Clase 28-05-21

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Ejemplo Construcción de una base ortonormal en $\mathbb{R}^3$

Sea la base $B = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \{\underset{\bar{v}_1}{(1,1,0)}, \underset{\bar{v}_2}{(0,1,1)}, \underset{\bar{v}_3}{(1,0,1)}\}$.

Paso 1 $\bar{u}_1 = \dfrac{\bar{v}_1}{|\bar{v}_1|}$

$$\bar{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

Paso 2 $\bar{v}_2' = \bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1)\bar{u}_1$

$$\bar{v}_2' = (0,1,1) - \left[(0,1,1)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (0,1,1) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\bar{v}_2' = \left(-\frac{1}{2}, \frac{1}{2}, 1\right)$$

Paso 3

$$\bar{u}_2 = \frac{\bar{v}_2'}{|\bar{v}_2'|}$$

$$\bar{u}_2 = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{4}{4}}} = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1\right)}{\sqrt{\frac{6}{4}}} = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1\right)}{\frac{\sqrt{6}}{2}}$$

$$\bar{u}_2 = \frac{2}{\sqrt{6}}\left(-\frac{1}{2}, \frac{1}{2}\ 1\right) = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

Paso 4 $\bar{v}_3' = \bar{v}_3 - (\bar{v}_3 \cdot \bar{u}_2)\bar{u}_2 - (\bar{v}_3 \cdot \bar{u}_1)\bar{u}_1$

$$\bar{v}_3' = (1,0,1) - \left[(1,0,1)\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\right]\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) - \left[(1,0,1)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (1,0,1) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (1,0,1) + \left(\frac{1}{6}, -\frac{1}{6}, -\frac{2}{6}\right) + \left(-\frac{1}{2}, -\frac{1}{2}, 0\right)$$

$$\bar{v}_3' = \left(\frac{4}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

Paso 5

$$\bar{u}_3 = \frac{\bar{v}_3'}{|\bar{v}_3'|}$$

$$\bar{u}_3 = \frac{\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)}{\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}}} = \frac{\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)}{\sqrt{\frac{12}{9}}}$$

$\sqrt{9}$ ...

$$\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} \qquad \sqrt{\frac{12}{9}}$$

$$= \frac{\sqrt{9}}{\sqrt{12}}\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{3}{2\sqrt{3}}\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$\bar{u}_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$\therefore$ La base ortonormal, B'' será:

$$B'' = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$$

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Construya una base ortonormal para el espacio o subespacio vectorial dado:

1) $B = \{\underbrace{(1,-3)}_{\bar{v}_1}, \underbrace{(3,0)}_{\bar{v}_2}\}$

$$\bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{(1,-3)}{\sqrt{1^2+(-3)^2}} = \frac{(1,-3)}{\sqrt{10}}$$

$$\bar{u}_1 = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$$

$$\bar{v}_2' = \bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1)\bar{u}_1$$

$$= (3,0) - \left[(3,0)\cdot\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)\right]\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$$

$$= (3,0) - \frac{3}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$$

$$= (3,0) + \left(-\frac{3}{10}, \frac{9}{10}\right)$$

$$= \left(\frac{27}{10}, \frac{9}{10}\right)$$

$$\bar{u}_2 = \frac{\bar{v}_2'}{|\bar{v}_2'|}$$

$$= \frac{\left(\frac{27}{10}, \frac{9}{10}\right)}{\sqrt{\frac{729}{100}+\frac{81}{100}}} = \frac{\left(\frac{27}{10}, \frac{9}{10}\right)}{\sqrt{\frac{810}{100}}} = \frac{\left(\frac{27}{10}, \frac{9}{10}\right)}{\frac{\sqrt{2\cdot 9^2\cdot 5}}{10}} = \frac{10}{9\sqrt{10}}\left(\frac{27}{10}, \frac{9}{10}\right)$$

$$\bar{u}_2 = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

Por lo tanto, la base ortonormal es: $B'' = \left\{\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right), \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right\}$

4) $H = \{(x,y) \in \mathbb{R}^2 : 2x+y=0\}$

Solución

H es el conjunto de puntos en $\mathbb{R}^2$ que están sobre una recta que pasa por el origen, por lo tanto se puede obtener una base ortonormal para H ya que es un subespacio.

$2x+y=0 \Rightarrow y=-2x$
$x \in \mathbb{R}$

Por lo que:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} x$$

Una base para H podría ser $B=\{(1,-2)\}$ cuando $x=1$

Como solo tenemos 1 vector en B:

$$\bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|}$$

$$\bar{u}_1 = \frac{(1,-2)}{\sqrt{1^2+(-2)^2}} = \frac{(1,-2)}{\sqrt{5}}$$

$$\bar{u}_1 = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right), \quad y \quad B'' = \left\{\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)\right\}$$

7) $\Pi = \{(x,y,z) : 2x-y-z=0\}$

Solución

Π es un subespacio porque Π es un plano que pasa por el origen.

Entonces podremos obtener una base ortonormal para Π.

De la ecuación $2x-y-z=0$ despejamos una variable, digamos z:

$z = 2x-y$
$x, y \in \mathbb{R} \leftarrow$ 2 variables libres

Entonces:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}^{\bar{v}_1} x + \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{\bar{v}_2} y$$

Una base para Π puede ser: $B=\{(1,0,2),(0,1,-1)\}$ para $x=1$ y $y=1$

Base ortonormal:

$\bar{v}_1 = (1,0,2)$; $|\bar{v}_1| = \sqrt{5}$

$$\therefore \bar{u}_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{1}{\sqrt{5}}(1,0,2) = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$$

$$\bar{v}_2' = \bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1)\bar{u}_1 \; ; \; \bar{v}_2 = (0,1,-1)$$
$$= (0,1,-1) - \left[(0,1,-1)\cdot\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$$

$$= (0,1,-1) + \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$$

$$= (0,1,-1) + \left(\frac{2}{5}, 0, \frac{4}{5}\right)$$

$$\bar{V}_2' = \left(\frac{2}{5}, 1, -\frac{1}{5}\right)$$

$$|\bar{V}_2'| = \sqrt{\frac{4}{25} + \frac{25}{25} + \frac{1}{25}} = \sqrt{\frac{30}{25}} = \frac{\sqrt{30}}{5}$$

$$\therefore\ \bar{U}_2 = \frac{\bar{V}_2'}{|\bar{V}_2'|} = \frac{\left(\frac{2}{5}, 1, -\frac{1}{5}\right)}{\frac{\sqrt{30}}{5}} = \frac{5}{\sqrt{30}}\left(\frac{2}{5}, 1, -\frac{1}{5}\right)$$

$$\bar{U}_2 = \left(\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}\right)$$

La base ortonormal de Π es $B'' = \left\{\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}\right)\right\}$

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Producto interno estandar en $\mathbb{C}^n$:

Sean $\bar{X} = (x_1, x_2, \dots, x_n)$ y $\bar{Y} = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n$, donde $x_k = a+bi$, $y_k = c+di$; $i = \sqrt{-1}$

Entonces:

$\langle \bar{X}, \bar{Y} \rangle = \bar{X} \cdot \bar{Y} = x_1\bar{y}_1 + x_2\bar{y}_2 + \dots + x_n\bar{y}_n$; $\bar{y}_k$ es el conjugado de y_k

Ejemplo

Sean $\bar{X} = (1+i, -3, 4-3i)$, $\bar{Y} = (2-i, -i, 2+i)$

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$\therefore\ \langle \bar{X}, \bar{Y} \rangle = (1+i)(2+i) + (-3)(i) + (4-3i)(2-i)$$

$$= 2+i+2i+i^2-3i+8-4i-6i+3i^2$$

$$= 2+3i-1-13i+8-3 = 6-10i$$

Producto interno en $C[0,1]$

Sean $f(t)$ y $g(t) \in C[0,1]$. El producto interno de $f(t)$ y $g(t)$ es:

$$\langle f, g \rangle = f(t) \cdot g(t) = \int_0^1 f(t)\, g(t)\, dt$$

Ejemplo Sean $f(t) = t^2 \in C[0,1]$ y $g(t) = 4-t \in C[0,1]$

$$\langle f, g \rangle = \int_0^1 t^2(4-t)\,dt = \int_0^1 (4t^2 - t^3)\,dt = \left(\frac{4}{3}t^3 - \frac{t^4}{4}\right)\Big|_0^1 = \frac{13}{12}$$

Producto interno estandar de matrices

Sean $A, B \in M_{mn}$. Entonces el producto interno de A y B es:

$\langle A,B \rangle = tr(AB^T)$; tr = traza = suma de los elementos de la diagonal principal

B^T = transpuesta de B

Ejemplo

Sean $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 0 & 2 \\ -2 & 1 \end{bmatrix}$

$$\langle A,B \rangle = tr\left\{ \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 2 & 2 & 1 \end{bmatrix} \right\} = tr \begin{bmatrix} -1 & 0 & -2 \\ -3 & -2 & -3 \\ 0 & 2 & -3 \end{bmatrix} = -1-2-3 = -6 \leftarrow$$ es un escalar

Producto interno complejo en M_{mn}

$$\langle A,B \rangle = tr(AB^*)$$

donde B^* es la transpuesta conjugada de B.

Ejemplo Obtener $\langle A,B \rangle$, si:

$$A = \begin{bmatrix} 1 & 1-i \\ 2-i & 3i \end{bmatrix}, \quad B = \begin{bmatrix} 2i & -i \\ 2i & 3i \end{bmatrix}$$

$$\langle A,B \rangle = tr\left\{ \begin{bmatrix} 1 & 1-i \\ 2-i & 3i \end{bmatrix} \begin{bmatrix} -2i & -2i \\ i & -3i \end{bmatrix} \right\} = tr \begin{bmatrix} 1-i & -3-5i \\ -5-4i & 7-4i \end{bmatrix}.$$

$$\langle A,B \rangle = (1-i) + (7-4i) = 8-5i$$

Ejemplo. Obtener $\langle A,B \rangle$ si:

$$A = \begin{bmatrix} 1-2i & 2i \\ -3i & -2-2i \end{bmatrix}, \quad B = \begin{bmatrix} 1+i & -1-i \\ -3i & -5-2i \end{bmatrix}$$

Solución

$$\langle A, B \rangle = tr(AB^*)$$

$$\langle A, B \rangle = tr\left\{\begin{bmatrix} 1-2i & 2i \\ -3i & -2-2i \end{bmatrix}\begin{bmatrix} 1-i & 3i \\ -1+i & -5+2i \end{bmatrix}\right\}$$

$$= tr\begin{bmatrix} -3-5i & 2-7i \\ 1-3i & 23+6i \end{bmatrix} = -3-5i+23+6i$$

$$\langle A, B \rangle = 20+i \rightarrow \text{escalar} \in \mathbb{C}$$