Clase 04-06-21

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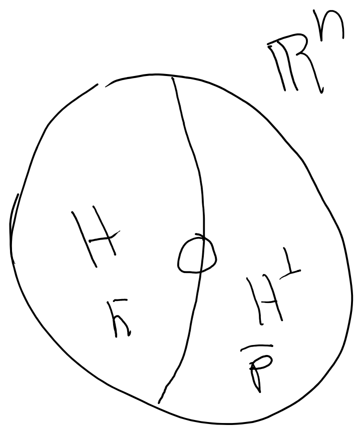

Definición 31 Complemento ortogonal

Sea H un subespacio de $\mathbb{R}^n$. El complemento ortogonal de H, denotado como $H^{\perp}$, está dado por:

$$H^{\perp} = \{ \bar{x} \in \mathbb{R}^n : \bar{x} \cdot \bar{h} = 0 \ \text{ para toda } \bar{h} \in H \}$$

Teorema 28 Si H es un subespacio de $\mathbb{R}^n$, entonces:

1) $H^{\perp}$ es un subespacio de $\mathbb{R}^n$
2) $H \cap H^{\perp} = \{\bar{0}\}$
3) $\dim H^{\perp} = n - \dim H$

Demostración de 1)

Sean $\bar{x}, \bar{y} \in H^{\perp}$ y $\alpha \in \mathbb{R}$; $\bar{h} \in H$

S1) $(\bar{x}+\bar{y}) \in H^{\perp}$

Si $(\bar{x}+\bar{y}) \in H^{\perp}$: $(\bar{x}+\bar{y}) \cdot \bar{h} = 0$

Verificación:

$(\bar{x}+\bar{y}) \cdot \bar{h} = 0$?

$\bar{x} \cdot \bar{h} + \bar{y} \cdot \bar{h} = 0 + 0 = 0$

$\therefore$ $(\bar{x}+\bar{y}) \in H^{\perp}$

M1) $\alpha \bar{x} \in H^{\perp}$

si $\alpha \bar{x} \in H^{\perp}$: $(\alpha \bar{x}) \cdot \bar{h} = 0$

Verificación:

$(\alpha \bar{x}) \cdot \bar{h} = \alpha \bar{x} \cdot \bar{h} = \alpha (\bar{x} \cdot \bar{h}) = \alpha (0) = 0$

$\therefore$ $\alpha \bar{x} \in H^{\perp}$

$\therefore$ $H^{\perp}$ es subespacio de $\mathbb{R}^n$

Demostración de 2)

Si $\bar{x} \in H \cap H^{\perp}$, entonces $\bar{x} \cdot \bar{x} = 0$, por tanto $\bar{x} = \bar{0}$, lo que muestra que $H \cap H^{\perp} = \{\bar{0}\}$

Teorema 29 Teorema de la proyección

Sea H un subespacio de $\mathbb{R}^n$ y sea $\bar{v} \in \mathbb{R}^n$. Entonces existe un par único de vectores $\bar{h}$ y $\bar{p}$ tales que $\bar{h} \in H$, $\bar{p} \in H^{\perp}$ y $\bar{v} = \bar{h} + \bar{p}$. En particular $\bar{h} = \text{proy}_H \bar{v}$ y $\bar{p} = \text{proy}_{H^{\perp}} \bar{v}$ de manera que:

$$\bar{v} = \bar{h} + \bar{p} = \text{proy}_H \bar{v} + \text{proy}_{H^{\perp}} \bar{v}$$

En los siguientes problemas se da un subespacio H y un vector $\bar{v}$. (Pág 435 Grossman)

a) Calcule $proy_H \bar{v}$

b) Encuentre una base ortonormal para $H^\perp$

c) Escriba $\bar{v}$ como $\bar{h}+\bar{p}$, donde $\bar{h} \in H$.

29) $H = \{(x,y) \in \mathbb{R}^2 ; x+y=0\}$; $\bar{v} = (-1,2)$

Solución:

base para H: $x+y=0$

$$x=-y \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} y \quad ; \quad B_H = \{(-1,1)\}$$
$$y \in \mathbb{R}$$

base ortonormal de H:

$$\bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{(-1,1)}{\sqrt{2}} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad ; \quad B''_H = \left\{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

a) $proy_H \bar{v} = (\bar{v} \cdot \bar{u}_1)\bar{u}_1$

$$= \left[(-1,2) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{3}{2}, \frac{3}{2}\right) = \bar{h}$$

b) $H^\perp = \{\bar{x} \in \mathbb{R}^2 : \bar{x} \cdot \bar{h} = 0, \ \bar{h} \in H\}$

$$(x,y) \cdot (-1,1) = -x+y = 0$$
$$x = y$$
$$y \in \mathbb{R}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y \quad \therefore B_{H^\perp} = \{(1,1)\} \ , \quad B''_{H^\perp} = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

c) $proy_{H^\perp} \bar{v} = \left[(-1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) = \bar{p}$$

$$\bar{v} = \bar{h} + \bar{p} = \left(-\frac{3}{2}, \frac{3}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) = (-1,2) \checkmark$$

33) $H = \{(x,y,z) \in \mathbb{R}^3 ; 3x+y-z=0\}$; $\bar{v} = (1,1,1)$

Solución: $\qquad \bar{n} = (3,1,-1)$

Base para H: $y = z-3x$

$x, z \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} z \quad ; \quad B_H = \{\overset{\bar{v}_1}{(1,-3,0)}, \overset{\bar{v}_2}{(0,1,1)}\}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} z \quad ; \quad B_H = \{(1,-3,0), (0,1,1)\}$$

$$\bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right)$$

$$\bar{v}_2' = \bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1)\bar{u}_1 = (0,1,1) - \left[(0,1,1) \cdot \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right)\right] \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right)$$

$$\bar{v}_2' = (0,1,1) + \frac{3}{\sqrt{10}} \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right) = (0,1,1) + \left(\frac{3}{10}, \frac{-9}{10}, 0\right) = \left(\frac{3}{10}, \frac{1}{10}, 1\right)$$

$$|\bar{v}_2'| = \sqrt{\frac{9+1+100}{100}} = \frac{\sqrt{110}}{10}$$

$$\bar{u}_2 = \frac{\left(\frac{3}{10}, \frac{1}{10}, 1\right)}{\frac{\sqrt{110}}{10}} = \frac{10\left(\frac{3}{10}, \frac{1}{10}, 1\right)}{\sqrt{110}} = \left(\frac{3}{\sqrt{110}}, \frac{1}{\sqrt{110}}, \frac{10}{\sqrt{110}}\right)$$

$$B'' = \left\{ \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right), \left(\frac{3}{\sqrt{110}}, \frac{1}{\sqrt{110}}, \frac{10}{\sqrt{110}}\right) \right\}$$

a) $\text{proy}_H \bar{v} = (\bar{v} \cdot \bar{u}_1)\bar{u}_1 + (\bar{v} \cdot \bar{u}_2)\bar{u}_2$

$$= \left[(1,1,1) \cdot \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right)\right] \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right) + \left[(1,1,1) \cdot \left(\frac{3}{\sqrt{110}}, \frac{1}{\sqrt{110}}, \frac{10}{\sqrt{110}}\right)\right] \left(\frac{3}{\sqrt{110}}, \frac{1}{\sqrt{110}}, \frac{10}{\sqrt{110}}\right)$$

$$= \frac{-2}{\sqrt{10}} \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right) + \frac{14}{\sqrt{110}} \left(\frac{3}{\sqrt{110}}, \frac{1}{\sqrt{110}}, \frac{10}{\sqrt{110}}\right) = \left(\frac{-2}{10}, \frac{6}{10}, 0\right) + \left(\frac{42}{110}, \frac{14}{110}, \frac{140}{110}\right)$$

$$= \left(-\frac{1}{5}, \frac{3}{5}, 0\right) + \left(\frac{21}{55}, \frac{7}{55}, \frac{14}{11}\right) = \left(\frac{2}{11}, \frac{8}{11}, \frac{14}{11}\right) = \bar{h}$$

b) $\dim \mathbb{R}^3 = 3$

$\dim H = 2$

$\dim H^\perp = 1$

Para obtener el vector de $H^\perp$:

$(x,y,z) \cdot (1,-3,0) = 0 \qquad x - 3y = 0$

$(x,y,z) \cdot (0,1,1) = 0 \qquad y + z = 0$

Resolvemos por Gauss:

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} x = -3z \\ y = -z \\ z = z \end{array} \qquad \begin{array}{l} \bar{n} = (-3,-1,1) \\ \text{ó } \bar{n} = (3,1,-1) \end{array}$$

Entonces:

$$B_{H^\perp} = \{(3,1,-1)\} \quad \text{y} \quad B''_{H^\perp} = \left\{\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right)\right\}$$

Otra manera de obtener $\bar{n}$ es directamente con la ecuación del plano: $\underset{a}{3}x + \underset{b}{y} \underset{c}{-z} = 0$

Otra manera de obtener $\bar{n}$ es directamente con la ecuación del plano: $3x+y-z=0$

$\bar{n}=(3,1,-1)$, $\quad B_{H^\perp}=\{(3,1,-1)\}$, $B''_{H^\perp}=\left\{\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)\right\}$

c) $\bar{v}=\bar{h}+\bar{p}$

$\bar{v}=(1,1,1)$

$\bar{h}=\left(\frac{2}{11},\frac{8}{11},\frac{14}{11}\right)$

$\bar{p}=\text{proy}_{H^\perp}\bar{v}=\left[(1,1,1)\cdot\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)\right]\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)$

$=\frac{3}{\sqrt{11}}\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)=\left(\frac{9}{11},\frac{3}{11},-\frac{3}{11}\right)$

Ahora:

$\bar{v}=\bar{h}+\bar{p}=\left(\frac{2}{11},\frac{8}{11},\frac{14}{11}\right)+\left(\frac{9}{11},\frac{3}{11},-\frac{3}{11}\right)=(1,1,1)$ ✓

<u>Tarea</u> Grossman pag 435 probs 34, 35, 36