Group Theory - Weeks 9 - Solvable Groups

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Contents

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1 Solvable Groups

1.1 Definition: Subnormal Series

> ***Subnormal series*** *are a* ***generalisation*** *of* ***composition series****.*
> *In particular, a* ***subnormal series*** *of G is a* ***chain*** *of subsequent* ***normal subgroups****:*
> $$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_s = G$$
> *(Definition 8.1.1)*

1.2 Definition: Solvable Group

> *A group G is* ***solvable****, provided that it has a* ***subnormal series****:*
> $$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_s = G$$
> *such that each* ***factor****:*
> $$G_{i+1}/G_i$$
> *is* ***abelian****.*
> *(Definition 8.1.2)*

1.2.1 Examples of Solvable Groups

- If A is **abelian**, A is **solvable**:
$$\{e\} \triangleleft A$$
is a **subnormal series**, whose only factor (A) is **abelian**
- S_3 is **solvable**, but not abelian:
$$\{e\} \triangleleft A_3 \triangleleft S_3$$
where recall:
$$A_3 = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$$
is **abelian**. Similarly, S_4 is also **solvable**
- A_5 is **not** solvable, since it is a simple group, and A_5 isn't abelian, so the subnormal series contains no abelian factor

1.2.2 Theorem: Finite p-groups are Solvable

> *Let G be a p-group, such that $|G| = p^n$. Then, G is* ***solvable****.*

Proof. Recall that if G is a p-group, it has a non-trivial centre:

> *Let G be a* ***non-trivial, finite p-group****. Then:*
>
> $$Z(G) \neq \{e\}$$
>
> *That is, the* ***centre*** *is* ***non-trivial****.*
> *(Theorem 4.2.12)*

We now proceed by induction on $|G| = p^n$.
1 $|G| = p^1$

Let $G_1 = Z(G)$. Clearly, G_1 is abelian, and thus, it is normal in G. Moreover, the quotient G/G_1 will be abelian, since $|G| = p$ implies that G is cyclic (and so abelian), and the quotient of an abelian group will be abelian. Hence, we have that:

$$\{e\} \triangleleft G_1 \triangleleft G$$

is a subnormal chain of G with abelian factors.

2 $|G| = p^k$

Assume that if G is a p-group with $|G| \leq p^k$, then G is solvable. That is:

$$\{e\} \triangleleft G_1 \triangleleft \ldots \triangleleft G$$

is a subnormal chain, such that G_{i+1}/G_i is abelian.

3 $|G| = p^{k+1}$

Since G is a p-group, G has a non-trivial centre $Z(G)$. If $G = Z(G)$, then G is abelian, and thus solvable.

Hence, assume that $Z(G)$ is a proper subgroup. In particular, we know that $Z(G) \triangleleft G$. Moreover, since $Z(G)$ is a subgroup of G, it must be a p-subgroup, and $|Z(G)| \leq p^k$. Hence, by the inductive hypothesis, $Z(G)$ is solvable:

$$\{e\} \triangleleft G_1 \triangleleft \ldots \triangleleft Z(G)$$

If $G/Z(G)$ is abelian, then we are done. Otherwise, since $|G/Z(G)| < p^{k+1}$, and $G/Z(G)$ is a p-group, $G/Z(G)$ will be solvable by inductive hypothesis, so:

$$Z(G) \triangleleft H_1 \triangleleft \ldots \triangleleft G/Z(G)$$

By the correspondence theorem, for each $H_j \triangleleft H_{j+1}$ there is a corresponding normal subgroup $K_j \triangleleft K_{j+1}$, where each K_j is contained in G. Since H_{j+1}/H_j is abelian, then K_{j+1}/K_j will also be abelian. In particular, this means that:

$$\{e\} \triangleleft G_1 \triangleleft \ldots \triangleleft Z(G) \triangleleft K_1 \triangleleft \ldots \triangleleft G$$

is a **subnormal chain**, where each factor is abelian. Thus, G is solvable.

□

1.3 Solvability from Composition Series

1.3.1 Lemma: Composition Factors of Abelian Groups

If A is a ***finite abelian group*** *of order:*

$$|A| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

then the composition factors of A are:

- *n_1 copies of C_{p_1}*
- *n_2 copies of C_{p_2}*
- *...*
- *n_k copies of C_{p_k}*

(Lemma 8.1.5)

Proof. □

1.3.2 Theorem: Solvability Iff Cyclic Composition Factors

A ***finite*** *group G is* ***solvable if and only if*** *all the* ***composition factors*** *of G are* ***cyclic***.
(Theorem 8.1.4)

Proof.

- ($\Longleftarrow$) Say G has a composition series, with all composition factors being cyclic. In particular, any composition series is a subnormal series, and every cyclic goup is abelian, so G must be solvable
- ($\Longrightarrow$) Say G is solvable. Then, it has a subnormal series, with each factor being abelian. We now induct on $|G|$.

  1 $|G| = 2$

  Then $G = C_2$, which is solvable (since abelian), and has composition series $\{e\} \triangleleft C_2$, with composition factor C_2, as required.

  2 $|G| = k$

Assume that if $|G| = k$ and G is solvable, then G has cyclic composition factors.

3 $|G| = k + 1$

Assume that G is solvable. Then, it has a subnormal series:

$$\{e\} \lhd G_1 \lhd \ldots \lhd G_{s-1} \lhd G$$

such that G_{i+1}/G_i is abelian. By the inductive hypothesis, since $|G_{s-1}| < k+1$, the composition factors of G_{s-1} are cyclic. Moreover, G/G_{s-1} is abelian, and by the Lemma above, it follows that the composition factors of G/G_{s-1} are cyclic.

Now, recall the Lemma:

Let G be a group, with $N \lhd G$.
Let:

$$\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd \lhd G_s = N$$

*be a **composition series** for N, and:*

$$N = H_0 \lhd H_1 \lhd \ldots \lhd \lhd H_r = G/N$$

*be a **composition series** for G/N.*

---

*Then, there is a **composition series** for G of length $s + r$, whose **composition factors** are:*

$$G_1, G_2/G_1, \ldots, G_s/G_{s-1}, H_1, H_2/H_1, \ldots, H_r/H_{r-1}$$

(Sublemma 7.2.2)

This means that the composition factors of G are precisely the composition factors of G_{s-1} and G/G_{s-1}. Hence, the composition factors of G must all be cyclic.

□

2 Solvable Groups from Subgroups

2.1 Theorem: Solvability Iff Normal Subgroup Solvable

*Let G be a **group**, and let $N \lhd G$. Then, G is **solvable if and only if**:*

- *N is **solvable***
- *G/N is **solvable***

(Theorem 8.1.6)

Proof. Let $N \triangleleft G$. By the Theorem above, G is solvable **if and only if** its composition factors are cyclic. By the sublemma:

> *Let G be a group, with $N \triangleleft G$.*
> *Let:*
> $$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft \triangleleft G_s = N$$
> *be a **composition series** for N, and:*
> $$N = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft \triangleleft H_r = G/N$$
> *be a **composition series** for G/N.*
>
> ---
>
> *Then, there is a **composition series** for G of length $s + r$, whose **composition factors** are:*
> $$G_1, G_2/G_1, \ldots, G_s/G_{s-1}, H_1, H_2/H_1, \ldots, H_r/H_{r-1}$$
> *(Sublemma 7.2.2)*

the composition factors of G are precisely those of G and G/N. Hence, G is solvable **if and only if** G has cylic composition factors **if and only if** N and G/N have cyclic composition factors, as required. ☐

2.1.1 Worked Exercise: Solvability of Groups of Order 40

Let G be a group of order 40. By Sylow I, G has a Sylow 5-subgroup, call it N. By Sylow III:

$$n_5 \mid 8 \qquad n_5 \equiv 1 \ (mod\ 5)$$

This only allows $n_5 = 1$, so N is normal in G. Since $|N| = 5$, N has prime order, and thus, is cyclic, so abelian, so solvable. Moreover, $|G/N| = 8 = 2^3$. Hence, G/N is a p-group, so it is solvable. Hence, since N and G/N are solvable, G must be solvable.

2.2 Theorem: Solvable Groups Have Solvable Subgroups

> *If G is **solvable** and $H \leq G$, then H is **solvable**.*
> *(Theorem 8.1.7)*

Proof. Let G have subnormal series:
$$\{e\} \lhd G_1 \lhd \ldots \lhd G$$
such that G_{i+1}/G_i is an abelian group.

If $H \leq G$, define:
$$H_i = H \cap G_i$$
Since $G_i \lhd G_{i+1}$, if $a \in G_i$:
$$\forall g \in G_{i+1}, gag^{-1} \in G_i$$
Now, let $b \in H_i$ and $h \in H_{i+1}$ and consider hbh^{-1}, since $h \in H_{i+1}$, in particular $h \in G_{i+1}$. Similarly, since $b \in H_i$, also $b \in G_i$, so:
$$hbh^{-1} \in G_i \cap H$$
(since $b.h$ are also in H). Hence, $hbh^{-1} \in H_i$, so $H_i \lhd H_{i+1}$.

Now, define:
$$\theta : H_{i+1} \to G_{i+1}/G_i$$
by the canonical map:
$$\theta(h) = hG_i$$
Then:
$$ker(\theta) = H_{i+1} \cap G_i = (G_{i+1} \cap H) \cap G_i = H \cap G_i = H_i$$
Hence, by the First Isomorphism Theorem:
$$H_{i+1}/ker(\theta) \cong im(\theta) \implies H_{i+1}/H_i \cong im(\theta) \leq G_{i+1}/G_i$$
But since G is solvable, G_{i+1}/G_i is abelian, and any subgroup of an abelian group is abelian. Hence, $im(\theta) \cong H_{i+1}/H_i$ is abelian.

Hence, we have found a subnormal series for H:
$$\{e\} \lhd H_1 \lhd \ldots \lhd H$$
such that H_{i+1}/H_i is abelian. Thus, H is solvable, as required.

□

3 Derived Subgroups

3.1 Commutators

3.1.1 Definition: The Commutator

*Let G be a **group**. The **commutator** of $a, b \in G$ is the element:*
$$[a, b] = aba^{-1}b^{-1}$$
(Definition 8.2.1)

3.1.2 Definition: The Derived/Commutator Subgroup

The ***derived subgroup*** *(or* ***commutator subgroup****) of G is the* ***subgroup*** *generated by* ***all*** *possible* ***commutators*** *in G:*

$$G' = \langle [a,b] \mid a, b \in G \rangle = [G, G]$$

(Definition 8.2.1)

3.1.3 Remark: Properties of the Derived Subgroup

1. ***Inverses*** *and* ***conjugates*** *of* ***commutators*** *are* ***commutators****:*

$$[a,b]^{-1} = [b,a] \qquad z[a,b]z^{-1} = [zaz^{-1}, zbz^{-1}]$$

2. *Every element in G' is a* ***product*** *of* ***commutators****. However,* ***it is not true that a product of 2 commutators is a commutator****: that is, the set of all commutators doesn't form a group*

3. *The* ***derived subgroup*** *is a* ***normal subgroup*** *in G:*

$$G' \triangleleft G$$

Proof.

(1) One can directly check:

$$[a,b][b,a] = (aba^{-1}b^{-1})(bab^{-1}a^{-1}) = e_G$$
$$[b,a][a,b] = (bab^{-1}a^{-1})(aba^{-1}b^{-1}) = e_G$$
$$[zaz^{-1}, zbz^{-1}] = zaz^{-1}zbz^{-1}(zaz^{-1})^{-1}(zbz^{-1})^{-1} = (zabz^{-1})(za^{-1}b^{-1}z^{-1}) = z[a,b]z^{-1}$$

(2) By definition, G' is generated by all commutators, so all of its elements are products of commutators.

$$[a,b], [c,d] \in G \implies [a,b][c,d]$$

However, any set containing commutators needn't be a group:

- What is a simple example of a group in which the product of commutators need not be a commutator?
- Commutator subgroup does not consist only of commutators
- Why is the set of commutators not a subgroup?

③

Let $x = [a, b] \in G'$, and let $g \in G$. Then:

$$gxg^{-1} = [gag^{-1}, gbg^{-1}] \in G'$$

so $\forall g \in G, gG'g^{-1} \subseteq G$, so $G' \triangleleft G$ as required. Alternatively, we have that:

$$gxg^{-1} = xx^{-1}gxg^{-1} = x[x^{-1}, g] \in G'$$

where we use the fact that $x, [x^{-1}, g]$ are both commutators.

□

3.2 Theorem: Abelian Factor Groups from Derived Subgroups

*Let G be a group. Then, N is a **normal subgroup** and G/N is **abelian** if and only if $G' \subseteq N$.*
In particular, $N = G'$ is the smallest subgroup, such that G/N is abelian.
(Theorem 8.2.2)

Proof.

- ($\implies$) Assume N is a normal subgroup, such that G/N is abelian. We seek to show that $G' \subseteq N$. Let $a, b \in G$. Then:

$$\begin{aligned}[a, b]N &= (aba^{-1}b^{-1})N \\ &= (aN)(bN)(a^{-1}N)(b^{-1}N) \\ &= (aN)(a^{-1}N)(bN)(b^{-1}N) \\ &= eN \\ &= N\end{aligned}$$

  so $[a, b] \in N \implies G' \subseteq N$ as required.

- ($\impliedby$). Assume that $G' \subseteq N$. We first show that $N \triangleleft G$, and then that G/N is abelian. The first part is similar to how we showed that G' is a normal subgroup. Indeed, let $g \in G$, and let $x \in N$. Then:

$$gxg^{-1} = gxg^{-1}x^{-1}x = [g, x]x$$

  Since $[g, x] \in G' \subseteq N$ and $x \in N$, it follows that $gxg^{-1} \in N$, so $N \triangleleft G$.

  Now, consider G/N. Let $a, b \in G$. Then:

$$(aN)(bN) = abN = (baa^{-1}b^{-1})(abN) = (ba[a^{-1}, b^{-1}])N = baN = (bN)(aN)$$

  where we have used the fact that $[a^{-1}, b^{-1}] \in N$

□

3.3 Derived Series

3.3.1 Definition: Derived Series of a Group

*Let G be a **group**. Set $G^0 = G$. Then, define:*

$$\forall i \geq 0, \quad G^{(i+1)} = (G^{(i)})' = [G^{(i)}, G^{(i)}]$$

*The **derived series** of G is the sequence:*

$$G = G^{(0)} \triangleright G^{(1)} = G' \triangleright G^{(2)} \triangleright \dots$$

(Definition 8.2.3)

3.3.2 Remark: Properties of Derived Series

1. $$\exists i \geq 0 \ : \ G^{(i+1)} = G^{(i)} \implies \forall j \geq i, G^{(j)} = G^{(i)}$$
2. *If $|G| < \infty$, then $\exists i \geq 0 \ : \ G^{(i+1)} = G^{(i)}$, but this doesn't necessarily mean that $G^{(i)} = \{e_G\}$*
3. *If there is some $n \geq 0$ such that $G^{(n)} = \{e_G\}$, then G is **solvable***

Proof.

1 Assume that for some $i \geq 0$, $G^{(i+1)} = G^{(i)}$. By definition, $(G^{(i)})' = G^{(i+1)} = G^{(i)}$. In other words, the derived subgroup of $G^{(i)}$ is itself, so if we continue computing its derived subgroup, we will continue obtain itself, as required.

2 The derived subgroup is a normal subgroup, so its order will be less than the original group. If the original group is finite, in particular this means that eventually there must exist an i such that $G^{(i)}$, upon taking its derived subgroup, can not decrease in order anymore.

To show that this last derived subgroup need not be trivial, consider $G = A_5$. A_5 is simple, so its only normal subgroups are trivial. In particular, $G' = \{e\}$ or $G' = A_5$. Since G‘ is a derived subgroup, it is the smallest subgroup such that G/G' is abelian. Since A_5 isn't abelian, $G' \neq \{e\}$, and so, $G' = A_5$. Thus, $\forall i \geq 0, G^{(i)} = A_5$. This argument works for any non-abelian simple group.

3 Assume there is some $n \geq 0$ such that $G^{(n)} = \{e_G\}$. Then, the derived series is:

$$G = G^{(0)} \triangleright G^{(1)} = G' \triangleright G^{(2)} \triangleright G^{(n)} = \{e_G\}$$

By definition of derived subgroups, $G^{(i)}/G^{(i+1)}$ is abelian (since $G^{(i+1)} = (G^{(i)})'$), so the derived series is a subnormal series with each factor abelian. Thus, G must be solvable.

$\square$

3.3.3 Theorem: Solvability from Derived Series

*Let G be a **group**. Then, G is **solvable** if and only if $\exists n \geq 0 \ : \ G^{(n)} = \{e_G\}$.*
(Theorem 8.2.4)

Proof.

- ($\implies$): assume that G is solvable. Then, there is some subnormal series:

  $$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{e\}$$

  where G_s/G_{s+1} is abelian. It is sufficient to show that $\forall s \in [0, n]$ we have that $G^{(s)} \subseteq G_s$. Then, taking $s = n$, we'd get that $G^{(n)} \subseteq \{e_G\} \implies G^{(n)} = \{e_G\}$, as required.

  We thus proceed by induction on s:

  1 **Base Case ($n = 0$)**

  Notice that $G^{(0)} = G = G_0$, so clearly $G^{(0)} \subseteq G_0$.

  2 **Inductive Hypothesis ($s = k$)**

  Assume that for $s \in [0, k]$, we have that $G^{(s)} \subseteq G_s$.

  3 **Inductive Step ($s = k + 1$)**

  Now, let $s = k + 1$. By definition of the derived series:

  $$G^{(k+1)} = (G^{(k)})'$$

  Since $G^{(k)} \subseteq G_k$ by the inductive hypothesis, we must have that $(G^{(k)})' \subseteq G'_k$. Moreover, G_k/G_{k+1} is abelian, so the derived subgroup of G_k must be a subgroup of G_{k+1}; in particular, $G'_k \subseteq G_{k+1}$. Thus, we have that:

  $$G^{(k+1)} \subseteq G_{k+1}$$

  as required.

- ($\impliedby$): this was 3 in the above Remark.

$\square$

3.3.4 Definition: Derived Length of a Group

*Let G be a **solvable** group. Then, $\exists n \geq 0 \ : \ G^{(n)} = \{e_G\}$. The least such n is the **derived length** of G.*
(Definition 8.2.5)

3.3.5 Example: Derived Length of Dihedral Groups

Let:

$$G = D_n \cong \left\langle g, h \mid g^n, h^2, (gh)^2 \right\rangle$$

where $n \geq 3$. The subgroup of D_n containing the rotations is $\langle h \rangle \cong C_n$. Since H is abelian (and thus normal), and $|G/H| = 2n/n = 2 \implies G/H \cong C_2$ is also abelian, they are both solvable, so G is solvable.

Notice:

- the fact that G/H is abelian implies that $G' \subseteq H$
- G' contains the commutator:

$$[g,h] = ghg^{-1}h^{-1} = hg^{-2}h^{-1} = g^2hh^{-1} = g^2$$

- if $K = \left\langle g^2 \right\rangle \leq G$, since $g^2 \in G'$, then $K \subseteq G'$

Now, we need to consider 2 cases:

1 n **is odd**

Then $n = 2k+1$, and:

$$H = \{g, g^2, \dots, g^{2k}, g^{2k+1} = e\}$$

Then, notice any element of $\left\langle g^2 \right\rangle$ has the form g^{2m+2} for some $m \in \mathbb{Z}$. In particular, when $m = k$:

$$g^{2k+2} = g^{2k+1+1} = g \in \langle g \rangle \implies \langle g \rangle \subseteq \left\langle g^2 \right\rangle$$

Moreover, since clearly $\left\langle g^2 \right\rangle \subseteq \langle g \rangle$, it follows that $K = \left\langle g^2 \right\rangle = \langle g \rangle = H$. Since $G' \subseteq H = K$ and $K \subseteq G'$, we must have that $G' = H$. Then, we have a derived series:

$$G = D_n \triangleright G^{(1)} = G' = H \triangleright G^{(2)}$$

where, since H is abelian, $G^{(2)} = \{e\}$, so the derived length of D_n is 2.

2 n **is even**

Since $n = 2k$ is even:

$$H = \{g, g^2, g^{2k-1}, g^{2k=e}\}$$

in particular, this means that $|\left\langle g^2 \right\rangle| = \frac{k}{2}$, so $H \neq K = \left\langle g^2 \right\rangle$. Now, $\left\langle g^2 \right\rangle$ commutes with any power of g, and:

$$hg^2h^{-1} = g^{-2} = g^{2k-2} \in \left\langle g^2 \right\rangle$$

Hence, $\left\langle g^2 \right\rangle$ is a normal subgroup, so $K \triangleleft G$. Then, $|G/K| = (2k)/(k/2) = 4$, which means that:

$$G/K \cong C_4 \qquad or \qquad G/K \cong C_2 \times C_2$$

But notice, elements of G/K can only have orders of 1 or 2:

$$G/K = \{K, hK, gK, (gh)K\}$$

Any odd power of g maps into the coset gK; any even power of g maps into the coset K. Any element of the form hg^{2m} maps into hK, and any element of the form hg^{2m+1} maps into $(gh)K$. K has order 1, whereas the remaining 3 elements have order 2, and are their own inverses. Hence, $G/K \cong C_2 \times C_2$, and is abelian, so $G' \subseteq K$. But we saw above that $K \subseteq G'$, so $G' = K$. Thus, we have a derived series:

$$G = D_n \triangleright G^{(1)} = G' = K \triangleright G^{(2)}$$

where, since K is abelian, $G^{(2)} = \{e\}$, so the derived length of D_n is 2.