10: ASYMPTOTIC PROPERTIES

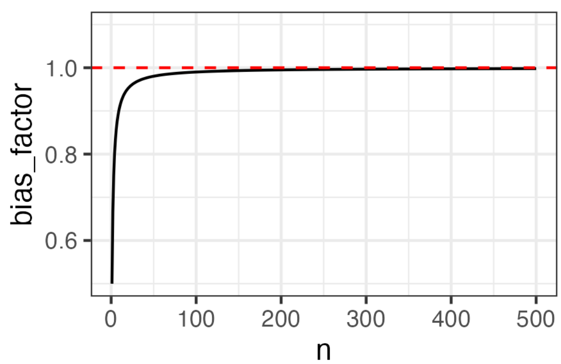
Stat250 S25

Prof Amanda Luby

When we've considered bias and variance of estimators, we've assumed that our data has a fixed sample size. This makes sense in the context of historical statistics: data was time-consuming and expensive to gather, and so experiments were very rigorously designed with a lot of consideration for sample sizes. As data has become easier and cheaper to gather, the *asymptotic* behavior of estimators has also become an important consideration. We may find, for example, that an estimator has a desired behavior *in the limit* that it fails to have for any fixed sample size.

Asymptotic bias

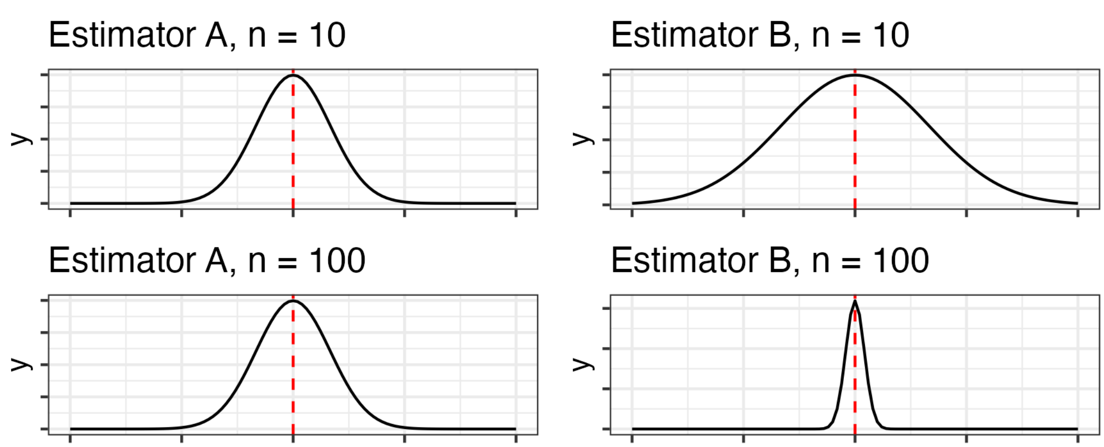
Example: Recall that for $X_i \sim \text{Unif}(0, \theta)$, $\hat{\theta}_{MLE} = X_{\max}$. We also showed that the MLE is *biased*: $E(\hat{\theta}_{MLE}) = \frac{n}{n+1}\theta$. Does this bias matter?



> **Asymptotically unbiased**
>
> Let $Y_1, ..., Y_n \sim f_y(y|\theta)$. If $E(\hat{\theta}_n) \to \theta$ as $n \to \infty$, then $\hat{\theta}_n$ is *asymptotically unbiased*

Consistency

Example: Which estimator do you prefer?



Consistent estimator

An estimator $\hat{\theta}_n$ is *consistent* if it converges in probability to θ:

Example: If an estimator is asymptotically unbiased, does that mean it is also consistent?

Code Example: Cauchy vs Normal

A detour into tail probability inequalities

Markov's Inequality

For any random variable W and any constant a,

Chebyshev's inequality

Let W be any random variable with mean μ and variance σ^2. For any $a > 0$,

Chernoff's inequality

Let W be any random variable and constants a and t,

Example: Let $X_1, \ldots, X_n$ an iid sample of discrete variables from $p_x(x|\theta)$ where $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Let $\hat{\mu}_n = \frac{1}{n} \sum X_i$. Is $\hat{\mu}$ a consistent estimator for μ?

Note:

An alternative method to check for consistency

Let $\{\hat{\theta}_n\}$ be a sequence of estimators for θ. If

$$\lim_{n\to\infty} E[\hat{\theta}_n] \to \theta \text{ and } \lim_{n\to\infty} \mathrm{Var}[\hat{\theta}_n] \to 0$$

Exercise: Let $X_1, ..., X_n \sim \mathrm{Unif}(0, \theta)$. Recall that $\hat{\theta}_{MoM} = 2\bar{X}$ and $E(\hat{\theta}_{MoM}) = \theta$ and $\mathrm{Var}(\hat{\theta}_{MoM}) = \frac{\theta^2}{3n}$. Is $\hat{\theta}_{MoM}$ consistent for θ? Check using both definitions of consistency.

Invariance

Transformation invariance

An estimation procedure is *transformation invariant* if it yields equivalent results for transformations of parameters. That is, if $\zeta = h(\theta)$ then $\hat{\zeta} = h(\hat{\zeta})$

Example: In class, we showed that the MLEs for μ and σ^2 in a $N(\mu, \sigma^2)$ distribution were $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$. If I changed my mind and instead want the MLE for the standard error σ, do I have to re-derive the MLE solution?

Note:

Asymptotic Properties of the MLE

I've mentioned a couple of times in class that the MLE is nice/optimal in some ways. Here's one of the major reasons: under smoothness conditions of f_x, the sampling distribution of the MLE is *approximately normal*

Sampling distribution of the MLE

Let $\hat{\theta}_{MLE} = h(Y)$ be the MLE for θ, where $Y \sim f_y(y|\theta)$. Then

$$\hat{\theta} \approx N(\theta, \frac{1}{nI(\theta)})$$

Example: Use the sampling distribution of the MLE to show that:

1. MLE Estimators are *asymptotically unbiased*
2. Under appropriate smoothness conditions of f_x, the MLE is *consistent*
3. MLE estimators are *asymptotically efficient*: for large n, other estimators do not have smaller variance