Manifolds and the Classification of Surfaces

Carlos Andrés Alvarado Álvarez

1 Manifolds

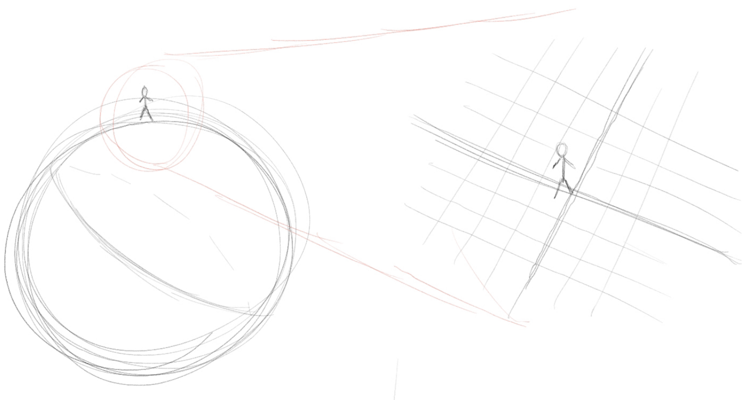

Figure 1: Person on the surface of the Earth

A manifold is something that locally looks like Euclidean space $\mathbb{R}^n$. This is similar to how the surface of the Earth looks flat to us (like $\mathbb{R}^2$) even if globally it looks like the 2-sphere S^2! This is because we are small compared to the surface of the Earth.

Slogan. A **manifold** is something that locally looks like $\mathbb{R}^n$.

Definition 1.1 (Manifold)**.** A **manifold** is a topological space M such that:

(1) For all $x \in M$, there exists a neighborhood $x \in U \overset{\text{open}}{\subset} M$ such that it is homeomorphic to $\mathbb{R}^n$.

(2) M is a Hausdorff space.

(3) M is second countable.

$\diamond$

Example (Sphere)**.** We define the n-sphere as follows:

$$S^n := \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + ... + x_{n+1}^2 = 1\}$$

where this inherits the subspace topology. This space is immediately Hausdorff and second countable, but we still need to show it is locally Euclidean. To do this, we use the stereographic projection.

Let $N = (0, ..., 0, 1) \in S^n$ be the North Pole and consider the plane $P = (x_1, ..., x_n, 0)$. For any point $x \in S^n \setminus \{N\}$, the unique line through N and x intersects P at exactly one point; similarly, the unique line between N and any $p \in P$ intersects S^n at exactly one point. One can check that this is in fact a homeomorphism $S^n \setminus \{N\} \cong P \cong \mathbb{R}^n$ and we are done (note that the choice of N was arbitrary.)

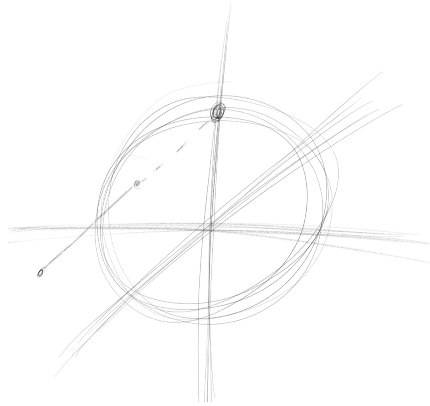

Figure 2: Stereographic Projection

In the last example we defined the sphere as something *living* in Euclidean space, not just locally Euclidean. However, in our definition we never said that this has to live in Euclidean space. In a sense, this definition of sphere makes it come with a choice of embedding into $\mathbb{R}^n$, but there are many different embeddings of a manifold into $\mathbb{R}^n$! For an example of the 2-sphere embedded in a funny way, look at the first page of Rolfsen [Rol03]! One way to rephrase how we study knots is to say that we are studying embeddings of the circle up to isotopy.

Example (Sphere again)**.** Another way to define the sphere is as follows:

Let $D^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 \leq 1\}$ be the closed n-disc. Then we have

$$S^n := D^n / \sim \text{ where } x \sim y \iff x, y \in S^{n-1}$$

is also the n-sphere.

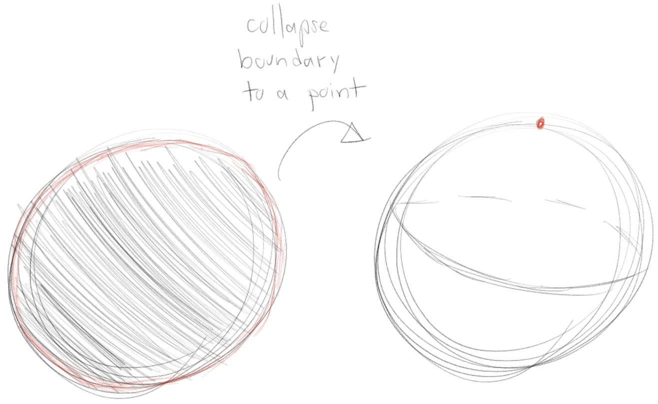

Figure 3: One point compactification

Before going further, let us generalize our definition of a manifold to allow more spaces. Recall our definition of D^n from the previous example. It is natural to ask: Is this a manifold? With our current definition, it almost is! For any point of the interior (norm strictly less than 1) there is always a locally Euclidean neighborhood. However, for points on the sphere (norm equal to 1) we can't find such a neighborhood.

One way to fix this that someone might suggest would be to ignore these edge cases and only look at those with norm less than 1. This is a perfectly fine way, to fix this but if we'd like to keep compactness (which I do want to keep), there is a natural way to generalize our definition to allow for this.

Definition 1.2 (Manifold with boundary)**.** Define $H^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. Then a **manifold with boundary** is the same as the definition of a manifold 1.1 but with open subsets of H^n instead of $\mathbb{R}^n$. ◇

Remark 1.3. We say the **boundary** are the points in the manifold that correspond to $(0, x_2, ..., x_n)$ under the homeomorphisms to H^n and we denote these ∂M. We call a boundaryless manifold a **closed** manifold.[1] ○

With our new definition, now we can have our usual definition of manifold as the special case where the boundary is empty! With this definition, we can realize D^n as a manifold with boundary $\partial D^n = S^{n-1}$. Let's give another example:

Example (Cylinder)**.** We can now realize the cylinder as a manifold with boundary.

$$C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z^2 \leq 1\}$$

The manifold of this boundary is the disjoint union of two circles $\partial C = S^1 \sqcup S^1$, one for $z = 1$ and another for $z = -1$.

We now introduce one last notion which is that of orientation. We won't properly define this, but morally speaking, it means a manifold has an inside and an outisde.

[1] Closed means compact and boundaryless, but we are taking all our manifolds to be compact.

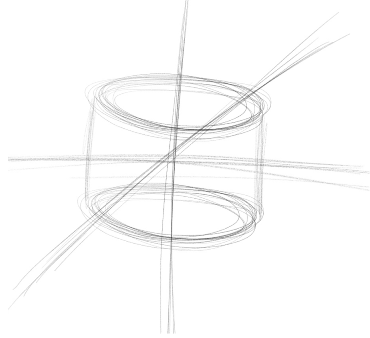

Figure 4: Cylinder

Slogan. A manifold is orientable if it has an inside and an outside.

For the rest of this talk we will only care about orientable manifolds, but let us quickly define an example of a manifold that is unorientable,

Example (Möbius Band)**.** Let $I^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i \in [0, 1] \text{ for all } i \in [n]\}$. Then we can define the Möbius band as follows:

$$M := I^2/\sim \text{ where } (0, y) \sim (1, -y)$$

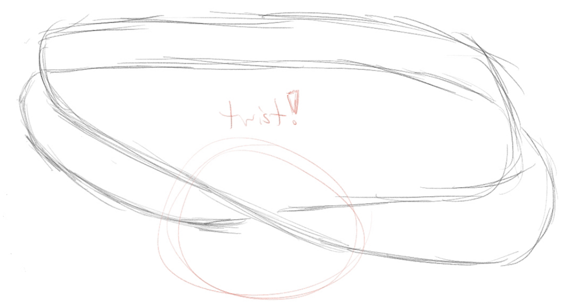

Figure 5: Mobius

The cool thing about the Möbius band is that it doesn't really have an outside or an inside, which is what makes it unorientable!

2 Classification of Surfaces

The main result of our talk relates to compact, orientable manifolds of dimension 2 which we call orientable surfaces.

Let's give another example of an orientable surface: the torus. This is homeomorphic to

$S^1 \times S^1$ which is also a manifold.[2] One way to see this is by considering it as a certain subspace of $\mathbb{R}^3$ which naturally looks like $S^1 \times S^1$. Here we will show a different construction:

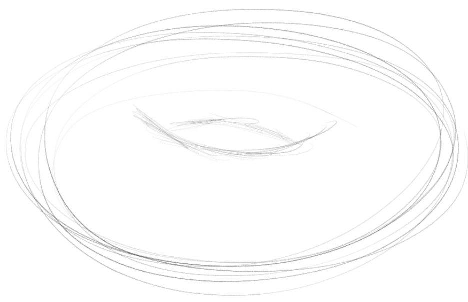

Figure 6: Torus

Example (Torus)**.** Let $I^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i \in [0, 1] \text{ for all } i \in [n]\}$ as before. Then we can define the torus as follows:

$$T := I^2/\sim \text{ where } (0, y) \sim (1, y) \text{ and } (x, 0) \sim (x, 1)$$

Remark 2.1. It turns out that the connected sum of manifolds of the same dimension gives us a new manifold. We can actually use this to get infinitely many more examples of 2 dimensional manifolds. ◦

Example (Genus g surface)**.** We can take the connected sum of two toruses (torii?). This then gives us a new manifold that looks like a torus with 2 holes now. We can sum this with a torus as many times as we want and we can get any surface with g holes.

We call the torus with g holes the **oriented surface of genus g**. So the 2-sphere is the oriented surface of genus 0 and the torus is the oriented surface of genus 1.

[2]If you haven't seen before that the product of two manifolds is a manifold, try to convince yourself that it's true. What should be the dimension of the resulting manifold?

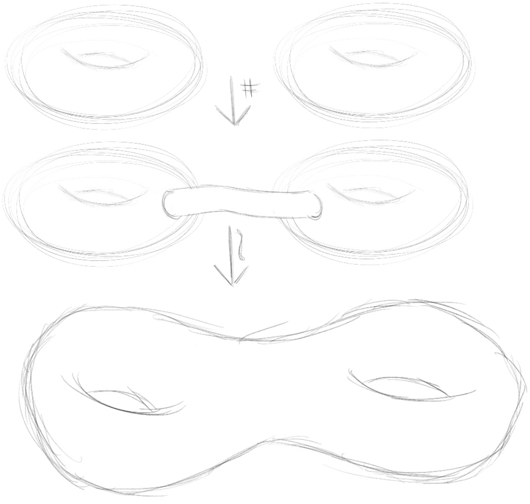

Figure 7: Connected sum of two torii

Now we can state our main theorem:

Theorem 2.2 (Classification of Surfaces)**.** *Any closed orientable surface is homeomorphic to a surface of genus g* Σ_g*. If we allow for manifolds with boundary, these come from puncturing* Σ_g *and are the same if they have the same number of boundary components.*

Example (Cylinder from sphere)**.** Let's give an example of how we can get our surfaces with boundary from this theorem. Say we have the cylinder, then if we consider the sphere without its north and south poles, it is homeomorphic to a cylinder! Similarly to this we can get the other surfaces.

Figure 8: Cylinder obtained from the twice punctured sphere

Appendices

A Non-examples

Here we take some time to add some examples of topological spaces that are **not** manifolds. This will be short of explanations and picture heavy.

Example (Figure X)**.** We can consider this to be the union of the x-axis and y-axis in $\mathbb{R}^2$.

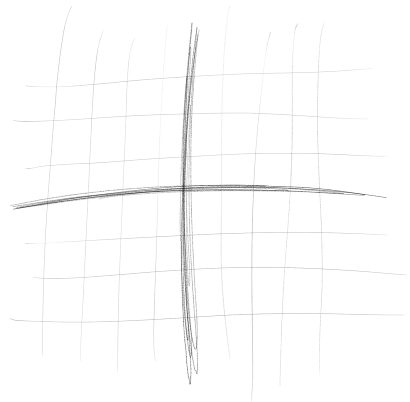

Figure 9: x and y axis in $\mathbb{R}^2$

A common misconception people have is that being locally Euclidean implies Hausdorff; it does guarantee a similar, weaker notion, but does *not* guarantee that it is Hausdorff!

Example (Line with two origins)**.** Consider the space that looks like a line with two origins. Take the disjoint union of two lines, and identify the corresponding points on both lines except for 0 (call this equivalence relation $\sim$.) Then we have

$$X := \mathbb{R} \sqcup \mathbb{R} / \sim$$

is not Hausdorff; any neighborhood of the first origin intersects any neighborhood of the other origin.

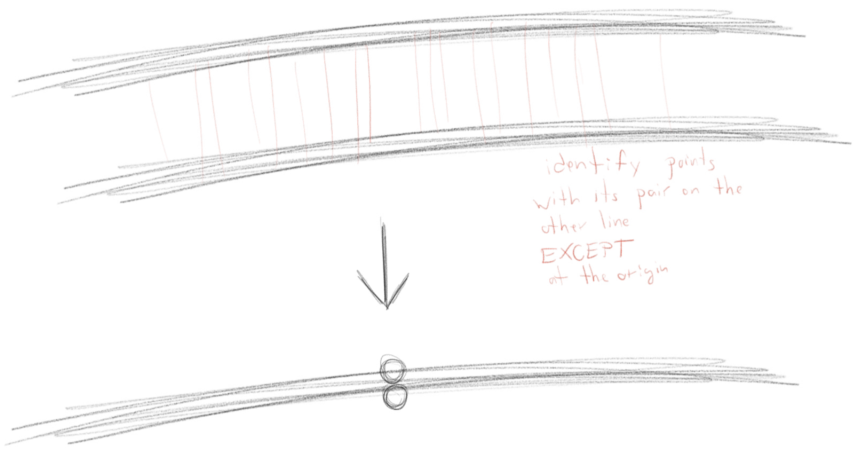

Figure 10: Cylinder obtained from the twice punctured sphere

We can also give an example of a space that is really big, such that it is not second-countable. This is called the long line which I don't know how I would draw so I will skip it, but feel free to look it up!

References

[Rol03] D. Rolfsen. *Knots and Links*. AMS Chelsea Publishing Series. AMS Chelsea Pub., 2003. 2