AMATH 351

Homework 7

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Problem 1

Find (i) the recurrence formula and (ii) the general solution of the given differential equation by the power series method around $x = 0$.

(a) $y'' - xy' + 2y = 0$

(i) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
Then, $y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.
Converting xy' into a Taylor series, $x\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^n$.
Since the $n = 0$ term of $\sum_{n=1}^{\infty} na_n x^n$ is 0, $\sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n$.
By substitution, the differential equation becomes $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$.
To make all the powers the same, shift $n \to n + 2$ in the first term.
Thus, $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$.
Combining, $\sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} + (2-n)a_n)x^n = 0$.
x^n varies, so $(n+2)(n+1)a_{n+2} + (2-n)a_n = 0$.
Rearranging, $(n+2)(n+1)a_{n+2} = -(2-n)a_n$.
Thus, the recurrence relation is $a_{n+2} = \frac{a_n(n-2)}{(n+2)(n+1)}$ for all $n \geq 0$.

(ii) Applying the recurrence relation, when $n = 0$, $a_{n+2} = -a_0$.
When $n = 1$, $a_3 = \frac{-1}{6}a_1$.
When $n = 2$, $a_4 = 0$.
When $n = 3$, $a_5 = \frac{-1}{120}a_1$.
When $n = 4$, $a_6 = 0$.
When $n = 5$, $a_7 = \frac{-a_1}{1680}$.
Let $a_0 = C_0$ and $a_1 = C_1$, where C_0 and C_1 are arbitrary constants.
Thus, $y_{even} = C_0(1 - x^2)$ and $y_{odd} = C_1(x - \frac{x^3}{6} - \frac{x^5}{120} - \frac{x^7}{1680} - ...)$.
Hence, the general solution is $y = C_0(1 - x^2) + C_1(x - \frac{x^3}{6} - \frac{x^5}{120} - \frac{x^7}{1680} - ...)$.

(b) $y'' - x^2y' - y = 0$

(i) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
Then, $y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.
Converting x^2y' into a Taylor series, $x^2y' = x^2\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^{n+1}$.
By substitution, the differential equation becomes $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n$.

To make all the powers the same, shift $n \to n+2$ in the first term and $n \to n-1$ in the second term.
Thus, $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $\sum_{n=1}^{\infty} na_n x^{n+1} = \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n$.
Thus, $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n - \sum_{n=0}^{\infty} a_n x^n = 0$.
Since x_n varies, $(n+1)(n+2)a_{n+2} - a_n = 0$ for $n < 2$ and $(n+1)(n+2)a_{n+2} - (n-1)a_{n-1} - a_n$ for $n \geq 2$.
First, considering $n < 2$, $(n+2)(n+1)a_{n+2} = a_n$.
Thus the recurrence relation is $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$ for all $n < 2$.
Now, considering $n \geq 2$, $(n+2)(n+1)a_{n+2} = (n-1)a_{n-1} + a_n$.
Thus the recurrence relation is $a_{n+2} = \frac{(n-1)a_{n-1}+a_n}{(n+2)(n+1)}$ for all $n \geq 2$.

(ii) First apply the recurrence relation $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$ for $n < 2$.
When $n = 0$, $a_2 = \frac{a_0}{2}$.
When $n = 1$, $a_3 = \frac{a_1}{6}$.
Now apply the recurrence relation $a_{n+2} = \frac{(n-1)a_{n-1}+a_n}{(n+2)(n+1)}$ for $n \geq 2$.
When $n = 2$, $a_4 = \frac{a_0}{24} + \frac{a_1}{12}$.
When $n = 3$, $a_5 = \frac{a_0}{20} + \frac{a_1}{120}$.
When $n = 4$, $a_6 = \frac{a_0}{720} + \frac{7a_1}{360} +$.
When $n = 5$, $a_7 = \frac{13a_0}{2520} + \frac{41a_1}{5040}$.
Let $a_0 = C_0$ and $a_1 = C_1$, where C_0 and C_1 are arbitrary constants.
Thus $y_0 = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^5}{20} + \frac{x^6}{720} + ...$ and $y_1 = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{7x^6}{360} + \frac{41x^7}{5040} + ...$
Hence, the general solution is $y = C_0(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^5}{20} + \frac{x^6}{720} + ...) + C_1(x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{7x^6}{360} + \frac{41x^7}{5040} + ...)$.

Problem 2

Determine whether the given values of x are ordinary points or singular points of the given differential equations. If it is a singular point, classify it further as a regular singular point or an irregular singular point.

(a) $x = 2$; $(x-2)y'' + 3(x^2 - 3x + 2)y' + (x-2)^2 y = 0$

Let $P(x) = x - 2$, $Q(x) = 3(x^2 - 3x + 2)$, and $R(x) = (x-2)^2$.
$P(x) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$.
At $x = 2$, $P(x)$ vanishes, so $x = 2$ is a singular point.
$\lim_{x\to\infty}(x-2)\frac{3(x^2-3x+2)}{x-2} = \lim_{x\to\infty} 3(x^2 - 3x + 2) \to 3(4 - 6 + 2) = 0$, which is finite.
$\lim_{x\to 2}(x-2)^2\frac{(x-2)^2}{x-2} = \lim_{x\to 2}(x-2)^3 \to 0$, which is finite.
Thus, $x = 2$ is a regular singular point.

(b) $x = -1$; $(x+1)^3 y'' + (x^2 - 1)(x+1)y' + (x-1)y = 0$

Let $P(x) = (x+1)^3$, $Q(x) = (x^2 - 1)(x+1)$, and $R(x) = (x-1)$.
$P(x) = 0 \Rightarrow (x+1)^3 = 0 \Rightarrow x = -1$.

At $x = -1$, $P(x)$ vanishes, so $x = -1$ is a singular point.
$\lim_{x\to -1}(x+1)\frac{(x^2-1)(x+1)}{(x+1)^3} = \lim_{x\to -1}\frac{x^2-1}{x+1} \to \frac{0}{0}$.
By L'Hôpital's Rule, $\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x^2-1)'}{(x+1)'} = \lim_{x\to -1}\frac{2x}{1} \Rightarrow -2$, which is finite.
$\lim_{x\to -1}(x+1)^2\frac{x-1}{(x+1)^3} = \lim_{x\to -1}\frac{x-1}{x+1} \Rightarrow \infty$, which is not finite.
Thus, $x = -1$ is an irregular singular point.

(c) $(\sin x)y'' + xy' + 4y = 0$

Let $P(x) = \sin x$, $Q(x) = x$, and $R(x) = 4$.
$P(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = c\pi$ for some constant c.
At $x = c\pi$, $P(x)$ vanishes, so all $x = c\pi$ are singular points.
$\lim_{x\to c\pi}(x-c\pi)\frac{x}{\sin x} \to \frac{0}{0}$.
By L'Hôpital's Rule, $\lim_{x\to c\pi}\frac{2x-c\pi}{\cos x} \to c\pi$, which is finite.
$\lim_{x\to c\pi}(x-c\pi)^2\frac{4}{\sin x} \to \frac{0}{0}$.
By L'Hôpital's Rule, $\lim_{x\to c\pi}(x-c\pi)^2\frac{4}{\sin x} = \lim_{x\to c\pi}\frac{8(x-c\pi)}{\cos x} \to 0$, which is finite.
Thus, all $x = c\pi$ are regular singular points.

Problem 3

Find all values of α for which all solutions of
$$x^2y'' + \alpha xy' + \frac{5}{2}y = 0$$
approach zero as $x \to 0$.

This is an Euler-Cauchy equation with $a = \alpha$ and $b = \frac{5}{2}$.
Thus, the solution is of the form $y = x^m$ and the indicial equation is $m^2 + (\alpha-1)m + \frac{5}{2} = 0$.
Hence, the roots of the equation are $m = \frac{1-\alpha\pm\sqrt{(\alpha-1)^2-10}}{2}$.
First, consider the case where $(\alpha-1)^2 > 10$.
This gives two distinct real roots $m_1 \neq m_2$ where $m_1 = \frac{1-\alpha+\sqrt{(\alpha-1)^2-10}}{2}$ and $m_2 = \frac{1-\alpha-\sqrt{(\alpha-1)^2-10}}{2}$.
Thus, the general solution is $y = C_1x^{\frac{1-\alpha+\sqrt{(\alpha-1)^2-10}}{2}} + C_2x^{\frac{1-\alpha-\sqrt{(\alpha-1)^2-10}}{2}}$.
Thus, $m_1 + m_2 = 1-\alpha$ and $m_1m_2 = \frac{5}{2} \Rightarrow 1-\alpha > 0 \Rightarrow \alpha < 1$.
Now, consider the case where $(\alpha-1)^2 = 10$.
This gives the repeated root $m = \frac{1-\alpha}{2}$.
Thus, the general solution is $y = C_1x^{\frac{1-\alpha}{2}} + C_2x^{\frac{1-\alpha}{2}}\ln x$.
For $y \to 0$ as $x \to 0$, we need $\frac{1-\alpha}{2} > 0 \Rightarrow 1-\alpha > 0 \Rightarrow \alpha < 1$.
Finally, consider the case where $(\alpha-1)^2 < 10$.
This gives two complex conjugate roots m_1 and m_2 where $m_1 = \frac{1-\alpha}{2} + \frac{(\alpha-1)^2-10}{2}$ and $m_2 = \frac{1-\alpha}{2} - \frac{(\alpha-1)^2-10}{2}$.
For $y \to 0$ as $x \to 0$, we need $\frac{1-\alpha}{2} > 0 \Rightarrow 1-\alpha > 0 \Rightarrow \alpha < 1$.
Thus, the solutions of $x^2y'' + \alpha xy' + \frac{5}{2}y = 0$ that approach zero as $x \to 0$ are all $\alpha < 1$.

Problem 4

Find the solution of the given initial-value problem. Describe how the solution behaves as $x \to 0$.

$$x^2y'' - 3xy' + 4y = 0,\ y(-1) = 2,\ y'(-1) = 3.$$

This is an Euler-Cauchy equation with $a = -3$ and $b = 4$.
Thus, the solution is of the form $y = x^m$ and the indicial equation is $m^2 - 4m + 4 = 0 \to (m-2)^2 = 0$.
This gives the repeated root $m = 2$.
Thus, the general solution is $y(x) = C_1x^2 + C_2x^2 \ln|x|$.
Considering the initial condition $y(-1) = 2$, $C_1(-1)^2 + C_2(-1)^2 \ln|-1| = 2 \Rightarrow C_1 + C_2 \ln 1 = 2 \Rightarrow C_1 = 2$.
Differentiating, $y'(x) = 2C_1x + \frac{C_2}{x} + \ln|x|$.
Considering the initial condition $y'(-1) = 3$, $2C_1(-1) + \frac{C_2}{(-1)} + \ln|-1| = 3 \Rightarrow -2C_1 - C_2 + \ln 1 = 3 \Rightarrow -2C_1 - C_2 = 3 \Rightarrow -2(2) - C_2 = 3 \Rightarrow C_2 = -7$.
Thus, the particular solution is $y(x) = 2x^2 - 7x^2 \ln|x|$.
As $x \to 0$, $y \to 2(0)^2 - 7(0)^2 \ln|0| = 0$.
Thus, $y \to 0$ as $x \to 0$.