04: CRAMER-RAO LOWER BOUND

Larsen & Marx 5.5

Prof Amanda Luby

1 Minimum-Variance Unbiased Estimators

Given two unbiased estimators for the parameters θ, $\hat{\theta}_1$ and $\hat{\theta}_2$, we've already established which is "better": the one with smaller variance. But what if there is a $\hat{\theta}_3$ that has smaller variance than both of them? How can we know if one exists?

The **Cramer-Rao Lower Bound** tells us exactly that. It gives a theoretical limit below which an unbiased estimator cannot fall. If the variance of an estimator $\hat{\theta}$ is equal to that bound, we know that $\hat{\theta}$ is *optimal* in a sense: no other unbiased estimator can estimate θ with greater precision.

Fisher Information

The Fisher Information is a way of measuring the amount of information that a random variable X carries about the unknown parameter θ.

$$I(\theta) = E[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2] = [-E(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta})]$$

Example: Find the Fisher information for X, where $X \sim \text{Bernoulli}(\pi)$

Cramer-Rao Lower Bound

Let $Y_1, ..., Y_n \sim f_y(y;\theta)$, where $f_y(y;\theta)$ is a continuous pdf with continuous first and second derivative (i.e. "smooth enough"). Also suppose the set of values where $f_y(y;\theta) \neq 0$ does not depend on θ.

Let $\hat{\theta} = h(Y_1, ..., Y_n)$ be any unbiased estimator of θ. Then,

$$Var(\hat{\theta}) \geq [nE[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2]^{-1} = [-nE(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta})]^{-1} = \frac{1}{nI(\theta)}$$

Example: Let $X_1, ..., X_n$ be n Bernoulli trials with probability of succeess π. Let $\hat{\pi} = \frac{\sum X_i}{n}$. How does $Var(\hat{\pi})$ compare with the Cramer-Rao lower bound?

Example: Let $Y_1, .., Y_n \sim f_y$, where $f_y = \frac{2y}{\theta^2}$ for $0 \leq y \leq \theta$. Compare the Cramer-Rao lower bound with the variance of the unbiased estimator $\frac{3}{2}\bar{Y}$. Discuss.

Minimum-variance unbiased estimator (aka "best" unbiased estimator)

Let Θ denote the set of all estimators that are unbiased for the paramter θ in the continuous pdf $f_y(y;\theta)$. We say $\hat{\theta}^*$ is the MVUE if $\hat{\theta}^* \in \Theta$ and

Efficient estimator

Let $Y_1, \ldots, Y_n \sim f_y(y;\theta)$. Let $\hat{\theta}$ be an unbiased estimator for θ.

1. $\hat{\theta}$ is said to be **efficient** if:

2. The **efficiency** of $\hat{\theta}$ is:

Note: