frac{1}{1-y}| = Ce^{\alpha t}$
$\frac{1}{1-y} = \pm Ce^{\alpha t}$
Since α, $y \geq 0$, $\frac{y}{1-y} = Ce^{\alpha t}$.
$y = Ce^{\alpha t}(1-y)$
$y = Ce^{\alpha t} - Ce^{\alpha t}y$
$y + Ce^{\alpha t}y = Ce^{\alpha t}$
$y(1 + Ce^{\alpha t}) = Ce^{\alpha t}$
Thus, $y = \frac{Ce^{\alpha t}}{1+Ce^{\alpha t}}$ is the general solution.
Considering the initial condition, $y(0) = y_0$, $y(0) = \frac{C}{1+C}$ by substitution.
$y_0(1+C) = C$
$y_0 = C(1-y_0)$
$C = \frac{y_0}{1-y_0}$
By substitution, $y = \frac{\frac{y_0}{1-y_0}e^{\alpha t}}{1+\frac{y_0}{1-y_0}e^{\alpha t}}$.
Rearranging, $\frac{y_0 e^{\alpha t}}{1+y_0(e^{\alpha t}-1)}$ is the particular solution.
As $t \to \infty$, $Ce^{\alpha t} \to \infty$ and $1 + Ce^{\alpha t} \to \infty$.
Since $y = \frac{Ce^{\alpha t}}{1+Ce^{\alpha t}}$, $y \to \frac{\infty}{\infty} = 1$ as $t \to \infty$. Thus, $y(t) \to 1$ as $t \to \infty$.
This implies that the equilibrium solution $y = 1$ is asymptotically stable, as all the solutions converge to it.
When $y = 0$, $y_0 e^{\alpha t} = 0$, which is not feasible, so this equilibrium solution is asymptotically unstable.

Problem 2

Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$\frac{dy}{dt} = -\beta y. \tag{2}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$\frac{dx}{dt} = -\alpha xy. \tag{3}$$

(a) Determine y at any time t by solving equation (2) subject to the initial condition $y(0) = y_0$.
Separating the variables, $\frac{1}{y}dy = -\beta dt$.
Integrating, $\int \frac{1}{y}dy = -\int \beta dt$.
$\ln|y| = -\beta t + c$.
$e^{\ln|y|} = e^{-\beta t + c}$.
$y = e^c e^{-\beta t}$

Thus, $y = Ce^{-\beta t}$ is the general solution.
Considering the initial condition, $y(0) = y_0$, $y_0 = C$ by substitution.
So $y(t) = y_0 e^{-\beta t}$ is the particular solution.

(b) Use the result of part (a) to find x at any time t by solving equation (3) subject to the initial condition $x(0) = x_0$.

Separating the variables, $\frac{1}{x}dx = -\alpha y dt$.
Substituting from (a), $\frac{1}{x}dx = -\alpha y_0 e^{-\beta t}dt$.
Integrating, $\int \frac{1}{x}dx = -\alpha y_0 \int e^{-\beta t}dt$.
$\ln|x| = -\alpha y_0(\frac{-1}{\beta}e^{-\beta t} + c)$
$\ln|x| = \frac{\alpha y_0}{\beta}e^{-\beta t} + c$
$x = e^{\frac{\alpha y_0}{\beta}e^{-\beta t}+c}$
$x = e^c e^{\frac{\alpha y_0}{\beta}e^{-\beta t}}$
Thus, $x = Ce^{\frac{\alpha y_0}{\beta}e^{-\beta t}}$ is the general solution.
Considering the initial condition, $x(0) = x_0$, $x_0 = Ce^{\frac{\alpha y_0}{\beta}}$ by substitution.
$C = x_0 e^{\frac{-\alpha y_0}{\beta}}$.
By substitution, $x = x_0 e^{\frac{-\alpha y_0}{\beta}} e^{\frac{\alpha y_0}{\beta}e^{-\beta t}}$.
$x = x_0 e^{\frac{-\alpha y_0}{\beta} + \frac{\alpha y_0}{\beta}e^{-\beta t}}$
Thus, $x(t) = x_0 e^{\frac{\alpha y_0}{\beta}(e^{-\beta t}-1)}$.

(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

From (b), $x(t) = x_0 e^{\frac{\alpha y_0}{\beta}(e^{-\beta t}-1)}$.
As $t \to \infty$, $e^{-\beta t} - 1 \to -1$.
Thus, $x \to x_0 e^{\frac{-\alpha y_0}{\beta}}$ as $t \to \infty$.
So $x_0 e^{\frac{-\alpha y_0}{\beta}}$ of the population escapes the epidemic.

Problem 3

Find all functions M such that the equation is exact.

(a) $M(x,y)dx + (x^2 - y^2)dy = 0$

Let $N(x,y) = (x^2 - y^2)$.
For the equation to be exact, $M_y = N_x$ must hold.
$N_x = \frac{d}{dx}(x^2 - y^2) = 2x$.
By substitution, $M_y = 2x$.
Thus, $M = \int M_y dy = \int 2x dy = 2xy + \phi(x)$, where $\phi(x)$ is an arbitrary function of x.

(b) $M(x,y)dx + 2xy \sin x \cos y dy = 0$

Let $N(x,y) = 2xy \sin x \cos y$.
For the equation to be exact, $M_y = N_x$ must hold.
$N_x = \frac{d}{dx}(2xy \sin x \cos y)$.

By the product rule, $N_x = (2xy)'(\sin x \cos y) + (2xy)(\sin x \cos y)'$.
$N_x = (2y)(\sin x \cos y) + (2xy)(\cos x \cos y)$
$N_x = 2y \sin x \cos y + 2xy \cos x \cos y = 2y \cos y(\sin x + x \cos x)$
By substitution, $M_y = 2y \sin x \cos y + 2xy \cos x \cos y$.
Thus, $M(x, y) = \int M_y dy = \int 2y \sin x \cos y dy + \int 2xy \cos x \cos y dy$
Evaluating the first integral, $2 \int y \sin x \cos y dy = \sin x \int 2y \cos y dy$.
Integrating by parts, $2 \sin x \int y \cos y dy = 2 \sin x(y \sin y - \int \sin y dy) = 2 \sin x(y \sin y + \cos y + c)$.
Evaluating the second integral, $\int 2xy \cos x \cos y dy = 2x \cos x \int y \cos y dy = 2x \cos x(y \sin y + cosy)$.
Combining, $M(x, y) = 2 \sin x(y \sin y + \cos y) + 2x \cos x(y \sin y + cosy) + \phi(x) = (2 \sin x + 2x \cos x)(y \sin y + \cos y) + \phi(x)$, where $\phi(x)$ is some arbitrary equation of x.

(c) $M(x, y)dx + (e^x - e^y sinx)dy = 0$

Let $N(x, y) = e^x - e^y sinx$.
For the equation to be exact, $M_y = N_x$ must hold.
By the product rule, $N_x = e^x - (e^y)(\sin x)' + (e^y)'(\sin x) = e^x - e^y \cos x$.
By substitution, $M_y = e^x - e^y \cos x$.
Thus, $M(x, y) = \int M_y dy = \int e^x - e^y \cos x dy = e^x y - e^y \cos x + \phi(x)$, where $\phi(x)$ is some function of x.

Problem 4

Check whether the following differential equations are exact. If they are not, determine an appropriate integrating factor and solve them.

(a) $(x + 2) \sin y + (x \cos y)y' = 0$

Let $M = (x + 2) \sin y$, $N = x \cos y$.
Then, $M_y = (x + 2)(\cos y)$ and $N_x = \cos y$.
Since $M_y \neq N_x$, the equation is not exact.
$\mu = \mu(x)$, so $\frac{M_y - N_x}{N} = \frac{x+1}{x} = 1 + \frac{1}{x}$.
Thus, $\mu(x) = e^{\int 1 + \frac{1}{x} dx} = e^{x + \ln x} = xe^x$.
So $M_{\text{new}} = x^2 e^x \sin y + 2xe^x \sin y$ and $N_{\text{new}} = x^2 e^x \cos y$.
Thus, $M_{\text{new, } y} = N_{\text{new, } x} = xe^x \cos y(x + 2)$.
Integrating M_{new} w.r.t. x, $f(x, y) = \int x^2 e^x \sin y dx + \int 2xe^x \sin y dx$.
Integrating by parts, $f(x, y) = x^2 e^x \sin y + \phi(x)$, where $\phi(y)$ is some arbitrary function of y.
Differentiating f w.r.t. y, $f_y = x^2 e^x \cos y + \phi'(y)$.
Comparing with $N_{\text{new}} = x^2 e^x \cos y$, $\phi'(x) = 0$.
Hence, $f(x, y) = x^2 e^x \sin y$.

(b) $y + (2xy - e^{-2y})y' = 0$

Let $M = y$, $N = 2xy - e^{-2y}$.
Then, $M_y = 1$ and $N_x = 2y$.

Since $M_y \neq N_x$, the equation is not exact.
$\mu = \mu(y)$, so $\frac{M_y - N_x}{M} = \frac{1-2y}{y} = \frac{1}{y} - 2$.
Thus, $\mu(y) = \int e^{\frac{1}{y}-2} dy = e^{\ln|y| - 2y} = \frac{e^{2y}}{y}$.
So $M_{\text{new}} = e^{2y}$ and $N_{\text{new}} = 2xe^{2y} - \frac{1}{y}$.
Thus, $M_{\text{new},\, y} = N_{\text{new},\, x} = 2e^{2y}$.
Integrating M_{new} w.r.t. x, $f(x,y) = \int e^{2y} dx = xe^{2y} + \phi(y)$, where $\phi(y)$ is some arbitrary function of y.
Differentiating f w.r.t. y, $f_y = 2xe^{2y} + \phi'(x)$.
Comparing with $N_{\text{new}} = 2xe^{2y} - \frac{1}{y}$, $\phi'(y) = -\frac{1}{y}$.
Thus, $\phi(y) = \ln y$.
Hence, $f(x,y) = xe^{2y} + \ln x$.