15: CONFIDENCE INTERVALS VIA PIVOTAL QUANTITIES

Stat250 S25

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Goal: develop an **interval estimate** of a population parameter

Example: Let $X_1, ..., X_n \sim N(\mu_1, \sigma_1^2)$ and let $Y_1, ..., Y_n \sim N(\mu_2, \sigma_2^2)$. Assume the X's are iid, Y's are iid, and all X's are independent of all Y's. Find the form of a $(1 - \alpha)$ confidence interval for $\mu_1 - \mu_2$

Let's break down what we've done today and last time:

1. Find a statistic that depends on parameter θ and estimator $\hat{\theta}$
2. Find the sampling distribution of that statistic and write down a probability statement that is true
3. Rearrange terms *within* the probability statement to solve for θ, giving us a lower/upper bound in terms of $\hat{\theta}$ (and other functions of the data $h(X_i, n)$)

Pivotal Quantity

Exercise: Assume $X_i, ..., X_n \sim N(\mu, \sigma^2)$ where σ^2 is *known*. Which of the following are pivotal quantities?

(a) $\bar{X}$

(b) $\bar{X} - \mu$

(c) $\frac{\bar{X}-\mu}{\sigma}$

(d) $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

(e) $\bar{X} \pm 1.96\frac{\sigma}{\sqrt{n}}$

Example: Let $X_1, ..., X_n \sim \text{Gamma}(r, \lambda)$ with λ *known*. Recall that

$$\hat{\lambda}_{MLE} = \frac{r}{\bar{X}}$$

(a) Use the MLE as a starting point to find a pivotal statistic for λ

(b) Use the pivotal statistic to construct a $100(1-\alpha)\%$ confidence interval for λ. Include R code for finding any necessary quantiles.

(c) Obtain an alternative pivotal quantity by using the CLT to approximate the distribution of $\bar{X}$ (Recall that $E(X) = r/\lambda$ and $\mathrm{Var}(X) = r/\lambda^2$ for a Gamma(r, λ) distribution)

(d) Find the form of an alternative $100(1-\alpha)\%$ confidence interval for λ using your pivotal quantity in (c)

(e) Suppose $n = 300$ and $\bar{X} = 1.6$. Construct two confidence intervals for λ using your results from (b) and (d) and compare them.