

# 09: EFFICIENCY AND CRLB

Stat250 S25

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## 1 Recap

- Observe data  $X_1, \dots, X_n \sim F_x(x|\theta)$ , where  $\theta$  is unknown
- Goal: *Estimate*  $\theta$  based on the values of  $X_i$  by formulating an *estimator*  $\hat{\theta}$
- One technique is to use the *maximum likelihood estimator*
- A second technique is to use the *method of moments estimator*
- There are lots of other ways to come up with estimators.
- We can compare estimators by comparing their *bias*, *variance*, and *mean squared error*
- Today: is there a way to know if we've found an "optimal" estimator?

## 2 Comparing unbiased estimators

### Efficiency

For two unbiased estimators,  $\hat{\theta}_1$  is more **efficient** than  $\hat{\theta}_2$  if

**Example:** Why efficiency matters

We now have two unbiased estimators for  $\theta$  in a  $\text{Unif}(0, \theta)$  distribution.  $\hat{\theta}_{MoM} = \frac{2}{n} \sum_{i=1}^n X_i$  and  $\hat{\theta}_3 = \frac{1}{n} \sum_{i=1}^n X_i^2$ .

- (a) If  $n = 10$ , what is the expectation and variance of each of these estimators?

(b) What  $n$  is needed for  $\hat{\theta}_{MoM}$  to reach the variance of  $\hat{\theta}_3$

### 3 Can we find a better estimator?

Is there another unbiased estimator with smaller variance?

#### Cramer-Rao Lower Bound

If  $X_1, \dots, X_n$  are an iid sample from a distribution with pdf  $f(x|\theta)$ , then any unbiased estimator  $\hat{\theta}$  of  $\theta$  satisfies:

$$V(\hat{\theta}) \geq \frac{1}{nI(\theta)}$$

where  $I(\theta)$  is the **Fisher Information** of  $X_i$

If the variance of an estimator is equal to the CRLB, then there is *no other unbiased estimator with more precision*

#### Fisher Information

The **Fisher Information** of an observation  $X$  is

$$I(\theta) = E\left[\left(\frac{d}{d\theta} \ln f(x|\theta)\right)^2\right] =$$

\*provided the domain of  $X$  does not depend on  $\theta$  (and a few other regularity conditions)

#### 3.1 Intuition: Fisher information

$$I(\theta) = E_X\left[\left(\frac{d}{d\theta} \ln f(x|\theta)\right)^2\right] = -E_X(l''(\theta))$$

- $l(\theta)$  is the log-likelihood function
- $l'(\theta)$  gives the slope of the log-likelihood at any point,  $l''(\theta)$  gives the curvature at any given point

- The Fisher information “averages out” over  $X$  and summarizes the overall curvature of the log-likelihood function



### 3.2 Practice with CRLB

**Example:** Find the *Fisher information* for  $Y$ , where  $X \sim \text{Exp}(\lambda)$

**Example:** Let  $Y_1, \dots, Y_n$  be  $n$   $\text{Exp}(\lambda)$  random variables. Let  $\hat{\lambda} = \frac{n}{\sum Y_i}$ . How does  $\text{Var}(\hat{\lambda})$  compare with the CRLB?

### 3.3 CRLB with simulation

**Example:** The median  $m$  of an  $\text{Exponential}(\lambda)$  distribution satisfies  $P(X \leq m) = 0.5$ . Solving  $1 - e^{-\lambda m} = 0.5$  gives  $m = \ln(2)/\lambda$ . This suggests an estimator for  $\lambda$  based on the median:  $\hat{\lambda} = \frac{\ln(2)}{m}$ . Finding the analytical variance of  $\hat{\lambda}$  is complicated. Finding the sampling distribution of  $m$  is complicated, and finding the non-linear transformation is also complicated.

Use simulation to see whether  $\frac{\ln(2)}{m}$  (a) is unbiased and (b) achieves the CRLB