Honours Analysis - Week 7 - The Lebesgue Integral (Continued) and the Riemann Integral

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Contents

1 Integrating Series of Functions **2**
- 1.1 Theorem: Integrability of a Convergent Series of Functions . . . 2
- 1.2 Theorem: Monotone Convergence Theorem . . . 3

2 The Riemann Integral **6**
- 2.1 Defining the Riemann Integral . . . 6
- 2.2 Theorem: Riemann Integrable Implies Boundedness . . . 6
- 2.3 Theorem: Riemann Integrability and Bounded Support . . . 8
- 2.4 Theorem: Computing Riemann Integral From Defintion . . . 9
- 2.5 Theorem: Connection Between Riemann and Lebesgue Integrability . . . 9
- 2.6 Lemma: Equivalent Criteria for Riemann Integrability . . . 12
- 2.7 Theorem: Riemann Integrability, Continuity and Monotonicity . . . 13
- 2.8 Corollary: Requirements For Both Riemann and Lebesgue Integrability . . . 15

3 Exercises **15**

4 Workshop **21**

1 Integrating Series of Functions

1.1 Theorem: Integrability of a Convergent Series of Functions

*We consider a theorem which tells us whether a **sequence of functions** f_n, which form a convergent series to some function $f(x)$, are **Lebesgue Integrable**. What is remarkable, is that we only require **pointwise convergence**, whilst previously, integrability of series of functions relied on **uniform convergence**.*

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*Suppose f_n is a sequence of functions, each of which is **integrable** on some I.*

*(a) **If**:*

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$

*(the sum of integrals of each function in the sequence is convergent) **and** f is a function on I, such that,*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

*for any x, such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$[a] , **then** f is **integrable** on I, and its integral is:*

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n < \infty$$

(b) If we further have that for any $n \in \mathbb{N}$, $f_n \geq 0$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in I$ (except for possibly finitely many points, in which we can allow that $\sum_{n=1}^{\infty} f_n(x) = \infty$).
*Then f is integrable on I **if and only if**:*

$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

[Theorem 4.3]

[a] this is just saying that we require the x to be such that the sum converges (to f)

In this theorem, we are basically saying that if:

$$\sum_{n=1}^{\infty} \int_I |f_n|$$

exists, and we can express $f(x)$ as a series of $f_n(x)$ (except possibly at finitely many points), then the integral of f exists, and can be expressed as an infinite sum of the integrals of $f_n(x)$.
*In part a), the convergence of $\sum_{n=1}^{\infty} \int_I |f_n|$ is a **sufficient** condition to prove convergence; in part b), and assuming f_n is non-negative, it is a **sufficient and necessary** condition (notice that since $f_n \geq 0$, then $\sum_{n=1}^{\infty} \int_I |f_n|$ is just $\sum_{n=1}^{\infty} \int_I f_n$).*
The proof for this is at the end of the notes.

1.2 Theorem: Monotone Convergence Theorem

*This theorem deals with the integrability of **sequences** of functions, as opposed to series of functions (as above).*

Suppose that f_n is a sequence of:

- ***monotone***
- ***non-decreasing***
- ***integrable***

functions on an interval I:

$$f_1(x) \leq f_2(x) \leq \dots$$

For any $x \in I$, define:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We allow that for some x, this limit diverges to infinity: we are not concerned with particular points. Notice, that if f_n is a bounded sequence, it will necessarily converge, since its monotone.
*Then, f must be **integrable** on I **if and only if**:*

$$\sup_{n\in\mathbb{N}} \int_I f_n = \lim_{n\to\infty} \int_I f_n < \infty$$

(this equivalence might not be immediately obvious, but it is due to the fact that f_n is non-decreasing, so $f_n \leq f_{n+1} \implies \int_I f_n \leq \int_I f_{n+1}$, so the supremum must coincide with the limit)
Moreover, we have that:

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

---

*There is an equivalent result if the sequence of functions is **monotone**, **non-increasing**, in which we just need to check that:*

$$\inf_{n\in\mathbb{N}} \int_I f_n$$

exists for

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

[Theorem 4.4]

Proof: Monotone Convergence Theorem. Without loss of generality, lets assume that $f_1 \geq 0$ (so every term is non-negative). If this is not the case, just consider the sequence formed by $f_n - f_1$ (this is guaranteed to always be non-negative), and at the end just add the integral of f_1.

Now, define a new sequence, g_n, such that:

$$g_n = \begin{cases} f_1, & n = 1 \\ f_n - f_{n-1}, & n > 1 \end{cases}$$

Notice, g_n is a sequence of **non-negative, integrable** functions, so in particular we can use part b) of the **Series Integrability Theorem** (1.1).

In order to apply the theorem, we just need to consider:

$$\sum_{n=1}^{\infty} g_n(x)$$

But then notice that for any $x \in I$:

$$\begin{aligned}
\sum_{n=1}^{\infty} g_n(x) &= \lim_{n\to\infty} \sum_{k=1}^{n} g_k(x) \\
&= \lim_{n\to\infty} (f_1(x) + (f_2(x) - f_1(x)) + \ldots + (f_n(x) - f_{n-1}(x)) \\
&= \lim_{n\to\infty} f_n(x) \\
&= f(x)
\end{aligned}$$

Thus, by the Theorem above, since $f(x) = \sum_{n=1}^{\infty} g_n(x)$, it must be the case that f is integrable if and only if:

$$\sum_{n=1}^{\infty} \int_I g_n(x) < \infty$$

So, we evaluate the series above:

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_I g_n(x) &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_I g_k(x) \\
&= \int_I g_1(x) + \int_I g_2(x) + \ldots + \int_I g_n(x) \\
&= \lim_{n\to\infty} \left(\int_I f_1(x) + \int_I (f_2(x) - f_1(x)) + \ldots + \int_I (f_n(x) - f_{n-1}(x)) \right) \\
&= \lim_{n\to\infty} \int_I f_n(x) \\
&= \sup_{n\in\mathbb{N}} \int_I f_n(x)
\end{aligned}$$

Thus, it follows that f is integrable, if and only if

$$\sum_{n=1}^{\infty} \int_I g_n(x) < \infty$$

And this is true if and only if:

$$\sup_{n\in\mathbb{N}} \int_I f_n(x)$$

converges, as required.

Knowing that f is integrable, and since $f(x) = \sum_{n=1}^{\infty} g_n(x)$, by the Theorem Above, it must be the case that:

$$\int_I f = \sum_{n=1}^{\infty} \int_I g_n(x)$$

But we have shown that:

$$\sum_{n=1}^{\infty} \int_I g_n(x) = \lim_{n\to\infty} \int_I f_n(x)$$

So it follows that:

$$\int_I f = \lim_{n\to\infty} \int_I f_n(x)$$

$\square$

2 The Riemann Integral

2.1 Defining the Riemann Integral

- **What is a Riemann Integrable function?**
  - consider a function $f : \mathbb{R} \to \mathbb{R}$
  - f is **Riemann Integrable** if $\forall \varepsilon > 0$, there exist 2 step functions ϕ, ψ such that:

$$\phi \leq f \leq \psi$$

and:

$$\int \psi - \int \phi < \varepsilon$$

2.2 Theorem: Riemann Integrable Implies Boundedness

*If f is **Riemann Integrable**, then:*

- *f is **bounded***
- *f has **bounded support**: $\exists E \subset \mathbb{R}$ such that E is **bounded**, and if $x \notin E$, then $f(x) = 0$; in particular,*

$$E = \{x \mid f(x) \neq 0\} \subset [a, b]$$

[Example 4.3]

Proof: Riemann Integrability and Boundedness. Let f be a Riemann Integrable Function. By the definition of Riemann Integrability, for any $\varepsilon = 1$, we can find step functions ϕ, ψ satisfying:

$$\phi \leq f \leq \psi$$

and

$$\int \psi - \int \phi < 1$$

We first show that f must be **bounded**.

Since ϕ, ψ are step functions, they are so with respect to some countable, bounded set of points. We consider the case for ϕ; ψ is analogous. We can define:

$$\phi(x) = \begin{cases} 0, & x < x_0 \\ 0, & x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

From this it follow that $\phi(x)$ is clearly bounded, since its supremum exists:

$$\sup_{x \in \mathbb{R}} |\phi(x)| = \max\{|c_0|, |c_1|, \ldots, |c_{n-1}|, |\phi(x_0)|, |\phi(x_1)|, \ldots, |\phi(x_n)|\}$$

where we need to consider the value of ϕ both on intervals (x_j, x_{j+1}) and at the endpoints of said intervals (which may be defined in any way). Since the set above is bounded, its supremum must exist.

If we apply the same logic on ψ, we observe that there must exist some $M \in \mathbb{R}$, such that for any $x \in \mathbb{R}$:

$$|\phi(x)| < M$$

$$|\psi(x)| < M$$

So in particular:

$$\phi \le f \le \psi \implies -M < f < M$$

so f is **bounded**. (Could've just argued that any step function is bounded, but this is an explicit statement of this)

We now show that f has **bounded support**. This follows very easily from the work below.

For ϕ, we know that if $x < x_0$ or $x > x_n$, $\phi(x) = 0$. Similarly, for ψ, we know that there exist y_0, y_k such that, if $x < y_0$ or $x > y_k$, $\psi(x) = 0$.

But then, it follows that if $x < a = \min\{x_0, y_0\}$ or $x > \max\{x_n, y_k\}$, we must have $f(x) = 0$, so it must be the case that the support E of f is bounded, since $E \subseteq [a, b]$.

$\square$

If we think about this, this is precisely the definition of Riemann Integral we know from school: ϕ, ψ represent the ***rectangles*** *which we use to approximate the function, and so their integral is representative of the area of the function. What this definition is saying is that a function f is integrable* ***if and only if*** *we can construct arbitrarily small rectangles which can bound the area of f.*
This also means that step functions are Riemann Integrable, since if f is a step function, we can choose $\phi = \psi = f$.

2.3 Theorem: Riemann Integrability and Bounded Support

*Let f be **Riemann Integrable**.*
If $f(x) = 0$ for all $x \notin [a, b]$, then we can take ϕ, ψ, such that ϕ and ψ are also 0 outside of $[a, b]$. [Example 4.3]

Proof: Bounded Support and Step Function for Riemann Integrable Functions. Let f be Riemann Integrable. Then, $\forall \varepsilon > 0$, there exist step functions ϕ, ψ, such that:

$$\phi \leq f \leq \psi \implies \int \psi - \int \phi < \varepsilon$$

Now, define 2 new step functions:

$$\phi^* = \phi \mathcal{X}_{[a,b]}$$
$$\psi^* = \psi \mathcal{X}_{[a,b]}$$

These are clearly 0 outside of $[a, b]$, and within $[a, b]$ have the same value as ϕ and ψ on said interval.

Now, we claim that $\phi \leq \phi^*$. This is easy to see. If we consider $x \in [a, b]$, then by definition of ϕ^*, we have $\phi(x) = \phi^*(x)$. Now, lets consider $x \notin [a, b]$. Since f is Riemann Integrable, for any x, we have:

$$\phi(x) \leq f(x)$$

But we know that $f(x) = 0$ for any $x \notin [a, b]$, so it follows that:

$$\phi(x) \leq 0$$

But $\phi^*(x) = 0$ precisely when $x \notin [a, b]$, so it follows that for any x:

$$\phi(x) \leq \phi^*(x)$$

Similarly, we claim that $\psi \geq \psi^*$. This is easy to see. If we consider $x \in [a, b]$, then by definition of ψ^*, we have $\psi(x) = \psi^*(x)$. Now, lets consider $x \notin [a, b]$. Since f is Riemann Integrable, for any x, we have:

$$f(x) \leq \psi(x)$$

But we know that $f(x) = 0$ for any $x \notin [a, b]$, so it follows that:

$$0 \leq \psi(x)$$

But $\psi^*(x) = 0$ precisely when $x \notin [a, b]$, so it follows that for any x:

$$\psi^*(x) \leq \psi(x)$$

From the work above, it follows that for any x:

$$\phi(x) \leq \phi^*(x) \leq f \leq \psi^*(x) \leq \psi(x)$$

So in particular:

$$\psi^*(x) - \phi^*(x) \leq \psi(x) - \phi(x) \implies \int \psi^*(x) - \phi^*(x) \leq \int \psi(x) - \phi(x) < \varepsilon$$

So it follows that

$$\int \psi^*(x) - \phi^*(x) < \varepsilon$$

Thus, we can use $\psi^*(x), \phi^*(x)$ for the Riemann Integrability of f, and these are 0 outside of $[a, b]$, as required. $\square$

2.4 Theorem: Computing Riemann Integral From Defintion

From the definition, we know that for an integrable function, we can define 2 step functions whose integrals get arbitrarily close to each other. Using this, we can find the value of a Riemann Integral.

> *A function $f : \mathbb{R} \to \mathbb{R}$ is* ***Riemann Integrable*** *if and only if, given 2 step functions ϕ, ψ, we have:*
>
> $$\sup \left\{ \int \phi \mid \phi \leq f \right\} = \inf \left\{ \int \psi \mid \psi \geq f \right\}$$
>
> *In particular, if such step functions exist, then we define the Riemann Integral of f as:*
>
> $$(R) \int f = \sup \left\{ \int \phi \mid \phi \leq f \right\} = \inf \left\{ \int \psi \mid \psi \geq f \right\}$$
>
> *[Theorem 4.5]*

2.5 Theorem: Connection Between Riemann and Lebesgue Integrability

> *Let f be a* ***Riemann Integrable*** *function.*
> *Then f is also* ***Lebesgue Integrable****, and both integrals are equal:*
>
> $$(R) \int f = \int f$$
>
> *[Theorem 4.6]*

Proof: Riemann Integrability Implies Lebesgue Integrability. We begin by noting that this holds for step functions. This is because the integral of a step function corresponds with the Lebesgue Integral of said function, and Riemann Integrals are found by bounding a function using step functions, and considering their integral.

The aim of the proof is to, from Riemann Integrability, use the **Monotone Convergence Theorem** (1.2), since this is a Theorem about Lebesgue Integrability. However, for this we need to consider a sequence of monotone, non-increasing/non-decreasing functions which converge to some function. In order to construct this sequence, we will use the step functions which help define Riemann Integrability.

The proof will follow the following steps:

1. Use Riemann Integrability to derive a sequence of step functions which bound f
2. From the above sequence, produce a sequence of monotone, step functions, as to apply the Monotone Convergence Theorem

3. Deduce that the step functions converge to step functions which bound f, and whose integral is equal to the Riemann Integral of f, by the MCT
4. We can then use the Thoerems about Lebesgue Integrability, to describe f as a sum of step functions, which implies Lebesgue Integrability
5. If we integrate our expression for f, we will notice that it reduces to the Riemann Integral of f

Indeed, a function is Riemann Integrable if there is a sequence of 2 step functions ϕ_n, ψ_n who's infimum/supremum coincide. In particular, for any $\varepsilon_n > 0$, we can find ϕ_n, ψ_n such that:

$$\phi_n \leq f \leq \psi_n \implies \int \psi_n - \phi_n < \varepsilon_n$$

and from properties of infimum and supremum, we must have:

$$(R)\int f = \lim_{n\to\infty} \int \phi_n = \lim_{n\to\infty} \int \psi_n$$

However, we don't have any guarantee that ϕ_n or ψ_n is monotone. In order to ensure this, we construct a new set of sequences of functions Φ, Ψ defined in the following manner:

$$\Phi_1 = \phi_1 \qquad \Phi_2 = \max\{\phi_1, \phi_2\} \qquad \Phi_3 = \max\{\phi_1, \phi_2, \phi_3\} \qquad \ldots$$ [1]

and

$$\Psi_1 = \psi_1 \qquad \Psi_2 = \min\{\psi_1, \psi_2\} \qquad \Psi_3 = \min\{\psi_1, \psi_2, \psi_3\} \qquad \ldots$$

By doing this, Φ_n and Ψ_n now represent monotone sequences of function, all memebers of which are step function. Moreover, notice that by construction, for any $n \in \mathbb{N}$:

$$\phi_n \leq \Phi_n$$

$$\Psi_n \leq \psi_n$$

Furthermore, we must also have (again by construction):

$$\phi_n \leq \Phi_n \leq f \leq \Psi_n \leq \psi_n$$

and by properties of integrals:

$$\int \phi_n \leq \int \Phi_n \leq (R)\int f \leq \int \Psi_n \leq \int \psi_n$$

But recall from the work above that:

$$(R)\int f = \lim_{n\to\infty} \int \phi_n = \lim_{n\to\infty} \int \psi_n$$

So taking the limit of the expression above, and applying Squeeze Theorem implies that:

$$(R)\int f = \lim_{n\to\infty} \int \Phi_n = \lim_{n\to\infty} \int \Psi_n$$

Lastly, notice that the following limits must exist:

$$\lim_{n\to\infty} \Phi_n(x) = \Phi(x)$$

[1] I advocate for the definition with $\Phi_1 = \phi_1$ and $\Phi_n = \max\{\Phi_{n-1}, \phi_n\}$, both because it is more succint, and more clearly evokes the monotonicity

$$\lim_{n\to\infty} \Psi_n(x) = \Psi(x)$$

since Φ_n, Ψ_n are monotone and bounded (step functions are always bounded). It also follows that:

$$\Phi(x) \leq f(x) \leq \Psi(x)$$

Since Φ_n, Ψ_n are monotone sequences of Lebesgue Integrable functions, by the Monotone Convergence Theorem, we have that:

$$\int \Phi(x) = \lim_{n\to\infty} \int \Phi_n(x)$$

$$\int \Psi(x) = \lim_{n\to\infty} \int \Psi_n(x)$$

and since:

$$(R) \int f = \lim_{n\to\infty} \int \Phi_n = \lim_{n\to\infty} \int \Psi_n$$

it follows that:

$$(R) \int f = \int \Phi = \int \Psi$$

Now that we have an expression for the Riemann Integral of f in terms of step functions, we need to show that f is Lebesgue Integrable, and that it is equal to the Riemann Integral. In order to do this, we can define a function:

$$h(x) = \Psi(x) - \Phi(x)$$

Notice that we must have $h(x) \geq 0$, and h is Lebesgue Integrable, since its the difference between 2 step functions. Moreover, it must be the case that:

$$0 \leq f(x) - \Phi(x) \leq h(x)$$

But now, recall a property of Lebesgue Integrability:

If $f \geq 0$ with $\int_I f = 0$ then any function h such that $0 \leq h \leq f$ on I is ***integrable*** *on I.*

Indeed, we have that $h(x) \geq 0$, and:

$$\int h(x) = \int \Psi(x) - \Phi(x) = 0$$

But then, by the Theorem above, and since $0 \leq f(x) - \Phi(x) \leq h(x)$, then $f(x) - \Phi(x)$ must be Lebesgue Integrable.

But since we can write:

$$f(x) = \Phi(x) - (\Phi(x) - f(x))$$

and both $\Phi(x)$ and $(\Phi(x) - f(x))$ are Lebesgue Integrable, f must be Lebesgue Integrable. Furthermore, from

$$0 \leq f(x) - \Phi(x) \leq h(x)$$

we know that $\int f(x) - \Phi(x) = 0$, so if we compute the Lebesgue Integral for our expression of f:

$$\int f(x) = \int \Phi(x) - (\Phi(x) - f(x)) = \int \Phi(x) = (R) \int f(x)$$

as required.

$\square$

2.6 Lemma: Equivalent Criteria for Riemann Integrability

*Let $f : \mathbb{R} \to \mathbb{R}$ be a **bounded** function with **bounded support** $[a, b]$. The following are equivalent:*

1. *f is **Riemann Integrable***
2. *$\forall \varepsilon > 0$, we can find $a = x_0 < x_1 < \ldots < x_n = b$, such that if M_j and m_j denote the infimum and supremum of f on (x_{j-1}, x_j) respectively, then:*
$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \varepsilon$$
3. *$\forall \varepsilon > 0$ there exist $a = x_0 < x_1 < \ldots < x_n = b$ such that if $I_j = (x_{j-1}, x_j)$ for $j \geq 1$, then:*
$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \varepsilon$$

[Lemma 4.1]

For the purposes of this lemma it's convenient to introduce some notation. For $f : \mathbb{R} \to \mathbb{R}$ a bounded function with bounded support $[a, b]$, and for $a = x_0 < \cdots < x_n = b$ we let $I_j = (x_{j-1}, x_j), m_j := \inf_{x \in I_j} f(x)$ and $M_j := \sup_{x \in I_j} f(x)$. We define the *lower step function of f with respect to* $\{x_0, \ldots, x_n\}$ as

$$\phi_*(x) = \sum_{j=1}^{n} m_j \chi_{I_j}(x) + \sum_{j=0}^{n} f(x_j) \chi_{\{x_j\}}(x)$$

and the *upper step function of f with respect to* $\{x_0, \ldots, x_n\}$ as

$$\phi^*(x) = \sum_{j=1}^{n} M_j \chi_{I_j}(x) + \sum_{j=0}^{n} f(x_j) \chi_{\{x_j\}}(x).$$

Notice that $\phi_*(x)$ and $\phi^*(x)$ are step functions, and that $\phi_*(x) \leq f \leq \phi^*(x)$.

Proof. (i) implies (ii). Suppose first that f is Riemann-integrable, let $\epsilon > 0$ and let ϕ and ψ be step functions as in the definition of Riemann-integrability, with $\int \psi - \int \phi < \epsilon$. We may assume that ϕ and ϕ are zero outside $[a, b]$. Enumerate the potential jump points of ϕ and ψ together as $a = x_0 < \cdots < x_n = b$. Let ϕ_* and ϕ^* be the lower and upper step functions of f with respect to $\{x_0, \ldots, x_n\}$. Then

$$\phi \leq \phi_* \leq f \leq \psi^* \leq \psi \quad \text{and} \quad \int \psi^* - \int \phi_* \leq \int \psi - \int \phi < \epsilon.$$

But $\int \psi^* - \int \phi_* = \sum_{j=1}^{n}(M_j - m_j)(x_j - x_{j-1}$, so (ii) holds.
(ii) implies (iii). Since for a nonempty bounded subset $A \subseteq \mathbb{R}$ we have $\sup\{|a - b| : a, b \in A\} = \sup A - \inf A$, it follows that $\sup_{x,y \in I_j} |f(x) - f(y)| = M_j - m_j$. So, assuming(ii),

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) = \int \sum_j M_j \chi_{I_j} - \int \sum_j m_j \chi_{I_j} = \int \phi^* - \phi_* < \epsilon.$$

(iii) implies (i). If there exist $a = x_0 < \cdots < x_n = b$ such that

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

holds, then the lower and upper step functions ϕ_* and ϕ^* of f with respect to $\{x_0, \ldots, x_n\}$ verify the definition of Riemann-integrability of f. ■

2.7 Theorem: Riemann Integrability, Continuity and Monotonicity

Suppose that

$$g : [a, b] \to \mathbb{R}$$

and that $g(x) = f(x)$ *for any* $x \in [a, b]$, *and* $f(x) = 0$ *elsewhere.*

1. *If* g *is continuous on* $[a, b]$, *then* f *is* ***Riemann Integrable***
2. *If* g *is a monotone function, then* f *is* ***Riemann Integrable***

The above holds even if g *is continuous on* (a, b) *and bounded. [Theorem 4.7]*

Proof: Riemann Integrability for Continuous and Monotone Functions. We first proof Theorem 1: continuity of g implies Riemann Integrability of f.

Clearly, f is bounded, by the Extreme Value Theorem, and has bounded support (by definition). In particular, since f is continuous on $[a, b]$, it must also be uniformly continuous on said interval:

> *Let I be an interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a function.*
> *We say that f is **uniformly continuous** on I if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x, y \in I$ and $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.*

Indeed, pick $\varepsilon > 0$, such that there exists a $\delta > 0$, such that for any $x, y \in [a, b]$, we have that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

We can exploit the above property by dividing the interval $[a, b]$ into subintervals of length less than δ, by considering the points:

$$a = x_0 < x_1 < \ldots < x_n = b$$

with $x_j - x_{j-1} < \delta$. This then means that if $x, y \in I_j = (x_{j-1}, x_j)$, then we must have $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Now, consider:

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j)$$

Since we know that for any $x, y \in I_j$, $|f(x) - f(y)| < \varepsilon$, it is easy to see that $\sup_{x,y \in I_j} |f(x) - f(y)| \leq \varepsilon$. So:

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) \leq \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \lambda(I_j) = \varepsilon$$

But then, from the Equivalence Theorem for Riemann Integrability (2.6), it follows that f is uniformly continuous.

We now prove part 2: if g is monotone, f is Riemann Integrable.

To do this, we again use the Equivalence Theorem (2.6). For simplicity, we can assume that f is monotone non-decreasing, so for any $x \in [a, b]$, we have $f(a) \leq f(x) \leq f(b)$. f must thus be **bounded**

If $f(a) = f(b)$, then f is constant, and it is a step function, which we know is Riemann Integrable.

Thus, consider $f(a) < f(b)$. If this is the case, we partition $[a, b]$ via

$$a = x_0 < x_1 < \ldots < x_n = b$$

which in particular means that:

$$\sup_{x,y \in (x_{j-1}, x_k)} |f(x) - f(y)| = f(x_j) - f(x_{j-1})$$

from non-decreasing monotonicity.

In particular, if we use any $\varepsilon > 0$ and define $\delta = \frac{\varepsilon}{f(b)-f(a)}$, such that $\lambda(I_j) = x_j - x_{j-1} < \delta$, we can again use the equivalence Theorem used for part 1:

$$\begin{aligned}
\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) &< \frac{\varepsilon}{f(b) - f(a)} \sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \\
&= \frac{\varepsilon}{f(b) - f(a)} \sum_{j=1}^{n} f(x_j) - f(x_{j-1}) \\
&= \frac{\varepsilon}{f(b) - f(a)} (f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n) - f(x_{n-1})) \\
&= \varepsilon
\end{aligned}$$

so f is Riemann Integrable.

$\square$

2.8 Corollary: Requirements For Both Riemann and Lebesgue Integrability

*Let $I = (a, b)$ be a **bounded** interval and suppose that there exist points*

$$a = x_0 < x_1 < \ldots < x_n = b$$

*such that a function $f : I \to \mathbb{R}$ is **bounded** and **continuous** on each subinterval (x_{j-1}, x_j), with $j = 0, 1, \ldots, n$.*
*Then, such function is both **Riemann** and **Lebesgue** integrable.*

Proof. Write f as $f_1 + f_2 + \ldots + f_n$, defining:

$$f_j = f \times \mathcal{X}_{(x_j, x_{j+1})}$$

Since each f_j is continuous and bounded on its subinterval (it is pointwise continuous), by the Theorem above, f_j must be Riemann Integrable. But then f must be Riemann Integrable, and so, Lebesgue Integrable.

$\square$

3 Exercises

1. **Show that $f(x) = 1$ is not integrable on $(1, \infty)$**

   We can easily write $f(x)$ by using characteristic functions. Indeed, $\forall n \in \mathbb{N}$, and for any $x > 1$, we have:

   $$f_n(x) = \mathcal{X}_{(n,n+1]}$$

   such that:

   $$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

   We now apply the Integrability of Series of Functions, and consider whether:

   $$\sum_{n=1}^{\infty} \int_{(1,\infty)} f_n(x)$$

   converges. But notice,

   $$\int_{(1,\infty)} \mathcal{X}_{(n,n+1]} = \lambda((n, n+1]) = 1$$

   so:

   $$\sum_{n=1}^{\infty} \int_{(1,\infty} f_n(x) = \sum_{n=1}^{\infty} 1 = \infty$$

   Since the above series diverges, it follows that $f(x) = 1$ is **not** integrable on the interval $(1, \infty)$.

2. **Let** $f(x) = x^m$ **if** $x \in [0,1)$**, and** $f(x) = 0$ **otherwise. Show that** f **is integrable, and compute** $\int x_m$

We want to express x^m as the limit of a sequence of functions (intuitively, as the interval over which we define each step functions decreases, the approximation will be better). This can be done in a natural way: we can split $[0,1)$ into subintervals, of length depending on $n \in \mathbb{N}$, such that the **left** endpoint of any interval I_J is given by$x_j = \frac{j}{n}$. We can define a sequence of functions f_n via:

$$f_n(x) = \begin{cases} (x_j)^m, & x \in I_j = [x_j, x_{j+1}), \forall j \in [0, n-1] \\ 0, & otherwise \end{cases}$$

Under this formulation, we can see that:

$$f(x) = \lim_{n \to \infty} f_n(x)$$

If x is outside $[0,1)$ this is clearly true, since $f_n(x) = 0$. Thus, consider $x \in [0,1)$. In particular, pick $x \in I_j$, it is then easy to see that:

$$(x_j)^m \leq f(x) < (x_{j+1})^m$$

We need to show that $|f(x) - x_j^m| \to 0$. To do so, we can consider the difference between the bounding terms:

$$\begin{aligned} (x_{j+1})^m - (x_j)^m &= \frac{(j+1)^m}{n^m} - \frac{j^m}{n^m} \\ &= \frac{1}{n^m}((j+1)^m - j^m) \\ &= \frac{1}{n^m}(j^m + mj^{m-1} + \mathcal{O}(m-2) - j^m) \\ &= \frac{1}{n^m}(mj^{m-1} + \mathcal{O}(m-2)) \\ &< \frac{1}{n^m}(mn^{m-1} + \mathcal{O}(n-2)) \quad \textit{Since } j \leq n-1 \\ &= \frac{1}{n}(m + \ldots)) \to 0 \end{aligned}$$

Since $(x_{j+1})^m - (x_j)^m \to 0$, and $|f(x) - (x_j)^m| \leq (x_{j+1})^m - (x_j)^m$, by Squeeze Theorem:

$$|f(x) - x_j^m| \to 0$$

as required.

Since we are dealing with a sequence of functions, we want to use the Monotone Convergence Theorem. However, for this we require that our f_n be monotone, which is not necessarily the case. We can see why diagramatically:

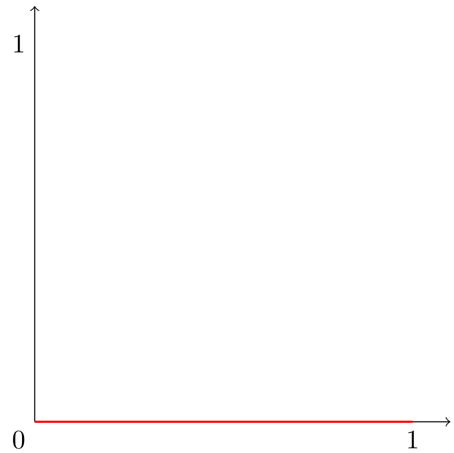


Figure 1: x^2 using $n = 1$ (1 interval)

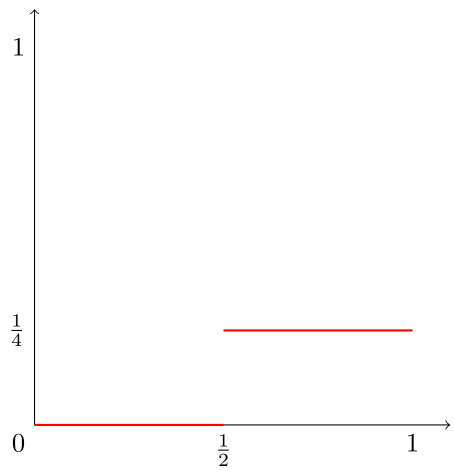


Figure 2: x^2 using $n = 2$ (2 intervals)

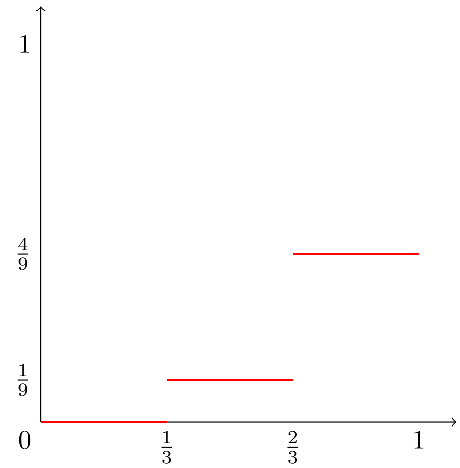


Figure 3: x^2 using $n = 3$ (3 intervals)

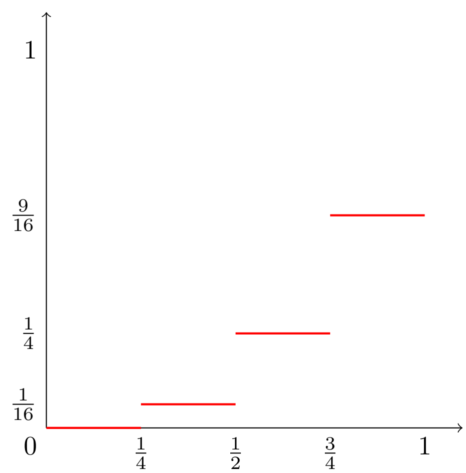


Figure 4: x^2 using $n = 4$ (4 intervals)

From the above we can see that, for example, when $n = 3$, on the interval $\left(\frac{1}{2}, \frac{2}{3}\right)$, we can see that $f_2 > f_3$. However, when we have $n = 2^k$, each subinterval of f_n is contained in the subintervals of $f_{2^{k+1}}$, which ensures that $f_{2^k} \leq f_{2^{k+1}}$.

Notice, since $f_n \to f$, and f_{2^k} is just a subsequence of f_n, we still have $f_{2^k} \to f$. Thus, the only step left required to show f is integrable on $[0, 1)$ is to show that:

$$\lim_{k\to\infty} \int_{[0,1)} f_{2^k} < \infty$$

In order to do this, we will consider our original f_n, and apply the general result to the particular integral above. Notice, we can write f_n as a finite sum:

$$f_n(x) = \sum_{j=0}^{n-1} (x_j)^m \mathcal{X}_{[x_j, x_{j+1})}$$

We can easily compute the integral of this finite sum, since characteristic functions are easily integrable:

$$\begin{aligned}\int f_n(x) &= \int \sum_{j=0}^{n-1} (x_j)^m \mathcal{X}_{[x_j, x_{j+1})} \\ &= \sum_{j=0}^{n-1} (x_j)^m \lambda([x_j, x_{j+1})) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{j}{n}\right)^m \\ &= \frac{1}{n^{m+1}} \sum_{j=0}^{n-1} j^m\end{aligned}$$

If we use induction, we get:

$$\frac{n^{m+1}}{m+1} \leq \sum_{j=0}^{n-1} j^m \leq \frac{(n+1)^{m+1}}{m+1}$$

So it follows that:

$$\frac{1}{m+1} \leq \int f_n(x) \leq \frac{1}{m+1} \left(\frac{n+1}{n}\right)^{m+1}$$

So by Squeeze Theorem,

$$\lim_{n \to \infty} \int f_n(x) = \frac{1}{m+1}$$

Hence, f is integrable, and by the Monotone Convergence Theorem, since:

$$f_{2^k} \to f$$

and f_{2^k} is a monotone, nondecreasing sequence, and:

$$\lim_{k \to infty} \int_{[0,1)} f_{2^k} = \frac{1}{m+1}$$

it follows that:

$$\int_{[0,1)} f = \frac{1}{m+1}$$

3. **Show that the Dirichlet Function, defined by:**

$$\mathcal{X}_{\mathbb{Q} \cap [0,1]}$$

is not Riemann Integrable.

We could just say that the Dirichlet Function is not Riemann Integrable by just showing that it is not continuous.

We can show this from the definition of Riemann Integrability. That is, assume that we can find 2 step functions ϕ, ψ such that, letting $F = \mathbb{Q} \cap [0,1]$:

$$\phi \leq \mathcal{X}_F \leq \psi \implies \int \psi - \int \phi < \varepsilon$$

for any $\varepsilon > 0$. Since the function has bounded support, we can assume that ϕ, ψ are 0 outside $[0, 1]$.

ϕ and ψ must be step function with respect to a set $\{x_0, x_1, \ldots, x_n\}$. ϕ can be defined by the values of c_j, and ψ defined by the values of d_j when $x \in (x_{j-1}, x_j$. But now, notice that on any open interval, there is always at least one rational and irrational number. Let y be a rational and z be an irrational on said interval.

Notice, $\mathcal{X}_F(y) = 1$, so it follows that $\psi(y) \geq 1$, so we must have that for any x, $d_j \geq 1$. Similarly, $\mathcal{X}_F(z) = 0$, so it follows that $\phi(z) \leq 0$, we we must have that for any x, $c_j \leq 0$.

Now, consider:

$$\begin{aligned}
\int \psi - \int \phi &= \sum_j d_j(x_j - x_{j-1}) - \sum_j c_j(x_j - x_{j-1}) \\
&\geq \sum_j (x_j - x_{j-1} - 0) \\
&= x_n - x_0 \\
&= 1
\end{aligned}$$

But this contradicts our initial claim that for any $\varepsilon > 0$, we must have:

$$\int \psi - \int \phi < \varepsilon[$$

so the Dirichlet Function can't be Riemann Integrable.

However, it is Lebesgue Integrable, since F is a countable set, and so we know that:

$$\int \mathcal{X}_F = 0$$

4. **We have seen that the Dirichlet Function is not Riemann Integrable. Is it true that if E is a countably infinite, bounded subset of the reals, $\mathcal{X}_E$ is not Riemann Integrable?**

We can show there is a counterexample. Let $E = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right)\}$. E is clearly countable (we can make a bijection to the natural numbers), and bounded (since $\forall n \in \mathbb{N}, 0 < \frac{1}{n} \leq 1$.

We can gain intuition by looking at how $\mathcal{X}_E$ looks like. It is 1 whenever $x = \frac{1}{n}$, and 0 otherwise.

Define 2 step functions:

$$\phi(x) = 0$$

$$\psi(x) = \begin{cases} 0, & x > 1, x < 0 \\ 0, & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n = 1, 2, \ldots, k-1 \\ 1, & x = \frac{1}{n}, n = 1, 2, \ldots, k \\ 1, & x \in \left[0, \frac{1}{k}\right) \end{cases}$$

where k is some positive integer. The above definition ensures that ψ is defined everywhere. We use the interval $\left[0, \frac{1}{k}\right)$ because naturally, using reciprocals we will never reach 0, so this interval solves said issue.

The above definition ensures that for any x:

$$\phi(x) \leq f(x) \leq \psi(x)$$

Consider the integrals of these step functions:

$$\int \phi = 0$$

$$\int \psi = 1\lambda\left(\left[0, \frac{1}{k}\right)\right) + \sum_{n=1}^{k} 1\lambda\left(\left[\frac{1}{n}\right]\right) + 0 + 0 = \frac{1}{k}$$

But then it follows that:

$$\int \psi - \int \phi = \frac{1}{k}$$

and since k is arbitrary, this difference can be made arbitrarily small, so for any $\varepsilon > 0$:

$$\int \psi - \int \phi < \varepsilon$$

Thus, $\mathcal{X}_E$ is Riemann Integrable.

4 Workshop

1. **Let $f(x) = [x]$ for all $x \in \mathbb{R}$. Calculate the integrals:**

$$\int_{(0,5)} f \qquad \int_{\left(-\frac{7}{3}, \frac{12}{5}\right]} f$$

For the first one, on $(0, 5)$ we can write f as a step function. If:

$$I_j = [j, j + 1)$$

then:

$$f(x) = \sum_{i=0}^{4} i\mathcal{X}_{I_j}(x)$$

Since f is a step function, it is integrable over all $\mathbb{R}$, and:

$$\int_{(0,5)} f = \sum_{i=0}^{4} i\lambda(I_j) = \sum_{i=0}^{4} i = 10$$

For the second one, we can also write f as a step function on $\left(-\frac{7}{3}, \frac{12}{5}\right]$. We first note that:

$$-\frac{7}{3} = -2.\dot{3} \qquad \frac{12}{5} = 2.4$$

If:

$$I_j = \begin{cases} \left(-\frac{7}{3}, -2\right), & j = -3 \\ \left[2, \frac{12}{5}\right], & j = 2 \\ [j, j+1), & otherwise \end{cases}$$

then:

$$f(x) = \sum_{j=-3}^{2} i\mathcal{X}_{I_j}(x)$$

f as a step function is integrable, and:

$$\int_{\left(-\frac{7}{3}, \frac{12}{5}\right]} f = -3\frac{1}{3} - 2 - 1 + 0 + 1 + 2\frac{2}{5} = -2.2$$

2. **Show that if $n \in \mathbb{Z}$ and $f(x) = [nx]^2$ for all $x \in \mathbb{R}$, then:**

$$\int_{(0,1)} f = \frac{1}{n}\sum_{j=1}^{n-1} j^2 = \frac{1}{6}(n-1)(2n-1)$$

We build some intuition. Notice, we integrate over $(0,1)$ so we restrict $x \in (0,1)$:

- $n = 1 \implies f(x) = [x]^2 = 0$
- $$n = 2 \implies f(x) = \begin{cases} 1, & \frac{1}{2} \leq x < 1 \\ 0, & 0 < x < \frac{1}{2} \end{cases}$$
- $$n = 3 \implies f(x) = \begin{cases} 2^2, & \frac{2}{3} \leq x < 1 \\ 1, & \frac{1}{3} \leq x < \frac{2}{3} \\ 0, & 0 < x < \frac{1}{3} \end{cases}$$

Thus, $\forall x \in (0,1)$, if $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right)$ we can write:

$$f(x) = \sum_{j=1}^{n-1} j^2 \mathcal{X}_{I_j}(x)$$

Thus, f is a step function, and so integrable (Corollary of Theorem 4.1):

$$\int_{(0,1)} f = \sum_{j=1}^{n-1} j^2 \lambda(I_j) = \sum_{j=1}^{n-1} j^2 \left(\frac{j}{n} - \frac{j-1}{n}\right) = \frac{1}{n}\sum_{j=1}^{n-1} j^2$$

3. **Let $f(x) = \frac{1}{[x]^2}$ for all $x \geq 1$. Show that f is integrable on the interval $[1, \infty)$ and:**

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

Define $I_j = [j, j+1]$. Then, $\forall x \geq 1$, we can write f as a step function:

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} \mathcal{X}_{I_j}(x)$$

By Definition 4.3, this is integrable, since:

$$\sum_{j=1}^{\infty} \left|\frac{1}{j^2}\right| \lambda(I_j) = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

by the p-series test.

Moreover, it thus follows that:

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

as required.

4. **Let:**

$$f(x) = \begin{cases} 1, & x \notin \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$$

Prove that f is integrable on every bounded interval I and:

$$\int_I f = \lambda(I)$$

> *For this one I required the solutions. As expected, the key is to use the countability of the rationals, and to define f in terms of step functions which include these rationals.*

Since $\mathbb{Q}$ is countable:

$$\mathbb{Q} = \{q_1, q_2, \ldots\}$$

where each q_i is a non-repeating, irreducible rational number.

Then, $\forall x \in I$ we can write:

$$f(x) = \mathcal{X}_I(x) - \sum_{j=1}^{\infty} \mathcal{X}_{[q_j, q_j]}(x)$$

Now, notice that since I is bounded, $\lambda(I)$ is finite. Moreover $[q_j, q_j]$ is a degenerate interval. Thus:

$$\lambda(I) + \sum_{j=1}^{\infty} |-1|\lambda([q_j, q_j]) = \lambda(I) + 0 = \lambda(I) < \infty$$

Hence, f is integrable on I and:

$$\int_I f = \lambda(I)$$

as required.

5. **For the purposes of this exercise, we will take for granted that continuous functions on closed bounded intervals are integrable. Let $f : [a, b] \to \mathbb{R}$ be continuous, and let:**

$$M = \sup_{x \in [a,b]} |f(x)|$$

Suppose that $M > 0$ and let $p > 0$.

(a) **Prove that $\forall \varepsilon > 0$ with $0 < \varepsilon < \frac{M}{2}$ there is a non-empty open interval $I \subseteq [a, b]$ such that:**

$$(M - \varepsilon)^p \lambda(I) \leq \int_a^b |f(x)|^p \, dx \leq M^p(b - a)$$

> *This is the type of question where definitions are critical. Need to ensure that every theorem is mentioned, as otherwise a lot of marks can be lost.*

Since f is continuous, then $|f|$ is continuous, and $[a, b]$ is a closed bounded interval, then $\exists c \in [a, b]$ such that, by the Extreme Value Theorem:

$$|f(c)| = M$$

Here, the solutions describe f as a step function, and I think this is more formal - I just used standard integration.

Now, $\forall x \in [a, b]$, we will have:

$$|f(x)| \leq M\mathcal{X}_{[a,b]} \implies |f(x)|^p \leq M^p\mathcal{X}_{[a,b]}$$

since taking powers is monotonic for positive numbers. Then, by Properties of the Lebesgue Integral (Theorem 4.2, Part b) it follows that integration over $[a, b]$ preserves the inequality:

$$\int_a^b |f(x)|^p \ dx \leq \int_a^b M^p\mathcal{X}_{[a,b]} \ dx = M^p(b-a)$$

($M^p\mathcal{X}_{[a,b]}$ is integrable since it is a step function, and $|f(x)|^p$ is integrable because it is continuous) Furthermore, since $|f|$ is continuous, it is continuous at c, so $\forall\varepsilon > 0$ we can always find $\delta > 0$ such that:

$$|x - c| < \delta \implies ||f(x)| - |f(c)|| = ||f(x)| - M| < \varepsilon$$

which in particular implies that for $x \in I = (c - \delta, c + \delta) \cap [a, b]$:

$$|f(x)| \geq M - \varepsilon$$

Again, by monotonicity of powers, we have that $\forall x \in I$:

$$|f(x)|^p \geq (M - \varepsilon)^p\mathcal{X}_I$$

So integrating via Theorem 4.2, part b and Theorem 4.8, part c:

$$\int_a^b |f(x)|^p \ dx \geq \int_I |f(x)|^p \ dx \geq \int_I (M - \varepsilon)^p\mathcal{X}_I \ dx = (M - \varepsilon)^p|I|$$

Hence, we get that:

$$(M - \varepsilon)^p\lambda(I) \leq \int_a^b |f(x)|^p \ dx \leq M^p(b-a)$$

(b) **Deduce that:**

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p \ dx\right)^{\frac{1}{p}} = M$$

Since

$$(M - \varepsilon)^p\lambda(I) \leq \int_a^b |f(x)|^p \ dx \leq M^p(b-a)$$

and all these elements are positive, taking powers preserves the inequality, so:

$$((M - \varepsilon)^p\lambda(I))^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p \ dx\right)^{\frac{1}{p}} \leq (M^p(b-a))^{\frac{1}{p}}$$

which implies that:

$$(M - \varepsilon)(\lambda(I))^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p \ dx\right)^{\frac{1}{p}} \leq M(b-a)^{\frac{1}{p}}$$

We know that:

$$\lim_{p\to\infty} \frac{1}{p} = 0$$

so by continuity of powers, and using the fact $\lambda(I), b-a > 0$:

$$\lim_{p\to\infty} M(b-a)^{\frac{1}{p}} = M(b-a)^{\lim_{p\to\infty}\frac{1}{p}} = M(b-a)^0 = M$$

$$\lim_{p\to\infty} (M-\varepsilon)(\lambda(I))^{\frac{1}{p}} = (M-\varepsilon)(\lambda(I))^{\lim_{p\to\infty}\frac{1}{p}} = (M-\varepsilon)(\lambda(I))^0 = M-\varepsilon$$

Hence, since limits preserve inequalities:

$$M-\varepsilon \leq \lim_{p\to\infty} \left(\int_a^b |f(x)|^p \ dx \right)^{\frac{1}{p}} \leq M$$

so by Squeeze Theorem, it follows that:

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p \ dx \right)^{\frac{1}{p}} = M$$

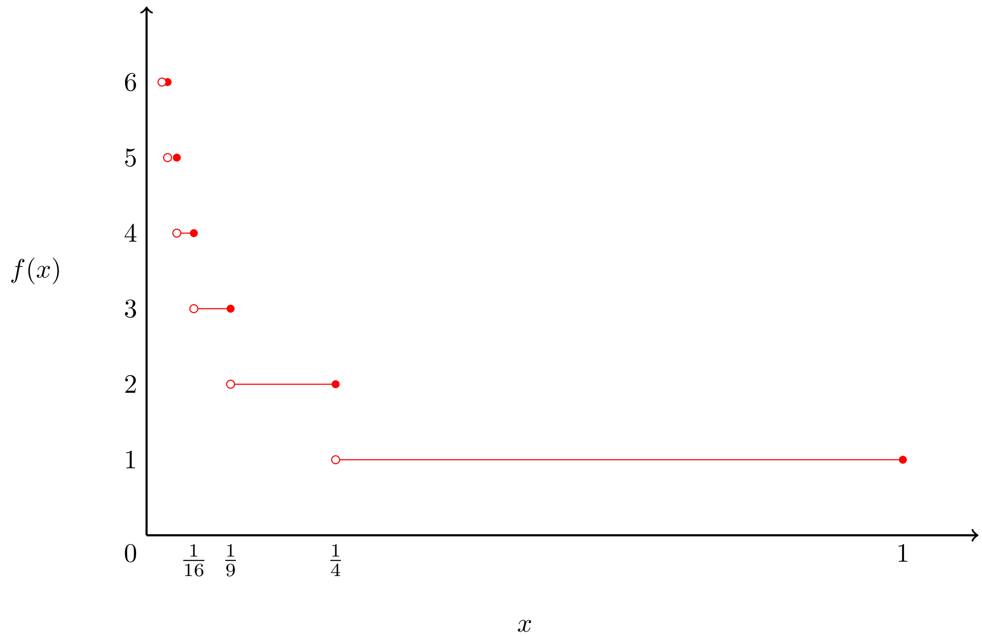
6. **Let $f(x) = n \ \forall x \in \left(\frac{1}{(n+1)^2}, \frac{1}{n^2}\right], n \in \mathbb{N}$. Sketch the graph of f. Prove that f is integrable on $(0,1]$ and show that:**

$$\int_{(0,1]} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

I pursued the most "straightforward" solution, which lead to a much longer solution. The solutions notice a key step, which allows a much simpler derivation. I include both for completeness.

If we consider some terms:

- if $n = 1$, we have that if $x \in \left(\frac{1}{4}, 1\right]$, then $f(x) = 1$
- if $n = 2$, we have that if $x \in \left(\frac{1}{9}, \frac{1}{4}\right]$, then $f(x) = 2$
- if $n = 3$, we have that if $x \in \left(\frac{1}{16}, \frac{1}{9}\right]$, then $f(x) = 3$



From the solutions:

We can write:

$$f(x) = \sum_{j=1}^{\infty} \mathcal{X}_{(0,j^{-2}]}(x), \qquad \forall x \in (0,1]$$

Now, since:

$$\sum_{j=1}^{\infty} |1|\lambda((0,j^{-2}]) = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

(by p-series test) then f is integrable on $(0,1]$ and:

$$\int_{(0,1]} f = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

Self:

If we define:

$$I_j = \left(\frac{1}{(j+1)^2}, \frac{1}{j^2} \right]$$

then we find that $\forall x \in (0,1]$:

$$f(x) = \sum_{j=1}^{\infty} j \mathcal{X}_{I_j}(x)$$

Thus, for f to be Lebesgue Integrable, we require that:

$$\sum_{j=1}^{\infty} |j|\lambda(I_j) = \sum_{j=1}^{\infty} j \left(\frac{1}{j^2} - \frac{1}{(j+1)^2} \right)$$

converges.

Using the above summation:

$$\begin{aligned}\sum_{j=1}^{\infty} j\left(\frac{1}{j^2}-\frac{1}{(j+1)^2}\right) &= \sum_{j=1}^{\infty} j\left(\frac{(j+1)^2-j^2}{j^2(j+1)^2}\right)\\ &= \sum_{j=1}^{\infty} \frac{j^2+2j+1-j^2}{j(j+1)^2}\\ &= \sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)^2}\end{aligned}$$

We now consider partial fraction decomposition. Let $A, B, C \in \mathbb{R}$, and assume that:

$$\frac{2j+1}{j(j+1)^2} = \frac{A}{j} + \frac{B}{j+1} + \frac{C}{(j+1)^2}$$

We solve for A, B, C:

$$\begin{aligned} & \frac{2j+1}{j(j+1)^2} = \frac{A}{j} + \frac{B}{j+1} + \frac{C}{(j+1)^2}\\ \implies\quad & \frac{2j+1}{j(j+1)^2} = \frac{A(j+1)^2 + Bj(j+1) + Cj}{j(j+1)^2}\\ \implies\quad & \frac{2j+1}{j(j+1)^2} = \frac{Aj^2 + 2Aj + A + Bj^2 + Bj + Cj}{j(j+1)^2}\end{aligned}$$

Which leads to the following equations:

$$\begin{gathered} A = 1\\ j^2(A+B) = 0 \implies B = -1\\ j(2A+B+C) = 2j \implies C = 1\end{gathered}$$

Thus, it follows that:

$$\sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)^2} = \sum_{j=1}^{\infty} \left(\frac{1}{(j+1)^2} + \frac{1}{j} - \frac{1}{j+1}\right)$$

The form above indicates that, if we consider partial sums, some terms might cancel, simplifying the expression. Hence, consider:

$$\begin{aligned} & \sum_{j=1}^{n} \left(\frac{1}{(j+1)^2} + \frac{1}{j} - \frac{1}{j+1}\right)\\ = \quad & \left(\frac{1}{2^2} + 1 - \frac{1}{2}\right) + \left(\frac{1}{3^2} + \frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n^2} + \frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{(n+1)^2} + \frac{1}{n} - \frac{1}{n+1}\right)\\ = \quad & \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right) - \frac{1}{n+1}\\ = \quad & \left(\sum_{j=1}^{n+1} \frac{1}{j^2}\right) - \frac{1}{n+1}\end{aligned}$$

But then, since an infinite series is the limit as $n \to \infty$ of its partial sums, it must be the case that:

$$\sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)^2} = \lim_{n\to\infty} \left[\left(\sum_{j=1}^{n+1} \frac{1}{j^2}\right) - \frac{1}{n+1}\right] = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^2}$ is a p-series with $p = 2$, it follows by the p-series test that it is a convergent series.

Overall, by the definition of Lebesgue Integrability, since:

$$\sum_{j=1}^{\infty} |j| \lambda(I_j) = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

and for any $x \in (0, 1]$ we have:

$$f(x) = \sum_{j=1}^{\infty} j \mathcal{X}_{I_j}(x)$$

it follows that f is Lebesgue Integrable, and:

$$\int_{(0,1]} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

as required.