$T: P_1 \to P_2$; $Tp(x) = (p(x))^2$

Sean $p(x) = a_1 + a_2 x$, $q(x) = b_1 + b_2 x \in P_1$ y $\alpha \in \mathbb{R}$

1) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

m. izq:

$T(\bar{u}+\bar{v}) = T[a_1 + a_2 x + b_1 + b_2 x] = T[a_1 + b_1 + (a_2+b_2)x]$ suma de polinomios

$= ((a_1+b_1) + (a_2+b_2)x)^2$ Aplicando la transformación

$= (a_1+b_1)^2 + 2(a_1+b_1)(a_2+b_2)x + (a_2+b_2)^2 x^2$ desarrollando

$= \underline{a_1^2} + 2a_1b_1 + \underline{b_1^2} + \underline{2a_1a_2x} + 2a_1b_2x + 2b_1a_2x + \underline{2b_1b_2x} + \underline{a_2^2x^2} + 2a_2b_2x^2 + \underline{b_2^2x^2}$

miembro derecho:

$T(\bar{u}) + T(\bar{v}) = (a_1+a_2x)^2 + (b_1+b_2x)^2 = \underline{a_1^2} + \underline{2a_1a_2x} + \underline{a_2^2x^2} + \underline{b_1^2} + \underline{2b_1b_2x} + \underline{b_2^2x^2}$

Comparando:

$T(\bar{u}+\bar{v}) \neq T(\bar{u}) + T(\bar{v})$

2) $T(\alpha\bar{u}) = \alpha T(\bar{u})$

m. izq

$T(\alpha\bar{u}) = T(\alpha a_1 + \alpha a_2 x)$ producto de escalar por polinomio

$= (\alpha a_1 + \alpha a_2 x)^2 = \alpha^2 a_1^2 + 2\alpha^2 a_1 a_2 x + \alpha^2 a_2 x^2$ Aplicando la transformación

m. derecho:

$\alpha T(\bar{u}) = \alpha T(a_1 + a_2 x)$ Sustituyendo

$= \alpha (a_1 + a_2 x)^2$ Aplicando T

$= \alpha (a_1^2 + 2a_1a_2x + a_2^2x^2)$

$= \alpha a_1^2 + 2\alpha a_1 a_2 x + \alpha a_2^2 x^2$

$T(\alpha\bar{u}) \neq \alpha T(\bar{u})$

$\therefore$ T no es lineal

2) $T: M_{22} \to M_{22}$; $T(A) = AB$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

a) núcleo y nulidad.

$nu\,T = \{ \bar{v} \in M_{22} : T(\bar{v}) = [0]_{2\times 2} \}$. Sea $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$

$$T(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b & a+b \\ c-d & c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} a = b \\ a = -b \\ c = d \\ c = -d \end{array} \quad \therefore\ nu\,T = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},\ \gamma(T) = 0$$

$\therefore a = b = c = d = 0$

b) Recorrido y rango:

$Rec\,T = \{ \bar{w} \in M_{22} ;\ \bar{w} = T(A) ;\ A \in M_{22} \}$

T sí es un isomorfismo

$\gamma(T) = 0$ T es 1-1

$\rho(A) = n = 4$ T es sobre

pero $\begin{array}{l} n = 4 \\ \gamma(T) = 0 \end{array} \Rightarrow \rho(T) = 4$, $\therefore Rec\,T = M_{22}$

3) $T: P_2 \to \mathbb{R}^2$; $T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1,\ a_1 + a_2 + a_3)$

$T(1) = (1, 0)$

$T(x) = (1, 1)$

$T(x^2) = (0, 1)$

$$A_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Desarrollando núcleo y recorrido simultáneamente:

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 0 & a-b \\ 0 & 1 & 1 & 0 & b \end{array}\right] ; \quad \begin{array}{l} a_0 - a_2 = 0 \\ a_1 + a_2 = 0 \end{array} \Rightarrow \begin{array}{l} a_0 = a_2 \\ a_1 = -a_2 \\ a_2 \in \mathbb{R} \end{array}$$

$nu\,A_T = \{ a_2 - a_2 x + a_2 x^2 \} = \{ (1 - x + x^2) a_2 \},\ a_2 \in \mathbb{R}$

$B_{nu\,A_T} = \{ 1 - x + x^2 \}$, $\gamma(A_T) = 1 \Rightarrow$ T no es 1-1

$n = \gamma(A_T) + \rho(T)$

$3 = 1 + \rho(T)$; $\rho(A_T) = 2 \Rightarrow Rec\,A_T = \mathbb{R}^2$ no hay renglón de ceros, hay 2 pivotes

$B_{Rec\,A_T} = \{ (1,0), (0,1) \}$ No es isomorfismo: $\gamma(A) \neq 0$

$\rho(A) \neq 3$

4.- $A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

$$p(\lambda) = (-2-\lambda)(1-\lambda) - 10$$
$$= -2 + 2\lambda - \lambda + \lambda^2 - 10$$
$$\det(A-\lambda I) = \lambda^2 + \lambda - 12 = (\lambda-3)(\lambda+4) = 0$$
$$\lambda_1 = 3 \quad ma = 1$$
$$\lambda_2 = -4 \quad ma = 1$$

para $\lambda_1 = 3 : (A-\lambda I)\bar{v} = \bar{0}$

$$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-5x_1 - 2x_2 = 0 \Rightarrow \quad x_1 = -\frac{2}{5}x_2 \quad$ para $x_2 = -5 : \bar{v}_1 = (2, -5)$

$x_2 \in \mathbb{R} \qquad E_\lambda = \{(-\frac{2}{5}x_2, x_2) ; x_2 \in \mathbb{R}\} \quad mg = 1$

para $\lambda_2 = -4 : (A-\lambda I)\bar{v} = \bar{0}$

$$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 2x_1 = 2x_2 \\ x_1 = x_2 \\ x_2 \in \mathbb{R} \end{matrix}$$

$\bar{v}_2 = (1,1) \quad E_\lambda = \{(x_2, x_2) ; x_2 \in \mathbb{R}\} \quad mg = 1$

$$C = \begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix}, \quad C^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$$

Comprobación:

$$C^{-1}AC = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & -3 \\ -20 & -8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -5 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 21 & 0 \\ 0 & -28 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

5). $A = \begin{bmatrix} 4 & 3-2i \\ 3+2i & 6 \end{bmatrix}$; $\bar{A}^t = \begin{bmatrix} 4 & 3-2i \\ 3+2i & 6 \end{bmatrix} = A \Rightarrow A$ es hermitiana

$$p(\lambda) = (4-\lambda)(6-\lambda) - (3-2i)(3+2i)$$
$$= 24-4\lambda-6\lambda+\lambda^2 - (9+6i-6i+4) = \lambda^2 - 10\lambda + 24 - 13 = \lambda^2 - 10\lambda + 11$$

$\det(A-\lambda I) = \lambda^2 - 10\lambda + 11 = 0 \Rightarrow \lambda_1 = 5+\sqrt{14}$ $ma = 1$

$\lambda_2 = 5-\sqrt{14}$ $ma = 1$

para $\lambda_1 = 5+\sqrt{14}$, $(A-\lambda I)\bar{v} = \bar{0}$

$$\begin{bmatrix} -1-\sqrt{14} & 3-2i \\ 3+2i & 1-\sqrt{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-1-\sqrt{14})x_1 + (3-2i)x_2 = 0$

$x_1 = \dfrac{-3+2i}{-1-\sqrt{14}} x_2$ $\quad \bar{v}_1 = \begin{bmatrix} -3+2i \\ -1-\sqrt{14} \end{bmatrix}$, con $x_2 = -1-\sqrt{14}$

$x_2 \in \mathbb{R}$

para $\lambda_2 = 5-\sqrt{14}$, $(A-\lambda I)\bar{v} = \bar{0}$

$$\begin{bmatrix} -1+\sqrt{14} & 3-2i \\ 3+2i & 1+\sqrt{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(3+2i)x_1 + (1+\sqrt{14})x_2 = 0$

$x_2 = \dfrac{-3-2i}{1+\sqrt{14}} x_1$; $\quad \bar{v}_2 = \begin{bmatrix} 1+\sqrt{14} \\ -3-2i \end{bmatrix}$ con $x_1 = 1+\sqrt{14}$

$x_1 \in \mathbb{R}$

ó $\bar{v}_2 = \begin{bmatrix} 3-2i \\ 1-\sqrt{14} \end{bmatrix}$