

23.08.2019

# Digital Image Processing (CSE/ECE 478)

## Lecture-8: Image Enhancement in Frequency Domain – FT of sampled functions, DFT

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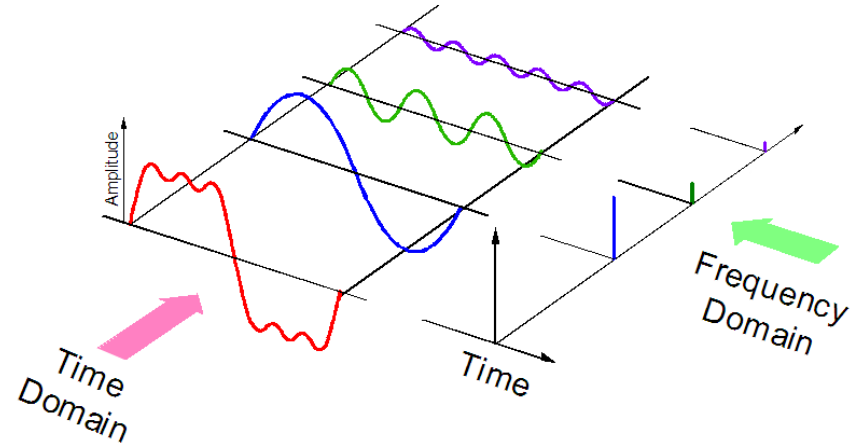


# Announcements

- Tutorial : Fourier Series and Transforms
- Quiz-1 : Wednesday, 9.00 am – 9.45 am, Room: H-205
  - Syllabus: Up to today's lecture
  - Format: **Derivations and Numericals**

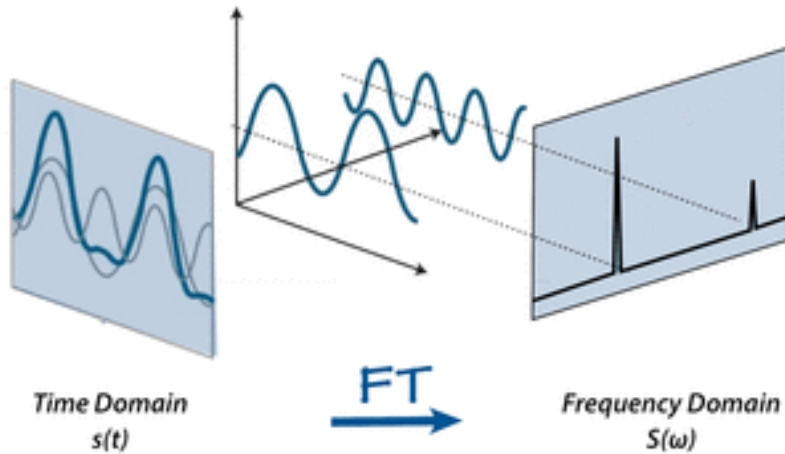
# Fourier Series

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n t}{T}}$$
$$c_n = \frac{1}{T} \int_0^T f(t) e^{-i \frac{2\pi n t}{T}} dt$$



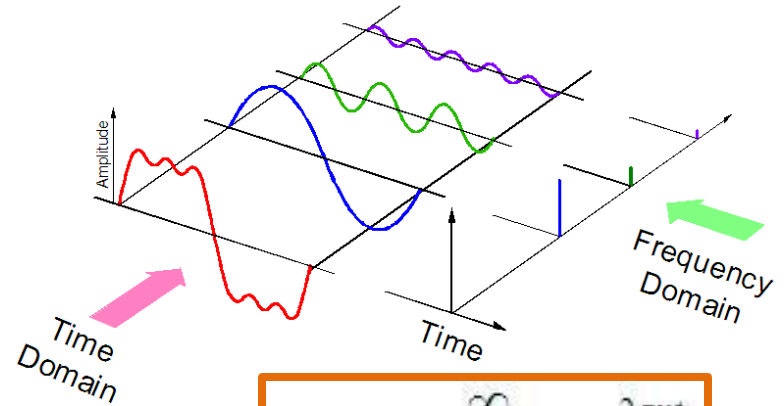
# Fourier Transform vs Series

Fourier Transform



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

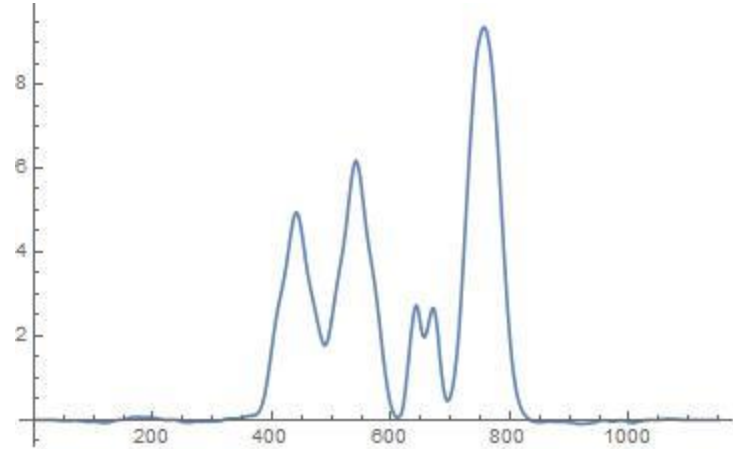
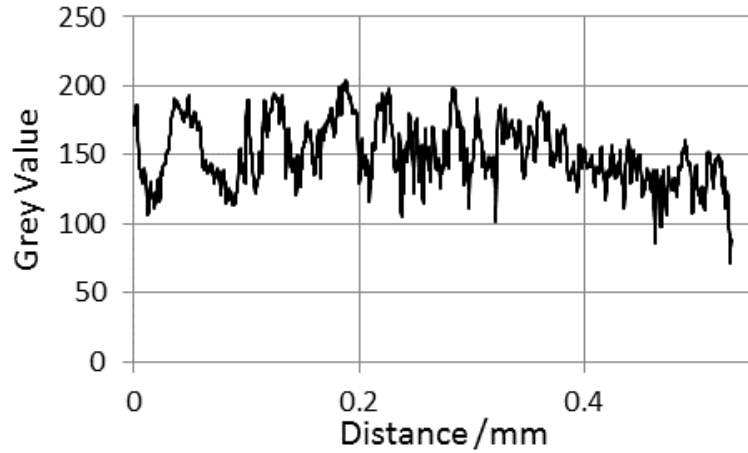
Fourier Series



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n t}{T}}$$

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-i \frac{2\pi n t}{T}} dt$$

# What if $x(t)$ is non-periodic ?



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n t}{T}}$$

# Fourier Transform

- Fourier Transform

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t) e^{-i\omega t} dt$$

*angular frequency*  
 $\omega = 2\pi f$

- Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} X(\omega) e^{i\omega t} d\omega$$

# Fourier Transform (G&W version)

- Fourier Transform

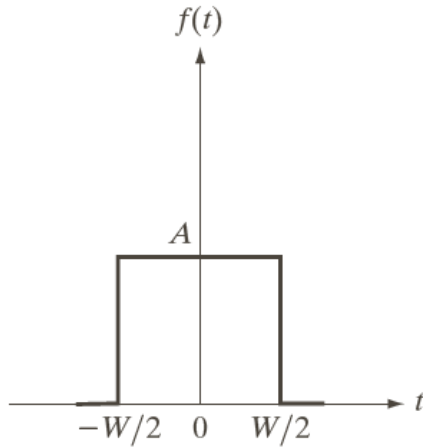
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j\underline{2\pi\mu t}} dt$$

- Inverse Fourier Transform

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j\underline{2\pi\mu t}} d\mu$$

# FT of box function

$$f(t) = \begin{cases} A & \text{if } -\frac{W}{2} \leq t \leq \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

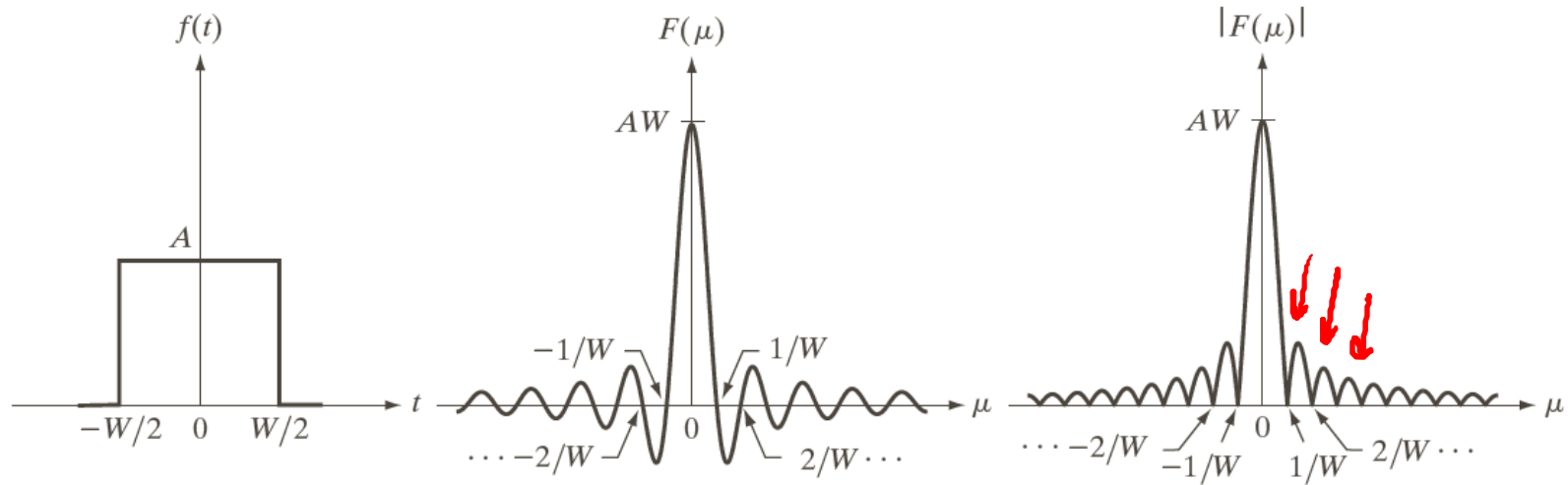


$$\downarrow$$
$$\frac{2A \sin(\pi u w)}{2\pi u}$$



# FT of box function

$$AW \frac{\sin(\pi\mu W)}{\pi\mu W}$$



a b c

**FIGURE 4.4** (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

# (Some) Properties of FT

	Name:	Condition:	Property:
1	Amplitude scaling	$f(t) \leftrightarrow F(\omega)$ , constant $K$	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$ , $g(t) \leftrightarrow G(\omega)$ , ...	$f(t) + g(t) + \dots \leftrightarrow F(\omega) + G(\omega) + \dots$
3	Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	Even	Real and even $f(t)$	Real and even $F(\omega)$
5	Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$
→ 6	Symmetry	$f(t) \leftrightarrow F(\omega)$	$F(t) \leftrightarrow 2\pi f(-\omega)$
7	Time scaling	$f(t) \leftrightarrow F(\omega)$ , real $s$	$f(st) \leftrightarrow \frac{1}{ s } F(\frac{\omega}{s})$
8	Time shift	$f(t) \leftrightarrow F(\omega)$	$f(t - t_o) \leftrightarrow F(\omega)e^{-j\omega t_o}$
9	Frequency shift	$f(t) \leftrightarrow F(\omega)$	$f(t)e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$
10	Modulation	$f(t) \leftrightarrow F(\omega)$	$f(t) \cos(\omega_o t) \leftrightarrow \frac{1}{2}F(\omega - \omega_o) + \frac{1}{2}F(\omega + \omega_o)$
11	Time derivative	Differentiable $f(t) \leftrightarrow F(\omega)$	$\frac{df}{dt} \leftrightarrow j\omega F(\omega)$
12	Freq derivative	$f(t) \leftrightarrow F(\omega)$	$-jtf(t) \leftrightarrow \frac{d}{d\omega} F(\omega)$
13	Time convolution	$f(t) \leftrightarrow F(\omega)$ , $g(t) \leftrightarrow G(\omega)$	$f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$
14	Freq convolution	$f(t) \leftrightarrow F(\omega)$ , $g(t) \leftrightarrow G(\omega)$	$f(t)g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$
15	Compact form	Real $f(t)$	$f(t) = \frac{1}{2\pi} \int_0^\infty 2 F(\omega)  \cos(\omega t + \angle F(\omega)) d\omega$
16	Parseval, Energy W	$f(t) \leftrightarrow F(\omega)$	$W \equiv \int_{-\infty}^\infty  f(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty  F(\omega) ^2 d\omega$

$b(x) = f(-x)$   
 $f(x) = -f(x)$

←

}

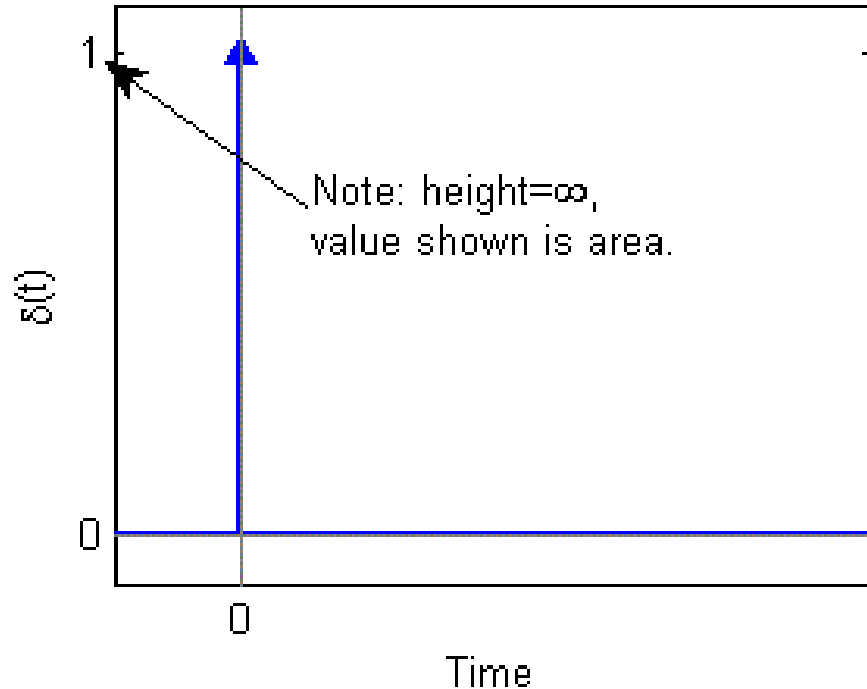
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←

# Unit Impulse Function

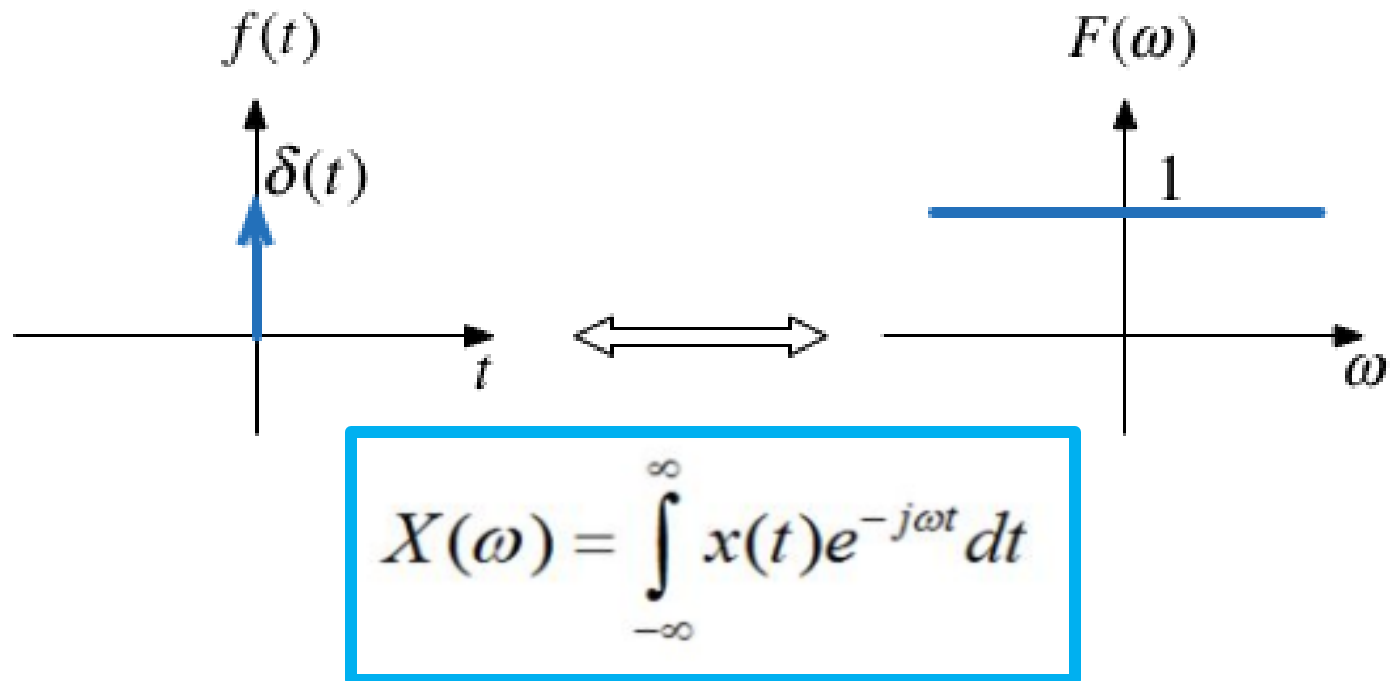
Impulse ( $\delta(t)$ ) = Derivative of step



$$\delta(t) = 0, \text{ for } t \neq 0.$$
$$\delta(0) = +\infty$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

# FT of impulse function



# Impulse: Sifting Property

$$\int_a^b \delta(t) dt = \begin{cases} 1 & a < 0 < b \\ 0 & \text{otherwise} \end{cases}$$

# Impulse: Sifting Property

$$\int_a^b \delta(t) dt = \begin{cases} 1, & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$



$$\int_a^b \delta(t) \cdot f(t) dt$$

$$\begin{aligned} & a < 0 < b \\ & \text{otherwise} \end{aligned}$$

# Time-shifted Impulse: Sifting Property

Time-shifted impulse

$$\delta(t - t_0)$$



# Time-shifted Impulse: Sifting Property

Time-shifted impulse

$$\int_a^b \delta(t - T) \cdot f(t) dt = \begin{cases} f(T) & a < T < b \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \int_a^b \delta(t) \cdot f(t) dt &= \int_a^b \delta(t) \cdot f(0) dt \\ &= f(0) \cdot \int_a^b \delta(t) dt \\ &= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$



# FT of impulse function

$$\int_a^b \delta(t) \cdot f(t) dt = \int_a^b \delta(t) \cdot f(0) dt$$

$$= f(0) \cdot \int_a^b \delta(t) dt$$

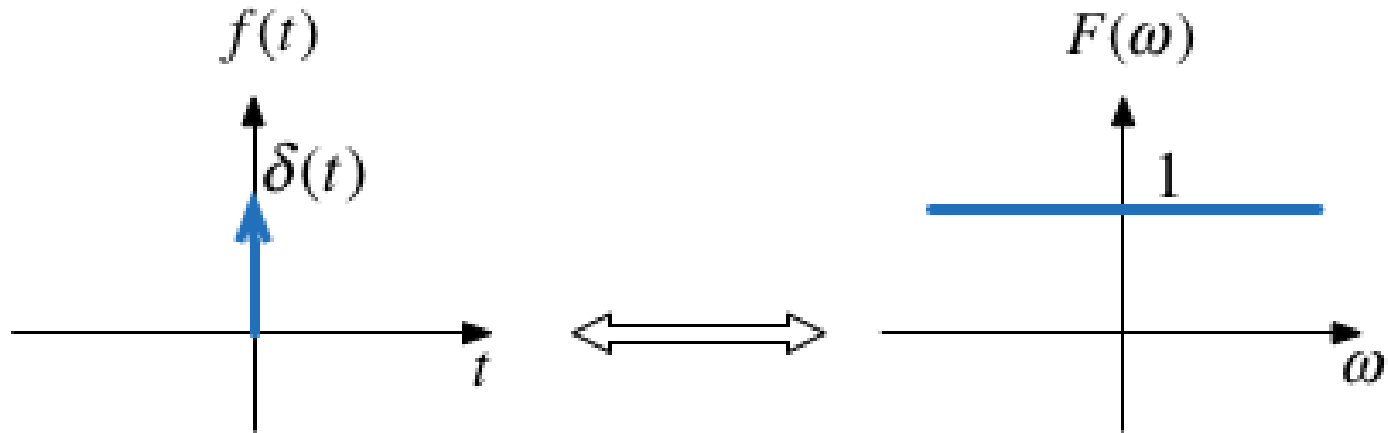
$$= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$



$$f(t) = \delta(t - T)$$

$$X(\omega) = e^{-j\omega T}$$

# FT of impulse function



# Time-shifted Impulse

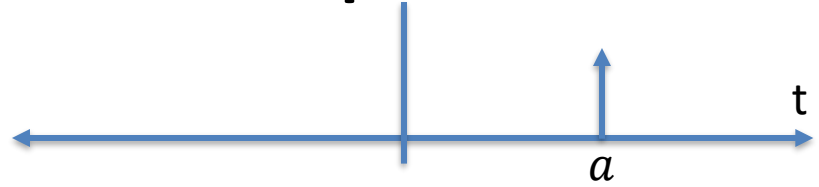
Time-shifted impulse

Sifting Property

$$\int_a^b \delta(t - T) \cdot f(t) dt = \begin{cases} f(T), & a < T < b \\ 0, & \text{otherwise} \end{cases}$$

# FT of time-shifted impulse

$$f(t) = \delta(t - a)$$



$$\mathcal{F}[\delta(t - a)] = F(\mu) = e^{-j2\pi\mu a}$$

# Symmetry property of FT

- $f(t) \leftrightarrow F(\mu)$
- $F(t) \leftrightarrow f(-\mu)$

# FT of complex exponential

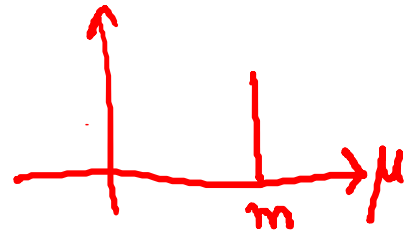
$$\delta(t-a) \longleftrightarrow e^{-j\omega a} \quad e^{-j2\pi\mu a}$$

$$\delta(t-a) \longleftrightarrow e^{-j2\pi\mu a}$$


$$e^{-j2\pi t a} \longleftrightarrow \delta(-\mu - a)$$

$$e^{j2\pi t m} \longleftrightarrow \delta(-\mu + m)$$

$$e^{j2\pi t m} \longleftrightarrow \delta(\mu - m)$$



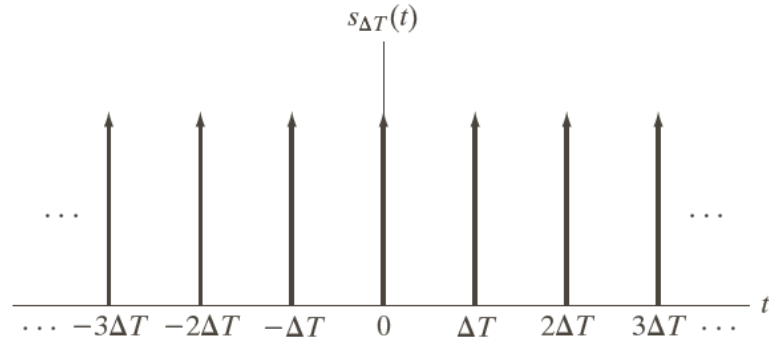
# FT of a periodic function

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n t / \Delta T}$$


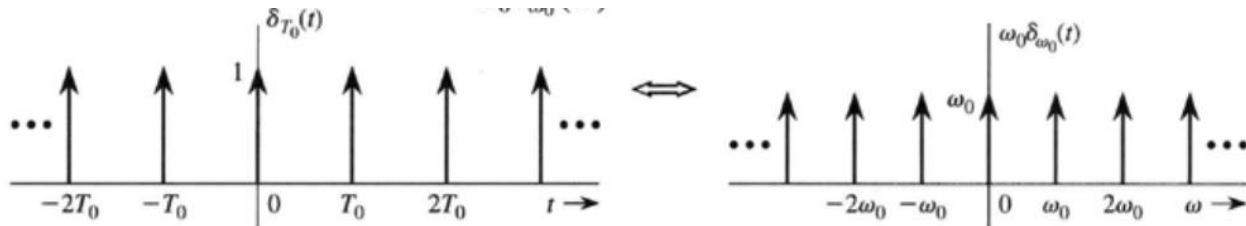
$$c_n = \frac{1}{\Delta T} \int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} s_{\Delta T}(t) e^{-j2\pi n t / \Delta T} dt$$

$$\mathcal{F}(s_{\Delta T}(t)) = \sum_{n=-\infty}^{\infty} c_n \delta(\mu - \frac{n}{\Delta T})$$

# FT of impulse train

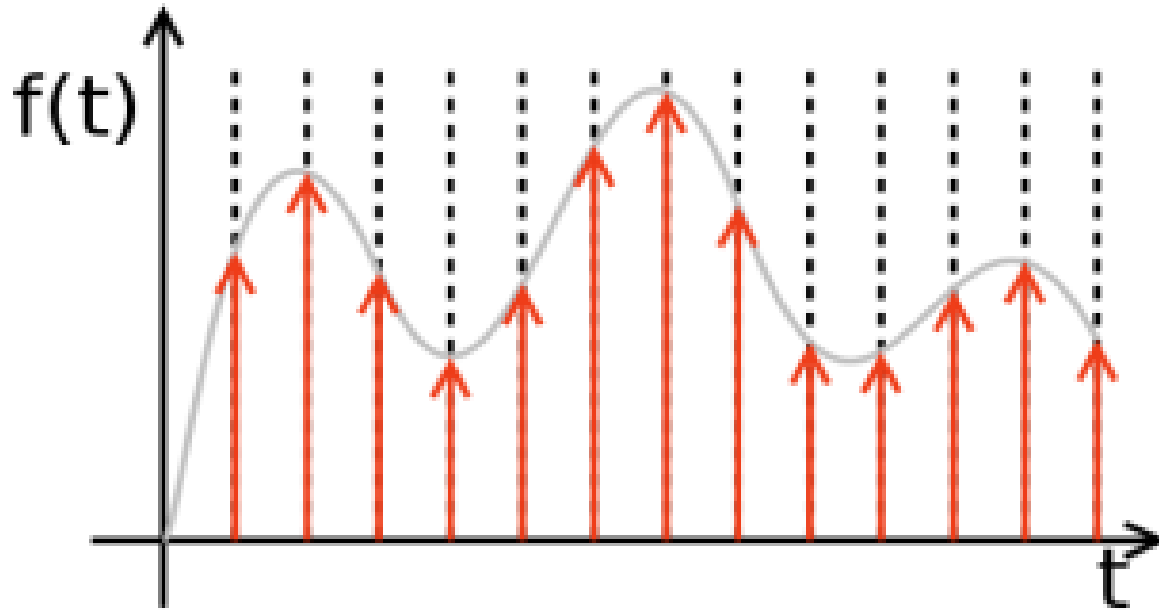


$$\mathcal{F}(s_{\Delta T}(t)) = \frac{1}{\Delta T} \sum_{n=-\infty}^{n=\infty} \delta(\mu - \frac{n}{\Delta T})$$

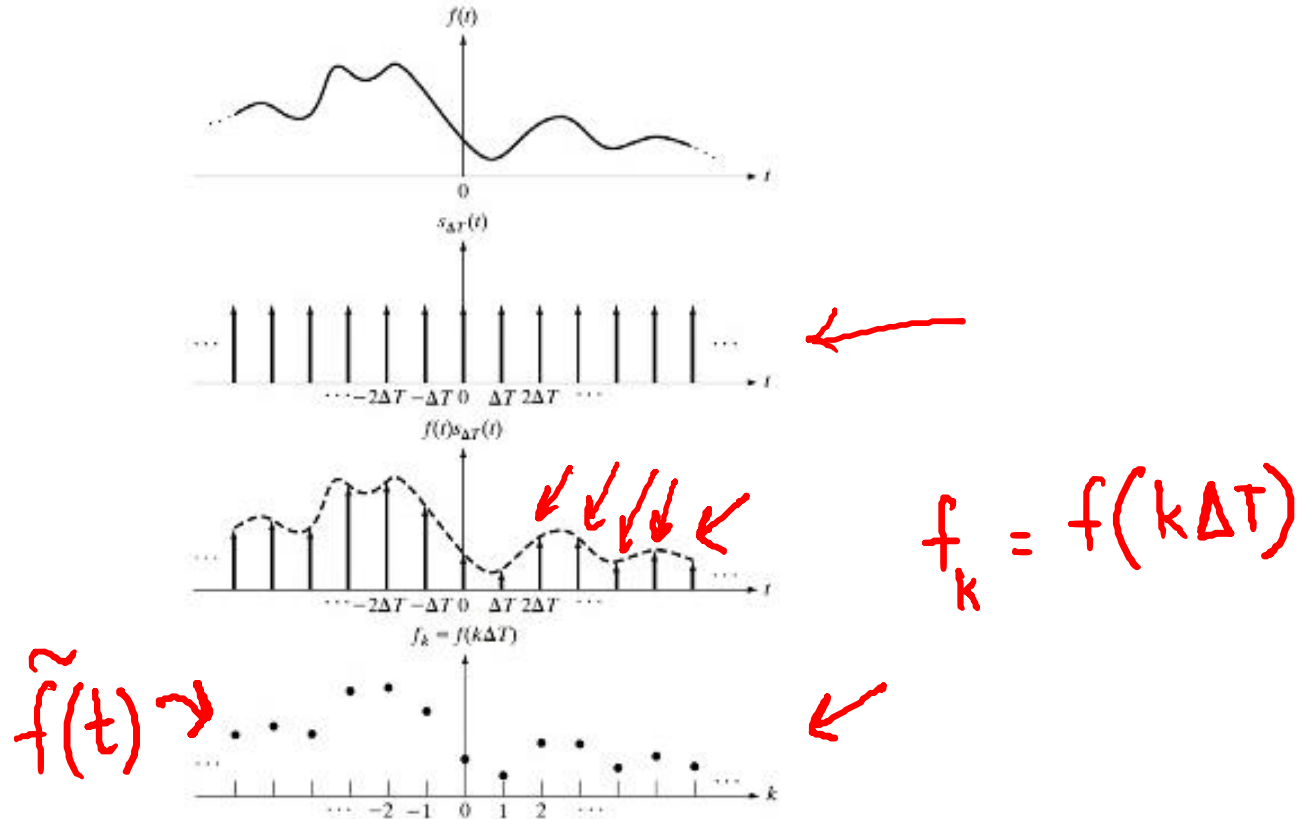




# Sampling



# Sampling = $f(t)$ x Impulse Train



# FT of sampled function

$$f(t) \leftrightarrow F(\mu)$$

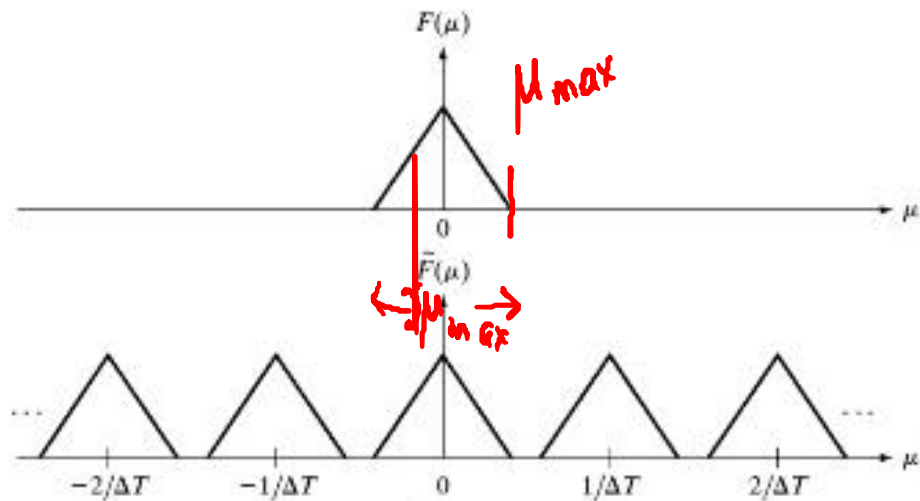


$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

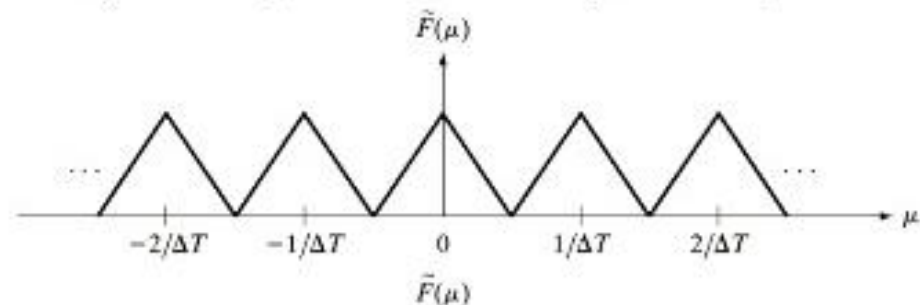
# FT of sampled function : properties

- Continuous
- Periodic (copies of  $f(t)$ 's FT)
- NOTE: FT is continuous

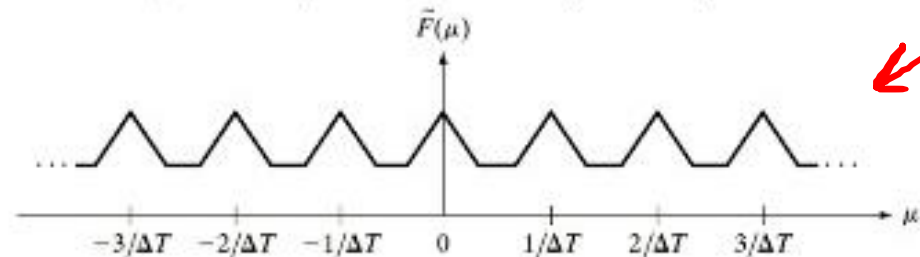
Oversampling



Critically sampling



Undersampling



# References & Fun Reading/Viewing

- GW DIP textbook, 3<sup>rd</sup> Ed.,
  - 4.2.4
  - 4.2.5
  - 4.3.1
  - 4.3.2 (FT of sampled functions)
- <http://www.thefouriertransform.com/>
- A visual introduction to Fourier Transform:  
<https://www.youtube.com/watch?v=spUNpyF58BY>
- Fourier Transform, Fourier Series and Frequency Spectrum:  
<https://www.youtube.com/watch?v=r18Gi8ISkfM>