HW12 is posted, due next wed

~~16~~7: MULTIPLE REGRESSION

Rice 14.4

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1 The Hat Matrix

$$\begin{bmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \vdots \\ \hat{\epsilon}_n \end{bmatrix} \quad \hat{\epsilon} = \mathbf{Y} - \hat{\mathbf{Y}}$$

$$= Y - X\hat{\beta}$$

$$= Y - X(X^TX)^{-1}X^TY$$

$$= Y - HY$$

$H = X(X^TX)^{-1}X^T$

$\hat{Y} = HY$

puts a "hat" on Y

$H^T = X[(X^TX)^{-1}]^T X^T$

$= X(X^TX)^{-1}X^T$

$= H$

H = constant, only a function of x's

Note: The "hat matrix" has some nice properties: $H = H^T = H^2$ and $(I-H) = (I-H)^T = (I-H)^2$.

2 Estimation of σ^2

$Y = X\beta + \epsilon$ $\quad \epsilon_i \sim N(0, \sigma^2)$

$var(Y_i) = \sigma^2$

In Notes 15, two of the properties that we worked with were:

SLR: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

(1) $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-2}$

(2) $S^2 = \frac{n}{n-2}\hat{\sigma}^2$

$\hat{\sigma}^2 = \frac{1}{n}\sum(Y_i - \hat{Y}_i)^2$

$S^2 = \frac{1}{n-2}\sum(Y_i - \hat{Y}_i)^2$

$E(S^2) = \sigma^2$ (unbiased)

MLR: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \dots + \hat{\beta}_p x_{ip}$

In matrix notation, we can write:

$$\sum \hat{\epsilon}_i^2 = \sum(Y_i - \hat{Y}_i)^2 = ||Y - HY||^2$$

$$= ||(I-H)Y||^2$$

$$= [(I-H)Y]^T(I-H)Y$$

$$= Y^T(I-H)^T(I-H)Y$$

$$= Y^T(I-H)(I-H)Y$$

$$= \underbrace{Y^T}_{\text{random vector}}\underbrace{(I-H)}_{\text{constant matrix}}\underbrace{Y}_{\text{random vector}}$$

Note: $||z||^2 = z^Tz$

$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-p}$

where p is # of columns in X (includes β_0)

Then, using some nice properties for finding means of matrices (see Rice 14.4), we can show that $E(||Y - \hat{Y}||^2) = (n-p)\sigma^2$. This leads to the unbiased estimate for σ^2 for the multiple regression case:

$$s^2 = \frac{||Y - \hat{y}||^2}{n-p} = \frac{1}{n-p} \sum (y_i - \hat{y}_i)^2$$

Errors vs Residuals:

Population model : $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + \epsilon$

Fitted Model : $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \dots + \hat{\beta}_k x_k$

Population / True error : $\epsilon_i \sim N(0, \sigma^2)$

Residuals : $\hat{\epsilon}_i = y_i - \hat{y}_i$

β_j's are pop. parameters (constant but unknown)

$\epsilon \sim N(0, \sigma^2)$

Covariance matrix of the residuals:

$\hat{\epsilon} = Y - \hat{y} = (I - H)Y$

$$\begin{aligned}\Sigma_{\hat{\epsilon}} &= (I-H)\Sigma_Y (I-H)^T \\ &= (I-H)\Sigma_\epsilon (I-H)^T \\ &= (I-H)\sigma^2 I (I-H)^T \\ &= \sigma^2 (I-H)(I-H)^T \\ &= \sigma^2 (I-H)\end{aligned}$$

$\leftarrow$ correlation between $\hat{\epsilon}_i, \hat{\epsilon}_j$ depends on $H = X(X^TX)^{-1}X^T$

In population model:

$Y = X\beta + \epsilon$

$$\Sigma_\epsilon = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 I$$

Cross-covariance matrix

Let X be a random vector of length n with covariance matrix Σ_X. If $Y = AX$ and $Z = BX$, where $A = p \times n$ and $B = m \times n$, then the cross-covariance matrix of Y and Z is given by:

$$\Sigma_{YZ} = A \Sigma_X B^T$$

$(p \times n)(n \times n)(n \times m)$ $\quad \Sigma_{YZ} = p \times m$

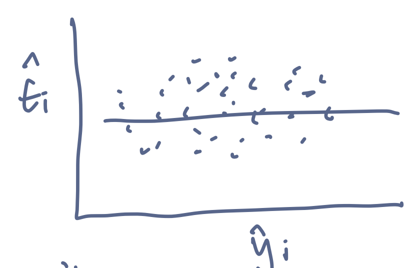
$cov(Y_1, Z_1)$, $cov(Y_1, Z_2)$

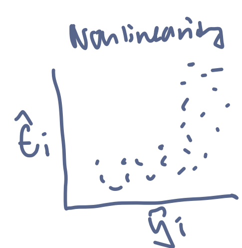
$$\Sigma_{YZ} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & & \vdots \\ \vdots & \vdots & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pm} \end{bmatrix}$$

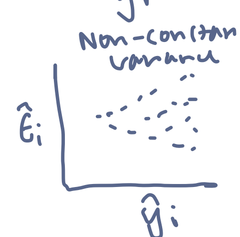
If the errors have covariance matrix $\sigma^2 I$, the residuals are uncorrelated with the predicted values

Proof: $\hat{\epsilon} = (I-H)Y \quad \hat{y} = HY \quad \Sigma_\epsilon = \sigma^2 I, \ \Sigma_Y = \sigma^2 I$

$$\begin{aligned}\Sigma_{\hat{\epsilon}\hat{y}} &= (I-H)\Sigma_Y H^T \\ &= (I-H)\sigma^2 I H^T \\ &= \sigma^2 [(I-H)H^T] \\ &= \sigma^2 [IH^T - HH^T] \\ &= \sigma^2 [H - H^2] \\ &= \sigma^2 [H - H] \\ &= \vec{0}\end{aligned}$$







3 CI's for β

Sampling distribution for $\hat{\beta}$

$$\hat{\beta} \sim MVN(\beta, \sigma^2 (X^T X)^{-1})$$ ← fun fact about MVN: each component has a marginal normal distribution

Each $\hat{\beta}_j \sim N(\beta_j, \sigma^2 c_{jj})$ $\quad C = (X^T X)^{-1}$

$$S^2/\sigma^2 \sim \chi^2_{n-p}$$

$$U_j = \frac{\left(\frac{\hat{\beta}_j - \beta_j}{\sigma \sqrt{c_{ii}}}\right)}{\sqrt{S^2/\sigma^2 (n-p)}} = \frac{\hat{\beta}_j - \beta_j}{S\sqrt{c_{jj}}} \sim t_{n-p}$$

In the simple LR case:

$$(X^T X)^{-1} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \sigma^2 \cdot \frac{n}{n\sum x_i^2 - (\sum x_i)^2}\right)$$

$$N\left(\beta_1, \sigma^2 \cdot \frac{n}{n\sum (x_i - \bar{x})^2}\right)$$

→ Same as SLR derivation

4 CI's and PI's for predictions

Let $x^T = (1, x_1, ..., x_p)$ be a vector of predictors for a new observation Y.

5 Multiple R^2

In the simple regression case, recall that

$R^2 =$

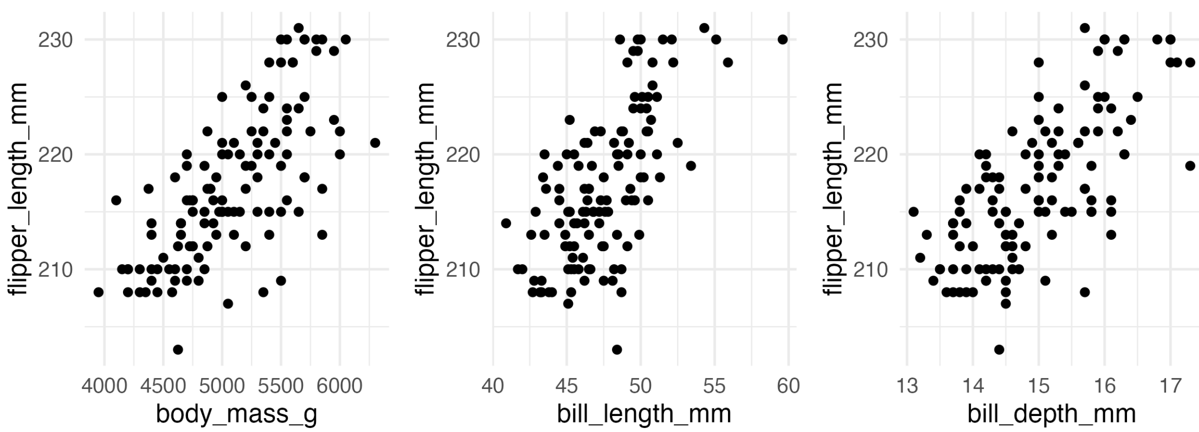
In simple linear regression, $R^2 = r^2$, where r is the sample correlation between X and Y. In the multiple regression case, we define $R = \text{Cor}(\hat{y}, y)$.

In multiple regression, whenever we add another predictor variable, R^2 *never gets worse*. The *Adjusted* R^2 is more often used in practice:

Adjusted $R^2 =$

as the number of predictors increase, what happens to the adjusted R^2?

6 Interpretation of β_i in Multiple Regression




```
Call:
lm(formula = flipper_length_mm ~ body_mass_g, data = gentoo)

Coefficients:
(Intercept)  body_mass_g
  1.713e+02    9.039e-03


Call:
lm(formula = flipper_length_mm ~ bill_length_mm, data = gentoo)

Coefficients:
   (Intercept)  bill_length_mm
       151.096           1.391
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Call:
lm(formula = flipper_length_mm ~ bill_depth_mm, data = gentoo)

Coefficients:
  (Intercept)  bill_depth_mm
       147.22           4.67


Call:
lm(formula = flipper_length_mm ~ body_mass_g + bill_length_mm +
    bill_depth_mm, data = gentoo)

Coefficients:
   (Intercept)     body_mass_g  bill_length_mm   bill_depth_mm
     139.99254         0.00382         0.52150         2.20463


Call:
lm(formula = flipper_length_mm ~ body_mass_g + bill_length_mm +
    bill_depth_mm, data = gentoo)

Residuals:
    Min      1Q  Median      3Q     Max
-12.440  -2.492   0.023   2.829   8.322

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)    1.400e+02  6.527e+00  21.448  < 2e-16 ***
body_mass_g    3.820e-03  1.153e-03   3.314 0.001217 **
bill_length_mm 5.215e-01  1.711e-01   3.047 0.002846 **
bill_depth_mm  2.205e+00  5.748e-01   3.836 0.000202 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.11 on 119 degrees of freedom
  (1 observation deleted due to missingness)
Multiple R-squared:  0.6082,    Adjusted R-squared:  0.5983
F-statistic: 61.58 on 3 and 119 DF,  p-value: < 2.2e-16
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