Continuous Optimization **Spring 2025**

Exercise 8: Accelerated Gradient Descent

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Problem 1 (Convergence of GD on quadratic functions):
Consider the quadratic function $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x}$ with bounded minimum and maximum eigenvalues: $\lambda_{\min}(\mathbf{A}) \geq \eta$ and $\|\mathbf{A}\| \leq L$.

1. Show that the gradient descent update can be written as

$$\mathbf{x}_{k+1} = \mathbf{V}(\mathbf{I} - \eta\mathbf{\Lambda})^k \mathbf{V}^{-1}\mathbf{x}_1,$$

   where $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}$ (eigen-decomposition).

2. What can you conclude about the convergence rate as a function of the eigenvalues?

3. Choose $\eta = \frac{1}{L}$. What is the convergence rate of gradient descent?

Problem 2 (Convergence of AGD on quadratic functions):
Consider the quadratic function $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x}$ with bounded minimum and maximum eigenvalues: $\lambda_{\min}(\mathbf{A}) \geq \eta$ and $\|\mathbf{A}\| \leq L$.

1. Write AGD as a recursion of the form $\mathbf{w}_k = \mathbf{G}\mathbf{w}_{k-1}$ and give the explicit form of the $\mathbf{G}$ matrix.

2. Following a similar analysis to Heavy ball (see lecture notes), write $\mathbf{G}$ as a block matrix and derive the eigenvalues of the $\mathbf{G}_i$ matrices that compose the blocks of $\mathbf{G}$.

3. Bonus (more difficult): Derive a rate of convergence for AGD based on your previous calculations. Compare this rate to the rate of GD obtained in Problem 1.

Problem 3 (Lyapunov analysis of AGD):
We optimize a function $f : \mathbb{R}^d \to \mathbb{R}$ that is convex and ℓ-smooth using Accelerated Gradient Descent (AGD) whose update is

$$\begin{aligned}\mathbf{y}_k &= \mathbf{x}_k + \gamma(\mathbf{x}_k - \mathbf{x}_{k-1}) \\ \mathbf{x}_{k+1} &= \mathbf{y}_k - \eta\nabla f(\mathbf{y}_k).\end{aligned}$$

Define the following Lyapunov function $L_k := f(\mathbf{x}_k) + \frac{1}{2\eta}\|\mathbf{x}_k - \mathbf{x}_{k-1}\|^2$. Prove that for $\eta \leq \frac{1}{\ell}$ and $\gamma \in [0, 1]$,

$$L_{k+1} - L_k \leq -\frac{1-\gamma^2}{2\eta}\|\mathbf{x}_k - \mathbf{x}_{k-1}\|^2.$$

Problem 4 (Programming exercise):
Fill in the `TODOs` in the associated Jupyter Notebook.