ENG1066 Solid Mechanics 1
Stress analysis in Mechanical Design

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Lecture 1 – Week 7

Overview

Week 7
- Uniform torsion of circular tubes and bars

Week 8
- Non-uniform torsion

Week 9
- Statically indeterminate torsional members
- Combined loading: torsion, bending, and axial loading
- Tutorial

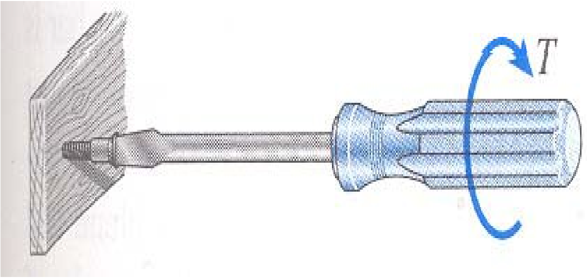
Week 10
- Combined loading: torsion, and bending in two planes
  - application to power-transmission shafts

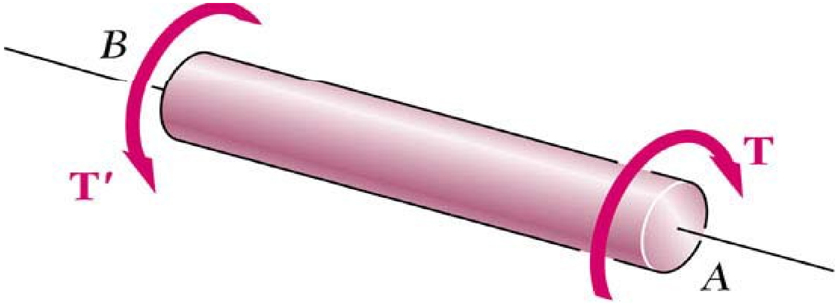
Week 11
- Revision
- Tutorial

Objectives of the lecture

On completion of this session you should:

- Understand concepts of uniform torsion of circular bars and tubes:
  - Derive basic equations for pure torsion
    - Shear stress
    - Angle of twist
  - Apply equations to solid and hollow shafts as torsional members



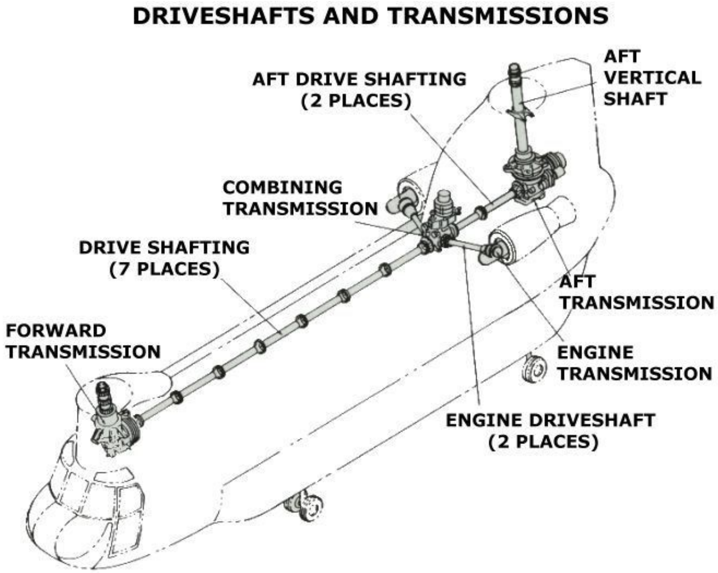


Applications of torsion

Steam turbine - largest are 600MW+

Ship propeller shaft

QE2 – 2 Propellers each driven by 44MW motor

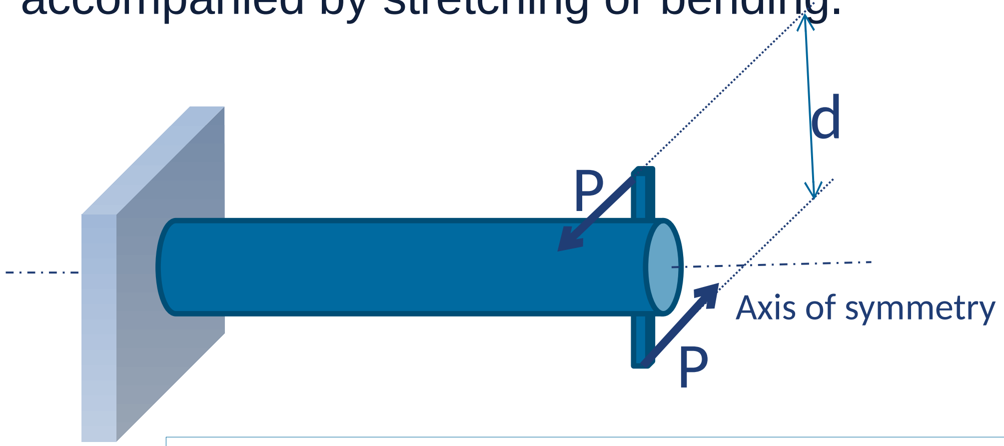


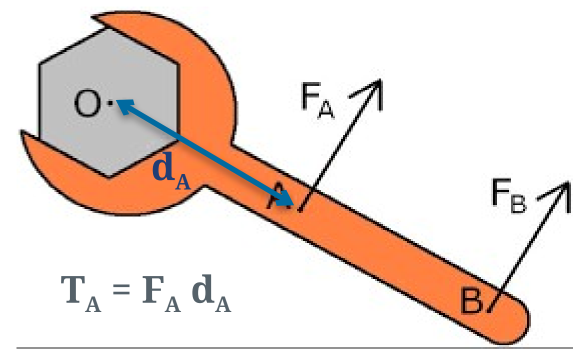
Chinook Helicopter

Definition of torsion

Torsion: Twisting of a bar loaded with moments (torques) that tend to produce rotation about its long axis.

- Torsion occurs when any shaft is subjected to a torque. This is true whether the shaft is rotating (e.g. drive shafts on engines, motors & turbines) or stationary (such as with a bolt or screw).
- The torque makes the shaft twist and one end rotates relative to the other inducing shear stress on any cross section.
- Failure may occur due to torsion alone or because the torsion is accompanied by stretching or bending.





Torque or twisting moment: **T = Pd**

Design for torsion

Need to know torsion loads

- Force, radius, power transmitted, rotational speed

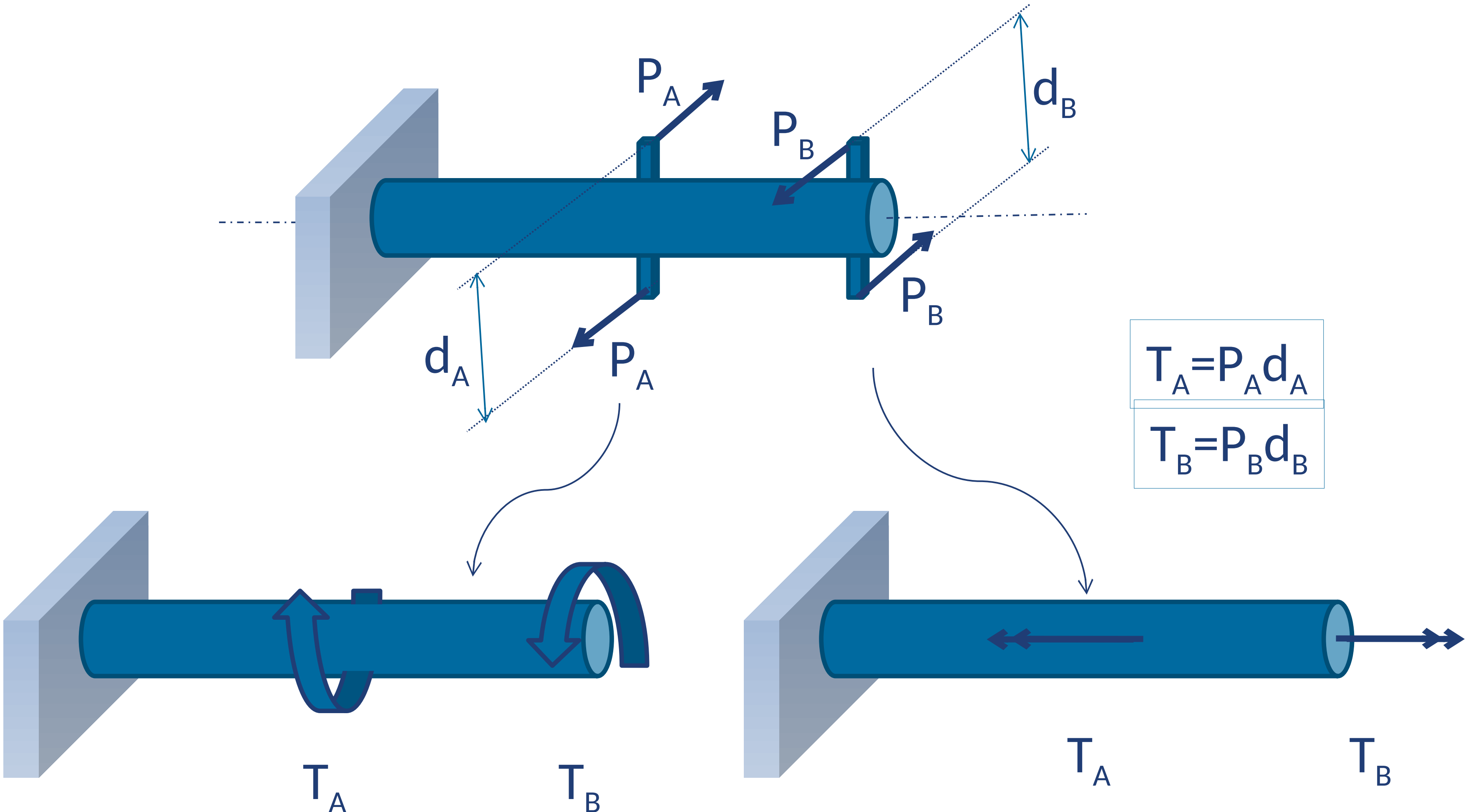
Are there other loads – bending stresses? Axial loads?

What are the limits on the material and/or design?

- Maximum shear stress?
  - (also need to consider principal stresses – alternative failure modes)
- Maximum twist or maximum twist per unit length?
  - Consider factors of safety

Let’s start with pure torsion and a single shaft!

Ways of representing torque



$T_A = P_A d_A$

$T_B = P_B d_B$

Right hand rule – thumb follows arrow, fingers show direction of twist.

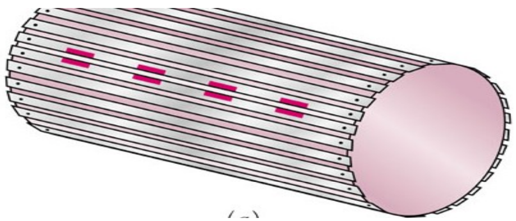
Torsion of circular shafts

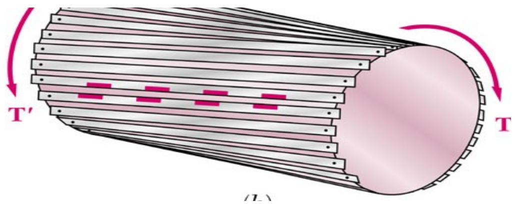
We assume

- Bar is in pure torsion
- Small rotations (the length and radius will not change)

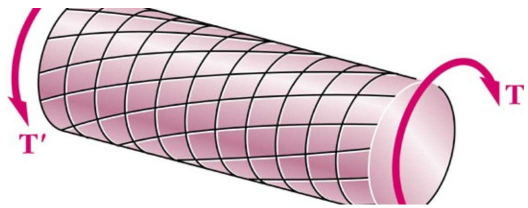
How does the bar deform?

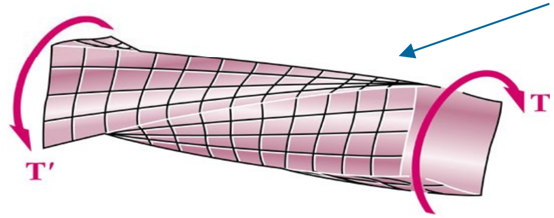
- Cross-section of the bar remains the same shape, bar is simply rotating.





- Cross-section remains perpendicular to axis of cylinder (cylinder does not warp).

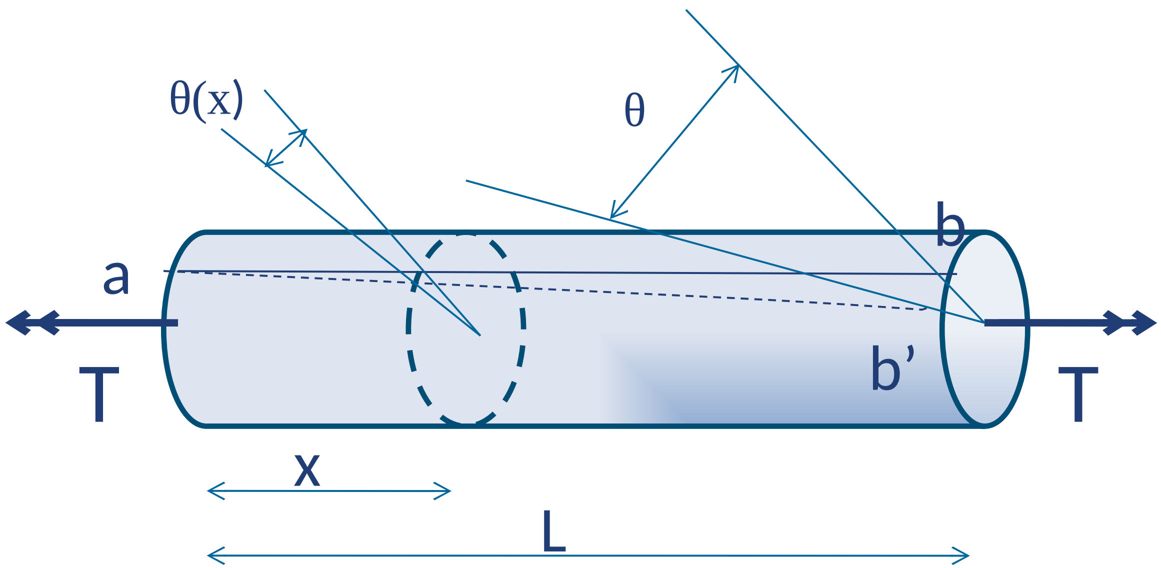




Not true for most non-circular bars

Uniform Torsion

- The internal torque is the same for the whole bar
- The bar has the same cross-section over the whole length



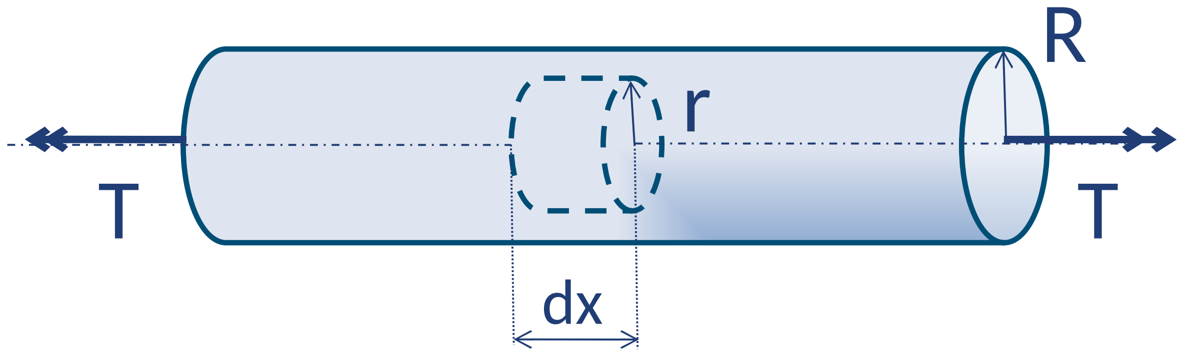
Rate of twist

$$\boxed{\frac{d\theta}{dx} = const}$$

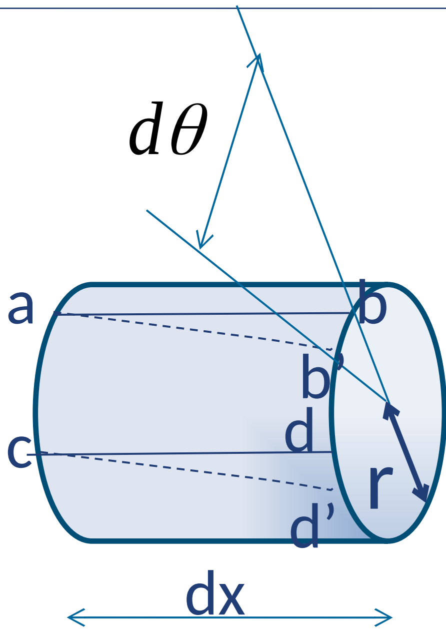
Rate of twist: Constant over the whole length.

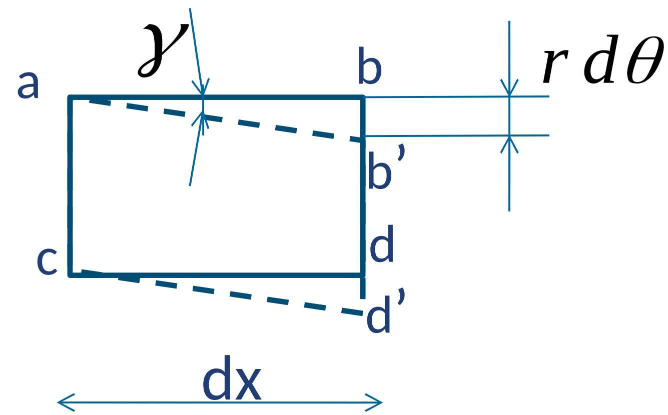
Angle of twist (ϕ): Increases linearly from one end to the other (in radians).

1) Relation between shear strain and angle of twist can be established by considering an element of the bar of radius r between two sections separated by a small distance (dx).



Angle of twist and rate of twist





Shear strain

$$\boxed{\gamma = \frac{bb'}{ab} = \frac{r\,d\theta}{dx}}$$

$$\boxed{\frac{d\theta}{dx} =}$$ Rate of change of the angle of twist θ with respect to distance x

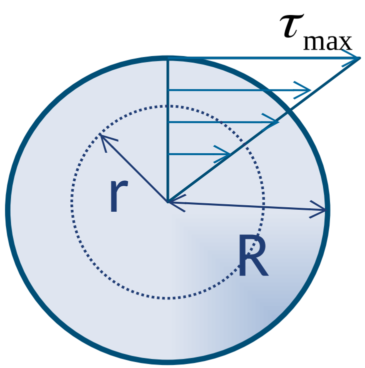
For pure torsion – rate of twist = the total angle of twist divided by the length L, therefore = / L

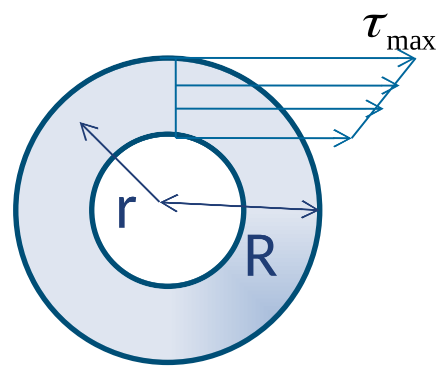
Distribution of shear strain and stress

For pure torsion
= / L

Shear strain $$\boxed{\gamma = r\frac{d\phi}{dx};\ \gamma_{\max} = R\frac{d\phi}{dx}}$$

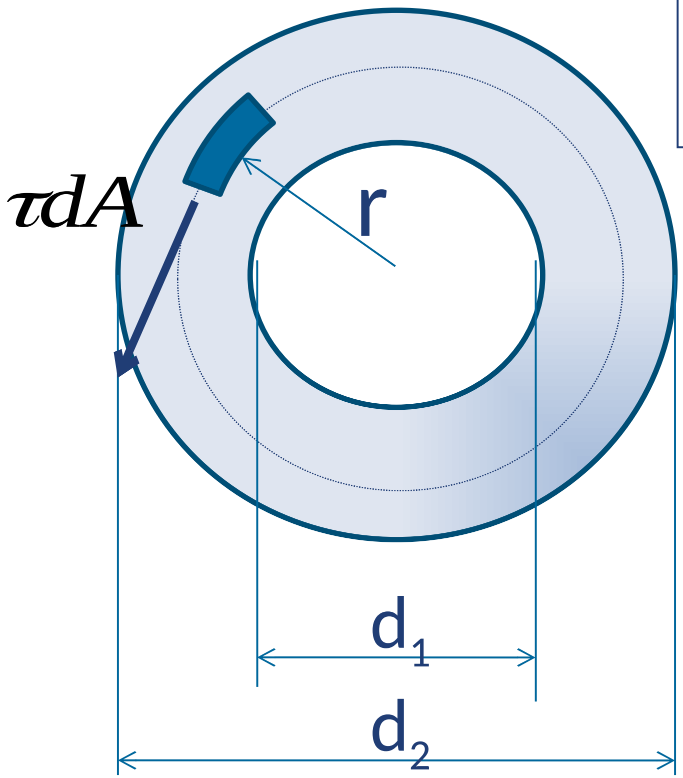
Shear stress $$\boxed{\tau = G\gamma = Gr\frac{d\phi}{dx};\ \tau_{\max} = GR\frac{d\phi}{dx}}$$





2) Distribution of shear strain and shear stress in a section of a solid and a hollow circular bar is such that stress/strain increases from zero at the centre (from minimum at inner radius) to its maximum value at the perimeter.

3) Relation between the rate of twist and the internal torque follows from the equilibrium of moments about the long axis.



Infinitesimal internal moment

$$dM = \tau dA \cdot r = G \frac{d\phi}{dx} r^2 dA$$

$$\int_A dM = T$$

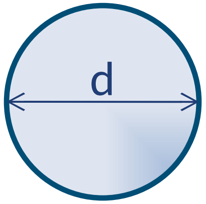
Equilibrium of moments

$$T = G \frac{d\phi}{dx} \int_A r^2 dA$$

Polar moment of area, J

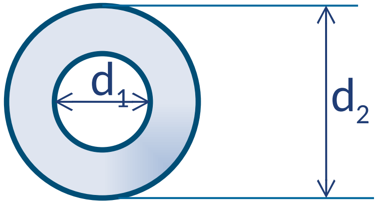
Polar moment of area: $J = \int_A r^2 dA \quad [\mathrm{m}^4]$

Solid circular bar



$$J = \frac{d^4 \pi}{32}$$

Hollow circular tube



$$J = \frac{\left(d_2^4 - d_1^4\right)\pi}{32}$$

Torsion equation

For pure torsion

Where

T = Torque (Nm)

J = Polar moment of area (m^4)

G = Modulus of rigidity (Nm^{-2})

θ = Angle of twist (radians)

τ = Shear stress (Nm^{-2})

r = radius (m)

Remember. These equations were derived for a circular bar; the material is homogeneous; the applied torques do not cause yielding and the material behaves in a linear elastic manner (obeys Hooke’s Law).

Worked Example 1

Example: A solid steel bar of circular cross-section
L=1.3m, G=80GPa, d=40mm is subjected to a torque T=340Nm. Find:

a) maximum shear stress
b) angle of twist between the ends

a)
$$\tau_{max} = \frac{T\,d/2}{J} = \frac{16T}{\pi d^3} = \frac{16 \cdot 340}{\pi \cdot 0.04^3} = 27.1\text{MPa}$$

b)
$$J = \frac{\pi d^4}{32} = \frac{\pi 0.04^4}{32} = 2.51 \times 10^{-7}\,\text{m}^4$$

$$\theta = \frac{TL}{GJ} = \frac{340 \cdot 1.3}{80 \times 10^9 \cdot 2.51 \times 10^{-7}} = 0.022\text{rad}$$

Example 1 - Extension

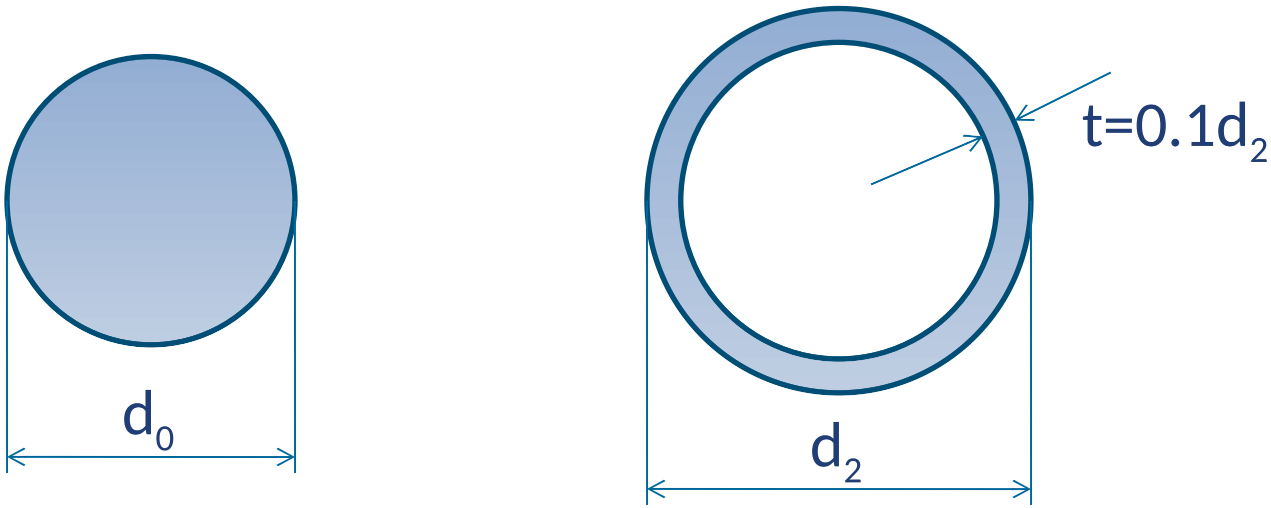
Considering the same bar (diameter 40mm, length 1.3m and G = 80MPa).

If the allowable shear stress is 42MPa and the allowable angle of twist is 2.5°, what is the maximum permissible torque?

Answer. 528Nm (from max shear stress) and 674Nm (from max twist).

Worked example 2

Example: A shaft is required to transmit a torque of 1200Nm without exceeding a stress of 40MPa and without exceeding a rate of twist of 0.75°/m (0.013 rad/m). Determine the minimum diameters of the solid and hollow shafts shown below that meet the requirements and compare their weights. G=78GPa for both shafts.



Worked example 2 - solution

Solid Bar

Diameter based on maximum allowed stress

$$\tau_{allow} = \tau_{\max} = \frac{T\, d_0/2}{J} = \frac{16T}{\pi d_0^3}$$

$$d_0 = \sqrt[3]{\frac{16T}{\pi \tau_{allow}}} = \sqrt[3]{\frac{16 \cdot 1200}{\pi \cdot 40 \times 10^6}} = 0.0535\text{m} = 53.5\text{mm}$$

Diameter based on maximum allowed rate of twist

$$\theta_{allow} = \frac{T}{GJ} \Rightarrow J = \frac{T}{G\theta_{allow}} = \frac{1200}{78 \times 10^9 \cdot 0.01308} = 1175 \times 10^{-9}\text{m}^4$$

$$d_0 = \sqrt[4]{\frac{32J}{\pi}} = \sqrt[4]{\frac{32 \cdot 1175 \times 10^{-9}}{\pi}} = 0.0588\text{m} = 58.8\text{mm}$$

Choose: d_0=58.8mm (60mm)

Worked example 2: Solution continued

Hollow tube

Diameter based on maximum allowed stress

$$J = \frac{\pi(d_2^4 - d_1^4)}{32} = \frac{\pi\left(d_2^4 - (0.8d_2)^4\right)}{32} = 0.058 d_2^4$$

$$\tau_{allow} = \frac{T\, d_2/2}{J} = \frac{T}{0.116 d_2^3} \Rightarrow d_2 = \sqrt[3]{\frac{T}{0.116\tau_{allow}}} = \sqrt[3]{\frac{1200}{0.116 \cdot 40 \times 10^6}} = 64\text{mm}$$

Diameter based on maximum allowed rate of twist

$$J = \frac{T}{G\theta_{allow}} \Rightarrow 0.058 d_2^4 = \frac{T}{G\theta_{allow}}$$

$$d_2 = \sqrt[4]{\frac{T}{0.058 G\theta_{allow}}} = \sqrt[4]{\frac{1200}{0.058 \cdot 78 \times 10^9 \cdot 0.01308}} = 0.067\text{m} = 67\text{mm}$$

Choose: d_0=67mm

d_2=67mm therefore d_1=0.8d_2

d_1=54mm

Worked example 2: Solution continued

Comparison of cross-sectional areas (weights per unit length) of the solid bar and hollow tube:

$$\frac{W_{hollow}}{W_{solid}} = \frac{A_{hollow}}{A_{solid}} = \frac{\pi(d_2^2 - d_1^2)/4}{\pi d_0^2/4} = \frac{d_2^2 - d_1^2}{d_0^2} = \frac{67^2 - 54^2}{58.8^2} = 0.45$$

The hollow tube requires a larger diameter, but has a (much) lower weight than the solid bar.

Solid bars vs hollow tubes as torsional elements:

The central portion of solid bars carries little load. Consequently, the same load-bearing capacity can be achieved with a much lighter hollow tube.

Must be careful to not make wall too thin to avoid buckling

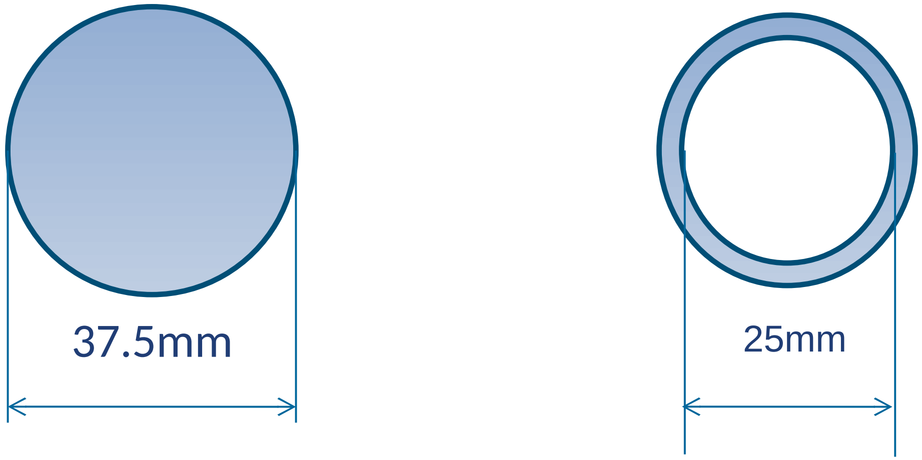
Summary

- We derived formulas for shear stress and angle of twist in circular bars and tubes under torsion
- In members under torsion, the stress increases from the centre towards the perimeter of the cross-section
- Hollow tubes are more efficient as torsional elements than solid bars
- Equations derived are limited to bars of circular cross section (either solid or hollow) that behave in a linearly elastic way (loads must be such that stresses do not exceed proportional limit of the material)
- Equations for stresses are only valid in parts of the bar away from stress concentrations (e.g. holes and changes in CS)

Review question

A shaft is made of a steel alloy having an allowable shear stress of 84MPa.

- If the diameter of the shaft is 37.5mm, determine the maximum torque that can be transmitted.
- What would be the maximum torque that can be transmitted if the shaft had a 25mm diameter hole bored through the shaft?
- What is the shear stress on the inner wall of the hollow shaft?
- Sketch the shear stress distribution along a radial line in each case.
- Compare the % reduction in torque and the % reduction in weight.



Reading

Gere and Goodno, “Mechanics of materials”, 7th Ed, Chapter 3.1-3.3, Pg 221-231

Other text books have titles such as “Strength of Materials” or “Solid Mechanics”; Good selection in library
e.g. Popov, Hibbeler

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4) Relation between rate of twist, angle of twist, shear stress, and the torque

Angle of twist

$$\phi = \frac{TL}{GJ}$$

Shear stress

$$\tau = \frac{Tr}{J}$$

Rate of twist (rad/m)

Design criteria for bars & tubes subjected to torsion

1) Maximum allowed shear stress:

$$\tau = \frac{TR}{J} = \frac{T\, d/2}{J}$$

2) Maximum allowed angle of twist between the ends:

$$\phi = \frac{TL}{GJ}$$