Friday, Sept 29

HW 4 is posted - due on wed

Today - CR CB

bit.ly/prof-ruby-oh ← weekly 1-1 meetings, usually on Fridays

04: CRAMER-RAO LOWER BOUND

Larsen & Marx 5.5

Prof Amanda Luby

1 Minimum-Variance Unbiased Estimators

Given two unbiased estimators for the parameters θ, $\hat{\theta}_1$ and $\hat{\theta}_2$, we've already established which is "better": the one with smaller variance. But what if there is a $\hat{\theta}_3$ that has smaller variance than both of them? How can we know if one exists?

The **Cramer-Rao Lower Bound** tells us exactly that. It gives a theoretical limit below which an unbiased estimator cannot fall. If the variance of an estimator $\hat{\theta}$ is equal to that bound, we know that $\hat{\theta}$ is *optimal* in a sense: no other unbiased estimator can estimate θ with greater precision.

Fisher Information : · pdf needs to meet regularity conditions
· range of x cannot depend on θ

The Fisher Information is a way of measuring the amount of information that a random variable X carries about the unknown parameter θ.

$$I(\theta) = E[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2] = [-E(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta^2})]$$

Example: Find the Fisher information for X, where $X \sim \text{Bernoulli}(\pi)$

$P(X) = \pi^x (1-\pi)^{1-x}$ $\quad E(X) = \pi$

$\ell(\pi) = x \ln \pi + (1-x)\ln(1-\pi)$

$\frac{\partial \ell}{\partial \pi} = \frac{x}{\pi} + \frac{(1-x)}{(1-\pi)} \cdot -1$

x is RV, θ constant

$\frac{\partial^2 \ell}{\partial \pi^2} = \frac{-x}{\pi^2} - \frac{(1-x)}{(1-\pi)^2} \cdot -1 \cdot -1$

Treating x as constant, θ RV

$E_x\left(\frac{\partial^2 \ell}{\partial \pi^2}\right) = E_x\left[\frac{-x}{\pi^2} - \frac{(1-x)}{(1-\pi)^2}\right] = \frac{-1}{\pi^2}E(X) - \frac{1}{(1-\pi)^2}(1-E(X))$

$= \frac{-\pi}{\pi^2} - \frac{(1-\pi)}{(1-\pi)^2} = -\left[\frac{\pi(1-\pi)^2 + \pi^2(1-\pi)}{\pi^2(1-\pi)^2}\right] = -\left[\frac{1-\pi+\pi}{\pi(1-\pi)}\right]$

$= \frac{-1}{\pi(1-\pi)}$

$I_x(\pi) = -\left[\frac{-1}{\pi(1-\pi)}\right]$

$= \frac{1}{\pi(1-\pi)}$

"information" that 1 draw from a Bernoulli tells us about π

Cramer-Rao Lower Bound

Let $Y_1, ..., Y_n \sim f_y(y;\theta)$, where $f_y(y;\theta)$ is a continuous pdf with continuous first and second derivative (i.e. "smooth enough"). Also suppose the set of values where $f_y(y;\theta) \neq 0$ does not depend on θ.

$I_n(\theta) = n I_1(\theta)$

$I(\theta)$ as 1 draw from f_y

if you instead found $I(\theta)$ for n draws, don't multiply by n in CRLB

Let $\hat{\theta} = h(Y_1, ..., Y_n)$ be any unbiased estimator of θ. Then,

$$Var(\hat{\theta}) \geq [nE[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2]^{-1} = [-nE(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta^2})]^{-1} = \frac{1}{nI(\theta)}$$

Example: Let $X_1, ..., X_n$ be n Bernoulli trials with probability of succeess π. Let $\hat{\pi} = \frac{\sum X_i}{n}$. How does $Var(\hat{\pi})$ compare with the Cramer-Rao lower bound?

① confirm unbiased

$E(\hat{\pi}) = E(\frac{1}{n}\sum X_i)$

$= \frac{1}{n} \cdot n \cdot E(X_i)$

$= E(X_i) = \pi$ ✓

② Find CRLB

$Var(\hat{\pi}) \geq \frac{1}{nI(\pi)}$

$= \frac{1}{n\left[\frac{1}{\pi(1-\pi)}\right]}$

$= \frac{\pi(1-\pi)}{n}$

③ Find $V(\hat{\pi})$

$V(\hat{\pi}) = V\left(\frac{1}{n}\sum X_i\right)$

$= \frac{1}{n^2}\sum V(X_i)$

$= \frac{1}{n^2} \cdot n\,(\pi(1-\pi))$

$= \frac{\pi(1-\pi)}{n}$

$\Rightarrow V(\hat{\pi})$ achieves CRLB, so there is no unbiased estimator that is more efficient!

Example: Let $Y_1, .., Y_n \sim f_y$, where $f_y = \frac{2y}{\theta^2}$ for $0 \leq y \leq \theta$. Compare the Cramer-Rao lower bound with the variance of the unbiased estimator $\frac{3}{2}\bar{Y}$. Discuss.

$E(Y) = \frac{2}{3}\theta \qquad V(Y) = \frac{\theta^2}{18}$

① Find $I(\theta)$

$L(\theta) = \frac{2y}{\theta^2}$

$\ell(\theta) = \ln 2y - 2\ln\theta$

$\frac{\partial \ell}{\partial \theta} = -\frac{2}{\theta}$

$I(\theta) = E\left[\left(\frac{\partial \ell}{\partial \theta}\right)^2\right] = E\left(\frac{4}{\theta^2}\right)$

$= \int_0^\theta \frac{4}{\theta^2} \cdot \frac{2y}{\theta^2}\, dy$

$= \frac{4y^2}{\theta^4}\Big|_0^\theta = \frac{4\theta^2}{\theta^4} - 0$

$= \frac{4}{\theta^2}$

② Find CRLB

$V(\hat{\theta}) \geq \frac{1}{nI(\theta)}$

$= \frac{1}{n(4/\theta^2)}$

$= \frac{\theta^2}{4n}$

③ Find $V(\hat{\theta})$

$V(\hat{\theta}) = V\left(\frac{3}{2}\bar{Y}\right)$

$= \frac{9}{4n^2}\sum V(Y_i)$

$= \frac{9}{4n^2} \cdot n \cdot \frac{\theta^2}{18}$

$= \frac{\theta^2}{8n}$

issue 1: $V(\hat{\theta}) <$ CRLB ???

issue 2: Different $I(\theta)$ depending on formula

range of Y depends on θ, so CRLB doesn't apply

Minimum-variance unbiased estimator (aka "best" unbiased estimator) MVUE

Let Θ denote the set of all estimators that are unbiased for the paramter θ in the continuous pdf $f_y(y;\theta)$. We say $\hat{\theta}^*$ is the MVUE if $\hat{\theta}^* \in \Theta$ and

$$\text{var}(\hat{\theta}^*) \leq \text{var}(\hat{\theta}) \quad \text{for all} \quad \hat{\theta} \in \Theta$$

If $\hat{\theta}$ achieves CRLB, then it is MVUE

Efficient estimator

Let $Y_1, ..., Y_n \sim f_y(y;\theta)$. Let $\hat{\theta}$ be an unbiased estimator for θ.

1. $\hat{\theta}$ is said to be **efficient** if:

$$V(\hat{\theta}) = \text{CRLB}$$

2. The **efficiency** of $\hat{\theta}$ is:

$$\frac{\text{CRLB}}{V(\hat{\theta})}$$

Note:

MVUE $\neq$ achieves CRLB

May be cases where <u>no</u> unbiased estimator achieves CRLB