Honours Algebra - Week 3 - Abstract Linear Mappings

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Contents

Based on the notes by Iain Gordon, Sections 2.3 - 2.4

1 Abstract Linear Mappings and Matrices

1.1 Generalising Representing Matrices

- **What is a representing matrix?**
  - we found a **bijection** linking homomorphisms to matrices:

$$M : Hom_{\mathbb{F}}(\mathbb{F}^m, \mathbb{F}^n) \to Mat(n \times m; \mathbb{F})$$

$$M : f \to [f]$$

  - the bijection was defined by defining a matrix with column vectors as $f(E) \subset \mathbb{F}^n$, where E is the set of standard bases of $\mathbb{F}^m$

- **What is an abstract linear mapping?**
  - a linear mapping $f : V \to W$, where V, W are (abstract) vector spaces, and $dim(V) = m, dim(W) = n$
  - we try to relate V, W to $\mathbb{F}^m, \mathbb{F}^n$

- **Can we represent abstract linear mappings as matrices?**
  - we know that if $dimV = n$, then there exists an isomorphism between $\mathbb{F}^n$ and V, namely:

$$\Phi : \mathbb{F}^n \to V$$

$$(\alpha_1, \ldots, \alpha_n) \to \alpha_1 \underline{v}_1 + \ldots + \alpha_n \underline{v}_n$$

    where $\underline{v}_1, \ldots, \underline{v}_n$ are **basis vectors** of V
  - it stands to reason from this isomorphism, that linear mappings $V \to W$, with **ordered bases**, can also be represented via matrices

1.2 Theorem: Abstract Linear Mappings and Matrices

*Let $\mathbb{F}$ be a **field**.*
*Let V, W be **vector spaces** over $\mathbb{F}$, with ordered bases:*

$$A = (\underline{v}_1, \dots, \underline{v}_m)$$

$$B = (\underline{w}_1, \dots, \underline{w}_n)$$

respectively.
For each linear mapping:

$$f : V \to W$$

*we can associate a **representing matrix of the mapping** f **with respect to the bases** A **and** B, which we denote as ${}_B[f]_A$.*
This is the matrix which turns basis elements in A to an element of W, expressed as a linear combination of basis elements in B.
In particular, the entries a_{ij} are given by:

$$f(\underline{v}_j) = \sum_{i=1}^{n} a_{ij}\underline{w}_i, \qquad f(\underline{v}_j) \in W$$

(since a_{ij} represent the coordinates in the space spanned by B).
*We again have a bijection (in fact, an **isomorphism** of vector spaces):*

$$M_B^A : Hom_{\mathbb{F}}(V, W) \to Mat(n \times m; \mathbb{F})$$

$$M_B^A : f \to {}_B[f]_A$$

[Theorem 2.3.1]

Proof. Define the isomorphisms:

$$\Phi_A : \mathbb{F}^m \to V$$

$$\Phi_B : \mathbb{F}^n \to W$$

as at the start of the section. The idea of this proof is summarised in the following diagram:

$$
\begin{tikzcd}
\boldsymbol{V} \arrow[r, "f"] \arrow[d, bend left, "\Phi_A^{-1}"] & \boldsymbol{W} \arrow[d, bend right, "\Phi_B^{-1}"'] \\
\mathbb{F}^{\boldsymbol{m}} \arrow[u, bend left, "\Phi_A"] \arrow[r, "\Phi_B^{-1} \circ f \circ \Phi_A"'] & \mathbb{F}^{\boldsymbol{n}} \arrow[u, bend right, "\Phi_B"']
\end{tikzcd}
$$

The idea is that we know how to map homomorphisms $\mathbb{F}^m \to \mathbb{F}^n$ to matrices, so if we want a matrix representation of $V \to W$, we can first map it to $\mathbb{F}^m \to \mathbb{F}^n$, and then get the corresponding matrix. To do this:

1. map $\mathbb{F}^m$ to V (we have an isomorphism for this)
2. map V to W (we have f for this)
3. map W to $\mathbb{F}^n$ (we have an inverse isomorphism for this)

It is then easy to see that we have:

$$_B[f]_A = [\Phi_B^{-1} \circ f \circ \Phi_A]$$

and the bijection is simply a composition of bijections:

$$Hom_{\mathbb{F}}(V, W) \to Hom_{\mathbb{F}}(\mathbb{F}^m, \mathbb{F}^n) \to Mat(n \times m; \mathbb{F})$$

$$f \to \Phi_B^{-1} \circ f \circ \Phi_A \to [\Phi_B^{-1} \circ f \circ \Phi_A]$$

$\square$

- **How can we represent mappings from or to the standard bases?**
  - the standard basis of $\mathbb{F}^n$ is:

    $$S(n)$$

  - whilst we could explicitly write:

    $$_{S(m)}[f]_{S(n)}$$

    $$_{S(m)}[f]_A$$

    $$_B[f]_{S(n)}$$

    it is more concise to use:

    $$[f]$$

    $$[f]_A$$

    $$_B[f]$$

- **How can we define the inverse of the bijection $\mathbb{F}^n \to V$?**
  - let Φ_A be the bijection:
  $$(\alpha_1, \ldots, \alpha_n) \to \alpha_1 \underline{v}_1 + \ldots + \alpha_n \underline{v}_n$$
  with $A = \{\underline{v}_1, \ldots, \underline{v}_n\}$
  - the **inverse** is given by:
  $$\Phi_A^{-1} : \underline{v} \to {}_A[\underline{v}]$$
  where ${}_A[\underline{v}] \in \mathbb{F}^n$ is a **column vector**
  - we call ${}_A[\underline{v}]$ the **representation of the vector $\underline{v}$ with respect to the basis** A, since depending on the basis vectors used by V, the elements of ${}_A[\underline{v}]$ will differ

1.3 Theorem: The Representing Matrix of a Composition of Linear Mappings

Let $\mathbb{F}$ be a field.
Let U, V, W be ***finite*** *dimensional vector spaces over F, with ordered bases A, B, C.*
If
$$f : U \to V$$
$$G : V \to W$$
are ***linear mappings****, then the* ***representing matrix*** *of the composition:*
$$g \circ f : U \to W$$
is the ***matrix product*** *of the* ***representing matrices*** *of f and g:*
$${}_C[g \circ f]_A = {}_C[g]_B \circ {}_B[g]_A$$
[Theorem 2.3.2]

Proof. The proof just relies on unpacking the notation:
$${}_C[g \circ f]_A = [\Phi_C^{-1} \circ (g \circ f) \circ \Phi_A]$$

$$\begin{aligned} & {}_C[g]_B \circ {}_B[g]_A \\ =& [\Phi_C^{-1} \circ g \circ \Phi_B] \circ [\Phi_B^{-1} \circ f \circ \Phi_A] \\ =& [\Phi_C^{-1} \circ g \circ \Phi_B \circ \Phi_B^{-1} \circ f \circ \Phi_A] \\ =& [\Phi_C^{-1} \circ (g \circ f) \circ \Phi_A] \end{aligned}$$

so both sides are equal.

□

1.4 Theorem: Representation of the Image of a Vector

> *Let $\mathbb{F}$ be a field.*
> *Let V, W be* ***finite*** *dimensional vector spaces over $\mathbb{F}$, with ordered bases A, B.*
> *Let*
> $$f : V \to W$$
> *be a* ***linear mapping****.*
> *For $\underline{v} \in V$:*
> $${}_B[f(\underline{v})] = {}_B[f]_A \circ {}_A[\underline{v}]$$
> *In other words, to get the image of ${}_A[\underline{v}]$ in the basis B of W, we just need to apply the representing matrix with respect to A and B. [Theorem 2.3.4]*

Proof. As above, we show that both sides are equal:

$${}_B[f(\underline{v})] = \Phi_B^{-1}(f(\underline{v})), \qquad f(\underline{v}) \in W$$

$$\begin{aligned} &{}_B[f]_A \circ {}_A[\underline{v}] \\ =&[\Phi_B^{-1} \circ f \circ \Phi_A] \circ \Phi_A^{-1}(\underline{v}) \\ =&\Phi_B^{-1}(f(\underline{v})) \end{aligned}$$

This can be shown more explicitly. Define:

$$A = (\underline{v}_1, \ldots, \underline{v}_m)$$
$$B = (\underline{w}_1, ldots, \underline{w}_n)$$

Define ${}_B[f]_A$ as the $n \times m$ matrix, given by the elements a_{ij} satisfying:

$$f(\underline{v}_j) = \sum_{i=1}^{n} a_{ij} \underline{w}_i$$

Since A is a basis of V, we can write any $v \in V$ as:

$$\underline{v} = \sum_{j=1}^{m} x_j \underline{v}_j$$

where $(x_1, \ldots, x_m) \in \mathbb{F}^m$.

Then:

$$\begin{aligned} f(\underline{v}) &= \sum_{j=1}^{m} x_j f(\underline{v}_j) \\ &= \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} a_{ij} \underline{w}_i \right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right) \underline{w}_i \end{aligned}$$

Notice, we are expressing $f(v)$ using the basis elements of W, having started with $\underline{v}$, defined using the basis elements of V. If we define:

$$y_i = \sum_{j=1}^{m} a_{ij} x_j$$

then the whole transformation can be summarised via:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = {}_B[f]_A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$\square$

1.4.1 Examples

- recall, in the previous week we define the linear mapping:

  $$f : \mathbb{R}^2 \to \mathbb{R}^2$$

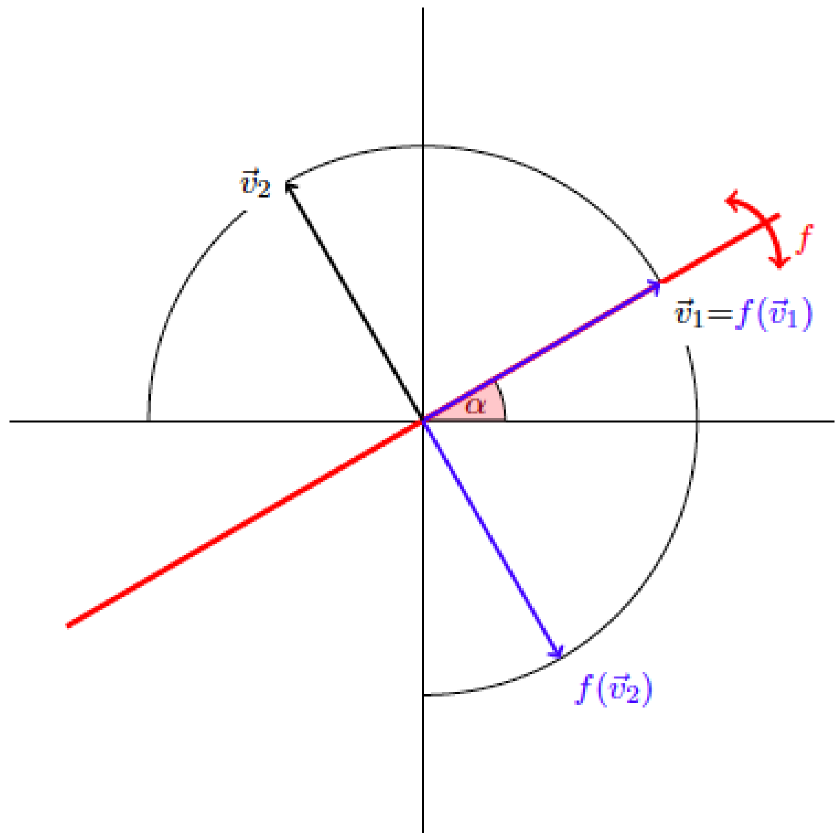
  such that it reflected on the straight line which makes an angle α with the x-axis. If we define $A = (\underline{v}_1, \underline{v}_2)$ with:

  $$\underline{v}_1 = (\cos\alpha, \sin\alpha)^T$$
  $$\underline{v}_2 = (-\sin\alpha, \cos\alpha)^T$$

  then:

  $${}_A[f]_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  To see why, it is easier to argue geometrically:



$\underline{v}_1$ is in the direction of the reflection line (just use the right-angled triangle), so when reflected it won't change. $\underline{v}_2$ is perpendicular to this line, so when reflected, it goes diametrically opposite. In other words:

$$f(\underline{v}_1) = \underline{v}_1 \qquad f(\underline{v}_2) = -\underline{v}_2$$

from which the matrix follows (bear in mind $\underline{v}_1 = (1,0)^T, \underline{v}_2 = (0,1)^T$ in the space which they span).

- consider the following vector spaces:

$$V = \mathbb{F}_{\leq 3}[x], \qquad A = \{\underline{v}_1 = 1, \underline{v}_2 = x, \underline{v}_3 = x^2, \underline{v}_4 = x^3\}$$
$$W = \mathbb{F}_{\leq 2}[x], \qquad B = \{\underline{w}_1 = 1, \underline{w}_2 = 1 + x, \underline{w}_3 = 1 + x^2\}$$

and define the linear mapping:

$$D : V \to W$$
$$D : v \to \frac{dv}{dx}$$

We want to find the matrix ${}_B[D]_A$ which performs the mapping D, from an eleemnt written via the basis A, to an element in W written via the basis B. For example, if:

$$\underline{v} = x^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in V$$

Then:

$$D(x^3) = 3x^2 = 3\underline{w}_3 - 3\underline{w}_1 = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \in W$$

(Technically, the column vector is **not** part of V, but rather of $\mathbb{F}^4$, but it is more useful to think as a column vector, particularly when thinking about D as a matrix) In other words, we want:

$$_B[D]_A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

We know that:

$$_B[D]_A = [\Phi_B^{-1} \circ D \circ \Phi_A]$$

Which is nothing but the matrix with column vectors:

$$_B[D(\underline{v}_i)]$$

(this is because $_B[D(\underline{v}_i)] = \Phi_B^{-1}(D(\underline{v}_i))$, and as column vectors we want to consider the basis elements)
Hence:

$$_B[D(\underline{v}_1)] = D(1) = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$_B[D(\underline{v}_2)] = D(x) = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$_B[D(\underline{v}_3)] = D(x^2) = 2x = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$_B[D(\underline{v}_4)] = D(x^3) = 3x^2 = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

Hence, we have that:

$$_B[D]_A = \begin{pmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Hence, if we consider any $\underline{v} = (\alpha, \beta, \mu, \omega)^T \in V$ (again, technically not in V), we can convert it to an element of W with basis B using:

$$
{}_B[D(\underline{v})] = {}_B[D]_A {}_A[\underline{v}] \implies {}_B[D(\underline{v})] = \begin{pmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \mu \\ \omega \end{pmatrix} = \begin{pmatrix} \beta - 2\mu - 3\omega \\ 2\mu \\ 3\omega \end{pmatrix}
$$

We can easily verify that if $\underline{v} = x^3$, this gives the right answer we obtained before. If we then actually want to convert it to an element in W (currently we just have a vector in $\mathbb{F}^3$), we just have to use:

$$
\Phi_B({}_B[D(\underline{v})]) \implies \begin{pmatrix} \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \end{pmatrix} \begin{pmatrix} \beta - 2\mu - 3\omega \\ 2\mu \\ 3\omega \end{pmatrix} = (\beta - 2\mu - 3\omega)\underline{w}_1 + 2\mu\underline{w}_2 + 3\omega\underline{w}_3
$$

Notice, if we put this back in terms of the basis A, we get:

$$
(\beta - 2\mu - 3\omega)(1) + 2\mu(1 + x) + 3\omega(1 + x^2) = \beta + 2\mu x + 3\omega x^2
$$

which is precisely the derivative of:

$$
\alpha + \beta x + \mu x^2 + \omega x^3
$$

as expected.

2 Changing Bases Using Matrices

2.1 Theorem: Change of Basis

- **What is the change of basis matrix?**
  - let V, W be vector spaces with respective bases A, B
  - the **change of basis matrix** is the representing matrix (with respect to A, B) defined by the **identity** mapping:

$$
{}_B[id_V]_A
$$

  - the entries are given by the a_{ij} satisfying:

$$
\underline{v}_j = \sum_{i=1}^{n} a_{ij}\underline{w}_i, \qquad \underline{v}_j \in A, \underline{w}_i \in B
$$

Let $\mathbb{F}$ be a field.
*Let V, W be **finite** dimensional vector spaces over $\mathbb{F}$.*
Let:

$$f : V \to W$$

be a linear mapping.
Suppose that V has ordered bases A, A'.
Similarly, suppose that W has ordered bases B, B'.
Then:

$$_{B'}[f]_{A'} = {}_{B'}[id_W]_B \circ {}_B[f]_A \circ {}_A[id_V]_{A'}$$

In other words, we can convert the representing matrix with respect to different bases, by applying the change of basis matrix.[Theorem 2.4.3]

Proof. From (1.3) we know that:

$$_C[g \circ f]_A = {}_C[g]_B \circ {}_B[g]_A$$

We also know that:

$$f = id_W \circ f \circ id_V$$

(since:

$$id_W(f(id_V(\underline{v})) = id_W(f(\underline{v}))f(\underline{v})$$

) Hence:

$$\begin{aligned}
&{}_{B'}[f]_{A'}\\
=&{}_{B'}[id_W \circ f \circ id_V]_{A'}\\
=&{}_{B'}[id_W \circ (f \circ id_V)]_{A'}\\
=&{}_{B'}[id_W]_B \circ {}_B[f \circ id_V]_{A'}\\
=&{}_{B'}[id_W]_B \circ {}_B[f]_A \circ {}_A[id_V]_{A'}
\end{aligned}$$

□

2.1.1 Examples

As above, define the linear mapping:

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

such that it reflected on the straight line which makes an angle α with the x-axis. Define $B = (\underline{v}_1, \underline{v}_2)$ with:

$$\underline{v}_1 = (\cos\alpha, \sin\alpha)^T$$

$$\underline{v}_2 = (-\sin\alpha, \cos\alpha)^T$$

and use $A = (\underline{e}_1, \underline{e}_2)$ as the standard basis. The change of basis matrix has entries satisfying:

$$\begin{pmatrix}1\\0\end{pmatrix} = a_{11}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + a_{21}\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = a_{12}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + a_{22}\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}$$

In other words:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Thus:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^{-1}$$

since we are just multiplying by the identity matrix. We know that (yeah, I used the determinant):

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

So then:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

We can then define the change of basis matrix:

$$_B[f]_A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

What this gives us is a form of converting a vector in A to its corresponding vector in B. For example, if we consider:

$$_A[\underline{v}_1] = (\cos\alpha, \sin\alpha)^T$$

we know that in terms of the basis B, $_B[\underline{v}_1] = (1, 0)^T$. Indeed:

$$_B[f]_{A\,A}[\underline{v}_1] = (1, 0)^T$$

2.2 Corollary: Change of Basis for Endomorphisms

This is a special case of the Theorem above, whereby instead of using different bases in a different vector space, we consider endomorphisms.

Let V be a ***finite*** *dimensional vector space.*
Define the endomorphism:

$$f : V \to V$$

Suppose that A, A' are ***ordered bases*** *of V.*
Then:

$$_{A'}[f]_{A'} = {}_A[id_V]^{-1}_{A'} \circ {}_A[f]_A \circ {}_A[id_V]_{A'}$$

[Corollary 2.4.4]

Proof. It is easy to see that:

$$_A[id_V]_A = \mathbb{I}_n$$

since, if $\underline{v}_i \in A$:

$$\underline{v}_i = \sum_{i=1}^{n} a_{ij}\underline{v}_i \iff a_{ij} = \delta_{ij}$$

Using (1.3), we know that:

$$_A[id_V]_A = \mathbb{I}_n \iff {}_A[id_V]_{A'} \circ {}_{A'}[id_V]_A = \mathbb{I}_n$$

Hence, it follows that:

$$_A[id_V]_{A'}^{-1} = {}_{A'}[id_V]_A$$

Thus, if we apply the Theorem above - (2.1) - using $A' = B'$ and $A = B$, we get:

$$_{A'}[f]_{A'} = {}_{A'}[id_V]_A \circ {}_A[f]_A \circ {}_A[id_V]_{A'} = {}_A[id_V]_{A'}^{-1} \circ {}_A[f]_A \circ {}_A[id_V]_{A'}$$

$\square$

- **What are similar matrices?**
  - consider:

$$N = {}_B[f]_B$$
$$M = {}_A[f]_A$$

- we say that N and M are **similar matrices** if:

$$N = T^{-1}MT$$

  where:

$$T = {}_A[id_V]_B$$

2.2.1 Examples

Consider $V = \mathbb{F}^2$, and the following bases:

$$A = \{(1,2)^T, (2,3)^T\} = \{\underline{v}_i\}$$
$$B = \{(1,5)^T, (3,2)^T\} = \{\underline{w}_i\}$$

We want to construct the change of basis matrix:

$$_B[id_V]_A$$

This matrix has coefficients a_{ij} given by:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + a_{21} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + a_{22} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

In matrix form:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Notice:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = {}_{S(2)}[id_v]_A$$

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} = {}_{S(2)}[id_v]_B$$

To find the change of basis matrix, we just need to invert $\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}^{-1} = -\frac{1}{13}\begin{pmatrix} 2 & -3 \\ -5 & 1 \end{pmatrix}$$

So it follows that:

$$_B[id_V]_A = -\frac{1}{13}\begin{pmatrix} 2 & -3 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 4 & 5 \\ 3 & 7 \end{pmatrix}$$

2.2.2 Exercises (TODO)

1. **Check that Corollary 2.4.4 agrees with the calculations made in the examples above, where we consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be the reflection on the line through the origin making an angle of α with the x-axis.**

2. **Let V be an F-vector space with ordered basis $A = (\underline{v}_1, \ldots, \underline{v}_n)$. Show that the change of basis matrices lead to a bijection:**

$$\{\textit{\textbf{ordered bases of }} V\} \to GL(n; \mathbb{F})$$

$$B \to {}_B[id_V]_A$$

   where $GL(n; \mathbb{F})$ is the group of $n \times n$ invertible matrices.

   To show this is a bijection, it is sufficient to show that it has an inverse, and the inverse is a bijection. In other words, we want a bijection of the form:

$$GL(n; \mathbb{F}) \to \{\textit{ordered bases of } V\}$$

$$g \to B$$

   We claim that this can be done by using:

$$B = \{g^{-1}\underline{v}_1, \ldots, g^{-1}\underline{v}_n\}$$

   If we show that:

   - B is a basis of V
   - $g = {}_B[id_V]_A$

   then we will have shown that the mapping $g \to B$ is indeed a bijection, and furthermore, an inverse of the original map. To see why this is, its because it allows us to do the following set of mappings:

$$B \to {}_B[id_V]_A := g \to B$$

   so clearly they are inverses.

We first show that $\{g^{-1}\underline{v}_1, \dots, g^{-1}\underline{v}_n\}$ is a basis. This is relatively straightforward.

To show linear independence, we can employ the linearity of g. Suppose that:

$$\sum_{i=1}^{n} \lambda_i (g^{-1}\underline{v}_i) = 0$$

Applying g, and knowing that as a linear map, $g(0) = 0$:

$$g\left(\sum_{i=1}^{n} \lambda_i (g^{-1}\underline{v}_i)\right) = g(0) \implies \sum_{i=1}^{n} \lambda_i \underline{v}_i = 0$$

Since A is a basis, we know that $\sum_{i=1}^{n} \lambda_i \underline{v}_i = 0$ only when $\lambda_i = 0$, so it follows that the set B is linearly independent.

Moreover, notice that V is such that $dim(V) = n$. Moreover, B has n elements, so it spans an n-dimensional subspace of V. Hence, it follows that B spans V. Hence, B must be a basis.

Now, if we compose the mappings, we'd get:

$$g \to B \to {}_B[id_V]_A$$

We have an inverse (and so a bijection) if we have $g = {}_B[id_V]_A$. Now, recall what ${}_B[id_V]_A$ "means": it is a matrix constructed by being able to write $A = \{\underline{v}_1, \dots, \underline{v}_n\}$ in terms of $B = \{g^{-1}\underline{v}_1, \dots, g^{-1}\underline{v}_n\}$ (i.e for each basis element $\underline{v}_i$, we can write it as a linear combination of elements in B).

If we consider the inverse mapping:

$${}_B[id_V]_A^{-1} = {}_A[id_V]_B$$

this is the matrix containing the coefficients which allow us to write elements in $B = \{g^{-1}\underline{v}_1, \dots, g^{-1}\underline{v}_n\}$ in terms of a linear combination of elements in $A = \{\underline{v}_1, \dots, \underline{v}_n\}$. But clearly, applying g^{-1} to $\underline{v}_i$ takes us to $g^{-1}\underline{v}_i$. In other words, we must have:

$${}_B[id_V]_A^{-1} = {}_A[id_V]_B = g^{-1}$$

Hence, it must be the case that, as required:

$$g = {}_B[id_V]_A$$

3. **We want to calculate the *order* of the *finite* group $GL(n;\mathbb{F})$ (recall, the *order* of a group is the number of elements in the group).**
   (a) **Show that $GL(n;\mathbb{F}_p)$ acts transitively on $\mathbb{F}_p^n \setminus \{0\}\}$. Recall, a *group acts transitively* on a set if for each pair of elements x, y in the set, there is a group element such that $g \cdot x = y$.**
   (b) **Determine the stabilizer of the vector $\underline{e}_1 \in \mathbb{F}_p^n$, and establish that:**

   $$|Stab_{GL}\ \underline{e}_1| = p^{n-1}|GL(n-1;\mathbb{F}|$$

   Recall, the *stabiliser* of an element x of a set is a *subgroup* of the group acting on a set. It contains all elements of the group which act on x, and do so by mapping it to itself.
   (c) **Using the Orbit Stabilizer Theorem, determine $|GL(n, \mathbb{F}_p)|$. Recall, the *orbit* of an element x is the set of all elements to which the group maps x. The orbit stabiliser theorem says that:**

   $$|G| = |Stab_G(x)| \times |Orb_G(x)|$$

2.3 The Trace

- **What is the trace of a matrix?**
  - the **trace** of a **square** matrix is the **sum** of its **diagonal** entries
  - it is denoted using:
  $$tr(A)$$
  - in terms of formulae:
  $$tr(A) = \sum_{i=1} na_{ii}$$
- **Are traces defined for infinite rank matrices?**
  - only if the sum converges
- **What is the trace of an endomorphism?**
  - we can define the **trace** of an endomorphism:
  $$f : V \to V$$
  as:
  $$tr(f) = tr(f|V) = tr_{\mathbb{F}}(f|V)$$
  - to compute it, we consider an ordered basis A of V, and define:
  $$tr(f) = tr({}_A[f]_A)$$
  - turns out, this definition is **independent** of the basis chosen (reason: $f(AB) = f(BA)$ and $tr(T^{-1}MT) = tr(M)$; this is proven below)

2.3.1 Exercises (TODO)

1. **Let:**
   - A **be an** $n \times m$ **matrix**
   - B **be an** $m \times n$ **matrix**

   Show that:
   $$tr(AB) = tr(BA)$$
   This is known as the *cyclicity of the trace.*

   The above exercise has a very nice implication. In particular, if we pick:
   $$A = T^{-1}M$$
   $$B = T$$
   then:
   $$tr(T^{-1}MT) = tr(M)$$
   *Hence, 2 matrices are similar **if and only if** they have the same trace.*

2. **Let** $A, B \in Mat(n, \mathbb{F})$ **and** $\lambda \in F$**.**

(a) **Show that:**

1. $Tr(\lambda A) = \lambda Tr(A)$
2. $Tr(A + B) = Tr(A) + Tr(B)$
3. $Tr(AB) = Tr(BA)$

(b) **Prove that, if:**

$$f : Mat(n; \mathbb{F}) \to F$$

and:

- f **is linear (for** $f(\lambda A + B) = \lambda f(A) + f(B)$
- $f(AB) = f(BA)$

then:

$$f(A) = \alpha Tr(A), \qquad \alpha \in \mathbb{F}$$

Moreover, show that if $f(\mathbb{I}_n) = n \neq 0$**, then:**

$$f(A) = tr(A)$$

The first part is given by dull calculations, so just check this link with proofs to all the properties (and the exercise above).

3. **Let** $f : V \to W$ **and** $g : W \to V$ **be 2 linear mappings (**V, W **are finite dimensional). Show that:**

$$tr(fg) = tr(gf)$$

4. **Let** V **be finite dimensional, and let** $f : V \to V$ **be idempotent (**$f^2 = f$**). Show that:**

$$tr(f) = dim(im(f))$$

Last week, in an exercise, we showed that:

$$ker(\phi \circ \phi) = ker(\phi) \iff V = ker(\phi) \oplus im(\phi)$$

Since f is idempotent, it must then be the case that:

$$V = ker(f) \oplus im(f)$$

Let $\{\underline{k}_1, \ldots, \underline{k}_s\}$ be a basis of $ker(f)$, and let $\{\underline{l}_1, \ldots, \underline{l}_t\}$ be a basis of $im(f)$. Then, a basis of V is given by:

$$B = \{\underline{k}_1, \ldots, \underline{k}_s, \underline{l}_1, \ldots, \underline{l}_t\}$$

(This next part I don't understand why) Hence, the representing matrix, written in block form, will be:

$${}_B[f]_B = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

Thus:

$$tr(f) = tr({}_B[f]_B) = dim(im(f))$$

5. **Let** V **be a finite dimensional F-vector space, and** $f : V \to V$ **a linear mapping. Show that:**

$$tr((f\circ)|End_F(V)) = (dim_F V)tr(f|V)$$

2.4 Mastering Calculations

1. **Define a linear map:**

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$f(x, y) = (10x - 21y, 4x - 9y)$$

Let A be the following basis of $\mathbb{R}^2$:

$$\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix} \right)$$

Determine:

$$_A[f]_A$$

We first need to determine where the basis vectors get mapped to under the transformation f:

$$f\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$f\left(\begin{pmatrix} -3 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} -9 \\ -3 \end{pmatrix}$$

As we have seen before, the matrix $_A[f]_A$ must satisfy:

$$\begin{pmatrix} -1 & -9 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} {_A[f]_A}$$

(that is, we can express the basis vectors in $f(A)$ using a linear combination of elements in A) We compute:

$$\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix}$$

So:

$$_A[f]_A = \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -9 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -1 & 3 \end{pmatrix}$$

Notice, if we go back to the theorems, we have done nothing else but apply (2.1) (technically Corollary 2.4.4 after):

$$_A[f]_A = {_A[id_{\mathbb{R}^2}]_{S(2)}} \circ {_{S(2)}[f]_{S(2)}} \circ {_A[id_{\mathbb{R}^2}]_{S(2)}}$$

where:

$$_{S(2)}[f]_{S(2)} = \begin{pmatrix} 10 & -21 \\ 4 & -9 \end{pmatrix}$$

(the matrix corresponding to the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$)

$$_{S(2)}[id_{\mathbb{R}^2}]_A = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$$

(the matrix of the basis elements A, in terms of the standard basis)

$$_A[id_{\mathbb{R}^2}]_{S(2)} = \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix}$$

(the inverse transformation, defining the standard basis in terms of A) Then the computation is automatic:

$$_A[f]_A = \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 10 & -21 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -1 & 3 \end{pmatrix}$$

The other method follows the more intuitive view.

2. **Let A and B be the following bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively:**

$$(-2,1)^T, (-3,2)^T$$

$$(-2,2,0)^T, (-2,1,0)^T, (4,-2,2)^T$$

The matrix $_B[f]_A$ representing the linear mapping:

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

with respect to the bases A and B is the following:

$$\begin{pmatrix} -2 & -2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}$$

Find the matrix which represents the mapping f with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

We seek $_{S(3)}[f]_{S(2)}$. By (2.1), we have:

$$_{S(3)}[f]_{S(2)} = {}_{S(3)}[id_{\mathbb{R}^3}]_B \circ {}_B[f]_A \circ {}_A[id_{\mathbb{R}^2}]_{S(2)}$$

Moreover, we have:

$$_{S(3)}[id_{\mathbb{R}^3}]_B = \begin{pmatrix} -2 & -2 & 4 \\ 2 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$_{S(2)}[id_{\mathbb{R}^2}]_A = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \implies {}_A[id_{\mathbb{R}^2}]_{S(2)} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$$

Thus:

$$
\begin{aligned}
{}_{S(3)}[f]_{S(2)} &= {}_{S(3)}[id_{\mathbb{R}^3}]_B \circ {}_B[f]_A \circ {}_A[id_{\mathbb{R}^2}]_{S(2)} \\
\implies {}_{S(3)}[f]_{S(2)} &= \begin{pmatrix} -2 & -2 & 4 \\ 2 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \\
\implies {}_{S(3)}[f]_{S(2)} &= \begin{pmatrix} -2 & -2 & 4 \\ 2 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 5 \\ 0 & 1 \end{pmatrix} \\
\implies {}_{S(3)}[f]_{S(2)} &= \begin{pmatrix} -10 & -10 \\ 7 & 7 \\ 0 & 2 \end{pmatrix}
\end{aligned}
$$

3 Workshop

1. **True or False. Let $\phi : V \to V$ be an endomorphism of a finite dimensional vector space V. Then,** $ker(\phi \circ \phi) = ker(\phi)$

   This is intuitively false. The key is to look for a counterexample by using matrices; in particular, if we can find a nilpotent matrix, such that ϕ^2 is the zero matrix, then it is likely that we can find a vector $\underline{v}$ such that $\phi^2(\underline{v}) = \underline{0}$ but $\phi(\underline{v}) \neq \underline{0}$.

   This is what is done in the solutions:

$$
[\phi] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies [\phi^2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

   so for example $\underline{e}_2 \in ker(\phi^2)$ but $\underline{e}_2 \notin ker(\phi)$.

   To do this, I began with a general matrix:

$$
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}
$$

   and then computed its square, alongside the result of applying them to a vector.

For the following exercise, I derived the following relation to compute representing matrices for different bases.

Say we have a mapping $f : V \to W$, with V having a basis $\mathcal{A} = \{\underline{v}_1, \dots, \underline{v}_n\}$ and W having a basis $\mathcal{B} = \{\underline{w}_1, \dots, \underline{w}_m\}$. We know that the representing matrix ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ has entries a_{ij} such that:

$$
f(\underline{v}_j) = \sum_{i=1}^{m} a_{ij} \underline{w}_i
$$

In terms of matrices, this is equivalent to having:

$$\begin{pmatrix} w_{11} & w_{21} & \dots & w_{m1} \\ w_{12} & w_{22} & \dots & w_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1n} & w_{2n} & \dots & w_{mn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (f(\underline{v}_1) \mid f(\underline{v}_2) \mid \dots \mid f(\underline{v}_n))$$

In other words, if:

$$X = \begin{pmatrix} w_{11} & w_{21} & \dots & w_{m1} \\ w_{12} & w_{22} & \dots & w_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1n} & w_{2n} & \dots & w_{mn} \end{pmatrix} = (\underline{w}_1 \mid \underline{w}_2 \mid \dots \mid \underline{w}_m)$$

and

$$Y = (f(\underline{v}_1) \mid f(\underline{v}_2) \mid \dots \mid f(\underline{v}_n))$$

Then we have that:

$$X\,_{\mathcal{B}}[f]_{\mathcal{A}} = Y \implies {}_{\mathcal{B}}[f]_{\mathcal{A}} = X^{-1}Y$$

Notice here that we can think of:

$$X = [id]_{\mathcal{B}} \implies X^{-1} = {}_{\mathcal{B}}[id]$$

(since X is expressing $f(\underline{w}_i) = \underline{w}_i$ using a linear combination of the standard basis vectors) and:

$$Y = [f]_{\mathcal{A}}$$

(since it expresses $f(\underline{v}_i)$ in terms of a linear combination of standard basis vectors) So indeed:

$$X^{-1}Y = {}_{\mathcal{B}}[id]\,[f]_{\mathcal{A}} = {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

---

2. **The linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by:**

$$f(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, x_1 - x_3))$$

In $\mathbb{R}^2$, $\mathcal{A}$ is the basis:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

and in $\mathbb{R}^3$, $\mathcal{B}$ is the basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Obtain:

(a) **The matrix of f with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$**

For this, we don't even need to use the formula: this is just the standard representing matrix obtained by applying f to the basis vectors of $\mathbb{R}^3$, and using the resulting vectors as our columns. Computing:

$$f(1,0,0) = (1,1) \qquad f(0,1,0) = (-1,0) \qquad f(0,0,1) = (2,-1)$$

Hence:

$$[f] = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

(b) **The matrix of f with respect to the standard basis of $\mathbb{R}^3$ and the basis $\mathcal{A}$ of $\mathbb{R}^2$**

We need to use the basis $\mathcal{A}$. We construct a matrix using its vectors:

$$X = [id]_{\mathcal{A}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which has inverse:

$$X^{-1} = {}_{\mathcal{A}}[id] = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then, we know that:

$${}_{\mathcal{A}}[f]_{\mathcal{S}(3)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 1 \\ 0 & -1 & 3 \end{pmatrix}$$

(c) **The matrix of f with respect to the basis $\mathcal{B}$ of $\mathbb{R}^3$ and the standard basis of $\mathbb{R}^2$**

We need to compute the value of f at the basis vectors $\mathcal{B}$:

$$f(1,1,0) = (0,1) \qquad f(0,1,1) = (1,-1) \qquad f(1,0,1) = (3,0)$$

so we have that:

$$[f]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$$

This is precisely what we need.

(d) **The matrix of f with respect to the basis $\mathcal{B}$ of $\mathbb{R}^3$ and the basis $\mathcal{A}$ of $\mathbb{R}^2$**

We already have all the ingredients:

$${}_{\mathcal{A}}[f]_{\mathcal{B}} = {}_{\mathcal{A}}[id]\,[f]_{\mathcal{B}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 3 \end{pmatrix}$$

(e) **Show that if the axis of rotation is the x-axis and you rotate by θ degrees, the matrix representing this linear transformation in standard coordinates is:**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Intuitively, since we rotate about the x axis, this is equivalent to just having a θ^{o} rotation on the yz plane, which the lower right matrix represents.

Computing, it is sufficient to show that the matrix has the desired result on the basis vectors:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

As expected, the x-axis remains fixed under rotation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$$

which is as expected.

(f) **Now prove, by a suitable change of basis, that there is a rotation in $\mathbb{R}^3$ with axis of rotation given by the line connecting $\underline{0}$ and $(1, 1, 1)$, which is represented by:**

$$\begin{pmatrix} \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1}{3} & \frac{1+\sqrt{3}}{3} \end{pmatrix}$$

What is the corresponding angle of rotation? It might help to consider the *orthonormal basis* for $\mathbb{R}^3$ given by:

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

We try to compute ${}_{\mathcal{B}}[f]_{\mathcal{B}}$. We have that:

$$[f] = \begin{pmatrix} \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1}{3} & \frac{1+\sqrt{3}}{3} \end{pmatrix}$$

Thus, we require ${}_{\mathcal{B}}[id]$ and $[id]_{\mathcal{B}}$.

To construct, $[id]_{\mathcal{B}}$, we use the basis vectors as column vector for the matrix:

$$[id]_{\mathcal{B}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}$$

Then (using our future knowledge of the fact that the inverse of an orthogonal matrix - such as the one above, constructed via an orthonormal basis - is its transpose):

$$_{\mathcal{B}}[id] = [id]_{\mathcal{B}}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

And so we can compute:

$$_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1+\sqrt{3}}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1}{3} & \frac{1+\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Thus, with respect to the basis $\mathcal{B}$, we have a rotation with axis $(1,1,1)$. In particular, for this rotation we must have:

$$\cos\theta = \frac{\sqrt{3}}{2} \qquad \sin\theta = -\frac{1}{2}$$

which corresponds to a rotation by $\theta = \frac{\pi}{6}$ clockwise

3. (a) **Work out the matrix $_{\mathcal{B}}[f]_{\mathcal{A}}$ for the linear map:**

$$f : \mathbb{C}^3 \to \mathbb{C}^2$$

$$f(x,y,z) = (-x-y+2z, 2x+2y-3z)$$

where:

$$\mathcal{A} = ((0,3,2),(1,1,1),(1,2,2))$$

is a basis of $\mathbb{C}^2$ and $\mathcal{B}$ is the standard basis of $\mathbb{C}^2$.

Since $\mathcal{B}$ is just the standard basis, we just need to compute $[f]_{\mathcal{A}}$, the matrix produced by using as columns the result of applying f to the basis vectors of $\mathcal{A}$.

We thus compute:

$$f(0,3,2) = (1,0)$$
$$f(1,1,1) = (0,1)$$
$$f(1,2,2) = (1,0)$$

Hence:

$$_{\mathcal{B}}[f]_{\mathcal{A}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) **Write down a basis for the kernel of f.**

This can be done in 2 ways.

From the solutions, notice that:

$$f(0,3,2) = f(1,2,2) = (1,0)$$

which means that:

$$(0,3,2)-(1,2,2) = (-1,-1,0) \in ker(f)$$

Notice, the rank of the representing matrix is 0 (2 linearly independent rows), so by Rank-Nullity, we expect a kernel of dimension 1, so $\{(-1,-1,0)\}$ is a basis for $ker(f)$

My approach, involving direct computation. If $\underline{v} = (x, y, z) \in \ker(f)$ then:

$$-x - y + 2z = 0 \qquad 2x + 2y - 3z = 0$$

Multiplying the first equation by 2, and adding it to the second one results in:

$$z = 0$$

So that we have:

$$-x - y = 0 \implies x = y$$

so $(1, 1, 0)$ is a basis for $ker(f)$.

4. **Let $\mathcal{S}(2) = (\underline{e}_1, \underline{e}_2)$ be the standard basis of $T = \mathbb{R}^2$ and let:**

$$\mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

Show that $\mathcal{B}$ is a basis of T. Now, suppose that a linear mapping $f : T \to T$ is represented with respect to $\mathcal{S}(2)$ by the matrix:

$$A = \begin{pmatrix} -6 & -9 \\ 4 & 6 \end{pmatrix}$$

Find the matrix $\mathcal{B}$ that represents f with respect to $\mathcal{B}$

It is clear that the vectors of $\mathcal{B}$ are linearly independent (can be verified by either using row reduction, or explicitly computing the linear combination of the vectors which leads to 0). Moreover, $\mathcal{B}$ contains 2 elements, and the dimension of T is 2, so $\mathcal{B}$ must be a basis.

We now need to compute ${}_{\mathcal{B}}[f]_{\mathcal{B}}$. There are 2 methods.

The first one from the solution involves computing the value of f when applied to the basis vectors of $\mathcal{B}$:

$$A(-3, 2) = (0, 0) \qquad A(2, -1) = (-3, 2)$$

The elements of the matrix are the coefficients required to write $(0, 0)$ and $(-3, 2)$ by using the basis $\mathcal{B}$, so it is easy to see that:

$${}_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Alternatively, we use the fact that:

$${}_{\mathcal{B}}[f]_{\mathcal{B}} = {}_{\mathcal{B}}[id][f][id]_{\mathcal{B}}$$

We have that:

$$[id]_{\mathcal{B}} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

(the coefficients are the ones used to write the basis elements of $\mathcal{B}$ in terms of the standard basis) It's inverse is:

$${}_{\mathcal{B}}[id] = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

So:

$$
\begin{aligned}
{}_{\mathcal{B}}[f]_{\mathcal{B}} &= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -6 & -9 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

5. **Consider the vector space $V = Mat(m \times n; F)$.**
   (a) **What is the dimension of $Mat(m \times n; F)$?**
   It is a mn dimensional space.
   (b) **Find a basis of this vector space.**
   Let E_{ij} be the matrix with a 1 in entry (i, j) and 0s elsewhere. Then, a basis for V will be:

$$\mathcal{B} = \{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

   It is clear that $\mathcal{B}$ spans the space. If $A \in Mat(m \times n, F)$ has entries $a_{ij} \in F$, then:

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$$

   Moreover, it is clear that $\mathcal{B}$ is linearly independent (each matrix has a 1 where the other $mn - 1$ have a 0). Thus, $\mathcal{B}$ is a basis.
   (c) **Let $p(z) \in F[z]$ be a polynomial whose coefficients belong to F. Given $A \in Mat(n; F)$, let $p(A) \in Mat(n; F)$ be the matrix you get by replacing each power of z in $p(z)$ by the corresponding power of A. Show that there exists a non-zero polynomial $p(z)$ such that $p(A)$ is the zero matrix.**
   Take a matrix $A \in Mat(n; F)$. Consider the set:

$$A^0, A^1, \ldots, A^{n^2}$$

   this is a set of $n^2 + 1$ elements, each of which is in $Mat(n; F)$. But this space is n^2 dimensional, so this must be a linearly dependent set. In other words, $\exists \lambda_i$, not all of which are non-zero, such that:

$$\sum_{i=0}^{n^2} \lambda_i A^i = 0$$

   Hence, the non-zero polynomial:

$$p(z) = \sum_{i=0}^{n^2} \lambda_i z^i$$

   evaluates to the 0-matrix when given A.
   (d) **Let:**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Find an explicit non-zero polynomial $p(z)$ for which $p(A)$ is the zero matrix.

(With future knowledge at hand, the Cayley-Hamilton Theorem tells us that a matrix always satisfies its characteristic polynomial, so:

$$p(z) = (z-1)(z-2)(z-3)$$

is a good answer)

(e) **Here is a fact, which you don't need to check. There is an invertible matrix Q such that:**

$$B = \frac{1}{2}\begin{pmatrix} 32 & -12 & 8 \\ 16 & 12 & -8 \\ 13 & -15 & 28 \end{pmatrix} = Q^{-1}AQ$$

Find a non-zero polynomial $p(z)$ for which $p(B)$ is the zero matrix.

(Again, future knowledge can tell us that the characteristic polynomial of similar matrices is identical, and so the $p(z)$ above works; however, it is nice to work without future knowledge)

Notice:

$$B^n = (Q^{-1}AQ)^n = (Q^{-1}AQ)(Q^{-1}AQ)\dots(Q^{-1}AQ) = Q^{-1}A^nQ$$

From work above, we know that there is a polynomial $p(z)$ such that $p(B)$ is the 0 matrix, so (for some t):

$$\begin{aligned}
p(B) &= \sum_{i=0}^{t} \lambda_i B^i \\
&= \sum_{i=0}^{t} \lambda_i (Q^{-1}AQ)^i \\
&= \sum_{i=0}^{t} \lambda_i Q^{-1}A^iQ \\
&= \sum_{i=0}^{t} Q^{-1}(\lambda_i A^i)Q \\
&= Q^{-1}\left(\sum_{i=0}^{t} \lambda_i A^i\right)Q \qquad \textit{(by applying distributivity)} \\
&= Q^{-1}p(A)Q
\end{aligned}$$

Thus, any polynomial $p(z)$ which evaluates to the 0 matrix under A will evaluate to the 0 matrix under B. Hence, we can pick $p(z) = (z-1)(z-2)(z-3)$ from above.