Honours Analysis - Week 10 - The L^2 Space and Orthogonality

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Contents

1 Recapping Complex Numbers

- **How can we define a complex function?**
  - a complex function f has the form:
$$f : [a,b] \to \mathbb{C}$$
  - we can write f as:
$$f = g + ih$$
    where:
$$g : [a,b] \to \mathbb{R}$$
$$h : [a,b] \to \mathbb{R}$$
- **When is a complex function Lebesgue Integrable?**
  - f is Lebesgue Integrable if g, h are Lebesgue Integrable
  - then, we can define:
$$\int_I f = \int_I f + i \int_I h$$
- **What is the modulus of a complex function?**
  - the modulus of f is $|f|$, which is a real valued function:
$$|f|^2 = f^2 + g^2$$
- **What is the complex conjugate of a complex function?**
  - if f is a complex function, then $\bar{f}$ is its complex conjugate, where:
$$\bar{f} = g - ih$$
  - extending what we know from complex numbers, we know that:
$$f\bar{f} = |f|^2$$
- **What is Euler's Formula?**
$$e^{ix} = \cos(x) + i\sin(x)$$

2 The Space L^2

2.1 Defining the Space L^2

- **What is the Space L^2?**
  - a **function space** defined over some interval:
$$L^2 = L^2([a,b])$$
  - contains **measurable** functions of the form:
$$f : [a,b] \to \mathbb{C}$$
  - $f \in L^2$ then $|f|^2$ is **Lebesgue Integrable**
- **What is the L^2-norm?**

- a quantity $\|f\|_2$ satisfying:

$$\|f\|_2^2 = \int_a^b |f(x)|^2 \ dx \ < \infty$$

- **When is a complex function L^2 normalised?**
  - f is L^2-normalised if its L^2-norm is 1:

$$\|f\|_2 = 1$$

- **Is the set L^2 a vector space?**
  - yes. If $f, g \in L^2$, then:

$$f + \lambda g \in L^2$$

    where $\lambda \in \mathbb{C}$

  - *Proof.* We firstly note that if f and g are measurable, then $f + \lambda g$ is also measurable. Consider:

$$\begin{aligned} |f + \lambda g|^2 &= (f + \lambda g)\overline{(f + \lambda g)} \\ &= f\bar{f} + \overline{\lambda g} f + \lambda g \bar{f} + (\lambda g)(\overline{\lambda g}) \\ &= |f|^2 + |\lambda g|^2 + \overline{\lambda g} f + \lambda g \bar{f} \end{aligned}$$

    Then, if we integrate:

$$\int_a^b |f + \lambda g|^2 = \int_a^b |f|^2 + |\lambda g|^2 + \overline{\lambda g} f + \lambda g \bar{f}$$

    Notice, since $f, g \in L^2$, clearly $\int_a^b |f|^2 + |\lambda g|^2$ converges. Hence, we just need to consider the convergence of:

$$\int_a^b \overline{\lambda g} f + \lambda g \bar{f}$$

    Theorem 4.15 in the notes states that if $f(x)$ is a measurable function on I, and we have $|f(x)| < g(x)$ for almost every $x \in I$ (in other words, the inequality doesn't hold only on a measurable set), then if g is integrable, f is also integrable (the proof of this involves construction a sequence of functions which converge to f, and are bounded above by g, and then applying the **Dominated Convergence Theorem**).

    Notice, if we take the absolute value of the complex functions:

$$\begin{aligned} |\overline{\lambda g} f + \lambda g \bar{f}| &\leq |\overline{\lambda g} f| + |\lambda g \bar{f}| \\ &= 2|\lambda||g||f| \end{aligned}$$

    A useful inequality to use is:

$$(a - b)^2 = a^2 + b^2 - 2ab \geq 0 \implies ab \leq \frac{a^2 + b^2}{2}$$

    It follows that:

$$\begin{aligned} |\overline{\lambda g} f + \lambda g \bar{f}| &\leq 2|\lambda||g||f| \\ &\leq |\lambda|(|g|^2 + |f|^2) \end{aligned}$$

    Hence, $|\overline{\lambda g} f + \lambda g \bar{f}|$ is dominated above by $|\lambda|(|g|^2 + |f|^2)$, which is integrable. Thus, by Theorem 4.15, it must be the case that $\overline{\lambda g} f + \lambda g \bar{f}$ is also integrable, so:

$$\int_a^b |f + \lambda g|^2$$

    must be integrable, and thus, $f + \lambda g \in L^2$. □

- **Are step functions in L^2?**
  - yes, since they are always positive, so $|f(x)| = f$, and f^2 is also a step function, which is Lebesgue Integrable
- **Are continuous functions on $[a, b]$ in L^2?**
  - yes, since $|f(x)|$ and $|f(x)|^2$ will be continuous, and so, Lebesgue Integrable

2.2 The Inner Product of Functions

- **What is the inner product of 2 functions in L^2?**
  - let $f, g \in L^2([a, b])$
  - define their **inner product** via:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

- **When are 2 functions orthogonal?**
  - whenever their inner product is 0:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx = 0$$

- **What is sesquilinearity of the inner product?**
  - the inner product is linear in the first term:

$$\langle f + \lambda g, h \rangle = \langle f, h \rangle + \lambda \langle g, h \rangle$$

  - the inner product is "semi-linear" in the second term:

$$\langle h, f + \lambda g \rangle = \langle h, f \rangle + \bar{\lambda} \langle h, g \rangle$$

- **What is antisymmetry of the inner product?**

$$\langle f, g \rangle = \overline{\langle g, f \rangle}$$

- **What is positivity of the inner product?**
  - if f is 0 almost everywhere (that is, $f = 0$ on a set of measure 0 only), then:

$$\|f\|_2^2 = \langle f, f \rangle > 0$$

  - otherwise,

$$\|f\|_2^2 = \langle f, f \rangle \geq 0$$

*These properties define the abstract notion of an **inner product space** of which L^2 is a prototypical example. In fact, L^2 is an example of a **Hilbert space** – that is an **inner product space**, which has the desirable property of completeness which means that every Cauchy sequence in L^2 converges to a limit in L^2.*

2.3 The Cauchy-Schwarz Inequality

We have defined the inner product as an integral for functions in L^2. However, we have no guarantee that said integral will be defined. The Cauchy-Schwarz inequality is useful in general, and in particular helps show that $\langle f, g\rangle$ is always defined, provided $f, g \in L^2$.

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*Let $f, g \in L^2([a, b])$. Then, the function $f\bar{g}$ is **Lebesgue Integrable** and:*

$$|\langle f, g\rangle| = \int_a^b |f\bar{g}| \le \|f\|_2\|g\|_2$$

[Theorem 5.1]

Proof: Cauchy-Schwarz Inequality. To show that:

$$\int_a^b f(x)\overline{g(x)}dx$$

exists, we once again apply Theorem 4.15:

*Let f_n be a sequence of **integrable** functions on an interval I, and assume that:*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

*Further assume that the sequence f_n is **dominated** by a function g:*

$$|f_n(x)| \le g(x), \qquad \forall x \in I, \forall n \ge 1$$

with:

$$\int_I g < \infty$$

*Then, the function f is **integrable** on I and:*

$$\int_I f = \int_I \left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} \int_I f_n$$

[Theorem 4.15]

We consider whether we can dominate $|f(x)\overline{g(x)}|$ with an integrable function. Indeed:

$$|f(x)\overline{g(x)}| = |f(x)||\overline{g(x)}| \le \frac{1}{2}(|f(x)|^2 + |g(x)|^2)$$

This is an integrable function, so it follows that $\int_a^b f(x)\overline{g(x)}dx$ exists. Moreover, notice that:

$$|\langle f, g\rangle| = \int_a^b |f(x)g(x)|dx$$

Using $\lambda > 0$, we can rewrite the integral as:

$$\int_a^b |f(x)g(x)|dx = \int_a^b |\lambda f(x)||\lambda^{-1}g(x)|dx \leq \frac{\lambda^2}{2}\int_a^b |f(x)|^2dx + \frac{1}{2\lambda^2}\int_a^n |g(x)|^2 = \frac{\lambda^2}{2}\langle f, f\rangle + \frac{1}{2\lambda^2}\langle g, g\rangle$$

If we want to make the inequality:

$$\int_a^b |f(x)g(x)|dx \leq \frac{\lambda^2}{2}\langle f, f\rangle + \frac{1}{2\lambda^2}\langle g, g\rangle$$

we minimise the RHS with respect to λ:

$$\lambda\langle f, f\rangle - \frac{1}{\lambda^3}\langle g, g\rangle = 0 \implies \lambda^2 = \sqrt{\frac{\langle g, g\rangle}{\langle f, f\rangle}}$$

Hence:

$$\begin{aligned}
&\int_a^b |f(x)g(x)|dx \leq \frac{\lambda^2}{2}\langle f, f\rangle + \frac{1}{2\lambda^2}\langle g, g\rangle \\
\implies &\int_a^b |f(x)g(x)|dx \leq \frac{1}{2}\left(\sqrt{\frac{\langle g, g\rangle}{\langle f, f\rangle}}\langle f, f\rangle + \sqrt{\frac{\langle f, f\rangle}{\langle g, g\rangle}}\langle g, g\rangle\right) \\
\implies &\int_a^b |f(x)g(x)|dx \leq \frac{1}{2}\left(\frac{\langle g, g\rangle\langle f, f\rangle + \langle f, f\rangle\langle g, g\rangle}{\sqrt{\langle f, f\rangle\langle g, g\rangle}}\right) \\
\implies &\int_a^b |f(x)g(x)|dx \leq \sqrt{\langle f, f\rangle\langle g, g\rangle} \\
\implies &\int_a^b |f(x)g(x)|dx \leq \|f\|_2\|g\|_2
\end{aligned}$$

□

2.3.1 Exercises (TODO)

1. **Show that if $f \in L^2([a, b])$ then $\exists C > 0$ such that:**

$$\|f\|_1 \leq C\|f\|_2$$

2. **Show that there does not exists $C \in (0, \infty)$ such that:**

$$\|f\|_2 \leq C\|f\|_1$$

   for every $f \in L^2$.

2.4 The Minkowski Inequality

The Cauchy-Schwarz Inequality allows us to generalise the triangle inequality for L^2 space, via the Minkowski Inequality:

> *If $f, g \in L^2([a,b])$ then:*
>
> $$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$$

Proof: Minkowski Inequality. Whilst the result might seem simple, it becomes less apparent if we consider the integral form:

$$\sqrt{\int_a^b |f+g|^2} \leq \sqrt{\int_a^b |f|^2} + \sqrt{\int_a^b |g|^2}$$

Notice, if $\|f+g\|_2^2 = 0$, since the L^2-norm is non-negative, we would be done. Hence, consider $\|f+g\|_2^2 > 0$. We have:

$$\begin{aligned}
\|f+g\|_2^2 &= \int_a^b |f+g|^2 \\
&= \int_a^b |f+g||f+g| \\
&\leq \int_a^b |f+g|(|f|+|g|) \\
&= \int_a^b |f+g||f| + \int_a^b |f+g||g|
\end{aligned}$$

Now, notice that:

$$\int_a^b |f+g||f| = |\langle f+g, f\rangle|$$

$$\int_a^b |f+g||g| = |\langle f+g, g\rangle|$$

Hence, by the Cauchy-Schwarz Inequality:

$$\begin{aligned}
\|f+g\|_2^2 &\leq |\langle f+g, f\rangle| + |\langle f+g, g\rangle| \\
&\leq \|f+g\|_2\|f\|_2 + \|f+g\|_2\|g\|_2 \\
&= \|f+g\|_2(\|f\|_2 + \|g\|_2)
\end{aligned}$$

But if we then divide by $\|f+g\|_2$ we get:

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$$

as desired.

$\square$

2.5 Convergence of Functions in L^2

- **When does a function converge in L^2?**
  - let $f_1, f_2, \ldots$ and f be functions in $L^2([a, b])$
  - sequence (f_n) **converges** to f in L^2 if the sequence:

    $$\|f_n - f\|_2 = \sqrt{\int_a^b |f_n(x) - f(x)|^2 dx}$$

    converges to 0 as $n \to \infty$
- **How is L^2 convergence related to normal function convergence?**
  - if $f_n \to f$ **uniformly** on $[a, b]$, then $f_n \to f$ on L^2
  - if $f_n \to f$ on L^2, then it doesn't mean that f_n converges. In fact, we can have convergence in L^2, but no convergence for any point on $[a, b]$
  - if f_n, f are all in L^2, and $f_n \to f$ for every $x \in [a, b]$, it doesn't mean that $f_n \to f$ on L^2 (for example, $f_n(x) = \sqrt{n}x^n$)
  - if however $|f_n| \leq 1$, then, if $f_n \to f$ on $[a, b]$, it follows that $f_n \to f$ on L^2

3 Orthonormal Systems

3.1 Defining Orthonormal Systems

- **What is an orthonormal system?**
  - consider a set of functions $\phi_n \in L^2$
  - it's an **orthonormal system** if the set is **mutually orthogonal**, and $\langle \phi_n, \phi_n \rangle = 1$:

    $$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(x)\overline{\phi_m(x)}dx = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

3.1.1 Examples

- if I_n are disjoint sets, then if $\phi_n = \mathcal{X}_{I_n}, n = 0, 1, \ldots, N-1$ forms an orthonormal system on $[0, N]$ (this is simple, since the sets are disjoint, the characteristic functions will be 1 if and only if both functions are defined over the same interval)
- the following are orthonormal systems on $[0, 1]$. Considering $n \in \mathbb{Z}$:
  - $e^{i2n\pi x}$
  - $\sqrt{2}\cos 2n\pi x$
  - $\sqrt{2}\sin 2n\pi x$
- the **Rademacher Function** for $n \in \mathbb{N}$ and $x \in [0, 1]$, defined as:

  $$\phi_n(x) = sgn(\sin(2^n \pi x))$$

3.2 Expanding Functions via Orthonormal Systems

One useful property of orthonormal systems is that we can use them as a basis for constructing arbitrary functions by using linear combinations. We motivate this by considering a best case scenario (i.e a function which is an actual linear combination of an orthonormal system), and then show how it generalises.

3.2.1 Exact Function as a Linear Combination

If we have an orthonormal system $(\phi_n)_n$, and we have a function f which is a linear combination of the functions in the system, we can write:

$$f(x) = \sum_n c_n \phi_n(x)$$

We can easily compute c_n by considering the fact that the ϕ_n are mutually orthonormal:

$$\begin{aligned}
& f(x) = \sum_n c_n \phi_n(x) \\
\implies & f(x)\overline{\phi_m(x)} = \sum_n c_n \phi_n(x)\overline{\phi_m(x)} \\
\implies & \int_a^b f(x)\overline{\phi_m(x)} = \int_a^b \sum_n c_n \phi_n(x)\overline{\phi_m(x)} \\
\implies & \langle f(x), \phi_m(x)\rangle = \sum_n c_n \int_a^b \phi_n(x)\overline{\phi_m(x)} \\
\implies & \langle f(x), \phi_m(x)\rangle = \sum_n c_n \langle \phi_n(x), \phi_m(x)\rangle \\
\implies & \langle f(x), \phi_m(x)\rangle = c_m
\end{aligned}$$

Hence, we can compute c_n via:

$$c_n = \langle f(x), \phi_n(x)\rangle$$

3.2.2 Theorem: Orthonormal Projection via Linear Combination

*What is remarkable is that, even if f is **not** expressed as a linear combination of ϕ_n, we can still **approximate** it using said linear combination, and the coefficients will be computed in the same way. In fact, we can show that if we approximate f via a linear combination of ϕ_n, $c_n = \langle f(x), \phi_n(x)\rangle$ are the **best** set of coefficients.*

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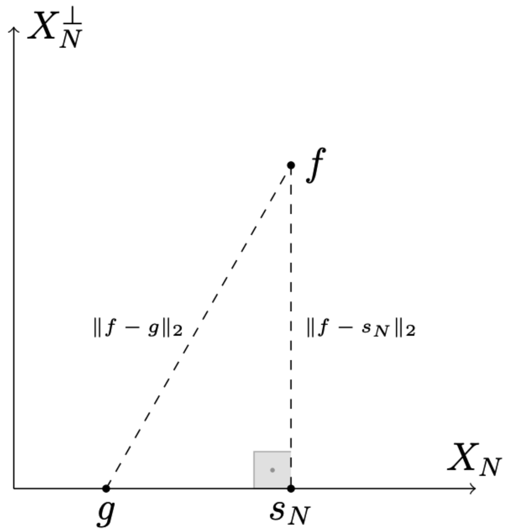
*Let $(\phi_n(x))$ be an **orthonormal system** on $[a, b]$, and consider $f \in L^2$. Define a linear combination of ϕ_n as:*

$$s_N = \sum_{n=1}^{N} \langle f, \phi_n\rangle \phi_n(x)$$

*Moreover, denote X_N to be the **span** of ϕ_n (i.e the set of all possible linear combinations of ϕ_n).*
*If g is **any** function in X_N (i.e $g = \sum_{n=1}^{N} b_n\phi_n(x)$), then:*

$$\|f - s_N\|_2 \leq \|f - g\|_2$$

*In other words, s_N is the **best possible approximation** to f (in L^2), out of all possible linear combinations in X_N. In fact, s_N is **unique**, so equality holds if and only if $g = s_N$. [Theorem 5.2]*



Proof: Orthonormal Projection. Define $g \in X_N$ via:

$$g(x) = \sum_{n=1}^{N} b_n \phi_n(x)$$

Furthermore, we have:

$$c_n = \langle f, \phi_n \rangle$$

and:

$$s_N = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

For this proof, we are interested in expressing $\|f - g\|_2$ and $\|f - s_N\|_2$. We can equivalently consider their squares, so:

$$\|f - g\|_2^2 = \int_a^b (f-g)\overline{(f-g)} = \langle f - g, f - g \rangle$$

Similarly:

$$\|f - s_N\|_2 = \langle f - s_N, f - s_N \rangle$$

Lets consider each expression at a time. If we employ sesquilinearity for

$$\langle f - g, f - g \rangle = \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle$$

Since f is arbitrary, we can't infer anything useful. For the remaining 3, we can compute expressions. Indeed:

$$\begin{aligned}
\langle g, g \rangle &= \left\langle \sum_{n=1}^{N} b_n \phi_n, \sum_{m=1}^{N} b_m \phi_m \right\rangle \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} b_n \overline{b_m} \langle \phi_n, \phi_m \rangle \\
&= \sum_{n=1}^{N} b_n \overline{b_n} \\
&= \sum_{n=1}^{N} |b_n|^2
\end{aligned}$$

Moreover:

$$\langle f, g\rangle = \left\langle f, \sum_{n=1}^{N} b_n\phi_n \right\rangle = \sum_{n=1}^{N} \overline{b_n}\langle f, \phi_n\rangle = \sum_{n=1}^{N} \overline{b_n}c_n$$

So by antisymmetry:

$$\langle g, f\rangle = \overline{\langle f, g\rangle} = \sum_{n=1}^{N} \overline{c_n}b_n$$

Thus,

$$\begin{aligned}
\langle f-g, f-g\rangle &= \langle f, f\rangle - \langle f, g\rangle - \langle g, f\rangle + \langle g, g\rangle \\
&= \langle f, f\rangle - \sum_{n=1}^{N} \overline{b_n}c_n - \sum_{n=1}^{N} \overline{c_n}b_n + \sum_{n=1}^{N} b_n\overline{b_n} \\
&= \langle f, f\rangle + \sum_{n=1}^{N} (b_n - c_n)\overline{(b_n - c_n)} - \sum_{n=1}^{N} c_n\overline{c_n} \\
&= \langle f, f\rangle + \sum_{n=1}^{N} |b_n - c_n|^2 - \sum_{n=1}^{N} |c_n|^2
\end{aligned}$$

We can perform similar computations for $\langle f - s_N, f - s_N\rangle$:

$$\langle f - s_N, f - s_N\rangle = \langle f, f\rangle - \langle f, s_N\rangle - \langle s_N, f\rangle + \langle s_N, s_N\rangle$$

Again, we compute each individual expression:

$$\langle s_N, s_N\rangle = \left\langle \sum_{n=1}^{\infty} c_n\phi_n, \sum_{m=1}^{\infty} c_m\phi_m \right\rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} c_n\overline{c_m}\langle \phi_n, \phi_m\rangle = \sum_{n=1}^{N} |c_n|^2$$

$$\langle f, s_N\rangle = \left\langle f, \sum_{n=1}^{\infty} c_n\phi_n \right\rangle = \sum_{n=1}^{N} \overline{c_n}\langle f, \phi_n\rangle = \sum_{n=1}^{N} \overline{c_n}c_n = \sum_{n=1}^{N} |c_n|^2$$

$$\langle s_N, f\rangle = \sum_{n=1}^{N} |c_n|^2$$

Hence:

$$\begin{aligned}
\langle f - s_N, f - s_N\rangle &= \langle f, f\rangle - \langle f, s_N\rangle - \langle s_N, f\rangle + \langle s_N, s_N\rangle \\
&= \langle f, f\rangle - 2\sum_{n=1}^{N} |c_n|^2 + \sum_{n=1}^{N} |c_n|^2 \\
&= \langle f, f\rangle - \sum_{n=1}^{N} |c_n|^2
\end{aligned}$$

In other words, we have:

$$\langle f-g, f-g\rangle = \langle f - s_N, f - s_N\rangle + \sum_{n=1}^{N} |b_n - c_n|^2$$

Thus, it follows that:

$$\langle f-g, f-g\rangle \geq \langle f - s_N, f - s_N\rangle$$

with equality holding **if and only if** $b_n = c_n$.

$\square$

3.3 Theorem: Bessel's Inequality

From the orthonormal projection above, a number of theorems can be derived.

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*If $(\phi_n)_{n=1,2,\ldots}$ is an **orthonormal system** on $[a,b]$, and $f \in L^2([a,b])$, then:*

$$\sum_n |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2$$

[Theorem 5.3]

Proof: Bessel's Inequality. Lets notice that:

$$\|f\|_2^2 = \langle f, f \rangle$$

$$\langle f, \phi_n \rangle = c_n$$

In other words, we can rewrite Bessel's Inequality as:

$$\langle f, f \rangle \geq \sum_n |c_n|^2$$

(without loss of generality we can assume $n = 1, 2, \ldots$).

But then notice that, in proving that an orthonormal projection is the best L^2 approximation to a function(3.2.2), we showed that:

$$\langle f - s_N, f - s_N \rangle = \langle f, f \rangle - \sum_{n=1}^{N} |c_n|^2$$

By positivity, $\langle f - s_N, f - s_N \rangle \geq 0$, so in particular:

$$\langle f, f \rangle - \sum_{n=1}^{N} |c_n|^2 \geq 0 \implies \langle f, f \rangle \geq \sum_{n=1}^{N} |c_n|^2$$

and this is true $\forall N$, so if we take $N \to \infty$:

$$\langle f, f \rangle \geq \sum_n |c_n|^2$$

as required. The series must converge, since it is bounded above, independent of n.

$\square$

3.4 Corollary: Riemann-Lebesgue Lemma

*Let $(\phi_n)_{n=1,2,\dots}$ be an **orthonormal system**, and let $f \in L^2$. Then:*

$$\lim_{n\to\infty} \langle f, \phi_n \rangle = \lim_{n\to\infty} c_n = 0$$

Proof: Riemann-Lebesgue Lemma. This is just a consequence of Bessel's Inequality: since $\sum_n |c_n|^2$ converges, it must be the case that $\sum_n |c_n|$ is also convergent (since $|c_n|$ is positive), so in particular $\sum_n c_n$ converges absolutely. But then, for this to converge, we must have:

$$\lim_{n\to\infty} c_n = 0$$

□

3.5 Complete Orthonormal Systems

Complete orthonormal systems are more robust, and particularly useful.

- **What is a complete orthonormal system?**
  - an **orthonormal system** satisfying **Parseval's Identity**:

$$\sum_n |\langle f, \phi_n \rangle|^2 = \|f\|_2^2$$

---

*Let (ϕ_n) be an **orthonormal system** in $[a, b]$.*
Define:

$$s_N = \sum_n c_n \phi_n$$

*We say the **orthonormal system** (ϕ_n) is **complete**, **if and only if***

$$s_N \to f$$

*on L^2 for **any** $f \in L^2$ [Theorem 5.4]*

Proof: Completeness and Convergence. Again using (3.2.2), we had:

$$\langle f - s_N, f - s_N\rangle = \langle f, f\rangle - \sum_{n=1}^{N} |c_n|^2 \implies \|f - s_N\|_2^2 = \|f\|_2^2 - \sum_{n=1}^{N}\langle f, \phi_n\rangle$$

If we take the limit as $N \to \infty$, notice that $\|f\|_2^2 - \sum_{n=1}^{N}\langle f, \phi_n\rangle \to 0$ if and only if the orthonormal system is complete (this is by definition). In other words:

$$\|f - s_N\|_2^2 \to 0 \implies s_N \to f$$

if and only if ϕ_n is complete.

□

4 Workshop

Define $L^2 = L^2([a,b])$ as the set of ***measurable functions****:*

$$f : [a,b] \to \mathbb{C}$$

so that:

$$\|f\|_2^2 := \int_a^b |f(x)|^2 \, dx < \infty$$

1. **Show that L^2 forms a vector space: if $f, g \in L^2$ and $\lambda \in \mathbb{C}$, then $f + \lambda g \in L^2$.**

Hints:

- *"Let f be a measurable function on I, and assume that:*

$$|f(x)| \leq g(x)$$

  for almost every $x \in I$, where g is an integrable function on I. Then, f is integrable on I." [Theorem 4.15]
- *if $z \in \mathbb{C}$, then $|z|^2 = z\bar{z}$*
- *if $x, y \geq 0$ then:*

$$xy \leq \frac{1}{2}(x^2 + y^2)$$

Since f, g are measurable, then $f + \lambda g$ are measurable.

From the definition of the modulus of a complex number:

$$\begin{aligned}
|f(x) + \lambda g(x)|^2 &= (f(x) + \lambda g(x))\overline{(f(x)\lambda g(x))} \\
&= f(x)\overline{f(x)} + \lambda g(x)\overline{f(x)} + \bar{\lambda} f(x)\overline{g(x)} + \lambda\bar{\lambda} g(x)\overline{g(x)} \\
&= |f(x)|^2 + |\lambda g(x)|^2 + g(x)\overline{f(x)} + f(x)\overline{g(x)}
\end{aligned}$$

Since $f, g \in L^2$, then $|f(x)|^2, |\lambda g(x)|^2$ are integrable, so it follows that $|f(x) + g(x)|^2$ is integrable if and only if we can show that:

$$\lambda g(x)\overline{f(x)} + \bar{\lambda} f(x)\overline{g(x)}$$

is integrable. Notice:

$$\overline{\lambda g(x)\overline{f(x)}} = \bar{\lambda} f(x)\overline{g(x)}$$

So it is sufficient to show (by linearity of the integral, and the fact that if $f \in L^2$, then $\bar{f} \in L^2$) that $\lambda g(x)\overline{f(x)}$ is integrable.

Now, consider Theorem 4.15. $\lambda g(x)\overline{f(x)}$ is a product of measurable functions, so it is measurable. Moreover:

$$|\lambda g(x)||\overline{f(x)}| \leq \frac{1}{2}(|\lambda g(x)|^2 + |\overline{f(x)}|^2)$$

Since $|\lambda g(x)|^2 + |\overline{f(x)}|^2$ is integrable (as $f, g \in L^2$), we have that Theorem 4.15 applies, and so, $\lambda g(x)\overline{f(x)}$ must be integrable.

Thus, if $f, g \in L^2$, it follows that $f + \lambda g \in L^2$.

2. **Let $f : [0,1] \to \mathbb{C}$, and $a \in \mathbb{R}$. For each of the following, decide if f is necessarily in L^2:**

   (a) $f(x) = e^{2\pi i a x}$

   We have:

$$\begin{aligned}\int_0^1 |e^{2\pi i a x}|^2 \, dx &= \int_0^1 (e^{2\pi i a x})(e^{-2\pi i a x}) \, dx \\ &= \int_0^1 1 \, dx \\ &= 1\end{aligned}$$

   Moreover, since f is continuous, it is Lebesgue Measurable. Thus, it follows that $e^{2\pi i a x} \in L^2$

   (b) $f(x) = x^a \mathcal{X}_{(0,1]}(x)$

   Firstly, $f(x)$ is continuous (except possibly at 0), so it is Lebesgue Measurable.
   Now, if $x \in (0,1]$, then:

$$|f(x)|^2 = x^{2a}$$

   We have that:

$$\begin{aligned}\int_0^1 |f(x)^2| &= \lim_{u \to 0^+} \int_u^1 x^{2a} \\ &= \begin{cases} \lim_{u \to 0^+} [\ln|x|]_u^1, & 2a = -1 \\ \lim_{u \to 0^+} \left[\frac{x^{2a+1}}{2a+1}\right]_u^1, & 2a \neq -1 \end{cases}\end{aligned}$$

   Now, if $2a = -1$, the integral won't be defined. If $2a \neq 1$ then:

$$\lim_{u \to 0^+} \left[\frac{x^{2a+1}}{2a+1}\right]_u^1 = \lim_{u \to 0^+} \left[\frac{1^{2a+1}}{2a+1} - \frac{u^{2a+1}}{2a+1}\right]$$

   If $2a < -1$, then $2a + 1 < 0$, so:

$$\lim_{u \to 0^+} \left[\frac{1^{2a+1}}{2a+1} - \frac{u^{2a+1}}{2a+1}\right] = \lim_{u \to 0^+} \left[\frac{1^{2a+1}}{2a+1} - \frac{1}{(2a+1)u^{-2a-1}}\right] = -\infty$$

   so the integral won't be defined.
   However, if $2a > 1$, then:

$$\lim_{u \to 0^+} \left[\frac{1^{2a+1}}{2a+1} - \frac{u^{2a+1}}{2a+1}\right] = \frac{1}{2a+1}$$

   and so, it follows that $f(x) = x^a \mathcal{X}_{(0,1]}(x) \in L^2$ only when $2a > -1 \implies a > -\frac{1}{2}$.

(c) f **is continuous**
If f is continuous, it is measurable. Moreover, $|f|^2$ will also be continuous, and so, integrable. Hence, $f \in L^2$.

(d) f **is a step function**
Step functions are measurable. Moreover, $|f|^2$ will also be a step function, and step functions are Lebesgue Integrable. Hence, $f \in L^2$.

(e) f **is Lebesgue Integrable**
Consider:

$$f(x) = x^{-\frac{1}{2}} \mathcal{X}_{(0,1]}(x)$$

Then:

$$\int_0^1 f(x)\ dx = \int_0^1 x^{-\frac{1}{2}}\ dx = [2x^{\frac{1}{2}}]_0^1 = 2$$

so f is Lebesgue Integrable. However, we showed above that that $|f(x)|^2$ won't be Lebesgue Integrable (since this is the case $a = -\frac{1}{2}$). Hence, f need not be in L^2.

If $f, g \in L^2([a, b])$, then their inner product is defined by:

$$\langle f, g \rangle := \int_a^b f(x)\overline{g(x)}\ dx$$

3. **Let us prove on the most important inequalities in analysis: the Cauchy-Schwarz inequality. For $f, g \in L^2$, we have:**

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$$

(a) **Show that the integral:**

$$\int_a^b f(x)\overline{g(x)}\ dx$$

exists. As a hint, you might want to use Theorem 4.15 of the notes, outlined above.
This is clear from the first question. Since f, g are measurable, then so is $f(x)\overline{g(x)}$. Moreover:

$$|f(x)||\overline{g(x)}| \leq \frac{1}{2}(|f(x)|^2 + |\overline{g(x)}|^2)$$

Since $|f(x)|^2 + |\overline{g(x)}|^2$ is integrable (as $f, g \in L^2$), we have that Theorem 4.15 applies, and so, $f(x)\overline{g(x)}$ must be integrable.
Thus, if $f, g \in L^2$, it follows that $f(x)\overline{g(x)}$ is integrable.

(b) **Let $\lambda > 0$. Show that:**

$$\int_a^b |f(x)g(x)| \leq \frac{\lambda}{2} \int_a^b |f(x)|^2\ dx + \frac{1}{2\lambda} \int_a^b |g(x)|^2\ dx$$

We have:

$$\begin{aligned} |f(x)g(x)| &= \left| \frac{\sqrt{\lambda}}{\sqrt{\lambda}} f(x)g(x) \right| \\ &= |\lambda f(x)| \left| \frac{1}{\sqrt{\lambda}} g(x) \right| \\ &\leq \frac{1}{2} \left(\lambda |f(x)|^2 + \frac{1}{\lambda} |g(x)|^2 \right) \end{aligned}$$

So indeed:

$$\int_a^b |f(x)g(x)| \leq \frac{\lambda}{2}\int_a^b |f(x)|^2 \; dx + \frac{1}{2\lambda}\int_a^b |g(x)|^2 \; dx$$

(c) **By using calculus, or otherwise, find the value of $lmabda$ that minimises the right hand side of the previous inequality (holding, f, g fixed) and finish the proof.**

Define:

$$F = \int_a^b |f(x)|^2 \; dx \qquad G = \int_a^b |g(x)|^2$$

Then:

$$\begin{aligned}
&\frac{d}{d\lambda}\left(\frac{\lambda}{2}F + \frac{1}{2\lambda}G\right) = 0\\
\implies &\frac{F}{2} - \frac{G}{2\lambda^2} = 0\\
\implies &\frac{F}{2} = \frac{G}{2\lambda^2}\\
\implies &\lambda = \sqrt{\frac{G}{F}}
\end{aligned}$$

Hence, we have that:

$$\begin{aligned}
\int_a^b |f(x)g(x)| &\leq \frac{\lambda}{2}F + \frac{1}{2\lambda}G\\
&= \frac{\sqrt{\frac{G}{F}}}{2}F + \frac{1}{2\sqrt{\frac{G}{F}}}G\\
&= \frac{\sqrt{G}\sqrt{F}}{2} + \frac{\sqrt{G}\sqrt{F}}{2}\\
&= \sqrt{G}\sqrt{F}
\end{aligned}$$

But now, notice:

$$\int_a^b |f(x)g(x)| = |\langle f, g\rangle| \qquad F = \int_a^b |f(x)|^2 \; dx = \|f\|_2^2 \qquad G = \int_a^b |g(x)|^2 = \|g\|_2^2$$

So:

$$|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$$

as required.

(d) **By examining the proof, determine when equality holds in the Cauchy-Schwarz inequality.**

The only inequality we considered was:

$$|f(x)g(x)| \leq \frac{1}{2}\left(\lambda|f(x)|^2 + \frac{1}{\lambda}|g(x)|^2\right)$$

Notice, we obtain equality if and only if:

$$\lambda|f(x)| = \frac{1}{\lambda}|g(x)|$$

If we integrate, equality of integration holds if and only if for almost every $x \in [a, b]$ (that is, $\lambda|f(x)| \neq \frac{1}{\lambda}|g(x)|$ only for x in a set of measure 0). In particular, we thus require that for some $c \in \mathbb{R}$, we have:

$$f(x) = cg(x)$$

for almost every $x \in [a, b]$.

4. **For integrable $f : [a, b] \to \mathbb{C}$, we write:**

$$\|f\|_1 = \int_a^b |f|$$

. Show that there exists a constant $C \in (0, \infty)$ such that:

$$\|f\|_1 \leq C\|f\|_2, \qquad \forall f \in L^2$$

Does the converse hold? That is, does there exist $C \in (0, \infty)$ such that:

$$\|f\|_2 \leq C\|f\|_1, \qquad \forall f \in L^2$$

Notice:

$$|\langle f, 1 \rangle| = \int_a^b |f \times 1| \; dx = \int_a^b |f| = \|f\|_1$$

Hence, by the Cauchy-Schwarz inequality:

$$\|f\|_1 \leq \|f\|_2 \|1\|_2 = \sqrt{b - a}\|f\|_2$$

as required.

For the second part, I had no idea what to do: none of the counterexamples I came up with worked.

Without loss of generality, consider the interval $[0, 1]$. Let $\varepsilon \in (0, 1)$ and define:

$$f = \mathcal{X}_{[0,\varepsilon]}$$

Then:

$$\|f\|_1 = \int_0^1 |\mathcal{X}_{[0,\varepsilon]}| \; dx = \varepsilon$$

$$\|f\|_2 = \sqrt{\int_0^1 |\mathcal{X}_{[0,\varepsilon]}|^2 \; dx} = \sqrt{\varepsilon}$$

But now, assume $\exists C > 0$ such that:

$$\|f\|_2 \leq C\|f\|_1$$

This would imply that:

$$\sqrt{\varepsilon} \leq C\varepsilon \implies \sqrt{\varepsilon} \geq \frac{1}{C} \implies \varepsilon \geq C^{-2}$$

However, ε was an arbitrary positive constant, so picking $\varepsilon < C^{-2}$ ensures that $\forall C > 0$ we don't have $\|f\|_2 \leq C\|f\|_1$.

Let $f, f_1, f_2, \ldots$ be functions in $L^2([a,b])$. We say that the sequence $(f_n)_n$ converges to f in L^2 if the sequence:

$$\|f_n - f\|_2 = \sqrt{\int_a^b |f_n(x) - f(x)|^2 \, dx}$$

converges to 0 as $n \to infty$. We write $f_n \to f$ in L^2-

5. **Show that if $f_n \to f$ uniformly on $[a,b]$, then $f_n \to f$ in L^2.**

I approached this using the standard definition, whilst the solutions use a much sleeker version of said definition.

Recall:

Let $f_n : E \to \mathbb{R}$ be a sequence of functions. Let $f : E \to \mathbb{R}$ be a function. Then, the following are equivalent:

1. *$f_n \to f$ uniformly on E*
2. *$\sup\limits_{x \in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$*
   - *in other words, $\forall \varepsilon > 0$ we can find some $N \in \mathbb{N}$ such that if $n \geq N$, then:*
     $$\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$$
   - *here, $\sup\limits_{x \in E} |f_n(x) - f(x)|$ is the sequence formed by:*
     - *for $n = 1$, consider the supremum of $|f_1(x) - f(x)|$ over all values of x*
     - *for $n = 2$, consider the supremum of $|f_2(x) - f(x)|$ over all values of x*
     - *$\ldots$*
3. *there exists a sequence $a_n \to 0$ such that for all $x \in E$, $|f_n(x) - f(x)| < a_n$*

[Proposition 2.1]

From solutions:

Since $f_n \to f$ uniformly, it follows that:

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| \to 0$$

as $n \to \infty$. But then:

$$\sqrt{\int_a^b |f_n(x) - f(x)|^2 \, dx} \leq \sqrt{\int_a^b \left(\underset{x \in [a,b]}{sup} |f_n(x) - f(x)| \right)^2 \, dx} = \underset{x \in [a,b]}{sup} |f_n(x) - f(x)| \sqrt{b-a} \to 0$$

so as required:

$$\|f_n - f\|_2 \to 0$$

From self:

By definition, since $f_n \to f$ uniformly on $[a,b]$, it follows that $\forall \varepsilon > 0, \exists N$ such that $\forall x \in [a,b], \forall n \geq N$ then:

$$|f_n(x) - f(x)| < \varepsilon$$

so if $n \geq N$:

$$\begin{aligned}
\|f_n - f\|_2 &= \sqrt{\int_a^b |f_n(x) - f(x)|^2 \, dx} \\
&\leq \sqrt{\int_a^b \varepsilon^2 \, dx} \\
&= \varepsilon \sqrt{b-a}
\end{aligned}$$

so indeed:

$$\|f_n - f\|_2 \to 0$$

6. **Suppose that $f_n(x) \to f(x), \forall x \in [0,1]$ for $f, f_1, f_2, \ldots \in L^2([0,1])$.**

   (a) **Show that not necessarily $f_n \to f$ in L^2.**

> *Again, we have 2 possible solutions, albeit with the same idea. One is from the solutions, and one is using a hint from the notes.*

From solutions:
Define:

$$f_n(x) = \sqrt{n} \mathcal{X}_{\left(0, \frac{1}{n}\right)}$$

Then,

$$f_n(x) \to 0$$

pointwise for $x \in [0,1]$ (since the N we pick will depend on the x aswell).
However:

$$\begin{aligned}
\|f_n(x) - f(x)\|_2 &= \sqrt{\int_0^1 |\sqrt{n} \mathcal{X}_{\left(0, \frac{1}{n}\right)}|^2 \, dx} \\
&= \sqrt{\int_0^1 n \mathcal{X}_{\left(0, \frac{1}{n}\right)} \, dx} \\
&= \sqrt{n \times \frac{1}{n}} \\
&= 1
\end{aligned}$$

and so:

$$\|f_n(x) - f(x)\|_2 \not\to 0$$

From self/notes:
Let:

$$f_n(x) = \sqrt{n}x^n$$

Then $f_n(x) \to 0$ pointwise on $[0, 1)$. However:

$$\begin{aligned}
\|f_n(x) - f(x)\|_2 &= \sqrt{\int_0^1 |\sqrt{n}x^n|^2 \ dx} \\
&= \sqrt{\int_0^1 nx^{2n} \ dx} \\
&= \sqrt{\left[n \times \frac{x^{2n+1}}{2n+1}\right]_0^1} \\
&= \sqrt{\left[n \times \frac{1^{2n+1}}{2n+1}\right]} \\
&= \sqrt{\frac{n}{2n+1}}
\end{aligned}$$

so:

$$\|f_n(x) - f(x)\|_2 \to \frac{1}{\sqrt{2}} \neq 0$$

(b) **Assume in addition that $|f_n(x)| \leq 1, \forall x \in [0, 1]$ and $n \geq 1$. Show that $f_n \to f$ in L^2.**
Since $f_n(x) \to f(x)$ pointwise, it follows that:

$$|f_n(x) - f(x)| \to 0 \implies |f_n(x) - f(x)|^2 \to 0$$

At this point, we could already claim that $|f_n(x) - f(x)|^2$ is bounded, but in the solutions they give an explicit bound.

Now, since $f_n \to f$, and $|f_n| \leq 1$, then $|f| \leq 1$, so:

$$\begin{aligned}
|f_n(x) - f(x)|^2 &\leq (|f_n(x)| + |f(x)|)^2 \\
&= |f_n(x)|^2 + 2|f_n(x)||f(x)| + |f(x)|^2 \\
&\leq |f_n(x)|^2 + (|f_n(x)|^2 + |f(x)|^2) + |f(x)|^2 \\
&= 2(|f_n(x)|^2 + |f(x)|^2) \\
&\leq 2(1 + 1) \\
&= 4
\end{aligned}$$

Hence, the sequence $|f_n(x) - f(x)|^2$ is bounded above by 4 for all n, x, and 4 is a constant, and so, integrable. Recall the Dominated Convergence Theorem:

> *Let f_n be a sequence of **integrable** functions on an interval I, and assume that:*
>
> $$f(x) = \lim_{n\to\infty} f_n(x)$$
>
> *Further assume that the sequence f_n is **dominated** by a integrable function g:*
>
> $$|f_n(x)| \leq g(x), \qquad \forall x \in I, \forall n \geq 1$$
>
> *with:*
>
> $$\int_I g < \infty$$
>
> *Then, the function f is **integrable** on I and:*
>
> $$\int_I f = \int_I \left(\lim_{n\to\infty} f_n \right) = \lim_{n\to\infty} \int_I f_n$$
>
> *[Theorem 4.12]*

It thus follows that:

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 \, dx = \int_a^b 0 = 0$$

and so, $f_n \to f$ on L^2-

7. **Construct a sequence $(f_n)_n$ of L^2 functions on $[0,1]$ so that $f_n \to 0$ in L^2, but the sequence $(f_n(x))_n$ does not converge for any $x \in [0,1]$.**

> *This is taken entirely from the solutions, unfortunately.*

We first show that the claim is true for the interval $\left[\frac{1}{2}, 1\right]$.

Define the subintervals:

$$I_n = \left[n2^{-k}, (n+1)2^{-k}\right]$$

where for each n, we pick k as the unique integer such that:

$$2^{k-1} \leq n < 2^k$$

Notice, by this definition, we always ensure that:

$$n2^{-k} \geq 2^{k-1}2^{-k} = \frac{1}{2}$$

$$(n+1)2^{-k} < 2^k 2^{-k} = 1$$

and so:

$$I_n \subset \left[\frac{1}{2}, 1\right]$$

Then, define:

$$f_n(x) = \mathcal{X}_{I_n}(x)$$

Then:

$$\|f_n(x) - 0\|_2 = \|\mathcal{X}_{I_n}(x)\|_2 = \sqrt{\lambda(I_n)} = 2^{-\frac{k}{2}}$$

Since as $n \to \infty$ we get $k \to \infty$, it follows that:

$$\lim_{n\to\infty} \|f_n(x) - 0\|_2 = \lim_{k\to\infty} 2^{-\frac{k}{2}} = 0$$

and so, $f_n \to f$ on L^2.

We now show that $f_n \not\to f$, and that in fact, it doesn't even converge. To do this, we show it isn't Cauchy. That is, for $N \geq 1$, we want to find $n_1, n_2 \geq N$ such that $x \in I_{n_1}$ but $x \notin I_{n_2}$, which then means that:

$$f_{n_1}(x) = 1 \qquad f_{n_2}(x) = 0$$

and so:

$$|f_{n_1}(x) - f_{n_2}(x)| = 1$$

implying that f_n won't be Cauchy.

To do this, pick $k \geq 2$ such that $2^{k-1} \geq N$. Then consider the intervals:

$$I_{2^{k-1}} = \left[2^{k-1}2^{-k}, (2^{k-1}+1)2^{-k}\right] = \left[\frac{1}{2}, \frac{1}{2} + 2^{-k}\right]$$

$$I_{2^{k-1}+1} = \left[(2^{k-1}+1)2^{-k}, (2^{k-1}+1+1)2^{-k}\right] = \left[\frac{1}{2} + 2^{-k}, \frac{1}{2} + 2^{-k+1}\right]$$

$$\vdots$$

$$I_{2^k-1} = \left[(2^k-1)2^{-k}, (2^k-1+1)2^{-k}\right] = \left[1 - 2^{-k}, 1\right]$$

Notice, these are all disjoint (except at the endpoints), and their union gives $\left[\frac{1}{2}, 1\right]$.

Since they cover the whole interval, it follows that $\exists n_1, 2^{k-1} \leq n_1 < 2^k$ such that $x \in I_{n_1}$.

Here I diverge from the solutions, and write what makes sense to me, as otherwise I'd just leave the proof as is.

Moreover, $\exists n_2, 2^{k-1} \leq n_2 < 2^k$ with $x \notin I_{n_2}$, as $I_{n_1} \cap I_{n_2} = \emptyset$. This proves the claim.

For the interval $[0,1]$, we just need to change each I_n by $2I_n - 1$.

*We say that 2 functions $f, g \in L^2([a,b])$ are **orthogonal** on $[a,b]$ if $\langle f, g\rangle = 0$.*

8. **Suppose that $f_1, f_2, \ldots, f_N$ are L^2 functions on $[a,b]$.**

   (a) **Show that:**

$$\left\|\sum_{k=1}^{N} f_k\right\|_2 \leq \sqrt{N}\sqrt{\sum_{k=1}^{N} \|f_k\|_2^2}$$

   and give an example to show that the constant $\sqrt{N}$ can't be improved.

For this I used an inductive proof. However, it requires getting a bit messy, so this is more elegant.

Recall the Cauchy-Schwarz Inequality (the normal one beyond our study of L^2 spaces):

$$\sum_{i=1}^{k} x_i y_i \leq \sqrt{\sum_{i=1}^{k} x_i^2}\sqrt{\sum_{i=1}^{k} y_i^2}$$

Then notice:

$$\left|\sum_{k=1}^{N} f_k\right| \leq \sum_{k=1}^{N} 1 \cdot |f_k|$$

If we apply the inequality, it follows that:

$$\left|\sum_{k=1}^{N} f_k\right| \leq \sqrt{\sum_{k=1}^{N} 1^2}\sqrt{\sum_{k=1}^{N} |f_k|^2} = \sqrt{N}\sqrt{\sum_{k=1}^{N} |f_k|^2}$$

So if we take the L^2 norm of both sides:

$$\left\|\sum_{k=1}^{N} f_k\right\|_2 \leq \sqrt{N}\left\|\sqrt{\sum_{k=1}^{N} |f_k|^2}\right\|_2$$

Now, notice:

$$\begin{aligned}\left\|\sqrt{\sum_{k=1}^{N} |f_k|^2}\right\|_2 &= \sqrt{\int_a^b \sum_{k=1}^{N} |f_k|^2} \\ &= \sqrt{\sum_{k=1}^{N} \int_a^b |f_k|^2} \\ &= \sqrt{\sum_{k=1}^{N} \|f_k\|_2^2}\end{aligned}$$

Hence:

$$\left\|\sum_{k=1}^{N} f_k\right\|_2 \leq \sqrt{N}\sqrt{\sum_{k=1}^{N} \|f_k\|_2^2}$$

as required.
To see that $\sqrt{N}$ can't be improved, consider:

$$f_1 = f_2 = \ldots = f_N = \mathcal{X}_{[a,b]}$$

Then:

$$\left\|\sum_{k=1}^{N} f_k\right\|_2 = \left\|\sum_{k=1}^{N} \mathcal{X}_{[a,b]}\right\|_2 = \left\|N\mathcal{X}_{[a,b]}\right\|_2 = N\sqrt{b-a}$$

$$\sqrt{N}\sqrt{\sum_{k=1}^{N}\|f_k\|_2^2} = \sqrt{N}\sqrt{\sum_{k=1}^{N}\|\mathcal{X}_{[a,b]}\|_2^2} = \sqrt{N}\sqrt{N(b-a)} = N\sqrt{b-a}$$

so we get:

$$\left\|\sum_{k=1}^{N} f_k\right\|_2 = \sqrt{N}\sqrt{\sum_{k=1}^{N}\|f_k\|_2^2}$$

and so, $\sqrt{N}$ can't be improved (in the sense that it is the smallest constant guaranteeing the inequality).

(b) **Suppose $f_1, \ldots, f_N$ are pairwise orthogonal on $[a,b]$. Show that:**

$$\left\|\sum_{k=1}^{n} f_k\right\|_2 \leq \sqrt{\sum_{k=1}^{N}\|f_k\|_2^2}$$

Again, this can be done using the standard definition of the L^2 norm, but it yields a long winded answer. Here, the properties of the inner product are your friend!

Notice:

$$\begin{aligned}\left\|\sum_{k=1}^{N} f_k\right\|_2^2 &= \left\langle \sum_{k=1}^{N} f_k, \sum_{k=1}^{N} f_k \right\rangle \\ &= \sum_{k=1}^{N}\sum_{j=1}^{n}\langle f_k, f_j\rangle\end{aligned}$$

where we have used symmetry and sesquilinearity.
But notice, since the functions are mutually orthogonal:

$$\langle f_k, f_j\rangle = 0 \iff k \neq j$$

and so:

$$\left\|\sum_{k=1}^{N} f_k\right\|_2^2 = \sum_{k=1}^{N}\sum_{j=1}^{n}\langle f_k, f_j\rangle = \sum_{k=1}^{N}\langle f_k, f_k\rangle = \sum_{k=1}^{N}\|f_k\|_2^2$$

as required.