1. Suppose you observe $X_1, ..., X_{10} \sim \text{Exp}(\theta)$ and plan to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

   (a) Write out the GLRT test statistic (you do *not* need to find an associated probability distribution) and explain why we would reject for large values of $\bar{X}$ when $\theta_1 < \theta_0$ and for small values if $\theta_1 > \theta_0$.

   (b) Suppose we use the test statistic $\bar{X}$. Note that $\bar{X} \sim Gamma(n, \frac{\theta}{n})$. Suppose $\theta_0 = 1$ and $\theta_1 = .5$. Find a $\alpha = .05$ rejection region for the test statistic $\bar{X}$.

   (c) The distribution of $W = 2n\theta\bar{X}$ is Gamma(n, 1/2), which is equivalent to a χ^2_{2n} random variable. Find a general formula for an $\alpha = .05$ rejection region using this representation. Give the rejection region in terms of $\bar{X}$ (that may depend on θ_0).

   (d) For the test in (c), suppose we observe $\bar{X} = 2.17$. What values of θ_0 would *not* be rejected at $\alpha = .05$?

(a) Since $X_i \sim Exp(\theta)$, $L(\theta) = \prod_{i}^{n} \theta e^{-\theta y} = \theta^n e^{-\theta \sum y_i}$

$$\lambda = \frac{L(\Omega_0)}{L(\Omega_1)} = \frac{\theta_0^n e^{-\theta_0 \sum y_i}}{\theta_1^n e^{-\theta_1 \sum y_i}} = \left(\frac{\theta_0^n}{\theta_1^n}\right) e^{-\theta_0 \sum y_i - (-\theta_1 \sum y_i)}$$

$$= \left(\frac{\theta_0}{\theta_1}\right)^n e^{-\theta_0 n\bar{y} - (-\theta_1 n \bar{y})} = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-n\bar{y}(\theta_0 - \theta_1)}$$

Recall we reject H_0 when $\lambda < k$, where $P(\lambda < k \mid \theta \in \Omega_0) = \alpha$ so we reject when λ is small.

If $\theta_1 < \theta_0$, $\left(\frac{\theta_0}{\theta_1}\right)^n > 1$ and $(\theta_0 - \theta_1) > 0$, so will reject when $e^{-n\bar{y}(\theta_0 - \theta_1)}$ is small compared to $\left(\frac{\theta_0}{\theta_1}\right)^n$, which means $\bar{y}$ would be large.

Reverse idea for $\theta_1 > \theta_0$.

(b) $\lambda = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-n\bar{y}(\theta_0 - \theta_1)} = 2^n e^{-n\bar{y}\left(\frac{1}{2}\right)} = 2^n e^{-n/2\, \bar{y}}$

$\bar{y} \sim Gamma(n, \frac{\theta}{n}) \Rightarrow \bar{y} \sim Gamma(n, \frac{2}{n})$ under H_0.

δ: Reject H_0 when $P(\lambda < k \mid \theta = 1) = \alpha$

$\lambda < k \Rightarrow 2^n e^{-n/2\, \bar{y}} < k$

$e^{-n/2\, \bar{y}} < k/2^n$

$-n/2\, \bar{y} < \ln k - n \ln 2$

$R = \bar{y} > -\frac{2}{n}(\ln k - n \ln 2)$

Set $k = \text{qgamma}(.95, \alpha = n, \beta = 1/n)$

$= 1.57$

1. Suppose you observe $X_1, ..., X_{10} \sim \text{Exp}(\theta)$ and plan to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

   (a) Write out the GLRT test statistic (you do *not* need to find an associated probability distribution) and explain why we would reject for large values of $\bar{X}$ when $\theta_1 < \theta_0$ and for small values if $\theta_1 > \theta_0$.

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   (d) For the test in (c), suppose we observe $\bar{X} = 2.17$. What values of θ_0 would *not* be rejected at $\alpha = .05$?

(c) From (b), $R = \bar{y} > -\frac{2}{n}(\ln k - n \ln 2)$

$= 2n\theta\bar{y} > -4\theta(\ln k - n \ln 2)$

$= w > -4\theta(\ln k - n \ln 2)$

Update based on upper tail

$W_\alpha = \text{qchisq}(.95, df = 20) = 31.41$

$\Rightarrow$ reject when $w > 31.41$

$2 \cdot 10 \cdot \theta_0 \bar{x} > 31.41$

$\bar{x} > \frac{31.41}{20 \cdot \theta_0} = 1.57$ (same as b!)

(d) $\bar{x} > \frac{31.41}{20 \cdot \theta_0}$

$2.17 > \frac{31.41}{20 \cdot \theta_0}$

$\theta_0 > \frac{31.41}{20 \cdot 2.17} = .73$

$\theta_0 \geq .73$ is rejection region $\Rightarrow$ acceptance region is $\theta < 0.73$

2. A study reported in the 2011 Journal of Clinical Sleep Medicine found that the rate of teenage auto accidents was much higher in one Virginia city than in a neighboring community where school started about an hour and a quarter later. The table below reports the 2007 and 2008 teen accident rates per 1000 drivers in Virginia Beach, where public high school started at 7:20am, and in Chesapeake, where school started at 8:45am. It also gives the average for each of the two sets of n = 2 numbers.

	Virginia Beach	Chesapeake
2007	71.2	55.6
2008	65.8	46.6
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Average	68.5	51.1

(a) First, define parameters and run a *two-sample* test for the average accident rate over the two years. Justify your choice of H_1.

(b) Since we have two years of data, we could also treat these data as $n = 2$ *matched pairs* (matching on year), where each data point represents the difference between the accident rate in the early-start town compared to the late-start town. Compute these differences and carry out a *one-sample t-test* using the differences. How does your conclusion compare to the conclusion in (a)?

(c) Would a significant result in (a) or (b) provide good evidence that starting high school later *causes* a lower rate of accidents? Explain.

(a) $H_0: p_{late} = p_{early}$ $\quad p_{late}$ = accident rate for 8:45 start

$H_1: p_{late} < p_{early}$ $\quad p_{early}$ = accident rate for 7:20 start

used 1-sided alternative to make argument in favor of late start

$$z = \frac{51.1/1000 - 68.5/1000}{\sqrt{\frac{.06(.94)}{1000} + \frac{.06(.94)}{1000}}} = -1.68$$

$$p_p = \frac{51.1 + 68.5}{2000} = \frac{119.6}{2000} = .06$$

p-value: pnorm(−1.68) = .046 ⇒ reject H_0 at $\alpha = .05$

(b) $d_1 = -15.6$, $d_2 = -19.2$ } new data points

$$t = \frac{\bar{d} - 0}{s_d/\sqrt{n}} = \frac{-17.4}{2.54/\sqrt{2}} = -9.68$$

$s_d = 2.54$

$\bar{d} = -17.4$

pt(−9.68, df = 1) = .005

→ reject H_0. "more" significant than (a)

(c) Not necessarily – could be other reasons why Chesapeake has sig. lower rate than VB. (e.g. # of stoplights, commuters, etc.)

asked to fuse the images. Their average time was $\bar{X}_1 = 8.560$ and $s_1^2 = 2745.7$. The second group of 35 subjects was shown a picture of the object, and their sample statistics were $\bar{X}_2 = 5.551$ and $s_2^2 = 783.9$. The null hypothesis is that the mean time of the first group is no larger than the mean mean time of the second group, while the alternative hypothesis is that the first group takes longer.

(a) Test the hypotheses at $\alpha_0 = .01$, assuming that the population variances are equal for the two groups.

(b) Now, test the same hypotheses assuming the population variances are not equal.

(a)

$H_0: \mu_1 \le \mu_2$ ← max(α) occurs at $\mu_1 = \mu_2 \Rightarrow \mu_{10} - \mu_{20} = 0$ in test stat,

$H_1: \mu_1 > \mu_2$

$$t = \frac{(\bar{X}_1 - \bar{X}_2)}{S_p \sqrt{1/43 + 1/35}}$$

$$= \frac{8.56 - 5.51}{\sqrt{\frac{42 \cdot 2745.7 + 34 \cdot 783.9}{43 + 35 - 2}} \sqrt{1/43 + 1/35}}$$

$$= .302$$

reject H_0 if $t_{43+35-2} > t_{.01,76} = 2.38$

→ fail to reject H_0.

(b) For Welch's approx,

$$v = \frac{\left(\frac{2745.7}{783.9} + \frac{43}{35}\right)^2}{\frac{1}{42}\left(\frac{2745.7}{783.9}\right)^2 + \frac{1}{34}\left(\frac{43}{35}\right)^2} = 66.55$$

reject H_0 if $t_{welch} > t_{.01,67} = 2.38$

$$t_{welch} = \frac{8.56 - 5.51}{\sqrt{\frac{2745.7}{43} + \frac{783.9}{35}}} = .328$$

→ slightly "more" significant, but still fail to reject