Clase 28-10-20

miércoles, 28 de octubre de 2020 11:00 a. m.

Tarea Determinar un número complejo z tal, que multiplicado por $\sqrt{2}$ cis 315° sea igual a 1 (En forma polar y en forma binómica).

Ejercicio Representar en el plano de Argand las soluciones de la ecuación $z^{1/5} = (2-2i)^{1/3}$

Solución:

$$z = \left(z^{1/5}\right)^5 = (2-2i)^{5/3} = \sqrt[3]{(2-2i)^5}$$

$$2-2i \xrightarrow{\text{polar}} \quad r = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = 360° - \varphi \; ; \quad \varphi = \tan^{-1}(1)$$

$$\theta = 360° - 45° = 315°$$

$$\therefore \; 2-2i = 2\sqrt{2} \text{ cis } 315° = \sqrt{8} \text{ cis } 315°$$

$$z = \sqrt[3]{(\sqrt{8} \text{ cis } 315°)^5} = \sqrt[3]{(\sqrt{8})^5} \text{ cis } 1575° = \sqrt[3]{(\sqrt{8})^5} \text{ cis } 135°$$

$$= \sqrt[3]{(\sqrt{8})^5} \text{ cis } \frac{135° + k(360°)}{3}, \quad k = 0,1,2$$

$$= \sqrt[3]{64(\sqrt{8})} \text{ cis } \frac{135° + k(360°)}{3}$$

Pero:

$$\sqrt[3]{64\sqrt{8}} = 4\sqrt[3]{\sqrt{8}} = 4\sqrt[3]{\sqrt{2^3}} = 4\sqrt[3]{2^{3/2}} = 4\left(2^{3/2}\right)^{1/3} = 4\sqrt{2}$$

$$\therefore \; z = 4\sqrt{2} \text{ cis } \frac{135° + k(360°)}{3}, \quad k = 0,1,2$$

Para $k=0$, $w_0 = 4\sqrt{2}$ cis 45°

$k=1$, $w_1 = 4\sqrt{2}$ cis 165°

$k=2$, $w_2 = 4\sqrt{2}$ cis 285°

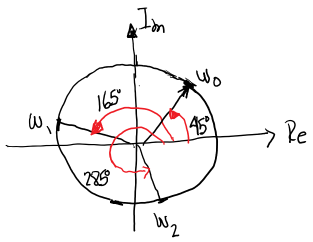


Diagrama de Argand

Tarea Historia del número e

Forma de Euler o Forma Exponencial

Euler: matemático suizo, s. XVIII. Estableció la relación:

$$\boxed{e^{\theta i} = \cos\theta + i \operatorname{sen}\theta}$$

Leonhard Euler

Por lo tanto, si $z = r \text{ cis } \theta$, $z = re^{\theta i}$

Definición 21 Forma de Euler

$$z = re^{\theta i},$$

donde: r es el módulo de z

θ es el argumento de z expresado en radianes.

$$0 \leq \theta \leq 2\pi \text{ rad}$$

Por ejemplo:

$z_1 = 2 \operatorname{cis} 225° = 2e^{\frac{5}{4}\pi i}$; $\begin{matrix} 180° - \pi \text{ rad} \\ 225° - x \text{ rad} \end{matrix} \Rightarrow x = \frac{225°}{180°}\pi \text{ rad}$

$x = \frac{5}{4}\pi \text{ rad}$

$z_2 = 3 \operatorname{cis} 180° = 3e^{\pi i}$

$z_3 = \sqrt{2} \operatorname{cis} 60° = \sqrt{2}\, e^{\frac{\pi}{3}i}$

Teorema 13

Sean $z_1 = r_1 e^{\theta_1 i}$, $z_2 = r_2 e^{\theta_2 i}$. Entonces:

$$z_1 = z_2 \iff \begin{matrix} r_1 = r_2 \\ \theta_1 = \theta_2 + k(2\pi),\ k = 0, 1, 2, \ldots \end{matrix}$$

Operaciones con Números Complejos en Forma de Euler

$$(r_1 e^{\theta_1 i})(r_2 e^{\theta_2 i}) = r_1 r_2\, e^{(\theta_1+\theta_2)i}$$ $r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)$

$$\frac{r_1 e^{\theta_1 i}}{r_2 e^{\theta_2 i}} = \frac{r_1}{r_2}\, e^{(\theta_1-\theta_2)i}$$ $\frac{r_1}{r_2} \operatorname{cis}(\theta_1-\theta_2)$

$$(re^{\theta i})^n = r^n e^{n\theta i}$$ $(r \operatorname{cis}\theta)^n = r^n \operatorname{cis} n\theta$ De Moivre

$$\sqrt[n]{re^{\theta i}} = \sqrt[n]{r}\, e^{\frac{\theta + k(2\pi)}{n} i} \quad ; \ k = 0, 1, 2, \ldots, (n-1)$$ $\sqrt[n]{r} \operatorname{cis} \frac{\theta + k(360°)}{n}$, $k = 0, 1, \ldots, n-1$

Ejemplos

Dados $z_1 = \sqrt{3}\, e^{\frac{\pi}{2}i}$, $z_2 = e^{\pi i}$, $z_3 = 8e^{3\pi i}$, $z_4 = 5e^{\frac{4}{3}\pi i}$.

Efectuar:

a) $z_1 z_2 = \left(\sqrt{3}\, e^{\frac{\pi}{2}i}\right)\left(e^{\pi i}\right) = \sqrt{3}\, e^{\frac{3}{2}\pi i}$ IV

b) $\frac{z_1}{z_2} = \frac{\sqrt{3}\, e^{\frac{\pi}{2}i}}{e^{\pi i}} = \sqrt{3}\, e^{-\frac{\pi}{2}i} = \sqrt{3}\, e^{(-\frac{\pi}{2}+2\pi)i} = \sqrt{3}\, e^{\frac{3}{2}\pi i}$

c) $(z_3)^{2/3} = \sqrt[3]{(8e^{3\pi i})^2} = \sqrt[3]{64 e^{6\pi i}} = \sqrt[3]{64 e^{2\pi i}}$

$$= \sqrt[3]{64}\; e^{\frac{(2\pi + k(2\pi))}{3} i}, \quad k=0,1,2$$

Para $k=0,\; w_0 = 4e^{\frac{2}{3}\pi i}$

$k=1,\; w_1 = 4e^{\frac{4}{3}\pi i}$

$k=2,\; w_2 = 4e^{2\pi i}$

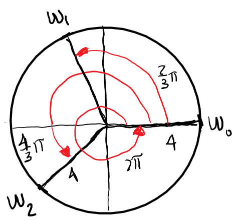
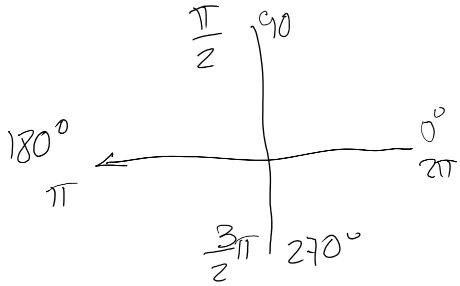


Diagrama de Argand

d) $\dfrac{z_1 + z_2}{z_4} = ?$



$z_1 = \sqrt{3}\, e^{\frac{\pi}{2} i} \xrightarrow{\text{a polar}} r = \sqrt{3}$

$\theta = 90^\circ$

$z_1 = \sqrt{3} \operatorname{cis} 90^\circ \xrightarrow{\text{a binómica}} a = \sqrt{3}\cos 90^\circ = 0$

$b = \sqrt{3}\operatorname{sen} 90^\circ = \sqrt{3}$

$\therefore z_1 = \sqrt{3}\, i$

Con z_2:

$z_2 = e^{\pi i} \xrightarrow{\text{polar}} z_2 = \operatorname{cis} 180^\circ$

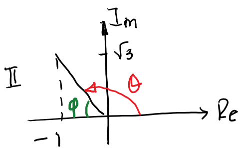
$z_2 = \operatorname{cis} 180^\circ \xrightarrow{\text{binómica}} a = \cos 180^\circ = -1$

$b = \operatorname{sen} 180^\circ = 0$

$\therefore z_2 = -1$

Entonces: $z_1 + z_2 = -1 + \sqrt{3}\, i \xrightarrow{\text{polar}} r = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$r = \sqrt{4} = 2$



$\theta = 180^\circ - \varphi$

$\varphi = \tan^{-1}(\sqrt{3}) = 60^\circ$

$\therefore \theta = 120^\circ$

$z_1 + z_2 = 2\operatorname{cis} 120^\circ \xrightarrow{\text{a Euler}} ?$

$120^\circ \xrightarrow{\text{a radianes}}$ $180^\circ — \pi$

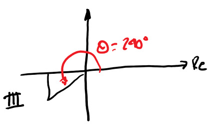
$120^\circ — x \text{ rad}$

$x = \dfrac{120^\circ}{180^\circ}\pi = \dfrac{2}{3}\pi \text{ rad}$

$\therefore z_1 + z_2 = 2e^{\frac{2}{3}\pi i}$

$$\therefore \frac{z_1 + z_2}{z_4} = \frac{2e^{\frac{2}{3}\pi i}}{5e^{\frac{4}{3}\pi i}} = \frac{2}{5} e^{-\frac{2}{3}\pi i} = \frac{2}{5} e^{(-\frac{2}{3}\pi + 2\pi) i}$$

$$\frac{z_1 + z_2}{z_4} = \frac{2}{5} e^{\frac{4}{3}\pi i}$$



Problema Obtener los valores de z, $z \in \mathbb{C}$ para los cuales se satisface la siguiente ecuación:

$$iz^2 - 1 + z^2 \operatorname{cis}\frac{\pi}{2} = 1 \operatorname{cis} 180^\circ + \frac{1}{i} e^{(2\pi i)^4}$$

Solución

Factorizando z^2:

$$z^2\left(i + \text{cis}\frac{\pi}{2}\right) = 1 + 1\,\text{cis}\,180° + \frac{1}{i}e^{(2\pi i)^4}$$

$$z^2(i+i) = 1-1 + \frac{1}{i}e^{(2\pi i)^4}$$

$$2i z^2 = \frac{1}{i}e^{(2\pi i)^4}$$

$$z^2 = \frac{1}{2i^2}e^{(2\pi i)^4} = -\frac{1}{2}e^{(2\pi i)^4}$$

$$z^2 = -\frac{1}{2}(1)^4 = -\frac{1}{2}$$

$$z = \pm\sqrt{\frac{-1}{2}} = \pm\sqrt{\frac{1}{2}}\,i$$

$$w_0 = \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}e^{\frac{\pi}{2}i}$$

$$w_1 = -\frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}e^{\frac{3}{2}\pi i}$$

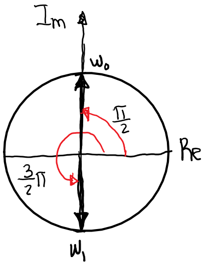


Diagrama de Argand