

# 15: CONFIDENCE INTERVALS VIA PIVOTAL QUANTITIES

Stat250 S25

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Goal: develop an **interval estimate** of a population parameter

**Example:** Let  $X_1, \dots, X_n \sim N(\mu_1, \sigma_1^2)$  and let  $Y_1, \dots, Y_n \sim N(\mu_2, \sigma_2^2)$ . Assume the X's are iid, Y's are iid, and all X's are independent of all Y's. Find the form of a  $(1 - \alpha)$  confidence interval for  $\mu_1 - \mu_2$

Let's break down what we've done today and last time:

1. Find a statistic that depends on parameter  $\theta$  and estimator  $\hat{\theta}$
2. Find the sampling distribution of that statistic and write down a probability statement that is true
3. Rearrange terms *within* the probability statement to solve for  $\theta$ , giving us a lower/upper bound in terms of  $\hat{\theta}$  (and other functions of the data  $h(X_i, n)$ )

#### Pivotal Quantity

**Exercise:** Assume  $X_1, \dots, X_n \sim N(\mu, \sigma^2)$  where  $\sigma^2$  is *known*. Which of the following are pivotal quantities?

- (a)  $\bar{X}$
- (b)  $\bar{X} - \mu$
- (c)  $\frac{\bar{X} - \mu}{\sigma}$
- (d)  $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$
- (e)  $\bar{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

**Example:** Let  $X_1, \dots, X_n \sim \text{Gamma}(r, \lambda)$  with  $\lambda$  *known*. Recall that

$$\hat{\lambda}_{MLE} = \frac{r}{\bar{X}}$$

- (a) Use the MLE as a starting point to find a pivotal statistic for  $\lambda$

- (b) Use the pivotal statistic to construct a  $100(1 - \alpha)\%$  confidence interval for  $\lambda$ . Include R code for finding any necessary quantiles.

- (c) Obtain an alternative pivotal quantity by using the CLT to approximate the distribution of  $\bar{X}$  (Recall that  $E(X) = r/\lambda$  and  $\text{Var}(X) = r/\lambda^2$  for a  $\text{Gamma}(r, \lambda)$  distribution)

(d) Find the form of an alternative  $100(1 - \alpha)\%$  confidence interval for  $\lambda$  using your pivotal quantity in (c)

(e) Suppose  $n = 300$  and  $\bar{X} = 1.6$ . Construct two confidence intervals for  $\lambda$  using your results from (b) and (d) and compare them.