

## Homework 05: Due 10/11

Stat061-F23

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1. Let  $X_1, X_2, X_3$  be drawn from a Bernoulli( $p$ ) distribution.
  - (a) Show that  $\hat{p}_1 = \sum X_i$  is sufficient for  $p$ .
  - (b) Show that  $\hat{p}_2 = X_1 + 2X_2 + 3X_3$  is *not* sufficient for  $p$ .
2. Suppose  $X_1, X_2, \dots, X_n$  are iid from a Gamma( $\alpha, \lambda$ ) distribution. That is,  $f_x(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ . In Homework 02, you showed that the likelihood function for a Gamma distribution can be written such that it depends on the data values only through  $\bar{X}$  and  $\bar{X}_g = (\prod X_i)^{1/n}$ . The work that you did will be helpful for this problem.
  - (a) If  $\alpha$  is known, show that the arithmetic mean  $T_1 = \frac{1}{n} \sum X_i$  is sufficient for  $\lambda$ .
  - (b) If  $\lambda$  is known, show that the geometric mean  $T_2 = \bar{X}_g = (\prod X_i)^{1/n}$  is sufficient for  $\alpha$ .
  - (c) If both  $\lambda$  and  $\alpha$  are unknown, show that  $T_1$  and  $T_2$  are jointly sufficient for  $\alpha$  and  $\lambda$ .
3. Suppose  $X_1, \dots, X_n$  are a random sample from a Poisson( $\lambda$ ) distribution. Let  $T = \sum X_i$  and recall that we showed  $T$  is sufficient for  $\lambda$  in class. Suppose we instead want to find an estimator for  $\theta = P(X_i = 0)$ .
  - (a) Show that  $\theta = e^{-\lambda}$
  - (b) Show that  $\hat{\theta} = \mathbb{1}\{X_1 = 0\}$  is unbiased for  $\theta$ .
  - (c) Use the Rao-Blackwell theorem to derive the new estimator  $\theta^* = E(\hat{\theta}|T = t)$  and show it is equal to  $(\frac{n-1}{n})^t$ .
  - (d) Explain why  $\theta^*$  is a “better” estimator than  $\hat{\theta}$ .
4. For each of the following families of distributions, show that it is an exponential family and deduce a sufficient statistic for the parameter:
  - (a) The family of negative binomial distributions for which the value of  $r$  is known and the value of  $p$  is unknown.
  - (b) The family of beta distributions for which the value of  $\alpha$  is unknown and the value of  $\beta$  is known
  - (c) The family of beta distributions for which the value of  $\alpha$  is known and the value of  $\beta$  is unknown
  - (d) The family of Pareto distributions, where

$$f_y(y; \theta) = \frac{\theta}{y^{\theta+1}}, \quad 1 < y \leq \infty; 0 \leq \theta \leq \infty$$