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Aplanatism in stigmatic optical systems

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The minimization of spherical and coma aberrations in optical imaging systems is currently accomplished through the use of corrective aspheric optical surfaces. In this work, we develop a new, to the best of our knowledge, theory for the design of rigorously aplanatic optical systems, considering as a starting point the rigorous stigmatism theory of optical systems composed of Cartesian surfaces. The main characteristic of these surfaces is their, *a priori*, zero spherical aberration. In a general parametric formulation for systems made up of a set of these surfaces, the Abbe sine condition is adapted to simultaneously obtain the stigmatism and aplanatism conditions. Thus, we achieved the design of optical systems that in theory are immune to both coma and spherical aberrations.

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The term aplanatism was first used by Herschel [1] to refer to optical systems free of spherical aberration, but after Abbe's discovery [2], this term has been used to refer to systems that are also free of comatic aberration [3]. So far, the only strictly aplanatic optical surfaces are the spheres, in the so-called aplanatic points or Young's points [4], discovered by Huygens [5] and then re-discovered by Young [6]. That is why the spherical refracting surfaces have been applied to the design of high-resolution and large-aperture microscope objectives [7,8].

Due to the ease of polishing and the aplanatism property, among other reasons, spherical surfaces are the main element of optical systems [9]. However, the aforementioned systems, when operating outside Young's points, cause the appearance of geometric aberrations [10]. The adopted strategies to treat this affectation are *a posteriori*, minimizing these aberrations using the combination of a set of spherical surfaces [11] or by corrective aspherical surfaces [12–15], which are calculated employing optimization algorithms for some region or angles of inclination of the rays that enter the system. However, recently, two aspherical optical surfaces have been used, allowing a stigmatic image formation of an object composed of separated points, without adapting a rigorous condition of aplanatism [16].

In contrast to the above, our approach to the problem is to attack the causes instead of the effects, and for this, we focus on the necessary conditions that *a priori* guarantee stigmatism, which strictly speaking is satisfied by Descartes' ovoids [17], to seek then the aplanatism. Thus, we have found an explicit expression for the conditions that optical systems must meet to fulfilling the Abbe sine condition, to the development of aplanatic optical systems in a general framework. To achieve this, we use a parametric expression for Cartesian surfaces and a set of shapes parameters that characterize them, both elements developed in previous works [18,19].

The theory of stigmatic optical systems is based on Descartes' ovoids, illustrated in Fig. 1(a). These ovoids, Σ_0, are surfaces of revolution, with vertex at V_0, that separate two media of refractive indices n_0 and n_1, which from the point object at P_0 produce a perfect image point at P_1. The proper combination of two of these surfaces allows us to design *stigmatic ovoid singlet lenses* (SOSL) [19] as is illustrated in Fig. 1(b). A combination of N surfaces produce in general stigmatic ovoid lenses (SOL), as shown in Fig. 1(c), guaranteeing the conservation of the stigmatism condition as light passes through each surface. The main feature of these systems is that all rays that pass through the system, departing from the stigmatic point object, are focused on the stigmatic point image, traveling the same optical path length. This feature is desirable because the stigmatic condition of these optical systems offers the conditions that allow us to reach an aplanatism condition and, thus in wave optics, the implementation of diffraction-limited systems [20].

For a system composed of N surfaces [see Fig. 1(c)], each surface within the system is identified by the index k; the kth surface (Σ_k) is located at $\zeta_k = \overline{\mathcal{O}V_k}$, and its stigmatic points, P_k and P_{k+1}, are located at $d_k = \overline{\mathcal{O}P_k}$ and $d_{k+1} = \overline{\mathcal{O}P_{k+1}}$, respectively. For these optical systems, the main difficulty lies in handling the formulations of these ovoids, which is why different expressions have been studied that facilitate some aspects [21–23], but still make their use difficult for optical design purposes. For this reason, we have developed a parametric formulation that simplifies the use of these Cartesian surfaces [18,19]. So, the coordinates of the kth surface are given by the parametric expressions,

$$z_k(\rho_k) = \zeta_k + \frac{(O_k + T_k\rho_k^2)\rho_k^2}{1 + S_k\rho_k^2 + \sqrt{1 + (2S_k - O_k^2 G_k)\rho_k^2}}, \tag{1}$$

$$r_k(\rho_k) = \pm\sqrt{\rho_k^2 - (z_k(\rho_k) - \zeta_k)^2}, \tag{2}$$

where z_k is the axial coordinate, r_k is the radial coordinate, $\rho_k = \sqrt{r_k^2 + (z_k - \zeta_k)^2}$ is the parameter (independent variable) given by the vertex-surface distance, and $(GOTS)_k$ are the

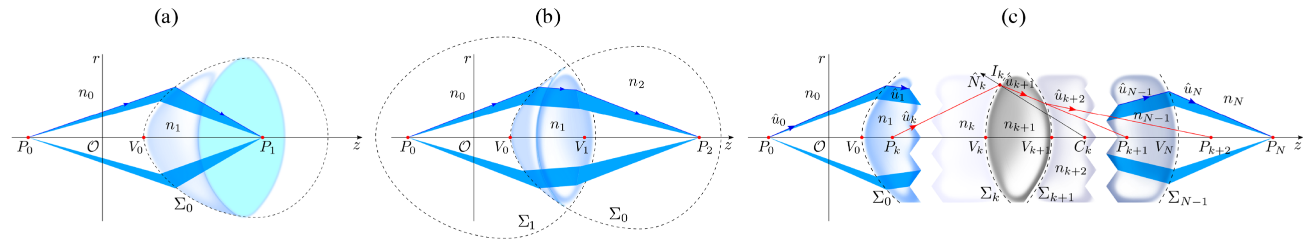

Fig. 1. The rigorously stigmatic system is composed of (a) Cartesian surfaces. Two of these surfaces form a (b) SOSL, and a set of them form a (c) SOL. The surfaces Σ_k and Σ_{k+1}, which belong to a system composed of N surfaces, are placed in such a way that the stigmatism condition is guaranteed.

shape parameters given by

$$G_k = \frac{\left(\frac{n_{k+1}^2}{d_k-\zeta_k} - \frac{n_k^2}{d_{k+1}-\zeta_k}\right)^2}{n_k n_{k+1}\left(\frac{n_{k+1}}{d_{k+1}-\zeta_k} - \frac{n_k}{d_k-\zeta_k}\right)\left(\frac{n_{k+1}}{d_k-\zeta_k} - \frac{n_k}{d_{k+1}-\zeta_k}\right)}, \tag{3}$$

$$O_k = \frac{\frac{n_{k+1}}{(d_{k+1}-\zeta_k)} - \frac{n_k}{(d_k-\zeta_k)}}{(n_{k+1}-n_k)}, \tag{4}$$

$$T_k = \frac{\left(\frac{n_{k+1}+n_k}{(d_{k+1}-\zeta_k)(d_k-\zeta_k)}\right)^2 (n_{k+1}-n_k)}{4n_{k+1}n_k\left(\frac{n_{k+1}}{(d_k-\zeta_k)} - \frac{n_k}{(d_{k+1}-\zeta_k)}\right)}, \tag{5}$$

$$S_k = \frac{\frac{(n_{k+1}+n_k)}{(d_{k+1}-\zeta_k)(d_k-\zeta_k)}\left(\frac{n_{k+1}^2}{(d_k-\zeta_k)} - \frac{n_k^2}{(d_{k+1}-\zeta_k)}\right)}{2n_{k+1}n_k\left(\frac{n_{k+1}}{(d_k-\zeta_k)} - \frac{n_k}{(d_{k+1}-\zeta_k)}\right)}, \tag{6}$$

where three of these parameters are independent due to the constraint $S_k^2 = G_k O_k T_k$ [19].

The shape parameters are useful and characterize these surfaces very simply. G_k allows us to describe the Cartesian ovoids, and it is a generalization of Schwarzschild constant, O_k is the curvature at the vertex (the paraxial curvature), and T_k and S_k characterize the different families of ovoids; for example, with $T_k = S_k = 0$, we have the family of conics. The handling of the shapes parameters is what allows the design of optical surfaces with special characteristics, and their selection determines the optical properties of these surfaces. Therefore, the problem of determining the shape of an optical system that satisfies an aplanatism condition must lead to a methodology that allows us to find the values for these parameters.

According to Fig. 1(c), $\hat{u}_k$ denotes the unit vector of the incident ray going through the kth surface, $\hat{u}_{k+1}$ denotes the unit vector of the refracted ray, and $\hat{N}_k$ is a normal unit vector in the intersection point between the ray and the surface. The aplanatism condition of the stigmatic system is expressed through the Abbe sine condition, written as

$$\frac{\sin\theta_0}{\sin\theta_N} = \frac{n_N}{n_0}M, \tag{7}$$

where M is the magnification, n_0 and n_N are the refractive indices of the object and image space, respectively, and the angles θ_0 and θ_N are the angles of the incident and emerging rays relative to the optical axis, respectively. Therefore, a system is aplanatic if the ratio of sines in Eq. (7) is a constant, which means M is constant.

To determine an explicit expression for this condition, Eq. (7), first, we define the point C_k as the intersection point between the line $\overline{C_k I_k}$ and the z axis, with the line $\overline{C_k I_k}$ perpendicular to the surface in the point I_k. If θ_k is the angle between the vector $\hat{u}_k$ and the z axis and α_k is the angle between the vectors $\hat{u}_k$ and $\hat{N}_k$, for the triangle $P_k I_k C_k$, the law of sines allows writing

$$\frac{\sin\theta_k}{\overline{C_k I_k}} = \frac{\sin\alpha_k}{\overline{V_k C_k} - \overline{V_k P_k}}, \tag{8}$$

and, if the angle θ_{k+1} is the angle between the vector $\hat{u}_{k+1}$ and the z axis and α_{k+1} is the angle between the vectors $\hat{u}_{k+1}$ and $\hat{N}_k$, by applying the law of sines in triangle $P_{k+1} I_k C_k$, we get

$$\frac{\sin\theta_{k+1}}{\overline{C_k I_k}} = -\frac{\sin\alpha_{k+1}}{\overline{V_k C_k} - \overline{V_k P_{k+1}}}, \tag{9}$$

where $\overline{V_k P_k} = d_k - \zeta_k$ and $\overline{V_k P_{k+1}} = d_{k+1} - \zeta_k$, such that $\overline{V_k P_k} < 0$ if $d_k < \zeta_k$ otherwise is a positive value, and $\overline{V_k P_{k+1}} < 0$ if $\zeta_k > d_{k+1}$ otherwise is a positive value.

The relation of the sines has to be fulfilled for the whole system; therefore, from the ratio between Eqs. (8) and (9) is obtained the following expression:

$$\frac{\sin\theta_k}{\sin\theta_{k+1}} = -\frac{\sin\alpha_k\left(\overline{V_k C_k} - (d_{k+1}-\zeta_k)\right)}{\sin\alpha_{k+1}\left(\overline{V_k C_k} - (d_k-\zeta_k)\right)}, \tag{10}$$

where the sines of the angles on the right-hand side of this expression are related through the Snell–Descartes law; that is

$$\frac{\sin\alpha_k}{\sin\alpha_{k+1}} = \frac{n_{k+1}}{n_k}. \tag{11}$$

This relation allows us to rewrite Eq. (10) as follows:

$$\frac{\sin\theta_k}{\sin\theta_{k+1}} = -\frac{n_{k+1}}{n_k}\frac{(d_{k+1}-\zeta_k)}{(d_k-\zeta_k)}\frac{\left(\frac{1}{d_{k+1}-\zeta_k} - \frac{1}{\overline{V_k C_k}}\right)}{\left(\frac{1}{d_k-\zeta_k} - \frac{1}{\overline{V_k C_k}}\right)}, \tag{12}$$

which is valid for each surface composing the system. Equation (12) is the contribution for the kth surface, but for the whole system, this law is presented as the product of a sequence of terms, written as

$$\frac{\sin\theta_0}{\sin\theta_N} = \frac{n_N}{n_0}M\mathcal{M}, \tag{13}$$

where

$$M = \left(\prod_{k=0}^{N-1} -\frac{n_k}{n_{k+1}} \frac{d_{k+1} - \zeta_k}{d_k - \zeta_k} \right), \tag{14}$$

is the magnification and

$$\mathcal{M}(\{\rho_k\}) = \prod_{k=0}^{N-1} \frac{n_{k+1}}{n_k} \left(\frac{\frac{1}{d_{k+1}-\zeta_k} - \frac{1}{\overline{V_k C_k}}}{\frac{1}{d_k - \zeta_k} - \frac{1}{\overline{V_k C_k}}} \right), \tag{15}$$

is the deviation of Eq. (13) from Eq. (7), and its value must be one to fulfill the Abbe sine condition.

Equation (13) is an explicit formulation of the Abbe sine condition in terms of $\overline{V_k C_k}$. It should be noted that for fixed parameters of the system, n_k, ζ_k, and d_k, Eq. (15) is not a constant for all the rays that go through the system, since $\overline{V_k C_k}$ depends on the set $\{\rho_k\}$ of each ray path.

Considering that the point I_k on each surface has coordinates (r_k, z_k) and the relation of the normal vector $\vec{N}_k = [(\vec{N}_k)_r, (\vec{N}_k)_z]$ with the derivatives of this coordinates with respect to ρ_k, written as

$$(\vec{N}_k)_z = -\frac{dr_k}{d\rho_k}, \quad (\vec{N}_k)_r = \frac{dz_k}{d\rho_k},$$

it is obtained from Fig. 1(c) the relationship,

$$\tan \beta_k = \frac{r_k}{(z_k - \zeta_k) - \overline{V_k C_k}} = \frac{(\vec{N}_k)_r}{(\vec{N}_k)_z}, \tag{16}$$

where β_k is the angle between $\vec{N}_k$ and $\hat{z}$. Thus, the following expression is derived:

$$\frac{1}{\overline{V_k C_k}} = \frac{\left(\frac{dz_k}{d\rho_k}\right)\left(\frac{dr_k}{d\rho_k}\right)^{-1}}{r_k + (z_k - \zeta_k)\left(\frac{dz_k}{d\rho_k}\right)\left(\frac{dr_k}{d\rho_k}\right)^{-1}}. \tag{17}$$

Calculating the derivatives of Eqs. (1) and (2) with respect to ρ_k and replacing in Eq. (17), we have

$$\frac{1}{\overline{V_k C_k}} = \frac{-\left(2S_k - G_k O_k^2\right) + 2S_k\sqrt{1 + \left(2S_k - G_k O_k^2\right)\rho_k^2}}{G_k O_k \sqrt{1 + \left(2S_k - G_k O_k^2\right)\rho_k^2}}. \tag{18}$$

Thus, by replacing Eq. (18) in Eq. (15), we obtain the explicit aplanatism condition for a system composed of N Cartesian surfaces, expressed as a function of the shape parameters of the system and $\{\rho_k\}$, written as

$$\mathcal{M}(\{\rho_k\}) = \prod_{k=0}^{N-1} \frac{n_{k+1}}{n_k} \frac{A_k - \left(2S_k - \frac{G_k O_k}{d_{k+1}-\zeta_k}\right)\sqrt{1 + A_k \rho_k^2}}{A_k - \left(2S_k - \frac{G_k O_k}{d_k - \zeta_k}\right)\sqrt{1 + A_k \rho_k^2}}, \tag{19}$$

with $A_k = 2S_k - G_k O_k^2$. This expression, as we mentioned above, is the deviation of systems composed of Cartesian surfaces from aplanatism. If its value is one for all the rays, such a system fulfills the Abbe sine condition.

Although Eq. (19) may seem complicated, the fact of being written in terms of the shape parameters, $(GOTS)_k$, makes its adaptation very simple to achieve aplanatism in a given system. This is also advantageous because this allows establishing mechanisms to vary the shapes of the surfaces composing the system until aplanatism is reached, guaranteeing the rigorous stigmatism.

For the system to be aplanatic, Eq. (19) must be a constant value determined by the axial ray. For such a ray, the set of values $\{\rho_k\}$ is zeros, and Eq. (19) is reduced to

$$\mathcal{M}(\{0\}) = \prod_{k=0}^{N-1} \frac{n_{k+1}}{n_k} \frac{O_k - \frac{1}{d_{k+1}-\zeta_k}}{O_k - \frac{1}{d_k - \zeta_k}}, \tag{20}$$

and it depends only on the paraxial curvature O_k shape parameter. Thus, by replacing Eq. (4) in Eq. (20), we obtaine that $\mathcal{M}(\{0\}) = 1$, which means that for the system to be aplanatic, Eq. (19) must be equal to 1, which leads Eq. (13) to be equal to Eq. (7). It is Eq. (19) that imposes the conditions on the shape parameters $(GOTS)_k$ so that an optical system composed of Cartesian surfaces is free of both spherical and coma aberrations.

Therefore, Eq. (19) constitutes the main contribution of this work, allowing us to propose a new way for the design of optical imaging systems, which we consider as the starting point of a new paradigm in optical systems design, where the Cartesian surfaces are the building blocks. A novelty of what is proposed here is that aplanatism can be achieved while maintaining rigorous stigmatism throughout the optimization process. Thus, in the optimization process, each surface maintains the condition of stigmatism, which is advantageous because there is no necessity to evaluate the spherical aberration affectation, resulting in less computational cost, and therefore, it is not necessary to limit the rays to a paraxial regime.

As an example, from an initial configuration of parameters resulting in a non-aplanatic SOL as shown in Fig. 2(a), we obtain an aplanatic SOL composed of six surfaces. The resulting aplanatic SOL is obtained optimizing the parameters d_1, d_2, d_3, d_4, d_5, n_1, n_3, n_5, ζ_1, ζ_2, ζ_3, and ζ_4 (degrees of freedom), which lead us to a set of shapes parameters for each surface (see Table 1). The criteria used to choose the best degrees of freedom is the *root mean square*(RMS) value of $\mathcal{M}(\{\rho_k\}) - 1$, given by

$$[\mathcal{M}(\{\rho_k\}) - 1]_{\mathrm{RMS}} = \sqrt{\frac{1}{N}\sum_i \left|[\mathcal{M}(\{\rho_k\}) - 1]_i\right|^2}, \tag{21}$$

where the index i indicates the ray that accesses the system. The closer this criteria approaches to zero, the more the system tends to an aplanatic optical system. As is shown in Fig. 2(a), the resulting system can produce an image of an extended object. The resulting system was obtained by the processing of 101 rays, for a physical aperture of 30 mm diameter, which returned a value of $[\mathcal{M}(\{\rho_k\}) - 1]_{\mathrm{RMS}} = 2.83786\mathrm{e} - 08$. Figure 2(b) shows the spot diagram of off-axis point objects with 2 mm and 1.4 mm height, placed at $z = -100$ mm, resulting in off-axis images with -1.29763 mm and 0.90834 mm height, respectively, placed in $z = 120$ mm. It is noticed that the bigger spot size is about 6 μm in the vertical direction and about 5 μm in the horizontal direction, indicating the presence of other affections that are considered as a subject for further work. This example results in a peak to valley of about 6 μm for a full-field of 2 mm as is shown in Fig. 2(c).

In conclusion, we have formulated an aplanatism condition for systems composed of N Cartesian surfaces. If a SOL

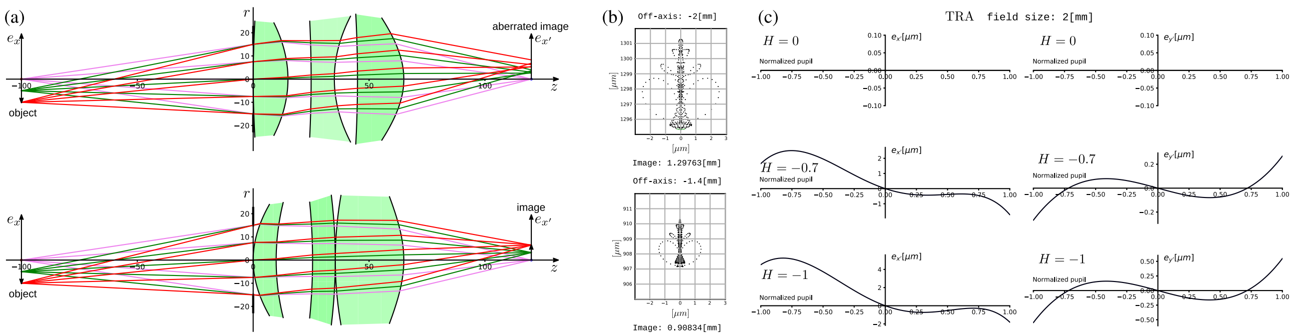

Fig. 2. (a) Initial non-aplanatic SOL (up), which, after optimization processes of the chosen degrees of freedom, we obtain an aplanatic SOL (down), through which is performed a ray-tracing technique for an extended object. The object is placed at $d_0 = -100$ mm, and the image at $d_6 = 120$ mm. In (b) are shown the spot diagrams of images, from off-axis point objects with 2 mm and 1.4 mm height, registered on a flat surface. In (c) are shown the transverse ray aberrations for on-axis point, 0.7 field, and full-field, considering the full-field an object with 2 mm height. The refractive indices of the object in image space are $n_6 = n_0 = 1.0$, and the other parameters of the resulting system are shown in Table 1.

Table 1. Parameters Resulting from Optimization Processes for the System Obtained in Fig. 2

k	ζ_k [mm]	n_{k+1}	G_k	O_k [mm^{-1}]	S_k [mm^{-2}]
0	0	1.71735	−1.73131	0.02504	−5.74018e − 5
1	9.9192	1.0	−0.02050	0.00970	1.67082e − 6
2	25.6283	1.78471	−1.52467	−0.00401	−1.68326e − 6
3	35.4319	1.0	−1.67370	−0.00438	−2.06442e − 6
4	35.4319	1.58912	−0.23011	0.00623	−6.75168e − 6
5	65		−3.64956	−0.02069	9.36804e − 5

satisfies this condition of aplanatism, Eq. (19), the fulfillment of the Abbe sine condition is guaranteed, and therefore the high-quality optical system design. Whereas standard aspherical surfaces need an infinite set of aspherical coefficients to achieve a system rigorously free from spherical aberration, whose limitation to a small number of coefficients penalize its performance, for these Cartesian surfaces systems, only three shape parameters by surface are required to achieve a system rigorously free from spherical and coma aberrations, that is, aplanatic, which offers advantages in a minimization scheme. Here we speak of rigorous aplanatism since this requires that the systems be rigorously stigmatic as the Descartes' ovoids that are the only systems that offer this property.

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See Supplement 1 for supporting content.

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