∞-Category seminar – Straightening and unstraightening

Rafah Hajjar, March 4, 2025

These notes are based on Markus Land's book *Introduction to Infinity-Categories*, section 3.3.

Last week we defined the notion of a cartesian and cocartesian fibrations of ∞-categories. We will focus on cocartesian fibrations for the purposes of this talk, but keep in mind that all statements can be dualized to cartesian fibrations. We briefly recall the notion:

Definition. A morphism of ∞-categories $p : \mathcal{E} \to \mathcal{C}$ is a *cocartesian fibration* if every lifting problem

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed, "f"] & \mathcal{C} \end{tikzcd}$$

has a solution $f : \Delta^1 \to \mathcal{E}$ which is *p-cocartesian*, meaning that the lifting problems

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

can be solved for all $n \geq 2$.

Remark. Note that left fibrations are cocartesian, and in fact are those in which every edge of $\mathcal{E}$ is p-cocartesian.

The goal of this talk is to show that there is an equivalence of ∞-cateogries

$$\mathrm{CoCar}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$$

this is, giving a cocartesian fibration over $\mathcal{C}$ corresponds to giving a functor from $\mathcal{C}$ to the ∞-category of ∞-categories. (This equivalence translates *existence data* into *coherence data.*)

Definition. Given a cocartesian fibration $p : \mathcal{E} \to \mathcal{C}$ and an object $x \in \mathcal{C}$, we denote by $\mathcal{E}_x = \mathcal{E} \times_{\mathcal{C}} \{x\} \in \mathbf{Cat}_\infty$ and call it the *fiber* of x in $\mathcal{E}$

Proposition 1. There is a functorial association $f \to f_!$ that sends a 1-morphism $f : \Delta_1 \to \mathcal{E}$ given by $f : x \to y$ on objects $x, y \in E$ to a map of ∞-categories $f_! : \mathcal{E}_x \to \mathcal{E}_y$. (Functorial in the sense that given $h \simeq gf$, we have $h_! \simeq g_! f_!$)

Construction and sketch of proof. We start with the following lemma:

Lemma. Denote by $\mathrm{Fun}^{cc}_\sigma(K, \mathcal{E})$ the simplicial set of K-simplices of $\mathcal{E}$ over a fixed $\sigma : K \to \mathcal{C}$ whose edges are all p-cocartesian. If $i : L \to K$ is left anodyne, then

$$\mathrm{Fun}^{cc}_\sigma(K, \mathcal{E}) \to \mathrm{Fun}^{cc}_{\sigma i}(L, \mathcal{E})$$

The simplicial set $\mathrm{Fun}^{cc}_f(\Delta^1, \mathcal{E})$ consists of all cocartesian edges of $\mathcal{E}$ over f. The canonical map $\mathrm{Fun}^{cc}_f(\Delta^1, \mathcal{E}) \to \mathcal{E}_x$ given by restriction to the source is a trivial fibration (this is the lemma applied to the left anodyne map Δ), so we can choose a section to produce the composite

$$f_! : \mathcal{E}_x \to \mathrm{Fun}^{cc}_f(\Delta^1, \mathcal{E}) \to \mathcal{E}_y,$$

where the second map is given by restriction to the target. Given a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$, $\sigma = (g, h, f)$, functoriality follows from the lemma applied to the restrictions

$$\mathrm{Fun}^{cc}_\sigma(\Delta^2, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}^{cc}_f(\Delta^{\{0,1\}}, \mathcal{E}), \quad \mathrm{Fun}^{cc}_\sigma(\Delta^2, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}^{cc}_h(\Delta^{\{0,2\}}, \mathcal{E})$$

□

Let $\Delta^{\mathrm{op}}_{/\mathcal{C}}$ be the category of simplicial objects of $\mathcal{C}$, and denote by $W_\mathcal{C}$ the set of morphisms that fix the vertex 0.

Proposition 2. To a cocartesian fibration $p : \mathcal{E} \to \mathcal{C}$ one can associate a functor $\Theta_p : \Delta^{\mathrm{op}}_{/\mathcal{C}} \to$ **sSet**, which sends any morphism in $W_\mathcal{C}$ to a Joyal equivalence. In particular, this localizes to a functor:

$$N(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_\mathcal{C}^{-1}] \to \mathbf{sSet}[\mathrm{Joy}^{-1}] \xrightarrow{\sim} \mathbf{Cat}_\infty$$

Proof. The functor Θ_p is defined by sending a simplicial object $\sigma : \Delta^n \to \mathcal{C}$ to the simplicial set $\mathrm{Fun}^{cc}_\sigma(\Delta^n, \mathcal{C})$. A morphism in $\Delta^{\mathrm{op}}_{/\mathcal{C}}$ is just a composite $\Delta^n \xrightarrow{f} \Delta^m \to \mathcal{C}$, and by definition $f \in W_\mathcal{C}$ iff f maps the vertex $0 \in \Delta^n$ to $0 \in \Delta^m$. For such an f, the diagram

$$\begin{tikzcd}
\mathrm{Fun}^{cc}_\sigma(\Delta^m, \mathcal{E}) \arrow[rr, "\Theta_p(f)"] \arrow[dr, "\sim"'] & & \mathrm{Fun}^{cc}_{\sigma f}(\Delta^n, \mathcal{E}) \arrow[dl, "\sim"] \\
& \mathrm{Fun}^{cc}_x(\Delta^0, \mathcal{E}) &
\end{tikzcd}$$

commutes, and both diagonal maps are trivial fibrations again by the lemma, so $\Theta_p(f)$ is a Joyal equivalence. □

Lemma. There is a canonical map $N(\Delta^{\mathrm{op}}_{/\mathcal{C}}) \to \mathcal{C}$ which sends $W_\mathcal{C}$ to degenerate edges. The induced functor of localizations

$$N(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_\mathcal{C}^{-1}] \xrightarrow{\sim} \mathcal{C}$$

is a Joyal equivalence.

Idea of proof. The map is defined by sending a k-simplex of $N(\Delta^{\mathrm{op}}_{/\mathcal{C}})$, which is given by a diagram:

$$\Delta^{n_0} \xrightarrow{\alpha_1} \Delta^{n_1} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} \Delta^{n_k} \to \mathcal{C}$$

to the k-simplex $\Delta^k \xrightarrow{\alpha} \Delta^{n_k} \to \mathcal{C}$, where the map α sends the vertex $i \in [k]$ to $\alpha(i) := \alpha_k \circ \cdots \circ \alpha_{k-i+1}(0)$. By definition, all morphisms of $W_\mathcal{C}$ are sent to a degenerate edge of $\mathcal{C}$, which are equivalences. The theorem is proven by showing that the composition

$$\mathcal{C} \to N(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_\mathcal{C}^{-1}] \to \mathcal{C}[\mathrm{deg}^{-1}] \simeq \mathcal{C}$$

preserves colimits and monomorphisms □

Theorem 1. By inverting the above equivalence we get a functor

$$\mathcal{C} \xleftarrow{\sim} N(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_{\mathcal{C}}^{-1}] \to \mathbf{Cat}_\infty$$

called the *straightening* of the cocartesian fibration $p : \mathcal{E} \to \mathcal{C}$. This construction extends to an equivalence

$$\mathrm{CoCar}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$$

which sends a cocartesian fibration $p : \mathcal{E} \to \mathcal{C}$ to its corresponding straightening functor.

There is a universal CoCartesian fibration $(\mathbf{Cat}_\infty)_{*//} \to \mathbf{Cat}_\infty$ corresponding to the identity functor $id \in \mathrm{Fun}(\mathbf{Cat}_\infty, \mathbf{Cat}_\infty)$. There is a way to define the pullback of a cocartesian fibration $\mathcal{E} \to \mathcal{D}$ along a map $F : \mathcal{C} \to \mathcal{D}$, which allows to express the inverse of the above equivalence (i.e. the *unstraightening*) as taking the pullback of $(\mathbf{Cat}_\infty)_{*//} \to \mathbf{Cat}_\infty$ along $F : \mathcal{C} \to \mathbf{Cat}_\infty$:

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathbf{Cat}_\infty)_{*//} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathbf{Cat}_\infty
\end{tikzcd}$$

The functor $\mathrm{Fun}(\mathcal{C} \to \mathbf{Spc}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$ is fully faithful ($\mathbf{Spc} \to \mathbf{Cat}_\infty$ is and $\mathrm{Fun}(K, -)$ preserves fully faithfulness).We have:

Theorem 2. The straightening-unstraightening equivalence restricts to an equivalence

$$\mathrm{LFib}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Spc})$$

Proof. Left fibrations are cocartesian fibrations where all fibres are ∞-groupoids. □

Theorem 3. For an ∞-groupoid X, $\mathrm{LFib}(X) \simeq \mathbf{Spc}_{/X}$, so the previous equivalence reads as

$$\mathbf{Spc}_{/X} \to \mathrm{Fun}(X, \mathbf{Spc})$$

Proof. Since X is an ∞-groupoid, any left fibration $\mathcal{E} \to X$ is a Kan fibration, and hence $\mathcal{E}$ is a Kan complex, so we have a natural map

$$\mathrm{LFib}(X) \to \mathbf{Spc}_{/X}$$

One can show that this is fully faithful and essentially surjective, hence an equivalence □