

Named Probability Distributions

Discrete Probability Distributions

pmf: $p(y)$ cdf: $F(y) = \sum_{z=-\infty}^y p(z)$
 $0 \leq p(y) \leq 1; \sum_{y=-\infty}^{\infty} p(y) = 1$
 $P(Y = y) = p(y); P(a \leq Y \leq b) = \sum_a^b p(y)$

Binomial – $Y \sim \text{Binom}(n, p)$

$$p(y) = \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}, y \in [0, n], p \in [0, 1]$$
$$\mathbb{E}[Y] = np$$
$$\mathbb{V}[Y] = np(1-p)$$
$$m(t) = [pe^t + (1-p)]^n$$

Geometric – $Y \sim \text{Geom}(p)$

$$p(y) = (1-p)^{y-1} p, y \in [1, \infty), p \in [0, 1]$$
$$\mathbb{E}[Y] = 1/p$$
$$\mathbb{V}[Y] = (1-p)/p^2$$
$$m(t) = \frac{p}{1-qe^t}$$

Hypergeometric – $Y \sim \text{HG}(N, K, n)$

$$p(y = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}, k \in \{\max(0, n+K-N), \dots, \min(n, K)\}, K \leq N; n \leq N$$
$$\mathbb{E}[Y] = \frac{nK}{N}$$
$$\mathbb{V}[Y] = \frac{nK(N-K)(N-n)}{N^2(N-1)}$$

Negative Binomial – $Y \sim \text{NBinom}(r, p)$

$$P(Y = k) = \binom{r+k-1}{r-1} p^r (1-p)^k, k \in [r, \infty), r \in \mathbb{Z}^+, p \in [0, 1]$$
$$\mathbb{E}[Y] = rq/p$$
$$\mathbb{V}[Y] = rq/p^2$$
$$m(t) = \left(\frac{p}{1-qe^t}\right)^r \text{ for } qe^t < 1$$

Poisson – $Y \sim \text{Poi}(\lambda)$

$$p(y) = \frac{\lambda^y}{y!} e^{-\lambda}, y \in [0, \infty);$$
$$\mathbb{E}[Y] = \mathbb{V}[Y] = \lambda$$
$$m(t) = e^{\lambda(e^t-1)}$$

Continuous Probability Distributions

pdf: $f(y) = \frac{d}{dy}(y)$ cdf: $F(y) = \int_{-\infty}^y f(z) dz$
 $f(y) \geq 0; \int_{-\infty}^{\infty} f(y) dy = 1; P(Y = y) = 0$
 $P(a \leq Y \leq b) = \int_a^b f(y) dy = F(b) - F(a)$

Uniform – $Y \sim \text{Uniform}(a, b)$

$$f(y) = (b-a)^{-1}, y \in [a, b]$$
$$\mathbb{E}[Y] = (a+b)/2$$
$$\mathbb{V}[Y] = (b-a)^2/12$$
$$m(t) = (e^{bt} - e^{at})/[t(b-a)]$$

Normal – $Y \sim N(\mu, \sigma^2)$

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} \quad y \in (-\infty, \infty), \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$$
$$\mathbb{E}[Y] = \mu;$$
$$\mathbb{V}[Y] = \sigma^2$$
$$m(t) = \exp(\mu t + t^2 \sigma^2/2)$$

If $Y \sim N(\mu, \sigma)$, then $Z = (Y - \mu)/\sigma; Z \sim N(0,1)$.

$$P(Y \leq y) = \Phi\left(\frac{y-\mu}{\sigma}\right) = \Phi(z) \text{ (non-analytic function)}$$

Exponential – $Y \sim \text{Exponential}(\lambda)$

$$f(y) = \lambda e^{-\lambda y}, y \in [0, \infty), \lambda \in \mathbb{R}^+$$
$$\mathbb{E}[Y] = 1/\lambda$$
$$\mathbb{V}[Y] = 1/\lambda^2$$
$$m(t) = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda$$

Gamma – $Y \sim \text{Gamma}(\alpha, \beta)$

$$f(y) = y^{\alpha-1} e^{-y/\beta} / [\beta^\alpha \Gamma(\alpha)], y \in [0, \infty), \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+$$
$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy = (\alpha-1)\Gamma(\alpha-1)$$

If n is a positive integer, $\Gamma(n) = (n-1)!$

$$\mathbb{E}[Y] = \alpha\beta$$
$$\mathbb{V}[Y] = \alpha\beta^2$$
$$m(t) = (1 - \beta t)^{-\alpha}$$

$\alpha = 1 \Rightarrow$ **exponential distribution**

$\beta = 2, \alpha = \nu/2, \nu \in \mathbb{Z}^+ \Rightarrow$ **chi-square distribution**

Beta – $Y \sim \text{Beta}(\alpha, \beta)$

$$f(y) = y^{\alpha-1} (1-y)^{\beta-1} / B(\alpha, \beta), y \in [0, 1], \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+$$
$$B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$$
$$\mathbb{E}[Y] = \alpha/(\alpha+\beta)$$
$$\mathbb{V}[Y] = \alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$$

T – $Y \sim T(\nu)$

$$f(y) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{(\sqrt{\nu\pi})\Gamma(\nu/2)} \left(1 + \frac{y^2}{\nu}\right)^{-(\nu+1)/2}, y \in (-\infty, \infty), \nu > 0$$
$$\mathbb{E}[Y] = 0 (\nu > 1)$$
$$\mathbb{V}[Y] = \frac{\nu}{\nu-2} (\nu > 2), \infty (1 < \nu \leq 2)$$

Properties of Estimators

Inequalities and Convergence

Cauchy-Schwarz: $|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$

Jensen: $E(g(X)) \geq g(E(X))$ if g convex; $E(g(X)) \leq g(E(X))$ if g concave.

Markov: $P(|W| > a) \leq E(|W|)/a$

Chebyshev: $P(|W - \mu| \geq \epsilon) \leq \sigma^2/\epsilon^2$

Chernoff: $P(W \geq a) \leq E(e^{tW})/e^{ta}$

$X_n \rightarrow_d X$ if $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ at all x

$X_n \rightarrow_p X$ if $\lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon) = 0$

Fisher Information

$$I(\theta) = E\left[\left(\frac{\partial \ln f_y(y; \theta)}{\partial \theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln f_y(y; \theta)}{\partial \theta^2}\right]$$

Cramer-Rao Lower Bound

$Y_1, \dots, Y_n \sim f_y, \{y : y \neq 0\}$ does not depend on $\theta, E(\hat{\theta}) = \theta$.
 $Var(\hat{\theta}) \geq \frac{1}{nI(\theta)}$

Consistency

$\hat{\theta}_n$ is consistent if $\hat{\theta}_n \rightarrow_p \theta$.

Invariance: If $\hat{\theta}_n$ is consistent for $\theta, g(\hat{\theta}_n)$ is consistent for $g(\theta)$

Sufficiency

$T = h(X_1, \dots, X_n)$ is sufficient for θ if $P(X_1, \dots, X_n | T = t)$ does not depend on θ .

T is sufficient if and only if $L(\theta) = g[h(X_1, \dots, X_n); \theta] \cdot b(X_1, \dots, X_n)$

Rao-Blackwell: Let $\hat{\theta}$ be an estimator of θ with $E(\hat{\theta}^2) < \infty$ and let T be a sufficient statistic. If $\theta^* = E(\hat{\theta} | T = t)$, then $MSE(\theta^*, \theta) \leq MSE(\hat{\theta}, \theta)$. Strict inequality unless $\hat{\theta} = f(T)$.

Exponential Families

$$f(x; \theta) = \exp[\eta(\theta)T(x) - A(\theta) + B(x)]$$

$$= h(x) \exp[\eta(\theta)T(x) - A(\theta)]$$

$$= h(x)g(\theta) \exp[\eta(\theta)T(x)]$$

$$E(Y) = \frac{\partial}{\partial \eta} A(\eta)$$

$$V(Y) = \frac{\partial^2}{\partial \eta^2} A(\eta)$$

Large-Sample Properties

WLLN: $\bar{X} \rightarrow_p \mu$

CLT: If $Y_i \sim f_y, E(Y) = \mu, V(Y) = \sigma^2$, then $\bar{Y} \sim N(\mu, \sigma^2/n)$

CLT: $\frac{\sum X_i - n\mu}{\sigma/\sqrt{n}} \rightarrow_d Z$, where $Z \sim N(0, 1)$.

Delta Method: If $Y_n \approx N(\mu, \frac{\sigma^2}{n})$ then $g(Y_n) \approx N(g(\mu), (g'(\mu))^2 \frac{\sigma^2}{n})$

$\hat{\theta}_{MLE} \sim N(\theta, \frac{1}{nI(\theta)})$ for large n

Inference

Sampling Distributions

If $Z \sim N(0, 1), Z^2 \sim \chi^2(1)$

$\sum_{i=1}^n Z_i^2 \sim \chi^2(n)$

$[(n-1)S^2/\sigma^2] \sim \chi^2(n-1)$

$[(\bar{Y} - \mu)/(\sigma/\sqrt{n})] \sim N(0, 1)$

$[(\bar{Y} - \mu)/(S/\sqrt{n})] \sim t(n-1)$

$Z \sim N(0, 1) \perp W \sim \chi^2(\nu)$, then $T = (Z/\sqrt{W/\nu}) \sim t(\nu)$

$$S_p^2 = \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}, T = \frac{\bar{X} - \bar{Y} - \mu_x + \mu_y}{S_p \sqrt{1/n+1/m}} \sim T_{n+m-2}$$

$$\frac{\bar{X} - \bar{Y} - \mu_x + \mu_y}{\sqrt{s_x^2/n + s_y^2/m}} \sim T_{[\nu]}, \nu = \frac{(s_x^2/s_y^2 + n/m)^2}{1/(n-1)(s_x^2/s_y^2)^2 + 1/(m-1)(n/m)^2}$$

$X \sim \text{Binom}(n, p_x), Y \sim \text{Binom}(m, p_y)$,

$X/n - Y/m \sim N(p_x - p_y, p_x(1-p_x)/n + p_y(1-p_y)/m)$

Goodness of Fit: $\sum_{i=1}^k \frac{(X_i - n\hat{p}_i)^2}{n\hat{p}_i} \sim \chi_{n-1-s}^2$

Power Function

Let δ be a test of statistic T and rejection region R .

$$\pi(\theta|\delta) = P(T \in R|\theta)$$

Likelihood Ratio Test (GLRT)

$$\lambda = \frac{\max_{\Omega_0} L(\theta)}{\max_{\Omega_1} L(\theta)}$$

$\delta = \text{Reject } H_0 \text{ when } \lambda \leq \lambda^*, \text{ where } P(\Lambda \leq \lambda^* | \theta \in \theta_0) = \alpha$