These are notes for Buntine & Jakulin paper: Discrete Component Analysis [1]

Gamma-Poisson (GP) model [2]:

$$\mathbb{E}_{w\sim p(w|l,\theta)}\left[w_j\right] = \sum_{k=1}^{K} \theta_{jk} l_k$$

- w_j word count of jth word in a document

$$w_j \sim \text{Po}(w_j; (\theta\mathbf{l})_j) = \frac{(\theta\mathbf{l})_j^{w_j} \exp(-(\theta\mathbf{l})_j)}{w_h!}$$

- l_k component scores (vector $\mathbf{l}$) that indicate amount of the component in the document

$$l_k \sim \text{Gamma}(l_k; \alpha_k, \beta_k) = \frac{l_k^{\alpha_k - 1} \beta_k^{\alpha_k} \exp(-\beta_k l_k)}{\Gamma(\alpha_k)}$$

- θ component loading matrix of size $J \times K$. θ_{jk} controls partition of the kth component in the jth word

The log-likelihood of this model:

$$\log p(\mathbf{w}, l|\theta,\text{GP},\text{K}) = \sum_{k=1}^{K} \left\{ \alpha_k \log(\beta_k) + (\alpha_k - 1)\log l_k - \beta_k l_k - \log\Gamma(\alpha_k) + \sum_{j=1}^{J} \left[w_j \log(\theta\mathbf{l})_j - (\theta\mathbf{l})_j - \log w_j! \right] \right\} \tag{1}$$

$$= \sum_{k=1}^{K} \text{log likelihood of } l_k + \sum_{j=1}^{J} \text{log likelihood of } w_j \text{ given } \mathbf{l}$$

Section 6: Components assignment for words.

Introducing a discrete latent vector $\mathbf{c}$ whose total count is $\sum_j w_j$. The count c_k gives the count of words in the document appearing in the kth component. It is derived from a latent matrix $\mathbf{V}$ of size $J \times K$ (entries v_{jk}).

$$\sum_{j=1}^{J} v_{jk} = c_k$$
$$\sum_{k=1}^{K} v_{jk} = w_j$$

The distribution underlying the GP model now becomes

$$l_k \sim \text{Gamma}(l_k; \alpha_k, \beta_k)$$
$$c_k \sim \text{Po}(c_k; l_k)$$
$$v_{j,k} \sim \text{Multinom}(v_{jk}; \theta_{jk}, c_k) = c_k! \prod_j \frac{\theta_{jk}^{v_{jk}}}{v_{jk}!}$$

Proof:

We have $p(c_k|l_k) = \text{Po}(c_k; l_k)$ and $p(v_{jk}|c_k) = \text{Binom}(v_{jk}; \theta_{jk}, c_k) = \binom{c_k}{v_{jk}} \theta_{jk}^{v_{jk}} (1-\theta_{jk})^{c_k - v_{jk}}$ (probability of having v_{jk} counts in c_k counts). Then:

$$
\begin{aligned}
p(v_{jk}|l_k) &= \sum_{c_k} p(v_{jk}|c_k)p(c_k|l_k) \\
&= \sum_{c_k=v_{jk}}^{\infty} \frac{c_k!}{v_{jk}!(c_k-v_{jk})!}\theta_{jk}^{v_{jk}}(1-\theta_{jk})^{c_k-v_{jk}} \times \frac{l_k^{c_k}\exp(-l_k)}{c_k!} \\
&= \frac{\exp(-l_k)\theta_{jk}^{v_{jk}}}{v_{jk}!}\sum_{c_k=v_{jk}}^{\infty}\frac{l_k^{c_k}(1-\theta_{jk})^{c_k-v_{jk}}}{(c_k-v_{jk})!} \qquad |\alpha_{jk}=c_k-v_{jk} \\
&= \frac{\exp(-l_k)(\theta_{jk}l_k)^{v_{jk}}}{v_{jk}!}\sum_{\alpha_{jk}=0}^{\infty}\frac{(l_k-\theta_{jk}l_k)^{\alpha_{jk}}}{(\alpha_{jk})!} \\
&= \frac{\exp(-l_k)(\theta_{jk}l_k)^{v_{jk}}}{v_{jk}!}\exp(l_k-\theta_{jk}l_k) \\
&= \frac{(\theta_{jk}l_k)^{v_{jk}}\exp(-\theta_{jk}l_k)}{v_{jk}!}
\end{aligned}
$$

and so $p(v_{jk}|l_k) \sim \mathrm{Po}(v_{jk};\theta_{jk}l_k)$.

Now sum of two independent Poisson distributed variables $Z = X_1 + X_2$ ($X_i \sim \mathrm{Po}(x;\lambda_i)$)is Poisson distributed:

$$
\begin{aligned}
p(Z) &= \sum_{x_1=0}^{z} p(X_1)p(Z-X_1) \\
&= \sum_{x_1=0}^{z}\frac{\lambda_1^{x_1}e^{-\lambda_1}}{x_1!}\frac{\lambda_2^{z-x_1}e^{-\lambda_2}}{(z-x_1)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\sum_{x_1=0}^{z}\frac{z!}{x_1!(z-x_1)!}\lambda_1^{x_1}\lambda_2^{z-x_1} \\
&= \frac{(\lambda_1+\lambda_2)^z e^{-(\lambda_1+\lambda_2)}}{z!}
\end{aligned}
$$

for more by induction.

So $w_j = \sum_{k=1}^{K} v_{jk}$ is Poisson distributed:

$$
w_j \sim \mathrm{Po}(w_j;\sum_{k=1}^{K}\theta_{jk}l_k)
$$

The joint distribution for v_{jk} (each is Poisson):

$$
\begin{aligned}
p(v_{1,k},v_{2,k}...v_{J,k}|l_k,\theta_{jk}) &= \prod_{j=1}^{J}\frac{(\theta_{jk}l_k)^{v_{jk}}\exp(-\theta_{jk}l_k)}{v_{jk}!} \\
&= e^{-l_k\sum_j\theta_{jk}}l_k^{\sum_j v_{jk}}\prod_j\frac{\theta^{v_{jk}}}{v_{jk}!} \qquad |\sum_j\theta_{jk}=1,\sum_j v_{jk}=c_k \\
&= \frac{l_k^{c_k}e^{-l_k}}{c_k!}c_k!\prod_j\frac{\theta^{v_{jk}}}{v_{jk}!} \\
&= \mathrm{Po}(c_k;l_k)\times\mathrm{Multinom}(v_{jk};\theta_{jk},c_k)
\end{aligned}
$$

The likelihood of GaP model with latent matrix V is then

$$
\begin{aligned}
p(V,l|\alpha,\beta,\theta,K) &= \prod_k p(l_k|\alpha_k,\beta_k)\prod_{jk}p(v_{1k},v_{2k}...v_{J,k}|l_k,\theta_{jk}) \\
&= \prod_k \mathrm{Gamma}(l_k;\alpha_k,\beta_k)\prod_{jk}\mathrm{Po}(c_k;l_k)\times\mathrm{Multinom}(v_{jk};\theta_{jk},c_k)
\end{aligned}
$$

explicitly:

$$p(V, l|\alpha, \beta, \theta, K) = \prod_k \frac{\beta_k^{\alpha_k} l_k^{c_k+\alpha_k-1} \exp(-(\beta_k+1)l_k)}{\Gamma(\alpha_k)} \prod_{jk} \frac{\theta_{jk}^{v_{jk}}}{v_{jk}!} \tag{2}$$

and

$$\log p(V, l|\alpha, \beta, \theta, K) = \sum_k \left\{ (c_k + \alpha_k - 1)\log l_k - (\beta_k+1)l_k + \alpha_k \log \beta_k - \log\Gamma(\alpha_k) + \sum_j [v_{jk}\log\theta_{jk} - \log v_{jk}!] \right\} \tag{3}$$

w_j is derived from V so it is not represented.
It is possible to integrate out l (not sure about discrete values...?):

$$\begin{aligned} p(V|\alpha, \beta, \theta, K) &= \int_0^\infty p(V, l|\alpha, \beta, \theta, K) dl \\ &= \prod_{jk} \frac{\theta_{jk}^{v_{jk}}}{v_{jk}!} \prod_k \frac{\beta_k}{\Gamma(\alpha_k)} \int_0^\infty \left[l_k^{c_k+\alpha_k-1} \exp(-(\beta_k+1)l_k) \right] dl_k \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \left[l_k^{c_k+\alpha_k-1} \exp(-(\beta_k+1)l_k) \right] dl_k &= \int_0^\infty l_k^{z-1} \exp(-(\beta_k+1)l_k) dl_k \quad |c_k + \alpha_k = z \\ &= \frac{1}{(\beta_k+1)^z} \int_0^\infty t^{z-1}\exp(-t) dt | (\beta_k+1)l_k = t \\ &= \frac{1}{(\beta_k+1)^z}\Gamma(z) \end{aligned}$$

so

$$p(V|\alpha, \beta, \theta, K) = \prod_k \frac{\beta_k}{(\beta_k+1)^{c_k+\alpha_k}} \frac{\Gamma(c_k+\alpha_k)}{\Gamma(\alpha_k)} \prod_{jk} \frac{\theta_{jk}^{v_{jk}}}{v_{jk}!}$$

EM algorithm

The term $l_k^{(c_k+\alpha_k-1)} = l_k^{(\sum_j v_{jk}+\alpha_k-1)}$ in Eq.(2) links together l_k and V and prevents simple evaluation of $\mathcal{Q}(\theta, \theta^{\text{old}}) = \mathbb{E}_{p(V,l|\theta^{\text{old}})}[\log p(V, l|\theta, ...)]$ in the EM algorithm because of the term $\mathbb{E}_{p(V,l|\theta^{\text{old}})}[v_{jk}]$.

In the likelihood Eq.(1) is problematic the term $w_k \log \sum_k \theta_{jk} l_k$. (In [2] is the term $\mathbb{E}_l[\log \sum_k \theta_{jk} l_k]$ approximated by $\log \mathbb{E}_l[\sum_k \theta_{jk} l_k]$ which might be quite crude.)

Section 7.1 Variational Approximation

Factorised approximate posterior distribution for latent variables:

$$p(l, V|w, \alpha, \beta, \theta, K) \approx q(l, V) = q_l(l) q_V(V)$$

Optimal solution [3]

$$\log q_l^*(l) = \mathbb{E}_{V\sim q_V}[\log p(V, l, w|\theta, \alpha, \beta)] + \text{const} \tag{4}$$

$$\log q_V^*(V) = \mathbb{E}_{l\sim q_l}[\log p(V, l, w|\theta, \alpha, \beta)] + \text{const} \tag{5}$$

The lower bound is given by [3]

$$\mathcal{L}(q, \theta) = \sum_z p(Z|X, \theta^{\text{old}}) \log p(X, Z|\theta) + C = \mathcal{Q}(\theta, \theta^{\text{old}}) + C$$

where

$$C = -\mathbb{E}_{l\sim q_l}[\log q_l] - \mathbb{E}_{V\sim q_V}[\log q_V] = H(q_l) + H(q_V)$$

are the entropy terms (independent on θ).

$$\log p(w|\theta,\alpha,\beta,K) \geq \mathbb{E}_{l,V\sim q(l,V)}[\log p(l,V,w|\theta,\alpha,\beta,K)] + C \tag{6}$$

The functional form of the complete likelihood suggests

$$q_l(l) = \prod_k \text{Gamma}(l_l;\alpha_k,\beta_k) = \prod_k \frac{l_k^{a_k-1} b_k^{a_k} \exp(-b_k l_k)}{\Gamma(a_k)} \tag{7}$$

$$q_V(V) = \prod_{jk} \text{Mutlinom}(v_{jk};n_{jk},w_j) = \prod_{jk} \frac{w_j!}{v_{jk}!} n_{jk}^{v_{jk}} \tag{8}$$

with $\sum_k n_{jk} = 1$.

Then from Eq.(4), (7) and (3) keeping terms dependent on l

$$(a_k - 1)\log l_k - b_k l_k + \text{const} = (c_k + \alpha_k - 1)\log l_k - (\beta_k + 1) l_k + \text{const}$$

and form Eq.(5), (8) and (3) keeping terms dependent on V

$$v_{jk}\log n_{jk} - \log v_{jk}! + \text{const} = v_{jk}\mathbb{E}_l[\log l_k] + v_{jk}\log\theta_{jk} - \log v_{jk}! + \text{const}$$

so the rewrite rules for the parameters:

$$\begin{aligned} n_{jk} &= \frac{1}{z_{jk}}\theta_{jk}\exp(\mathbb{E}_l[\log l_k]) \\ a_k &= \sum_j n_{jk} w_j + \alpha_k \\ b_k &= 1 + \beta_k \end{aligned} \tag{9}$$

where z_{jk} is the normalisation constant ($\sum_k n_{jk} = 1$) so $z_{jk} = \sum_k \theta_{jk}\exp(\mathbb{E}_l[\log l_k])$ and $\sum_j n_{jk} w_j = c_k$. (n_{jk} is the proportion of the w_j in the kth component). $\mathbb{E}_{l\sim q_l}[\log l_k] = \psi_0(a_k) - \log b_k$ where ψ_0 is digamma function (logarithmic derivation of the gamma function...)

Now recompute model parameter θ by maximising lower bound Eq.(6) (keeping constraints $\sum_j \theta_{jk} = 1$). Keeping only term dependent on θ_{jk}:

$$\begin{aligned} \mathcal{L}(\theta) &= \sum_{j,k} \mathbb{E}_{q_V(V)}[v_{jk}]\log\theta_{jk} + \text{const} \\ &= \sum_{j,k} n_{jk} w_j \log\theta_{jk} + \text{const} \end{aligned}$$

(from Eq.(8) $\mathbb{E}_{q_V(V)}[v_{jk}] = w_j n_{jk}$)

$$0 = \frac{\partial}{\partial\theta_{mn}}\left[\sum_{j,k} n_{jk} w_j \log\theta_{jk} + \lambda_k(1 - \sum_p \theta_{pk})\right]$$

we get

$$\theta_{mn} = \frac{n_{mn} w_m}{\lambda_n}$$

and from normalisation constraints $\lambda_n = \sum_m n_{mn} w_m$.

If we take likelihood function over all documents ($i = 1 : L$) each $w_j \to w_{j(i)}$ and $n_{jk} \to n_{jk(i)}$ then we get

$$\theta_{mn} = \frac{\sum_i n_{mn(i)} w_{m(i)}}{\lambda_n} \tag{10}$$

Buntine [1] even introduce prior on $\theta_{jk} \sim \text{Dirichlet}(\theta_{\text{jk}}; \gamma, \text{J}) = \text{C}(\gamma_{\text{j}}) \prod_{\text{j=1}}^{\text{J}} \theta_{\text{jk}}^{\gamma_{\text{j}}-1}$. This is incorporated into the complete log-likelihood function $p(V, l, w, \theta|\alpha, \beta, K)$ so that lower bound $\mathbb{E}_{l,V\sim q(l,V)} [\log p(l, V, w, \theta|\alpha, \beta, K)]$ and terms dependent on θ:

$$\begin{aligned}\mathcal{L}(\theta) &= \sum_{i,j,k} \mathbb{E}_{q_V(V)} \left[v_{jk(i)}\right] \log \theta_{jk} + (\gamma_j - 1) \log \theta_{jk} + \text{const} \\ &= (\sum_{i,j,k} n_{jk(i)} w_{j(i)} + \gamma_j - 1) \log \theta_{jk} + \text{const}\end{aligned}$$

and by maximising with normalisation constraints:

$$\theta_{mn} \propto \sum_i n_{mn(i)} w_{m(i)} + \gamma_j \tag{11}$$

The lower bound Eq.(6)

$$\begin{aligned}\mathcal{L}(\theta) &= \mathbb{E}_{l,V\sim q(l,V)} \left[\sum_k (c_k + \alpha_k - 1) \log l_k - (\beta_k + 1) l_k + \log \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)} + \sum_j \left[v_{jk} \log \theta_{jk} - \log v_{jk}!\right]\right] + C \\ &= \sum_k \mathbb{E}_l \left[\log l_k\right] (\sum_j \mathbb{E}_V \left[v_{jk}\right] + \alpha_k - 1) - (\beta_k + 1) l_k + \log \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)} \\ &+ \sum_j \left[\mathbb{E}_V \left[v_{jk}\right] \left(\log n_{jk} + \log z_j - \mathbb{E}_l \left[\log l_k\right] - \mathbb{E}_V \left[\log v_{jk}!\right]\right)\right] + C \\ &= \sum_k \mathbb{E}_l \left[\log l_k\right] (\alpha_k - 1) - (\beta_k + 1) l_k + \log \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)} + \sum_j \left[\mathbb{E}_V \left[v_{jk}\right] \left(\log n_{jk} + \log z_j - \log v_{jk}!\right)\right] + C\end{aligned}$$

where Eq.(9) for θ was used and $c_k = \sum_j v_{jk}$.

Including entropy terms $C = H(q_l) + H(q_V)$ from Eq.(6)

$$\begin{aligned}H(q_l) &= -\sum_k \left\{(a_k - 1)\mathbb{E}_l \left[\log l_k\right] - b_k \mathbb{E}_l \left[l_k\right] - \log \frac{b_k^{a_k}}{\Gamma(a_k)}\right\} \\ H(q_V) &= -\sum_{jk} \left\{-\mathbb{E}_V \left[\log v_{jk}!\right] + \mathbb{E}_V \left[v_{jk}\right] \log n_{jk} + \log w_j!\right\}\end{aligned}$$

we get

$$\mathcal{L} = \sum_k \mathbb{E}_l \left[\log l_k\right] (\alpha_k - a_k) + \sum_j w_j \log z_j + \sum_k \log \frac{\Gamma(a_k) \beta_k^{\alpha_k}}{\Gamma(\alpha_k) b_k^{a_k}} - \log \prod_j w_j! \tag{12}$$

where Eq.(9) for b_k and $\sum_k n_{jk} = 1$ was used.

After initialisation the algorithm then repeats until convergence:

1. For each document: update n_{jk} and a_k according to Eq.(9) (variational E step).
2. Update θ according to Eq.(10) or (11) (variational M step).
3. Compute lower bound on log-probability Eq.(12) and check for convergence.

NMF

add correspondance to NMF...

References

[1] Buntine W, Jakulin A. Discrete component analysis. In: Saunders C, Grobelnik M, Gunn S, Shawe-Taylor J, eds. Subspace, Latent Structure and Feature Selection. Springer; 2006:1â33. Available at: http://www.springerlink.com/index/d53027666542q3v7.pdf [Accessed January 26, 2011].

[2] Canny, J. (2004). GaP: a factor model for discrete data. Proceedings of the 27th annual international ACM SIGIR conference on Research and development in information retrieval (p. 122–129). ACM. Retrieved January 25, 2011, from http://portal.acm.org/citation.cfm?id=1009016.

[3] Bishop, C. M. (2006). Pattern Recognition and Machine Learning. (M. Jordan, J. Kleinberg, & B. Scholkopf, Eds.)Pattern Recognition (p. 738). Springer. doi: 10.1117/1.2819119.