Homework 03: Due 9/27 (completion based)

Stat061-F23

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In HW1, we found the MOM and MLE estimates for each of the probability distributions below. For Q1-Q4, find (a) the posterior distribution for an iid sample of $X_1, X_2, \dots, X_n$ and (b) the posterior mean (our Bayesian estimate).

1. The parameter λ for a Poisson distribution where $P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$ for $k = 0, 1, 2, \dots$ and we assume the prior distribution for λ is Gamma(α, β). (This should be a named distribution; be sure to specify the parameters)

2. The parameter p in the Geometric distribution where $P(X = k) = p(1-p)^{k-1}$ for $k = 1, 2, 3, \dots$ and we assume the prior distribution for p is Beta(a, b). (This should be a named distribution; be sure to specify the parameters)

3. The parameter α in the distribution with pdf $f(x|\alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}[x(1-x)]^{\alpha-1}$ where $x \in [0, 1]$ and we assume the prior distribution for α is Unif(0,1). (You will probably not be able to solve the integral to determine the posterior mean, but write out the integral you would have to solve.)

4. The parameter β in the Pareto distribution with pdf $f(x|\beta) = \frac{\beta}{x^{\beta+1}}$ where $x > 1$ and we assume the prior distribution for β is Gamma(α, λ). (This should be a named distribution; be sure to specify the parameters)

5. For a Binomial(n, π) observation y, consider the Bayes estimator of π using a Beta(α, β) prior distribution.

    (a) For large n, show that the posterior distribution of π has approximate mean $\hat{\pi} = \frac{y}{n}$ (it also has approximate variance $\frac{\hat{\pi}(1-\hat{\pi})}{n}$. Relate this result to classical estimation.

    (b) Show that the MLE estimator is a limit of Bayes estimators, for a certain sequence of $\alpha = \beta$ values.

6. In class on Friday, we defined the posterior distribution for θ as:

$$f_{\theta|X}(\theta|x) = \frac{f_x(x|\theta)f_\theta(\theta)}{\int f_x(x|\theta)f_\theta(\theta)d\theta}$$

which is true if we observe 1 draw from the data model and have $X \sim f_x$.

If we have n IID observations $X_1, \dots, X_n$, we replace $f_x(x|\theta)$ with the *joint pdf*:

$$f_{X^n}(x_1, \dots, x_n|\theta) = \prod_{i=1}^{n} f_x(x_i|\theta) = L_n(\theta)$$

where x^n denotes the set of $(x_1, \dots, x_n)$, and $L_n(\theta)$ is the same likelihood function that is so near and dear to our hearts.

Then, the posterior distribution is:

$$f(\theta|x^n) = \frac{f_{x^n}(x^n|\theta)f_\theta(\theta)}{\int f_{x^n}(x^n|\theta)f_\theta(\theta)d\theta} = \frac{L_n(\theta)f_\theta(\theta)}{c_n} \propto L_n(\theta)f_\theta(\theta)$$

(a) Why can we write the joint pdf as $\prod_{i=1}^{n} f_x(x_i|\theta)$?

(b) What is c_n and how do we know that it is a constant?

(c) Explain what the $\propto L_n f_\theta(\theta)$ means.

(d) Do you think the Bayes estimator will generally be more similar to the MLE or to the MoM? Why?

7. Wrap up lab activity

8. Review for quiz on Wednesday!