08: BIAS AND EFFICIENCY

Stat250 S25

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1 Recap

- Observe data $X_1, ..., X_n \sim F_x(x|\theta)$, where θ is unknown
- Goal: *Estimate* θ based on the values of X_i by formulating an *estimator* $\hat{\theta}$
- One technique is to use the *maximum likelihood estimator*, which finds the value of θ that maximizes the joint probability $\prod_{i=1}^{n} f_x(x_i|\theta)$
- A second technique is to use the *method of moments estimator*, which finds the value of θ that make the theoretical moments equal to the sample moments
- Today: if we have multiple estimators, how do we decide which is better?

2 Example: exponential distribution

Warm up: Suppose we take a random sample of size 50 from an exponential distribution with rate is $\lambda = 1/10$. What is $\mu = E(Y)$? If we want to design an estimator for the mean, $\hat{\mu}$, what are some intuitive estimators?

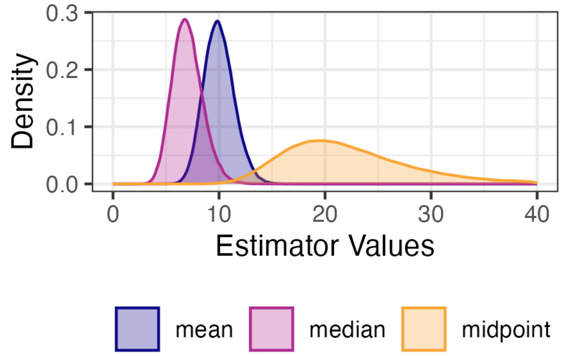
Example: Suppose we take a random sample of size 50 from an exponential distribution with mean 10 (rate is $\lambda = 1/10$). Consider three estimators of μ:

$$\hat{\mu}_1 =$$
$$\hat{\mu}_2 =$$
$$\hat{\mu}_3 =$$

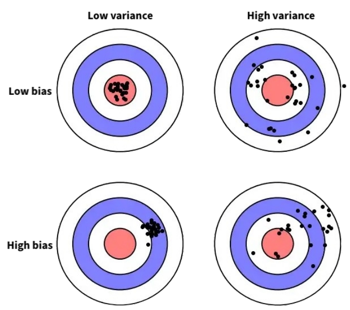
Aside: estimators are also random variables. $X_1, ..., X_n \sim F_x(\theta)$ and each $\hat{\theta} = g(X_1, ..., X_n)$ is a function of the data, so each $\hat{\theta}$ is itself a random variable and has:

3 Comparing estimators: simulation

```
n <- 50
N_sims <- 100000
theta <- 3
est1 <- numeric(N_sims)
est2 <- numeric(N_sims)
est3 <- numeric(N_sims)
for(i in 1:N_sims){
  x <- rexp(n, rate = .1)
  est1[i] <- mean(x)
  est2[i] <- median(x)
  est3[i] <- (max(x) - min(x))/2
}
```



4 Properties of Estimators



Bias of an estimator

If $\hat{\theta}(X)$ is an estimator of θ, then the bias of the estimator is equal to

Note: the expected value is computed from the sampling distribution of $\hat{\theta}(X)$

Variance of an estimator

The variance of an estimator is

MSE

The MSE of an estimator is

Exercise: Your task is to compare the estimators

$$\hat{\theta}_{MLE} = X_{\max} \qquad\qquad \hat{\theta}_{MoM} = 2\bar{X}$$

(a) What is the bias of each estimator? (A helpful fact is that $f_{X_{max}}(x) = n[F(x)]^{n-1} f_X(x)$)
(b) What is the variance of each estimator?
(c) What is the MSE of each estimator?
(d) When does $\hat{\theta}_{MLE}$ "beat" $\hat{\theta}_{MoM}$ in terms of MSE?

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(e) Since $\hat{\theta}_{MLE}$ beats $\hat{\theta}_{MoM}$ in terms of MSE but is biased, can we "unbias" the MLE? Call this third estimator $\hat{\theta}_3$

(f) Does $\hat{\theta}_3$ ever "beat" $\hat{\theta}_{MLE}$ in terms of MSE?

5 Comparing unbiased estimators

Efficiency

For two unbiased estimators, $\hat{\theta}_1$ is more **efficient** than $\hat{\theta}_2$ if

Cramer-Rao Lower Bound

If $X_1, ..., X_n$ are an iid sample from a distribution with pdf $f(x|\theta)$, then any unbiased estimator $\hat{\theta}$ of θ satisfies:

where $I(\theta)$ is the **Fisher Information** of X_i

Fisher Information

The **Fisher Information** of an observation X is

$$I(\theta) = E[(\frac{d}{d\theta} \ln f(x|\theta))^2] =$$

(f) Does $\hat{\theta}_3$ meet the CRLB?