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SOME QUESTIONS LOOKING FOR ANSWERS IN DYNAMICAL SYSTEMS

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Dedicated to my friend, professor Rafael de la Llave Canosa, for his 60th birthday

Abstract. Dynamical systems appear in many models in all sciences and in technology. They can be either discrete or continuous, finite or infinite dimensional, deterministic or with random terms.

Many theoretical results, the related algorithms and implementations for careful simulations and a wide range of applications have been obtained up to now. But still many key questions remain open. They are mainly related either to global aspects of the dynamics or to the lack of a sufficiently good agreement between qualitative and quantitative results.

In these notes a sample of questions, for which the author is not aware of the existence of a good solution, are presented. Of course, it is easy to largely extend the list.

1. **Introduction.** Dynamical systems, either ordinary or partial differential equations or discrete maps, are used as models in physics, chemistry, biology, human sciences and many technical domains. They can be fully deterministic or contain random terms.

After a validation of the model, based on the comparison of the predictions which follow from the model and experimental data, they can be used to make long time predictions, to describe the possible dynamics and the changes on it due to bifurcations, as depending on initial conditions and parameters. A deep knowledge can allow also to control the evolution in the desired way, either by using the suitable parameters or by adding external terms to the model.

Some of the questions deal with theoretical problems in analysis, geometry, topology, algebra, number theory, statistics, etc. Other questions are related to understanding key points of the dynamics which can allow to clarify global aspects of it. That is: what the orbits of the system do. A general point of view is that it would be desirable to have a good agreement between qualitative and quantitative approaches.

To understand the behavior of the orbits one needs to have information on the skeleton of the system: fixed points, periodic orbits, invariant curves/tori, invariant manifolds of hyperbolic or partially hyperbolic objects, perhaps in a weak sense, and their relative position.

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Changes in the dynamics are produced by bifurcations when the parameters of the model are changed. They can be local (around a given point), semi-local (around an orbit or a simple family of orbits) and global. For last ones it is not clear a priori where they occur.

In the sequel a sample of questions is proposed. The first three of them were presented in [131] and additional details can be found in this publication. Some of the questions deal with very concrete, simple, models while other appear in a quite general context.

For simplicity we shall consider we are in the analytic context.

Another point to take into account is the relation between the ranges of validity of a mathematical result and its applicability to concrete problems. As a very simple example we can mention that in [61] it was proven that the motion around a mass, under the Newtonian attraction plus the J_2 term due to oblateness, is non-integrable by using Ziglin's theory [141]. However, in [126] (written after [61] but published before) it was shown that for the Earth, with realistic values of mass and J_2 and taking into account that only motion outside the Earth's surface is possible, the amount of chaos produced by the lack of integrability is negligible.

In general, it can be nice, for a perturbation of a Hamiltonian system appearing in some application, to produce accurate results for some measure of the chaotic domain, depending on parameters like energy level, size of the perturbation and other. If it is below what can be detected experimentally, these effects can be neglected. This idea can be extended to a large variety of systems.

A similar example concerning the interpretation of the lack of integrability was shown in [109]. A very simple analytic Hamiltonian system was proved to be non-integrable. But the numerical simulations were showing a very smooth dynamics. The reason of this was that, indeed, a large amount of chaos exists in the complex phase space, but it is away from the real phase space.

Before entering into the list of questions we recall in this Introduction some topics that will be used along the paper. They concern KAM theory, Diophantine approximations, Gevrey expansions and chaos. Readers familiar with these topics can skip them.

- KAM: Consider a Hamiltonian H_0 of n degrees of freedom (dof) which is integrable in the Liouville-Arnold sense. That is, there exist n first integrals functionally independent almost everywhere and in involution. Then, under suitable conditions, the system can be put in angle-action variables $(\phi, I), \phi \in \mathbb{T}^n, I \in K \subset \mathbb{R}^n$ (or $K \subset \mathbb{C}^n$), being K a compact set, and one has $H_0 = H_0(I)$ (at least theoretically; the effective reduction can be hard depending on the Hamiltonian). The motion is elementary: $\dot{\phi}_i = \partial H_0/\partial I_i =: \omega_i,\ \dot{I}_i = 0,\ i = 1,\dots,n$. For given I the dynamics takes place in a torus with frequencies ω_i. Assume we perturb to $H_0(I) + \varepsilon H_1(\phi, I)$. If the frequencies satisfy a Diophantine condition and the passage from actions to frequencies is invertible (that is, the Hessian $D_I^2(H_0)$ is regular, a non-degeneracy condition) then, for ε small enough, the perturbed system has an invariant torus with the same frequencies, close to the torus of H_0.

  The basic idea is to do canonical transformations to cancel the perturbation. At every step in the proof one has to solve an equation of the form

$$\frac{\partial v}{\partial \phi}(\phi, I) = \varepsilon f(\phi, I) = \varepsilon \sum_{k \in \mathbb{Z}^n \setminus \{0\}} f_k(I) e^{\mathrm{i}\,(k,\phi)},$$

where the function f is assumed to be analytic in some complex domain $I \in \mathcal{D}$ in the actions, around the actions of unperturbed torus, times a complex strip around $\mathbb{T}^n$. Using Fourier expansions the solvability of this equation requires lower bounds for $|(k,\omega)|$ to prevent from too small divisors, producing a large increase in the Fourier coefficients of v. This is the reason to ask for the Diophantine condition. Even if the effect of too small divisors can be partially compensated by the Cauchy estimates on the size of the f_k coefficients, the final result can be a large decrease in the size of the analyticity domain or functions v. After the transformation the actions have changed, and to recover the previous frequencies one has to do little changes in the actions. This is why one asks for the non-degeneracy condition (also known as torsion condition and, in the case of area-preserving maps, known as twist condition). The domain of analyticity is reduced but in a controlled way under Diophantine conditions. The process is iterated until the perturbation is canceled. This is why one requires small values of ε.

Similar ideas apply to symplectic maps in dimension $2n$. See [75, 2, 3, 4, 5, 112, 113, 80]. In particular in the last Appendix in [5] there is a clear presentation of the steps in the proof of KAM theorem.

A different approach can be found in [36] for symplectic maps. One starts with an approximated representation of the invariant torus using a parametrization method and one requires an invariance condition: the image of the torus under the map should be the same torus with the angles shifted by some vector ω. With an iterative process one shows the existence of the torus with the desired ω. All the estimates are made rigorous using Computer Assisted Proofs (CAP). This allows quite sharp rigorous estimates of the parameter for which the torus breaks down, but there is no need to be in a perturbative context.

For KAM results under weaker conditions see, e.g., [121, 122, 123].

- Diophantine: The condition to avoid small divisors in KAM theory (and other domains) is typically written in the form $|(k,\omega)| > c/||k||^\tau$ for all $k \in \mathbb{Z}^n \setminus \{0\}$ for some $c > 0$ and where τ should be greater than or equal to n. The norm $||k||$ can be $||k||_1 = \sum_{i=1}^n |k_i|$ or other equivalent norm, eventually with weights adapted to the exponential decrease of the coefficients of the Fourier expansions of the perturbation. We recall that if $\tau > n$ the set of vectors ω satisfying that condition for some $c > 0$ has full measure. For symplectic maps the condition is replaced by $|(k,\omega) - k_0| > c/||k||^\tau$.

  We note that other conditions can be used, like $|(k,\omega)| > c/(||k||^n \log(||k||)^\tau)$ with $c > 0$, $\tau \geq 1$. In that case one has also full measure if $\tau > 1$. Weaker Diophantine conditions can also be found in [73], or using the so-called Brjuno numbers. See also [64, 65] for some intermediate conditions.
- Gevrey: Consider a formal power series around the origin. For simplicity we comment on the case of a single variable $\sum_{n\geq 0} a_n \varepsilon^n$, but the idea is easily extended to several variables. One says that it is of Gevrey class $\alpha > 0$ if the coefficients of order n satisfy $a_n = \mathcal{O}(n!^\alpha z^n)$ for some $\alpha > 0$, $z > 0$. In an equivalent way one can say that the series $\sum_{n\geq 0}(a_n/n!^\alpha)\varepsilon^n$ is convergent with positive radius of convergence. If the series is asymptotic to a function g (i.e., the difference between $g(\varepsilon)$ and the truncation of the series to some order n is less than a constant, M, times the norm of the first neglected term or times the sum of the norms of the first k neglected terms, for a constant k)

then, an optimal selection of the truncation order as a function of ε, which is of the form $n = \mathcal{O}(\varepsilon^{-1/\alpha})$, produces errors of the form $\mathcal{O}(\exp(-c\varepsilon^{-1/\alpha}))$ for some $c > 0$.

A formal power series of class $\alpha > 0$ is not convergent for any value of ε. The case $\alpha = 0$ corresponds to analytic functions. In some definitions of Gevrey expansions the value of α is shifted by one unit and then it is $\alpha = 1$ what corresponds to analyticity. Note that to obtain sharp estimates of α (that is: the series is of class α but not of class α^* for any $\alpha^* < \alpha$) can be a difficult question. See Sections 8 and 9 for some "intermediate" questions.

- Chaos: There are many possible definitions of chaos in dynamical systems. For concreteness we shall consider a discrete system f acting on a compact set K. In the case of a flow (autonomous or autonomized) one can replace f by the time-T-map of the flow, for some fixed time T. We can consider a system as having chaos if there exists a compact subdomain $\hat{K} \subset K$ which is the Ω-limit set of a set of positive measure, and f restricted to $\hat{K}$ has sensitive dependence to initial conditions (SDIC) and topological transitivity (TT).

  We recall that a discrete system f in $\hat{K}$ has SDIC if for a fixed value $c > 0$ (perhaps small) and every $x \in \hat{K}$ there exists $y \in \hat{K}$ such that the distance between them satisfies $d(x, y) < \varepsilon$ for arbitrarily small values of ε but there exist values $n = n(\varepsilon, c)$ such that $d(f^n(x), f^n(y)) > c$. Concerning TT the condition to be imposed is that for any two open sets U and V in $\hat{K}$ there exists n such that $f^n(U) \cap V \neq \emptyset$.

  It can happen that $\hat{K}$ is the union of several sets $\hat{K}_1, \hat{K}_2, \dots$ each one of them invariant under f. Then the conditions of being $\hat{K}_j$ the Ω-limit set of a set of positive measure and TT are required for each one of them.

  The requirement to have positive measure ensures that the set of points with chaotic dynamics will be "detectable". Instead of SDIC one requires in many cases to have positive maximal Lyapunov exponent, easy to detect numerically, but not necessary. It is important to notice that SDIC can be found even in the Kepler problem if we consider an interval of energies. Nearby bodies with different energies have different periods, say τ_1 and τ_2 and the distance between them will be of the order of the semimajor axes when the time is $\mathcal{O}(|\tau_1 - \tau_2|^{-1})$. But Kepler problem do not satisfies TT.

  An old sample of examples, concerning only strange attractors, can be found in [42].

2. **Standard-like maps: "Last" invariant curve.** In many problems, for instance in Celestial Mechanics or in particle accelerators, we can apply the usual methods to obtain, from the splitting of suitable manifolds and the return time to a fundamental domain, a separatrix map (that is, the return map to this domain) which can be approximated by a standard-like map in a subdomain not too close to the broken separatrix. See, e.g., [83, 95, 117] and references therein.

The classical standard map in $\mathbb{T}^2$ or in $\mathbb{S} \times \mathbb{R}$ (see [25, 140]):

$$SM_\kappa \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \bar{u} = u + \bar{v} \\ \bar{v} = v + \kappa \sin(u) \end{pmatrix},$$

has invariant rotational curves, IRC, for which the variable v depends on u in a periodic way, provided $|\kappa| \leq \kappa_G \approx 0.971635406$, the so-called Greene's parameter, [51, 87, 33, 74]. The last IRC to be destroyed has as rotation number the golden

$\omega_g = (\sqrt{5}-1)/2$. The parameter κ reflects the role of the splitting, due to a perturbation, of related manifolds of an integrable system, after a suitable scaling. In the present context of 2D symplectic maps, we consider a map F as integrable if there exists a non-constant continuous or smooth function G such that $G(F(u,v)) = G(u,v)$.

Typically the splitting is exponentially small in a perturbation parameter with respect to an integrable system, in the case of the analytic setting [37, 38, 43, 45, 52, 114, 127]. A geometric mechanism for the breakdown of IRC can be found in [116, 82]. Before the breakdown we can consider periodic hyperbolic orbits located at both sides of the IRC. When the perturbation parameter increases it can give rise to heteroclinic connections of the invariant manifolds of these periodic orbits. This prevents from the existence of the IRC. One can imagine that these manifolds produce holes in the IRC, which becomes a Cantor set, also called Cantorus or Aubry-Mather set (see Section 5 for a relation with the renormalization theory).

Consider general standard-like maps, like it was done in [70], but using the formulation

$$SM_{\kappa,\alpha}\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \bar{u} = u + \bar{v} \\ \bar{v} = v + \kappa f_\alpha(u) \end{pmatrix},$$

where f_α is 2π-periodic with zero average. The case $f_\alpha(u) = \cos(2\pi\alpha)\sin(u) + \sin(2\pi\alpha)\sin(2u)$ has been considered in [131] where $\alpha \in (-1/4, 1/4]$ is an additional parameter (it is enough to restrict α to this range). The case $\alpha = 0$ (resp. $\alpha = 1/4$) is the classical one (resp. the classical one with 2κ instead of κ). See [131] for details on several references, like [18, 81, 41, 49, 84], where similar models are studied and different tools are introduced.

For these standard-like maps and based on many evidences using, in particular, the $SM_{\kappa,\alpha}$ map with the two-harmonics f_α just mentioned, the following questions seem natural:

- For which set of κ do there exist IRC, as a function of α? Using the $SM_{\kappa,\alpha}$ one observes that the set of parameters (κ, α) for which there exist IRC is connected.
- Which are the rotation numbers of the last IRC which subsist? Both from theoretical considerations and based on many numerical simulations there is evidence that these rotation numbers must be *noble numbers* (see Section 6 for definition).
- Which geometric objects are involved in the IRC breakdown? Are there other mechanisms beyond the existence of heteroclinic intersections of manifolds of hyperbolic periodic orbits located at both sides of the curve?
- Using again the $SM_{\kappa,\alpha}$ it can happen, for a given α, that there are IRC for $\kappa \in [0, \kappa_1]$. Then, for $\kappa \in (\kappa_1, \kappa_2)$ there are no IRC and they reappear at κ_2 and keep existing for some nearby ranges of κ. This is observed for $\alpha = 0.042$ and values $\kappa_1 \approx 0.651$, $\kappa_2 \approx 1.211$. The rotation numbers of the last IRC for these critical values of κ seem to be completely unrelated. Some other structures, concerning the values of (κ, α) for which there exist IRC, have a fractal structure, in agreement with [70]. How to predict big jumps in the parameter κ for the existence of IRC?
- Which is the measure of the set of points in $\mathbb{T}^2$ with chaotic dynamics as a function of α and κ? For simplicity we can consider the measure of the set of points with positive Lyapunov exponent.

Similar questions can be posed for families of volume-preserving maps, defined in $\mathbb{T}^3$ or on $\mathbb{T}^2 \times \mathbb{R}$ (two angular and one action variables) which can be seen as a generalization of $SM_{\kappa,\alpha}$. See [15, 16, 23, 24, 31, 137], and also [101] for work in progress.

Note that for volume-preserving maps in $\mathbb{T}^2 \times \mathbb{R}$ the preservation of invariant tori, in a kind of KAM theorem, has a difficulty: one has two angle variables and a single action variable. Hence, the non-degeneracy condition that one requires in that case does not allow to recover the frequencies of the torus in the unperturbed case. One obtains tori with nearby frequencies. Similar ideas can be used when the number of angles exceeds the number of actions in general.

3. **A driven standard map.** One can modify the classical standard map by using a non-constant parameter driven in a quasiperiodic way (defined on $\mathbb{T}^3$ or in $\mathbb{S} \times \mathbb{R} \times \mathbb{S}$). In this way one obtains a skew product like:

$$SM_{\kappa_0,\kappa_1,\gamma} : \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \to \begin{pmatrix} \bar{u} = u + \bar{v} \\ \bar{v} = v + \kappa(\bar{\theta})\sin(u) \\ \bar{\theta} = \theta + 2\pi\gamma \end{pmatrix},$$

where $\kappa(\bar{\theta}) = \kappa_0 + \kappa_1 \sin(\bar{\theta})$. More general periodic functions of $\bar{\theta}$ can also be considered instead of just the sinus function.

For $\kappa_0 < \kappa_G$ and if κ_1 is small one can expect 2D invariant tori coming from an IRC with rotation number ρ, provided (ρ, γ) satisfy a Diophantine condition. For elliptic or hyperbolic periodic points, coming from the case $\kappa_1 = 0$, the parametrization methods in [55, 56, 58, 54] provide theoretical and numerical tools for the invariant curves which replace these points when $\kappa_1 \neq 0$. The theory and the related algorithms allow also to obtain the center or stable/unstable invariant manifolds associated to the fixed and periodic points and other possible invariant objects.

The IRC which appear when $\kappa_1 = 0$ can be broken after the introduction of the driving terms. But this depends not only on the size of κ_1 but also on the Diophantine properties of the frequencies of the subsisting tori. See again Section 6 where the question is considered in a general setting. One of them is γ and the other can change as a function of κ_1. In [131] there is strong evidence of the different effects produced by the value of γ on the destruction of invariant tori for a value of κ_0 close to the Greene's one in two extreme cases: when γ is close to the golden mean ω_g (hence in 1-1 resonance with the rotation number of the last IRC of the non-driven case) and when γ satisfies a good Diophantine condition, like $|k_1\gamma + k_2\omega_g + k_0| > c||k||^{-2}$ for $(k_1, k_2, k_0) \neq (0,0,0)$ up to $||k|| = 2^{20}$ and for $c \approx 0.233$.

See also [139, 53] for related discussions.

This leads to the following questions:

- For a given κ_0, up to which value of κ_1 some invariant $\mathbb{T}^2$ survive? Can some invariant $\mathbb{T}^2$, which is destroyed for a value of κ_1, reappear when increasing κ_1?
- How this destruction and possible reappearance depends on γ?
- Which are the geometrical mechanisms which produce the destruction of invariant $\mathbb{T}^2$? Are they related to invariant manifolds of some invariant curves?
- What one can expect if the map instead of being driven by an angle is driven by a set of angles? That is, θ in the previous formula becomes a vector of m angles in $\mathbb{T}^m$ which change as $\bar{\theta} = \theta + 2\pi\hat{\gamma}$ where $\hat{\gamma}$ is an m-dimensional vector satisfying a Diophantine condition.

4. **Strange attractors with large Hausdorff dimension.** Consider dissipative (rather weakly dissipative) systems. We take the simplest version of the forced and damped Duffing model

$$\ddot{x} = x - x^3 - \delta\dot{x} + \gamma\cos(\omega t),$$

where, for simplicity, we assume $\omega = 1$ and the parameters δ and γ are assumed to be positive. A simple way to obtain plots starting at some initial condition $(x_0, \dot{x}_0)$ is to compute the time-2π map.

Classical parameter values like $\delta = 0.2, \gamma = 0.3$ give rise to strange attractors, reminiscent of the Hénon one. As it happens in the case of the Hénon map, the attractors have a Hausdorff dimension between 1 and 2 which is not close to 2.

Alternatively, to measure the degree of chaoticity of the orbits and how large is the "thickness" of the attractors in the phase space one can use the Lyapunov dimension D_L. In the present case it is equal to $1 + \lambda_1/(\lambda_1 + 2\pi\delta)$, where λ_1 is the maximal Lyapunov exponent, assumed to be positive. See, e.g., [79] where relatively "thick" attractors are shown.

Some experiments, scanning the set of parameters (δ, γ) using a fine grid, suggest that there exist values of the parameters such that $D_L = 2 - \mathcal{O}(\delta)$ [131], as it can be expected from the definition of D_L. This has been found, for instance, for $\gamma = 2.4$, very small values of δ and a number of iterates of the time-2π map which exceeds 10^9.

The following questions can be raised:

- Is it possible to prove such a behavior for the values of D_L for the parameters for which there is numerical evidence of a large thickness?
- Would it be possible to obtain the same results if the Hausdorff dimension is used instead of the Lyapunov dimension?
- Which are the geometrical objects producing this behavior?
- What one should expect for the limit, conservative case, when $\delta = 0$? Are there domains where the iterates of a single point under the time-2π map produce a set of Hausdorff dimension 2?

Concerning the last question, the evidence that follows from the simulations, in the conservative case, is that there is a domain in which using a 1024×1024 set of pixels, all the pixels of the domain are visited by the first 2^{26} iterates of an initial point. It is not excluded that some islands appear in the domain, but they should be very small. Furthermore the domain is surrounded by smooth invariant curves, a quite surprising fact. See [131] for additional details.

5. **Breakdown of invariant tori for symplectic maps, $n > 1$.** Consider symplectic maps in dimension $2n$, $n > 1$ and related problems, like Poincaré maps of Hamiltonian systems with $n+1$ degrees of freedom on a given level of energy. When the maps (or the Hamiltonian systems) are non-degenerate and are a perturbation of an integrable system, the KAM theory (see Introduction) ensures the existence of invariant n-dimensional tori.

In the $n = 1$ case the renormalization theory [85, 86, 88, 89, 1] allows for a very good description of the breakdown of invariant rotational curves which, just after the breakdown, are replaced by invariant Cantor sets [97, 98]. See Section 2 for geometric aspects. Some works in the direction of renormalization for $n > 1$ can be found in [71, 72] and their references.

In contrast with the case $n = 1$ many items highly relevant for the dynamics are open for $n > 1$. There are many questions related to the breakdown of these tori when the size of the perturbation increases or the non-degeneracy condition is weakened.

- Which are the geometric objects leading to breakdown? In the case $n = 1$ the obstructions come from heteroclinic phenomena. But in that case the invariant manifolds of periodic hyperbolic points have codimension 1. One expects that also codimension 1 objects play a role in the destruction of n-dimensional tori. For a symplectic map in dimension 4, $n = 2$, one could expect that fixed or periodic points of center-saddle type, that is, having a 2-dimensional normally hyperbolic center manifold W^c, and hence their invariant manifolds $W^{u,s}(W^c)$ have codimension 1, play a relevant role.
- In the case $n = 2$ the remnant, after the torus has been destroyed, is usually of type Cantor $\times$ segment or Cantor $\times$ Cantor. Which is the reason? How to predict which one of them one has to expect?
- Which kinds on remnants appear in higher dimension? I am not aware of any result or simulation in this direction.
- Is there a suitable renormalization theory for arbitrary frequencies which allows to predict which kind of remnant has to appear?
- If instead of a symplectic map we consider a volume preserving map in dimension 3 with two angles and one action variable, like the problems studied in [23, 24], which is the structure of the Cantor sets after destruction of the invariant tori? As a simple example one can consider discrete versions of the Michelson system [32] and other more general models [101].
- In all cases (including symplectic maps with $n = 1$) when there is a codimension 1 torus and it is replaced by a Cantor set after breakdown, which is the probability density of the required number of iterates to escape through the Cantor set? In the case $n = 1$ the renormalization allows to predict the behavior of the average number of iterates needed to cross the Cantor as a function of the distance of the present parameter to the critical one. The critical parameter is the last one for which the invariant curve subsists. How this average number of iterates behaves in larger dimension as a function of the distance of the present parameter to the critical one?
- For $2n$-dimensional symplectic maps with $n > 1$, as the invariant tori are sticky (see Section 12), it is also relevant to study the number of iterations required to escape from the vicinity of the corresponding Cantorus when a given torus is destroyed. This is also related to effective stability, Section 15.

6. **Dynamical questions coming from Diophantine approximation.** For $n = 1$ the best Diophantine number is the golden number ω_g, also known as golden mean. It has minimal value of $\tau(= 1)$ and (asymptotically) maximal value of $c(= 1/\sqrt{5})$ among the irrational numbers with $\tau = 1$. But it can happen, for a given problem, that even if an IRC with golden rotation number exists for small enough perturbation, it is destroyed while other IRC, with different rotation number, still subsist. Which are the arithmetic properties of the ω for which the IRC subsists in this case? They are the *noble numbers*. That is, numbers whose continued fraction expansion, after a transient, has all the quotients equal to 1. They belong to $\mathbb{Q}(\sqrt{5})$, like the golden mean. This gives, asymptotically, a good behavior of the small divisors (k, ω). Numerical evidences give also strong support of this property of the noble numbers. As an example in [102] one studies how some properties of the standard map compare to properties of the area-preserving Hénon map written in the form

$$HP_c : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + 2y + \frac{c}{2}(1-(x+y)^2) \\ y + \frac{c}{2}(1-(x+y)^2) \end{pmatrix},$$

where it is enough to consider $c > 0$. In [102] one displays a plot showing the measure $\mu(c)$ of the set of confined points as a function of c. That is, the points inside the last IRC. A careful determination of the rotation number ω of the last IRC allows to obtain between 25 and 30 validated quotients of the continued fraction expansion (CFE) of ω for several values of c. In all these cases the last 8 or 10 quotients are equal to 1.

But the extensions to higher dimensional cases seem to be hard to obtain. They are related, from one side, to the question posed on Section 3 on driven standard maps, and from the other to the breakdown of invariant tori for symplectic maps and Hamiltonian systems, like in Section 5 and other cases like the volume preserving ones mentioned in Sections 2 and 5. As an example, if $\omega = (\Omega, \Omega^2)$, where Ω is the real zero of $x^3 + x - 1$ (the so-called cubic golden $\Omega \approx 0.682327803828$), very nice results are obtained. See [26] for details and applications.

Some of the questions which appear in a natural way are:

- For a given n which is the best set $\omega = (\omega_1, \ldots, \omega_n)$ with $\tau = n$ in the Diophantine condition, in the sense of having a maximal c? It is not restrictive to assume $\omega_i > 0$ for all i and some normalization, like $\max_{i=1,\ldots,m}(\omega_i) = 1$. This would be the equivalent to the golden mean which appears if $n = 1$. Answering this question would be extremely useful to help to locate the most robust invariant n-dimensional tori.
- Given $n, 0 < k < n$ and $\omega_1, \ldots, \omega_k$, how to select $\omega_{k+1}, \ldots, \omega_n$ in an optimal way? (i.e., minimal τ, maximal c). This would be the case when a system has a k-dimensional invariant torus and it is perturbed by a external quasiperiodic perturbation (for instance, a driven action) with frequencies $\omega_{k+1}, \ldots, \omega_n$.
- Is there some kind of sets of frequencies which are analogous to the noble numbers in higher dimension?

In [39, 40] a simple problem is presented, leading to a quasiperiodic behavior of the splitting of invariant manifolds. They depend on two phases. Under the assumption of an exponential decrease of the coefficients of the perturbing function one has changes in the dominant harmonics of the splitting as a function of a small parameter. But these dominant harmonics strongly depend on the arithmetic properties of the ratio of the frequencies. In [40] one considers also some transcendental numbers, like π and some exponentials of algebraic numbers chosen in a rather arbitrary way, to see how the rational approximations behave concerning some limits, e.g., if they tend to Khinchin constant and to Lévy constant, and also the sharper Diophantine properties. This leads to the following questions:

- Which are the typical properties of splitting functions when they display a quasiperiodic behavior?
- Which kind of limit behavior have the rational approximations of a set of n frequencies? How this affects the dynamics?

7. **On the abundance of invariant tori in Hamiltonian systems.** Consider Hamiltonian systems like $H = H_0(p) + \varepsilon H_1(p, q, \varepsilon)$ or $\hat{H} = H(q, p, \mu)$, with a given dependence in μ, $p \in K, q \in \mathbb{T}^n$ being $K \subset \mathbb{R}^n$ an invariant compact. They are integrable if $\varepsilon = 0$ or (we assume) $\mu = 0$, respectively. KAM theory implies that most of the invariant tori subsist under perturbation if some non-degeneracy condition is satisfied. See Introduction.

In principle, for H, most means that the complement of the tori has relative measure $\mathcal{O}(\sqrt{\varepsilon})$. This is due to estimates of the size of the resonant zones. A similar result has been proved in [100] for the complement of Lagrangian tori near resonances. But this is a rough upper estimate. In particular there appears in [6] the conjecture that the complement of the tori has a relative measure $\mathcal{O}(\varepsilon)$. Here we consider not only the KAM tori but also the ones which appear in resonant zones when the perturbation is introduced. Hence:

- Which is the shape of sharp estimates? Should some logarithmic terms also be introduced? This is suggested by [11] and in more recent preprints of the same authors on that topic. If, for simplicity, one considers that the complement of the tori has positive Lyapunov exponents, is there some numerical evidence of the shape of the estimates? That is, for a given compact set, how the measure of the set of points with positive Lyapunov exponents behaves as a function of the size of the perturbation? See [130] for some results and related topics.
- What happens around a totally elliptic fixed point whose frequencies satisfy a $DC(c, \tau)$ condition (or other conditions satisfied by most of the frequencies) as a function of the energy and the Birkhoff coefficients? For concreteness one can assume, first, that the Hamiltonian is positive definite around the fixed point and obtain estimates as a function of the energy level. Later on one can consider the general case. See [27] for nice results in this direction, where the complement of the tori is seen to be exponentially small in the distance to the elliptic fixed point.
- In the previous case, which are the bounds if the frequencies at the elliptic fixed point are in simple or multiple resonance?

Similar questions can be posed for symplectic diffeomorphisms and other kinds of maps, like the volume preserving maps mentioned at the end of Section 2 and Section 5.

8. **On the regularity of center manifolds for Hamiltonian systems.** Consider a Hamiltonian system around a fixed point with eigenvalues $\pm\mathrm{i}\,\omega_j$, $j = 1, \ldots, k$, $\pm\lambda_j$, $j = k+1, \ldots, n$, being $\omega_j > 0, \lambda_j > 0$, $1 < k < n$. It has a $2k$-dimensional center manifold W^c (perhaps non unique). Assume that the frequencies ω satisfy a Diophantine condition or, perhaps, some slightly weaker condition. The regularity of W^c depends on the DC condition and on non-degeneracy conditions. This W^c can be a normally hyperbolic invariant manifold, NHIM, and the regularity can also depend on the part of W^c that we consider. See also Chapter 2 in [54].

As an example of the loss of regularity of the center manifold, when going away from the fixed point, we can consider a very simple case. It consists on a symplectic map in dimension four. Assume the eigenvalues at the fixed point are $\exp(\pm 2\pi\mathrm{i}\,\omega)$ and $\exp(\pm\lambda)$, being $\omega > 0$ and satisfying a DC, and $\lambda > 0$. To fix ideas we assume that the map restricted to the center manifold satisfies a twist condition. There is a Cantor set of IRC around the fixed point in W^c. Between them there are resonant zones where, generically, one will find periodic hyperbolic orbits. But due to the Diophantine condition the periods of these orbits, when they are very close to the fixed point, are very big and the eigenvalues at them, inside the center manifold, say $\exp(\pm\mu)$, are rather close to 1. That is, μ is very small. We can assume that the eigenvalues transversal to W^c are close to the ones at the fixed point. The regularity of the NHIM depends on the ratio of the logarithms of the eigenvalues outside and inside the NHIM. It will be $\approx \lambda/\mu$, rather large. But going away from the fixed point there appear resonances of lower order. This produces an increase of μ and a decrease on the regularity of W^c.

Formally one can obtain an approximation of the center manifold, in the Hamiltonian case, by computing the normal form to some order and setting then the hyperbolic variables equal to zero. This will produce a truncated approximation of the manifold. A simpler method can be found, e.g., in [129]. It is possible to obtain a partial normal form in which the successive changes of variables are done so that only the terms in which the hyperbolic variables appear with total degree equal to 1, are canceled. Then the center manifold is obtained when the hyperbolic variables are set equal to zero.

In general one can compute formally the center manifold to all orders but it is not analytic. This suggests different questions.

- Which is the degree of regularity of W^c at a distance ρ from the fixed point? As a value of ρ one can take the energy or the values of suitable action variables.
- How is this related to the rate of increase of the coefficients in a formal expansion of W^c?
- Is it possible to show that the center manifold has an expansion around the fixed point which is of some Gevrey class (see Introduction)?
- In some examples there is symbolical/numerical evidence that if we denote as a_m the sum of the modulus of all the coefficients of degree m in the Hamiltonian restricted to the center manifold, then $a_m^{1/m}$ behaves as $\log(m)$. This would give sharper estimates. Is it possible to prove this kind of behavior, at least for some class of Hamiltonian systems?

9. **The Gevrey character of weakly stable/unstable manifolds.** It is well known that invariant objects of parabolic type can have weak invariant manifolds. They are usually not analytic (except if the fixed point is excluded, like in [99]) but of some Gevrey class α, see Introduction. See [7, 8, 9, 10] for a sample of theoretical results.

- How to obtain sharp upper and lower estimates of α? For the upper estimates it is enough to produce upper bounds of the rate of increase of the coefficients of the expansions. Lower estimates are harder and they strongly depend on the problem under consideration. In the formal expansions some cancellations can occur and this fact could produce a decrease in the rate of increase of the coefficients.
- Which is the dependence on the kind of invariant objects? That is, whether the invariant object is a point, an invariant curve or a torus or other more general invariant manifolds.
- Are there non-analytic manifolds whose coefficients have a rate of increase slower than the one of a Gevrey class for any $\alpha > 0$? This would be the case in the last item in Section 8. In that case, assuming that the series is asymptotic, the optimal value of the truncation order would be $\mathcal{O}(\exp(1/\varepsilon - 1))$ and the bound of the error would be doubly exponentially small, of the form $\mathcal{O}(\exp(-\varepsilon \exp(1/\varepsilon)))$.
- How is the character of the expansion changing as a function of the dimension of the manifold?
- Based on symbolic computations there is evidence in different model problems in Celestial Mechanics, like the Sitnikov problem [136], that the stable and unstable manifolds, going to or escaping from infinity in a parabolic way, are of Gevrey class 1/3 (a sharp estimate). Is this also true for general three-body

problems when a binary escapes to (or comes from) infinity? Is it possible to prove that this fact is also true for some class of general N-body problems?

Having a good information on the class α of a Gevrey expansion can be very useful, for not too small values of ε, to select the optimal order of symbolic expansions to minimize the error, assuming that one proves also the asymptotic character of the expansion. This allows to have a good starting point to do, next, the numerical continuation of the invariant objects.

10. **Regularity of invariant manifolds of weakly hyperbolic tori.** Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\varphi \in \mathbb{T}^p$ and consider the system

$$\dot{x} = A\,x + G_1\,(x, y, \varphi), \quad \dot{y} = B\,y + G_2\,(x, y, \varphi), \quad \dot{\varphi} = \omega + G_3\,(x, y, \varphi),$$

where $\mathrm{Re}\,(\lambda) > 0$ for $\lambda \in \mathrm{Spec}\,(A)$, $\mathrm{Re}\,(\lambda) < 0$ for $\lambda \in \mathrm{Spec}\,(B)$, ω satisfies a DC and $G_1, G_2, G_3, D_x\,G_1, D_y\,G_1, D_x\,G_2$ and $D_y\,G_2$ are zero at $(0,0,\varphi)$, $\forall\,\varphi \in \mathbb{T}^p$.

The linear approximation of W^u is given by $y = 0$ and one can obtain $y = h\,(x, \varphi)$ for the nonlinear case in a simple recurrent way. To this end y is expanded in Fourier series in φ with coefficients which are polynomials in x. To determine the coefficients of the polynomials one has to solve, at every order, a simple linear system. See, e.g., [125].

Now we slow down the x, y variables. To this end we multiply the equations of x and y by some non-negative weight function $f(x, y)$ of order $r \in \mathbb{N}$ with $k_1|(x, y)^\top|^r \le f(x, y) \le k_2|(x, y)^\top|^r$, being $0 < k_1 < k_2$.

But the recurrence now allows to determine the constant terms, independent on φ, at order s as a condition to obtain the periodic terms at order $r + s$. Indeed, one has to request that the average of the periodic terms at order $r + s$ be equal to zero to avoid unsuitable secular terms. It can happen, depending on p and on the dimension of φ, that the number of conditions exceeds the number of free parameters. See [125] for details.

- What happens when this formal obstruction appears? Is there still a smooth invariant manifold but not analytical?
- Is there a way to overcome this difficulty and obtain a formal expansion of the invariant manifold? Are there alternatives to expansions which are Fourier in φ and polynomial in x?
- How does the value of r, the order of the weight, affects the regularity of W^u? Would this gives raise again to Gevrey manifolds?

Similar questions can be posed for W^s.

11. **Meandering and labyrinthine tori.** For area-preserving maps, the failure of the twist condition can lead to the appearance of invariant curves of meandering type [128] which are not graphs in the usual Cartesian or polar coordinates. See [60, 21, 22, 111] for detailed descriptions of the phenomenology and for its relevant importance in many physical processes and [28] for some mathematical results. Using an action variable $I \in \mathbb{R}$ and an angle variable $\varphi \in \mathbb{S}$ the values of the angle, as a function of a suitable parameter $s \in [0, 1)$ to describe the curve, are 1-periodic, but they have ranges in which they increase and ranges in which they decrease. The geometrical reason is that one has to avoid the vicinity of the invariant manifolds of upper and lower periodic hyperbolic orbits which have the same rotation number, close to the rotation number of the meandering IRC.

They appear already in simple problems, like the planar restricted three-body problem. For examples in Hill's problem see [133]. The existence of meandering curves in Hill's problem implies also the existence of the same kind of curves in the planar restricted three-body problem for small mass ratios.

These meandering curves appear generically under weak assumptions. It is enough to have an extreme of the rotation number (hence the twist condition do not longer holds) and that the extreme be close to some rational number, to create periodic hyperbolic orbits in both sides.

In some more degenerate cases curves of labyrinthine type can appear, which can show arbitrarily complicated patterns in a geometric sense. See [128].

When going to higher dimension different types of degeneracy can occur. Still some tori can subsist under degenerate conditions. See, e.g., [48] for some results. On the other hand the torsion can have either one or several eigenvalues equal to zero. Also the higher order terms can be zero up to some order. These cases suggest the following questions.

- Which are the possible invariant objects which appear in higher dimension when the non-degeneracy condition fails? How this depends on the kind of degeneracy?
- It seems natural to expect that meandering/labyrinthine tori show up. See [128] for a simple illustration. How to prove this? Which are the typical shapes? For instance, in the simple case of 2-dimensional tori, can they have a meandering shape in both angles?

12. **Quantitative estimates on diffusion.** In Hamiltonian systems of $n \geq 3$ degrees of freedom, even close to integrable and non-degenerate, the n-dimensional tori have codimension $n-1$ inside the levels of energy.

They do not separate the phase space and can give rise to diffusion, which is commonly known as Arnold diffusion when it appears for arbitrarily small values of the perturbation. It is quite different from classical diffusion processes because it is, typically, strongly anisotropic and heterogeneous.

It occurs along resonances among the local frequencies and the diffusion coefficient can change by many orders or magnitude as a function of the distance to invariant tori.

Close to the tori the stickiness effect of these tori are relevant. Thanks to [115] it is known that the time of stability close to them is exponentially large and in [110, 13] it is proved that it can be even super exponentially large. Of course, if a point is close to an invariant torus the probability to approach to a small distance from it is also small. But if this happens, the dynamics can be confined to a vicinity of the torus for a long time. Explicit bounds of the time of confinement based on estimates of the error of normal forms can be found in [124, 46].

To understand the diffusion in Hamiltonian systems there are several natural questions which remain open:

- Given a set of points U in the phase space (with a given density), which is the density of the image points after a given time T? Even if we restrict the attention to a single resonance in which there is an action variable which diffuses, is it possible to derive a diffusion partial differential equation or, perhaps, some integro-differential equation, which describes the long term behavior of the density? How this could be done in the case of a network of relatively low order resonances?
- Let us keep our attention to the case in which the diffusion takes place along a single resonance channel. Assume that an action variable I takes initially a

value I_1. We are interested on the first time in which I takes a value I_2, not too close to I_1. Which is the probability that the traveling time is less than a given value t? How this depends on the parameters of the problem? See [29, 30] for models in this case.
- Given ε, δ and assuming that a set of points starts at a given value of the actions, up to which value of the time, $T(\varepsilon, \delta)$, the probability that the set of points diffusing more than δ in time $T(\varepsilon, \delta)$ is less than ε?
- Which is the role of the secondary resonances at a crossing of resonances? It can be difficult to enter in secondary resonances, but if it happens, the time of stay near a high order resonance can be large. They can also contribute to pass from a low order resonance to another low order one. How to characterize this role?
- If the system is a perturbation of size μ of an integrable system, how do the answers to the previous questions depend on μ?

The same questions can be posed concerning symplectic maps in dimension $2n$, $n > 1$.

13. **Links between obstructions to integrability based on differential Galois theory and estimates of the measure of chaotic dynamics.** Consider a Hamiltonian system, with n degrees of freedom, extended to the complex phase space and let $z(t)$ be a solution, where t is seen as a complex time, which defines a Riemann surface Γ.

We can consider the first order variational equations $\dot{A} = DX^H(z(t))A$ along closed paths on Γ, starting with $A(0) = \mathrm{Id}$. The coefficients belong to some field of functions K. The entries of $A(t)$ belong to a larger field L. We can consider the Galois group $G_1 = G(L/K)$ of this extension, the automorphisms of L which leave K invariant. It is an algebraic group and we can consider the identity component $(G_1)^0$ of $G(L/K)$ under Zariski's topology.

The Morales-Ramis theorem and its extensions [103, 105, 106, 107, 108] assert that if the system is integrable in the Liouville-Arnold (LA) sense, see Introduction, with meromorphic first integrals, then $(G_1)^0$ is Abelian. These results include several previous studies concerning necessary conditions for integrability, as the well know Ziglin's theorem [141]. Under LA integrability, the flows, taking as Hamiltonian each one of the first integrals, commute.

For the extensions one has to consider not only the first order variational equations, but the variational equations to any order k. Then, it is possible to define an increasing sequence of generalized Galois groups, G_k, and to consider again its identity component $(G_k)^0$. All of them must be Abelian if the system is integrable. Of course, the higher order variational equations depend in a nonlinear way on the solutions of the lower order ones. But it is possible to introduce auxiliary variables so that the system of variational equations up to order $k > 1$ is linear (see [108]). Then one falls again in the setting of the differential Galois theory.

The commutativity of the identity component $(G_k)^0$ of the extensions of the variational equations to any order k can be seen as the algebraic consequence of the commutativity of the flows of the n independent first integrals.

Some additional examples can be found in [92, 93, 109].

The lack of commutativity is related to singularities of $z(t)$. They also have geometrical consequences like the splitting of separatrices leading to dynamical consequences: existence of chaotic dynamics. See [104] for a preliminary study in this direction.

- Which quantitative information on the lack of commutativity of the identity components $(G_k)^0$ allows to predict the amount of chaos? That is, is there some measure of the lack of commutativity which allows to predict the size of the splittings?
- Is there some quantitative difference, in the dynamical consequences, if the lack of commutativity appears for the first time in some $(G_k)^0, k > 1$, instead of showing up in $(G_1)^0$?

14. **Dynamics of non-integrable Hamiltonian systems when no obstructions are found via differential Galois theory.** As mentioned in the previous Section the result for a Hamiltonian H with n degrees of freedom was extended from $(G_1)^0$ to $(G_k)^0$ for all k. All these groups must be Abelian if H is integrable. Of course, it can happen that for a given Hamiltonian, the Abelian character of the groups $(G_k)^0$ holds for all k for many solutions $z(t)$ despite the system is non-integrable. This means that one has to select suitable solutions $z(t)$ to detect the non-integrability, see [93]. Some of the obstructions can be found by looking around a single singularity of $z(t)$, but in other cases the interaction between several singularities is the key point.

In some examples the necessary condition for $k = 1$ is satisfied, but they are not at some higher order. See, e.g., [108, 92, 93]. It is easy to produce examples in which the obstructions to integrability, due to the lack of Abelian character of $(G_k)^0$, start to appear at any value of $k > 1$.

But there are systems known to be non-integrable for which all the $(G_k)^0$ associated to different solutions $z(t)$ are Abelian, see [108]. This suggests questions like:

- Is this lack of algebraic detection of the non-integrability related, for instance, to the fact that there exist n formal first integrals, functionally independent almost everywhere and in involution, in a vicinity of Γ but they are in some Gevrey class?
- In an example of this kind of behavior, how would be possible to compute, formally, these n first integrals?

15. **Effective boundaries of stability.** In several problems, despite diffusion exists, there are *effective boundaries of stability*. That is, a collection of invariant manifolds of different invariant objects such that to cross them requires a very large time. This is related to the fact that a measure of the lack of coincidence of some manifolds (i.e., a suitable splitting measure) is quite small. In these cases one can talk about *practical stability*. For the applications one should consider that the life of any system is finite or that the system is modified after long time intervals, so that one has to change the mathematical model of the phenomenon under consideration.

Typical examples appear in the vicinity of $L_{4,5}$ in three-body problems and generalizations either planar or spatial. Classical cases, using just the restricted three-body problem (RTBP) are simple models of the dynamics of the Trojan asteroids. See [132] and the Applications to Celestial Mechanics in the third Chapter of [83].

In case of the spatial RTBP there is evidence that both the unstable and stable manifolds of a one parameter (Cantor) family of two-dimensional normally hyperbolic invariant tori, born at a family of center-saddle periodic orbits, and the unstable and stable manifolds of the center manifold around the Eulerian point L_3 play a key role. The periodic orbits appear by a bifurcation of the so-called vertical family of Lyapunov orbits when the amplitude of these orbits is large.

In other cases, like the orbits replacing $L_{4,5}$ in the Earth-Moon RTBP perturbed by the Sun, it turns out that these orbits are unstable, but at some distance in the normal direction to the plane of the orbit of the Moon around the Earth, there appear practical stability zones. One can speculate about the possible relation between these stable zones (where the orbits move up and down crossing quickly the Earth-Moon plane) with the Kordylewski clouds.

In another context, like the planar restricted three-body problem, there appear invariant curves which prevent from the escape of the orbits at moderate values of the Jacobi constant (see [94]). After some perturbations, either by considering non-planar orbits of by introducing eccentricity in the motion of the massive bodies, the escape becomes possible, but it requires an extremely large time.

- How to identify all the relevant objects which play a role as effective boundaries of stability? Typically one can expect that these objects are codimension 1 stable and unstable manifolds of center manifolds, having a small splitting.
- How to obtain good estimates of the practical stability time T in a concrete problem? More concretely, which is the probability to escape in a given time?
- If it is possible to improve the non-degeneracy condition, using some technological modification of the structure, how this affects the practical stability time in a concrete device? We can think of suitable magnetic fields in a particle accelerator.

16. **Phenomena associated to the loss of reducibility under quasiperiodic time dependence.** Consider a linear ODE like $\dot{x}(t) = A(t,\varepsilon)x(t)$ where A is an $n\times n$ matrix, the dependence of A on t is quasiperiodic with basic frequencies $\omega \in \mathbb{R}^n$ satisfying some non-resonance conditions and it depends on some parameter ε, being $A(t,0)$ a constant matrix. All the time-depending terms in A have frequencies which are linear combinations of the basic ones.

In contrast with the periodic case, in which the system can always be reduced to constant coefficients via a change of variables periodic in time (according to Floquet's theorem), this is no longer true in the quasiperiodic case.

The problem can be extended to nonlinear equations (including in particular the Hamiltonian case) like

$$\dot{x}(t) = (A_0 + \varepsilon A_1(t,\varepsilon))x + \varepsilon b(t,\varepsilon) + c(x,t,\varepsilon),$$

where the matrix A_0 is elliptic, A_1 and b are quasiperiodic in t and c is of second order in x and quasiperiodic in t. The basic frequencies for A_1, b and c are assumed to be the same. The linear problem is recovered when b and c are set equal to zero. Under suitable assumptions on analyticity, non-resonance and non-degeneracy with respect to ε, as well as suitable Diophantine conditions, there exist quasiperiodic solutions, with the same basic frequencies, whose amplitudes tend to zero as ε tends to zero, for ε in a Cantor set. The intersection of this set with the interval $[0,\varepsilon_0]$ has a complement whose relative measure tends to zero exponentially in ε_0. See [64, 65] for details and [14] for the special case of Hill's equation with quasiperiodic forcing. See also [118, 119, 120] for families of reducible equations and for the role of the number of frequencies in the case of Schrödinger operators.

These results hold true, in principle, for small values of ε. Hence

- How to predict the values of ε for which reducibility to constant coefficients is lost in the linear case?
- What happens to the solutions in the case that the reducibility is lost? Is it possible, as it happens in other cases, to detect the loss of reducibility by

anomalous behavior of the Lyapunov exponents as a function of the parameter?
- How to extend this to the nonlinear case and which are the dynamical consequences?

As additional information it should be mentioned that the loss of reducibility is strongly related to fractalization. This can affect an attracting invariant torus, a saddle torus by collision of the stable and unstable bundles or by collision of tangent and (un)stable bundles and other phenomena. These can be detected by looking at several observables, like the angles between bundles, the Lyapunov multipliers or the blow up of the Sobolev norm. See [19, 34, 35, 57, 58, 68].

17. **Two-dimensional symplectic maps in complex variables.** Consider simple area-preserving maps like the standard map and other standard-like maps but looking at them in $\mathbb{C}^2$. Then the homoclinic points which exist in $\mathbb{R}^2$ and which disappear at an homoclinic tangency, continue in fact as homoclinic points in $\mathbb{C}^2$ as seen in examples in [44].

In some cases it has been found experimentally that fixed points, which seen in $\mathbb{R}^2$ are of elliptic type, appear in $\mathbb{C}^2$ at the boundary of the complex invariant manifolds of some saddle point sitting also in $\mathbb{R}^2$ [78]. That is, the closure of the complex invariant manifolds seems to contain the real elliptic fixed points.

These maps can be seen in the real phase space as symplectic maps in $\mathbb{R}^4$. But they turn out to be doubly symplectic: they preserve two symplectic real forms, corresponding to the real and imaginary parts of the complex form.

- How to characterize and predict the occurrence of the persistence of the homoclinic points which simply move to the complex phase space after an homoclinic tangency? Which is the domain in the parameter space in which these complex homoclinic points subsist? What happens at the boundary of that domain, if any?
- How to characterize and prove the existence of real elliptic fixed points belonging to the boundary of the complex manifolds of a real saddle point?
- The same question can be posed if real elliptic fixed points are replaced by real elliptic periodic orbits.

18. **Regularity of simultaneous binary collisions for the N-body problem in the general case.** Consider a Newtonian N-body problem consisting of k sets of binaries approaching collision ($k > 0$) and $N - 2k$ other bodies away from collision. It is elementary that in the case of two bodies the binary collision can be regularized just using the Levi-Civita transformation in dimension 2 (or the Kuustanheimo-Stiefel transformation in dimension 3). Furthermore the regularizations are analytical. If we consider the passage from same analytic section, transversal to the flow, before the collision, to another similar section after the collision, the map going from one section to the other is analytical.

It was proved that, in general, the simultaneous binary collisions can be topologically regularized [76, 77]. Also for some simple, concrete examples with symmetry, they can be regularized at order $r < 8/3$ (r being as close to $8/3$ as desired) being this value of the regularity a sharp estimate [90, 91]. That is the passage from some suitable section before collision to another suitable section after collision is just of class $\mathcal{C}^r$.

In the two-body problem some results have also been obtained using, instead of the Newtonian r^{-1} potential other central, non-Newtonian, potentials of the form

$r^{-\alpha}$, for which the binary collisions are regularizable for a countable set of values of α and also for the logarithmic potential.

One should note that unless some exceptional symmetries occur, the three-body collision is not regularizable. See [96] for different examples of regularizable and non-regularizable collisions.

- Which is the degree of regularity of simultaneous binary collisions for arbitrary values of the number of bodies, N, and the number of binaries, k, without symmetries and with arbitrary positive masses?
- Are there other central force potentials for which simultaneous binary collisions can be regularized? Which would be in this case the degree of regularity?

This would be useful to predict the behavior of bodies passing close to collision and in which cases a strong sensitivity to initial conditions appears. It is a simple exercise, even for three massive bodies moving on a line, to prove that when two of them collide there is a loss of regularity in the motion of the third one.

19. **Heteroclinic connections between lower and upper Lagrange points in the total collision manifold for three bodies.** It is well known that triple collisions on the three-body problem, with positive arbitrary masses, require zero angular momentum. Using blow up techniques one can construct the (non-rotating) triple collision manifold. On it there are 10 fixed points, corresponding to the five relative equilibria of the problem. Five of the fixed points are related to approaching collision and five are related to going away from it. On the triple collision manifold the Lagrangian equilibria L_4 and L_5 give rise to saddles.

But approaching a collision close to an equilateral configuration leads, generically, to escape close to a collinear configuration or to produce an escape in which a binary has been formed. In this case the binary escapes in some direction while the third body escapes in opposite direction.

It would be interesting to see if for some masses it is possible to pass from the vicinity of a Lagrange point in the lower part of the triple collision manifold to a Lagrange point in the upper part. This requires a heteroclinic connection between both points. There is a strong evidence that part of this connecting orbit is very close to collinear motion.

- Which are the masses for which approaching total collision near a Lagrange configuration can lead to escape near a Lagrange configuration?
- In which cases the Lagrange configurations are of the same type (i.e., passage from lower L_4 to upper L_4 or from lower L_5 to upper L_5 and in which cases the connected lower and upper configurations are different?

Partial information in this direction can be found in [134, 135].

20. **Effect of adding perturbations to strange non-chaotic attractors SNA.** There are systems for which the dynamics looks as having a strange attractor but which are non-chaotic, in the sense that the Lyapunov exponents are negative. But in the iterations the successive points seems to go up and down in a chaotic way.

Typically they appear in skew-products. This kind of phenomena are known as strange non-chaotic attractors. See [50, 69, 62, 63, 12, 66, 67, 59] for background and a variety of results.

But in some cases the true attractor is smooth; even analytic. What happens is that the attractor can be described by a function which has very large derivatives. Locally the errors can increase by a big factor.

The observed phenomenon is then due to the lack of accuracy of the computations. We can imagine that, alternatively, there are ranges of 10^8 iterates in which the distance between nearby orbits decreases by a factor 10^{-100}, followed by ranges of 10^8 iterates for which it increases by a factor 10^{30}. On the average there is a strong decrease, but the roundoff errors in the decreasing period can prevent from a correct estimate of the decrease of the distance. For a concrete problem this difficulty can be avoided by doing the computations with the required number of digits.

There are many examples of this behavior for skew products and for 3D maps. We can refer to [17] where the use of a sufficiently large number of digits allows to clarify the phenomenon. See also [20] for numerical evidence of the existence of strange attractors, due to the fact that the computations are just done in double precision, while it has been proved that a normally hyperbolic attracting curve exists. The idea is quite simple. Assume that for a given range of parameters a system has an attractor (negative maximal Lyapunov exponent) and for a different range it has a repeller (positive Lyapunov exponent). One can change the parameter in a periodic way. Then the character of the dynamics will depend on the integral of the maximal Lyapunov exponent in one period.

- Is it possible to predict in these cases which is the minimal size of a perturbation which can produce a true strange chaotic attractor? In the case that the attractor is described by a function with large derivatives, how the required size of the perturbation is related to the size of the derivatives?
- How to prove that it is indeed a strange chaotic attractor? Probably the most suitable method can be a Computer Assisted Proof (CAP).
- Under which conditions a strange chaotic attractor, in a general model, subsists for an open set in the parameter space?

21. **Three-dimensional dissipative maps with strange attractors of Hausdorff dimension greater than 2.** Dissipative maps in dimension three (or higher) use to display strange attractors which look very much as the Hénon attractor. This is due to the fact that some fixed or periodic points have just unstable manifolds of dimension 1.

But it seems natural that one could expect attractors associated to unstable manifolds of larger dimension. Evidences of this behavior are found in [47] where maps which can be seen as three-dimensional Hénon-like maps are studied. They are quadratic dissipative maps with constant Jacobian. The basic idea is to find invariant unstable objects whose unstable manifolds have dimension 2. There are attractors for which orbits on them have two positive Lyapunov exponents.

- The simplest question is: do exist three-dimensional maps with strange attractors of Hausdorff dimension greater than 2 and, in an analogous way to Section 4, even close to 3?
- Do they appear in quite natural models and how are they characterized?
- Which are possible extensions to arbitrary dimensions of the phase space and of the attractor? Concretely, is it possible to have dissipative maps in dimension n having strange attractors whose Hausdorff dimension is as close to n as desired?

22. **Bounding the manifolds of L_3 in the restricted three-body problem.** Consider the Restricted Three-Body Problem [138] and the libration point L_3 (located opposite to the secondary with respect to the primary).

The point is of center × saddle type in the planar problem and center × center × saddle type in the spatial one. It has one-dimensional stable and unstable manifolds W^s, W^u.

The manifolds (1-dimensional) do not coincide, as expected, and they have a splitting which can be measured as the distance in the phase space the first time that the upper branches reach, say, $r = 1$ to the left of L_5. By the symmetry of the problem the same value is obtained if the lower branches are used. This distance is exponentially small in $\sqrt{\mu}$.

A long continuation of W^s, W^u leads to escape, in the sense that they go either to small or large values of the radius r or come very close to the secondary. This has been reported in [132].

But this seems only to happen up to a value $\mu \approx 0.00043$. Below that value W^s, W^u seem to be confined, even for extremely long simulations, while for larger values of μ the escape is fast or happens for moderate values of the integration time.

- Which are the objects which confine the manifolds of L_3 for sufficiently small μ?
- How to predict the critical value?

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