Homework 04: Due 10/4

Stat061-F23

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1. Let $X_1, ..., X_n \sim Pois(\lambda)$ and let $\hat{\lambda} = \bar{X}$ be an estimator for λ (recall this is the MLE).

   (a) Find the Fisher Information for X_i
   (b) Find the Cramer-Rao Lower Bound
   (c) Find the variance of $\hat{\lambda}$ and show that it is an efficient estimator.

2. Prove the equivalence of

$$E[(\frac{\partial \ln f_y(y;\theta)}{\partial \theta})^2] = -E(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta})$$

   used in Cramer-Rao. *Hint:* Start with the second form and work towards the first form. If you are stuck on where to start, try $E(\frac{\partial^2 \ln f_y(y;\theta)}{\partial \theta}) = \int \frac{\partial}{\partial \theta}(\frac{\partial \ln f_y}{\partial \theta}) f_y dy$ and apply the chain rule. A "trick" in this proof is to twice differentiate $\int f_y(y) dy = 1$ with respect to θ: this will eventually be useful for showing that something in the resulting expression is equal to 0.

3. When Y has a positively skewed distribution over the positive real line, statisticians often treat $\ln Y$ as having a $N(\mu, \sigma^2)$ distribution. Then Y has the *log-normal distribution* which has pdf for $y > 0$:

$$f(y;\mu,\sigma) = \frac{1}{y\sigma\sqrt{2\pi}} e^{\frac{-[\ln y - \mu]^2}{2\sigma^2}}$$

   (a) For n independent observations, find the MLE for μ and σ^2.
   (b) Find the approximate variance of $\hat{\mu}_{MLE}$
   (c) Using the invariance property of the MLE, find the MLE for the mean and variance of this distribution, which are $E(Y) = e^{\mu + \sigma^2/2}$ and $V(Y) = (e^{\sigma^2} - 1)E(Y)^2$.

4. If $2n + 1$ random observations are drawn from a continuous and symmetric pdf with mean μ and if $f_Y(\mu;\mu) \neq 0$, then the *sample median* $\tilde{\mu}_n = Y_{n+1}$ is unbiased for μ, and $V(\tilde{\mu}_n) = \frac{1}{8n[f_y(\mu,\mu)]^2}$. Show that $\tilde{\mu}_n$ is consistent for μ.

5. Let $X_1, ..., X_n \sim N(\mu, \sigma^2)$. There are two different commonly-used estimators for σ^2:

$$\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2 \text{ and } s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$

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   We showed $\hat{\sigma}^2$ is the MLE in Notes01, and s^2 is commonly referred to as the "sample variance". When we use `var(x)` in R, s^2 is the output.

   (a) Show that the MLE $\hat{\sigma}^2$ is biased, and explain why s_n^2 corrects that bias.
   (b) Show that s_n^2 is a consistent estimator for σ^2. (*Hint:* Use Chebyshev's inequality. You may also find it useful that $Z_n = \frac{(\sum X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$.)
   (c) Use Jensen's inequality to show that s_n is biased for estimating σ.