

14: CLT-BASED CONFIDENCE INTERVALS

Stat250 S25

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Goal: develop an **interval estimate** of a population parameter

Today, we'll use the **asymptotic method**, which uses large-sample theory to approximate the sampling distribution, primarily through the CLT.

(1 - α) Confidence Interval

$$P(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) = 1 - \alpha$$

Central Limit Theorem

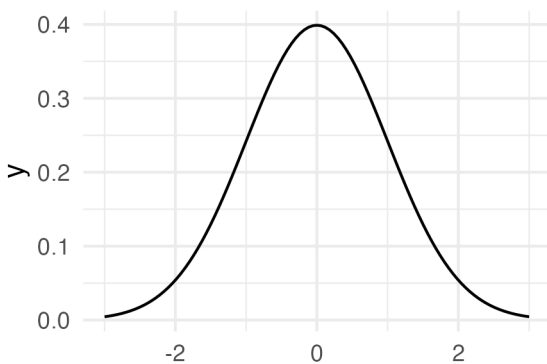
If $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, then

1 R detour

```
rnorm(100)
```

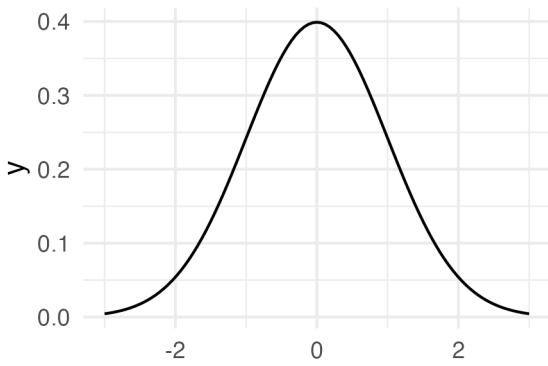
```
dnorm(1)
```

```
[1] 0.2419707
```



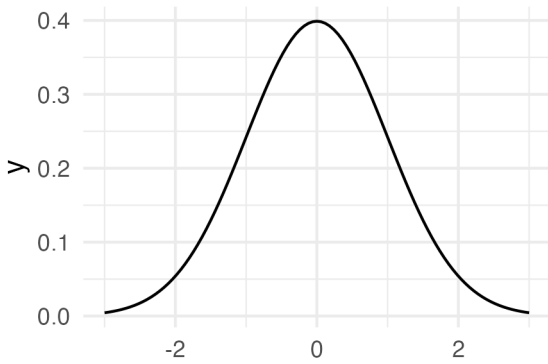
```
pnorm(1)
```

```
[1] 0.8413447
```



```
qnorm(.8413447)
```

```
[1] 0.9999998
```



Example: Find the value of q that is needed for the following $(1 - \alpha)100\%$ normal-based CIs:

1. 90%
2. 95%
3. 97%

Example: Find a 90% confidence interval for the mean bill length of Gentoo penguins. Assume that $\sigma = 3.08$. We also have $\bar{X} = 47.5$ and $n = 123$

2 Plug-in principle

Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$. Then $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

PROBLEM: $\bar{x} \pm z_{1-\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$

Proposed solution:



Student's t-distribution

$$T = \frac{Z}{\sqrt{V/df}}$$

where $Z \sim N(0, 1)$, $V \sim \chi_{df}^2$, and $Z \perp V \implies T \sim t_{df}$



- Symmetric around 0
- For $df = 1$, mean doesn't exist (Cauchy distribution)
- For $df \geq 2$, $E(T) = E(Z)E(1/\sqrt{V/n}) = 0$
- Heavier tails than normal distribution
- $t_{df} \rightarrow N(0, 1)$ as $df \rightarrow \infty$

Example: Find the value of q that is needed for the following $(1 - \alpha)100\%$ t-based CIs:

1. 90%, $n = 123$
2. 95%, $n = 25$
3. 99%, $n = 34$

Example: Find a 90% confidence interval for the mean bill length of Gentoo penguins. Assume that $\sigma = 3.08$.

Sample statistics:

- $n = 123$
- $\bar{x} = 47.5$
- $s = 3.08$

3 Assumptions

- Random sample from a _____ population distribution
- _____ observations

Robustness

If the a procedure “perform well” even if some of the assumptions under which they were developed do not hold, then they are called **robust**.

To check whether a procedure is robust, we can use simulation:

1. Simulate data from a variety of different probability distributions
2. Run the procedure (e.g., build a one-sample t-interval)
3. Compare the results of the procedure to what should have happened.

n	Bell-shaped	Short-tailed	Long-tailed	Mild Skew	Moderate Skew	Strong Skew
5	95.3	94	96.3	91.6	91.8	89.8
10	95.9	94	96.3	93.3	93.2	90.8
25	95.3	95.4	95.9	93.8	93.5	90.3
50	94.8	94.3	96.3	94.1	94	93.8
100	95.3	95.7	94.9	95.1	95.9	94.6

Robustness for 1-sample t procedure:

- If the population distribution is roughly _____ and _____, then the procedure works well for sample sizes of at least _____ (just a rough guide)
- For _____ population distributions, the t-procedure can be substantially affected, depending on the severity of the _____ and the sample size.
- t-procedures are not resistant to _____
- If observations are not _____, everything breaks