Continuous Optimization **Spring 2025**

Homework 2: Due 28/04/2025 before 23.55

Lecturer: Aurelien Lucchi

Problem 1 (Subgradient Descent on a Maximum Function, 10 points):

a. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \;=\; \max\Big\{\, |x-1|,\; \sqrt{|x|}\,\Big\}.$$

Derive the subdifferential $\partial f(x)$ in the following regions:

i) For $x < 0$.

ii) For $0 < x < 1$, distinguishing the subcases where $1 - x$ and $\sqrt{x}$ dominate.

iii) For $x > 1$, distinguishing the subcases where $\sqrt{x}$ and $x - 1$ dominate.

iv) At the switching points $x = x_1 := \frac{3-\sqrt{5}}{2}$ and $x = x_2 := \frac{3+\sqrt{5}}{2}$ (i.e. where $|1 - x| = \sqrt{x}$) and at $x = 0$ and $x = 1$.

b. Assume we perform subgradient descent with update

$$x^{k+1} \;=\; x^k \;-\; \alpha\, g^k, \quad \text{with } g^k \in \partial f(x^k),$$

and choose $x^0 = 0.5$ with constant step size $\alpha = 0.1$. Select an appropriate subgradient g^0 at x^0 and compute x^1.

Problem 2 (Proximal Gradient Descent for l_1-Regularized Quadratic in $\mathbb{R}^d$, 10 points):

Consider the optimization problem

$$\min_{x\in\mathbb{R}^d} F(x) = \frac{1}{2}\|x\|^2 + \lambda\|x\|_1,$$

where $\lambda > 0$. We can decompose F as $F(x) = f(x) + g(x)$ with

$$f(x) = \frac{1}{2}\|x\|^2 \quad \text{and} \quad g(x) = \lambda\|x\|_1.$$

i) Derive the proximal operator for g; that is, show that for any $z \in \mathbb{R}^d$

$$\mathrm{prox}_{\alpha g}(z) = \mathrm{sign}(z) \odot \max\{|z| - \alpha\lambda,\, 0\},$$

where the operations are applied elementwise. (6 pts)

ii) The proximal gradient update is given by

$$x^{k+1} = \mathrm{prox}_{\alpha g}\Big(x^k - \alpha\nabla f(x^k)\Big).$$

Given $x^0 = (3, -1, 2)^\top \in \mathbb{R}^3$, $\alpha = 0.5$, and $\lambda = 1$, compute x^1. (4 pts)

Problem 3 (Constrained Optimization with a Nonlinear Equality Constraint, 10 points):

a. Consider the problem

$$\min_{x,y\in\mathbb{R}} \quad f(x, y) = (x-2)^2 + (y+1)^2,$$

$$\text{subject to} \quad h(x, y) = x^2 + y^2 - 5 = 0.$$

i) Formulate the Lagrangian function $\mathcal{L}(x, y, \lambda)$ for this problem. (2 Pts)

ii) Derive the first-order (KKT) conditions. (2 Pts)

b. Solve the KKT conditions to determine all candidate optimal solutions (x, y) and the corresponding Lagrange multipliers λ. (4 Pts)

c. Briefly discuss how the constraint affects the solution compared to the unconstrained minimizer of $f(x, y)$. (2 Pts)