14: CLT-BASED CONFIDENCE INTERVALS

Stat250 S25

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Goal: develop an **interval estimate** of a population parameter

Today, we'll use the **asymptotic method**:, which uses large-sample theory to approximate the sampling distribution, primarily through the CLT.

(1 - α) Confidence Interval

$$P(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) = 1 - \alpha$$

Central Limit Theorem

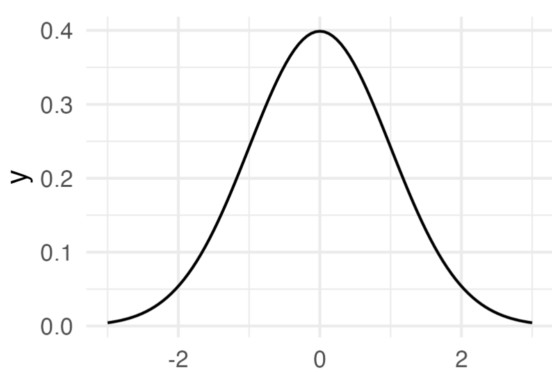
If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, then

1 R detour

```r
rnorm(100)
```

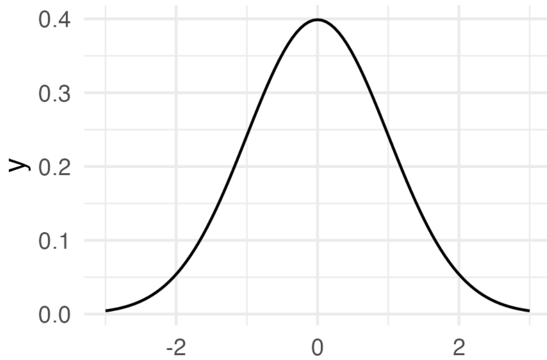
```r
dnorm(1)
```

```
[1] 0.2419707
```



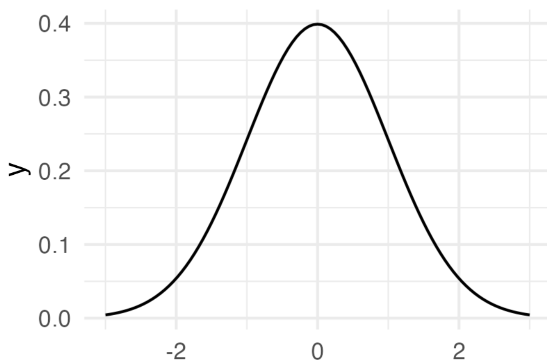

```r
pnorm(1)
```

```
[1] 0.8413447
```




```
qnorm(.8413447)
```

```
[1] 0.9999998
```



Example: Find the value of `q` that is needed for the following $(1 - \alpha)100\%$ normal-based CIs:

1. 90%
2. 95%
3. 97%

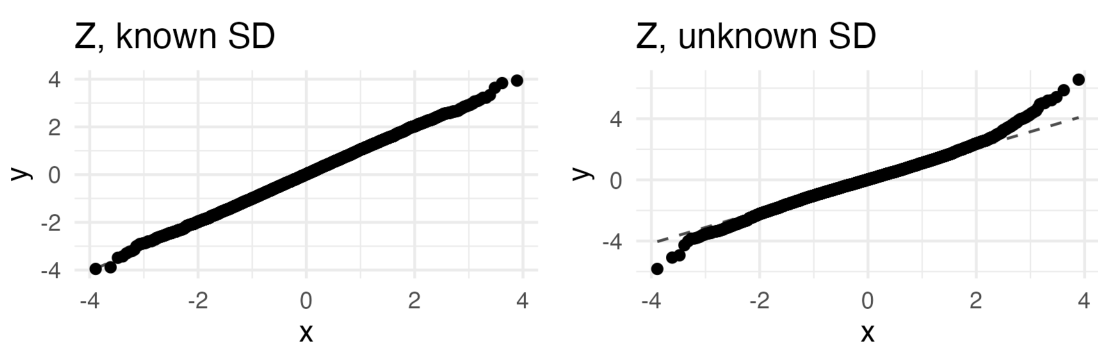
Example: Find a 90% confidence interval for the mean bill length of Gentoo penguins. Assume that σ = 3.08. We also have $\bar{X}$ = 47.5 and n = 123

2 Plug-in principle

Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$. Then $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

PROBLEM: $\bar{x} \pm z_{1-\alpha/2}\left(\dfrac{\sigma}{\sqrt{n}}\right)$

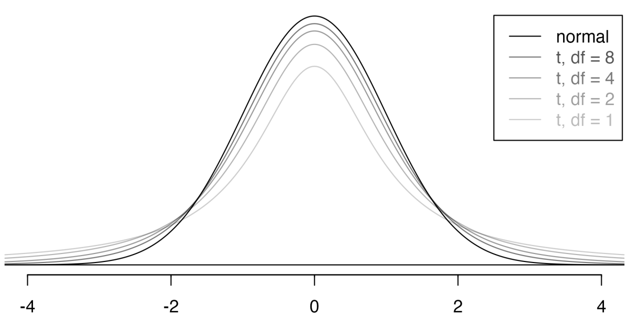
Proposed solution:



Student's t-distribution

$$T = \frac{Z}{\sqrt{V/df}}$$

where $Z \sim N(0,1)$, $V \sim \chi^2_{df}$, and $Z \perp V \implies T \sim t_{df}$



- Symmetric around 0
- For $df = 1$, mean doesn't exist (Cauchy distribution)
- For $df \geq 2$, $E(T) = E(Z)E\left(1/\sqrt{V/n}\right) = 0$
- Heavier tails than normal distribution
- $t_{df} \to N(0,1)$ as $df \to \infty$

Example: Find the value of q that is needed for the following $(1-\alpha)100\%$ t-based CIs:

1. 90%, n = 123
2. 95%, n = 25
3. 99%, n = 34

Example: Find a 90% confidence interval for the mean bill length of Gentoo penguins. ~~Assume that = 3.08~~.

Sample statistics:

- $n = 123$
- $\bar{x} = 47.5$
- $s = 3.08$

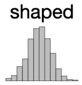
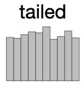
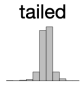
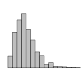
3 Assumptions

- Random sample from a ________________ population distribution
- __________________ observations

> **Robustness**
>
> If the a procedure "perform well" even if some of the assumptions under which they were developed do not hold, then they are called **robust**.

To check whether a procedure is robust, we can use simulation:

1. Simulate data from a variety of different probability distributions
2. Run the procedure (e.g., build a one-sample t-interval)
3. Compare the results of the procedure to what should have happened.

n	Bell-shaped 	Short-tailed 	Long-tailed 	Mild Skew 	Moderate Skew 	Strong Skew 
5	95.3	**94**	96.3	**91.6**	**91.8**	**89.8**
10	95.9	**94**	96.3	**93.3**	**93.2**	**90.8**
25	95.3	95.4	95.9	**93.8**	**93.5**	**90.3**
50	94.8	94.3	96.3	**94.1**	**94**	**93.8**
100	95.3	95.7	94.9	95.1	95.9	**94.6**

Robustness for 1-sample t procedure:

- If the population distribution is roughly __________________ and ____________________, then the procedure works well for sample sizes of at least __________________ (just a rough guide)
- For __________________ population distributions, the t-procedure can be substantially affected, depending on the severity of the _____________ and the sample size.
- t-procedures are not resistant to __________________
- If observations are not ____________________, everything breaks