Introduction to MCMC

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Overview

Introduction to Bayesian inference

Basic Monte Carlo methods

Markov Chain Monte Carlo methods

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Intro to Bayesian Inference ("inverse probability")

A controversial formula

"The theory of inverse probability is founded upon an error, and must be wholly rejected!" Fisher, 1925

"Reduces all probability to a subjective judgement." Fisher, 1956

"Fallacious rubbish!" Fisher about Laplace, 1958

"The more I consider it, the more clearly it would appear that I have been doing almost exactly what Bayes had done in the 18th century." Fisher, 1959

Intro to Bayesian Inference ("inverse probability")

Thomas Bayes' formula (one more time)

$$\underbrace{p(\theta|y)}_{\textit{posterior}} = \frac{\overbrace{p(y,\theta)}^{\text{joint probability}}}{\int_\theta p(y,\theta)} = \frac{\overbrace{p(y|\theta)}^{\textit{likelihood}}\ \overbrace{p(\theta)}^{\textit{prior}}}{\int_\theta p(y|\theta)p(\theta)} \propto \overbrace{p(y|\theta)}^{\textit{likelihood}}\ \overbrace{p(\theta)}^{\textit{prior}} \qquad (1)$$

- Imagine we want to use Bayes in a non-trivial problem...
- ...where the denominator is not tractable (**usually**)
- ...but we can **always** evaluate the likelihood and the prior.
- How can I get access to the posterior?

Intro to Bayesian Inference

Computers to the rescue

$$\underbrace{p(\theta|y)}_{posterior} \propto \overbrace{p(y|\theta)}^{likelihood}\ \overbrace{p(\theta)}^{prior} \tag{2}$$

- The denominator was just a normalizing factor.
- Therefore drawing samples from $\overbrace{p(y|\theta)}^{likelihood}\ \overbrace{p(\theta)}^{prior}$ is like drawing samples directly from the posterior.
- "*I wish I had an Intel Core i7 :_(*"
  Thomas Bayes, circa 1750.

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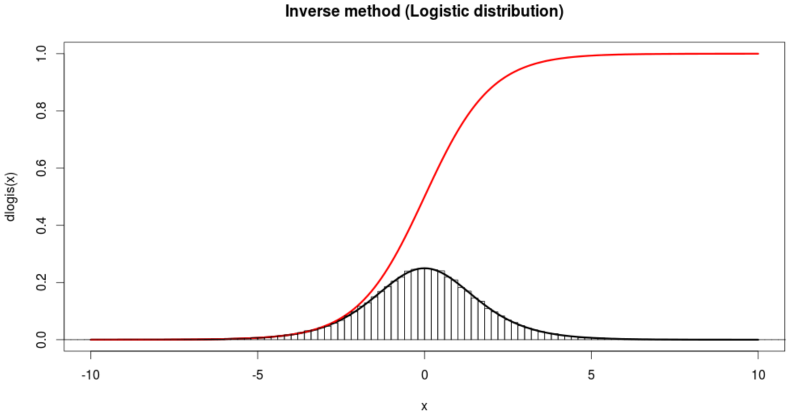
Inverse method

Exploiting the Cumulative Distribution Function

1. Get the CDF: $\mathbb{R} \rightarrow [0,1]$
2. Draw samples from uniform distribution.
3. Map uniform samples to final samples through $CDF^{-1} : [0,1] \rightarrow \mathbb{R}$



Inverse method

Pros and cons

Pros:

- Faster possible method.

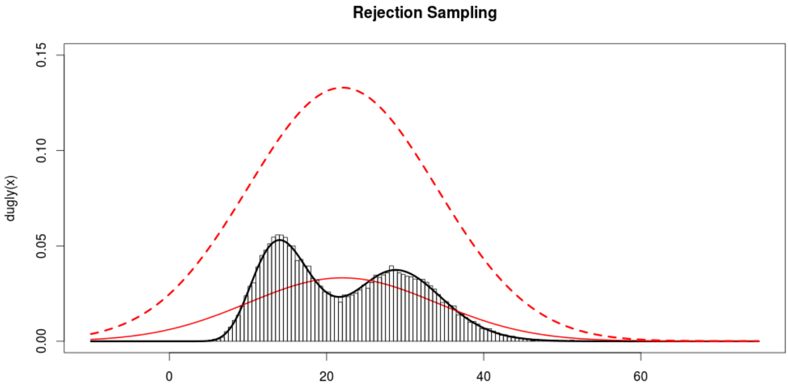
Cons:

- We don' have the CDF of most distributions.
- Only for unidimensional distributions.

Accept-Reject sampling

Sample from a comfort zone

1. Look for some pretty density q that you know how to can sample from.
2. Make it bigger than your ugly density p. (Kq)
3. Draw sample x_i from the pretty one.
4. Accept samples with probability $p(x_i)/(Kq(x_i))$



Accept-Reject sampling

Pros and cons

Pros:

- We can usually find a pretty envelope Kq.

Cons:

- If Kq is too wide or too big, lots of wasted (rejected) samples.
- Not always easy to find a good Kq.

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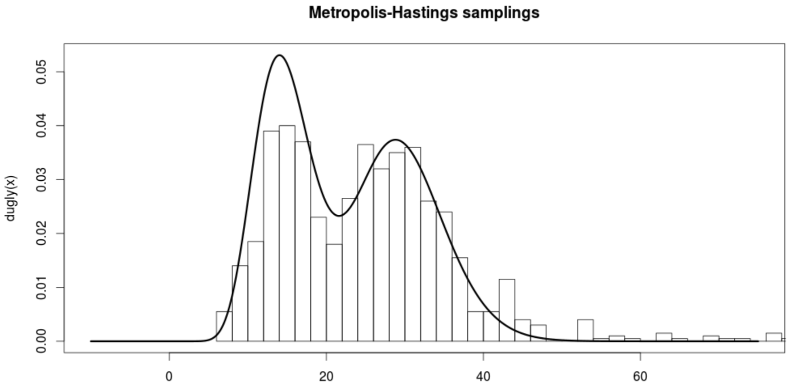
Metropolis-Hastings

Jumping from last sample

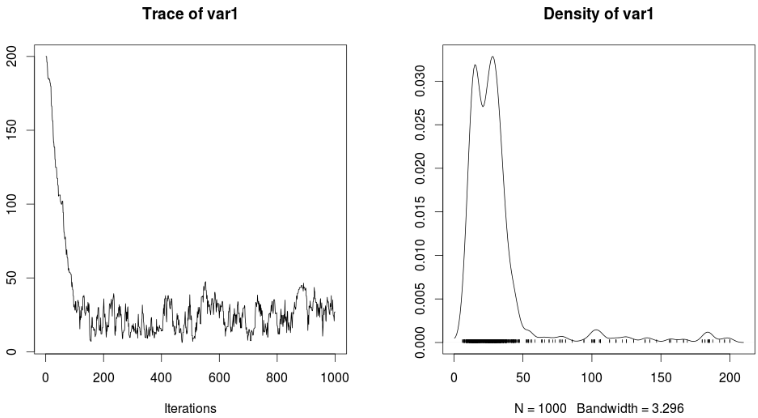
1. Start at some initial sample x_0
2. Random jump $x_{i+1} \sim x_i + \mathcal{N}(0, step)$
3. Accept with probability $p(x_{i+1})/p(x_i)$. If rejected, $x_{i+1} = x_i$
4. Repeat 2 and 3 N times (N is the length of your Markov Chain)



Metropolis-Hastings

About the Markov Chain

- First samples depend on x_0. We drop them (burn-in).
- After burn-in, we are sampling from the true distribution.
- Statistical convergence checks (see `coda` package in R).



Metropolis-Hastings

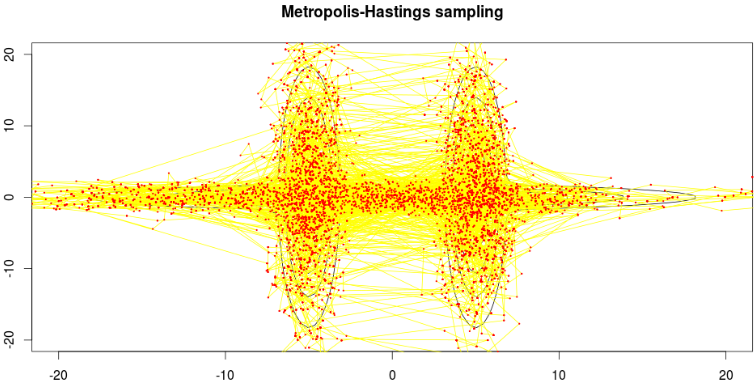
Pros and cons

Pros:

- Always works! (see ugly density below)
- Step size needs to be carefully tuned (for efficiency).

Cons:

- Rejects $\sim 70 - 80\%$ of samples (recommended).

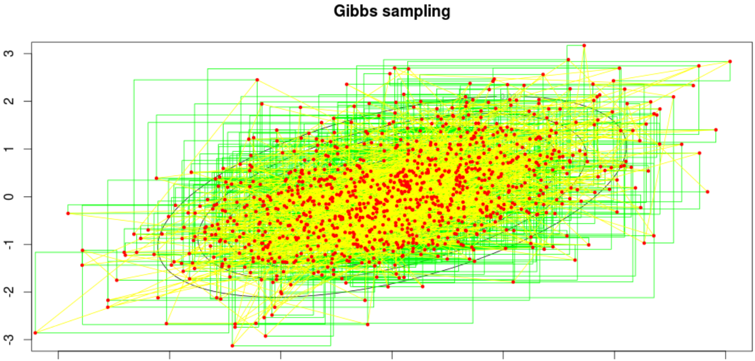


Gibbs sampling

Sample from pretty individual conditional probabilities.

Ugly $p(x1, x2, x3)$

1. Choose a initial x_1, x_2, x_3
2. Draw $x_1 \sim p(x_1|x_2, x_3)$ (easy!)
3. Draw $x_2 \sim p(x_2|x_1, x_3)$ (easy!)
4. Draw $x_3 \sim p(x_2|x_2, x_1)$ (easy!)
5. Repeat 1,2 and 3 N times.



Gibbs sampling

Prons and cons

Pros:

- Just a smart Metropolis-Hastings where all samples are accepted.

Cons:

- Only possible if conditional probabilities are pretty.
- (... and for that we need to use *conjugate priors*)

Wrap-up

- We sample because we can't do the maths analytically.
- Gibbs and Metropolis are a must in a Bayesian toolbox.

Caveat:

- burn-in can take toooo much.

Alternatives:

- Laplace approximations.
- **Variational Inference (approximation)** (very popular, very scalable)
- Expectation Propagation (approximation).
- ...

Readings

Historical debate:

- Aldrich, *R. A. Fisher on Bayes and Bayes' theorem*
- Fisher. *The Design of Experiments* (page 6)

MC Books and tutorials

- Kruschke. *Doing Bayesian Data Analysis.*
- Bishop. *Pattern Recognition and Machine Learning* (ch.11)
- Robert et Casella. *Méthodes de Monte Carlo avec R.*
- Gellman. *Bayesian Data Analysis.*
- Cam Davidson-Pilon. *Bayesian Methods for Hackers.*

Divulgation:

- Salsburg, D. *The lady tasting tea.*
- Silver, N. *The Signal and the Noise: Why So Many Predictions Fail –but Some Don't.*