# The Holographic Universe

International Summer School for Young Physicists

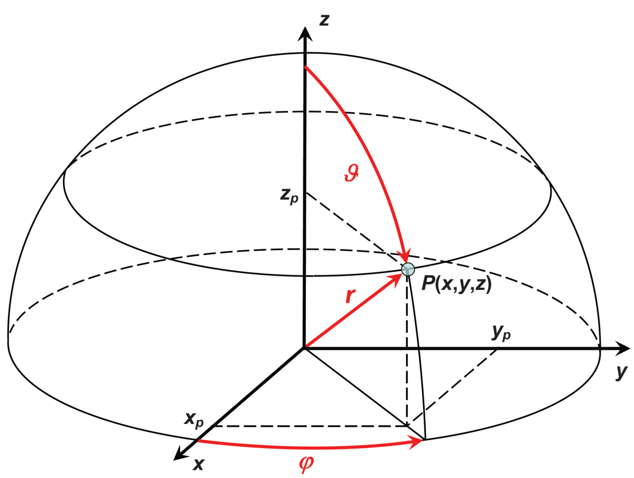
July 24, 2019

## Chapter 1: General Relativity and Anti-de Sitter Space

### Exercise 1: Geometry of a 2-dimensional Spherical Shell

As we saw today, the sphere (of unit radius) is a two-dimensional manifold. It is defined as the set

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3; \quad x^2 + y^2 + z^2 = 1\right\}, \tag{1}$$




(a) **Spherical Coordinates.** Show that the following parametrization

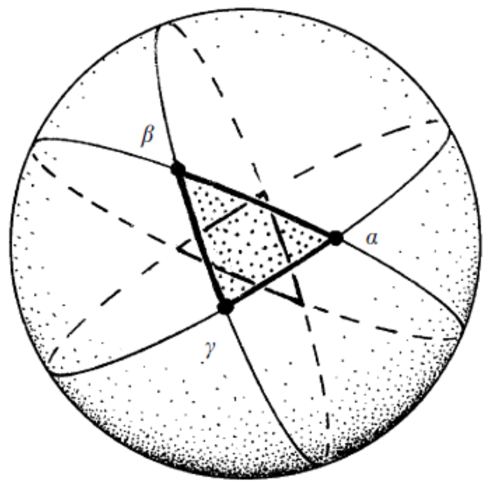
$$x = \sin\vartheta\cos\varphi; \quad y = \sin\vartheta\sin\varphi; \quad z = \cos\vartheta \tag{2}$$

satisfies the condition $x^2 + y^2 + z^2 = 1$ independent of the angles $\vartheta$ and $\varphi$. Look at the image and the equations and explain what these angles represent.

(b) **Timeless North Pole.** Now focus on the North Pole, that is, the point $(x, y, z) = (0, 0, 1)$. What are the angles $\vartheta$ and $\varphi$ that represent this point? Do you find something weird about

$\varphi$? Indeed, this means that this parametrization is not mathematically useful at this point. This is equivalent to trying to say what the time is at the geographic north pole: the time zone there is not defined!

(c) **Area of a triangle on a sphere.**




The geodesics on a sphere (straight lines for someone walking on it) are actually arcs of great circles, meaning that they form part of a circle whose center is also the center of the sphere (like the equator on Earth). We would like to find the area of a triangle made of geodesics on a sphere of radius $R$, with angles $\alpha$, $\beta$ and $\gamma$ (measured in radians). This can be done using elementary geometry, and I will guide you through this in this exercise.

(i) Find the area of a segment of a sphere bounded by two great circles with intersecting angle $\alpha$ connecting a pair of antipodal points on the sphere. If this is not clear, I mean the surface area of an orange slice like this one:

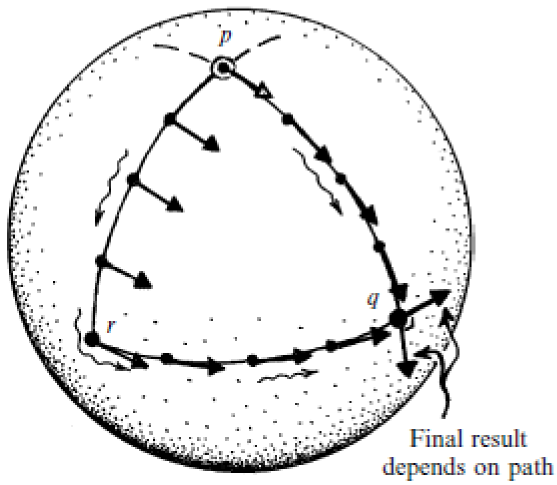
(ii) Now stare at the figure with the triangle on the sphere and find the area of all the "orange slices" you see there. Argue that the sum of all these areas is the area of the sphere plus four times the area of the triangle we want to find (observe that we count the area of the triangle four more times than needed).

(iii) Combine the results of (a) and (b) to deduce Hariot's formula for the area of the spherical triangle

$$A = R^2(\alpha + \beta + \gamma - \pi). \tag{3}$$

Of course, this area must be positive. What does this tell you about the sum of the angles of a spherical triangle?

(d) **Parallel transport on a sphere.**




When we parallel transport a vector around a closed curve on a sphere, the angle difference between the initial and final vector is proportional to the area of the (smallest) region enclosed inside the curve

(i) Can you confirm this for the case of a spherical triangle, using Hariot's formula derived in the previous exercise? You should obtain, for the angle difference $\delta$

$$\delta = \pi - \alpha - \beta - \gamma. \tag{4}$$

Therefore they are proportional with proportionality constant $1/R^2$.

(ii) If a person were to parallel transport a stick around the constant latitude of 30° around the Earth, what would be the angle difference once he completes his trajectory? Assume that the proportionality constant is the same as for the spherical triangle.

(e) **Metric Tensor.**

The matrix representation of the metric tensor for a 2-dimensional sphere in the spherical coordinates we studied before is given by

$$g_{ij} = \begin{pmatrix} g_{\vartheta\vartheta} & g_{\vartheta\varphi} \\ g_{\varphi\vartheta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin\vartheta^2 \end{pmatrix} \tag{5}$$

Also in two dimensions, the Ricci scalar $\mathcal{R}$ turns out to be twice the proportionality constant found in (d)(i), so we have

$$\mathcal{R} = 2/R^2. \tag{6}$$

The Riemann tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ only has one independent component, given by

$$\mathcal{R}_{\vartheta\varphi\vartheta\varphi} = \frac{\mathcal{R}}{2} \det g_{ij}. \tag{7}$$

The other non-zero components are dependent on this one and can be found by using the symmetry properties

$$\mathcal{R}_{\mu\nu\rho\sigma} = -\mathcal{R}_{\nu\mu\rho\sigma} = -\mathcal{R}_{\mu\nu\sigma\rho} \tag{8}$$

(i) Recall (or learn) how to compute the determinant $\det M$ of a matrix $M$ and calculate it for the metric $g_{ij}$. Then write down the only independent component of the Riemann tensor. This is related to the deviation of parallel transported vectors around a parallelogram as we saw today. In fact, the determinant of the metric calculates the area of such a small parallelogram, so this makes sense in two dimensions.

(ii) Recall (or learn) how to compute the inverse of a matrix. Compute the inverse of the metric

$$g^{ij} = \begin{pmatrix} g^{\vartheta\vartheta} & g^{\vartheta\varphi} \\ g^{\varphi\vartheta} & g^{\varphi\varphi} \end{pmatrix} \tag{9}$$

(iii) Use the fact that the Ricci tensor $\mathcal{R}_{\nu\sigma} = g^{\mu\rho}\mathcal{R}_{\mu\nu\rho\sigma}$ can be calculated by

$$\mathcal{R}_{\vartheta\vartheta} = g^{\varphi\varphi}\mathcal{R}_{\varphi\vartheta\varphi\vartheta} \tag{10}$$

$$\mathcal{R}_{\varphi\varphi} = g^{\vartheta\vartheta}\mathcal{R}_{\vartheta\varphi\vartheta\varphi} \tag{11}$$

$$\mathcal{R}_{\varphi\vartheta} = R_{\vartheta\varphi} = 0 \tag{12}$$

to show that $\mathcal{R}_{\mu\nu} = \frac{1}{R^2} g_{\mu\nu}$.

(iv) Finally, show that the sphere satisfies the vacuum Einstein field equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0 \tag{13}$$

In fact, this happens for every 2-dimensional surface. This implies that, in two dimensions, gravity (as proposed by Einstein) cannot exist.

### Exercise 2: Anti-de Sitter Spacetime

In dimensions higher than 2, only certain spacetime metrics are solutions to the vacuum Einstein equations. Moreover, it allows for a modification

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu} = 0 \tag{14}$$

where $\Lambda$ is the cosmological constant. When $\Lambda < 0$, the so-called “maximally symmetric” solution is the Anti-de Sitter spacetime. In three (spacetime) dimensions, it is given by the following set

$$AdS_3 = \left\{ (X^0, X^1, X^2, X^3) \in \mathbb{R}^4; \quad -(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 = -L^2 \right\}. \tag{15}$$

Please note that $X^0$, $X^1$, $X^2$ and $X^3$ do not represent powers of some variable $X$. They just represent four different variables. We will now study two different sets of coordinates of this three-dimensional manifold.

(a) **Global Coordinates.**

(i) To write this set of coordinates, we will first need to learn what the hyperbolic trigonometric functions are (if you know this, feel free to skip this part). The hyperbolic cosine is defined via

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{16}$$

and the hyperbolic sine is given by

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{17}$$

Remember that the ordinary trigonometric functions satisfy $\cos^2 x + \sin^2 x = 1$. These new ones are called hyperbolic functions because they satisfy a similar identity. Show that

$$\cosh^2 x - \sinh^2 x = 1 \tag{18}$$

(ii) Show that we can parametrize the three dimensional Anti-de Sitter spacetime using

$$X^0 = L\cosh\rho\cos\tau \tag{19}$$

$$X^1 = L\sinh\rho\cos\theta \tag{20}$$

$$X^2 = L\sinh\rho\sin\theta \tag{21}$$

$$X^3 = L\cosh\rho\sin\tau \tag{22}$$

This means you need to show that $-(X^0)^2+(X^1)^2+(X^2)^2-(X^3)^2 = -L^2$ for any values of the parameters $\rho$, $\tau$ and $\theta$. These three parameters are enough to describe $AdS_3$, so this should convince you that it is a three-dimensional manifold.

(b) **Poincaré coordinates.** Show that another possible parametrization of Anti-de Sitter spacetimes is

$$X^0 = \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(x^2 - t^2 + L^2\right)\right) \tag{23}$$

$$X^1 = \frac{rx}{L} \tag{24}$$

$$X^2 = \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(x^2 - t^2 - L^2\right)\right) \tag{25}$$

$$X^3 = \frac{rt}{L}. \tag{26}$$

Unlike the global coordinates, the Poincaré coordinates $(r, x, t)$ do not cover all of $AdS_3$, because we require $r > 0$. The part of $AdS_3$ that is covered by these coordinates is called the Poincaré patch. (If this is taking too much time for you, feel free to skip it).

(c) **Metric tensors and Einstein equations.** For the $AdS_3$ spacetime, the metric is given by

$$g_{ij} = \begin{pmatrix} g_{tt} & g_{tr} & g_{tx} \\ g_{rt} & g_{rr} & g_{rx} \\ g_{xt} & g_{xr} & g_{xx} \end{pmatrix} = \begin{pmatrix} -r^2/L^2 & 0 & 0 \\ 0 & L^2/r^2 & 0 \\ 0 & 0 & r^2/L^2 \end{pmatrix} \tag{27}$$

in Poincaré coordinates.

(i) Find the inverse metric $g^{ij}$.

(ii) The Ricci tensor is given by

$$\mathcal{R}_{\mu\nu} = \begin{pmatrix} 2r^2/L^4 & 0 & 0 \\ 0 & -2/r^2 & 0 \\ 0 & 0 & -2r^2/L^4 \end{pmatrix} \tag{28}$$

Compute the Ricci scalar $\mathcal{R}$ by using

$$\mathcal{R} = g^{tt}\mathcal{R}_{tt} + g^{rr}\mathcal{R}_{rr} + g^{xx}\mathcal{R}_{xx}. \tag{29}$$

You should get that $\mathcal{R} = -6/L^2$.

(iii) By plugging into the vacuum Einstein equations , show that the cosmological constant is related to the anti-de Sitter radius $L$ via

$$\Lambda = -\frac{1}{L^2}. \tag{30}$$

This also confirms that $AdS_3$ is a solution to Einstein's equations with a negative cosmological constant.

## Chapter 2: Symmetries in Physics and Conformal Field Theories

### Exercise 3: Field Lagrangians and Scattering Amplitudes

Consider the following Lagrangian that describes the interaction between two scalar fields $\phi$ and $\psi$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}m^2\psi^2 - \frac{A}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 - \frac{y}{2}\phi\psi^2 \tag{31}$$

Here we assume $M$ is much larger than $m$, and that the constants $\hbar$ and $c$ have been set to 1.

(a) **Physical constants and dimensional analysis.** What mass dimensions do length and time have when we set $\hbar$ and $c$ equal to 1? Using the fact that in four spacetime dimensions the Lagrangian must have a mass dimension of 4, what are the mass dimensions of the fields $\phi$ and $\psi$? What are the mass dimensions of the couplings $A$, $\lambda$ and $y$?

(b) **Interaction vertices.** Draw all the interaction vertices allowed by this Lagrangian, identifying the amplitude of each vertex. Note that the denominators in the action terms are just symmetry factors and need not be accounted for in the interaction strength. Then proceed to draw some Feynman diagrams using these vertices, and write down the process that they represent.

(c) **Decay process.** Draw the simplest Feynman diagram that mediates the decay $\phi \to \psi + \psi$. Can you draw a one-loop diagram that mediates this process. What about a two-loop diagram? Use dimensional analysis to estimate the lifetime of the $\phi$ particle to leading order.

(d) **Scattering process.** Now consider the scattering $\phi + \phi \to \phi + \phi$. Draw the simplest Feynman diagram that mediates this process. Feel free to draw any more complicated (higher-loop) diagrams you can come up with. Use dimensional analysis to estimate the scattering cross-section of this process to leading order in $\lambda$.

## Exercise 4: Conformal symmetry and correlation functions.

In this exercise we are going to compute the correlation functions in a Conformal Field Theory (CFT), which is a Quantum Field Theory that is symmetric under conformal transformations. As we talked about, all physical quantities in a Quantum Field Theory reduce to computing the correlation functions of the fields. In the case of a CFT, a lot of these can be calculated without the need of a Lagrangian. Consider two operators, which are functions of fields, with conformal dimension $\Delta$, which means two things:

$$\mathcal{O}_\Delta(\lambda \vec{x}) = \lambda^{-\Delta} \mathcal{O}(\vec{x}) \tag{32}$$

$$\mathcal{O}_\Delta\left(\frac{\vec{x}}{x^2}\right) = \left(\frac{1}{x^2}\right)^{-\Delta} \mathcal{O}(\vec{x}). \tag{33}$$

In this exercise we are going to consider the correlation function of two operators at different points $\vec{x}_1$ and $\vec{x}_2$, namely

$$f(\vec{x}_1, \vec{x}_2) = \langle \mathcal{O}_{\Delta_1}(\vec{x}_1) \mathcal{O}_{\Delta_2}(\vec{x}_2) \rangle \tag{34}$$

(a) **Poincaré Invariance.** Argue that if the theory is to be invariant under rigid rotations and translations, then the function $f$ must be of the form

$$f(\vec{x}_1, \vec{x}_2) = f(|\vec{x}_1 - \vec{x}_2|). \tag{35}$$

(b) **Dilation Invariance.** Show that dilation invariance as stated in equation (32) implies that $f$ must be of the form

$$f(\vec{x}_1, \vec{x}_2) = \frac{C_{12}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1 + \Delta_2}} \tag{36}$$

where $C_{12}$ is a constant.

(c) **Full Conformal Invariance.** Finally, show that inversion invariance as stated in equation (33) implies that $f$ must be of the form

$$f(\vec{x}_1, \vec{x}_2) = \frac{C_{12} \delta_{\Delta_1 \Delta_2}}{|\vec{x}_1 - \vec{x}_2|^{2\Delta_1}} \tag{37}$$

where $\delta_{\Delta_1 \Delta_2} = 1$ if $\Delta_1 = \Delta_2$ and $\delta_{\Delta_1 \Delta_2} = 0$ if $\Delta_1 \neq \Delta_2$.

## Chapter 3: The holographic principle and the AdS-CFT correspondence

### Exercise 5: Lie groups and Gauge Symmetries

In the previous sections we derived how spacetime symmetries constrain the form of, for example, essential physical quantities such as the correlation functions, as in the case of a Conformal Field Theory. A lot of theories are not conformal: only massless theories can be conformal. However, there is another type of symmetry, called a gauge symmetry, that can help us constrain how our physical theories look like. Let us start with some simple mathematical facts.

(a) **The complex exponential.** You might already know that the exponential can be expressed as an infinite series as follows

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{38}$$

(i) What would $e^{i\alpha}$ look like, if we expand it as an infinite series? Be mindful of your signs (remember that $i^2 = -1$).

(ii) Two other well-known functions, the sine and the cosine, can also be expanded as infinite series. Namely,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{39}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \tag{40}$$

Here, $x$ is measured in radians. Use these two expressions and your answer to (i) to prove Euler's identity

$$e^{ix} = \cos x + i \sin x. \tag{41}$$

In particular, show that this implies that $e^{i\pi} + 1 = 0$. Isn't this beautiful?

(iii) The set

$$U(1) = \left\{ z \in \mathbb{C}; \quad z = e^{i\alpha}, \text{ for some } \alpha \in \mathbb{R} \right\} \tag{42}$$

is called the unitary group $U(1)$. Verify that $U(1)$ is a group under the operation of complex multiplication.

(b) **Scalar Electromagnetic Lagrangian.** The scalar electromagnetic Lagrangian is given by

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi + m^2 \phi^* \phi \tag{43}$$

where $D_\mu \phi(x) \equiv \partial_\mu \phi_x + i A_\mu \phi(x)$. Show that

$$D_\mu \left( e^{i\alpha(x)} \phi(x) \right) = e^{i\alpha(x)} D_\mu \phi(x) \tag{44}$$

provided that $A_\mu$ transforms as $A_\mu \to A_\mu + \partial_\mu \alpha$. and use this fact to deduce that the scalar electromagnetic Lagrangian remains invariant under the gauge transformation $\phi(x) \to$

$e^{i\alpha(x)}\phi(x)$. In fact, this is the only (renormalizable) scalar Lagrangian that is invariant under the $U(1)$ group. The requirement of $U(1)$ symmetry has basically determined the entire physics of electromagnetism.

(c) **Two dimensional special orthogonal group.** In the previous exercise, we observed how all the elements of $U(1)$ are of the form $e^{i\alpha}$, for some real number $\alpha$. When all the elements of a Lie group $G$ can be expressed as an exponential of an element of a set $\mathfrak{g}$, we say that the Lie Algebra of $G$ is $\mathfrak{g}$. Let us study the case of the matrices that represent orthogonal rotations

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{45}$$

(i) Show that these matrices form an abelian group under matrix multiplication. In particular, you should find that $R(\theta)R(\phi) = R(\theta + \phi)$. This group is called special orthogonal group $SO(2)$. It is an example of a matrix group.

(ii) Define the exponential of a matrix in the same way as for numbers. Let $X$ be some matrix, then

$$\exp(X) = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots \tag{46}$$

Now consider $X$ to be the antisymmetric 2-dimensional matrix

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{47}$$

Find $X^2$, $X^3$ and $X^4$. Do you see a pattern? Give a formula for $X^n$.

(iii) Show that $\exp(i\theta X) = R(\theta)$, where $X$ is the antisymmetric matrix given above. This means that the set of antisymmetric matrices is the Lie algebra of the group $SO(2)$ (note that any 2 dimensional antisymmetric matrix is a multiple of $X$).

The fields in the Lagrangians are not always scalars, they could be vectors, spinors or even matrices. This is key to understanding most of the physics we know. The Lagrangians for the weak force and the strong force contain fields that are matrices, and they are invariant under the groups $SU(2)$ and $SU(3)$. This basically determines the whole physics and even tells us which particles we should observe in nature. And we do observe them! Again, **symmetries are key** in Physics.

### Exercise 6: Large N Gauge Theory as a String Theory

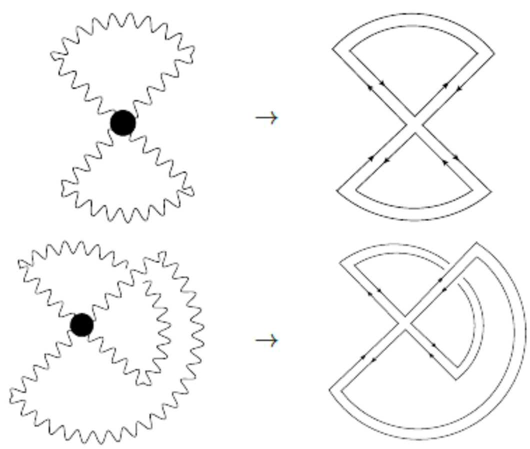
In more general gauge theories, the fields can be matrices and the Lagrangian is invariant under some gauge group $U(N)$. Here $N$ represents the dimensions of the matrices of the fields (an $N \times N$ matrix), but it is also related to the number of generators of the gauge Lie group. The groups we studied before were only 1-dimensional. For example, to describe rotation in 2-dimensional space we needed only one parameter (one angle), so this group is 1-dimensional. But to describe rotations in 3-dimensional space we need three angles, so this group is 3-dimensional. The gauge group, under which the Lagrangian is invariant, can in principle have any dimension. Here, we will study the case where the gauge group is $U(N)$, where $N$ is a very large number. $U(N)$ is the group of complex matrices with determinant $\pm 1$, and its dimension is $N^2$. Our object of study is the Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left((\partial\Phi)^2 + \frac{1}{4}\Phi^4\right) \tag{48}$$

where $\Phi$ is an $N \times N$ matrix.

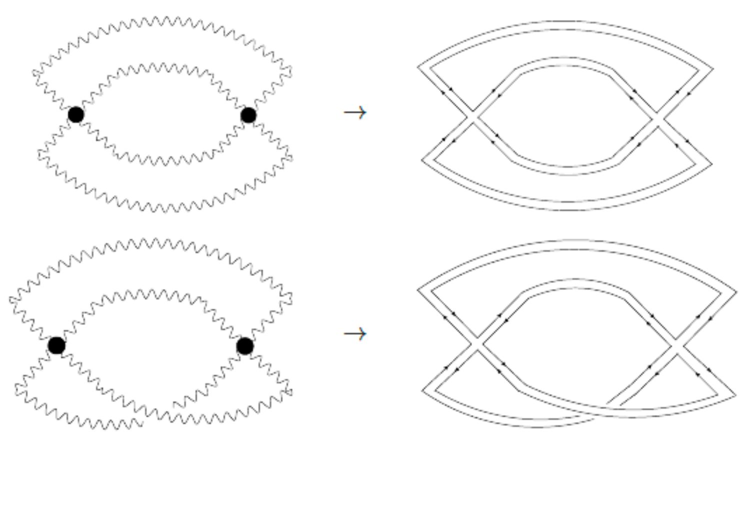
(a) **Vacuum bubbles.** The Feynman rules are a bit different in this case, but here is how they work. For a given diagram, we write in t'Hooft's double-line notation. Then, each propagator contributes to the diagram as $g^2$, each vertex as $1/g^2$ and each fully contracted line in the double-line notation as $N$.

(i) For the following diagrams



what is the contribution of each diagram to the vacuum correlation function? This is equivalent to the contribution to the vacuum energy density.

(ii) To next order in perturbation theory, we have the following two diagrams



What is the contribution of each diagram? Observe that "planar" diagrams are the ones that contribute to leading order.

(b) **Structure of large $N$ expansion.** We would like to get a general counting for the diagrams. The contribution is

$$A \sim (g^2)^E (g^2)^{-V} N^F \tag{49}$$

where $F$ is the number of faces of the diagram straightened out on some topological surface of genus $h$. To get an $N$ counting, we will take the limit where $\lambda = g^2 N$ is fixed but $N \to \infty$, so $g$ must go to zero. In this case, use the facts that the Euler characteristic is $\chi = F + V - E = 2 - 2h$ and that for a general diagram $E - V = L - 1$, where $L$ is the number of loops, to get

$$A \sim \lambda^{L-1} N^{2-2h}. \tag{50}$$

This means that planar diagrams contribute as $N^2$, toroidal ones as order unity, and so on. We then have, for the vacuum diagrams:

$$A \sim N^2 f_0(\lambda) + N^0 f_1(\lambda) + N^{-2} f_2(\lambda) + \dots \tag{51}$$

(c) **Analogy with string theory.** With more advanced methods, we can calculate that the correlation functions for non-vacuum diagrams have the following expansion

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle \sim \sum_{h=0} N^{2-n-2h} F_n^{(h)}(x_1, \dots, x_n; \lambda). \tag{52}$$

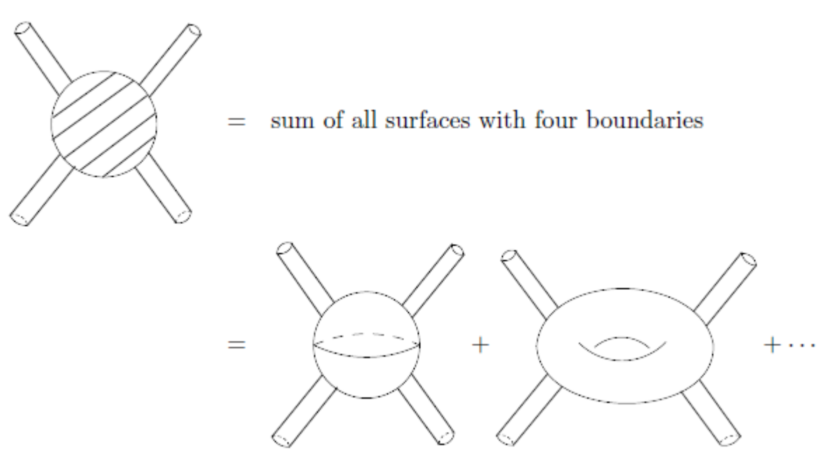
Notice that all that we gained is a power of $-n$ in the exponent of $N$, and that the functions of $\lambda$ now gain a dependence on the positions where we insert the operators (particles or "glueballs"). We will see that we can get the same result from string theory. The rules of string "Feynman" diagrams are that

(i) The creation or annihilation of a string contributes as $1/g_s$.

(ii) The separation or merging of strings contributes as $g_s$.

(iii) Each external string contributes a power of $g_s$.

Here $g_s$ represents the string coupling constant. For example, for the scattering process string + string → string + string the following diagrams must be taken into account



What is the contribution of each string diagram? Argue that for $n$ external strings, we must have

$$A_n \sim \sum_{h=0} g_s^{n-2+2h} f_n^{(h)}. \tag{53}$$

Compare this formula with the one in (b) for the scattering of glueballs. Do you see any similarities? If these expressions are to be equivalent, what identification between $N$ and $g_s$ should we make? This identification is at the core of the AdS-CFT correspondence: **A large $N$ strongly coupled gauge theory is dynamically equivalent to a weakly coupled string theory.**

## Exercise 7: The Holographic Principle

In fact, there is an extra parameter in string theory that we kind of shoved under the rug before. That parameter is the string length $l_s = \sqrt{\alpha'}$ (string theorists use $\alpha'$ because it represents the tension of the string, we will also prefer $\alpha'$). So in fact, we should have written

$$A_n \sim \sum_{h=0} g_s^{n-2+2h} f_n^{(h)}(\alpha'). \tag{54}$$

Comparing with the amplitudes for the large $N$ theory, we must have that $\alpha'$, which has dimensions of length squared, must be related to $\lambda = g_{YM}^2 N$ in some way. By using dimensional analysis, and given that $\lambda$ is dimensionless, we must have

$$\lambda \sim \frac{L^4}{\alpha'^2}, \tag{55}$$

where $L$ is the AdS radius (the only other length parameter we have available). We also know that $g_s \sim 1/N$ (yes, I just gave you the answer to a previous question). The precise relation between the parameters is given in the statement of the AdS-CFT conjecture, which I now state.

> $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $SU(N)$ and Yang-Mills coupling constant $g_{YM}$ in 3+1 dimensions is dynamically equivalent to
> type IIB superstring with string length $l_s = \sqrt{\alpha'}$ and string coupling $g_s$ on $AdS_5 \times S^5$ with AdS radius $L$. The parameters are related by
>
> $$g_{YM}^2 = 2\pi g_s \quad \text{and} \quad 2g_{YM}^2 N = L^4/\alpha'^2 \tag{56}$$

(a) **Values of the parameters.** What limit did we take for $N$ and $\lambda$ when we used the t'Hooft prescription? What does this mean for the values of the dimensionless quantities $g_s$ and $L^4/\alpha'^2$?

(b) **Counting the degrees of freedom.** So far, in what we have seen, there is nothing to tell us about the dimension of the spacetime where the theories are defined. We can see what they are by comparing the degrees of freedom. For an $SU(N)$ theory, the degrees of freedom should be proportional to $N^2$. The holographic principle states that the degrees of freedom $\mathbb{N}$ inside some region of surface area $A$ is given by

$$\mathbb{N} = \frac{A}{4G}. \tag{57}$$

Now, the area of $AdS$ is infinite, but instead we can write it as

$$A = \frac{R^3 L^3}{\delta} \tag{58}$$

where $\delta$ is a very small number and $R$ is some arbitrary number. This means we are considering an area as big as we want of Anti-de Sitter space, but not an infinite one. According to string theory, the gravitational constant is given by

$$G = \frac{g_s^2 \alpha'^4}{L^5}. \tag{59}$$

Using this and the relation between the parameters given by the correspondence, show that

$$\mathbb{N} \sim \frac{R^3 N^2}{\delta^3} \tag{60}$$

which is precisely what we expect for a large $N$ gauge theory. So we have (kind of) verified, using the holographic principle, that the number of degrees of freedom match. This means that the AdS-CFT correspondence is a realization of the holographic principle.

## Chapter 4: Applications of the AdS-CFT Correspondence

### Exercise 8: Scalar Fields in AdS.

Another possible metric for the Poincaré patch 5 dimensional Anti-de Sitter spacetime is the following

$$g_{\mu\nu} = \frac{L^2}{z^2} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{61}$$

where the first diagonal entry is $g_{tt}$ and the second one is $g_{zz}$. The coordinate $z$ is the radial coordinate and it is equal to zero at the boundary of AdS.

(a) **Klein-Gordon equation in curved spacetime.** A scalar field $\phi = \phi(z)$ defined in this anti-de Sitter background must satisfy the Klein-Gordon equation in curved spacetime, which reads

$$\frac{1}{\sqrt{-g}} \partial_z \left( \sqrt{-g} g^{zz} \partial_z \phi \right) - m^2 \phi = 0 \tag{62}$$

Here, $g$ represents the determinant of the metric.

(i) Calculate the matrix for the inverse metric $g^{\mu\nu}$.

(ii) Calculate the determinant of the metric and then evaluate $\sqrt{-g}$.

(iii) Show that the Klein-Gordon equation in AdS reduces to

$$z^2\phi'' - 4z\phi' - m^2L^2\phi = 0. \tag{63}$$

(b) **Near boundary solutions.** In (a), I made the assumption that $\phi = \phi(z)$. This is a safe assumption when we are close to the boundary $z \to 0$, so the above equation is only valid in such location. Assume that $\phi(z) = z^\Delta$ for some $\Delta$. Show that this is a solution of the near-boundary equation if we have

$$\Delta = \Delta_\pm = 2 \pm \sqrt{4 + m^2L^2}. \tag{64}$$

This implies that the general solution is of the form

$$\phi(z) = \phi_{(0)}z^{\Delta_-} + \phi_{(+)}z^{\Delta_+}. \tag{65}$$

In the AdS-CFT correspondence, the field $\phi(z)$ turns out to be dual to an operator $\mathcal{O}$ on the boundary theory. The coefficient $\phi_{(+)}$ corresponds to the vacuum expectation value of the dual operator, while $\phi_{(0)}$ corresponds to the source of such operator (you can think of this as a coupling constant). In fact, this is enough information for us to calculate the correlation functions of the operator $\mathcal{O}$. They turn out to be exactly as prescribed by conformal invariance. For example the two-point function is

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}}, \tag{66}$$

and the same happens for higher-order $n$-point functions, but this was calculated by solving an equation in AdS spacetime. This is in fact a robust test of the AdS-CFT correspondence, it tells us that this conjecture is probably correct.

## Exercise 9: Black Hole Thermodynamics in AdS

The matrix of the metric for a Schwarzschild black hole in a 5-dimensional Anti-de Sitter spacetime of radius $L$ is given by

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{f(r)} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & r^2\sin^2\theta\sin^2\varphi \end{pmatrix} \tag{67}$$

with $f(r) = 1 - \frac{2\mu}{r^2} + \frac{r^2}{L^2}$. Here, $\mu$ is related to the mass $M$ of the black hole by the expression

$$M = \frac{3\pi\mu}{4G}. \tag{68}$$

(a) **Thermodynamic quantities.** The Schwarzschild radius $r_h$ of the black hole, where the event-horizon is located, is found by setting $f(r_h) = 0$. Of course, we expect larger black holes to have a larger mass and a larger entropy.

(i) Find the mass of the black hole in terms of $r_h$.

(ii) The event horizon of the Schwarzschild black hole is a three-dimensional spherical surface. The area of a 3-sphere of radius $R$ is

$$A = 2\pi^2 R^3 \tag{69}$$

Use this fact and the Hawking-Bekenstein formula

$$S = \frac{A}{4G} \tag{70}$$

to find the entropy of this black hole in terms of $r_h$.

(iii) The temperature of the black hole can be found by using the formula

$$T = \frac{f'(r_h)}{4\pi}. \tag{71}$$

What is the temperature of the black hole as a function of $r_h$. Is this function increasing or decreasing with $r_h$?

(b) **Smarr relation.** The volume of a 3-sphere of radius $R$ is given by

$$V = \frac{1}{2}\pi^2 R^4. \tag{72}$$

In 5-dimensional Anti-de Sitter spacetime, the cosmological constant $\Lambda$ and the $AdS$ radius $L$ are related by $\Lambda = -6/L^2$. Taking this into account and using your answers to (a), verify the Smarr relation

$$2M = 3TS + 2PV \tag{73}$$

where the pressure of spacetime is given by $P = -\Lambda/8\pi G$ and $V$ is the spatial volume of the black hole. Can you believe this formula was discovered only 10 years ago? It is still subject to a lot of research, including mine.

(c) **First Law of Black Hole Thermodynamics.** Using your previous results, verify that

$$\frac{dM}{dr_h} = T\frac{dS}{dr_h} \tag{74}$$

or $dM = TdS$ for short. This is known as the First Law of Black Hole Thermodynamics, and it is very similar to the First Law of Thermodynamics

$$dU = TdS. \tag{75}$$

This law (and the temperature formula) were two of Stephen Hawking's greatest discoveries.

(d) **Hawking-Page Phase transition.** The free energy $F$ of a spacetime can be calculated as

$$F = M - TS \tag{76}$$

(i) What is the free energy of pure AdS spacetime (without any black holes)? What is the free energy of a Schwarzschild-AdS spacetime with a black hole of radius $r_h$? At this point, I should tell you the answer so that you know your calculations are correct so far. It is

$$F = \frac{\pi r_h^2}{8G}\left(1 - \frac{r_h^2}{L^2}\right). \tag{77}$$

(ii) When is the free energy of Schwarzschild-AdS smaller than that of pure AdS? You should find that this happens for $r_h$ larger than some critical $r_c$. In Physics, states with the lowest free energy are preferred. This result shows that large black holes are stable, while small black holes decay into the so-called thermal AdS.

(iii) What is the critical temperature? This means you should evaluate the temperature $T$ at $r = r_c$.

This means that there is a phase transition between spacetimes. For large temperatures, black hole spacetimes are stable, while for smaller ones, thermal AdS is stable. If we observe such a phase transition, we should also observe one in the dual boundary theory. And we do! In Quantum Chromodynamics, there is a transition between confinement and deconfinement of quarks. What we are observing here is precisely this transition, and gravity calculations allow us to find the exact critical temperature at which this happens!