Honours Algebra - Week 5 - Equivalence Relations, The First Isomorphism Theorem & Modules

Antonio León Villares

February 2022

Contents

Based on the notes by Iain Gordon, Sections 3.5 - 3.7

1 Equivalence Relations

1.1 Defining Equivalence Relations

- **What is a relation?**
  - a **relation** R on a set X is a **subset**:
  $$R \subset X \times X$$
  - we describe an element $(x, y) \in R$ via:
  $$xRy$$
- **What is an equivalence relation?**
  - a **relation**, typically denoted $\sim$, satisfying:
    1. **Reflexivity**
    $$x \sim x$$
    2. **Symmetry**
    $$x \sim y \iff y \sim x$$
    3. **Transitivity**
    $$x \sim y \wedge y \sim z \implies x \sim z$$

1.1.1 Examples

- simple equivalence relations include:
$$x \sim y \iff x = y \qquad x \sim y \iff x^2 = y^2$$
The first one is more "restrictive", since in the second one tuples like $(x, -y)$ and $(-x, y)$ are allowed.
- **congruence modulo** m also defines an equivalence relation:
$$x \sim y \iff x \equiv y \ (mod\ m)$$
  - $x \equiv x \ (mod\ m)$ (reflexivity)
  - $x \equiv y \ (mod\ m) \iff y \equiv x \ (mod\ m)$ (symmetry)
  - $x \equiv y \ (mod\ m) \wedge y \equiv z \ (mod\ m) \implies x \equiv z \ (mod\ m)$ (transitivity)
- a more interesting example is that of matrix conjugacy:
$$A \sim B \iff \exists X : B = XAX^{-1}, \qquad A, X, B \in Mat(n; F)$$
  - $IAI^{-1} = A \implies A \sim A$
  - $B = XAX^{-1} \implies A = YBY^{-1}, \qquad Y = X^{-1}$
  - $B = XAX^{-1}, C = YBY^{-1} \implies C = ZAZ^{-1}, \qquad Z = YX$

  This relates to the notion of **similar matrices**, discussed in W2, whereby basis matrices were similar. That is, if $N = {}_B[f]_B$ and $N = {}_A[f]_A$, then N, M are similar in the sense that with $T = {}_A[id_V]_B$:
$$N = T^{-1}MT$$

1.1.2 Exercises (TODO)

1. **Show that the relation $\sim$ on $Mat(n \times m; F)$, defined by:**

$$A \sim B \iff \exists P \in GL(n;F), Q \in GL(m;F) : B = PAQ$$

   is an equivalence relation.

2. **Show that isomorphism is an equivalence relation on finite dimensional vector spaces over a field F.**

1.2 Equivalence Classes

- **What is an equivalence class?**
  - consider a set X with **equivalence relation** $\sim$
  - an **equivalence class** for $x \in E$ is a subset $E \subseteq X$ such that:

$$E(x) := \{z \mid x \sim z,\ x \in X\}$$

- **What is a representative of an equivalence class?**
  - an element $e \in E(x)$
- **What is a system of representatives?**
  - a subset $Z \subseteq X$
  - it contains excatly **one** element from each **equivalence class** $E(x), x \in X$
- **What are some properties of equivalence classes?**
  - the following notions are **equivalent**:
    1. $x \sim y$
    2. $E(x) = E(y)$ (this follows from (1) + symmetry)
    3. $E(x) \cap E(y) \neq \emptyset$ (this follows from (1) + reflexivity, which means that $x \in E(x), x \in E(y)$)

1.2.1 Examples

- if X is a set of students, with equivalence relation "same degree", each equivalence class contains all students which prusue the same degree
- if $X = \mathbb{R}$, then the equivalence relation:

$$x \sim y \iff x - y \in \mathbb{Z}$$

  has equivalence classes like:

$$E(1.2) = \{\dots, -2.8, -1.8, -0.8, 0.2, 1.2, 2.2, \dots\}$$

  More generally, if $z \in \mathbb{Z}$, the equivalence relation tells us that:

$$y \in E(x) \implies y - x = z \implies y = x + z \implies E(x) = \{x + z \mid x \in X, z \in \mathbb{Z}\}$$

  (here we have used symmetry)

- if we define:

$$x \sim y \iff x \equiv y \ (mod \ m)$$

then the equivalence classes are familiar:

$$E(x) = \bar{x} = \{x + mz \mid z \in \mathbb{Z}\}$$

Furthermore, if $m > 0$, a **system of representatives** will contain an element of each **equivalence class**, which can be:

$$\{0, 1, 2, \ldots, m-1\}$$

where $0 \in E(0), 1 \in E(1)$, and so on. However, in general, we can pick:

$$\{a, a+1, a+2, \ldots, a+m-1\}$$

where $a \in \mathbb{Z}$

1.2.2 Exercises (TODO)

1. **Show that the $n \times m$ matrices over a field F in Smith-Normal Form form a system of representatives for the equivalence relation:**

$$A \sim B \iff \exists P \in GL(n; F), Q \in GL(m; F) : B = PAQ$$

2. **Show that the set:**

$$\{F^n \mid n \in \mathbb{Z}_{\geq 0}\}$$

is a system of representatives for the equivalence relation defined by an isomorphism on finite dimensional vector spaces over a field F. Show that another system of representatives for this equivalence relation is:

$$\{F[X]_{<n} \mid n \in \mathbb{Z}_{\geq 0}\}$$

1.3 The Set of Equivalence Classes

- **What is the set of equivalence classes?**
  - let X be a set, with equivalence relation $\sim$
  - the **set of equivalence classes** is a **subset** of the **power set** $\mathcal{P}(X)$
  - it is the set containing all equivalence classes of X:

$$(X/\sim) := \{E(x) \mid x \in X\}$$

  - this is also known as the **quotient set**
- **What canonical mapping arises from this definition?**

> *A **canonical map** is a map or morphism between objects that arises naturally from the **definition** or the **construction** of the objects. In general, it is the map which preserves the widest amount of structure, and it tends to be unique.*

  - in this case, a **canonical map** is of the form:

$$can : X \to (X/\sim)$$

$$can(x) = E(x)$$

  - this is a **surjection**, since each equivalence class $E(x)$ contains at least one element in X (so "worst case", each $x \in X$ maps to a unique $E(x)$)

1.3.1 Examples: Canonical Mappings Preserving Structure (as Homomorphisms)

1. **Abelian Groups**
   - A is an **abelian group**; B is a **subgroup** of A
   - define an equivalence relation.
     $$x \sim y \iff x - y \in B$$
   - the equivalence classes are:
     $$y - x = b \in B \implies E(x) = \{x + b \mid b \in B\}$$
     (here w use the fact that the group is abelian, so $x + b = b + x$)
   - the **quotient set** is $A/B \equiv A/\sim$, an **abelian group**, defined by:
     $$E(x) + E(y) = E(x + y) = \{x + y + b \mid b \in B\}$$
   - the canonical mapping $can : A \to A/B$ is a **surjective homomorphism**, with kernel being B (since $0 \in A/B = \{0 + b \mid b \in B\} = B$, and any element $b \in B$ will get mapped to this set)
   - A/B is the **quotient abelian group** of A by the subgroup B
2. **Non-Abelian Group**
   - G is a **group**, and H is a **normal subgroup**: that is, if $h \in H$ and $g \in G$, then:
     $$ghg^{-1} \in H$$
   - define an equivalence relation:
     $$x \sim y \iff xy^{-1} \in H$$
   - the equivalence classes are given by the **left** and **right** cosets:
     $$E(x) = xH = Hx \subseteq G$$
     This is because if $xy^{-1} \in H$, by symmetry, $yx^{-1} = h \in H$, so $y = hx$, meaning that $E(x) = \{hx \mid h \in H\}$. the equivalence relation is defined by
     $$xy^{-1} \in H$$
     . Moreover, since G is a group, and H is a normal subgroup, we know that $g^{-1}hg \in H$. Thus, if $xy^{-1} \in H$, we must also have $y^{-1}(xy^{-1})y = y^{-1}x \in H$. Hence, $y^{-1} \sim x^{-1}$, and again by symmetry, $x^{-1} \sim y^{-1} \implies x^{-1}y \in H \implies y = xh \implies E(x) = xH$.
   - the quotient set $G/\sim \equiv G/H$ is the group with operations:
     $$E(x)E(y) = E(xy)$$
   - the canonical **surjective homomorphism** $can : G \to G/H$ has kernel H, since $hH = Hh = H$, so $\forall h \in H, can(h) = H$, and H is the identity element in G/H
   - this relates to **Lagrange's Theorem**, which states that:
     $$|G| = |G/H||H|$$
     which is proved by noting that each coset $E(x)$ has exactly $|H|$ elements (since it is given by $xH = Hx$), and that G is the disjoint union of $|G/H|$ cosets (the union is disjoint because otherwise we'd have elements belonging to more than 1 equivalence classes; there are $|G/H|$ cosets because G/H is the set of all cosets).

-
- G/H is the **quotient group** of G by the normal subgroup H

3. **F-Vector Spaces**
   - let V be a **F-Vector Space**, with subspace W
   - define an equivalence relation:
   $$x \sim y \iff x - y \in W$$
   - as in the first case, the equivalence classes are:
   $$y - x = w \implies E(x) = \{x + w \mid w \in W\} = x + W$$
   - the **quotient set** is an **F-Vector Space** with:
   $$\lambda E(x) = E(\lambda x)$$
   - the canonical **surjective homomorphism** $can : V \to V/W$ has kernel W (same reason as in case (1))
   - by the exercise below, we can show that if V is finite-dimensional:
   $$dim(V/W) = dim(V) - dim(W)$$
   - V/W is the **quotient vector space** of V by the subspace W

1.3.2 Examples

- if $\sim$ defines the congruence equivalence relation, modulo m, then:
$$(\mathbb{Z}/\sim) = \mathbb{Z}_m$$
This is easy to see, since as discussed above, the equivalence classes of $\sim$ are the sets $\bar{0}, \bar{1}, \ldots, \overline{m-1}$, and the set of all these elements is precisely $\mathbb{Z}_m$

1.3.3 Exercises (TODO)

1. **Let $R = F$ be a field, V and F-vector space, and $W \subseteq V$ a subspace of V. The quotient V/W is the *quotient vector space*, and $can : V \to V/W$ is a linear mapping. Assume that $dimV = m < \infty$. By the *Dimension Estimate for Vector Subspaces*, $dimW = n \leq m$. Let:**
$$\{\underline{v}_1, \ldots, \underline{v}_n\}$$
be a basis for W. Using the *Steinitz Exchange Theorem*, we can extend it to a basis of V:
$$\{\underline{v}_1, \ldots, \underline{v}_n, \underline{v}_{n+1}, \ldots, \underline{v}_m\}$$
Show that:
$$\{\underline{v}_{n+1} + W, \ldots, \underline{v}_m + W\}$$
is a basis for the vector space V/W. Hence, deduce that:
$$dim(V/W) = dimV - dimW$$

1.4 Remark: A Very Important Remark At That

Consider $\sim$ as an equivalence relation on X, and let $f : X \to Z$ be a mapping, such that:

$$x \sim y \implies f(x) = f(y)$$

*In other words, whatever the equivalence relation is, it is such that all elements in the same **equivalence class** are mapped to the same value under f*

*Then, there exists a **unique** mapping:*

$$\bar{f} : (X/\sim) \to Z$$

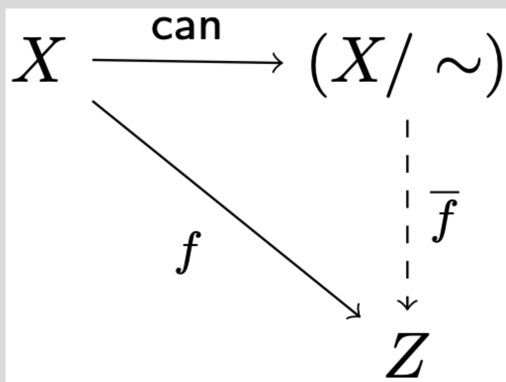
This mapping is simple to define:

$$\bar{f}(E(x)) = f(x)$$

such that:

$$f = \bar{f} \circ can$$

This can be summarised by the following diagram:



*This is known as the **universal property of the set of equivalence classes***

A more interesting case occurs when

$$f : X \to Z$$

is ***any*** *mapping, and we define:*

$$x \sim y \iff f(x) = f(y)$$

Then, we will have that:

$$\bar{f} : (X/\sim) \to imf$$

is a ***bijection****. This bijection is a prelude to the* ***First Isomorphism Theorem****.*

1.5 A Well-Defined Mapping

- **When is a mapping well-defined?**
  - consider a mapping:
  $$g : (X/\sim) \to Z$$
  - g is **well-defined** if there exists a mapping:
  $$f : X \to Z$$
  such that:
  $$x \sim y \implies f(x) = f(y)$$
  - here we recognise $g = \bar{f}$
- **Why are well-defined mappings important?**
  - they **solidify** the notion of **equivalence**
  - they ensure that elements in the **same equivalence class** are mapped to the **same** value
  - this means that equivalent elements in X are the same in Z
  - more on this in Proof-Wiki

1.5.1 Examples

- recall the equivalence relation:
  $$a \sim b \iff a - b \in \mathbb{Z}$$
  with equivalence classes:
  $$E(a) = \{\dots, a-2, a-1, a, a+1, a+2, \dots\}$$
  Further consider:
  $$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \cos(x)$$
  $$g : \mathbb{R} \to \mathbb{R} \qquad f(x) = \cos(2\pi x)$$
  Then, f is **not** well-defined, since:
  $$0 \sim 1 \qquad f(0) = 1 \neq f(1)$$

However, g **is** well-defined. Indeed:

$$a \sim b \implies a = b + z, z \in \mathbb{Z}$$

So:

$$g(a) = \cos(2\pi a) = \cos(2\pi b + 2\pi z) = \cos(2\pi b) = g(b)$$

where we exploit the fact that cos is 2π periodic. Moreover, we can define:

$$\bar{g} : (\mathbb{R}/\sim) \to \mathbb{R}$$

via:

$$\bar{g}(E(a)) = g(a) = cos(2\pi a)$$

1.5.2 Exercises (TODO)

1. **Define a relation $\sim$ on $\mathbb{X} \times \mathbb{N}$ by:**

$$(x, y) \sim (a, b) \iff x + b = y + a$$

   (a) **Show that $\sim$ is an equivalence relation**

   (b) **Let $\bar{\mathbb{N}} = (\mathbb{N} \times \mathbb{N}/\sim)$. Show that addition on $\mathbb{N}$ induces a *well-defined* addition on $\bar{\mathbb{N}}$**

   (c) **Show that with this addition, $\bar{\mathbb{N}}$ is an abelian group**

   (d) **Show that**

$$nat : \mathbb{N} \to \bar{\mathbb{N}}$$

   is an additive mapping, where:

$$nat(a) = E((a + n, n)), \forall n \in \mathbb{N}$$

   That is:

$$nat(a + b) = nat(a) + nat(b)$$

   (e) Show that $\bar{\mathbb{N}}$ is *isomorphic* as a group to $(\mathbb{Z}, +)$

2 Factor Rings

2.1 Motivation 1: Equivalence Relations From Kernels

We just showed that ***mappings between sets*** *generate* ***equivalence relations****. In particular, consider a* ***ring homomorphism****:*

$$f : R \to S$$

We can define an ***equivalence relation*** *on R:*

$$x \sim y \iff f(x) = f(y)$$

By properties of homomorphism:

$$\begin{aligned} x \sim y &\iff f(x) = f(y) \\ &\iff f(x) - f(y) = 0_S \\ &\iff f(x - y) = 0_S \\ &\iff x - y \in ker(f) \end{aligned}$$

This then defines the ***equivalence classes*** *via:*

$$y - x = k \in ker(f) \implies y = x + k$$

so in particular:

$$E(x) = x + ker(f) = \{x + k \mid k \in ker(f)\}$$

2.2 Motivation 2: Equivalence Relations From Ideals

In fact, all the above generalises easily to **ideals**:

If I is an ideal of a ring R, and $f : R \to S$, then the following is an equivalence relation:

$$r_1 \sim r_2 \iff r_1 - r_2 \in I$$

1. $r_1 - r_1 = 0_R \in I \iff r_1 \sim r_1$ *(since 0 is always part of an ideal). Hence,* ***reflexivity*** *holds.*
2. *if $r_1 \sim r_2$, then:*

$$r_1 - r_2 \in I$$

Since ideals are closed under substraction, it follows that:

$$-(r_1 - r_2) = r_2 - r_1 \in I$$

so we have $r_2 \sim r_1$. Thus, ***symmetry*** *holds.*

3. *if $r_1 \sim r_2$ and $r_2 \sim r_3$, then:*

$$r_1 - r_2 \in I$$

$$r_2 - r_3 \in I$$

Ideals are closed under substraction/addition, so:

$$(r_1 - r_2) + (r_2 - r_3) = r_1 - r_3 \in I$$

so we have $r_1 \sim r_3$. Thus, ***transitivity*** *holds.*

As above, the equivalence classes then become:

$$E(r_1) = r_1 + I = \{r_1 + i \mid i \in I\}$$

and the ***quotient of*** *R* ***by*** *I (the set of* ***all equivalence classes****) is:*

$$R/I$$

We have constructed R/I. We now go back, and discover that it is a ring.

2.3 Motivation 3: Quotients From Ideals are Rings

*R/I is the set of all equivalence classes, constructed from the **equivalence relation**:*

$$r_1 \sim r_2 \iff r_1 - r_2 \in I$$

But if we think about it, this equivalence relation was originally defined as:

$$r_1 \sim r_2 \iff f(r_1) = f(r_2)$$

But now recall, such an equivalence relation lead to the following bijection:

$$\bar{f} : (R/I) \to im(f)$$

$$\bar{f}(E(r)) = f(r)$$

*The existence of this bijection tells us that, since $im(f)$ is a **subring** of S, we should expect that R/I should also have a ring-like structure, since we have a one-to-one correspondance between elements in R/S and a subring (in fact, if $\bar{f}$ is an **isomorphism**, R/S would indeed be **isomorphic** to the subring $im(f)$).*
*So if we have R/I as a ring, we better endow it with **addition** and **multiplication**:*

$$E(r_1) + E(r_2) = E(r_1 + r_2)$$

$$E(r_1 r_2) = E(r_1)E(r_2)$$

*This section focuses on formalising the notion of R/I as a **factor ring**, defines the **Universal Property of Factor Rings** results, and has a grand finale in the **First Isomorphism Theorem**.*

2.4 Cosets of Rings

- **What is a coset of a ring?**
  - let R be a ring, and I an ideal of R
  - the **coset of** x **with respect to** I **in** R is the subset of R:

$$x + I = \{x + i \mid i \in I\}$$

  - cosets in **rings** are special cases of cosets in **groups**

- **Are cosets equivalence classes?**
  - as we saw above,

$$x \sim y \iff x - y \in I$$

    defines an equivalence class:

$$E(x) = x + I$$

  - so **cosets** are **equivalence classes**

- **Given 2 cosets, how can they be related?**
  - we saw that if $x \sim y$, then:

$$E(x) = E(y) \qquad E(x) \cap E(y) \neq \emptyset$$

  - hence, depending on whether $x \sim y$, 2 cosets $x + I$ and $y + I$ are related in one of 2 ways:
    * $x + I = y + I$
    * or $(x + I) \cap (y + I) = \emptyset$

2.5 Defining the Factor Ring

- **What is a factor ring?**
  - let:
    * R be a ring
    * I be an ideal of R
    * $\sim$ the equivalence relation on R:

$$x \sim y \iff x - y \in I$$

- the **factor ring of** R **by** I is nothing but the **quotient of** R **by** I
- hence, the factor ring is R/I:
  - the set of **equivalence classes** under $\sim$
  - the set of **cosets** of I in R

2.6 Theorem: Operations on Factor Rings

For the **factor ring** to be a ring, we need to provide ring operations.

> *Let R be a **ring**, and I an **ideal**.*
> *Then, R/I is a **ring**, with **addition** defined as:*
>
> $$(x+I)\dot{+}(y+I) = (x+y)+I, \forall x, y \in R$$
>
> *and **multiplication** defined as:*
>
> $$(x+I)\cdot(y+I) = xy+I, \forall x, y \in R$$
>
> *[Theorem 3.6.4]*

For the proof we need to be careful, and show that:

- R/I is an **abelian** group under **addition**
- addition is **well-defined**
- R/I is a **monoid** under **multiplication**
- multiplication is **well-defined**
- R/I satsifies the **distributive axioms**

This proves that R/I is a ring. It is important to emphasise the need to show that the operations are **well-defined**:

> *Consider $R = \mathbb{Z}$ and $I = 15\mathbb{Z} = \{15z \mid z \in \mathbb{Z}\}$. The **equivalence classes**, as discussed, are of the form:*
>
> $$E(r) = \{r + 15z \mid z \in \mathbb{Z}\}$$
>
> *The factor ring is our well known:*
>
> $$\mathbb{Z}/15\mathbb{Z} = \mathbb{Z}_{15}$$
>
> *Now, consider the following products:*
>
> $$E(7)\cdot E(9) = E(63) \qquad E(22)\cdot E(9) = E(198)$$
>
> *Now, we know that 22 and 7 are congruent modulo 15, so obviously $E(7) = E(22)$ (they are the same equivalence class). The question now becomes: are $E(63)$ and $E(198)$ congruent modulo 15? How can we be sure? This is the importance of having **well-defined** operations: we are working over **equivalence classes**, so we need to ensure that any arithmetic we do doesn't depend of our choice of **representative** of the equivalence class.*
> *In this example, any arithmetic we do shouldn't depend on the number we choose (i.e 7 and 22), but rather the **remainder** when dividing by 15.*

Proof. 1. **Addition is Well-Defined**

To show that addition is well-defined, we need to show that if $x, x', y, y' \in R$ and:

$$E(x) = E(x') \qquad E(y) = E(y')$$

(we use $E(x)$ instead of $x + I$ for ease of reading and writing) then:

$$E(x) + E(y) = E(x') + E(y')$$

Notice, by how addition is defined, this is equivalent to showing that:

$$E(x + y) = E(x' + y')$$

which is equivalent to showing that the two are the **same equivalence class**. In other words, we need to show that:

$$(x + y) \sim (x' + y') \implies (x + y) - (x' + y') \in I$$

We consider:

$$(x + y) - (x' + y') = (x - x') + (y - y')$$

By assumption, $E(x) = E(x')$, so $x \sim x'$, so $x - x' \in I$. Similarly, $y - y' \in I$. Since ideals are closed under addition/subtraction, it is clear that:

$$(x - x') + (y - y') \in I$$

Hence, addition is well-defined.

2. R/I **is an Abelian Group Under Addition**

(a) **Existence of Identity**

$$E(0) + E(x) = E(0 + x) = E(x) = E(x + 0) = E(x) + E(0)$$

(b) **Existence of Inverse**

$$E(-x) + E(x) = E(-x + x) = E(0) = E(x - x) = E(x) + E(-x)$$

(c) **Associativity**

$$(E(x)+E(y))+E(z) = E(x+y)+E(z) = E(x+y+z) = E(x)+E(y+z) = E(x)+(E(y)+E(z))$$

(d) **Closure** This follows directly from the definition of addition.

(e) **Abelian**

$$E(x) + E(y) = E(x + y) = E(y + x) = E(y) + E(x)$$

(using commutativity of R under addition)

Hence, R/I is an abealian group under addition.

3. **Multiplication is Well-Defined** Again ,consider x, x', y, y' with:

$$E(x) = E(x') \qquad E(y) = E(y')$$

we need to show that:

$$E(x)E(y) = E(x')E(y')$$

Since $E(x) = E(x')$, we know that:

$$x \sim x' \implies x - x' \in I \implies x = x' + i, i \in I$$

Similarly:

$$y = y' + j, j \in I$$

Hence:

$$\begin{aligned} E(x)E(y) &= E(xy) \\ &= E((x'+i)(y'+j)) \\ &= E(x'y' + x'j + iy' + ij) \\ &= E(x'y') + E(x'j) + E(iy') + E(ij) \end{aligned}$$

Now, notice that:

$$E(x'j) = E(iy') = E(ij) = E(0)$$

Since $ij, iy', x'j \in I$ (since multiplying elements in R by elements in I produces elements in I), then:

$$ij - 0 \in I \qquad iy' - 0 \in I \qquad x'j - 0 \in I$$

So $ij \sim iy' \sim x'j \sim 0$, from which $E(x'j) = E(iy') = E(ij) = E(0)$ follows. Hence, we have shown that:

$$E(x)E(y) = E(x'y') = E(x')E(y')$$

so multiplication is well-defined.

4. R/I **is a Monoid Under Multiplication**

   (a) **Closure** This follows directly from the definition of multiplication.

   (b) **Associativity**

   $$(E(x)E(y))E(z) = E(xy)E(z) = E(xyz) = E(x)E(yz) = E(x)(E(y)E(z))$$

   (c) **Existence of Identity**

   $$E(x)E(1) = E(x \cdot 1) = E(x) = E(1 \cdot x) = E(1)E(x)$$

   Hence, R/I is a monoid under multiplication.

5. R/I **Satisfies Distributivity**

   $$E(x)(E(y)+E(z)) = E(x)E(y+z) = E(x(y+z)) = E(xy+xz) = E(xy)+E(xz) = E(x)E(y)+E(x)E(z)$$

Hence, by all of the above, R/I is a ring, with **well-defined** operations.

□

2.6.1 Examples

- this is another way of seeing that $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m$ is a **ring**
- consider the ring $R = \mathbb{F}_2[X]$ (the ring of polynomials with coefficients 0 or 1), and the following 2 ideals:

  $$I = {}_R\langle X^2 \rangle = \{pX^2 \mid p \in R\}$$
  $$J = {}_R\langle X^2 + X + 1 \rangle = \{p(X^2 + X + 1) \mid p \in R\}$$

  How do we describe R/I and R/J?

  1. **Elements in the Factor Rings**

(a) **Elements in** R/I For $p \in R$, We denote:

$$E_I(p) = p + I$$

Then, we can consider the equivalence classes, given by the relation:

$$p_1 \sim p_2 \iff p_1 - p_2 \in I$$

that is $p_1 \sim p_2$ if $p_1 - p_2$ has X^2 as a factor. Consider the constant polynomials first (only 2 of them)

$$E_I(0) = \{i \mid i \in I\} \qquad E_I(1) = \{1 + i \mid i \in I\}$$

Clearly, $0 \not\sim 1$, since $0 - 1 = 1$ (we operate in $\mathbb{F}_2$) which is not divisible by X^2. Hence, $E_I(0)$ and $E_I(1)$ are different equivalence classes. Now considering linear polynomials:

$$E_I(X) = \{X + i \mid i \in I\} \qquad E_I(X+1) = \{X + 1 + i \mid i \in I\}$$

Then:

- $X - 0 \notin I$
- $X - 1 \notin I$
- $(X + 1) - 0 \notin I$
- $(X + 1) - 1 \notin I$
- $(X + 1) - X \notin I$

Hence, we have 2 more equivalence classes:

$$E_I(X) \qquad E_I(X+1)$$

Now, consider a polynomial $p \in R$ with $deg(p) \geq 2$. We can factorise it as:

$$p(X) = q(X)(X^2) + r(X)$$

such that $deg(r) < deg(X^2) = 2$. Then notice that:

$$p - r = qX^2 \in I \implies p \sim r$$

Thus, any polynomial is in the equivalence class of its remainder. Since the remainder r has degree 0 or 1, it means that p is in one of $E_I(0), E_I(1), E_I(X), E_I(X+1)$. Thus:

$$(R/I) = \{E_I(0), E_I(1), E_I(X), E_I(X+1)\}$$

(b) **Elements in** R/J Working in a similar, we will see that:

$$(R/J) = \{E_J(0), E_J(1), E_J(X), E_J(X+1)\}$$

2. **Behaviour of Elements** We have reduced the elements of R/I to a set of 4 elements. We now need to see hwo they interact with each other througha multiplication table:

	$E_I(0)$	$E_I(1)$	$E_I(X)$	$E_I(X+1)$
$E_I(0)$	$E_I(0 \cdot 0) = E_I(0)$	$E_I(0 \cdot 1) = E_I(0)$	$E_I(0 \cdot X) = E_I(0)$	$E_I(0 \cdot (X+1)) = E_I(0)$
$E_I(1)$	$E_I(1 \cdot 0) = E_I(0)$	$E_I(1 \cdot 1) = E_I(1)$	$E_I(1 \cdot X) = E_I(X)$	$E_I(1 \cdot (X+1)) = E_I(X+1)$
$E_I(X)$	$E_I(X \cdot 0) = E_I(0)$	$E_I(X \cdot 1) = E_I(X)$	$E_I(X^2) = E_I(0)$	$E_I(X^2 + X) = E_I(X)$
$E_I(X+1)$	$E_I((X+1) \cdot 0) = E_I(0)$	$E_I((X+1) \cdot 1) = E_I(X+1)$	$E_I(X^2 + X) = E_I(X)$	$E_I(X^2 + 2X + 1) = E_I(1)$

Table 1: Here we use facts like $X^2 \sim 0$ and $X^2 + X \sim X$ to simplify. Also, don't forget that $2 = 0$ in $\mathbb{F}_2$.

	$E_J(0)$	$E_J(1)$	$E_J(X)$	$E_J(X+1)$
$E_J(0)$	$E_J(0 \cdot 0) = E_J(0)$	$E_J(0 \cdot 1) = E_J(0)$	$E_J(0 \cdot X) = E_J(0)$	$E_J(0 \cdot (X+1)) = E_J(0)$
$E_J(1)$	$E_J(1 \cdot 0) = E_J(0)$	$E_J(1 \cdot 1) = E_J(1)$	$E_J(1 \cdot X) = E_J(X)$	$E_J(1 \cdot (X+1)) = E_J(X+1)$
$E_J(X)$	$E_J(X \cdot 0) = E_J(0)$	$E_J(X \cdot 1) = E_J(X)$	$E_J(X^2) = E_J(X+1)$	$E_J(X^2+X) = E_J(1)$
$E_J(X+1)$	$E_J((X+1) \cdot 0) = E_J(0)$	$E_J((X+1) \cdot 1) = E_J(X+1)$	$E_J(X^2+X) = E_J(1)$	$E_J(X^2+2X+1) = E_J(X)$

Table 2: Here we use facts like $X^2 \sim X+1$ (since $X^2-(X+1) = X^2-X-1 = X^2+X+1 \in J$ and $X^2+X \sim 1$ (since $X^2+X-1 = X^2+X+1 \in J$) to simplify. Also, don't forget that $2=0$ in $\mathbb{F}_2$.

> Now notice: in R/J every non-zero element has an inverse, so R/J is a **field** with 4 elements. On the other hand, R/I is **not**, since it has a **zero-divisor** (for example $E_I(X)$).

2.6.2 Exercises (TODO)

1. **Let R be a ring, and I an ideal of R. Show that if R is commutative, then so is R/I.**
2. **Let R be a ring, and I an ideal of R. Show that R/I is a non-zero ring if and only if $I \neq R$.**
3. **Let R be a ring, and I a *proper* ideal of R (so $I \neq R$. Show that if $r \in R^\times$, then $E(r) \in (R/I)^\times$, and $(E(r))^{-1} = E(r^{-1})$.**

2.7 Theorem: The Universal Property of Factor Rings

In the **Very Important Remark** (1.4) (the **universal property of the set of equivalence classes**, we showed how there are 2 ways to go between sets X, Z: one that is direct ($f : X \to Z$) and one that is indirect ($g : X \to X/\sim \to Z$, with $g = \bar{g} \circ can$, where $\bar{g}$ is a unique mapping $\bar{g}(E(r)) = f(r)$). This theorem considers the same case, but adapted to **rings**.

*Let R be a **ring**, and I an **ideal** of R.*

1. *The **canonical mapping**:*

$$can : R \to (R/I) \qquad can(r) = E(r), \forall r \in R$$

*is a **surjective ring homomorphism**, with **kernel**:*

$$ker(can) = I$$

2. *If:*

$$f : R \to S$$

*is a **ring homomorphism** and:*

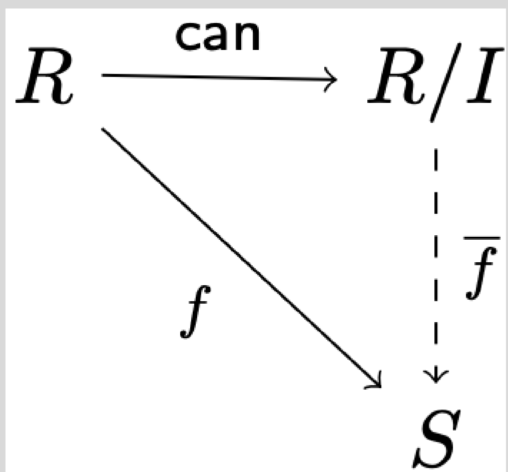
$$f(I) = \{0_S\}$$

*so that $I \subseteq ker(f)$, then there is a **unique ring homomorphism**:*

$$\bar{f} : (R/I) \to S$$

such that:

$$f = \bar{f} \circ can$$



[Theorem 3.6.7]

Proof. 1. **The Canonical Mapping is a Surjective Ring Homomorphism With Kernel I**

(a) **Surjective Mapping** This easy to see. Any $E(r)$ is produced by at least one element in R. By the pigeonhold principle, every possible $E(r)$ must be mapped to by some element in R.

(b) **The Kernel is I** If $i \in I$, then by properties of ideals:

$$i - 0_R \in I \implies i \sim 0 \implies E(i) = E(0) = 0_{R/I}$$

Any other $i \notin I$ won't have an equivalence relation with 0_R. Hence, $ker(can) = I$.

(c) **The Mapping is a Ring Homomorphism** This follows from how **addition** and **multiplication** are defined in the **factor ring** R/I:

$$can(x) + can(y) = E(x) + E(y) = E(x + y) = can(x + y)$$
$$can(x)can(y) = E(x)E(y) = E(xy) = can(xy)$$

2. **Existence of Unique Ring Homomorphism $\bar{f}$**

(a) **Existence of Unique Mapping $\bar{f}$** Since $f(I) = \{0_S\}$, then:

$$f(E(x)) = \{f(x + i) \mid i \in I\} = \{f(x) + f(i) \mid i \in I\} = \{f(x)\}$$

Define:

$$\bar{f}(E(x)) = f(x)$$

such that:

$$f(E(x)) = \{\bar{f}(E(x))\}$$

Then, $\bar{f}$ is the only mapping satisfying $f = \bar{f} \circ can$.

(b) **$\bar{f}$ is a Ring Homomorphism**

$$\bar{f}(E(x) + E(y)) = \bar{f}(E(x + y)) = f(x + y) = f(x) + f(y) = \bar{f}(E(x)) + \bar{f}(E(y))$$
$$\bar{f}(E(x)E(y)) = \bar{f}(E(xy)) = f(xy) = f(x)f(y) = \bar{f}(E(x))\bar{f}(E(y))$$

□

2.8 Theorem: First Isomorphism Theorem for Rings

Let R and S be ***rings****.*
Then, every ***ring homomorphism****:*

$$f : R \to S$$

induces a ***ring isomorphism****:*

$$\bar{f} : (R/ker(f)) \to im(f)$$

This ***isomorphism*** *is nothing but:*

$$\bar{f}(r + ker(f)) = f(r)$$

[Theorem 3.6.9]

Proof. Notice, $ker(f)$ is an ideal, so R/I is a ring; similarly, $im(f)$ is a subring, so a **ring**. Moreover, by definition of the kernel, we must have that $f(ker(f)) = \{0_S\}$. Hence, by the **universal property of factor rings**, we have that $\bar{f}$ is a homomorphism.

Clearly, it is also **surjective** (each $f(r) \in im(f)$ is produced by at least one element in each equivalence class $r + ker(f)$).

Moreover, $ker(\bar{f}) = 0 + ker(f) = ker(f)$ (recall $0 + ker(f)$ is nothing but $E(0)$. If $E(r) = ker(f)$, clearly $\bar{f}(E(r)) = f(r) = 0_S$, by definition of the kernel. No other equivalence class achieves this. Hence, since the kernel only contains the additive identity, the homomorphism must be **injective**.

Thus, $\bar{f}$ is a bijective homomorphism - an isomorphism.

□

2.8.1 Examples

- if $R = \mathbb{R}[X]$ and $I = {}_R\langle X^2 + 1\rangle$ (the **ideal** generated by $X^2 + 1$, or in other words, the set of all polynomials with $X^2 + 1$ as a factor), then R/I is not only a **ring**, but it is **isomorphic** to the **complex numbers**
  - we can **factorise** polynomials uniquely ($P = AQ + B$, with $P, Q \in R$, $deg(B) < deg(Q)$)
  - in particular, we can write $P \in R$ as:
    $$P = A(X^2 + 1) + B$$
  - since $Q = X^2 + 1$, and $deg(B) < deg(Q)$ we must have:
    $$B = a + bX, \qquad a, b \in \mathbb{R}$$
  - now consider the **evaluation homomorphism**:
    $$f : \mathbb{R}[X] \to \mathbb{C}$$
    defined by evaluation $P \in \mathbb{R}[X]$ at $\sqrt{-1}$
  - clearly, $f(P) = f(B)$, since $\sqrt{-1}$ is a root of $X^2 + 1$, so:
    $$f(P) = f(B) = a + b\sqrt{-1}$$
  - clearly, f is surjective
  - moreover, $P \in ker(f)$ **if and only if** $a = b = 0$, which in particular means that:
    $$ker(f) = {}_{\mathbb{R}}[X]\langle X^2 + 1\rangle$$
  - by the **first isomorphism theorem for rings**, we thus have an **isomorphism**:
    $$\bar{f} : (\mathbb{R}[X]/_{\mathbb{R}[X]}\langle X^2 + 1\rangle) \to \mathbb{C}$$

3 Modules

Just as **rings** are the generalisation of fields, we introduct **modules** as the generalisation of **vector spaces**.

3.1 Defining Modules

- **What is a left module?**
  - a **left module** is defined over **rings**
  - it consists of an **abelian group**:
    $$M = (M, \dot{+})$$
    armed with a mapping:
    $$R \times M \to M$$
    $$(r, a) \to r \cdot a$$
  - **left modules** must satisfy:
    $$r(a \dot{+} b) = (ra) \dot{+} (rb)$$
    $$(r + s)a = (ra) \dot{+} (sa)$$
    $$r(sa) = (rs)a$$
    $$1_R a = a$$

- **What is an R-Module?**
  - a module defined over the **ring** R
- **How do right modules differ from left modules?**
  - a module in which multiplication by rings is defined via:

$$(r, a) \to a \cdot r$$

- **What is the trivial module?**
  - the singleton $\{0\}$ for any ring R
- **What is a direct sum?**
  - given R-modules:

$$M_1, M_2, \ldots, M_n$$

    their **cartesian product**:

$$M_1 \times M_2 \times \ldots \times M_n$$

    alongside:

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$
$$r(a_1, \ldots, a_n) = (ra_1, \ldots, ra_n)$$

    is an R-module
  - denoted:

$$M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

    is the **direct sum**
- **How do R-Modules and F-Vector Spaces differ?**
  - since modules are defined over **rings**, multiplication by R might **not** have inverses defined
  - this means that if for example:

$$rm = 0$$

    we can't assume that $r = 0$ or $m = 0$. For example, if:

$$R = \mathbb{Z}, \qquad (M, +) = (\mathbb{Z}_4, +)$$

    then:

$$2 \cdot \bar{2} = \bar{4} = \bar{0}$$

  - amongst other things, this means that the notion of **linear independence** no longer makes sense in **modules**, since **linear combinations** can be 0, with not all ring scalars being 0

3.1.1 Examples

- F-vector spaces are just R-modules in which the **ring** R is a field F
- $\mathbb{Z}$-modules are **abelian groups**. Indeed, any abelian group M is a $\mathbb{Z}$-module.
- if I is an ideal of a ring R, then I is an R-module, under multiplication in the ring. In fact, R is an R-module.
  - for example, $\mathbb{Z}$ and $\mathbb{Z}_6$ are both modules
  - ideals exploit the fact that if an element of $r \in R$ multiplies $i \in R$, then $ir, ri \in I$

3.1.2 Exercises (TODO)

1. **Let S be a ring, and let $R = Mat(n; S)$. Let $M = S^n$. Show that M is an R-module under the operations of componentwise addition and amtrix multiplication.**

2. **Let V be an F-vector space for some field F and let $\phi \in End(V)$ be an endomorphism of V. Show that V is an $F[X]$-module under the operation:**

$$\left(\sum_{i=0}^{m} a_i X\right) \underline{v} = \sum_{i=0}^{m} a_i \phi^m(\underline{v})$$

   For better understanding, we are multiplying a vector $\underline{v}$ by a polynomial with coefficients in a field. This multiplication then needs to result in a vectors in the vector space. We denote the $F[X]$-module via V_ϕ.

   As an example for this, consider:

$$R = \mathbb{C}[X] \qquad (M, +) = \mathbb{C}^n$$

   We can define the endomorphism ϕ as:

$$\phi(\underline{v}) = A\underline{v} \qquad \underline{v} \in m, A \in Mat(n; \mathbb{C})$$

   and then define ring multiplication as:

$$p(X)\underline{v} := p(A)\underline{v}, \qquad p(X) \in R$$

   As a concrete example, define:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   and

$$q(X) = X^2 + 2X + 3$$

   Then:

$$q(A) = A^2 + 2A + 3I = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$$

   Such that:

$$q(X) \cdot (0\ 1)^T = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

3.2 Lemma: Module Hygiene

Let R be a ring, and M an R-module.

1. $0_R \cdot a = 0_M, \qquad \forall a \in M$
2. $r0_M = 0_M, \qquad \forall r \in R$
3. $(-r)a = r(-a) = -(ra), \qquad \forall r \in R, a \in M$

3.3 Module Homomorphisms

- **What is a module homomorphism?**

- let R be a ring, and let M, N be R-modules
- a R-**homomorphism** is a mapping:

$$f : M \to N$$

satisfying:

$$f(a + b) = f(a) + f(b)$$
$$f(ra) = rf(a)$$

- **What results from composing module homomorphisms?**
  - you obtain another **homomorphism**
- **What is an R-Module Isomorphism?**
  - a **bijective** homomorphism
- **What is the kernel of an R-homomorphism?**
  - the set:

$$ker(f) = \{a \in M \mid f(a) = 0_N\} \subseteq M$$

- **What is the image of an R-homomorphism?**
  - the set:

$$im(f) = \{f(a) \mid a \in M\} \subseteq N$$

3.3.1 Examples

- the mapping $f(a) = 0_N$ is **always** an R-homomorphism
- if R is a field, module homomorphism are the standard **linear mappings**
- any **group** homomorphism between abelian groups is also a $\mathbb{Z}$-homomorphism
- consider $R = \mathbb{C}[X]$, and consider 2 modules:

$$M = \mathbb{C}^2_A \qquad N = \mathbb{C}^2_B$$

where:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$$

We can then define a homomorphism:

$$f : \mathbb{C}^2_A \to \mathbb{C}^2_B$$

defined by:

$$\underline{v} \to T\underline{v}, \qquad \underline{v} \in M, T \in Mat(2; \mathbb{C})$$

Recall, multiplication in the module is defined by replacing each X in the polynomial in $\mathbb{C}[X]$ by the given matrix (A or B). Hence, the homomorphism is defined by:

$$f(X\underline{v}) = f(A\underline{v}) = T(A\underline{v})$$

But notice, this must be an element in N. By properties of homomorphisms:

$$f(X\underline{v}) = Xf(\underline{v}) = B(T\underline{v})$$

Hence, if T exists, it must satisfy:

$$TA = BT$$

Let:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then:

$$AT = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$$

$$TB = \begin{pmatrix} 2a - c & 2b - d \\ 4a - 2c & 4b - 2d \end{pmatrix}$$

Hence, we have 4 variables, and 4 sets of linear equations:

$$2a - c = 0 \implies c = 2a$$
$$4a - 2c = 0 \implies c = 2a$$
$$4b - 2d = c$$
$$2b - d = a$$

Notice, the last 2 equations coincide with the fact that $c = 2a$. Overall, this system has infinitely many solutions, such that:

$$T = \begin{pmatrix} 2b - d & b \\ 4b - 2d & d \end{pmatrix}$$

3.3.2 Exercises (TODO)

1. **Let F be a field, and let $V = F^2$ and $W = F^3$ be F-vector spaces. Define:**

$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and consider the $F[X]$-modules V_ϕ, W_ψ. Show that:

$$f : V_\phi \to W_\psi, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$f : W_\psi \to V_\phi, \qquad \begin{pmatrix} x \\ y \\ x \end{pmatrix} \to \begin{pmatrix} z \\ 0 \end{pmatrix}$$

are $F[X]$-homomorphisms

3.4 Submodules

- **What are submodules?**
  - non-empty **subsets** of an R-module, which are themselves R-modules, with respect to the operations in the R-module

3.4.1 Examples

- a **submodule** of an F-vector space is a subspace
- the **submodules** of a $\mathbb{Z}$-module are the subgroups of its corresponding group
- the **submodules** of a ring are its ideals
- consider a field F and the $F[X]$-module W_ψ defined using the matrix:

$$\psi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(recall W_ψ is an $F[X]$-module, where its elements are in F^3, and multiplication by a matrix $F[X]$ is defined as multiplication of $\underline{v} \in F^3$ by $F[\psi]$). Then the subspaces:

  - $\langle \underline{e}_1 \rangle = \{0\}$
  - $\langle \underline{e}_1, \underline{e}_2 \rangle = \{0\} \cup \{k\underline{e}_1 \mid k \in F\}$ (since $\phi \underline{e}_2 = \underline{e}_1$, and $\phi \underline{e}_1 = 0$)

are $F[X]$-submodules of W_ψ, but $\langle \underline{e}_2 \rangle$ (since $\phi \underline{e}_2 = \underline{e}_1$, but $\underline{e}_1$ is not part of the generating set).

- again, consider:

$$R = \mathbb{C}[X] \qquad M = \mathbb{C}^2_A$$

where multiplication by polynomials in M is defined by multiplying $\underline{v} \in M$ by:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The question to consider is: the 1-d subspace of $\mathbb{C}^2$ given by:

$$L = \{\lambda(x, y) \mid \lambda \in \mathbb{C}\}$$

is definitely closed under addition and scalar multiplication; is it closed under multiplication by polynomials? That is, is it a submodule of M? Consider $p(X) = \sum_{i=0}^{n} p_i X \in \mathbb{C}[X]$. Moreover, notice that:

$$A^2 = Mat(0) \implies A^k = Mat(0), \qquad \forall k \in [2, n]$$

Thus:

$$p(A) = p_0 I_2 + p_1 A = \begin{pmatrix} p_0 & p_1 \\ 0 & p_0 \end{pmatrix}$$

Hence, we ask whether:

$$p(A)(x, y) = \begin{pmatrix} p_0 & p_1 \\ 0 & p_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_0 x + p_1 y \\ p_0 y \end{pmatrix} \in L$$

We thus need to find suitable x, y, such that:

$$\begin{pmatrix} p_0 x + p_1 y \\ p_0 y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

Since the second entry only depends on y, we focus on that first. There are 2 cases to consider:

1. $\boldsymbol{y \neq 0}$ In this case, and since $\mathbb{C}$ is an integral domain, we must have that $p_0 = \lambda$. Thus, in the first entry:
$$p_0 x + p_1 y = \lambda x \implies \lambda x + p_1 y = \lambda x \implies p_1 y = 0$$
Now, since $y \neq 0$, this is only possible if $p_1 = 0$. But we need to consider every possible polynomial in $\mathbb{C}[X]$, so this is not possible. Hence, the only alternative is that $y = 0$.
2. $\boldsymbol{y = 0}$ In this case, p_0 can be anything in $\mathbb{C}$. Then:
$$p_0 x + p_1 y = \lambda x \implies p_0 x = \lambda x \implies p_0 = \lambda$$

Thus, for any x, and for $y = 0$, L defines a submodule.

3.5 Proposition: Test for a Submodule

*Let R be a **ring**, and M a module over R.*
*A subset $M' \subseteq M$ is a **submodule** if and only if:*

1. $0_M \in M'$
2. $a, b \in M' \implies a - b \in M'$
3. $r \in R, a \in M' \implies ra \in M'$

[Proposition 3.7.20]

Proof. If M' is a submodule, these properties hold (since they are properties of modules). Alternatively, assume that M' satisfies the conditions. Then, recall the test of a (finite) subgroup. If G is a group, H is a subgroup if and only if:

- $H \neq \emptyset$
- $h, k \in H \implies hk^{-1} \in H$

Condition (1) means that M is not empty, and condition (2) ensures that $a - b \in M'$. Hence, M' is a subgroup of M. By (3), we know that we have:
$$R \times M' \to M'$$
The remaining properties of a module are satisfied by the fact that M' is a subset of M. Hence, M' must be a submodule.

□

3.6 Lemma: Kernel and Image as Submodules

Let:
$$f : M \to N$$
*be a **module homomorphism**. Then:*

- *$ker(f)$ is a **submodule** of M*
- *$im(f)$ is a **submodule** of N*

[Lemma 3.7.21]

Proof. 1. **The Kernel is a Submodule**

- since $f(0_M) = 0_N$, $0_M \in ker(f)$
- if $a, b \in ker(f)$ then:

$$f(a) - f(b) = 0_M \implies f(a-b) = 0 \implies a - b \in ker(f)$$

- if $r \in R, a \in ker(f)$:

$$rf(a) = r0_M = 0_M \implies f(ra) = 0_M \implies ra \in ker(f)$$

Hence, by the test for a submodule, the kernel is a submodule.

2. **The Image is a Submodule**

- since $f(0_M) = 0_N$, $0_N \in im(f)$
- if $a, b \in im(f)$, then $\exists a', b' \in M$ such that:

$$f(a') = a \qquad f(b') = b$$

But then, by properties of the homomorphism:

$$f(a' - b') = a - b$$

Since $a' - b' \in M$ (by definition of a module), it follows that $a - b \in N$

- if $r \in R, a \in im(f)$, then $\exists a' \in M$ such that:

$$f(a') = a$$

But then:

$$rf(a') = ra \implies f(ra') = ra$$

so $ra \in im(f)$

Hence, by the test for a submodule, the image is a submodule.

□

3.7 Lemma: Injectivity and Kernel

*Let R be a **ring**, with M, N as R-**modules**.*
Let:

$$f : M \to N$$

*be a **module homomorphism**. Then, f is injective **if and only if**:*

$$ker(f) = \{0_M\}$$

[Lemma 3.7.22]

Proof. This follows directly from the fact that this property is true for group homomorphisms. □

3.8 Generating Submodules

- **What is a generated module?**
  - consider a ring R, with R-module M, and a subset $T \subseteq M$
  - the **submodule of** M **generated by** T is the submodule:

$$_R\langle T\rangle = \left\{\sum_{i=1}^{m} r_i t_i \mid r_i \in R, t_i \in T\right\}$$

  - if $T = \emptyset$, then $_R\langle T\rangle$ contains 0_M
- **What is a finitely generated module?**
  - a **module** generated by a **finite set**:

$$M = {}_R\langle T\rangle$$

- **What is a cyclic module?**
  - a **module** generated by a **single** element:

$$M = {}_R\langle t\rangle, \qquad t \in M$$

3.8.1 Examples

- a **cyclic** $\mathbb{Z}$-module is equivalent to a **cyclic abelian group**
- the **ideal** generated by a subset T of a **commutative ring** R is equivalent to a **submodule** of R generated by T
- a **principal ideal** of R is equivalent to a **cyclic submodule** of R
- if F is a field, and W_ψ is defined as above, then w_ψ is a **cyclic** $F[X]$-module generated by $\underline{e}_3$
- 0_M is **generates** a **cyclic submodule** $\{0_M\}$ of **any** module

3.9 Lemma: Smallest Submodule Containing a Subset

If $T \subseteq M$, then:

$$_R\langle T\rangle$$

*is the **smallest submodule** of M containing T. [Lemma 3.7.28]*

3.10 Lemma: Intersection of Submodules

*The **intersection** of any **collection of modules** is a **module**. [Lemma 3.7.29]*

3.11 Lemma: Addition of Submodules

*If M_1, M_2 are **submodules** of M, then:*

$$M_1 + M_2 := \{m_1 + m_2 \mid m_1 \in M_1, m_2 \in M_2\}$$

*is also a **submodule** of M. [Lemma 3.7.30]*

3.12 Theorem: Factor Modules

- **What are cosets in modules?**
  - let:
    * R be a **ring**
    * M be a **module**
    * N a **submodule** of M
  - similarly to before, we can define an **equivalence relation**:

$$a \sim b \iff a - b \in N, \qquad a, b \in M$$

  - for $a \in M$, the **equivalence class** of this relation is the **coset of a with respect to N in M**:

$$E(a) = a + N = \{a + n \mid n \in N\}$$

- **How are factor modules defined?**
  - the **factor of M by N** (or the **quotient of M by N** is the set:

$$M/N = M/\sim$$

    of **all cosets/equivalence classes** of N in M.

3.12.1 Examples

Let $R = \mathbb{R}, M = \mathbb{R}^4$ and:

$$N = \{(x_1, x_2, x_3, x_4) \mid x_1 = 2x_3, x_2 = 4x_4\}$$

What is M/N? We know its an R module over a field, so M/N is a vector space. First, lets consider what the bases of M, N are. For N its simple:

$$\{(2, 0, 1, 0), (0, 4, 0, 1)\}$$

For M, we can extend the basis for N:

$$\{(2, 0, 1, 0), (0, 4, 0, 1), (1, 0, 0, 0), (0, 1, 0, 0)\}$$

Indeed, this is a basis, since the vectors are linearly independent, and:

$$(x_1, x_2, x_3, x_4) = x_3(2, 0, 1, 0) + x_4(0, 4, 0, 1) + (x_1 - 2x_3)(1, 0, 0, 0) + (x_2 - 4x_4)(0, 1, 0, 0)$$

Now, recall that:

$$a \sim b \iff a - b \in N$$

We claim that a basis for M/N is given by:

$$\{(1, 0, 0, 0) + N, (0, 1, 0, 0) + N\}$$

For this we need 2 things:

1. **Generation** Take any element in M/N:

$$(x_1, x_2, x_3, x_4) + N$$

Then, it is clear that:

$$((x_1, x_2, x_3, x_4)+N)-(((x_1-2x_3)(1,0,0,0)+N)+((x_2-4x_4)(0,1,0,0)+N)) = (x_3(2,0,1,0)+N)+(x_4(0,4,0,1)+N)$$

But notice, $\{(2, 0, 1, 0), (0, 4, 0, 1)\}$ is a basis for N, so:

$$((x_1, x_2, x_3, x_4) + N) - (((x_1 - 2x_3)(1, 0, 0, 0) + N) + ((x_2 - 4x_4)(0, 1, 0, 0) + N)) \in N$$

or in other words, $(x_1, x_2, x_3, x_4) + N$ and $((x_1 - 2x_3)(1, 0, 0, 0) + N) + ((x_2 - 4x_4)(0, 1, 0, 0) + N$ are equivalent in the cosets, so:

$$(x_1, x_2, x_3, x_4) + N = ((x_1 - 2x_3)(1, 0, 0, 0) + N) + ((x_2 - 4x_4)(0, 1, 0, 0) + N)$$

Hence, any element in M/N is generated by our claimed basis.

2. **Linear Independence** It is clear that if:

$$(\alpha(1, 0, 0, 0) + \beta(0, 1, 0, 0)) + N = (0, 0, 0, 0) + N$$

we can only have $\alpha = \beta = 0$, so the generating set is lienarly independent.

3.13 Theorem: Factor Module Operations

*Let R be a ring, and let M, N be R-moduls. For $a, b \in M$ and $r \in R$. For the **factor module** M/N, define **addition** via:*

$$(a + N) + (b + N) = (a + b) + N \qquad E(a) + E(b) = E(a + b)$$

*and **multiplication** via:*

$$r(a + N) = (ra) + N$$

[Theorem 3.7.31]

Proof. As before, we need to show that not only M/N is a module, but also that **addition** and **multiplication** are **well-defined**.

Addition is well-defined, since additively, modules are abelian groups, so the proof for **factor rings** applies.

For multiplication, consider $a, b \in M$, such that:

$$E(a) = E(b)$$

We need to show that:

$$rE(a) = rE(b)$$

By properties of modules, this means that $a \sim b \implies a - b \in N$. Again by properties of modules, $r(a - b) \in N \implies ra - rb \in N$. Hence:

$$E(ra) = E(rb) \iff rE(a) = rE(b)$$

Thus, multiplication is **well-defined**.

Lastly, we check that addition defines a group:

- $E(0) + E(a) = E(0 + a) = E(a) = E(a + 0) = E(a) + E(0)$
- $E(a) + E(-a) = E(a - a) = E(0)$

Hence, M/N is indeed a module.

$\square$

3.14 Theorem: The Universal Property of Factor Modules

*Let R be a **ring**, with L and M as R-modules. Let N be a **submodule** of M.*
Then:

1. *The **canonical mapping**:*

$$can : M \to M/N$$
$$can(a) = E(a) = a + N, \qquad \forall a \in M$$

*is a **surjective** R**-module homomorphism**, with:*

$$ker(can) = n$$

2. *If:*

$$f : M \to L$$

*is an R**-homomorphism** with:*

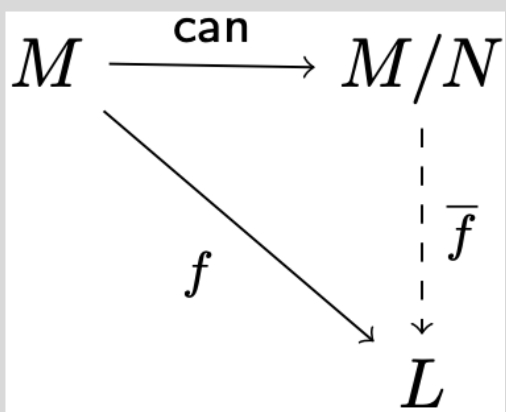
$$f(N) = \{0_L\}$$

*(so then $N \subseteq ker(f)$), then there is a **unique homomorphism**:*

$$\bar{f} : M/N \to L$$
$$\bar{f}(E(a)) = f(a), \qquad \forall a \in M$$

such that:

$$f = \bar{f} \circ can$$



[Theorem 3.7.32]

Proof. The proof is completely analogous to the proof of the **universal property of factor rings** (2.7) □

3.15 Theorem: First Isomorphism Theorem for Modules

*Let R be a **ring**, and let M, N be R**-modules***
*Then, every R**-homomorphism**:*

$$f : M \to N$$

*induces an R**-isomorphism**:*

$$\bar{f} : (M/ker(f)) \to im(f)$$

[Theorem 3.7.33]

Proof. Again, completeley analogous to that of the **first isomorphism theorem for rings** (2.8) $\square$

3.16 Remark: First Isomorphism Theorem for Vector Spaces

*If we pick $R = F$ to be a field, then the above gives us the **First Isomorphism Theorem for Vector Spaces**.*
Similarly to before, we can show that:

$$dim(M/ker(f)) = dim(M) - dim(ker(f))$$

Moreover, due to the isomorphism $\bar{f} : (M/ker(f)) \to im(f)$ we know that:

$$dim(M/ker(f)) = dim(im(f))$$

*which gives us another proof of the **rank-nullity theorem**. [Remark 3.7.34]*

3.17 Remark: First Isomorphism Theorem for Abelian Groups

*If we pick $R = \mathbb{Z}$, then the above gives us the **First Isomorphism Theorem for Abelian Groups**, a special case of the **First Isomorphism Theorem for Groups**. [Remark 3.7.35]*

1. **Let N, K be submodules of an R–module M. Show that K is a submodule of $N + K = \{b + c \mid b \in N, c \in K\}$ and $N \cap K$ is a submodule of N. Show further that:**
$$\frac{N+K}{N} \cong \frac{N}{N \cap K}$$
This is the *Second Isomorphism Theorem for Modules*

2. **Let N, K be submodules of an R–module M, where $K \subseteq N$. Show that N/K is a submodule of M/K, and that:**
$$\frac{M/K}{N/K} \cong M/N$$

This is the *Third Isomorphism Theorem for Modules*

4 Workshop

1. **True or false. Although $\mathbb{Q}$ is not algebraically closed, the set:**

$$\mathbb{Q}[\sqrt{-1}] = \{a + b\sqrt{-1} \mid a, b \in \mathbb{Q}\}$$

(a subset of $\mathbb{C}$ is algebraically closed.

This is false. Consider $X^2 - 2$. This only has roots $X = \pm\sqrt{2}$, but neither of these roots are in $\mathbb{Q}[\sqrt{-1}]$, so this field isn't algebraically closed.

2. **Define an equivalence relation $\sim$ on $\mathbb{R}$ by:**

$$x \sim y \textbf{\textit{ if and only if }} x - y \in \mathbb{Z}$$

Let $E(x)$ denote the equivalence class containing $x \in \mathbb{R}$. Which of the following operations are well-defined where $x, y \in \mathbb{R}$?

(a) $E(x) \to e^{2\pi\sqrt{-1}x}$

Assume that $x \sim y$ (that is, $E(x) = E(y)$). When $\exists z \in \mathbb{Z}$ such that $y = x + z$. Then, this mapping is well defined if:

$$e^{2\pi\sqrt{-1}x} = e^{2\pi\sqrt{-1}y}$$

We compute:

$$e^{2\pi\sqrt{-1}y} = e^{2\pi\sqrt{-1}(x+z)} = e^{2\pi\sqrt{-1}x}e^{2\pi\sqrt{-1}z} = e^{2\pi\sqrt{-1}x}$$

where we use the fact that $z \in \mathbb{Z}$ and so $e^{2\pi\sqrt{-1}z} = 1$

(b) $(E(x), E(y)) \to E(x + y)$

This mapping is well defined if we can show that:

$$(E(x), E(y)) = (E(x'), E(y')) \implies E(x + y) = E(x' + y')$$

Notice:

$$E(x) = E(x') \implies x - x' = z \in \mathbb{Z}$$
$$E(y) = E(y') \implies y - y' = w \in \mathbb{Z}$$

Now, consider:

$$(x + y) - (x' + y') = (x - x') + (y - y') = z + w \in \mathbb{Z}$$

Thus:

$$(x + y) - (x' + y') \in \mathbb{Z} \implies E(x + y) = E(x' + y')$$

(c) $(E(x), E(y)) \to E(xy)$

Operating similarly as above, consider:

$$xy - x'y'$$

If this difference is an integer, then $E(xy) = E(x'y')$, where:

$$E(x) = E(x') \implies x - x' = z \in \mathbb{Z}$$

$$E(y) = E(y') \implies y - y' = w \in \mathbb{Z}$$

Thus:

$$x'y' = (x + z)(y + w) = xy + zy + zw + xw$$

Hence:

$$xy - x'y' = -(zy + xw + zw)$$

This need not be an integer. For example, picking rational x, y can ensure this. Indeed, if $x = \frac{3}{2}$ and $y = \frac{1}{2}$, then $x - y \in \mathbb{Z}$. Then:

$$\left(E\left(\frac{1}{2}\right), E\left(\frac{1}{2}\right)\right) \to E\left(\frac{1}{4}\right)$$

but

$$\left(E\left(\frac{3}{2}\right), E\left(\frac{1}{2}\right)\right) \to E\left(\frac{3}{4}\right)$$

and:

$$E\left(\frac{3}{4}\right) \neq E\left(\frac{1}{4}\right)$$

since:

$$\frac{3}{4} - \frac{1}{4} = \frac{1}{2} \notin \mathbb{Z}$$

3. **Let:**

$$I = {}_{\mathbb{C}[X]}\langle X^2 + 1\rangle$$

the principal ideal of $\mathbb{C}[X]$ generated by X^2+1. Is the factor ring $\mathbb{C}[X]/I$ an integral domain?

> *This question relies on having a strong understanding of all the concepts involved.*
>
> - *I is the ideal containing all the polynomials which have $X^2 + 1$ as a factor*
> - *$\mathbb{C}[X]/I$ is a quotient ring, with 0 element $E(0)$ given by I: that is, $E(0)$ is the set of polynomials with $X^2 + 1$ as a factor*
> - *an **integral domain** is a non-zero commutative ring that has no zero-divisors; that is, multiplying non-zero elements together never produces the 0 element*

This claim is False. This is because we can find non-zero elements in $\mathbb{C}[X]/I$ which when multiplied produce $E(0)$.

Notice, we can write:

$$X^2 + 1 = (X + \sqrt{-1})(X - \sqrt{-1})$$

This means that:

$$E(X + \sqrt{-1})E(X - \sqrt{-1}) = E(X^2 + 1) = E(0)$$

We just need to show that neither of the two are 0. This is clear, since these are both polynomials of degree 1. The ideal I contains only elements of degree at least 2 (since they are obtained by multiplying non-zero polynomials by X^2+1, and since $\mathbb{C}$ is an integral domain, if $P = Q(X)(X^2+1)$ then $deg(P) = deg(Q) + 2 \geq 2$). Hence, $E(X + \sqrt{-1}) \neq E(0), E(X - \sqrt{-1}) \neq E(0)$.

4. **Let $n \in \mathbb{Z}$ with $n \geq 2$ and let $I = {}_{\mathbb{Z}[X]}\langle n, X\rangle$, an ideal of $\mathbb{Z}[X]$. Show that $\mathbb{Z}[X]/I$ is isomorphic to $\mathbb{Z}_n$**

We need to realise 2 things:

- *I is an ideal generated by using combinations of the constant polynomial n and the linear polynomial X*
- *to show the isomorphic nature of the 2 rings, we first need to come up with a ring homomorphism ($f : \mathbb{Z}[X] \to \mathbb{Z}_n$) which must be surjective, so that $im(f) = \mathbb{Z}_n$. Then, if we can show that $I = ker(f)$ then f leads to an isomorphism from $\mathbb{Z}[X]/I$ to $\mathbb{Z}_n$.*

$\mathbb{Z}[X]$ is the ring of polynomials with integer coefficients. It makes intuitive sense to define a mapping:

$$f\left(a_n X^n + \ldots + a_0\right) = \overline{a_0}$$

We verify that it is a ring homomorphism. Consider 2 polynomials:

$$P(X) = \sum_{i=0}^{n} a_i X^i \qquad P(X) = \sum_{i=0}^{n} b_i X^i$$

Then:

$$f(P+Q) = f\left(\sum_{i=0}^{n}(a_i + b_i)X^i\right) = \overline{a_i + b_i} = \overline{a_i} + \overline{b_i} = f(P) + f(Q)$$

$$f(PQ) = \overline{a_i b_i} = \overline{a_i}\,\overline{b_i} = f(P)f(Q)$$

Moreover, f is clearly surjective, since if $\bar{x} \in \mathbb{Z}_n$ then the constant polynomial $P(X) = x$ is such that:

$$f(P) = \bar{x}$$

The final step is to show that $I = ker(f)$. It is clear that $I \subseteq ker(f)$, since:

$$f(n) = \bar{n} = \bar{0}$$

$$f(X) = f(X+0) = \bar{0}$$

Now, suppose that:

$$P(X) = \sum_{i=0}^{n} a_i X^i \qquad P(X) = \sum_{i=0}^{n} b_i X^i \in ker(f)$$

This means that:

$$a_0 = nz, \qquad z \in \mathbb{Z}$$

since then $f(nz) = \bar{0}$. But then:

$$P(X) = nz + X\left(\sum_{i=0}^{n-1} a_{i+1} X^i\right)$$

so clearly:

$$P(X) \in I \implies ker(f) \subseteq I$$

Hence, $I = ker(f)$. Then, by the first isomorphism Theorem we have that:

$$\mathbb{Z}[X]/ker(f) \cong im(f) \implies \mathbb{Z}[X]/I \cong \mathbb{Z}_n$$

5. **Let V be the real vector space of polynomials $\mathbb{R}[X]_{<4}$ of degree less than 4. Let:**

$$U = \{P \in V \mid P(3) = 0\}$$

(a) **Show that U is a subspace of V.**

We check the 3 properties of a subspace:

1 **Contains 0 Element**

If $P(X) = 0$, then clearly $P(3) = 0$, so $0 \in U$.

2 **Closed Under Addition**

Let $P(X), Q(X) \in U$. Then:

$$(P + Q)(3) = P(3) + Q(3) = 0 \implies P + Q \in U$$

3 **Closed Under Scalar Multiplication**

Let $P(X) \in U, \lambda \in \mathbb{R}$. Then:

$$(\lambda P)(3) = \lambda P(3) = 0 \implies \lambda P \in U$$

Hence, it follows that U is a subspace of U.

(b) **Find a basis for U and extend it to a basis for V. Express $P \in V$ explicitly in terms of this basis.**

When I did this, as a basis I picked:

$$\{(X - 3), (X - 3)^2, (X - 3)^3\}$$

which certainly worked, but it makes the calculations a bit harder. The solutions pick a simpler basis, so I will use their ansers below.

As a basis we can pick:

$$\{(X - 3), X(X - 3), X^2(X - 3)\} = \{X - 3, X^2 - 3X, X^3 - 3X^2\}$$

Clearly, each element is linearly independent (they differ by factors of X). Intuitively, it will span, since any linear combination of these will have 3 as a root. In particular, consider $P \in U$, then, we can write:

$$\begin{aligned} P(X) &= (aX^2 + bX + c)(X - 3) \\ &= aX^3 - 3aX^2 + bX^2 - 3bX + cX - 3c \\ &= a(X^3 - 3X^2) + b(X^2 - 3X) + c(X - 3) \end{aligned}$$

so the set is spanning.

Again, it is intuitive that to produce a basis of P, we just need control over the constant term (since the current basis already "handles" all the powers of X, except for 0). Hence, for P, we consider the basis:

$$\{1, X - 3, X^2 - 3X, X^3 - 3X^2\}$$

Now, consider $P \in V$. Then:

$$\begin{aligned}
P &= aX^3 + bX^2 + cX + d \\
&= aX^3 - 3aX^2 + 3aX^2 + bX^2 + cX + d \\
&= a(X^3 - 3X^2) + (3a+b)X^2 + cX + d \\
&= a(X^3 - 3X^2) + (3a+b)X^2 - 3(3a+b)X + 3(3a+b)X + cX + d \\
&= a(X^3 - 3X^2) + (3a+b)(X^2 - 3X) + (9a+3b+c)X + d \\
&= a(X^3 - 3X^2) + (3a+b)(X^2 - 3X) + (9a+3b+c)X - 3(9a+3b+c) + 3(9a+3b+c) + d \\
&= a(X^3 - 3X^2) + (3a+b)(X^2 - 3X) + (9a+3b+c)(X-3) + (27a+9b+3c+d)
\end{aligned}$$

So we can see that this basis is LiD and spans V, as required.

> *Using my basis, these calculations were a true pain, but I got very similar results (albeit not checked, so just stick to the above).*

(c) **Write down a basis for V/U.**

> *This is very similar to an exercise for the notes, which tells us that the basis for V/U is obtained by applying the canonical mapping to the elements used to extend U to V.*

We claim that:

$$\{1 + U\}$$

is a basis for V/U.

To verify this, consider $P + U \in V/U$. In particular, by the exercise above, we know that we can find $a, b, c, d \in \mathbb{R}$ such that:

$$P = a(X^3 - 3X^2) + b(X^2 - 3X) + c(X-3) + d = (X-3)(aX^2 + bX + c) + d$$

Then, it is clear that:

$$P + U = d + U = d(1 + U)$$

Hence, the basis spans V/U. Moreover, it contains a single element, so it is linearly independent.

(d) **Write down the matrix that represents the canonical mapping:**

$$can : V \to V/U$$

which sends P to $P+U$, in terms of the (ordered) basis $\{X^3, X^2, X, 1\}$ of V, and the one you chose in (3) for V/U.

We have that $(1+U)$ is the basis vector of V/U. Recall, the representing matrix of the mapping is defined by the coefficients used to write $can(P)$ in terms of $(1+U)$. Notice, in part b), we showed that:

$$P(X) = aX^3 + bX^2 + cX + d$$

can be written as:

$$P(X) = a(X^3 - 3X^2) + (3a+b)(X^2 - 3X) + (9a+3b+c)(X-3) + (27a+9b+3c+d)$$

This great, since then:

$$can(P) = P + U = (27a + 9b + 3c + d) + U$$

Hence, we compute:

$$can(1) = 1 + U = 1(1 + U)$$

If we set $c = 1$, and everything else to 0:

$$can(X) = X + U = 3 + U = 3(1 + U)$$

If we set $b = 1$, and everything else to 0:

$$can(X^2) = X^2 + U = 9 + U = 9(1 + U)$$

If we set $a = 1$, and everything else to 0:

$$can(X^3) = X^3 + U = 27 + U = 27(1 + U)$$

Hence, we get that the representing matrix is:

$${}_{\{1+U\}}[can]_{\{X^3, X^2, X, 1\}} = \begin{pmatrix} 27 & 9 & 3 & 1 \end{pmatrix}$$