Announcements

Wednesday: Quiz 2 / HW6 also due (completion)

Today: Review

OH: today 11:30 - 12:30

reschedule wed OH to Tuesday 3 - 4:15 or so

Friday: start new stuff, project

Note: updated 10/24 9:30 am.
New stuff/corrections are highlighted

① $X_1, \ldots, X_n \sim Poi(\theta)$

(a) $E(X_i) = \theta = V(X_i)$

$V(X_i) = E(X_i^2) - E(X_i)^2$

$\rightarrow E(X_i^2) = V(X_i) + E(X_i)^2$

$= \theta + \theta^2$

$\hat{\theta}_1 \rightarrow E(X_i) = \frac{1}{n}\sum X_i$

$\hat{\theta} = \bar{X}$

$\hat{\theta}_2 \rightarrow E(X_i)^2 = \frac{1}{n}\sum X_i^2$

$\theta + \theta^2 = \frac{1}{n}\sum X_i^2$

$\rightarrow$ solve for θ

(b) $L(\theta) = \prod \frac{\theta^{y_i}}{y_i!} e^{-\theta}$

$= \theta^{\sum y_i} e^{-n\theta} \prod \frac{1}{y_i!}$

$\ell(\theta) = \sum y_i \ln\theta - n\theta \ln e + \ln \prod \frac{1}{y_i!}$

$\frac{\partial \ell}{\partial \theta} = \frac{\sum y_i}{\theta} - n = 0$

$\sum y_i = n\theta$

$\hat{\theta} = \frac{1}{n}\sum y_i = \bar{y}$

(c) $I_n(\theta) = -E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right)$

$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{-\sum y_i}{\theta^2} \rightarrow -E\left(\frac{-\sum y_i}{\theta^2}\right) = \frac{1}{\theta^2}\sum E(y_i)$

$= \frac{1}{\theta^2} \cdot n\theta = \frac{n}{\theta}$

↑ since we did $\frac{\partial \ell}{\partial \theta}$ in (b)

(d) CRLB applies (support(f) does not depend on θ, $E(\hat{\theta}) = \theta$)

$var(\hat{\theta}) \geq \frac{1}{nI(\theta)}$ or $I_n(\theta)$ $= \frac{1}{n/\theta} = \frac{\theta}{n}$ ←

$V(\bar{x}) = V\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2} \cdot n \cdot \theta = \frac{\theta}{n} \Rightarrow \bar{x}$ meets CRLB and is MVUE

used $I_n(\theta)$ based on n samples. Since we already had $\ell(\theta)$ based on n samples. Since we accounted for n samples, don't need to multiply by n in CRLB.

CRLB: $V(\hat{\theta}) \geq \frac{1}{nI(\theta)}$ where $I(\theta) = -E\left[\frac{\partial^2 f_x}{\partial \theta^2}\right]$

$V(\hat{\theta}) \geq \frac{1}{I_n(\theta)}$ where $I_n(\theta) = -E\left[\frac{\partial^2 \ell}{\partial \theta^2}\right]$

Note $I_n(\theta) = nI(\theta)$

② $X_1, \dots, X_n \sim \text{Gamma}(\alpha, \frac{1}{2})$

(a) $L(\alpha) = \prod y_i^{\alpha-1} e^{-2y_i} \cdot \left[\left(\frac{1}{2}\right)^{\alpha} \Gamma(\alpha) \right]^{-1}$

$$= \underbrace{\frac{2^{n\alpha}}{[\Gamma(\alpha)]^n} \left(\prod y_i\right)^{\alpha-1}}_{g(\prod y_i, \alpha)} \underbrace{e^{-2\sum y_i}}_{b(y_i)}$$

By factorization thm, $\prod y_i$ is suff. for α

Can also find sufficient stat using exponential family form $T(X)$ – just make sure to use n samples instead of 1 in final answer. (Should also be a function of $T = \prod y_i$)

(b) $E(X_i) = \alpha\beta = \frac{\alpha}{2} = \bar{X}$

$$\Rightarrow \hat{\alpha} = 2\bar{X}$$

(c) If $\sum X_i \sim \text{Gamma}(n\alpha, \frac{1}{2})$, then $E(\hat{\alpha}) = E\left(\frac{2}{n}\sum X_i\right)$

$$= \frac{2}{n} E(\sum X_i)$$

$$= \frac{2}{n} \cdot n\frac{\alpha}{2} = \alpha$$

$V(\hat{\alpha}) = V\left(\frac{2}{n}\sum X_i\right) = \frac{2^2}{n^2} \cdot n \cdot V(X_i) = \frac{4}{n} \cdot \alpha \cdot \frac{1}{4}$

$$= \frac{\alpha}{n}$$

(d) By CLT, $\bar{X} \sim N\left(\alpha, \frac{\alpha}{4n}\right)$

By delta method, $2\bar{X} \sim N\left(2\alpha, 4 \cdot \frac{\alpha}{4n}\right)$

$$\sim N\left(2\alpha, \frac{\alpha}{n}\right)$$

asymp. variance (same as exact in this case)

(e) $\text{bias}^2 + V = 0^2 + \frac{\alpha}{n} = \frac{\alpha}{n}$

(f) By Rao-Blackwell, $\bar{X}$ not a function of $T = \prod X_i$

$\Rightarrow \alpha^* = E(\bar{X} \mid T = t)$ will have strictly smaller MSE

3 (a) $L(\lambda) = \prod \lambda^2 x_i^{-3} e^{-\lambda/x_i}$

(b) $L(\lambda) = \underbrace{\lambda^{2n} e^{-\lambda \sum \frac{1}{x_i}}}_{g(\sum \frac{1}{x_i}, \lambda)} \cdot \underbrace{\prod \frac{1}{x_i^3}}_{b(x_i)}$

* can also find via exponential family. Not necessarily = but should be a function of $\sum \frac{1}{x_i}$

By fact theorem, $\sum x_i^{-1}$ suff for λ

(c) $\ell(\lambda) = 2n \ln \lambda - \lambda \sum \frac{1}{x_i} \ln e + \sum \ln \frac{1}{x_i^3}$

$$\frac{\partial \ell}{\partial \lambda} = \frac{2n}{\lambda} - \sum \frac{1}{x_i} = 0$$

$$2n = \lambda \sum x_i^{-1}$$

$$\hat{\lambda} = \frac{2n}{\sum x_i^{-1}}$$

(d) MLE has large sample dist $N\left(\lambda, \frac{1}{I(\theta)}\right)$

$$I(\lambda) = -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) = -E\left(-\frac{2n}{\lambda^2}\right) = \frac{2n}{\lambda^2} \quad N\left(\lambda, \frac{\lambda^2}{2n}\right)$$

(e) By invariance of MLE,

$$\hat{\theta}_{MLE} = \log(\hat{\lambda}_{MLE}) = \log\left(\frac{2n}{\sum x_i^{-1}}\right)$$

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(a) False. MLE is asymptotically unbiased

(b) False. LLN tells us $\bar{x} \to_p \mu$

gets arbitrarily close to μ, but not $= \mu$

(c) False - $\bar{X}$ is consistent for μ, so

only true if $E(X_i) = \theta$

(d) CRLB : $V(\hat{\theta}) \geq \frac{1}{nI(\theta)} = \frac{\sigma^2}{n}$ for any $\hat{\theta}$

$V(\bar{X}) = V(\frac{1}{n} \sum X_i) = \frac{1}{n^2} \cdot n \quad V(X_i) = \frac{\sigma^2}{n}$

$\bar{X}$ meets CRLB, so it is MUVE.

↳ and is unbiased.

True.

(e) True. This is def. of CRLB