# 02: PROPERTIES OF ESTIMATORS I

Larsen & Marx 5.4

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We've seen two methods of estimating parameters: the MLE and the MoM. Both give very reasonable criteria to identify estimators for unknown parameters, but they do not always yield the same answer.

For example, on your homework, you showed that the MLE estimator for $\theta$ in a (continuous) $\text{Unif}(0, \theta)$ distribution is $\hat{\theta} = 2\bar{Y}$.

**Example:** Find the MLE for $Y_1, Y_2, ..., Y_n \sim \text{Unif}(0, \theta)$. (Recall $f_y(y) = \frac{1}{\theta}, 0 \leq y \leq \theta$))

Implicit in the two estimators for the same parameters is the obvious question: which one should we use?

There are actually an infinite number of estimators for any given parameter, and this requires that we have a principled way of evaluating the statistical properties associated with any given estimator. What qualities should a "good" estimator have? Is it possible to find a "best" $\hat{\theta}$? This set of notes, and the second unit of the course, is going to begin to address these questions.

**Note:** Every estimator is a function of a set of random variables (ie $\hat{\theta} = g(Y_1, Y_2, ..., Y_n)$)) and is itself a random variable.

**Notation for $\hat{\theta}$**

1

# 1 Unbiasedness

Lab01 ended with the idea of *random samples*, and noting that every sample is going to give a slightly different estimate for $\theta$. Here are eight random samples of size 20 of $N(0, 1)$ random variables, with the sample mean overlaid on each facet.
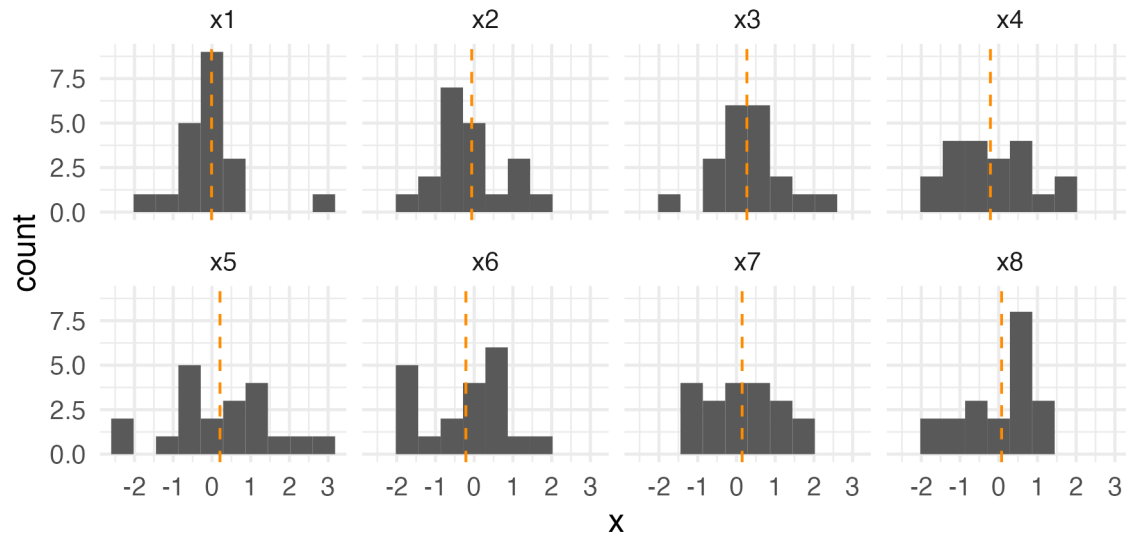


Ideally, we want the overestimates to "balance out" the underestimates: $\hat{\theta}$ should not systematically err in either direction.

When the mean of the estimator $\hat{\theta}$ is equal to the true parameter $\theta$, we say tat the estimator is **unbiased**.

> **Definition**
>
> Let $W_1, W_2, ..., W_n$ be a random sample from $f_w(w, \theta)$. An estimator $\hat{\theta} = g(W_1, W_2, ..., W_n)$ is said to be **unbiased for** $\theta$ if

**Example:** is the MoM for the Unif$(0, \theta)$ distribution, $\hat{\theta}_1 = 2\bar{X}$ unbiased?

**Example:** is the MLE $\hat{\theta}_2 = \max(X_i)$ unbiased?

**Example:** Construct an estimator, $\hat{\theta}_3$ based on $\max(X_i)$ that is unbiased.

# 2 Efficiency

We now have two estimators, $\hat{\theta}_1$ and $\hat{\theta}_3$, that are unbiased estimators for $\theta$. Does it matter which one we choose?

**Idea:**

> **Definition**
>
> Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators for a parameter $\theta$. If $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$, we say that

**Example:** Let $Y_1, Y_2, Y_3 \sim N(\mu, \sigma^2)$. Which is a more efficient estimator for $mu$:

$$\hat{\mu}_1 = \frac{1}{4}Y_1 + \frac{1}{2}Y_2 + \frac{1}{4}Y_3$$

or

$$\hat{\mu}_1 = \frac{1}{3}Y_1 + \frac{1}{3}Y_2 + \frac{1}{3}Y_3?$$

**Exercise:** Which of our two unbiased estimators for the uniform distribution is more efficient?

# 3 The Bias-Variance Tradeoff

> **Mean Square Error (MSE)**
>
> $$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 = V(\hat{\theta}) + \text{bias}^2$$

*Note:* We'll come back to this, and show that it is true, later in the course

**Idea:**