The quintic, the icosahedron, ~~and elliptic curves~~ another day!

1 Arnold's topological proof of the unsolvability of the quintic

quadratic $\quad X^2 + aX + b = 0$

$$x = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

cubic (Cardano, 1545) $\quad X^3 + aX + b = 0$

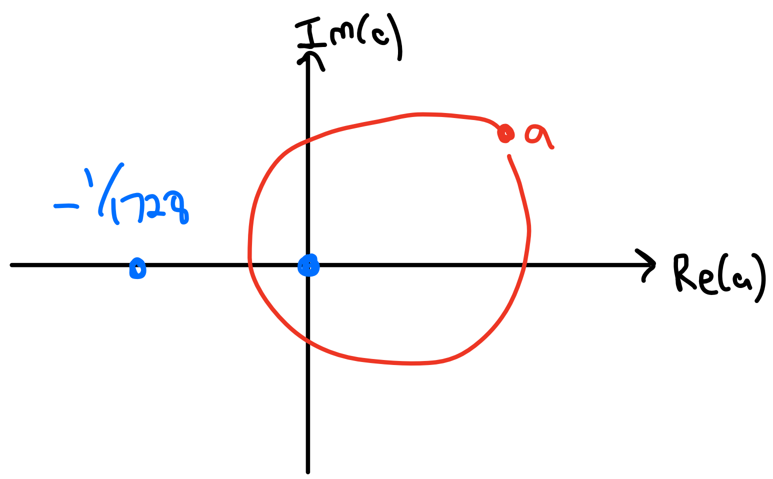
$$x = c - \frac{a}{3c}, \quad c = \sqrt[3]{-b/2 + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

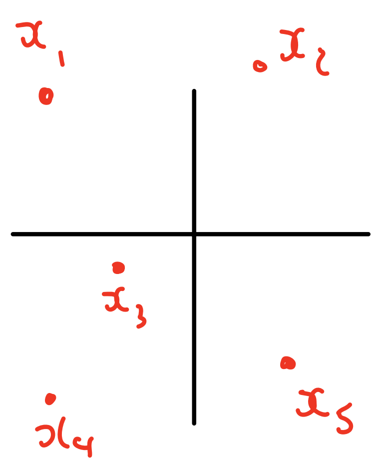
quartic (Ferrari, 1545)

quintic

$+, -, \cdot, \div, \sqrt{\ }$
$\downarrow$
No radical formula for roots! (Ruffini, Abel 1824)
Galois 1832

Arnold's proof (1963):





$$P_a = X^5 + 10aX^3 + 45a^2X + a^2 = 0$$

all even permutations possible! (A_5)

$\therefore$ dance of the roots too complex

$\therefore$ x_i can't be a radical function of a.

got up to here

Groups

A group encodes the idea of an "abstract collection of symmetries" in maths.

Defn A group G is a set equipped with a multiplication operation, and a special element $e \in G$, satisfying:

(associative) 1. $(gh)k = g(hk) \quad \forall\, g,h,k \in G$

(unit) 2. $eg = ge = g \quad \forall\, g \in G$

(inverses). 3. For every $g \in G$, there is a $g^{-1} \in G$, such that $gg^{-1} = e$ and $g^{-1}g = e$

Example $\{e, g, h\}$ with:

multiplication

	e	g	h
e	e	g	h
g	g	h	e
h	h	e	g

$\{e, g, h\} \cong \{0, 1, 2\}$

multiplication $= +\ (\text{mod } 3)$.

<u>Example</u> If X is a set of things, then

$$G := \mathrm{Perm}(X) \leftarrow \text{invertible maps from } X \text{ to } X$$

is a group. Group multiplication ?
Associative ?
Inverses ?

eg. $X =$ 

$$\mathrm{Perm}(X) = \left\{ \underbrace{\begin{pmatrix} \text{dog} & \text{cat} \\ \downarrow & \downarrow \\ \text{dog} & \text{cat} \end{pmatrix}}_{e}, \underbrace{\begin{pmatrix} \text{dog} & \text{cat} \\ \downarrow & \downarrow \\ \text{cat} & \text{dog} \end{pmatrix}}_{g} \right\}$$

or $\begin{pmatrix} \text{dog} & \text{cat} \\ \text{dog} & \text{cat} \end{pmatrix}$.

When $X = \{1, 2, \dots, n\}$, we write S_n for Perm_n.
"symmetric group of order n".

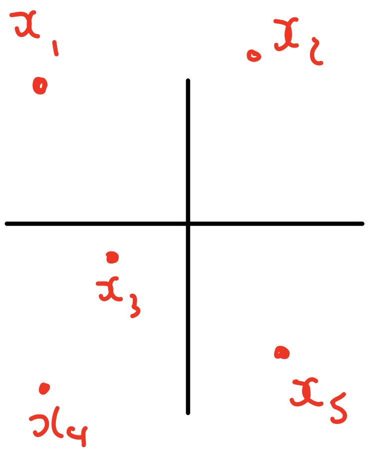
<u>Even permutations</u>

A <u>transposition</u> is a permutation which swaps two elements, leaving the rest fixed.

2. Enter the icosahedron

Will show:

Evenly ordered 5-tuple of roots $(x_1, \dots, x_5)$ of a Brioschi quintic $\longleftrightarrow$ a point on the icosahedron



$\longleftrightarrow$