fuzzycreator:
Measures on Fuzzy Sets within the Toolkit

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This document lists the equations and references of all measures provided as part of the fuzzycreator toolkit. All measures can be found within the appropriate file in the measures/ directory.

Constants:
`n = global_settings.global_x_disc`
`m = global_settings.global_alpha_disc`
`k = global_settings.global_zlevel_disc`

Contents

1 Similarity

1.1 Type-1

pappis1 [1]

$$s_{p1}^{T1}(A, B) = 1 - \max_i |\mu_A(x_i) - \mu_B(x_i)| \tag{1}$$

pappis2 [1]

$$s_{p2}^{T1}(A, B) = 1 - \frac{\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)|)}{\sum_{i=1}^{n}(\mu_A(x_i) + \mu_B(x_i))} \tag{2}$$

pappis3 [1]

$$s_{p3}^{T1}(A, B) = 1 - \frac{\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)|)}{n} \tag{3}$$

jaccard [2]

$$s_j^{T1}(A, B) = \frac{\sum_{i=1}^{n} \min(\mu_A(x_i), \mu_B(x_i))}{\sum_{i=1}^{n} \max(\mu_A(x_i), \mu_B(x_i))} \tag{4}$$

dice [3]

$$s_d(A, B) = \frac{2\sum_{i=1}^{n} \min(\mu_A(x_i), \mu_B(x_i))}{\sum_{i=1}^{n} \mu_A(x_i) + \sum_{i=1}^{n} \mu_B(x_i)}. \tag{5}$$

zwick [4]

$$s_z^{T1}(A, B) = \sup_i \mu_{A\cap B}(x_i) \tag{6}$$

chen [5]

$$s_c^{T1} = \frac{\sum_{i=1}^{n} \mu_A(x_i) \cdot \mu_A(x_i)}{\max\{\sum_{i=1}^{n} \mu_A(x_i)^2, \sum_{i=1}^{n} \mu_A(x_i)^2\}} \tag{7}$$

vector [6]

$$s_w^{T1}(A, B) = s_j^{T1}(A, B) s_{w_d}(A, B) \tag{8a}$$

$$s_{w_d}(A, B) = e^{-rd(A,B)} \tag{8b}$$

where $d(A, B) = |c(A) - c(B)|$, $c(A)$ refers to the centroid of set A, and where $r \equiv 4/|X|$ is a positive constant in which $|X|$ is the length of support of $A \cup B$.

1.2 Interval Type-2

zeng_li [7]

$$s_{zl}^{IT2}(\tilde{A}, \tilde{B}) = 1 - \frac{1}{2n}\sum_{i=1}^{n}(|\underline{\mu}_{\tilde{A}}(x_i) - \underline{\mu}_{\tilde{B}}(x_i)| + |\overline{\mu}_{\tilde{A}}(x_i) - \overline{\mu}_{\tilde{B}}(x_i)|), \tag{9}$$

`gorzalczany` [8]

$$S_G(\tilde{A},\tilde{B}) = [S^L(\tilde{A},\tilde{B}),\ S^U(\tilde{A},\tilde{B})] \tag{10a}$$

$$S^L(\tilde{A},\tilde{B}) = \min\{S_1(\tilde{A},\tilde{B}), S_2(\tilde{A},\tilde{B})\} \tag{10b}$$

$$S^U(\tilde{A},\tilde{B}) = \max\{S_1(\tilde{A},\tilde{B}), S_2(\tilde{A},\tilde{B})\} \tag{10c}$$

$$S_1(\tilde{A},\tilde{B}) = \frac{\max_i(\min[\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i)])}{\max_i \underline{\mu}_{\tilde{A}}(x_i)} \tag{10d}$$

$$S_2(\tilde{A},\tilde{B}) = \frac{\max_i(\min[\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{B}}(x_i)])}{\max_i \overline{\mu}_{\tilde{A}}(x_i)} \tag{10e}$$

`bustince` [9]

$$s_b^{IT2}(\tilde{A},\tilde{B}) = [s_{b_L}(\tilde{A},\tilde{B}), s_{b_U}(\tilde{A},\tilde{B})] \tag{11a}$$

$$s_{b_L}(\tilde{A},\tilde{B}) = \Upsilon_L(\tilde{A},\tilde{B}) \star \Upsilon_L(\tilde{B},\tilde{A}) \tag{11b}$$

$$s_{b_U}(\tilde{A},\tilde{B}) = \Upsilon_U(\tilde{A},\tilde{B}) \star \Upsilon_U(\tilde{B},\tilde{A}) \tag{11c}$$

$$\Upsilon_L(\tilde{A},\tilde{B}) = \inf_i\{1, \min(1 - \underline{\mu}_{\tilde{A}}(x_i) + \underline{\mu}_{\tilde{B}}(x_i), 1 - \overline{\mu}_{\tilde{A}}(x_i) + \overline{\mu}_{\tilde{B}}(x_i))\} \tag{11d}$$

$$\Upsilon_U(\tilde{A},\tilde{B}) = \inf_i\{1, \max(1 - \underline{\mu}_{\tilde{A}}(x_i) + \underline{\mu}_{\tilde{B}}(x_i), 1 - \overline{\mu}_{\tilde{A}}(x_i) + \overline{\mu}_{\tilde{B}}(x_i))\} \tag{11e}$$

where $\star$ is the minimum t-norm.

`jaccard` [10, 11]

$$s_j^{IT2}(\tilde{A},\tilde{B}) = \frac{\sum_{i=1}^n \min(\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{B}}(x_i)) + \sum_{i=1}^n \min(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))}{\sum_{i=1}^n \max(\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{B}}(x_i)) + \sum_{i=1}^n \max(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))}. \tag{12}$$

`zheng` [12]

$$s_{zh}^{IT2}(\tilde{A},\tilde{B}) = \frac{1}{2}\left(\frac{\sum_{i=1}^n \min(\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{B}}(x_i))}{\sum_{i=1}^n \max(\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{B}}(x_i))} + \frac{\sum_{i=1}^n \min(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))}{\sum_{i=1}^n \max(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))}\right) \tag{13}$$

`vector` [10]
Same as the vector similarity measure on type-1 fuzzy sets, using the Jaccard measure for interval type-2 fuzzy sets.

1.3 General Type-2

`jaccard` [13]

$$s_j^{GT2}(\tilde{A},\tilde{B}) = \frac{\sum_{i=1}^k z_i s_j^{IT2}(\tilde{A}_{z_i}, \tilde{B}_{z_i})}{\sum_{i=1}^k z_i} \tag{14}$$

zhao_crisp [14]

$$s_{zh}^{GT2}(A,B) = \frac{1}{\Delta+1} \sum_{\alpha=0,\frac{1}{\Delta},\frac{2}{\Delta}\ldots,\frac{\Delta-1}{\Delta},1} s_j^{IT2}(\tilde{A}_{z_\alpha}, \tilde{B}_{z_\alpha}) \tag{15}$$

hao_fuzzy [15]
Note, this represents similarity as a discrete type-1 fuzzy set.

$$s_h^{GT2-F}(\tilde{A},\tilde{B}) = \bigcup_{\forall z_i} z_i / s_j^{IT2}(\tilde{A}_{z_i}, \tilde{B}_{z_i}), \tag{16}$$

hao_crisp [15]
Calculate the centroid of hao_fuzzy.

yang_lin [16, 17]

$$s_{yl}^{GT2}(\tilde{A},\tilde{B}) = \frac{1}{N} \sum_{x\in X} \frac{\sum_{u\in J_x} \min\{u\cdot f_x(u), u\cdot g_x(u)\}}{\sum_{u\in J_x} \max\{u\cdot f_x(u), u\cdot g_x(u)\}} \tag{17}$$

where $f_x(u)$ and $g_x(u)$ are the secondary membership functions $\mu_{\tilde{A}}(x,u)$ and $\mu_{\tilde{B}}(x,u)$, respectively.

mohamed_abdaala [18]

$$s_{ma}^{GT2}(\tilde{A},\tilde{B}) = \frac{1}{N} \sum_{x\in X} \frac{\min\{\sum_{u\in J_x} 1-u\cdot f_x(u), \sum_{u\in J_x} 1-u\cdot g_x(u)\}}{\max\{\sum_{u\in J_x} 1-u\cdot f_x(u), \sum_{u\in J_x} 1-u\cdot g_x(u)\}} \tag{18}$$

where $f_x(u)$ and $g_x(u)$ are the secondary membership functions $\mu_{\tilde{A}}(x,u)$ and $\mu_{\tilde{B}}(x,u)$, respectively.

hung_yang [19]

$$s_{hy}^{GT2}(\tilde{A},\tilde{B}) = 1 - d^N(\tilde{A},\tilde{B}) \tag{19a}$$

$$d^N(\tilde{A},\tilde{B}) = \frac{\sum_{i=1}^{n} H_f(\tilde{A}(x_i), \tilde{B}(x_i))}{n} \tag{19b}$$

$$H_f(A,B) = \frac{\sum_{i=1}^{m} \alpha_i H(A_{\alpha_i}, B_{\alpha_i})}{\sum_{i=1}^{n} \alpha_i} \tag{19c}$$

$$H(U,V) = \max\{L(U,V), L(V,U)\} \tag{19d}$$

$$L(U,V) = \inf\{\lambda \in [0,\infty] | U^\lambda \supset V\} \tag{19e}$$

2 Distance

2.1 Type-1

Hausdorff distance between alpha-cuts:

$$h(A_\alpha, B_\alpha) = \max\{|A_{\alpha_L} - B_{\alpha_L}|, |A_{\alpha_R} - B_{\alpha_R}|\} \tag{20}$$

where $A_\alpha = [A_{\alpha_L}, A_{\alpha_R}]$

`ralescu1` [20]

$$d_{r1}^{T1}(A, B) = \int_{\alpha=0}^{1} h(A_\alpha, B_\alpha) d\alpha \tag{21}$$

`ralescu2` [20]

$$d_{r2}^{T1}(A, B) = \sup_{\alpha>0} h(A_\alpha, B_\alpha) \tag{22}$$

`chaudhuri_rosenfeld` [21]

$$d_{cr}^{T1}(A, B) = \frac{\sum_{\alpha=1}^{m} y_\alpha \; h(A_\alpha, B_\alpha)}{\sum_{\alpha=1}^{m} y_\alpha} \tag{23}$$

`ban` [22]

$$d_b^{T1}(A, B) = \sqrt{\int_0^1 (A_{\alpha_L} - B_{\alpha_L})^2 \; d\alpha + \int_0^1 (A_{\alpha_R} - B_{\alpha_R})^2 \; d\alpha} \tag{24}$$

`allahviranloo` [23]

$$d_a^{T1}(A, B) = \sqrt{[I(A) - I(B)]^2 + [D(A) - D(B)]^2} \tag{25a}$$

$$I(A) = \int_0^1 (cA_{\alpha_L} + (1 - c)A_{\alpha_R}) \; d\alpha \tag{25b}$$

$$D(A) = \int_0^1 (A_{\alpha_R} - A_{\alpha_L} f(\alpha)) \; d\alpha \tag{25c}$$

Optional parameters:
$0 \leq c \leq 1$ *denotes optimism/pessimism in the operation (default:* `c=0.5`*)*
$f(\alpha)$ *is a function which satisfies* $f(0) = 0$, $f(1) = 1$ *and* $\int_0^1 f(\alpha)d\alpha = 1/2$ *(default:* `f=lambda a:a`*; i.e.,* $f(\alpha) = \alpha$*).*

`Grzegorzewski` [24]
Grzegorzewski established two methods of measuring the distance between two fuzzy numbers for the purpose of ranking and decision making as follows

$$d_{pq}(A, B) = \begin{cases} \sqrt[p]{(1 - q) \int_0^1 |B_{\alpha_L} - A_{\alpha_L}|^p \; d\alpha + q \int_0^1 |B_{\alpha_R} - A_{\alpha_R}|^p \; d\alpha} & \text{if } 1 \leq p < \infty \\ (1 - q) \sup_{0<\alpha\leq 1}(|B_{\alpha_L} - A_{\alpha_L}|) + q \sup_{0<\alpha\leq 1}(|B_{\alpha_R} - A_{\alpha_R}|) & \text{if } p = \infty \end{cases} \tag{26}$$

and

$$d_p(A,B) = \begin{cases} \max\left\{ \sqrt[p]{\int_0^1 |B_{\alpha_L} - A_{\alpha_L}|^p \, d\alpha},\ \sqrt[p]{\int_0^1 |B_{\alpha_R} - A_{\alpha_R}|^p \, d\alpha} \right\} & \text{if } 1 \le p < \infty \\ \max\left\{ \sup_{0<\alpha\le 1}(|B_{\alpha_L} - A_{\alpha_L}|),\ \sup_{0<\alpha\le 1}(|B_{\alpha_R} - A_{\alpha_R}|) \right\} & \text{if } p = \infty \end{cases} \tag{27}$$

where the properties of the above two measure depend on the value of p [24]. The parameter q of (26) may be used to weight the sides of the fuzzy numbers. However, if there is no reason to weight one side more than the other then q may be set as $1/2$ or (27) may be used instead [24].

`grzegorzewski_non_inf_pq`: Calculate d_{pq}. Optional parameter default values: $p = 2$, $q = 0.5$.

`grzegorzewski_non_inf_p`: Calculate d_p. Optional parameter default value: $p = 2$.

`grzegorzewski_inf_q`: Calculate d_{pq} where $p = \infty$. Optional parameter default value: $q = 0.5$.

`grzegorzewski_inf`: Calculate d_p where $p = \infty$.

`yao_wu` [25]

$$d_{yw}^{T1}(A,B) = 1/2 \int_0^1 [A_{\alpha_L} + A_{\alpha_R} - B_{\alpha_L} - B_{\alpha_R}] \, d\alpha \tag{28}$$

Note, this is a directional distance measure where $d(A,B) < 0$ if $A < B$ and $d(A,B) \ge 0$ if $A \ge B$.

`mcculloch`

$$d_m^{T1}(A,B) = \frac{\sum_{\alpha \in [0,\lambda]]} y_\alpha \ \dot{\bar{d}}_p(A_\alpha, B_\alpha)}{\sum_{\alpha \in [0,\lambda]} y_\alpha}, \tag{29}$$

where $\lambda = \max\{\alpha \mid A_\alpha \ne \emptyset \vee B_\alpha \ne \emptyset,\ \alpha \in (0,1]\}$ and $\bar{d}_p$ is

$$\dot{\bar{d}}_p(A_\alpha, B_\alpha) = \begin{cases} \bar{\bar{d}}_p(A_\alpha, B_\alpha) & A_\alpha \ne \emptyset \wedge B_\alpha \ne \emptyset \\ \bar{\bar{d}}_p(A_{\alpha k}, B_\alpha) & A_\alpha = \emptyset \wedge B_\alpha \ne \emptyset \\ \bar{\bar{d}}_p(A_\alpha, B_{\alpha k}) & A_\alpha \ne \emptyset \wedge B_\alpha = \emptyset \end{cases} \tag{30}$$

where $A_{\alpha_k} = \max\{A_\alpha \mid A_\alpha \ne \emptyset, \forall \alpha \in (0,1]\}$, and $\bar{\bar{d}}_p(A_\alpha, B_\alpha)$ is given as

$$\bar{\bar{d}}_p(A_\alpha, B_\alpha) = \frac{1}{||A_\alpha|| ||B_\alpha||} \sum_{i=1}^{||A_\alpha||} \sum_{j=1}^{||B_\alpha||} \bar{d}_p(A_{\alpha_i}, B_{\alpha_j}) \tag{31}$$

Note, this is a directional distance measure where $d(A,B) > 0$ if $A < B$ and $d(A,B) \le 0$ if $A \ge B$.
Additionally, unlike the above listed measures, d_m^{T1} has no restriction on the normality or convexity of the fuzzy sets A and B.

2.2 Interval Type-2

Definition 1 *The α-cut of an interval type-2 fuzzy set may be represented by the α-cuts of the upper and lower membership functions; throughout this thesis this is denoted* $\tilde{A}_\alpha = \left\{\overline{\tilde{A}_{\alpha W}}, \overline{\tilde{A}_{\alpha U}}\right\}$ *for* $\tilde{A} \in IT2(X)$ *where* $\overline{\tilde{A}_{\alpha W}}$ *and* $\overline{\tilde{A}_{\alpha U}}$ *are the α-cuts of the lower and upper membership functions of $\tilde{A}$, respectively.*

Definition 2 *The centre of sets of an interval type-2 fuzzy set $\tilde{A}$ is denoted* $C(\tilde{A}) = [C_L(\tilde{A}), C_R(\tilde{A})]$.

`figueroa_garcia_alpha` [26]

$$d^{IT2}_{fg\alpha}(\tilde{A}, \tilde{B}) = \int_0^1 \Big[\begin{array}{l} |\overline{\tilde{A}_{\alpha_{i_U}L}} - \overline{\tilde{B}_{\alpha_{i_U}L}}| + |\overline{\tilde{A}_{\alpha_{i_W}L}} - \overline{\tilde{B}_{\alpha_{i_W}L}}| + \\ |\overline{\tilde{A}_{\alpha_{i_W}R}} - \overline{\tilde{B}_{\alpha_{i_W}R}}| + |\overline{\tilde{A}_{\alpha_{i_U}R}} - \overline{\tilde{B}_{\alpha_{i_U}R}}| \Big] d\alpha \end{array} \tag{32}$$

Note that both the upper and lower membership functions of $\tilde{A}$ and $\tilde{B}$ must be normal as (32) does not account for non-normality.

`figueroa_garcia_centres_hausdorff` [26]

$$d^{IT2}_{fg_h}(\tilde{A}, \tilde{B}) = \max\left\{|C_L(\tilde{A}) - C_L(\tilde{B})|, |C_R(\tilde{A}) - C_R(\tilde{B})|\right\}. \tag{33}$$

`figueroa_garcia_centres_minkowski` [26]

$$d^{IT2}_{fg_m}(\tilde{A}, \tilde{B}) = |C_L(\tilde{A}) - C_L(\tilde{B})| + |C_R(\tilde{A}) - C_R(\tilde{B})|. \tag{34}$$

`mcculloch`

$$d^{IT2}_m(\tilde{A}, \tilde{B} = \frac{\sum_{\alpha\in[0,\gamma_U]} y_\alpha \, \dot{\bar{d}}_p(\overline{\tilde{A}_{\alpha U}}, \overline{\tilde{B}_{\alpha U}}) + \sum_{\alpha\in[0,\gamma_W]} y_\alpha \, \dot{\bar{d}}_p(\overline{\tilde{A}_{\alpha W}}, \overline{\tilde{B}_{\alpha W}})}{\sum_{\alpha\in[0,\gamma_U]} y_\alpha \sum_{\alpha\in[0,\gamma_W]} y_\alpha}, \tag{35}$$

where y_α is the y-coordinate (or u value) for the given α-cut, $\dot{\bar{d}}_p$ is described in (30), $\gamma_U = \max\left\{\alpha \mid \overline{\tilde{A}_{\alpha U}} \neq \emptyset \vee \overline{\tilde{B}_{\alpha U}} \neq \emptyset,\ \alpha \in [0,1]\right\}$, and γ_W is the same for the lower membership functions of $\tilde{A}$ and $\tilde{B}$.
Note, this is a directional distance measure where $d(\tilde{A}, \tilde{B}) > 0$ if $\tilde{A} < \tilde{B}$ and $d(\tilde{A}, \tilde{B}) \leq 0$ if $\tilde{A} \geq \tilde{B}$.
Additionally, unlike the above listed measures, d^{IT2}_m has no restriction on the normality or convexity of the fuzzy sets $\tilde{A}$ and $\tilde{B}$.

2.3 General Type-2

`mcculloch`

$$d^{GT2}_m(\tilde{A}, \tilde{B}) = \frac{1}{\sum_{i\in\mathscr{L}(\tilde{A},\tilde{B})} z_i}$$

$$\sum_{i\in\mathscr{L}(\tilde{A},\tilde{B})} z_i \frac{\sum_{\alpha\in[0,\gamma_{z_{i_U}}]} y_\alpha \, \dot{\bar{d}}_p(\overline{\tilde{A}_{z_{i\alpha_U}}}, \overline{\tilde{B}_{z_{i\alpha_U}}}) + \sum_{\alpha\in[0,\gamma_{z_{i_W}}]} y_\alpha \, \dot{\bar{d}}_p(\overline{\tilde{A}_{z_{i\alpha_W}}}, \overline{\tilde{B}_{z_{i\alpha_W}}})}{\sum_{\alpha\in[0,\gamma_{z_{i_U}}]} y_\alpha \sum_{\alpha\in[0,\gamma_{z_{i_W}}]} y_\alpha}, \tag{36}$$

where $\dot{\tilde{d}}_p$ is described in (30), $\gamma_{z_{i_U}} = \max\left\{\alpha \mid \overline{\tilde{A}_{z_{i_{\alpha_U}}}} \neq \emptyset \vee \overline{\tilde{B}_{z_{i_{\alpha_U}}}} \neq \emptyset,\ \alpha \in [0,1]\right\}$ (i.e., the maximum α-cut where at least one of the upper membership functions of the zSlices $\tilde{A}_{z_i}$ or $\tilde{B}_{z_i}$ is non-empty), and $\gamma_{z_{i_W}}$ is the same for the lower membership functions of $\tilde{A}_{z_i}$ and $\tilde{B}_{z_i}$. Also, $\mathscr{L}(\tilde{A},\tilde{B})$ is

$$\mathscr{L}(\tilde{A},\tilde{B}) = \tilde{A}_Z \cup \tilde{B}_Z, \tag{37}$$

where $\tilde{A}_Z$ is the set of zLevels in the zSlices general type-2 fuzzy set $\tilde{A}$ defined as

$$\tilde{A}_Z = \{z_i \mid \forall i \in \{1,2,...,I\}\}, \tag{38}$$

where I is the total number of zLevels in $\tilde{A}$.

3 Entropy

3.1 Type-1

kosko [27]

$$e(A) = \frac{\sum_{x\in X} \min(\mu_A(x), 1-\mu_A(x))}{\sum_{x\in X} \max(\mu_A(x), 1-\mu_A(x))} \tag{39}$$

3.2 Interval Type-2

szmidt_pacprzyk [28]

$$e(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n} \frac{1-\max(1-\overline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{A}}(x_i)}{1-\min(1-\overline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{A}}(x_i)} \tag{40}$$

zeng_li [7]

$$e(\tilde{A}) = 1 - \frac{1}{n}\sum_{i=1}^{n} |\overline{\mu}_{\tilde{A}}(x_i) + \underline{\mu}_{\tilde{A}}(x_i) - 1| \tag{41}$$

4 Inclusion (subsethood)

4.1 Type-1

sanchez [29]

$$i(A,B) = \frac{\sum_{i=1}^{n} \min(\mu_A(x_i), \mu_B(x_i)}{\sum_{i=1}^{n} \min(\mu_A(x_i)} \tag{42}$$

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