

Homework 04: Due 10/4

Stat061-F23

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1. Let $X_1, \dots, X_n \sim \text{Pois}(\lambda)$ and let $\hat{\lambda} = \bar{X}$ be an estimator for λ (recall this is the MLE).

- (a) Find the Fisher Information for X_i
- (b) Find the Cramer-Rao Lower Bound
- (c) Find the variance of $\hat{\lambda}$ and show that it is an efficient estimator.

2. Prove the equivalence of

$$E\left[\left(\frac{\partial \ln f_y(y; \theta)}{\partial \theta}\right)^2\right] = -E\left(\frac{\partial^2 \ln f_y(y; \theta)}{\partial \theta^2}\right)$$

used in Cramer-Rao. *Hint:* Start with the second form and work towards the first form. If you are stuck on where to start, try $E\left(\frac{\partial^2 \ln f_y(y; \theta)}{\partial \theta^2}\right) = \int \frac{\partial}{\partial \theta} \left(\frac{\partial \ln f_y}{\partial \theta}\right) f_y dy$ and apply the chain rule. A “trick” in this proof is to twice differentiate $\int f_y(y) dy = 1$ with respect to θ : this will eventually be useful for showing that something in the resulting expression is equal to 0.

3. When Y has a positively skewed distribution over the positive real line, statisticians often treat $\ln Y$ as having a $N(\mu, \sigma^2)$ distribution. Then Y has the *log-normal distribution* which has pdf for $y > 0$:

$$f(y; \mu, \sigma) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}$$

- (a) For n independent observations, find the MLE for μ and σ^2 .
 - (b) Find the approximate variance of $\hat{\mu}_{MLE}$
 - (c) Using the invariance property of the MLE, find the MLE for the mean and variance of this distribution, which are $E(Y) = e^{\mu + \sigma^2/2}$ and $V(Y) = (e^{\sigma^2} - 1)E(Y)^2$.
4. If $2n + 1$ random observations are drawn from a continuous and symmetric pdf with mean μ and if $f_Y(\mu; \mu) \neq 0$, then the *sample median* $\tilde{\mu}_n = Y_{n+1}$ is unbiased for μ , and $V(\tilde{\mu}_n) = \frac{1}{8n[f_Y(\mu, \mu)]^2}$. Show that $\tilde{\mu}_n$ is consistent for μ .
5. Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. There are two different commonly-used estimators for σ^2 :

$$\hat{\sigma}^2 = \frac{1}{n} \sum (X_i - \bar{X})^2 \text{ and } s^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$$

We showed $\hat{\sigma}^2$ is the MLE in Notes01, and s^2 is commonly referred to as the “sample variance”. When we use `var(x)` in R, s^2 is the output.

- (a) Show that the MLE $\hat{\sigma}^2$ is biased, and explain why s_n^2 corrects that bias.
- (b) Show that s_n^2 is a consistent estimator for σ^2 . (*Hint:* Use Chebyshev’s inequality. You may also find it useful that $Z_n = \frac{\sum (X_i - \bar{X})^2}{\sigma^2} \sim \chi_{n-1}^2$.)
- (c) Use Jensen’s inequality to show that s_n is biased for estimating σ .