# Linear Regression

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# Weight-Height example

Dataset: heights and weights of different people.

Task: build a model that predict the height given the weight.

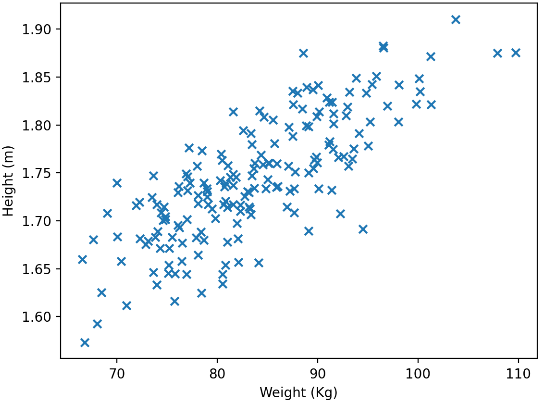


Figure: Data plot

# A solution - Linear regression model

Some remarks on data.

- Regression problem (continuous output).
- Data with different order of magnitude.

A possible solution to this problem is represented by **linear regression** (LR).

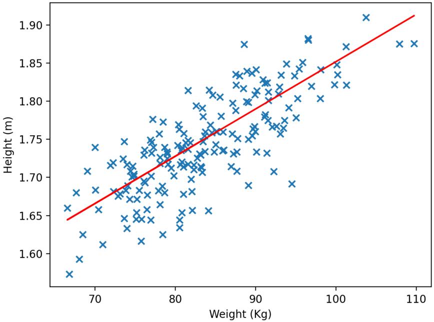


Figure: Trained model (in red)

# General ingredients

Notation:

- $x$: a data sample.
- $y$: the data target corresponding to $x$
- $N$: number of data.

Model/hypothesis: $h_{\boldsymbol{w}}(x) = w_1 x + w_0$, where $\boldsymbol{w} = [w_0, w_1]$ is the vector of parameter that has to be learned.
In our example, $x$ is the weight of a single sample and $h_{\boldsymbol{w}}(x)$ corresponds to the prediction of its height.

Usually the vector $\boldsymbol{w}$ is called **weights vector** and the set $\mathcal{H} := \{h_{\boldsymbol{w}} | \boldsymbol{w} \in \mathbb{R}^2\}$ is called **hypothesis space**.

How to learn $w$ from data?

# Mean squared error (MSE)

Given a training sample $x_i$ and a model $h_{\boldsymbol{w}}$ we can predict the target computing $h_{\boldsymbol{w}}(x_i)$. To evaluate how good is the prediction we compute the error $(h_{\boldsymbol{w}}(x_i) - y_i)^2$.

$(h_{\boldsymbol{w}}(x_i) - y_i)^2 \geq 0$ and $(h_{\boldsymbol{w}}(x_i) - y_i)^2 = 0$ if and only if $h_{\boldsymbol{w}}(x_i) = y_i$. The **mean squared error** (MSE) is:

$$E(\boldsymbol{w}) := \frac{1}{N} \sum_{i=1}^{N} (h_{\boldsymbol{w}}(x_i) - y_i)^2.$$

To find the best model we minimize the training error, hence in this case the MSE.

$$\boldsymbol{w} \in \underset{\tilde{\boldsymbol{w}} \in \mathbb{R}^2}{\arg\min}\, E(\tilde{\boldsymbol{w}}).$$

# $n$-dimensional LR

*Dataset samples.*

Previous case: $x \in \mathbb{R}, y \in \mathbb{R}$.

Now: $\boldsymbol{x} \in \mathbb{R}^n, y \in \mathbb{R}$.

Notation: $x_j^i$ is the $j$-th coordinate of the $i$-th sample.

*Hypothesis.*

Previous case:

$$h_w(x) = w_1 x + w_0,$$

where $w = [w_0, w_1]$.

Now:

$$\begin{aligned} h_{\boldsymbol{w}}(\boldsymbol{x}) &= w_n x_n + w_{n-1} x_{n-1} + \cdots + w_1 x_1 + w_0 \\ &= \sum_{i=0}^{n} w_i \tilde{x}_i = \boldsymbol{w}^T \tilde{\boldsymbol{x}}, \end{aligned}$$

where $\boldsymbol{w} = [w_0, \ldots, w_n]$ and $\tilde{\boldsymbol{x}} = [1, x_1, \ldots, x_n]$.

# n-dimensional LR

*MSE.*

Previous case:

$$E(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N}(h_{\boldsymbol{w}}(x_i) - y_i)^2.$$

Now:

$$\begin{aligned} E(\boldsymbol{w}) &= \frac{1}{N}\sum_{i=1}^{N}(h_w(\boldsymbol{x}^i) - y^i)^2 \\ &= \frac{1}{N}(\mathrm{X}\boldsymbol{w} - \boldsymbol{y})^T(\mathrm{X}\boldsymbol{w} - \boldsymbol{y}) \\ &= \frac{1}{N}||\mathrm{X}\boldsymbol{w} - \boldsymbol{y}||^2 \end{aligned}$$

where

$$\mathrm{X} := \begin{bmatrix} \tilde{\boldsymbol{x}}^1 \\ \vdots \\ \tilde{\boldsymbol{x}}^N \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}.$$

# Spot the minimum - Gradient descent

How to find $\boldsymbol{w} \in \arg\min_{\tilde{\boldsymbol{w}} \in \mathbb{R}^2} E(\tilde{\boldsymbol{w}})$?
Main idea:

- Start with a random $\boldsymbol{w}^0$.
- For $j \geq 0$, update $\boldsymbol{w}^{j+1} := \boldsymbol{w}^j + \boldsymbol{d}^j$, where $\boldsymbol{d}^j$ is such that

$$E(\boldsymbol{w}^{j+1}) \leq E(\boldsymbol{w}^j)$$

Gradient descent: $\boldsymbol{d}^j = -\alpha \nabla E(\boldsymbol{w}^j)$. $\alpha$ is called **learning rate**.

# Gradient descent - 3D visualization

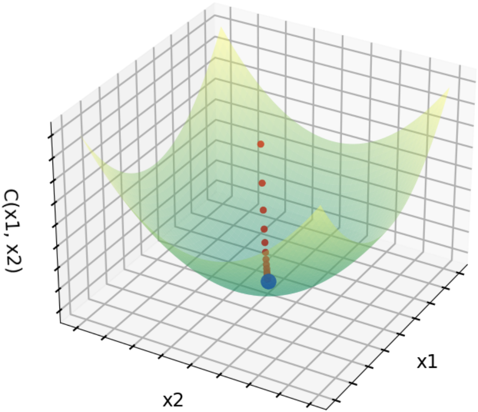


Figure: In blue the global minimum, in red the iteration points.

# Gradient descent - 2D visualization

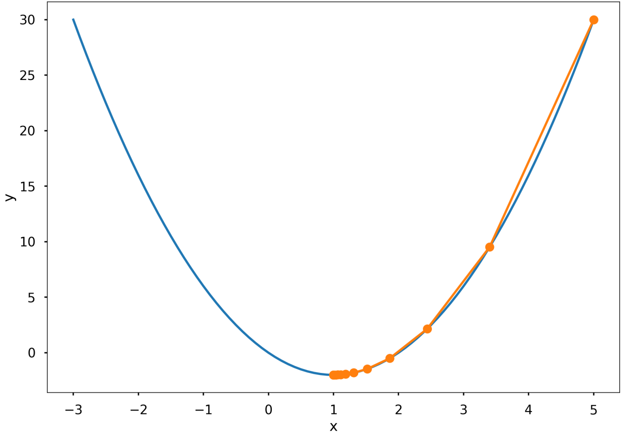


Figure: Learning rate = 0.1

# Gradient descent - 2D visualization

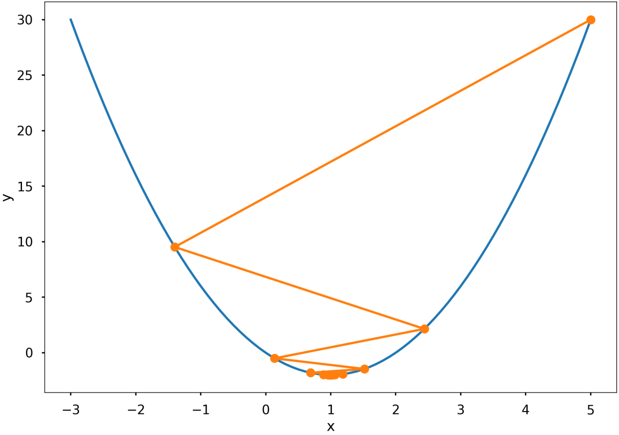


Figure: Learning rate = 0.4

# Gradient descent - 2D visualization

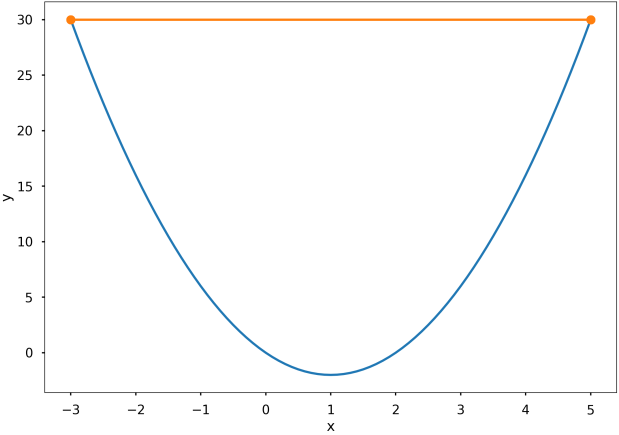


Figure: Learning rate $= 0.5$

# Batch,SGD and Mini-Batch

Notation: $E(\boldsymbol{w}) = 1/N \sum_{i=1}^{N} E_i(\boldsymbol{w})$

**Batch**

- Start with a random $\boldsymbol{w}^0$.
- For $j \geq 0$, update $\boldsymbol{w}^{j+1} := \boldsymbol{w}^j - \alpha \nabla E(\boldsymbol{w}^j)$.

**SGD** (or online)

- Start with a random $\boldsymbol{w}^0$.
- For $j \geq 0$ and for each pattern $1 \leq i \leq N$ update $\boldsymbol{w}^{j+1} := \boldsymbol{w}^j - \alpha \nabla E_i(\boldsymbol{w}^j)$.

**Mini-Batch**. Fix an integer $1 \leq \mathrm{mb} \leq N$(mini-batch size).

- Start with a random $\boldsymbol{w}^0$.
- For $j \geq 0$ and for each pattern $0 \leq i < \frac{N}{\mathrm{mb}}$ update

$$\boldsymbol{w}^{j+1} := \boldsymbol{w}^j - \alpha \nabla \sum_{k=i\cdot\mathrm{mb}+1}^{(i+1)\cdot\mathrm{mb}} E_k(\boldsymbol{w}^j)$$

# Tips and Tricks - How to choose?

- Batch: usually more stable and provide a more accurate estimation of the gradient, but very slow.
- SGD: very fast, stochastic approximation of the gradient implies possible instability (Zig-zag effect)
- Mini-Batch: a trade-off (parallelism available).

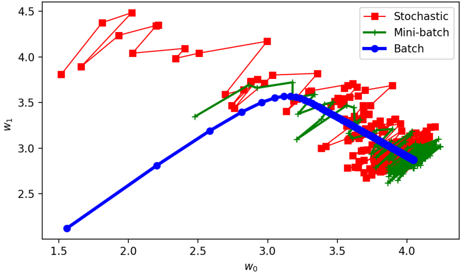


Figure: Batch vs SGD vs Mini-batch

# Gradient descent and normal equation for LR

We have $E(\boldsymbol{w}) = \frac{1}{N}||\mathrm{X}\boldsymbol{w} - \boldsymbol{y}||^2$, hence

$$\nabla E(\boldsymbol{w}) = \frac{1}{N}\nabla(||\mathrm{X}\boldsymbol{w} - \boldsymbol{y}||^2) = \frac{2}{N}\mathrm{X}^T(\mathrm{X}\boldsymbol{w} - \boldsymbol{y})$$

Normal equation ( $\iff$ holds if $\mathrm{X}^T\mathrm{X}$ is invertible):

$$\begin{aligned}
\nabla E(\boldsymbol{w}) = 0 &\iff \frac{2}{N}\mathrm{X}^T(\mathrm{X}\boldsymbol{w} - \boldsymbol{y}) = 0 \\
&\iff \mathrm{X}^T\mathrm{X}\boldsymbol{w} = \mathrm{X}^T\boldsymbol{y} \\
&\iff \boldsymbol{w} = (\mathrm{X}^T\mathrm{X})^{-1}\mathrm{X}^T\boldsymbol{y}
\end{aligned}$$

Gradient descent main iteration for LR:

$$\boldsymbol{w}^{j+1} := \boldsymbol{w}^j - \frac{2\alpha}{N}\mathrm{X}^T(\mathrm{X}\boldsymbol{w}^j - \boldsymbol{y})$$

# Normal equation vs gradient descent

Normal equation:

- No hyperparameter (explicit solution).
- No need to iterate.
- $\mathcal{O}(n^3)$, hence slow when $n$ is large.

Gradient descent:

- Need to choose the learning rate $\alpha$.
- Needs many iterations.
- $\mathcal{O}(kn^2)$, hence fast when $n$ is large.

# Tips and Tricks - Standardization

General (not only for LR): features must be on a similar scale!

- Speed up the convergence of gradient descent.
- Try to have (on average) $-1 \leq x^i \leq 1$.

Common techniques:

- **Feature scaling**. Compute the max $\boldsymbol{M}$ and the min $\boldsymbol{m}$ data value. Then normalize each feature as follows

$$\boldsymbol{x}^i_{\text{norm}} = \frac{\boldsymbol{x}^i - \boldsymbol{m}}{\boldsymbol{M} - \boldsymbol{m}}$$

- **Mean normalization**. Compute mean $\mu$ and standard deviation $\sigma$ of the data. Then normalize each feature as follows

$$\boldsymbol{x}^i_{\text{norm}} = \frac{\boldsymbol{x}^i - \boldsymbol{\mu}}{\boldsymbol{\sigma}}$$

# Tips and Tricks - Invertibility of $X^T X$

What happens when $X^T X$ is not invertible?
Invertibility of $X^T X =$ column of $X$ linearly independent (preprocessing information)

If a column is linearly dependent to the other then a feature is correlated with others (redundant feature).

Solution: discard that feature. The information carried by that feature is contained in some of the others.

# Polynomial regression (PR)

LR corresponds to linear hypothesis, i.e. of the form

$$h_{\boldsymbol{w}}(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

PR corresponds to polynomial hypothesis, i.e. of the form

$$h_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{j=0}^{n} w_j x_j^j.$$

More in general: linear basis expansion (LBE)

$$h_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{j=0}^{n} w_j \phi_j(\boldsymbol{x}),$$

where $\phi_j : \mathbb{R}^n \to \mathbb{R}$.