Continuous Optimization **Spring 2025**

Exercise 1: Fundamentals

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Problem 1 (Smooth functions):

Assume that the function $f(\mathbf{x})$ is L-smooth (i.e. it has L-Lipschitz continuous gradients). Show that if $g(\mathbf{x}) = \frac{L}{2}\mathbf{x}^\top\mathbf{x} - f(\mathbf{x})$ is convex, then

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y}-\mathbf{x}) + \frac{1}{2L}\|\nabla f(\mathbf{y}) - \nabla f(\mathbf{x})\|^2, \quad \forall \mathbf{x}, \mathbf{y}.$$

Problem 2 (Functions with Lipschitz-continuous Hessian):

If f has an L-Lipschitz-continuous Hessian, show that

1.
$$\|\nabla f(\mathbf{y}) - \nabla f(\mathbf{x}) - \nabla^2 f(\mathbf{x})(\mathbf{y}-\mathbf{x})\| \leq \frac{L}{2}\|\mathbf{y}-\mathbf{x}\|^2 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{1}$$

2.
$$|f(\mathbf{y}) - f(\mathbf{x}) - (\mathbf{y}-\mathbf{x})^\top \nabla f(\mathbf{x}) - (\mathbf{y}-\mathbf{x})^\top \nabla^2 f(\mathbf{x})(\mathbf{y}-\mathbf{x})| \leq \frac{L}{6}\|\mathbf{y}-\mathbf{x}\|^3 \ \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \tag{2}$$

Problem 3 (Convex sets):

1. Show that every ball $B(\mathbf{a}, r) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}-\mathbf{a}\| \leq r$ for $\mathbf{a} \in \mathbb{R}^d$ and $r \geq 0$ is convex.
2. Let $\mathbf{A} \in \mathbb{R}^{m \times d}$ and $C = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq 0\}$. Prove that C is a convex cone. Recall that a cone C is a convex cone if $\alpha\mathbf{x} + \beta\mathbf{y} \in C$ for any $\mathbf{x}, \mathbf{y} \in C$ and $\alpha, \beta > 0$.
3. Is the union of two convex sets convex?

Problem 4 (Convex functions):

1. Show that the exponential function $\exp(x)$ is convex.
2. Show that a sum of convex functions is convex.
3. Show that if f is μ-strongly-convex and L-smooth, then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we have:

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x}-\mathbf{y} \rangle \geq \frac{\mu L}{\mu + L}\|\mathbf{x}-\mathbf{y}\|^2 + \frac{1}{\mu + L}\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2 \tag{3}$$