

# 14: CLT-BASED CONFIDENCE INTERVALS

Stat250 S25

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Goal: develop an **interval estimate** of a population parameter

Today, we'll use the **asymptotic method**, which uses large-sample theory to approximate the sampling distribution, primarily through the CLT.

## (1 - $\alpha$ ) Confidence Interval

$$P(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) = 1 - \alpha$$

## Central Limit Theorem

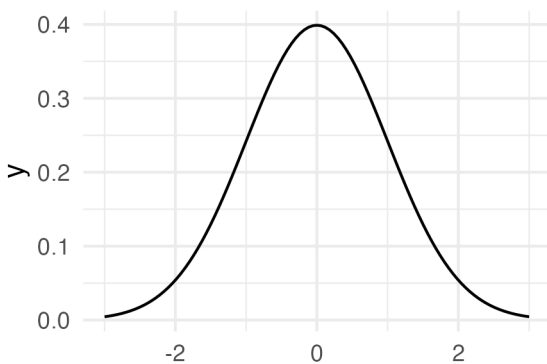
If  $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ , then

## 1 R detour

```
rnorm(100)
```

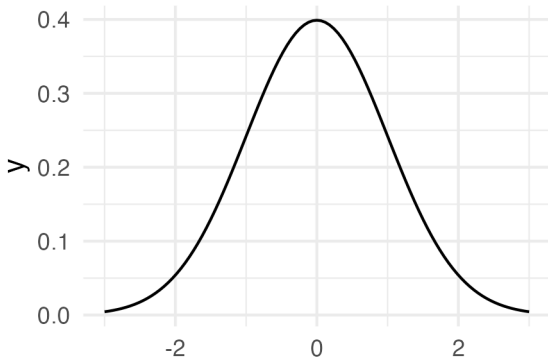
```
dnorm(1)
```

```
[1] 0.2419707
```



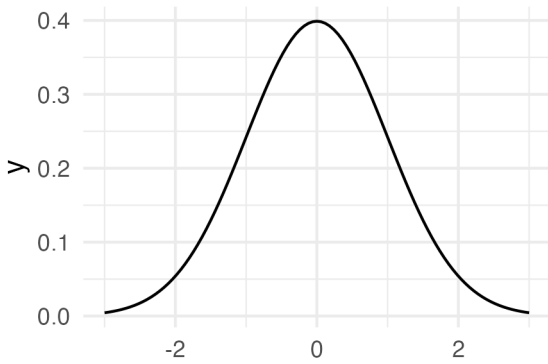
```
pnorm(1)
```

```
[1] 0.8413447
```



```
qnorm(.8413447)
```

```
[1] 0.9999998
```



**Example:** Find the value of  $q$  that is needed for the following  $(1 - \alpha)100\%$  normal-based CIs:

1. 90%
2. 95%
3. 97%

**Example:** Find a 90% confidence interval for the mean bill length of Gentoo penguins. Assume that  $\sigma = 3.08$ . We also have  $\bar{X} = 47.5$  and  $n = 123$

## 2 Plug-in principle

Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$ . Then  $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

PROBLEM:  $\bar{x} \pm z_{1-\alpha/2} \left( \frac{\sigma}{\sqrt{n}} \right)$

Proposed solution:



### Student's t-distribution

$$T = \frac{Z}{\sqrt{V/df}}$$

where  $Z \sim N(0, 1)$ ,  $V \sim \chi_{df}^2$ , and  $Z \perp V \implies T \sim t_{df}$



- Symmetric around 0
- For  $df = 1$ , mean doesn't exist (Cauchy distribution)
- For  $df \geq 2$ ,  $E(T) = E(Z)E(1/\sqrt{V/n}) = 0$
- Heavier tails than normal distribution
- $t_{df} \rightarrow N(0, 1)$  as  $df \rightarrow \infty$

Example: Find the value of  $q$  that is needed for the following  $(1 - \alpha)100\%$  t-based CIs:

1. 90%,  $n = 123$
2. 95%,  $n = 25$
3. 99%,  $n = 34$

**Example:** Find a 90% confidence interval for the mean bill length of Gentoo penguins. Assume that  $s = 3.08$ .

Sample statistics:

- $n = 123$
- $\bar{x} = 47.5$
- $s = 3.08$

### 3 Assumptions

- Random sample from a \_\_\_\_\_ population distribution
- \_\_\_\_\_ observations

#### Robustness

If the a procedure “perform well” even if some of the assumptions under which they were developed do not hold, then they are called **robust**.

To check whether a procedure is robust, we can use simulation:

1. Simulate data from a variety of different probability distributions
2. Run the procedure (e.g., build a one-sample t-interval)
3. Compare the results of the procedure to what should have happened.

n	Bell-shaped	Short-tailed	Long-tailed	Mild Skew	Moderate Skew	Strong Skew
5	95.3	<b>94</b>	96.3	<b>91.6</b>	<b>91.8</b>	<b>89.8</b>
10	95.9	<b>94</b>	96.3	<b>93.3</b>	<b>93.2</b>	<b>90.8</b>
25	95.3	95.4	95.9	<b>93.8</b>	<b>93.5</b>	<b>90.3</b>
50	94.8	94.3	96.3	<b>94.1</b>	<b>94</b>	<b>93.8</b>
100	95.3	95.7	94.9	95.1	95.9	<b>94.6</b>

**Robustness for 1-sample t procedure:**

- If the population distribution is roughly \_\_\_\_\_ and \_\_\_\_\_, then the procedure works well for sample sizes of at least \_\_\_\_\_ (just a rough guide)
- For \_\_\_\_\_ population distributions, the t-procedure can be substantially affected, depending on the severity of the \_\_\_\_\_ and the sample size.
- t-procedures are not resistant to \_\_\_\_\_
- If observations are not \_\_\_\_\_, everything breaks