LEARNING SEMINAR ON MANIFOLDS AND HOMOTOPY THEORY

ABSTRACT. As the goal of life of homotopy theorists is to compute the homotopy groups of spheres, geometric topologists on the other hand dream to compute the cohomology groups of the moduli of surfaces $\mathcal{M}_g$. There are tons of other families of groups (or spaces) whose homology is still unknown. Fortunately, the homology of some of these stabilize in a range: this phenomena is called homological stability. In particular, the groups $H^*(\mathcal{M}_g)$ are independent of g for degrees smaller than an affine function of g.

We plan to cover the cellular $\mathbb{E}_k$ algebra method developed by Galatius-Kupers-Randal-Williams to prove homological stability as well as the necessary homotopy-theoretic techniques. As an application, we will discuss the cohomology of general linear groups of fields. From this point, we could look into more surprising interactions between algebraic K-theory and Geometric Topology. Depending on the participants' interests, we may also discuss applications to arithmetic statistics or representation theory.

FIGURE 1. *What freedom!, 1903, Russian Museum, Saint Petersburg. The painting represents Geometric Topologists and Homotopy Theorists walking together, holding hands.*

1. Topic Ideas

We have roughly 14 talks.
This is a non-exhaustive list of topics we could cover:

1. Introduction: kinda announce main results and characters we want to discuss in this seminar
2. What is M_g? From AG to AT
3. Stable homology of moduli spaces of surfaces
4. Proof of Dundas-McCarthy Theorem and Goodwillie Calculus

(5) Homological Stability and algebraic K-theory
(6) Power Operations on E_k-algebras
(7) Stable h-cobordisms and K-theory
(8) Dwyer-Weiss-Williams stuff (always wanted to read this, something with K theory and manifolds)
(9) Some spectral sequences
(10) (G)-stuff: unstable equivariant homotopy, genuine, naive
(11) Rognes filtration of K-theory, as in: https://arxiv.org/pdf/2512.19128 and Rogne's connectivity conjecture
(12) Koszul's duality, bar-cobar constructions
(13) Unstable K-theory, reductive Borel-Serre compactifications like in Jansen and Clausen work, eg see [Jan24]

2. Introduction

Speaker: Azélie Picot

Abtsract: In this introductory talk, I will define homological stability and discuss examples. In particular, I will discuss the case of the moduli space of surfaces $\mathcal{M}_g$. Finally, I will present the possible directions the seminar can take and we will choose depending on the participants' interests.

3. A detailed example

We would see the proof of homological stability for one classic family of groups. We would also discuss the stable homology of this family. Possible choices are: mapping class groups, symmetric groups, automorphisms of free groups,...

4. A general method to prove homological stability

4.1. The splitting complexes. Speaker:

Abstract: Search for a general method to prove homological stability, or rather set of conditions so that a family of groups satisfy homological stability. Overview of Randal-Williams-Wahl method?

4.2. What are $\mathbb{E}_k$-algebras? Speaker:

Abstract: Define operads, little k-disks operads, algebras over these operads, discuss commutativity and examples. If time permits, define cellular $\mathbb{E}_k$-algebras and give examples of $\mathbb{E}_2$-algebras.

References:

4.3. Power Operations and free $\mathbb{E}_k$-algebras. Speaker:

Abstract: the multiplication on on an $\mathbb{E}_k$-algebra X endows its homology $H_*(X;E)$ with some algebraic structure. Discuss power operations, Dyer-Lashof operations and Browder bracket, as well as the connection to the homology of free $\mathbb{E}_k$-algebras. A theorem of Cohen shows that for a free $\mathbb{E}_k$-algebra, the homology is also a type of "free" algebra. This allows us to deduce homological stability for braid groups and symmetric groups.

4.4. Derived indecomposables.

4.5. **Secondary Stability and other phenomena. Speaker**: Azélie **Abstract**: So far, we've looked at the range in which $H_*(G_n, G_{n-1})$ vanishes. However, we could ask whether the relative homology groups $H_*(G_n, G_{n-1})$ also stabilize. This phenomena is called secondary stability and we will discuss the case of mapping class groups, as in [GKR19]. If time permits, we will discuss higher order stability as in [Ran25].

4.6. **Koszul Duality. Speaker**: **Abstract**:

5. Examples in Number Theory or Representation Theory

Possible topics include:

(1) Homological stability for Hurwitz spaces and applications to arithmetic statistics: [LL25]
(2) Homological stability for classical sequences of groups with coefficients being representations of arithmetic groups and applications to arithmetic statistics, see [Mil+25].

6. General linear groups and Algebraic K-theory

6.1. **Homological stability for general linear groups of fields. Speaker**:
Abstract: See [GKR25] and [GKR25]. Stable homology and algebraic K-theory.

6.2. **Filtrations on Algebraic K-theory. Speaker**: **Abstract**: See [Rog21], [CKZ25], [MPW25].

6.3.

7. More on Algebraic K-theory

7.1. **Another type of stability result: the parametrized h-cobordism theorem.**

7.2. **Dwyer-Weiss Williams paper.**

7.3. **The Dundas-Goodwillie-McCarthy Theorem and its proof.** Algebraic K-theory is notoriously hard to compute, but we have better methods for computing THH. THH is a sort of linearization of K theory (is this right?) and one can rationally get a cartesian square relating these for nilpotent ring extensions which can then be used to compute K theory. A similar result at a prime exists when one instead uses TC. A modern proof can be found in [Ras18].

References

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