04: CRAMER-RAO LOWER BOUND

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1 Minimum-Variance Unbiased Estimators

Given two unbiased estimators for the parameters θ , $\hat{\theta}_1$ and $\hat{\theta}_2$, we've already established which is "better": the one with smaller variance. But what if there is a $\hat{\theta}_3$ that has smaller variance than both of them? How can we know if one exists?

The **Cramer-Rao Lower Bound** tells us exactly that. It gives a theoretical limit below which an unbiased estimator cannot fall. If the variance of an estimator $\hat{\theta}$ is equal to that bound, we know that $\hat{\theta}$ is optimal in a sense: no other unbiased estimator can estimate θ with greater precision.

Fisher Information

The Fisher Information is a way of measuring the amount of information that a random variable X carries about the unknown parameter θ .

$$I(\theta) = E[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2] = [-E(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta})]$$

Example: Find the Fisher information for X, where $X \sim \text{Bernoulli}(\pi)$

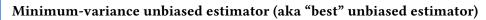
Cramer-Rao Lower Bound

Let $Y_1,...,Y_n \sim f_y(y;\theta)$, where $f_y(y;\theta)$ is a continuous pdf with continuous first and second derivative (i.e. "smooth enough"). Also suppose the set of values where $f_y(y;\theta) \neq 0$ does not depend on θ .

Let
$$\hat{\theta} = h(Y_1,...,Y_n)$$
 be any unbiased estimator of θ . Then,
$$Var(\hat{\theta}) \geq [nE[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2]^{-1} = [-nE(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta})]^{-1} = \frac{1}{nI(\theta)}$$

Example: Let $X_1,...,X_n$ be n Bernoulli trials with probability of succeess π . Let $\hat{\pi}=\frac{\sum X_i}{n}$. How does $Var(\hat{\pi})$ compare with the Cramer-Rao lower bound?

Example: Let $Y_1,..,Y_n\sim f_y$, where $f_y=\frac{2y}{\theta^2}$ for $0\leq y\leq \theta$. Compare the Cramer-Rao lower bound with the variance of the unbiased estimator $\frac{3}{2}\bar{Y}$. Discuss.



Let Θ denote the set of all estimators that are unbiased for the paramter θ in the continuous pdf $f_y(y;\theta)$. We say $\hat{\theta^*}$ is the MVUE if $\hat{\theta^*} \in \Theta$ and

Efficient estimator

Let $Y_1,...,Y_n \sim f_y(y;\theta).$ Let $\hat{\theta}$ be an unbiased estimator for $\theta.$

- 1. $\hat{\theta}$ is said to be **efficient** if:
- 2. The **efficiency** of $\hat{\theta}$ is:

Note: