

09: EFFICIENCY AND CRLB

Stat250 S25

Prof Amanda Luby

1 Recap

- Observe data $X_1, \dots, X_n \sim F_x(x|\theta)$, where θ is unknown
- Goal: *Estimate* θ based on the values of X_i by formulating an *estimator* $\hat{\theta}$
- One technique is to use the *maximum likelihood estimator*
- A second technique is to use the *method of moments estimator*
- There are lots of other ways to come up with estimators.
- We can compare estimators by comparing their *bias*, *variance*, and *mean squared error*
- Today: is there a way to know if we've found an "optimal" estimator?

2 Comparing unbiased estimators

Efficiency

For two unbiased estimators, $\hat{\theta}_1$ is more **efficient** than $\hat{\theta}_2$ if

Example: Why efficiency matters

We now have two unbiased estimators for θ in a $\text{Unif}(0, \theta)$ distribution. $\hat{\theta}_{MoM} = \frac{2}{n} \sum_{i=1}^n X_i$ and $\hat{\theta}_3 = \frac{1}{n} \sum_{i=1}^n X_i^2$.

- (a) If $n = 10$, what is the expectation and variance of each of these estimators?

(b) What n is needed for $\hat{\theta}_{MoM}$ to reach the variance of $\hat{\theta}_3$

3 Can we find a better estimator?

Is there another unbiased estimator with smaller variance?

Cramer-Rao Lower Bound

If X_1, \dots, X_n are an iid sample from a distribution with pdf $f(x|\theta)$, then any unbiased estimator $\hat{\theta}$ of θ satisfies:

$$V(\hat{\theta}) \geq \frac{1}{nI(\theta)}$$

where $I(\theta)$ is the **Fisher Information** of X_i

If the variance of an estimator is equal to the CRLB, then there is *no other unbiased estimator with more precision*

Fisher Information

The **Fisher Information** of an observation X is

$$I(\theta) = E\left[\left(\frac{d}{d\theta} \ln f(x|\theta)\right)^2\right] =$$

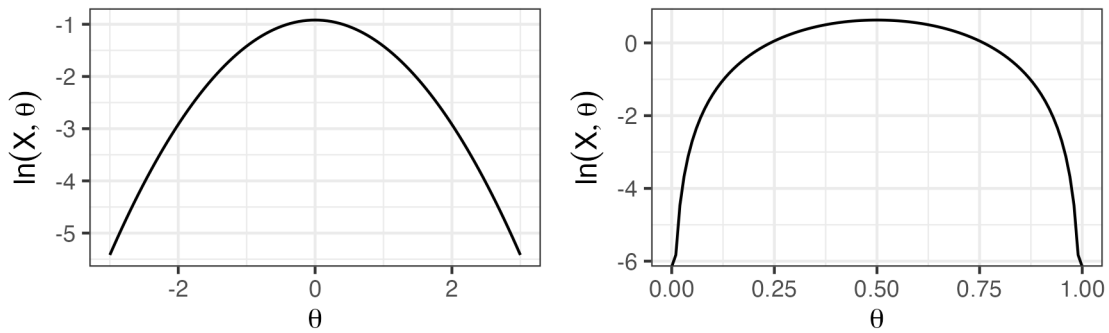
*provided the domain of X does not depend on θ (and a few other regularity conditions)

3.1 Intuition: Fisher information

$$I(\theta) = E_X\left[\left(\frac{d}{d\theta} \ln f(x|\theta)\right)^2\right] = -E_X(l''(\theta))$$

- $l(\theta)$ is the log-likelihood function
- $l'(\theta)$ gives the slope of the log-likelihood at any point, $l''(\theta)$ gives the curvature at any given point

- The Fisher information “averages out” over X and summarizes the overall curvature of the log-likelihood function



3.2 Practice with CRLB

Example: Find the *Fisher information* for Y , where $X \sim \text{Exp}(\lambda)$

Example: Let Y_1, \dots, Y_n be n $\text{Exp}(\lambda)$ random variables. Let $\hat{\lambda} = \frac{n}{\sum Y_i}$. How does $\text{Var}(\hat{\lambda})$ compare with the CRLB?

3.3 CRLB with simulation

Example: The median m of an $\text{Exponential}(\lambda)$ distribution satisfies $P(X \leq m) = 0.5$. Solving $1 - e^{-\lambda m} = 0.5$ gives $m = \ln(2)/\lambda$. This suggests an estimator for λ based on the median: $\hat{\lambda} = \frac{\ln(2)}{m}$. Finding the analytical variance of $\hat{\lambda}$ is complicated. Finding the sampling distribution of m is complicated, and finding the non-linear transformation is also complicated.

Use simulation to see whether $\frac{\ln(2)}{m}$ (a) is unbiased and (b) achieves the CRLB