SOLUTIONS Homework 00: Probability & Calculus Review

Stat061-F23

Prof Amanda Luby

The prerequisites for Stat61 are probability (Stat51) and multivariable calculus (Math 33, 34, 35). I will also assume some familiarity with matrix algebra. If you do not know your calculus and probability reasonably well, you may have a hard time following the course material and keeping up with assignments. In particular, you will need to be comfortable with limits, infinite series, differentiation, and integration. In addition, it is recommended that you've seen some elementary statistics at the level of Stat 11.

To give you an idea of what you're expected to know, try out the following calculus and probability questions. This is not homework, you do not need to turn it in! It's OK if you don't immediately remember how to solve everything, but you should be able to complete most of these problems relatively quickly with a small amount of looking back at prior notes or googling. This is *not* an exhaustive list of topics, but a sample of the mechanics that I'll assume you can mostly figure out on your own (but it's OK if it takes some time!) If you have trouble with most of the questions, please see me during the first two weeks of class.

1 Calculus Review

1. Simplify $\sum_{k=0}^{\infty} x^k$ if $0 < x < 1$.

Solution

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + x \\
S_3 &= 1 + x + x^2 \\
S_k &= 1 + x + x^2 + \ldots + x^{k-1} \\
xS_k &= x + x^2 + x^3 + \ldots + x^k \\
S_k - xS_k &= 1 - x^k \\
S_k(1-x) &= 1 - x^k \\
S_k &= \frac{1 - x^k}{1 - x}
\end{aligned}$$

when $0 < x < 1$, as $k \to \infty$,

$$S_k = \frac{1}{1-x}$$

2. Simplify $\sum_{k=0}^{\infty} kx^k$ if $0 < x < 1$. (*Hint*: use the result from the previous question)

Solution

From the first problem, we know $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ when $0 < x < 1$. Taking the derivative of $\sum_{k=0}^{\infty} x^k$ with respect to x we find

$$\begin{aligned} \frac{d}{dx}\sum_{k=0}^{\infty} x^k &= \sum_{k=0}^{\infty} \frac{d}{dx} x^k \text{ From absolute convergence.} \\ &= \sum_{k=0}^{\infty} k x^{k-1} \end{aligned}$$

We also see

$$\begin{aligned} \frac{d}{dx}\sum_{k=0}^{\infty} x^k &= \frac{d}{dx}\frac{1}{1-x} \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

So

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

meaning

$$\sum_{k=0}^{\infty} k x^{k} = \frac{x}{(1-x)^2}.$$

3. What is the value of $\sum_{k=1}^{\infty} \frac{1}{k}$?

Solution

This is the harmonic series and there are lots of different ways to show that it diverges - here's one based on a cleverly-chosen comparison series.

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_k &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} \end{aligned}$$

Using the comparison test, we see that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots$$

And note that the RHS can be grouped in sets that add up to 1/2:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

So

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

4. Show that $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

Solution

Using a Taylor Series expansion we see

$$e^x = e^a + e^a(x-a) + \frac{e^a(x-a)^2}{2!} + \frac{e^a(x-a)^3}{3!} + \dots$$

At $a = 0$

$$\begin{aligned} e^x &= e^0 + e^0(x-0) + \frac{e^0(x-0)^2}{2!} + \frac{e^0(x-0)^3}{3!} + \dots \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

1. Find the derivative with respect to x of $f(x) = xe^{-x}$.

Solution

Use the product rule:

$$\begin{aligned}
\frac{d}{dx}xe^{-x} &= e^{-x}\frac{d}{dx}x + x\frac{d}{dx}e^{-x} \\
&= e^{-x} + xe^{-x}(-1) \\
&= e^{-x}(1-x)
\end{aligned}$$

1. Find the antiderivative of $f(x) = xe^{-x}$.

Solution

Using integration be parts, let $u = x$, $du = dx$, $v = -e^{-x}$, $dv = e^{-x}$. Then

$$\begin{aligned}
\int xe^{-x}dx &= -xe^{-x} - \int -e^{-x}dx \\
&= -xe^{-x} + \int e^{-x}dx \\
&= -xe^{-x} - e^{-x} + c \\
&= -e^{-x}(x+1) + c
\end{aligned}$$

1. Find the partial derivatives of $f(x, y) = y^x$ where $x > 1, y > 1$.

Solution

$$\frac{df}{dy} = xy^{x-1}$$

\ To find the partial derivative with respect to x we will use logarithmic differentiation,

$$\begin{aligned}
f(x) &= y^x \\
\ln f(x) &= x\ln y \\
\frac{f'(x)}{f(x)} &= \ln y \\
f'(x) &= f(x)\ln y \\
f'(x) &= y^x \ln y
\end{aligned}$$

1. Find the value of μ (in terms of the x_i's) that maximizes the function

$$L(\mu) = \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2/2\right].$$

Solution

Because a logarithm is a monotonic transformation, we can find the value of μ that maximizes the $ln(L(\mu))$ to simplify the problem.

$$\begin{aligned} l(\mu) &= ln(L(\mu)) \\ &= -\frac{1}{4}\sum_{i=1}^{n}(x_i-\mu)^2 \\ &= -\frac{1}{4}(\sum_{i=1}^{n}x_i^2 - 2\mu\sum_{i=1}^{n}x_i + n\mu^2) \end{aligned}$$

Take the derivative

$$l'(\mu) = \frac{1}{2}\sum_{i=1}^{n}x_i - \frac{1}{4}(2n)\mu$$

Setting the derivative equal to 0 we find

$$\begin{aligned} \frac{1}{2}(\sum_{i=1}^{n}x_i - n\mu) &= 0 \\ \sum_{i=1}^{n}x_i &= n\mu \\ \frac{\sum_{i=1}^{n}x_i}{n} &= \mu \end{aligned}$$

Checking the second derivative to determine if we have found a minimum or a maximum, we see,

$$l"(\mu) = -\frac{n}{2}$$

which is negative, so we have found a maximum.

1. Show that

$$\sum_{i=1}^{n}(x_i-\mu)^2 = n(\bar{x}-\mu)^2 + \sum_{i=1}^{n}(x_i-\bar{x})^2, \qquad \text{where } \bar{x} = \frac{1}{n}\sum_{i=1}^{n}x_i.$$

Solution

$$\begin{aligned}
n(\bar{x}-\mu)^2 + \sum_{i=1}^{n}(x_i-\bar{x})^2 &= n\bar{x}^2 - 2n\bar{x}\mu + n\mu^2 + \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2 \\
&= 2n\bar{x}^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2 + \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i \\
&= \sum_{i=1}^{n} x_i^2 + 2(\bar{x}\sum_{i=1}^{n} x_i - \mu\sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} x_i) + n\mu^2 \\
&= \sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2 \\
&= \sum_{i=1}^{n}(x_i-\mu)^2
\end{aligned}$$

2 Probability Review

1. Target's Market Pantry sells a mixed fruit variety of "fruit" snacks that my nephew loves. Each packet contains 15 fruit snacks with five flavors (grape, strawberry, orange, peach, and raspberry). In one recent packet he exclaimed delight that he got 4 strawberry (his favorite), 4 orange (second favorite), 3 raspberry, 3 peach and 1 grape (least favorite.)

    a) If my nephew reaches into the recent packet and picks out 5 fruit snacks at once what is the probability he will select at least one strawberry?
    b) If he reaches in and picks out 5 fruit snacks at once, what is the probability he will select one of each flavor?
    c) If he samples 5 *with replacement*, what is the probability he will select at least one strawberry?

Solution

(a) It is easiest to do this problem by using the complement rule.

$$\begin{aligned}
P(\text{at least 1 strawberry}) &= 1 - P(\text{no strawberry}) \\
&= 1 - \frac{\binom{4}{0}\binom{11}{5}}{\binom{15}{5}} \\
&= 1 - \frac{(1)\frac{11!}{5!6!}}{\frac{15!}{5!10!}} \\
&= 1 - \frac{(10)(9)(8)(7)}{(15)(14)(13)(12)} \\
&= 0.846
\end{aligned}$$

(b) We will again use the complement rule, but this time we will take into account the fact that we are sampling with replacement.

$$\begin{aligned}
P(\text{at least 1 strawberry}) &= 1 - P(\text{no strawberry}) \\
&= 1 - (\frac{11}{15})^5 \\
&= 0.788
\end{aligned}$$

(c) Recall there are 4 strawberry and 4 orange, 3 raspberry and 3 peach and 1 grape.

$$\begin{aligned}
P(\text{1 of each flavor}) &= \frac{\binom{4}{1}\binom{4}{1}\binom{3}{1}\binom{3}{1}\binom{1}{1}}{\binom{15}{5}} \\
&= 0.0480
\end{aligned}$$

1. Let X be the number on a standard die roll (so X is chosen uniformly from the set $\{1, 2, 3, 4, 5, 6\}$).

   a) What is the moment generating function (MGF) of X?
   b) Suppose that ten dice are rolled independently and Y is the sum of the numbers on all the dice. What is the moment generating function (MGF) of Y?

Solution

(a) Recall the definition of an MGF ($M_x(t) = E(e^{tx})$). Using this definition, we can solve for the MGF of X\

$$\begin{aligned} M_x(t) &= \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}) \\ &= \frac{1}{6}\sum_{i=1}^{6} e^{it} \end{aligned}$$

(b) So $Y = \sum_{i=1}^{10} X_i$. Using this, the independence of the rolls, and part a, we can find the MGF of Y.\

$$\begin{aligned} M_y(t) &= M_{\sum x}(t) \\ &= [M_x(t)]^{10} \text{ by independence} \\ &= [\frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})]^{10} \end{aligned}$$

1. Suppose X has a continuous probability distribution f_X such that $E(X) = 10$ and $Var(X) = 10$. Let $X_1, X_2, ...X_5$ be continuous independent random variables with the probability distribution f_X. Let $Y = \frac{\sum_{i=1}^{5} X_i}{5}$. Using the Central Limit Theorem, find the approximate probability $P(5 \leq Y \leq 15)$.

Solution

We will start by finding the mean and variance of Y. Knowing the X's are independent makes our calculations much easier.\

$$\begin{aligned} E(Y) &= E(\frac{\sum_{i=1}^{5} X}{5}) \\ &= \frac{1}{5}\sum_{i=1}^{5} E(X) \\ &= \frac{1}{5}(10)(5) \\ &= 10 \end{aligned}$$

$$\begin{aligned} var(Y) &= var(\frac{\sum_{i=1}^{5} X}{5}) \\ &= \frac{1}{5^2}\sum_{i=1}^{5} var(X) \\ &= \frac{1}{25}(10)(5) \\ &= 2 \end{aligned}$$

By the CLT, then $Y \sim N(10, 2)$. Thus we can find the $P(5 \leq Y \leq 15)$.

$$\begin{aligned} P(5 \leq Y \leq 15) &= P(Y \leq 15) - P(Y \leq 5) \\ &= P(Z \leq \frac{15-10}{\sqrt{2}}) - P(Z \leq \frac{5-10}{\sqrt{2}}) \\ &= P(Z \leq 3.54) - P(Z \leq -3.54) \\ &\approx 1 \end{aligned}$$

1. Suppose the measurement error of a certain scale is known to follow a continuous distribution whose probability density function is

$$f(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the probability density function of $|X|$, the absolute measurement error.
   b) Find $E(X)$ and $Var(X)$.

Solution

a) To solve this problem, we will use the CDF method.\ \ Let $ Y = |X| $, then

$$\begin{aligned} P(Y \leq y) &= P(|X| \leq y) \\ &= P(-y \leq X \leq y) \\ &= F_X(y) - F_X(-y) \end{aligned}$$

Using the fact that the pdf is the derivative of the CDF, we can find the pdf of Y.

$$\begin{aligned} \frac{d}{dy} P(Y \leq y) &= \frac{d}{dy}[F_X(y) - F_X(-y) \\ f_Y(y) &= f_X(y) - f_X(-y)(-1) \\ &= f_X(y) + f_X(-y) \\ &= (1-y) + 1 + (-y) \\ &= 2(1-y)\text{where } 0 < y < 1 \end{aligned}$$

Note that Y must be positive which is what allows us to do the penultimate step.

b) We will solve for $E(X)$ first.

$$\begin{aligned} E(X) &= \int x f_X(x) dx \\ &= \int_0^1 x(1-x)dx + \int_{-1}^0 x(1+x)dx \\ &= (\frac{x^2}{2} - \frac{x^3}{3})|_0^1 + (\frac{x^2}{2} + \frac{x^3}{3})|_{-1}^0 \\ &= 0 \end{aligned}$$

Now we will solve for $Var(X)$. To do this, we must find the second moment, because $Var(X) = E(X^2) - [E(X)]^2$. Since $E(X) = 0$ in this case, we know $Var(X) = E(X^2)$.

$$\begin{aligned} Var(X) &= E(X^2) \\ &= \int x^2 f_X(x) dx \\ &= \int_0^1 x^2(1-x)dx + \int_{-1}^0 x^2(1+x)dx \\ &= (\frac{x^3}{3} - \frac{x^4}{4})|_0^1 + (\frac{x^3}{3} + \frac{x^4}{4})|_{-1}^0 \\ &= \frac{1}{6} \end{aligned}$$

1. Suppose that the random variables X, Y, and Z have the multivariate PDF for $0 < x < 1, 0 < y < 1,$

and $z > 0$.

$$f_{X,Y,Z} = (x+y)e^{-z}$$

a) Find $f_{X,Y}(x, y)$
b) $f_{Y,Z}(y, z)$
c) $f_Z(z)$

Solution