Monday : Review
Wed : Quiz 2 ; HW due at night / completion

08: LARGE SAMPLE PROPERTIES

Rice 5.3, 8.5.2

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Reminder: $\lim_{n \to \infty} P(|\hat{\theta} - \theta| < \epsilon) = 1$

"convergence in probability"

$\hat{\theta}_n \xrightarrow{p} \theta$

1 Sampling Distribution of a Statistic

Convergence in distribution

Let $X_1, \ldots, X_n$ be a sequence of random variables with CDFs $F_1, \ldots, F_n$ and let X be a random variable with CDF F. We say X_n converges in distribution to F if:

$$\lim_{n \to \infty} F_n(x) = F(x)$$

$X_n \xrightarrow{d} X$

at every point x where F is continuous

Sampling distribution of a statistic

Let $X_1, \ldots, X_n$ be a random sample with pdf $f_x(\theta)$. Let $T = h(X_1, \ldots, X_n, \theta)$. Then, the distribution of T (given θ) is called the *sampling distribution* of T. $T \xrightarrow{d} F_T$

Example: In lab on Wed, χ^2 example

1,000 times

$X_1, \ldots, X_{25} \sim N(\mu, \sigma^2)$

$\bar{X}_{25} \sim N(\mu, \frac{\sigma^2}{25})$ → $N(\mu, \frac{\sigma^2}{25})$ is the sampling distribution for $\bar{X}$

$Z_{25} = \frac{\sum (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{24}$

χ^2_{24} is the sampling distribution for Z_n

Example:

(skipped)

2 Central Limit Theorem

Central Limit Theorem

For a large number n of iid observations $Y_i \sim f_y$, with mean μ and finite variance σ^2, the sampling distribution of $\bar{Y}$ is approximately:

$$N\left(\mu, \frac{\sigma^2}{n}\right)$$

- $E[\bar{Y}]$ $V(\bar{Y})$ come directly from rules of E + V $E\left[\frac{1}{n}\sum X_i\right]$ $V\left[\frac{1}{n}\sum X_i\right]$
- CLT gives us the shape of the sampling distribution

Central Limit Theorem II

For a large number n of iid observations $Y_i \sim f_y$, with mean μ and variance σ^2, then for each fixed x:

$$\lim_{n \to \infty} \underbrace{P\left(\frac{\bar{Y}_n - \mu}{\sqrt{\sigma^2/n}} \leq x\right)}_{\text{CDF of } \frac{\bar{Y}_n - \mu}{\sqrt{\sigma^2/n}}} = \Phi(x)$$

$\Phi(x) \leftarrow$ CDF of $N(0,1)$ $P(\underset{\sim}{X} \leq x)$

3 Delta Method

$$\frac{\bar{Y}_n - \mu}{\sqrt{\sigma^2/n}} \xrightarrow{d} N(0,1)$$

$$\left(\sqrt{\frac{n}{\sigma^2}}\right)(\bar{Y}_n - \mu) \xrightarrow{d} N(0,1)$$

Delta Method

Let $Y_1, Y_2, \ldots$ be a sequence of random variables, and let F^* be a continuous c.d.f. Let θ be a real number, and let $a_1, a_2, \ldots$ be a sequence of positive numbers that increase to ∞. Suppose that $a_n(Y_n - \theta)$ converges in distribution to F^*. Let α be a function with continuous derivative such that $\alpha'(\theta) \neq 0$. Then, $\alpha: \mathbb{R} \to \mathbb{R}$

$$\frac{a_n[\alpha(Y_n) - \alpha(\theta)]}{\alpha'(\theta)} \xrightarrow{d} F^*$$

Proof outline: know $a_n(Y_n - \theta) \xrightarrow{d} F^*$ and $a_n \to \infty$. So $(Y_n - \theta) \to 0$ with high probability as $n \to \infty$ $\Rightarrow$ ① $Y_n \to \theta$ with high probability. Otherwise, $a_n[Y_n - \theta] \to \infty$ instead of F^*.

② Since α is continuous, $\alpha(Y_n) \to \alpha(\theta)$ with high probability since ①.

③ Taylor expansion (1st term only):

$$\alpha(Y_n) \approx \alpha(\theta) + \alpha'(\theta)(Y_n - \theta)$$

$$\alpha(Y_n) - \alpha(\theta) \approx \alpha'(\theta)(Y_n - \theta)$$

$$\frac{a_n}{\alpha'(\theta)}[\alpha(Y_n) - \alpha(\theta)] \approx \frac{a_n}{\alpha'(\theta)}\alpha'(\theta)(Y_n - \theta)$$

$$\approx F^*$$

$$\frac{a_n[\alpha(Y_n) - \alpha(\theta)]}{\alpha'(\theta)} \xrightarrow{d} F^* \Big\} \text{ delta method}$$

Example: Variance-stabilizing transformation

$Y_1, \dots, Y_n \sim \text{Pois}(\lambda)$

MLE: $\bar{Y}$

CLT: $\bar{Y} \sim N(\lambda, \frac{\lambda}{n}) \rightarrow \sqrt{\frac{n}{\lambda}}(\bar{Y} - \lambda) \xrightarrow{d} N(0,1)$

$a_n(\bar{Y} - \lambda) \xrightarrow{d} N(0,1)$

In order to "stabilizing variance"

Let $\alpha(\lambda) = \sqrt{\lambda}$ $\qquad$ $\alpha'(\lambda) = \frac{1}{2\sqrt{\lambda}}$

By delta method, $\frac{a_n}{\alpha'(\lambda)} \cdot (\sqrt{\bar{Y}} - \sqrt{\lambda}) \xrightarrow{d} F^*$

$$\frac{\sqrt{n/\lambda}}{1/2\sqrt{\lambda}}(\sqrt{\bar{Y}} - \sqrt{\lambda}) \xrightarrow{d} N(0,1)$$

$$2\sqrt{n}(\sqrt{\bar{Y}} - \sqrt{\lambda}) \xrightarrow{d} N(0,1)$$

$$\sqrt{\bar{Y}} \sim N(\sqrt{\lambda}, \frac{1}{4n})$$

4 Large-Sample Properties of the MLE

1. MLE estimators are *sufficient*
2. MLE estimators are *invariant*
3. MLE estimators are *asymptotically unbiased.*
4. Under appropriate smoothness conditions of f_x, the MLE from an iid sample is *consistent.*
5. MLE estimators are *asymptotically efficient*: for large n, other estimators do not have smaller variance
6. Under smoothness conditions of f_x, the MLE has a *normal sampling distribution* for large samples

Sampling distribution of the MLE

Let $\hat{\theta} = h(Y)$ be the MLE for θ, where $Y \sim f_y(\theta)$.

$$\hat{\theta} \sim N(\theta, \frac{1}{nI(\theta)})$$

Example: (skipped)

5 Large-Sample Properties of the Bayes Estimator

The large-sample properties of the MLE generally extend to the Bayes estimator:

1. Bayes estimators are asymptotically unbiased
2. Bayes estimators are asymptotically efficient
3. Bayes estimators are consistent
4. Bayes estimators are sufficient
5. Bayes estimators have normal sampling distributions for large n

Idea: Bayes estimator is mean of posterior

$$f_{\theta|x} \propto \underbrace{f_{x|\theta}}_{\text{likelihood}} \cdot f_\theta$$

likelihood = same likelihood function that MLE optimizes

when n large, $f_{x|\theta}$ has n "things" prior: 1 "thing"

f_θ gets "washed out" by the likelihood