· Short thr due Wed on written
· Quizzes hopefully Mon 6
· Project info on Monday

09: UNCERTAINTY INTERVALS

Larsen & Marx 5.3, 5.9

Prof Amanda Luby

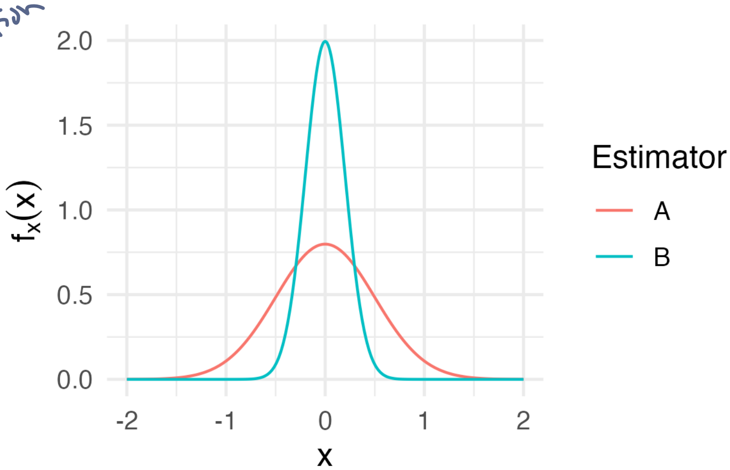
Up until this point, our focus has been on *point estimation*: if we have a parameter, what's the single "best guess" for that parameter, and how do we evaluate how good of a guess it is?

Consider two estimators A and B that both have large-sample normal sampling distributions.

sampling distribution

$\hat{p}, \bar{x}$

$\frac{n}{n-1} \sum x_i$



$A \sim N(\theta, .5^2)$

$B \sim N(\theta, .25^2)$

smaller variance = more precision in the estimator

1 Confidence Intervals

Since we know the *shape* and *parameters* of the *sampling distribution*, we know that:

$B \sim N(\theta, .25^2)$

$$z = \frac{B-\theta}{.25} \sim N(0,1)$$

$$P\left(-2 \le \frac{B-\theta}{.25} \le 2\right) = .95$$

By inverting the terms in the probability statement, this is equivalent to:

$$P(-2 \cdot .25 \le B-\theta \le 2 \cdot .25) = .95$$

$$P(-2 \cdot .25 - B \le -\theta \le 2 \cdot .25 - B) = .95$$

$$P(B - 2 \cdot .25 \le \theta \le B + 2 \cdot .25) = .95$$

95% CI for θ is

$[B - 2 * .25, B + 2 * .25]$

$[A - 2 * .5, A + 2 * .5]$

→ A is wider → more uncertainty about θ if we use A compared to B

$X_i \sim F_X \quad \mu, \sigma^2$

$\to \frac{1}{n-1}\sum(x_i-\bar{x})^2$

From CLT: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \to N(\mu, \frac{\sigma^2}{100}) \to N(\mu, \frac{s^2}{100}) \to N(\mu, \frac{56.33^2}{100})$

Example: Among a random sample of 100 recent college graduates, the average monthly student loan payment was \$287, with a standard deviation of \$56.33. Construct a 95% confidence interval for μ, the average monthly student loan payment among the population.

$[174, 400]$

$[275, 297]$

$$\left[287 - 2\cdot\frac{56.33}{10},\ 287 + 2\cdot\frac{56.33}{10}\right]$$

$$Z = \frac{\bar{X}-\mu}{56.33/10} \sim N(0,1)$$

$$P\left(\bar{X} - 2\cdot\frac{56.33}{10} \le \mu \le \bar{X} + 2\cdot\frac{56.33}{10}\right)$$

$[275, 298] =$ 95% CI for μ

Caution: What can we conclude?

(a) 95% of recent college grads have monthly student loan payments in this range $\to$ NO - CI doesn't say anything about data points X_i

(b) There is a .95 probability that $\bar{X}$ falls in this range $\to$ 100% chance $\bar{X}$ is in range

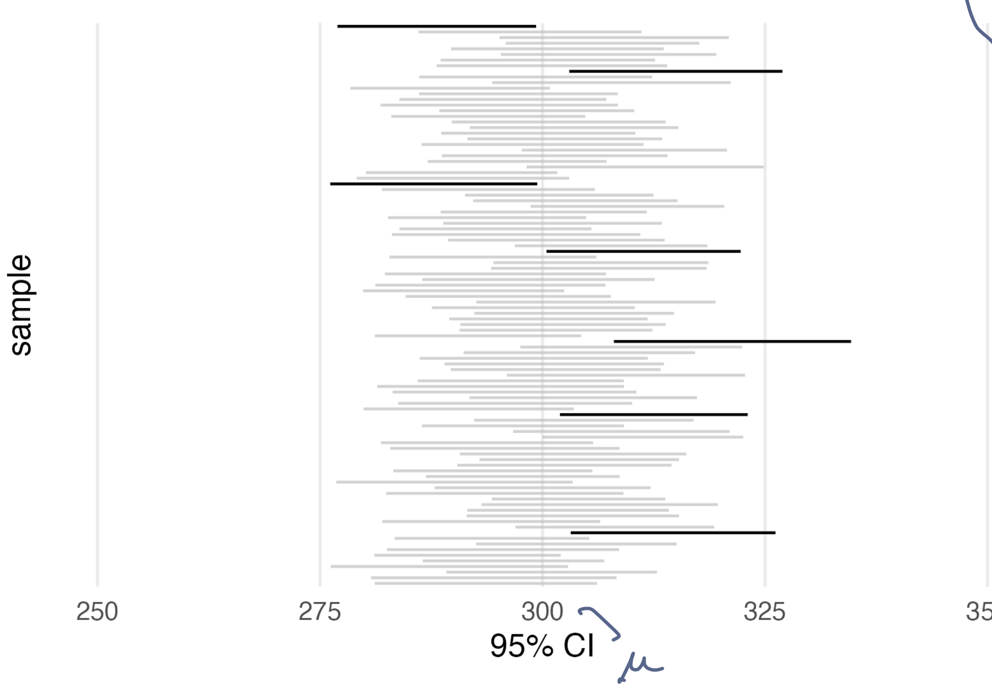
(c) 95% of samples with $n = 100$ would fall in this range $\to$ NO - same as above

(d) There is a .95 probability that μ falls within this range $\to$ NO - once we build interval, μ is either in it or not. BUT this is true before we see any data

(e) 95% of samples with $n = 100$ would give an interval that contains μ $\to$ YES!!!

Example: Below are the CI's from 100 samples where each $X_i \sim N(300, 60^2)$.

$\bar{X} \sim N(300, \frac{60^2}{100})$



$$95\%: \left[\bar{X} - 2 \cdot \frac{S}{\sqrt{n}}, \bar{X} + 2 \frac{S}{\sqrt{n}}\right]$$

Example: Find a 90% confidence interval for μ.

$\rightarrow$ detour to back of notes

$$90\%: \left[\bar{X} - 1.645 \cdot \frac{S}{\sqrt{n}}, \bar{X} + 1.645 \cdot \frac{S}{\sqrt{n}}\right]$$

$$\left[287 - 1.645 \cdot \frac{56.33}{10}, 287 + 1.645 \cdot \frac{56.37}{10}\right]$$

Example: Suppose we want to precisely estimate μ such that our confidence interval is no wider than \$15.

Example: For a Pew Research survey of a representative sample of $n = 2500$ adults, $X = 1300$ said that they played video games. Let θ be the true proportion of adults who play video games. Give (a) the *exact* sampling distribution of X, (b) the approximate sampling distributions for X and X/n. Use your answer from (b) to set up 95% CI's for X and X/n.

A *conservative* CI for X/n:

2 Bayesian Intervals

In the Bayesian estimation framework, uncertainty intervals are no longer based on long-term *coverage* but are instead based on *uncertainty in the posterior*.

Recall from Notes03 that if $X_1, ..., X_n \sim \text{Bernoulli}(p)$ and $p \sim \text{Beta}(\alpha, \beta)$, then $p|\sum X \sim \text{Beta}(\sum X_i + \alpha, n - \sum X_i + \beta)$.

Example: Using the Pew Research sample above, what is the resulting posterior distribution $p|\sum X_i$? Assume a uniform prior distribution: $p \sim Beta(1, 1)$. How could we construct a 95% posterior probability interval?

Recall from Notes03 that if $X_1, ..., X_n \sim N(\theta, \sigma^2)$ (σ^2 known) and $\theta \sim N(a, b^2)$, then $\theta|X \sim N(\frac{b^2 \sum X_i + \sigma^2 a}{nb^2 + \sigma^2}, \frac{\sigma^2 b^2}{nb^2 + \sigma^2})$.

Example: From our student loan payment example, $\sum X_i = \$287, n = 100$ and we'll assume $\sigma^2 = 60^2 = 3600$. Let's also assume a "flat" prior: $\theta \sim N(250, 100^2)$. What's the resulting posterior distribution? What's a 95% posterior probability interval?

3 Z Tables

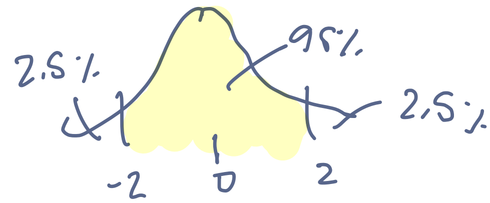
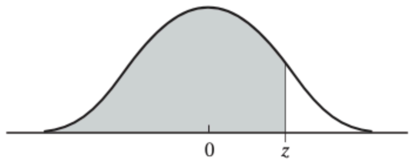


Table A.1 Cumulative Areas under the Standard Normal Distribution

Φ



z	0	1	2	3	4	5	6	7	8	9
−3.	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
−1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0570	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

(*cont.*)

2.5th percentile

$Z = -1.96$

5th perc.

$Z = -1.645$

Table A.1 Cumulative Areas under the Standard Normal Distribution (*cont.*)

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

4 Alternative to a z-table

density

```
dnorm(-1)
```

→ value of $f_x(-1)$

```
[1] 0.2419707
```

prob / CDF

```
pnorm(-1)
```

→ value $\Phi(-1)$

→ gives $P(Z \le x)$

inverses of eachother

```
[1] 0.1586553
```

quantile

```
qnorm(.1586)
```

→ gives x where $.1586 = \Phi(x)$

gives z-score for a certain probability

```
[1] -1.000228
```


90% – qnorm(.05)

= -1.645

Common z-scores

```r
qnorm(.005)
```

```
[1] -2.575829
```

```r
qnorm(.025)
```

```
[1] -1.959964
```

```r
qnorm(.05)
```

```
[1] -1.644854
```

```r
qnorm(.0975)
```

```
[1] -1.295929
```

```r
qnorm(.095)
```

```
[1] -1.310579
```