09: EFFICIENCY AND CRLB

Stat250 S25

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1 Recap

- Observe data $X_1, ..., X_n \sim F_x(x|\theta)$, where θ is unknown
- Goal: *Estimate* θ based on the values of X_i by formulating an *estimator* $\hat{\theta}$
- One technique is to use the *maximum likelihood estimator*
- A second technique is to use the *method of moments estimator*
- There are lots of other ways to come up with estimators.
- We can compare estimators by comparing their *bias*, *variance*, and *mean squared error*
- Today: is there a way to know if we've found an "optimal" estimator?

2 Comparing unbiased estimators

Efficiency

For two unbiased estimators, $\hat{\theta}_1$ is more **efficient** than $\hat{\theta}_2$ if

Example: Why efficiency matters

We now have two unbiased estimators for θ in a Unif$(0, \theta)$ distribution. $\hat{\theta}_{MoM}$ = ____________ and $\hat{\theta}_3$ = ____________________.

(a) If $n = 10$, what is the expectation and variance of each of these estimators?

(b) What n is needed for $\theta_{\hat{MoM}}$ to reach the variance of $\hat{\theta}_3$

3 Can we find a better estimator?

Is there another unbiased estimator with smaller variance?

Cramer-Rao Lower Bound

If $X_1, ..., X_n$ are an iid sample from a distribution with pdf $f(x|\theta)$, then any unbiased estimator $\hat{\theta}$ of θ satisfies:

$$V(\hat{\theta}) \geq \frac{1}{nI(\theta)}$$

where $I(\theta)$ is the **Fisher Information** of X_i

If the variance of an estimator is equal to the CRLB, then there is *no other unbiased estimator with more precision*

Fisher Information

The **Fisher Information** of an observation X is

$$I(\theta) = E[(\frac{d}{d\theta} \ln f(x|\theta))^2] =$$

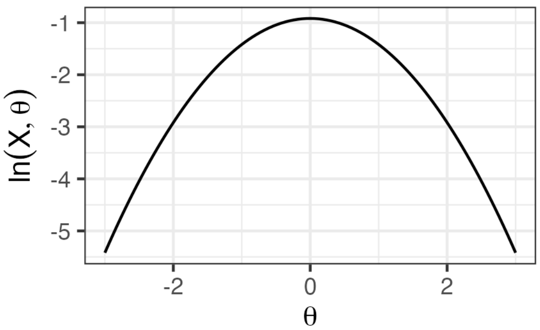
*provided the domain of X does not depend on θ (and a few other regularity conditions)

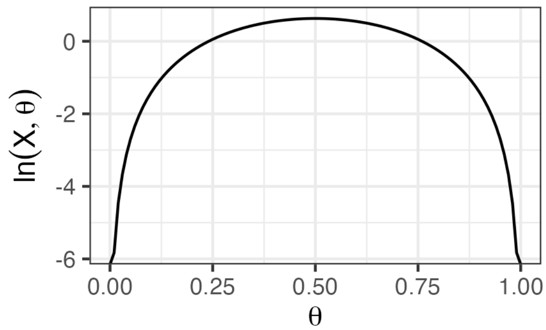
3.1 Intuition: Fisher information

$$I(\theta) = E_X[(\frac{d}{d\theta} \ln f(x|\theta))^2] = -E_X(l''(\theta))$$

- $l(\theta)$ is the log-likelihood function
- $l'(\theta)$ gives the slope of the log-likelihood at any point, $l''(\theta)$ gives the curvature at any given point

- The Fisher information "averages out" over X and summarizes the overall curvature of the log-likelihood function





3.2 Practice with CRLB

Example: Find the *Fisher information* for Y, where $X \sim Exp(\lambda)$

Example: Let $Y_1, ..., Y_n$ be n Exp(λ) random variables Let $\hat{\lambda} = \frac{n}{\sum Y_i}$. How does Var($\hat{\lambda}$) compare with the CRLB?

3.3 CRLB with simulation

Example: The median m of an Exponential(λ) distribution satisfies P(X ≤ m) = 0.5. Solving $1 - e^{-\lambda m} = 0.5$ gives $m = ln(2)/\lambda$. This suggests an estimator for λ based on the median: $\hat{\lambda} = \frac{ln(2)}{m}$. Finding the analytical variance of $\hat{\lambda}$ is complicated. Finding the sampling distribution of m is complicated, and finding the non-linear transformation is also complicated.

Use simulation to see whether $\frac{ln(2)}{m}$ (a) is unbiased and (b) achieves the CRLB