· Short HW due Wed on usual
· Quizzes hopefully Mon 6
· Project info on Monday

09: UNCERTAINTY INTERVALS

Larsen & Marx 5.3, 5.9

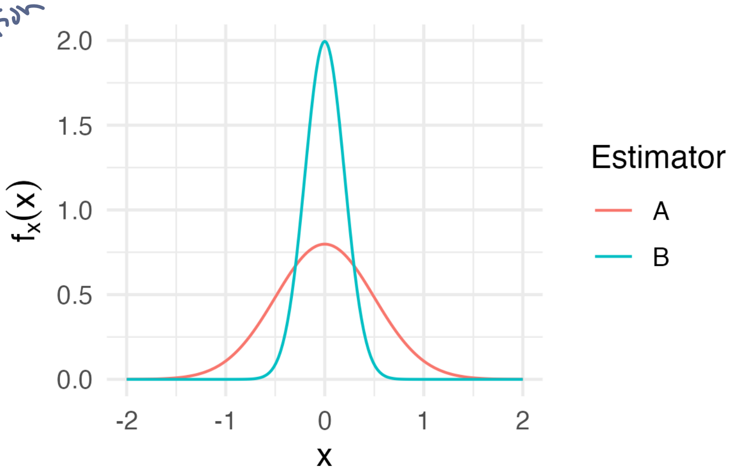
Prof Amanda Luby

Up until this point, our focus has been on *point estimation*: if we have a parameter, what's the single "best guess" for that parameter, and how do we evaluate how good of a guess it is?

Consider two estimators A and B that both have large-sample normal sampling distributions.

sampling distribution

$\hat{p}$, $\bar{X}$, $\frac{n}{n-1}\sum X_i$



$A \sim N(\theta, .5^2)$

$B \sim N(\theta, .25^2)$

↑

smaller variance = more precision in the estimator

1 Confidence Intervals

Since we know the *shape* and *parameters* of the *sampling distribution*, we know that:

$B \sim N(\theta, .25^2)$

$$z = \frac{B - \theta}{.25} \sim N(0,1)$$

$$P\left(-2 \le \frac{B-\theta}{.25} \le 2\right) = .95$$

By inverting the terms in the probability statement, this is equivalent to:

$$P(-2 \cdot .25 \le B - \theta \le 2 \cdot .25) = .95$$

$$P(-2 \cdot .25 - B \le -\theta \le 2 \cdot .25 - B) = .95$$

$$P(B - 2 \cdot .25 \le \theta \le B + 2 \cdot .25) = .95$$

95% CI for θ is

$[B - 2 * .25, B + 2 * .25]$

$[A - 2 * .5, A + 2 * .5]$

→ A is wider → more uncertainty about θ if we use A compared to B

$X_i \sim F_X \quad \mu, \sigma^2$

$\to \frac{1}{n-1} \sum (X_i - \bar{X})^2$

From CLT: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \to N(\mu, \frac{\sigma^2}{100}) \to N(\mu, \frac{s^2}{100}) \to N(\mu, \frac{56.33^2}{100})$

Example: Among a random sample of 100 recent college graduates, the average monthly student loan payment was \$287, with a standard deviation of \$56.33. Construct a 95% confidence interval for μ, the average monthly student loan payment among the population.

[174, 400]

[275, 297]

$$\left[287 - 2 \cdot \frac{56.33}{10},\ 287 + 2 \cdot \frac{56.33}{10}\right]$$

$$z = \frac{\bar{X} - \mu}{56.33/10} \sim N(0,1)$$

$$P\left[\bar{X} - 2 \cdot \frac{56.33}{10} \le \mu \le \bar{X} + 2 \cdot \frac{56.33}{10}\right]$$

$[275, 298] =$ 95% CI for μ

Caution: What can we conclude?

(a) 95% of recent college grads have monthly student loan payments in this range → NO - CI doesn't say anything about data points X_i

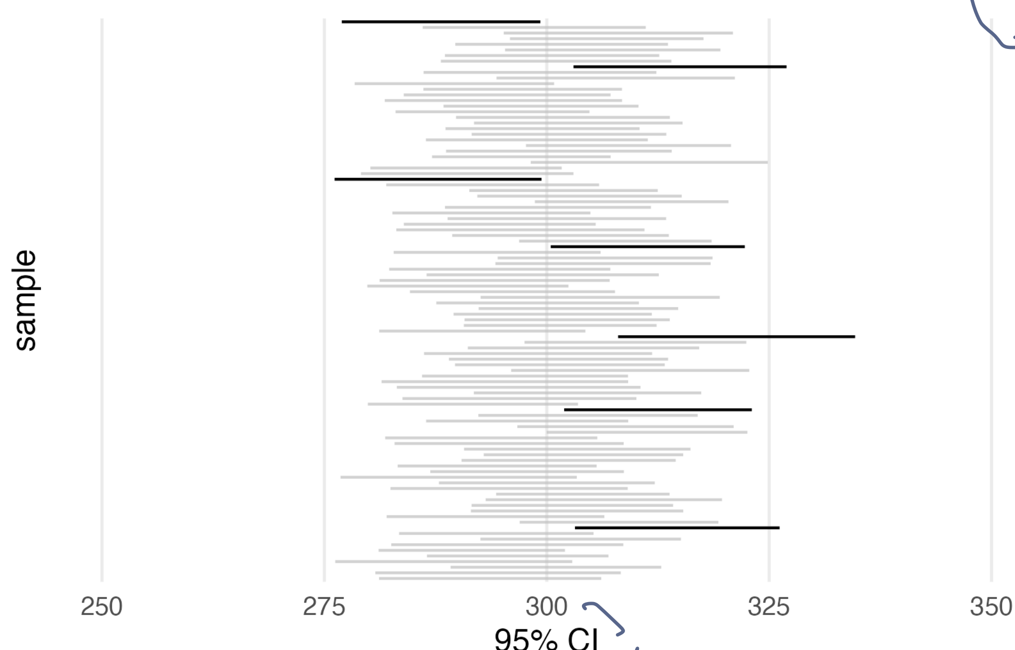
(b) There is a .95 probability that $\bar{X}$ falls in this range → 100% chance $\bar{X}$ is in range

(c) 95% of samples with $n = 100$ would fall in this range → NO - same as above

(d) There is a .95 probability that μ falls within this range → NO - once we build interval, μ is either in it or not. BUT this is true before we see any data

(e) 95% of samples with $n = 100$ would give an interval that contains μ → YES!!!

Example: Below are the CI's from 100 samples where each $X_i \sim N(300, 60^2)$.



μ

$\bar{X} \sim N(300, \frac{60^2}{100})$

$$95\% : \left[\bar{x} - 2 \cdot \frac{s}{\sqrt{n}}, \bar{x} + 2 \frac{s}{\sqrt{n}}\right]$$

Example: Find a 90% confidence interval for μ.

→ detour to back of notes

$$90\% : \left[\bar{x} - 1.645 \cdot \frac{s}{\sqrt{n}}, \bar{x} + 1.645 \cdot \frac{s}{\sqrt{n}}\right]$$

$$\left[287 - 1.645 \cdot \frac{56.33}{10}, 287 + 1.645 \cdot \frac{56.33}{10}\right]$$

$$[277.7, 296.26]$$

Monday 10/30
- HW 7 due Wed – OH after class
- Wed – no class, but zoom OH
- No Wed afternoon OH
- Friday – lab on your own time

Example: Suppose we want to precisely estimate μ such that our confidence interval is no wider than \$15. what sample size would we need?

let's find a general formula: $\left(\bar{x} + z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}\right) - \left(\bar{x} - z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}\right) \leq w$

$$2 \cdot z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} \leq w$$

$$\sqrt{n} \geq \frac{2 z_{\alpha/2} \cdot s}{w}$$

$$n \geq \left(\frac{2 z_{\alpha/2} \cdot s}{w}\right)^2$$

In this case: $n \geq \frac{4 \cdot 1.645^2 \cdot 56.33^2}{15^2}$

≥ 152.6

≥ 153

* width vs. margin of error

width ↑ total length of CI

margin of error ↑ $z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$

width = 2 · margin of error

Example: For a Pew Research survey of a representative sample of $n = 2500$ adults, $X = 1300$ said that

X_i = i-th response – 1/0 ← Bernoulli

$n = 2500$ $\quad$ $X = \sum X_i$ ← Binomial

$X = 1300$

they played video games. Let θ be the true proportion of adults who play video games. Give (a) the *exact* sampling distribution of X, (b) the approximate sampling distributions for X and X/n. Use your answer from (b) to set up 95% CI's for X and X/n.

(a) $X \sim \text{Binomial}(n = 2500, \theta)$

(b) $\frac{X}{n} \sim N\left(\mu = \theta, \sigma^2 = \frac{\theta(1-\theta)}{n}\right)$

$X \sim N(n\theta, n\theta(1-\theta))$ ← delta method
linearity of Normal dist.

$X = n\bar{x}$

(c) 95% CI for $\bar{x}$: $\bar{x} \pm 1.96\sqrt{\frac{\theta(1-\theta)}{n}}$ / $\bar{x} \pm 1.96\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}$

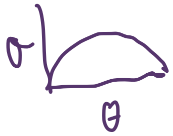
[.5004, .53958]

95% CI for X: $X \pm 1.96\sqrt{n\bar{x}(1-\bar{x})}$

A *conservative* CI for X/n: Problem! The "plug in" estimate $\hat{\theta} = \frac{X}{n}$ could be incorrect, making our CI too small

"conservative" = wide enough to guarantee $1-\alpha$ % coverage even in "worst case scenario"

↳ matters the most when n is small or $\bar{x}$ is close to 0 or 1



$\sigma = \sqrt{\frac{\theta(1-\theta)}{n}}$ ← exploit the fact that $0 \le \theta \le 1$

$\Rightarrow \sigma$ is maximized when $\theta = 1/2$

conservative binomial CI: $\sigma = \sqrt{\frac{1/2 \cdot 1/2}{n}} = \sqrt{\frac{1}{4n}}$

$\bar{X} \pm z_{\alpha/2} \cdot \sqrt{\frac{1}{4n}}$

For example above:

$\frac{1300}{2500} \pm 1.96\sqrt{\frac{1}{4 \cdot 2500}} = [.5004, .5396]$

$\theta \sim f_\theta$ prior distribution

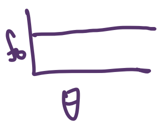
$x | \theta \sim f_{x|\theta}$ likelihood

$\theta | x \propto f_{x|\theta} \cdot f_\theta$ posterior

2 Bayesian Intervals

In the Bayesian estimation framework, uncertainty intervals are no longer based on long-term *coverage* but are instead based on *uncertainty in the posterior.*

Recall from Notes03 that if $X_1, ..., X_n \sim$ Bernoulli($\overset{\theta}{p}$) and $\overset{\theta}{p} \sim$ Beta(α, β), then $\overset{\theta}{p} | \sum X \sim$ Beta$(\sum X_i + \alpha, n - \sum X_i + \beta)$.



Example: Using the Pew Research sample above, what is the resulting posterior distribution $p | \sum X_i$? Assume a uniform prior distribution: $p \sim Beta(1, 1)$. How could we construct a 95% posterior probability interval?

$n = 2500$

$\sum X_i = 1300$

$\alpha = 1$

$\beta = 1$

$\theta | x \sim \text{Beta}(1300 + 1, 2500 - 1300 + 1)$

$\sim \text{Beta}(1301, 1201)$

Find 2.5% & 97.5% quantiles

qbeta(.025, 1301, 1201) = .5004

qbeta(.975, 1301, 1201) = .5395

95% posterior probability intervals

→ credible intervals

→ after observing our data, there's a 95% probability that θ is between .5004 and .5395

Recall from Notes03 that if $X_1, ..., X_n \sim N(\theta, \sigma^2)$ (σ^2 known) and $\theta \sim N(a, b^2)$, then $\theta | X \sim N(\frac{b^2 \sum X_i + \sigma^2 a}{nb^2 + \sigma^2}, \frac{\sigma^2 b^2}{nb^2 + \sigma^2})$.

Example: From our student loan payment example, $\sum X_i = \$287, n = 100$ and we'll assume $\sigma^2 = 60^2 = 3600$. Let's also assume a "flat" prior: $\theta \sim N(250, 100^2)$. What's the resulting posterior distribution? What's a 95% posterior probability interval?

$\theta | x \sim N\left(\frac{100^2 \cdot 100 \cdot 287 + 60^2 \cdot 250}{100 \cdot 100^2 + 60^2}, \frac{60^2\, 100^2}{100 \cdot 100^2 + 60^2}\right)$

$\sim N(286.87, 35.87)$

$\sim N(286.87, 5.989^2)$

lower : qnorm(.025, 286.87, 5.89) = 275.33

upper : qnorm(.975, 286.87, 5.89) = 298.41

After observing our data, there's a 95% probability that θ is between [275.33, 298.41]

careful about sd vs var

3 Z Tables

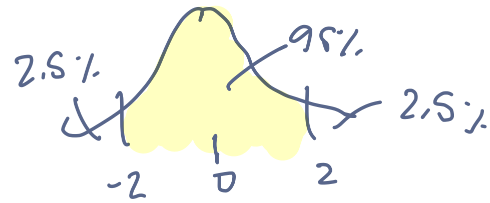
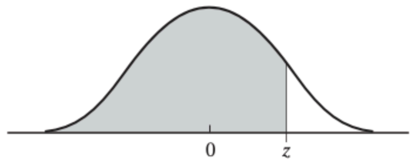


Table A.1 Cumulative Areas under the Standard Normal Distribution

Φ



z	0	1	2	3	4	5	6	7	8	9
−3.	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
−1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0570	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

(*cont.*)

2.5th percentile

$z = -1.96$

5th perc.

$z = -1.645$

Table A.1 Cumulative Areas under the Standard Normal Distribution (*cont.*)

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Source: From Samuels/Witmer, *Statistics for Life Sciences*, Table 3, p. 675, © 2003 Pearson Education, Inc. Reproduced by permission of Pearson Education, Inc.

4 Alternative to a z-table

density

```
dnorm(-1)
```

→ value of $f_x(-1)$

```
[1] 0.2419707
```

prob / CDF

```
pnorm(-1)
```

→ value $\Phi(-1)$

→ gives $P(Z \le x)$

inverses of eachother

```
[1] 0.1586553
```

quantile

```
qnorm(.1586)
```

→ gives x where $.1586 = \Phi(x)$

gives z-score for a certain probability

```
[1] -1.000228
```


90% – qnorm(.05)

= -1.645

common z-scores

```
qnorm(.005)
```

```
[1] -2.575829
```

```
qnorm(.025)
```

```
[1] -1.959964
```

```
qnorm(.05)
```

```
[1] -1.644854
```

```
qnorm(.0975)
```

```
[1] -1.295929
```

```
qnorm(.095)
```

```
[1] -1.310579
```