Homework 05: Due 10/11

Stat061-F23

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1. Let X_1, X_2, X_3 be drawn from a Bernoulli(p) distribution.
   (a) Show that $\hat{p}_1 = \sum X_i$ is sufficient for p.
   (b) Show that $\hat{p}_2 = X_1 + 2X_2 + 3X_3$ is *not* sufficient for p.

2. Suppose $X_1, X_2, ..., X_n$ are iid from a Gamma(α, λ) distribution. That is, $f_x(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$. In Homework 02, you showed that the likelihood function for a Gamma distribution can be written such that it depends on the data values only through $\bar{X}$ and $\bar{X}_g = (\prod X_i)^{1/n}$. The work that you did will be helpful for this problem.
   (a) If α is known, show that the arithmetic mean $T_1 = \frac{1}{n} \sum X_i$ is sufficient for λ.
   (b) If λ is known, show that the geometric mean $T_2 = \bar{X}_g = (\prod X_i)^{1/n}$ is sufficient for α.
   (c) If both λ and α are unknown, show that T_1 and T_2 are jointly sufficient for α and λ.

3. Suppose $X_1, ..., X_n$ are a random sample from a Poisson(λ) distribution. Let $T = \sum X_i$ and recall that we showed T is sufficient for λ in class. Suppose we instead want to find an estimator for $\theta = P(X_i = 0)$.
   (a) Show that $\theta = e^{-\lambda}$
   (b) Show that $\hat{\theta} = \mathbb{1}\{X_1 = 0\}$ is unbiased for θ.
   (c) Use the Rao-Blackwell theorem to derive the new estimator $\theta^* = E(\theta|T = t)$ and show it is equal to $(\frac{n-1}{n})^t$.
   (d) Explain why θ^* is a "better" estimator than $\hat{\theta}$.

4. For each of the following families of distributions, show that it is an exponential family and deduce a sufficient statistic for the parameter:
   (a) The family of negative binomial distributions for which the value of r is known and the value of p is unknown.
   (b) The family of beta distributions for which the value of α is unknown and the value of β is known
   (c) The family of beta distributions for which the value of α is known and the value of β is unknown
   (d) The family of Pareto distributions, where

$$f_y(y;\theta) = \frac{\theta}{(1+\theta)^{\theta+1}}, \; 0 \leq y \leq \infty; 0 \leq \theta \leq \infty$$