Honours Analysis - Week 11 - Formalising Fourier Series

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Contents

Throughout this, we consider what is known as the **trigonometric system** on $[0, 1]$:

$$\phi_n(x) = e^{2\pi i n x}, \qquad n \in \mathbb{Z}$$

1 Trigonometric Polynomials

1.1 Defining Trigonometric Polynomials

- **What is a trigonometric polynomial?**
  - a function of the form:
    $$f(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$$
    where:
    * $x \in \mathbb{R}$
    * $N \in \mathbb{N}$
    * $c_n \in \mathbb{C}$
  - since $e^{2\pi i n x}$ is continuous $\forall x \in \mathbb{R}$, so is $f(x)$
- **What is the degeree of a trigonometric polynomial?**
  - the highest index in the summation (N) (given that $c_N \neq 0$ or $c_{-N} \neq 0$)
- **What is an alternative way of writing a trigonometric polynomial?**
  - applying **Euler's Identity**:
    $$e^{ix} = \cos(x) + i\sin(x)$$
    then $\exists a_n, b_n \in \mathbb{C}$ such that:
    $$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x))$$
  - we can define a_n, b_n explicitly in terms of c_n:
    $$\begin{aligned} f(x) &= \sum_{n=-N}^{N} c_n e^{2\pi i n x} \\ &= c_0 + \sum_{n=1}^{N} c_n e^{2\pi i n x} + c_{-n} e^{-2\pi i n x} \\ &= c_0 + \sum_{n=1}^{N} c_n(\cos(2\pi n x) + i\sin(2\pi n x) + c_{-n}(\cos(-2\pi n x) + i\sin(-2\pi n x) \\ &= c_0 + \sum_{n=1}^{N} (c_n + c_{-n})\cos(2\pi n x) + i(c_n - c_{-n})\sin(2\pi n x) \end{aligned}$$
    so that:
    $$a_n = \begin{cases} c_0, & n = 0 \\ c_n + c_{-n}, & n \geq 1 \end{cases} \qquad b_n = i(c_n - c_{-n})$$
- **Are trigonometric polynomials periodic?**

– they are **1-periodic**, since:

$$f(x+1) = \sum_{n=-N}^{N} c_n e^{2\pi i n(x+1)} = \sum_{n=-N}^{N} c_n e^{2\pi i n x} e^{2\pi i n} = \sum_{n=-N}^{N} c_n e^{2\pi i n x} = f(x)$$

where we use the fact that:

$$e^{2\pi i n} = e^{i0n} = 1$$

Because of the ***1-periodic*** *nature of $f(x)$, throughout we will use L^2 to refer to the L^2 space of 1-periodic functions $f : \mathbb{R} \to \mathbb{C}$.*
The L^2-norm can be simplified to be:

$$\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 \, dx}$$

Notice, ***periodicity*** *ensures that integration gives the same result, independently of the interval of length 1 which we use to integrate such that, for example:*

$$\int_0^1 |f(x)|^2 \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 \, dx$$

Explicitly, if we have a L periodic function $g(x)$, and using $y = x - L$:

$$\begin{aligned}
\int_c^{c+L} g(x) \, dx &= \int_c^L g(x) \, dx + \int_L^{c+L} g(x) \, dx \\
&= \int_c^L g(x) \, dx + \int_{y=L-L}^{y=(c+L)-L} g(y) \, dy \\
&= \int_c^L g(x) \, dx + \int_0^c g(x) \, dx \\
&= \int_0^L g(x) \, dx
\end{aligned}$$

Any theory developed for 1 periodic functions can be adapted for L-periodic functions. For instance, the trigonometric system can be adapted:

$$\frac{1}{\sqrt{L}} e^{\frac{2\pi i n x}{L}}$$

to yield L-periodic functions

1.2 Lemma: Trigonometric Polynomials as an Orthonormal System

The set:

$$\{e^{2\pi inx \ | \ n\in\mathbb{Z}}\}$$

forms an ***orthonormal system*** *on* $[0,1]$.
In particular:

1.
$$\int_0^1 e^{2\pi inx}\,dx = \begin{cases} 0, & n\neq 0 \\ 1, & n = 0 \end{cases} \qquad \forall n\in\mathbb{Z}$$

2. *if*
$$f(x) = \sum_{n=-N}^{N} c_n e^{2\pi inx}$$
is a ***trigonometric polynomial****, then:*
$$c_n = \langle f, \phi_n\rangle = \int_0^1 f(t)e^{-2\pi int}$$
[Lemma 5.1]

The key aim of this whole week is to show that the orthonormal system $\{e^{2\pi inx \ | \ n\in\mathbb{Z}}\}$ is in fact **complete**:

An orthonormal system $(\phi_n)_n$ *is called* ***complete*** *if:*

$$\sum_n |\langle f, \phi_n\rangle|^2 = \|f\|_2^2$$

[Definition 5.5]
This is of particular importance, since: if (ϕ_n) *is an orthonormal system in* $[a,b]$*, define* $s_N = \sum_n c_n\phi_n$*. We say the orthonormal system* (ϕ_n) *is* ***complete*****,** ***if and only if*** $s_N \to f$ *on* L^2 *for* ***any*** $f\in L^2$*. [Theorem 5.4]*
That is, showing that this system is complete then tell us that we can use $\{e^{2\pi inx \ | \ n\in\mathbb{Z}}\}$ *to construct series which* ***converge*** *to given functions* f *on* L^2*.*

Proof. The first property relies on standard integration, by noting we can use the substitution $y = 2\pi inx$ and further using the fact that $e^{i2\pi n} = e^0 = 1$.

The second property uses the orthonormality of the system: just take the inner product of f with ϕ_n; orthonormality ensures that $c_n = \langle f, \phi_n \rangle$.

$\square$

1.3 Fourier Series

- **What is a Fourier Coefficient?**
  - consider a **1-periodic** and **integrable** function f
  - if $n \in \mathbb{Z}$ the ***n*th Fourier Coefficient** is:

$$\hat{f}(n) = \langle f, \phi_n \rangle = \int_0^1 f(t) e^{-2\pi int} \, dt$$

  - this exists, since f is integrable
- **What is a Fourier Series?**
  - consider a **1-periodic** and **integrable** function f
  - its **Fourier Series** is:

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \phi_n(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi inx}$$

  - currently, we haven't established **any** property of this Fourier Series: we don't know if it even converges, or is representative of anything
- **When is a doubly infinite series convergent?**
  - we consider a **doubly infinite series**:

$$S = \sum_{n=-\infty}^{\infty} a_n$$

  - the convergence of S can be discussed in 2 ways:
    * **Convergent**: if both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} a_{-n}$ are **convergent** (in this case, S is the sum of the composing series)
    * **Convergent in the Principle Value Sense**: a weaker notion - only require that the sequence of partial sums:

$$S_n = \sum_{n=-N}^{N} a_n$$

      converges (we use this sense when discussing convergence)

We now define the following sequence of partial sums:

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$$

Recall, last week we defined the ***orthonormal projection*** *of a function f onto an* ***orthonormal system*** *via:*

$$s_N = \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n(x)$$

In particular, we showed that s_N is the best possible approximation to f (in L^2). Notice, because of this, we can see that $S_N f(x)$ is precisely the orthonormal projection of f onto the space of ***trigonometric polynomials*** *of degree at most N. This is the best possible approximation, in the sense that:*

$$\|f - S_N f\|_2 \leq \|f - g\|_2$$

This is some intuitive evidence that a ***Fourier Series*** *can be thought of as an appropriate approximation of f (although we still don't even know if it converges!).*
It is important to remark that this is not generally true for other norms beyond L^2.

1.3.1 Exercises (TODO)

1. **Show that if a doubly infinite series converges, then it also converges in the principal value sense.**
2. **Give an example of a doubly infinite series:**

$$\sum_{n=-\infty}^{\infty} a_n$$

which does not converge, but converges in the principal value sense.

3. **Show that if $a_n \geq 0, \forall n \in \mathbb{Z}$ then:**

$$\sum_{n=-\infty}^{\infty} a_n$$

converges *if and only if* it converges in the principal value sense.

2 Introducing Convolutions

2.1 Defining Convolutions

- **What is a convolution?**
  - an operation between functions
  - consider 2 **1-periodic** functions $f, g \in L^2$

- their **convolution** is defined by:

$$f * g(x) = \int_0^1 f(y)g(x-y)\ dy$$

- seems overly complex, but it has very nice properties
- think of it like addition or multiplication of functions: a way of combining functions to produce other functions

2.1.1 Examples: Intuitive Meaning of Convolutions

Convolutions can be thought in 2 senses.

1. If $\int_0^1 g = 1$, g is used to compute a **weighted average** of f, centered at x

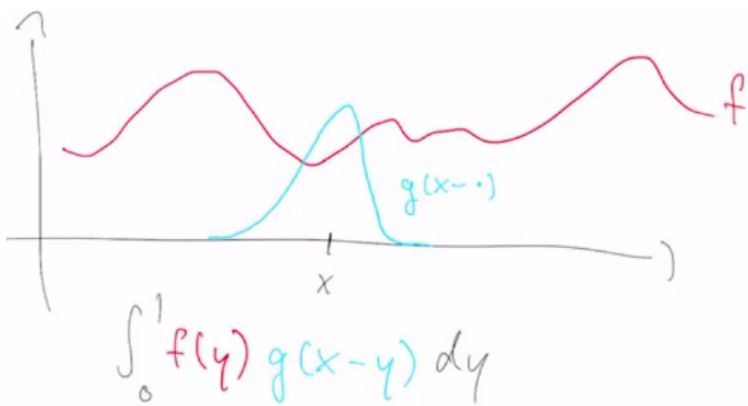


Figure 1: The convolution involves centering $g(y)$ at $y = x$ (so we get $g(y - x)$), and then multiplying $f(y)$ by this, effectively weighting f according to g. The integral then acts as taking an average.

   In fact, if $g = 1$, we actually get the average of f.

   If $I = \left[-\frac{1}{N}, \frac{1}{N}\right]$, and $g = \frac{N}{2}\mathcal{X}_I$, then $f * g(x)$ is the average value of f over the interval $\left[x - \frac{1}{N}, x + \frac{1}{N}\right]$ (the key here is that g is such that $\int_0^1 g = 1$, so it doesn't add extra weight, it just redistributes it)

2. In this video, Dr. Peyam discusses how $f * g(n)$ can be thought as a continuous case of **multiplication**; in particular, $f * g(n)$ can be thought of as the coefficient of x^n in a polynomial expansion, where $n \in \mathbb{R}$

2.2 Lemma: Properties of Convolutions

*Let $f, g, h \in L^2$ be **1-periodic**. Then:*

1. $f * g \in L^2$
2. $f * g = g * f$
3. $(f + \lambda g) * h = f * h + \lambda(g * h)$

[Lemma 5.2]

Proof. For the first one, I have a proof, but I am not too sure, so I won't add it in case it causes confusion.

For commutativity, we note that:

$$\begin{aligned}
f * g(x) &= \int_0^1 f(y)g(x-y)\ dy \\
&= -\int_{z=x-(0)}^{z=x-(1)} f(x-z)g(z)\ dz \\
&= \int_{x-1}^{x} g(z)f(x-z)\ dz \\
&= \int_0^1 g(z)f(x-z)\ dz \\
&= g * f(x)
\end{aligned}$$

The last one follows directly from linearity of integration.

$\square$

3 Dirichlet and Fejér Kernels

3.1 Lemma: The Dirichlet Kernel

- **Why are convolutions important for the study of Fourier Analysis?**
  - we can use them to express the partial sum $S_N f$ of a **Fourier Series**:

$$\begin{aligned}
S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(x) e^{2\pi i n x} \\
&= \sum_{n=-N}^{N} \left(\int_0^1 f(y) e^{-2\pi i n y}\ dy \right) e^{2\pi i n x} \\
&= \int_0^1 f(y) \left(\sum_{n=-N}^{N} e^{-2\pi i n y} e^{2\pi i n x} \right)\ dy \qquad \textit{(since the sum is finite)} \\
&= \int_0^1 f(y) \left(\sum_{n=-N}^{N} e^{2\pi i n (x-y)} \right)\ dy
\end{aligned}$$

Thus, if we define:

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x}$$

It follows that:

$$S_N f(x) = \int_0^1 f(t) D_N(x-y)\ dy = f * D_N(x)$$

The sequence of functions:

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx}$$

is called the ***Dirichlet kernel****.*
We can write the ***Dirichlet kernel*** *as:*

$$D_N(x) = \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

[Lemma 5.3]

Proof. Just "follow your nose".

Notice, we can rewrite:

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx} = e^{-2\pi iNx} \sum_{n=0}^{2N} e^{2\pi inx}$$

In this form, we can think of $D_N(x)$ as a geometric series, with first term 1, and common ratio $e^{2\pi ix}$:

$$D_N(x) = e^{-2\pi iNx} \sum_{n=0}^{2N} \left(e^{2\pi ix}\right)^n$$

Recall the sum of a (finite) geometric series:

$$\sum_{n=0}^{N} a_0 \times r^n = \frac{a(1-r^{N+1})}{1-r}$$

Thus, we can write:

$$\begin{aligned}
D_N(x) &= e^{-2\pi iNx}\frac{1-\left(e^{2\pi ix}\right)^{2N+1}}{1-e^{2\pi ix}} \\
&= \frac{e^{-2\pi iNx}-e^{2\pi i(2N+1)x}e^{-2\pi iNx}}{1-e^{2\pi ix}} \\
&= \frac{e^{-2\pi iNx}-e^{2\pi i(N+1)x}}{1-e^{2\pi ix}} \\
&= \frac{e^{-2\pi iNx}-e^{2\pi iNx}e^{2\pi ix}}{1-e^{2\pi ix}} \\
&= \frac{e^{-\pi ix}}{e^{-\pi ix}}\times\frac{e^{-2\pi iNx}-e^{2\pi iNx}e^{2\pi ix}}{1-e^{2\pi ix}} \\
&= \frac{e^{-2\pi iNx}e^{-\pi ix}-e^{2\pi iNx}e^{\pi ix}}{e^{-\pi ix}-e^{\pi ix}} \\
&= \frac{e^{-2\pi i\left(N+\frac{1}{2}\right)x}-e^{2\pi i\left(N+\frac{1}{2}\right)x}}{e^{-\pi ix}-e^{\pi ix}} \\
&= \frac{e^{2\pi i\left(N+\frac{1}{2}\right)x}-e^{-2\pi i\left(N+\frac{1}{2}\right)x}}{e^{\pi ix}-e^{-\pi ix}}
\end{aligned}$$

But recall the formula (derived from Euler's Identity):

$$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

So it follows that:

$$D_N(x) = \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

as required.

It must be noted that in this form, $D_N(0)$ is undefined; however, if we look at the original formula, we can see that:

$$D_N(0) = \sum_{n=-N}^{N} e^0 = 2N+1$$

which will in fact be the maximum of the function. We can visualise $D_N(x)$:

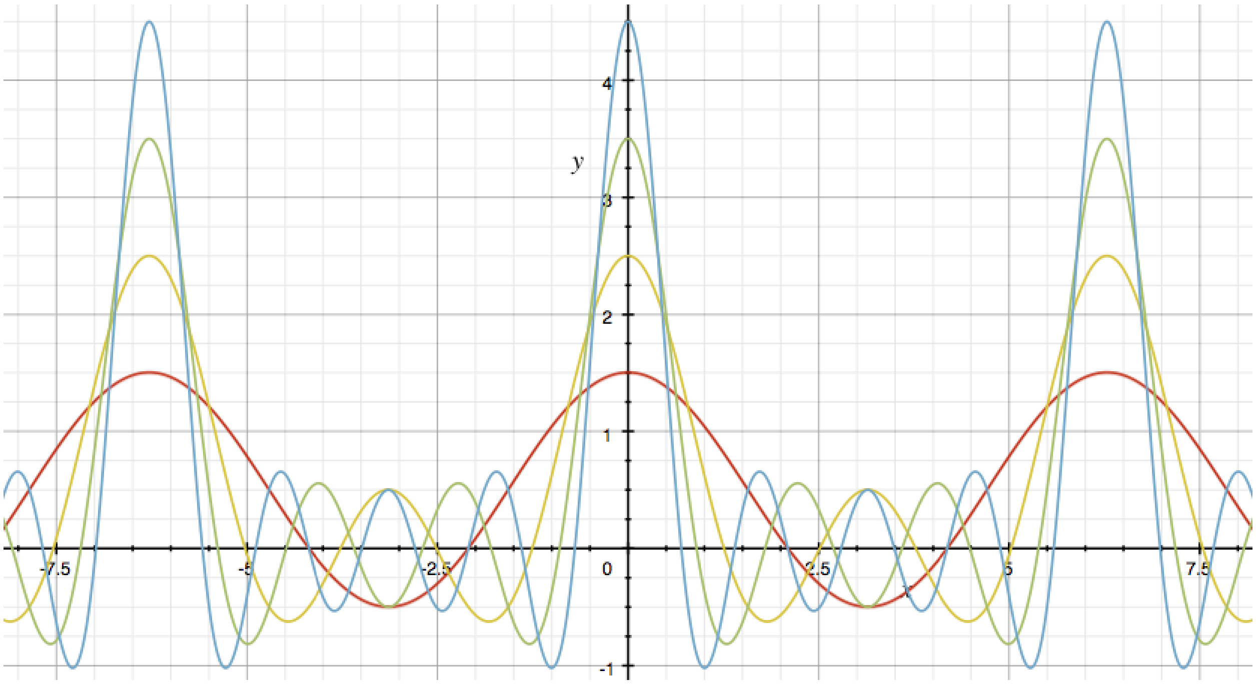


Figure 2: As N increases, the frequency of $D_N(x)$ also increases. Moreover, its maximum $2N+1$ will also increase. The result is taht the sequence gets progressively more squashed towards the center.

$\square$

3.1.1 Lemma: Integral of Dirichlet Kernel

$$\int_0^1 D_N(x)\ dx = 1, \qquad N \in \mathbb{N}$$

Proof. This is a straightforward calculation:

$$\begin{aligned} \int_0^1 D_N(x)\ dx &= \int_0^1 \left(\sum_{n=-N}^{N} e^{2\pi i n x} \right)\ dx \\ &= \sum_{n=-N}^{N} \left(\int_0^1 e^{2\pi i n x}\ dx \right) \end{aligned}$$

But recall, by Lemma 5.1, since $e^{2\pi i n x}$ is part of the trigonometric system:

$$\int_0^1 e^{2\pi i n x}\ dx = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \qquad \forall n \in \mathbb{Z}$$

Such that:

$$\int_0^1 D_N(x)\ dx = \sum_{n=-N}^{N} \left(\int_0^1 e^{2\pi i n x}\ dx \right) = 1$$

□

3.2 Intuition: Approximations via Dirichlet Kernel

From the above, the Dirichlet Kernel has some nice properties, namely:

- $\int_0^1 D_N(x)\ dx = 1$
- most of it's "mass" gets concentrated on the origin as $N \to \infty$

Because of this, it seems reasonable that we can approximate any function f via:

$$f(x) \approx f * D_N(x)$$

(the convolution acts as a weighted average of f at x; but most of the probability mass $D_N(x)$ will assign to the value of f at x)

However, $D_N(x)$ has a problem: it is an oscillating function, with oscillations getting very out of hand as $N \to \infty$. In particular this means that for some functions (including well-behaved, continuous ones):

$$f * D_N(x)$$

can diverge as $N \to \infty$.

3.2.1 Exercises: Cesàro Summation (TODO)

Consider a sequence $(a_k)_{k\in\mathbb{N}}$.
*It's **Nth Cesàro Sum** or **Nth Cesàro Mean** is the sequence obtained by taking an average of the first N partial sums of a_k:*

$$\sigma_N = \frac{\sum_{i=1}^N S_i}{N} = \frac{\sum_{i=1}^N \sum_{k=1}^i a_k}{N}$$

*The series $\sum_{k=1}^\infty a_k$ is called **Cesàro Summable** to S if σ_N converges to $S < \infty$.*

1. **Prove that if $a_k \to L$ then:**
$$\lim_{n\to\infty} \frac{S_n}{n} = L$$
2. **Prove that if $S = \sum_{k=1}^\infty a_k$ with $S < \infty$, then $\sum_{k=1}^\infty a_k$ is Cesàro Summable to S.**
3. **Prove that the series $\sum_{k=1}^\infty (-1)^{k-1}$ does *not* converge, but is nonetheless Cesàro Summable to S. Find this S.**

3.3 Lemma: The Fejér Kernel

We now introduce the Fejér Kernel. It is constructed by employing Cesàro Sums, with the hopes (well justified) that taking the average will smoothen out the oscillations. As we can see, this tends out to solve our problem, and allow us to prove our claims with regards to the convergence of Fourier Series.

*The **Fejér Kernel** is the **Cesàro Sum** of the **trigonometric system**:*

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x) = \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} e^{2\pi i k x}$$

Furthermore, we have that:

$$K_N(x) = \frac{1}{2N+1}\frac{1-\cos(2\pi(N+1)x)}{\sin(\pi x)^2} = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2$$

In particular, this means that $K_N(x) \geq 0$ for any $x \in \mathbb{R}$. [Lemma 5.4]

Proof. Recall, we have that:

$$D_N(x) = \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

The key is to make use of the following identity:

$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$$

This is immediately derived from the use of the sum of angle formulae:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \qquad \cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

To use it, we require a product of sines, which can be easily achieved (here we assume that $x \neq 0$):

$$\begin{aligned}
D_N(x) &= \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)} \\
&= \frac{2\sin(\pi x)}{2\sin(\pi x)} \times \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{2\sin(\pi x)} \\
&= \frac{2\sin(\pi x)\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{2\sin^2(\pi x)} \\
&= \frac{\cos\left(\pi x - 2\pi\left(N+\frac{1}{2}\right)x\right) - \cos\left(\pi x + 2\pi\left(N+\frac{1}{2}\right)x\right)}{2\sin^2(\pi x)} \\
&= \frac{\cos\left(\pi x - 2\pi N x - \pi x\right) - \cos\left(\pi x + 2\pi N x + \pi x\right)}{2\sin^2(\pi x)} \\
&= \frac{\cos\left(-2\pi N x\right) - \cos\left(2\pi(N+1)x\right)}{2\sin^2(\pi x)} \\
&= \frac{\cos\left(2\pi N x\right) - \cos\left(2\pi(N+1)x\right)}{2\sin^2(\pi x)}
\end{aligned}$$

since the cosine is an even function.

Now, for the Féjer Kernel:

$$
\begin{aligned}
K_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} D_n(x) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{\cos(2\pi N x) - \cos(2\pi(N+1)x)}{2\sin^2(\pi x)} \\
&= \frac{1}{2\sin^2(\pi x)(N+1)}\sum_{n=0}^{N} \cos(2\pi N x) - \cos(2\pi(N+1)x) \\
&= \frac{1}{2\sin^2(\pi x)(N+1)}(\cos(2\pi 0 x) - \cancel{\cos(2\pi x)} + \cancel{\cos(2\pi x)} - \cancel{\cos(2\pi 2)} \\
&+ \ldots + \cancel{\cos(2\pi N x)} - \cos(2\pi(N+1)x)) \\
&= \frac{1-\cos(2\pi(N+1)x)}{2\sin^2(\pi x)(N+1)}
\end{aligned}
$$

This is the first of the formulae. For the second one, we employ the identity:

$$1 - \cos(2x) = 2\sin^2(x)$$

such that:

$$K_N(x) = \frac{1-\cos(2\pi(N+1)x)}{2\sin^2(\pi x)(N+1)} = \frac{2\sin^2(\pi(N+1)x)}{2\sin^2(\pi x)(N+1)} = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2$$

□

4 Approximations of Unity

4.1 Defining an Approximation of Unity

- **What is the identity function for the convolution operator?**
  - if $*$ had an identity g, this would mean that for any **1-periodic** function $f \in L^2$ we would have that:
  $$f * g(x) = \int_0^1 f(y)g(x-y)\ dy = f(x)$$
  - turns out, no such g can exists

Proof. We recall the Riemann-Lebesgue Lemma:

> *Let $(\phi_n)_{n\in\mathbb{N}}$ be an **orthonormal system**, and let $f \in L^2$. Then:*
>
> $$\lim_{n\to\infty}\langle f, \phi_n\rangle = 0$$
>
> *[Corollary 5.2]*

Now, lets assume that an identity g exists, and consider the **trigonometric system**. Then:

$$\phi_n * g(x) = g * \phi_n(x) = \int_0^1 \phi_n(x-y)g(y)\ dy = e^{2\pi i n x}\int_0^1 g(y)e^{-2\pi i n y}\ dy = \phi_n(x)\int_0^1 g(y)\overline{\phi_n(y)}\ dy = \phi_n(x)\langle g, \phi_n\rangle$$

But if g is the identity, then we must have that:

$$\phi_n * g(x) = \phi_n(x)\langle g, \phi_n \rangle = \phi_n(x) \iff \langle g, \phi_n \rangle = 1$$

But this contradicts the Riemann-Lebesgue Lemma, so no such g can exist.

$\square$

- **What is an approximation of unity?**
  - the "next best thing" in terms of having a unity for convolution
  - an **approximation of unity** is a **sequence** of **1-periodic** and **integrable** functions $(k_n)_{n\in\mathbb{N}}$
  - the $(k_n)_{n\in\mathbb{N}}$ are such that:
  $$k_n * f \to f$$
  uniformly on $\mathbb{R}$ for any **1-periodic** and **continuous** f
  - in other words:
  $$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f * k_n(x) - f(x)| = 0$$

4.2 Theorem: Properties of Approximations of Unity

*Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of **1-periodic** and **integrable** functions satisfying:*

1. $\forall x \in \mathbb{R},\ k_n(x) \geq 0$
2. $$\int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t)\ dt = 1$$
3. $\forall \delta \in \left(0, \frac{1}{2}\right]$ *we have that:*
$$\lim_{n\to\infty} \left(\int_{-\delta}^{\delta} k_n(t)\ dt \right) = 1$$

*Then, $(k_n)_{n\in\mathbb{N}}$ is an **approximation of unity**. [Theorem 5.6]*

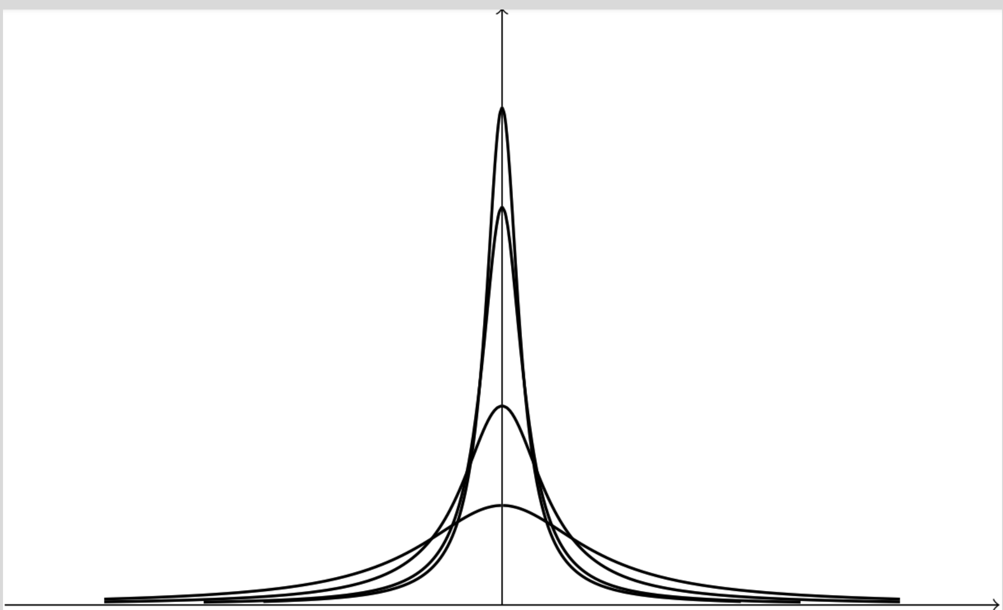
The conditions (2) and (3) can be alternatively formulated as:

$$\lim_{n\to\infty}\left(\int_{\delta\leq|t|\leq\frac{1}{2}} k_n(t)\ dt\right)=0$$

What these conditions tell us is that approximations of unity:

- *are always positive*
- *have constant area under the curve*
- *most of the area under the curve is concentrated near the origin $x=0$; alternatively, away from its centre, there is very little area under the curve*

This allows us to get an idea of what approximations of unity look like:



Proof. Consider f which is **1-periodic** and **continuous**. In particular, this means that on $\left[-\frac{1}{2},\frac{1}{2}\right]$, f is bounded and uniformly continuous, so by periodicity, this is the case over all $\mathbb{R}$.

By definition of uniform continuity, $\exists\delta>0$ such that $\forall\varepsilon>0$, whenever $|t|<\delta$ we have that:

$$|f(x-t)-f(x)|\leq\frac{\varepsilon}{2}$$

where $x\in\mathbb{R}$.

Now, consider the expression:

$$f*k_n(x)-f(x)$$

Using the definition of convolution:

$$\begin{aligned} f * k_n(x) - f(x) &= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t) f(x-t)\ dt\right) - f(x) \\ &= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t) f(x-t)\ dt\right) - f(x)\int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t)\ dt \qquad \textit{(by property (2), } \int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t)\ dt = 1\textit{)} \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t)(f(x-t) - f(x))\ dt \end{aligned}$$

Now, we can split this integral into 2 integrals, depending on whether $|t| < \delta$ or not (since in such a case, uniform continuity applies, whilst in the other case it doesn't). Hence, define:

$$A = \int_{|t|<\delta} k_n(t)(f(x-t) - f(x))\ dt$$

$$B = \int_{\delta \leq |t| \leq \frac{1}{2}} k_n(t)(f(x-t) - f(x))\ dt$$

such that:

$$f * k_n(x) - f(x) = A + B$$

For A, the uniform continuity condition applies, so we know that:

$$|f(x-t) - f(x)| \leq \frac{\varepsilon}{2}$$

Thus:

$$|A| = \left|\int_{|t|<\delta} k_n(t)(f(x-t) - f(x))\right| \leq \int_{|t|<\delta} |k_n(t)||f(x-t) - f(x)| \leq \frac{\varepsilon}{2}\int_{|t|<\delta} k_n(t)\ dt \leq \frac{\varepsilon}{2}\int_{|t|\leq\frac{1}{2}} k_n(t)\ dt = \frac{\varepsilon}{2}$$

where we have used the positivity of $k_n(t)$ so that $|k_n(t)| = k_n(t)$. Overall, we have shown that:

$$|A| \leq \frac{\varepsilon}{2}$$

Now, we consider B. Notice, since f is bounded, it follows that $\exists C > 0$ such that $\forall x \in \mathbb{R}$:

$$|f(x)| \leq C$$

In particular, this implies that:

$$|f(x-t) - f(x)| \leq |f(x-t)| + |f(x)| \leq 2C$$

Now, recall, by assumption of the theorem we have that:

$$\lim_{n\to\infty}\left(\int_{\delta \leq |t| \leq \frac{1}{2}} k_n(t)\ dt\right) = 0$$

That is, $\exists N$ such that whenever $n \geq N$, $\int_{\delta \leq |t| \leq \frac{1}{2}} k_n(t)\ dt$ is arbitrarily small. In other words, we can find N such that if $n \geq N$:

$$\int_{\delta \leq |t| \leq \frac{1}{2}} k_n(t)\ dt \leq \frac{\varepsilon}{4C}$$

This means that:

$$|B| \leq \int_{\delta \leq |t| \leq \frac{1}{2}} |k_n(t)||(f(x-t) - f(x))| \, dt \leq 2C \int_{\delta \leq |t| \leq \frac{1}{2}} k_n(t) \leq 2C \frac{\varepsilon}{4C} = \frac{\varepsilon}{2}$$

Thus, we have shown that whenever $n \geq N$, we have a $\varepsilon > 0$ such that:

$$|f * k_n(x) - f(x)| = |A + B| \leq |A| + |B| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

That is:

$$f * k_n(x) \to f(x)$$

so $k_n(x)$ is an approximation of unity, as required.

□

4.3 Corollary: Fejér Kernel as Approximation of Unity

*The **Fejér Kernel**:*

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x)$$

*is an **approximation of unity**.*
*However, the **Dirichlet Kernel** is **not** (it doesn't satisfy positivity).*

Proof. We just need to verify the 3 properties above:

1. $\forall x \in \mathbb{R}, \; k_n(x) \geq 0$

   This follows immediately from the definition of the Fejér Kernel.

2. $$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_N(t) \, dt = 1$$

   This is clear, since we know that $\int_0^1 D_N(x) \, dx = 1, \qquad N \in \mathbb{N}$ so:

   $$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_N(t) \, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N+1} \sum_{n=0}^{N} D_n(t) \, dt = \frac{1}{N+1} \sum_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} D_n(t) \, dt = \frac{1}{N+1} \sum_{n=0}^{N} 1 = 1$$

3. $$\lim_{n \to \infty} \left(\int_{\delta \leq |t| \leq \frac{1}{2}} K_N(t) \, dt \right) = 0$$

   Recall we can write the Fejér Kernel explicitly as:

   $$K_N(x) = \frac{1}{2N+1} \frac{1 - \cos(2\pi(N+1)x)}{\sin(\pi x)^2} = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)} \right)^2$$

Now, let $\delta \in \left(0, \frac{1}{2}\right)$ and let $|x| \geq \delta$. Now, we know that $\forall x \in \mathbb{R}$:

$$0 \leq (\sin(\pi(N+1)x))^2 \leq 1$$

Moreover, on $\left[0, \frac{\pi}{2}\right]$, $\sin^2(x)$ is an increasing function, so $\sin^2(\pi x)$ is increasing on $\left[0, \frac{1}{2}\right]$. In particular, this means that if $|x| \geq \delta$ with $-\frac{1}{2} \leq x \leq \frac{1}{2}$, then:

$$\sin^2(\pi x) \geq \sin^2(\sigma x) \implies \frac{1}{\sin^2(\pi x)} \leq \frac{1}{\sin^2(\sigma x)}$$

Thus, we have an upper bound on $K_N(x)$:

$$K_N(x) == \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2 \leq \frac{1}{N+1}\frac{1}{\sin^2(\pi\sigma)}$$

So then:

$$\int_{\delta \leq |t| \leq \frac{1}{2}} K_N(t)\ dt \leq \int_{\delta \leq |t| \leq \frac{1}{2}} \frac{1}{N+1}\frac{1}{\sin^2(\pi\sigma)}\ dt \leq \frac{1}{N+1}\frac{1}{\sin^2(\pi\sigma)}$$

which converges to 0 as $N \to \infty$.

Hence, it follows that the Fejér Kernel is an approximation of unity.

□

4.3.1 Exercises (TODO)

1. **Show that there exists a constant $c > 0$ such that:**

$$\int_0^1 |D_N(x)|\ dx \geq c\log(2+N), \qquad \forall N \geq 0$$

5 L^2 Convergence of Fourier Series: The Grand Finale

5.1 Theorem: Fejér's Theorem

*For every **1-periodic, continuous** function f:*

$$K_N * f \to f$$

uniformly on $\mathbb{R}$ as $N \to \infty$. [Theorem 5.5]

An alternative statement is that Fourier series of continuous functions are uniformly Cesàro Summable.

Proof. Since the Fejér Kernel is an approximation of unity, by definition:

$$K_N * f \to f$$

uniformly on $\mathbb{R}$ as $N \to \infty$

□

5.2 Corollary: Corollary of Fejér's Theorem

*Every **1-periodic, continuous** function can be **uniformly** approximated by **trigonometric polynomials**.*
*That is, for every **1-periodic, continuous** f, there exists a sequence:*

$$(f_n)_n$$

*of **trigonometric polynomials**, such that:*

$$f_n \to f$$

uniformly.

Proof. If we can show that $K_N * f$ is a **trigonometric polynomial**, then Fejér's Theorem tells us that:

$$K_n * f \to f$$

uniformly, as we require.

A **trigonometric polynomial** has the form:

$$\sum_{n=-N}^{N} c_n e^{2\pi i n x}$$

We can write Kejér's Kernel as:

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{2\pi i k x}$$

We then compute:

$$\begin{aligned} K_n * f(x) &= \int_0^1 f(t) K_n(x-t)\ dt \\ &= \int_0^1 f(t) \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{2\pi i k (x-t)}\ dt \\ &= \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{2\pi i k x} \int_0^1 f(t) e^{-2\pi i k t}\ dt \\ &= \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x} \end{aligned}$$

Now notice, for any n, we have that $\sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x}$ is a trigonometric polynomial. It then follows that $K_n * f(x)$ is a finite sum of trigonometric polynomials, so it must also be a trigonometric polynomial, as required.

□

5.3 Lemma: L^2 Convergence of Periodic and Continuous Functions

> *Let f be a* ***1-periodic*** *and* ***continuous*** *function. Then, $S_N f$ converges to f on L^2. That is:*
> $$\lim_{N\to\infty} \|S_N f - f\|_2 = 0$$
> *[Lemma 5.5]*

Proof. The notes have a longer, more spectacular proof, but I include a tiny, succint one below too.

Notice, $S_N f$ is a trigonometric polynomial, so by the Corollary of Fejér's Theorem, we know that:

$$S_N f \to f$$

uniformly. But then, we show that uniform convergence implies L^2 convergence, so it follows that

$$\lim_{n\to\infty} \|S_N f - f\|_2 = 0$$

For the proof in the notes we make use of **Minkowski's Inequality**:

> *If $f, g \in L^2([a,b])$ then:*
> $$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$$

and **Bessel's Inequality**:

> *If $(\phi_n)_{n=1,2,\ldots}$ is an orthonormal system on $[a,b]$, and $f \in L^2([a,b])$, then:*
> $$\sum_n |\langle f, \phi_n\rangle|^2 \leq \|f\|_2^2$$

Now, consider $\varepsilon > 0$. Since f is 1-periodic and continuous, the corollary of Fejér's Theorem says that we can find a sequence $(p_n)_{n\in\mathbb{N}}$ of trigonometric polynomials such that:

$$p_n \to f$$

uniformly. In particular, we can always find a trigonometric polynomial p satisfying:

$$|f(x) - p(x)| < \frac{\varepsilon}{2}, \qquad x \in \mathbb{R}$$

In particular, and using the fact that we integrate over a unit interval, we know that:

$$\|f-p\|_2 = \sqrt{\int_0^1 |f(x)-p(x)|^2 \, dx} < \sqrt{\int_0^1 \frac{\varepsilon^2}{4} \, dx} = \frac{\varepsilon}{2}$$

Let N denote the degree of p. Since p is a trigonometric polynomial, in particular:

$$S_N p = p$$

It follows that:

$$S_N f - f = S_N f - S_N p + S_N p - f = S_N(f-p) + (p-f)$$

Applying Minkowski's Inequality:

$$\|S_N f - f\|_2 \leq \|S_N(f-p)\|_2 + \|p-f\|_2$$

Now consider:

$$\begin{aligned}
\|S_N g\|_2^2 &= \langle S_N g, S_N g \rangle \\
&= \left\langle \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi i n x}, \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi i n x} \right\rangle \\
&= \sum_{n=-N}^{N} \sum_{k=-N}^{N} \left\langle \hat{g}(n) e^{2\pi i n x}, \hat{g}(k) e^{2\pi i k x} \right\rangle \\
&= \sum_{n=-N}^{N} \left\langle \hat{g}(n) e^{2\pi i n x}, \hat{g}(n) e^{2\pi i n x} \right\rangle \qquad \textit{(by orthogonality of trigonometric system)} \\
&= \sum_{n=-N}^{N} |\hat{g}(n)|^2
\end{aligned}$$

So it follows by Minkowski's Inequality that:

$$\|S_N g\|_2 = \sqrt{\sum_{|n| \leq N} |\hat{g}(n)|^2} \leq \|g\|_2$$

Hence, we have that:

$$\|S_N(f-p)\|_2 \leq \|f-p\|_2$$

Thus:

$$\|S_N f - f\|_2 \leq 2\|f-p\|_2 \leq \varepsilon$$

The assumption that f is continuous can be dropped, since every L^2 function can be approximated by continuous functions.

$\square$

5.4 Lemma: L^2 Convergence of Periodic Function in L^2

*If f is a **1-periodic** L^2 function, then there exists a sequence $(f_n)_n$ of **continuous, 1-periodic** functions so that:*

$$f_n \to f$$

in L^2. That is:

$$\|f_n - f\|_2 \to 0$$

[Exercise 5.12]

Proof. *Hint: First show the claim if f is a step function. Then, use that an L^2 function can be approximated by step functions in the L^2 norm.*

□

The difference between these 2 theorems is that the first on considered general 1-periodic functions, and showed L^2 convergence via $S_N f$.
In this theorem, we specifically consider 1-periodic function in L^2, and show that there is a sequence of periodic functions which produce L^2 convergence.
In the next theorem, we show that in fact, $S_N f$ are one such sequence of functions, thus showing that Fourier Series converge in L^2 to any 1-periodic function.

5.5 Theorem: Completeness of Trigonometric System

*The **trigonometric system** is complete:*

$$\sum_n |\langle f, e^{2\pi inx}\rangle|^2 = \|f\|_2^2$$

*In particular, as shown in Theorem 5.4, this is true **if and only if** for any 1-periodic $f \in L^2$ we have that $S_N f = \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx}$ converges to f in L^2. That is:*

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0$$

*Hence, the **Fourier Series** of f converges to f in the L^2 sense. [Theorem 5.7]*

Proof. Let $f \in L^2$ be 1-periodic, and let $\varepsilon > 0$.

By Exercise 5.12 above, there is a sequence of continuous, 1-periodic functions which converge to f in L^2. Pick g such that:

$$\|f - g\|_2 \le \varepsilon$$

Moreover, since g is continuous, Lemma 5.5 applies, such that:

$$\|S_N g - g\|_2 \to 0$$

in L^2. In particular, this means that $\forall N \geq M$, we have that:

$$\|S_N g - g\|_2 \leq \varepsilon$$

Now, we can write:

$$S_N f - f = S_N f - S_N g + S_N g - g + g - f$$

Applying Minkowski's Inequality, it follows that:

$$\|S_N f - f\|_2 \leq \|S_N f - S_N g\|_2 + \|S_N g - g\|_2 + \|g - f\|_2$$

We know that:

$$\|S_N f - S_N g\|_2 = \sqrt{\sum_{n=-N}^{N} |\widehat{(f-g)}(n)|^2}$$

So by Bessel's Inequality, it follows that:

$$\|S_N f - S_N g\|_2 \leq \|f - g\|_2 \leq \varepsilon$$

Thus, we have shown that:

$$\|S_N f - f\|_2 \leq 3\varepsilon$$

so in particular, $S_N f$ converges to f in L^2, so the trigonometric system is complete, as required.

□

5.6 Corollary: Parseval's Theorem

*Let $g, f \in L^2$ be **1-periodic**.*
Then, we have that:

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

In particular, this means that:

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

Proof. Notice that by (sesqui)linearity of the inner product:

$$\langle S_N f, g \rangle = \sum_{n=-N}^{N} \hat{f}(n)\langle e^{2\pi i n x}, g \rangle = \sum_{n=-N}^{N} \hat{f}(n)\overline{\hat{g}(n)}$$

Now consider:

$$|\langle S_N f, g \rangle - \langle f, g \rangle| = |\langle S_N f - f, g \rangle|$$

But by the Cauchy-Schwarz Inequality, alongside the fact that $S_N f \to f$ in L^2, it follows that::

$$|\langle S_N f, g\rangle - \langle f, g\rangle| \leq \|\langle S_N f - f\|_2 \|g\|_2 \to 0$$

Hence, we have that:

$$\langle S_N f, g\rangle \to \langle f, g\rangle$$

So in particular:

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

$\square$

5.6.1 Exercises (TODO)

1. **Let f be the 1-periodic function satisfying $f(x) = x, x \in [0, 1)$. Using Parseval's Theorem, derive the formula:**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2. **Using Parseval's Theorem for a suitable 1-periodic function, determine the value of:**

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

6 Workshop

> *We say that the series:*
>
> $$\sum_{n=0}^{\infty} a_n, \qquad a_n \in \mathbb{C}$$
>
> *is* ***Abel summable to*** S *if the series:*
>
> $$A(r) = \sum_{n=0}^{\infty} a_n r^n$$
>
> *converges for every* $r \in (0, 1)$ *and:*
>
> $$\lim_{r \to 1^-} A(r) = S$$

1. **We now prove *Abel's Theorem*:**

> *If the series* $\sum_{n=0}^{\infty} a_n$ *converges to* S*, then it is also* ***Abel summable*** *to* S*.*

(a) **Show that for complex numbers $a_0, \dots, a_N, b_0, \dots, b_N$:**

$$\sum_{n=0}^{N}(a_n - a_{n-1})b_n = a_N b_N + \sum_{n=0}^{N-1} a_n(b_n - b_{n+1})$$

We can think of this as *summation by parts*.

We expand the sum, with $a_{-1} = 0$. Then:

$$\begin{aligned}
\sum_{n=0}^{N}(a_n - a_{n-1})b_n &= (a_0 - a_{-1})b_0 + (a_1 - a_0)b_1 + (a_2 - a_1)b_2 + \ldots + (a_{N-1} - a_{N-2})b_{N-1} + (a_N - a_{N-1})b_N \\
&= a_0 b_0 + a_1 b_1 - a_0 b_1 + a_2 b_2 - a_1 b_2 + \ldots + a_{N-1}b_{N-1} - a_{N-2}b_{N-1} + a_N b_N - a_{N-1}b_N \\
&= a_0(b_0 - b_1) + a_1(b_1 - b_2) + \ldots + a_{N-1}(b_{N-1} - b_N) + a_N b_N \\
&= a_N b_N + \sum_{n=0}^{N-1} a_n(b_n - b_{n+1})
\end{aligned}$$

(b) **If:**

$$s_n = \sum_{k=0}^{n} a_k$$

show that:

$$\sum_{n=0}^{N} a_n r^n = s_N r^N + (1-r)\sum_{n=0}^{N-1} s_n r^n$$

> *This is a clear example of an easy question with good exam technique. For some reason, I completely ignored what we just showed above, and proceeded to prove this by induction, which is completely unnecessary.*

Notice, we can write:

$$a_n = s_n - s_{n-1}, \qquad a_0 = s_0$$

Then, applying summation by parts:

$$\sum_{n=0}^{N} a_n r^n = \sum_{n=0}^{N}(s_n - s_{n-1})r^n = s_N r^N + \sum_{n=0}^{N-1} s_n(r^n - r^{n+1}) = s_N r^N + (1-r)\sum_{n=0}^{N-1} s_n r^n)$$

(c) **For $r \in (0,1)$, show that $A(r) = (1-r)\sum_{n=0}^{\infty} s_n r^n$. To conclude, show that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $1 - r < \delta$ then:**

$$|A(r) - S| < \varepsilon$$

> *This becomes straightforward once we realise that:*
>
> $$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \implies (1-r)\sum_{k=0}^{\infty} r^k = a$$
>
> *as this allows us to apply the fact that that $s_n \to S$. Failing to realise this makes it extremely hard to prove.*

Notice:

$$A(r) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n r^n = \lim_{n\to\infty} \left(s_N r^N + (1-r) \sum_{n=0}^{N-1} s_n r^n) \right)$$

Notice, since $s_n \to S$, we have that s_N is bounded; since $r \in (0,1)$, it follows that $s_N r^N \to 0$, and so:

$$A(r) = (1-r) \sum_{n=0}^{\infty} s_n r^n$$

as required.

Moreover, we have that $s_n \to S$, so $\forall \varepsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$ then:

$$|s_n - S| < \varepsilon$$

Now, consider:

$$|A(r) - S| = \left| (1-r) \sum_{n=0}^{\infty} s_n r^n - S \right| = \left| (1-r) \sum_{n=0}^{\infty} s_n r^n - (1-r) \sum_{n=0}^{\infty} S r^n \right| = \left| (1-r) \sum_{n=0}^{\infty} (s_n - S) r^n \right|$$

Now, we can split the summation:

$$\left| (1-r) \sum_{n=0}^{\infty} (s_n - S) r^n \right| \leq (1-r) \sum_{n=0}^{\infty} |s_n - S| r^n = (1-r) \left(\sum_{n=0}^{N} |s_n - S| r^n + \sum_{n=N+1}^{\infty} |s_n - S| r^n \right)$$

By convergence of s_n, it follows that:

$$(1-r) \sum_{n=N+1}^{\infty} |s_n - S| r^n < (1-r) \sum_{n=N+1}^{\infty} \varepsilon |s_n - S| r^n = \varepsilon$$

Hence, we have that:

$$|A(r) - S| < (1-r) \sum_{n=0}^{N} |s_n - S| r^n + \varepsilon$$

But since $\sum_{n=0}^{N} |s_n - S| r^n$ is a finite sum, it is finite, and so, if $1 - r < \delta$, with δ small enough (that is, by making r as close to 1 as possible), it follows that we can ensure that:

$$(1-r) \sum_{n=0}^{N} |s_n - S| r^n < \varepsilon$$

Such that:

$$|A(r) - S| < 2\varepsilon$$

and so:

$$A(r) \to S$$

as required.

2. **For each of the following, decide if the series is Abel summable and if so, compute the corresponding limit:**

   (a) $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$

   We have:

$$A(r) = \sum_{n=0}^{\infty} (-1)^n r^n = \sum_{n=0}^{\infty} (-r)^n$$

This is a geometric series, with first term 1, and common ratio $-r$. Since $r \in (0,1)$:

$$A(r) = \frac{1}{1+r}$$

We still don't know the value of the Abel summation: we need to take a limit!

Taking the limit:

$$\lim_{r \to 1^-} A(r) = \lim_{r \to 1^-} \frac{1}{1+r} = \frac{1}{2}$$

Hence, $\sum_{n=0}^{\infty} (-1)^n$ is Abel summable to $\frac{1}{2}$.

(b) $\sum_{n=0}^{\infty} (-1)^n n = -1 + 2 - 3 + 4 + \dots$

This is quite tricky, and it requires some ingenuity.

We have:

$$A(r) = \sum_{n=0}^{\infty} (-1)^n n r^n$$

Notice, if we define:

$$f(r) = \sum_{n=0}^{\infty} (-1)^n r^n$$

we have a power series. We can compute its radius of convergence:

$$\left| \frac{(-1)^{n+1} r^{n+1}}{(-1)^n r^n} \right| = |-r|$$

so we require:

$$|r| < 1$$

Hence, $f(r)$ will be differentiable on $(0,1)$, and we can obtain the derivative by termwise differentiation:

$$f'(r) = \sum_{n=0}^{\infty} (-1)^n n r^{n-1} =$$

Hence, we have that:

$$A(r) = r f'(r)$$

Now, notice that:

$$f(r) = \sum_{n=0}^{\infty} (-1)^n r^n = \frac{1}{1+r}$$

so:

$$f'(r) = \frac{d}{dr}\left(\frac{1}{1+r}\right) = -\frac{1}{(1+r)^2}$$

Thus:

$$A(r) = r f'(r) = -\frac{r}{(1+r)^2}$$

so $A(r)$ converges, and:

$$\lim_{r \to 1^-} A(r) = \lim_{r \to 1^-} -\frac{r}{(1+r)^2} = -\frac{1}{4}$$

Hence, $\sum_{n=0}^{\infty} (-1)^n n$ is Abel summable to $-\frac{1}{4}$.

(c) $\sum_{n=0}^{\infty}(-1)^n 2^n = 1 - 2 + 4 - 8 + \dots$
We have:

$$A(r) = \sum_{n=0}^{\infty}(-1)^n 2^n r^n = \sum_{n=0}^{\infty}(-2r)^n$$

This only converges when:

$$|-2r| < 1 \implies |r| < \frac{1}{2}$$

so $A(r)$ won't converge $\forall r \in (0,1)$. Hence, $\sum_{n=0}^{\infty}(-1)^n 2^n$ is not Abel summable.

(d) $\sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots$
We have:

$$A(r) = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

But notice:

$$\lim_{r \to 1^-} A(r) = \lim_{r \to 1^-} \frac{1}{1-r} = \infty$$

so the limit is not finite, and so, $\sum_{n=0}^{\infty} 1$ is not Abel summable.

Recall, the ***Fourier Series*** *of a 1-periodic function f is given by:*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

where:

$$\hat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t}\ dt$$

A ***Fourier Series*** *is* ***Abel summable*** *if:*

$$A_r f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{2\pi i n x}$$

converges $\forall r \in (0,1)$, and the limit:

$$\lim_{r \to 1^-} A_r f(x)$$

exists.

3. **Show that if f is integrable, then the series defining $A_r f(x)$ converges absolutely $\forall r \in (0,1)$ and $\forall x \in \mathbb{R}$. You might want to use the fact that:**

$$|\hat{f}(n)| = \left| \int_0^1 f(t) e^{-2\pi i n t}\ dt \right| \leq \int_0^1 |f(t)|\ dt$$

Consider:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \left|\hat{f}(n) r^{|n|} e^{2\pi i n x}\right| &\leq \sum_{n=-\infty}^{\infty} \left|\int_0^1 f(t)\ dt\right| r^{|n|} |e^{2\pi i n x}| \\
&= \sum_{n=-\infty}^{\infty} \left|\int_0^1 f(t)\ dt\right| r^{|n|} \\
&= \left|\int_0^1 f(t)\ dt\right| \sum_{n=-\infty}^{\infty} r^{|n|}
\end{aligned}$$

Now, since f is integrable, $\left|\int_0^1 f(t)\ dt\right| < \infty$. Moreover, $\sum_{n=-\infty}^{\infty} r^{|n|}$ is a convergent geometric series, so it follows that:

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n) r^{|n|} e^{2\pi i n x}\right| \leq \sum_{n=-\infty}^{\infty} \left|\int_0^1 f(t)\ dt\right| r^{|n|} < \infty$$

and so, $A_r f(x)$ is absolutely convergent.

In the solutions, they use power series. In particular, they use the fact that:

$$|e^{2\pi i n x}| = 1 \qquad |\hat{f}(n)| \leq \int_0^1 |f(t)|\ dt$$

so the radius of convergence of:

$$\sum_{n=0}^{\infty} \hat{f}(n) r^n e^{2\pi i n x} \qquad \sum_{n=1}^{\infty} \hat{f}(-n) r^n e^{-2\pi i n x}$$

will be 1, and so, both are absolutely convergent for $r \in (0, 1)$, implying that:

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n) r^{|n|} e^{2\pi i n x}\right|$$

is absolutely convergent $\forall r \in (0, 1)$.

4. (a) **Show that for all integrable f with $r \in (0,1)$ and $x \in \mathbb{R}$, we have:**

$$A_r f(x) = (f * P_r)(x)$$

where:

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n x}$$

Notice, $P_r(x)$ is an absolutely convergent series. It is known as the *Poisson kernel.*

From definition:

$$\begin{aligned}
A_r f(x) &= \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{2\pi i n x} \\
&= \sum_{n=-\infty}^{\infty} \left(\int_0^1 f(t) e^{-2\pi i n t} \, dt \right) r^{|n|} e^{2\pi i n x} \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(t) e^{-2\pi i n t} r^{|n|} e^{2\pi i n x} \, dt \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(t) e^{2\pi i n (x-t)} r^{|n|} \, dt
\end{aligned}$$

Now, recall:

> *Suppose f_n is a sequence of functions, each of which is* ***integrable*** *on some I.*
> ***If****:*
>
> $$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$
>
> *(the sum of integrals of each function in the sequence is convergent)*
> ***and*** *f is a function on I, such that,*
>
> $$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
>
> *for any x, such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$[a], **then** f is **integrable** on I, and its integral is:*
>
> $$\int_I f = \sum_{n=1}^{\infty} \int_I f_n < \infty$$
>
> *[Theorem 4.3]*
>
> [a] this is just saying that we require the x to be such that the sum converges (to f)

Indeed, we have that:

$$\sum_{n=-\infty}^{\infty} \int_0^1 |f(t) e^{2\pi i n (x-t)} r^{|n|}| \, dt = \sum_{n=-\infty}^{\infty} r^{|n|} \left| \int_0^1 f(t) \, dt \right| = \left| \int_0^1 f(t) \, dt \right| \sum_{n=-\infty}^{\infty} r^{|n|} < \infty$$

since $\left| \int_0^1 f(t) \, dt \right|$ is finite, and $\sum_{n=-\infty}^{\infty} r^{|n|} \, dt$ is a geometric series with $|r| < 1$.
Hence, the theorem applies, and we can swap the integral and the summation:

$$A_r f(x) = \int_0^1 f(t) \sum_{n=-\infty}^{\infty} e^{2\pi i n (x-t)} r^{|n|} = (f * P_r)(x)$$

as required.

(b) **Show that:**

$$P_r(x) = \frac{1-r^2}{1-2r\cos(2\pi x)+r^2}, \qquad \forall r \in (0,1)$$

Since $P_r(x)$ is absolutely convergent, we can "choose" the order of summation, and so:

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{2\pi inx} = \sum_{n=0}^{\infty} r^n e^{2\pi inx} + \sum_{n=1}^{\infty} r^n e^{-2\pi inx} = \sum_{n=0}^{\infty} \left(re^{2\pi ix}\right)^n + \sum_{n=1}^{\infty} \left(re^{-2\pi ix}\right)^n$$

These are 2 geometric series, with common ratio less than 1, so:

$$\begin{aligned}
P_r(x) &= \frac{1}{1-re^{2\pi ix}} + \frac{re^{-2\pi ix}}{1-re^{-2\pi ix}} \\
&= \frac{1-re^{-2\pi ix}}{1-re^{-2\pi ix}}\frac{1}{1-re^{2\pi ix}} + \frac{1-re^{2\pi ix}}{1-re^{2\pi ix}}\frac{re^{-2\pi ix}}{1-re^{-2\pi ix}} \\
&= \frac{1-re^{-2\pi ix}+re^{-2\pi ix}-r^2}{1-r(e^{2\pi ix}+e^{-2\pi ix})+r^2} \\
&= \frac{1-r^2}{1-2r\cos(2\pi x)+r^2}
\end{aligned}$$

as required

(c) **Use:**

> *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of **1-periodic** and **integrable** functions satisfying:*
>
> 1. $\forall x \in \mathbb{R},\ k_n(x) \geq 0$
> 2. $$\int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t)\ dt = 1$$
> 3. $\forall \delta \in \left(0, \frac{1}{2}\right]$ *we have that:*
> $$\lim_{n\to\infty}\left(\int_{-\delta}^{\delta} k_n(t)\ dt\right) = 1$$
>
> *Then, $(k_n)_{n\in\mathbb{N}}$ is an **approximation of unity**. [Theorem 5.6]*

to show that $P_{r_n}(x)$ with $r_n = 1 - \frac{1}{n+1}$ provides an approximation of unity.
Notice:

$$1-r^2>0 \qquad 1-2r\cos(2\pi x)+r^2 \geq 1-2r+r^2 = (1-r)^2 > 0 \qquad \forall r \in (0,1)$$

and since:

$$P_r(x) = \frac{1-r^2}{1-2r\cos(2\pi x)+r^2}$$

it follows that $P_r(x) > 0$ as required.
Moreover:

$$\begin{aligned}\int_0^1 P_r(x)\ dx &= \int_0^1 \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n x}\ dx \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \int_0^1 e^{2\pi i n x}\ dx \qquad\qquad = r^0 \\ &= 1\end{aligned}$$

since $\int_0^1 e^{2\pi i n x}\ dx$ is non-zero only when $n = 0$.
For the third property, we use the alternative form, and we try to show that:

$$\lim_{n\to\infty}\left(\int_{\delta\le|t|\le\frac{1}{2}} P_{r_n}(t)\ dt\right) = 0$$

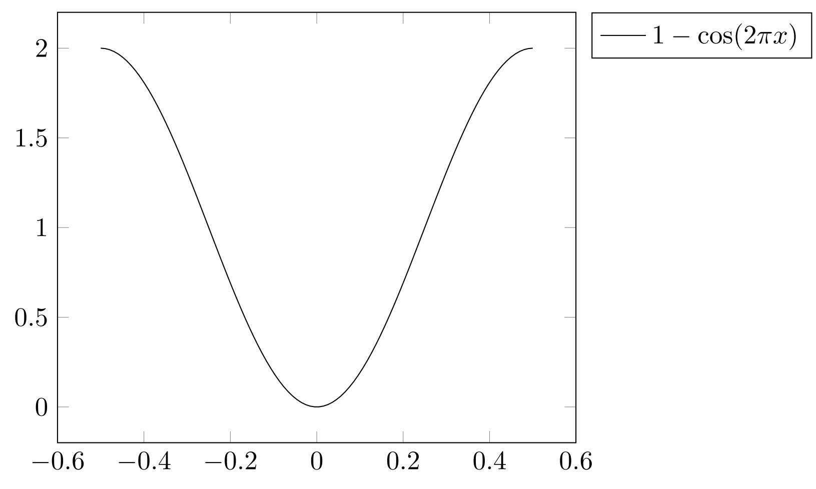
Consider $0 < \delta \le \frac{1}{2}$. Then, if $\delta \le |x| \le \frac{1}{2}$ (that is, either $x \in \left[\delta, \frac{1}{2}\right]$ or $x \in \left[-\frac{1}{2}, -\delta\right]$). Then.

$$P_r(x) = \frac{1-r^2}{1-2r\cos(2\pi x)+r^2} < \frac{1-r^2}{1-\cos(2\pi x)}$$

since:

$$1-\cos(2\pi x) < 1-\cos(2\pi x)+r^2$$

We can plot $1-\cos(2\pi x)$ for $x \in [-0.5, 0.5]$:



Now, if $x \in \left[\delta, \frac{1}{2}\right]$, we can see that no matter what δ we pick, $1-\cos(2\pi x)$ will be increasing, and so:

$$1-\cos(2\pi x) \ge 1-\cos(2\pi\delta)$$

Similarly, if $x \in \left[-\frac{1}{2}, -\delta\right]$, we have that $1-\cos(2\pi x)$ is decreasing, and so, no matter what δ we pick:

$$1-\cos(2\pi x) \ge 1-\cos(2\pi(-\delta)) = 1-\cos(2\pi\delta)$$

Thus, for any x satisfying $\delta \le |x| \le \frac{1}{2}$, we have that:

$$1-\cos(2\pi x) \ge 1-\cos(2\pi\delta)$$

and so for these x:

$$P_r(x) < \frac{1-r^2}{1-\cos(2\pi x)} \leq \frac{1-r^2}{1-\cos(2\pi\delta)}$$

Thus:

$$\lim_{n\to\infty}\left(\int_{\delta\leq|t|\leq\frac{1}{2}} P_{r_n}(t)\ dt\right) < \lim_{n\to\infty}\left(\int_{\delta\leq|t|\leq\frac{1}{2}} \frac{1-r_n^2}{1-\cos(2\pi\delta)}\ dt\right) = \lim_{n\to\infty}\left(\frac{(1-2\delta)(1-r_n^2)}{1-\cos(2\pi\delta)}\right) = 0$$

where we use the fact that $\lim_{n\to\infty} r_n = 1$.

> *Consider a sequence $(a_k)_{k\in\mathbb{N}}$.*
> *It's **Nth Cesàro Sum** or **Nth Cesàro Mean** is the sequence obtained by taking an average of the first N partial sums of a_k:*
>
> $$\sigma_N = \frac{\sum_{i=1}^N S_i}{N} = \frac{\sum_{i=1}^N \sum_{k=1}^i a_k}{N}$$
>
> *The series $\sum_{k=1}^\infty a_k$ is called **Cesàro Summable** to S if σ_N converges to $S < \infty$.*

5. **Prove that if $\sum_{k=1}^\infty a_k$ is summable to S, then $\sum_{k=1}^\infty a_k$ is Cesàro summable to S.**

Define $s_n = \sum_{k=1}^n a_k$. Since s_n converges, in particular:

1. s_n is *bounded*: $\exists M \in \mathbb{Z}^+ : \forall n \in \mathbb{N}\ :\ |a_n| < M$
2. $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ : \forall\ n \geq N \implies |s_n - S| < \frac{\varepsilon}{2}$

We claim that $\frac{s_1+s_2+\ldots+s_n}{n} \to a$ so, by the definition of the limit, we require that:

$$\forall\varepsilon > 0,\ \exists N^* \in \mathbb{N}\ :\ \forall n \geq N^* \implies \left|\frac{s_1+s_2+\ldots+s_n}{n} - S\right| < \varepsilon$$

Now, let:

$$N^* = \max\left\{N, \frac{4M(N-1)}{\varepsilon}\right\}$$

and let $n \geq N^*$.

Using the triangle inequality, we can split the LHS into 2 summations: since we are considering $n \geq N^*$, in particular $n \geq N$, so we can have one summation with terms $(s_i)_{1\leq i<N}$, and another one with terms $(s_i)_{N\leq i\leq n}$:

$$\begin{aligned}\left|\frac{s_1+s_2+\ldots+s_n}{n} - S\right| &= \frac{1}{n}\left|(s_1-S)+(s_2-S)+\ldots+(s_n-S)\right| \\ &\leq \frac{1}{n}|s_1-S|+|s_2-S|+\ldots+|s_n-S| \\ &= \frac{1}{n}\left(\sum_{i=1}^{N-1}|s_i-S| + \sum_{i=N}^{n}|s_i-S|\right)\end{aligned}$$

Now, since s_n is bounded, its limit is also bounded, so $\exists M \in \mathbb{Z}^+$:

$$\forall i \in \mathbb{N},\ |s_i| < M \quad \text{and} \quad |S| < M$$

For $i < N$, the largest possible value of $|s_i - S|$ must be $2M$ (for example if $s_i = M, S = -M$). This also follows from the triangle inequality ($|s_i - S| < |s_i| + |S| < M + M$). Thus:

$$\sum_{i=1}^{N-1} |s_i - S| \leq 2M(N-1)$$

For $i \geq N$, we can impose a tighter bound, as we know that $\forall i \geq N,\ |s_i - S| < \frac{\varepsilon}{2}$, so:

$$\sum_{i=N}^{n} |s_i - S| < \frac{\varepsilon(n - N + 1)}{2}$$

Thus, it follows that:

$$\frac{1}{n}\left(\sum_{i=1}^{N-1} |s_i - S| + \sum_{i=N}^{n} |s_i - S|\right) < \frac{2M(N-1)}{n} + \frac{\varepsilon(n-N+1)}{2n}$$

Since $n \geq N^*$, then either $n \geq N \geq \frac{4M(N-1)}{\varepsilon}$ or $n \geq \frac{4M(N-1)}{\varepsilon} \geq N$.

Since $n \geq \frac{4M(N-1)}{\varepsilon}$, then:

$$\frac{2M(N-1)}{n} \leq \frac{2M(N-1)}{\frac{4M(N-1)}{\varepsilon}} = \frac{\varepsilon}{2}$$

Moreover, since $n - N + 1 \leq n \implies \frac{n-N+1}{n} \leq 1$ then:

$$\frac{\varepsilon(n-N+1)}{2n} \leq \frac{\varepsilon}{2}$$

By choosing $n \geq N^*$, we thus ensure that:

$$\begin{aligned}
\left|\frac{s_1 + s_2 + \ldots + s_n}{n} - S\right| &= \frac{1}{n}\left|(s_1 - S) + (s_2 - S) + \ldots + (s_n - S)\right| \\
&\leq \frac{1}{n}\left(\sum_{i=1}^{N-1} |s_i - S| + \sum_{i=N}^{n} |s_i - S|\right) \\
&< \frac{2M(N-1)}{n} + \frac{\varepsilon(n-N+1)}{n} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, $\forall \varepsilon > 0,\ \exists N^* \in \mathbb{N}$ such that for all $n \geq N^*$:

$$\left|\frac{s_1 + s_2 + \ldots + s_n}{n} - S\right| < \varepsilon$$

By the definition of the limit, it follows that:

$$\frac{s_1 + s_2 + \ldots + s_n}{n} = \sigma_n \to S$$

So $\sum_{k=1}^{\infty} a_k$ is Cesàro summable to S, as required.

6. **Prove that the sum $\sum_{k=0}^{\infty}(-1)^k$ does not converge, but is Cesàro summable to some limit S, and determine S.**

> *The key here is to find a formula with which to define the sequence of partial sums.*

Looking at partial sums:

$$s_1 = 1 \qquad s_2 = 0 \qquad s_3 = 1 \qquad \ldots$$

We can write this as:

$$s_n = \frac{1}{2}((-1)^n + 1)$$

since when n is odd, we get $s_n = 0$, and if n is even we get $s_n = 1$.

Notice, s_n diverges, since $\lim_{n\to\infty}(-1)^n$ is undefined. However:

$$\begin{aligned}
\sigma_{N+1} &= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2}((-1)^n + 1) \\
&= \frac{1}{2(N+1)}\left(\sum_{n=0}^{N}((-1)^n + 1)\right) \\
&= \frac{1}{2(N+1)}\left(s_n + N + 1)\right) \\
&= \frac{\frac{1}{2}((-1)^n + 1) + N + 1}{2(N+1)} \\
&= \frac{1}{2} + \frac{(-1)^n + 1}{4(N+1)}
\end{aligned}$$

Hence:

$$\lim_{N\to\infty}\sigma_{N+1} = \lim_{N\to\infty}\frac{1}{2} + \frac{(-1)^n + 1}{4(N+1)} = \frac{1}{2}$$

so $\sum_{k=0}^{\infty}(-1)^k$ is Cesàro summable to $\frac{1}{2}$.

As an alternative, one can notice that in the partial sums, only the even terms contribute to $s_0 + s_1 + \ldots + s_N$, so:

$$s_0 + s_1 + \ldots + s_N = 1 + \left\lfloor\frac{N}{2}\right\rfloor$$

So then:

$$\sigma_{N+1} = \frac{1 + \lfloor\frac{N}{2}\rfloor}{N+1} \leq \frac{1 + \frac{N}{2}}{N} = \frac{N+2}{2N} \to \frac{1}{2}$$

However, I feel that this is less rigorous.

7. **Show that if $\sum_{k=1}^{\infty} a_k$ is Cesàro summable, then:**

$$\lim_{n\to\infty}\frac{a_n}{n} = 0$$

Since the series is Cesàro summable, in particular σ_N converges, and so, it must be Cauchy. This means that:

$$\sigma_N - \sigma_{N-1} \to 0$$

But we compute:

$$\begin{aligned}
\sigma_N - \sigma_{N-1} &= \frac{\sum_{n=1}^{N} s_n}{N} - \frac{\sum_{n=1}^{N-1} s_n}{N-1} \\
&= \frac{(N-1)\sum_{n=1}^{N} s_n - N\sum_{n=1}^{N-1} s_n}{N(N-1)} \\
&= \frac{N\sum_{n=1}^{N} s_n - \sum_{n=1}^{N} s_n - N\sum_{n=1}^{N-1} s_n}{N(N-1)} \\
&= \frac{Ns_N - \sum_{n=1}^{N} s_n}{N(N-1)} \\
&= \frac{(N-1)s_N - \sum_{n=1}^{N-1} s_n}{N(N-1)} \\
&= \frac{s_N}{N} - \frac{\sigma_{N-1}}{N}
\end{aligned}$$

Notice, since σ_N converges, in particular it is bounded, so:

$$\frac{\sigma_{N-1}}{N} \to 0$$

But then:

$$\sigma_N - \sigma_{N-1} \to 0 \iff \frac{s_N}{N} \to 0$$

In particular, this again means that $\frac{s_N}{N}$ is Cauchy, so:

$$\frac{s_N}{N} - \frac{s_{N-1}}{N-1} \to 0$$

But we compute:

$$\begin{aligned}
\frac{s_N}{N} - \frac{s_{N-1}}{N-1} &= \frac{\sum_{n=1}^{N} a_n}{N} - \frac{\sum_{n=1}^{N-1} a_n}{N-1} \\
&= \frac{(N-1)\sum_{n=1}^{N} a_n - N\sum_{n=1}^{N-1} a_n}{N(N-1)} \\
&= \frac{N\sum_{n=1}^{N} a_n - \sum_{n=1}^{N} a_n - N\sum_{n=1}^{N-1} a_n}{N(N-1)} \\
&= \frac{Na_N - \sum_{n=1}^{N} a_n}{N(N-1)} \\
&= \frac{(N-1)a_N - \sum_{n=1}^{N-1} a_n}{N(N-1)} \\
&= \frac{a_N}{N} - \frac{s_{N-1}}{N(N-1)}
\end{aligned}$$

We know that $\frac{s_{N-1}}{N-1} \to 0$ so $\frac{s_{N-1}}{N(N-1)} \to 0$, and so:

$$\frac{s_N}{N} - \frac{s_{N-1}}{N-1} \to 0 \iff \frac{a_N}{N} \to 0$$

as required.

8. **Give an example of a series which is Abel summable, but not Cesàro summable.**

   We saw above that $\sum_{n=1}^{\infty}(-1)^n n$ is Abel summable. However, it won't be Cesàro summable. If it were, then:
$$\frac{a_n}{n} \to 0$$
   but:
$$\frac{a_n}{n} = (-1)^n$$
   which doesn't even converge.

9. **Show that if a series is Cesàro summable, then it is Abel summable (to the same value).**

   Let $\sum_{n=1}^{\infty} a_n$ be Cesàro summable, and assume that:
$$\sigma_N \to 0$$
   If not, we can modify a_1 so that this is the case, since:
$$\sigma_N = \frac{\sum_{n=1}^{N} s_n}{N} = \frac{Na_1 + \sum_{n=1}^{N}\sum_{k=2}^{n} a_k}{N} = a_1 + \frac{\sum_{n=1}^{N}\sum_{k=2}^{n} a_k}{N}$$
   so modifying a_1 accordingly ensures that we can make σ_N converge to 0.

   Now, consider the series:
$$A(r) = \sum_{n=1}^{\infty} a_n r^n$$
   this will converge for $r \in (0, 1)$.

   Moreover, from 1)b) above:
$$\sum_{n=1}^{\infty} a_n r^n = (1-r)\sum_{n=1}^{\infty} s_n r^n$$
   which further implies:
$$\sum_{n=1}^{\infty} s_n r^n = (1-r)\sum_{n=1}^{\infty} n\sigma_n r^n$$
   Hence:
$$A(r) = \sum_{n=1}^{\infty} a_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$
   Since $\sigma_n \to 0$, it follows that $\forall \varepsilon > 0\ \exists N$ such that if $n \geq N$:
$$|\sigma_n| \leq \varepsilon$$
   Then:
$$|A(r)| = \left|(1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n\right| \leq (1-r^2)\sum_{n=1}^{\infty} n|\sigma_n| r^n \leq (1-r^2)\left(\sum_{n=1}^{N} n|\sigma_n| r^n + \sum_{n=N+1}^{\infty} n\varepsilon r^n\right)$$
   Notice, if we let $f(r) = \sum_{n=N*1}^{\infty} r^n$, we have aconvergent power series, so $f'(r) = \sum_{n=N*1}^{\infty} nr^{n-1}$ is also a convergent power series, and:
$$rf'(r) = \sum_{n=N+1}^{\infty} nr^n$$
   But:
$$f(r) = \frac{r^{N+1}}{1-r} \implies f'(r) = \frac{(N+1)r^N(1-r) + r^{N+1}}{(1-r)^2} = r^N\left(\frac{N - Nr + 1}{(1-r)^2}\right)$$

Thus:

$$(1-r)^2 \sum_{n=N+1}^{\infty} n\varepsilon r^n = \varepsilon r^{N+1}(N - Nr + 1) < \varepsilon$$

Moreover, $\sum_{n=1}^{N} n|\sigma_n|r^n$ is a finite sum, so if we make r as close as we want to 1, we ensure that:

$$(1-r)^2 \sum_{n=1}^{N} n|\sigma_n|r^n \leq \varepsilon$$

Thus, we ensure that:

$$|A(r)| \leq 2\varepsilon$$

so:

$$\lim_{r \to 1^-} A(r) = 0$$

as required.

10. **Prove the following Theorem of Tauber:**

> *If* $\sum_{n=1}^{\infty} a_n$ *is* ***Abel summable*** *to* S *and:*
>
> $$na_n \to 0$$
>
> *then* $\sum_{n=1}^{\infty} a_n = S$.