16: CONFIDENCE INTERVALS VIA BOOTSTRAP T DISTRIBUTIONS

Stat250 S25

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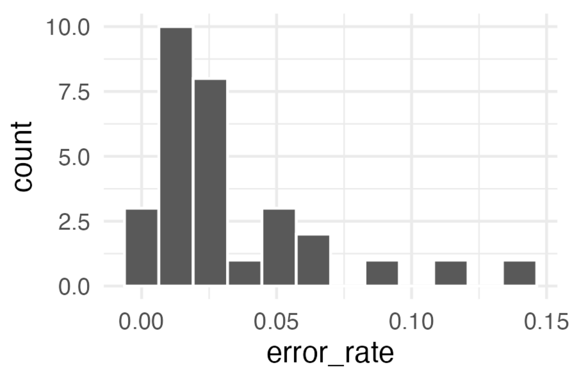
Goal: develop an **interval estimate** of a population parameter

Example: Suppose we observe $X_1, ..., X_n$, which represent the error rates of n different forensic examiners. I am interested in finding a confidence interval for μ, the average error rate in the population. We have a sample of size 30 shown below:



mean	sd	n
0.0325	0.0343	30

Let's review our confidence interval toolkit so far:

1. Construct a bootstrap distribution and find a **percentile-based** confidence interval
2. Either (a) assume a normal population or (b) use the CLT to construct a **formula t-based** confidence interval
3. Assume a different population distribution and construct a **pivot-based** confidence interval

Bootstrap percentile interval

```
error_rates <- examiner_error_rates$error_rate
n = length(error_rates)
N = 10^4
boot_means = numeric(N)

for(i in 1:N){
  x <- sample(error_rates, size = n, replace = TRUE)
  boot_means[i] <- mean(x, na.rm = TRUE)
}

quantile(boot_means, probs = c(.025, .975))
```

```
      2.5%      97.5%
0.02147176 0.04576585
```

CLT-based t confidence interval

```r
qt(.975, df = 29)
```

```
[1] 2.04523
```

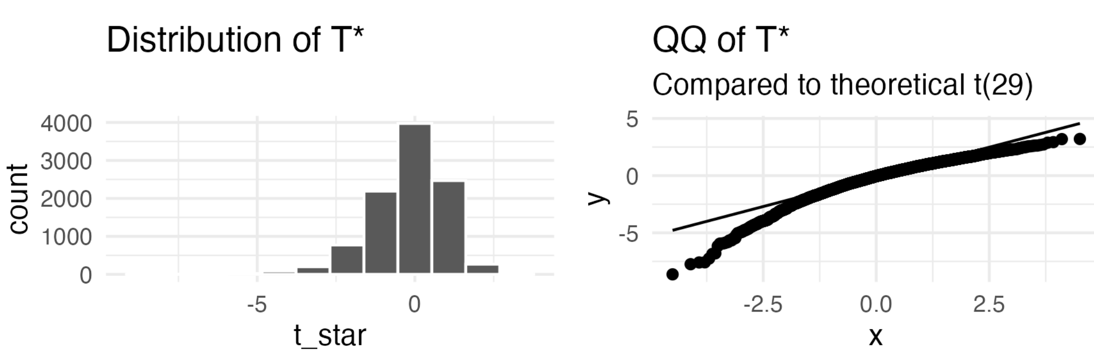
```r
t.test(error_rates, conf.level = .95)$conf
```

```
[1] 0.01969500 0.04531505
attr(,"conf.level")
[1] 0.95
```

When do we trust this confidence interval?

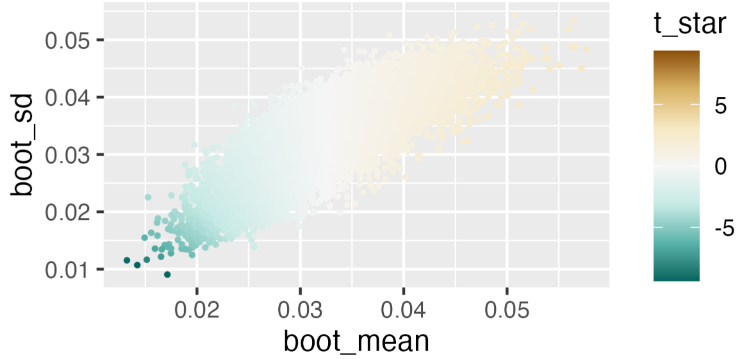
```r
t_star <- numeric(N)

for(i in 1:N){
  x <- sample(error_rates, size = n, replace = TRUE)
  t_star[i] <- (mean(x) - mean(error_rates))/(sd(x)/sqrt(n))
}
```



- Distribution of $T*$ is slightly **left-skewed**
- Tails of $T*$ don't seem to match the theoretical t

Why does this happen?



Bootstrap t confidence interval

The CLT-based confidence interval relies on:

$$1 - \alpha = P(t_{\alpha/2} < \frac{\bar{X} - \mu}{s/\sqrt{n}} < t_{1-\alpha/2})$$

Idea: generate "better" quantiles using the bootstrap distribution:

Then solve for μ:

to obtain the bootstrap t confidence interval:

> **Watch out!**
>
> Lower bound is computed from the **upper** percentile and upper bound is computed from the **lower** percentile

Bootstrap t algorithm

1. Repeat the bootstrap procedure many times:
    - Take bootstrap sample
    - Compute mean ($\bar{X}^*$) and sd (s^*) of each bootstrap sample
    - Compute t-statistic for each bootstrap sample: $T^* = \frac{\bar{X}^* - \bar{X}}{s^*/\sqrt{n}}$
2. Find quantiles $Q^*_{\alpha/2}$ and $Q^*_{1-\alpha/2}$ using distribution of T^*
3. Compute confidence interval using the bootstrap t quantiles:

$$(\bar{x} - Q^*_{1-\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} - Q^*_{\alpha/2}\frac{s}{\sqrt{n}})$$

Example: Let's find a bootstrap t confidence interval for the 30 forensic examiners

```r
n = length(error_rates)
N = 10^4
boot_means <- numeric(N)
boot_tstar <- numeric(N)
for(i in 1:N){
  x <- sample(error_rates, size = n, replace = TRUE)
  boot_means[i] <- mean(x)
  t_star[i] <- (mean(x) - mean(error_rates))/(sd(x)/sqrt(n))
}

quantile(t_star, probs = c(.025, .975))
```

```
     2.5%     97.5%
-2.934144  1.717921
```

```r
mean(error_rates) - quantile(t_star, probs = c(.025, .975))*sd(error_rates)/sqrt(n)
```

```
      2.5%      97.5%
0.05088264 0.02174506
```

Summary of results

	Lower	Upper
Percentile Bootstrap	.021	.0458
CLT t-based	.020	.0453
Bootstrap t	.022	.0509