Invariant Manifolds of Dynamical Systems and an application to Space Exploration

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1 Abstract

In this paper we go over the basics of stable and unstable manifolds associated to the fixed points of a dynamical system. We will start with an overview of stable and unstable sets in general, and then look at some simple examples from linear systems, mostly taken from Proctor (2010). To study some non-linear examples we will use (without proof) the stable manifold theorem and will conclude with an application of these concepts to space exploration following Cain (2003).

2 Motivation

The stable (unstable) sets of a fixed point in a dynamical system are loosely defined as those points which converge to said point in forwards (backwards) time[4]. Their study is an important tool in understanding the local behavior of non-linear systems near their fixed points. For example, one could use them to examine trajectories near the Lagrange points of a reduced three body problem. This is done through a linearization of the system, which under relatively mild conditions allows us to relate the (un)stable sets (which turn out to be manifolds) of the non-linear system to those of the linearized system. Thus we can study the stability of a fixed point (though this will not be addressed in this paper) and the geometric properties of trajectories near it[5]. Such a geometric approach was used by Hadamard to construct the unstable manifold of a diffeomorphism of the plane [2]. Analytical methods have historically been less used but in recent years many algorithms have been developed to compute (un)stable manifolds [4]. Accordingly, such techniques have been used in space missions in order to make them more fuel-efficient[3].

3 Basic definitions

First, we define a dynamical system as an ordinary differential equation

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = f(x,t)$$

Where $\mathbf{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a vector field. We could also define f on a subset $\mathrm{E} \subset \mathbb{R}^n$. Either way the domain of f is called the **phase space** of the system. For this paper, our focus will be on systems where f does not depend on time (called **autonomous systems**), in which case we can define a flow as seen in class, which we denote $\phi_{\mathrm{t}}(\mathrm{x})$. We will focus on points where f vanishes, that is, for which $\mathbf{x}(t) = \mathbf{x}(0)$ for all t. These we call **equilibrium points** of the system. To study these points, we will require the notion of **invariant sets** which we define now.

Definition an invariant set is a subset $\mathrm{A} \subset \mathrm{E}$ of the phase space s.t for any $\mathrm{x} \in \mathrm{A}$ and $\mathrm{t} \in \mathbb{R}$ $\phi_{\mathrm{t}}(\mathrm{x}) \in \mathrm{A}$.

. In particular, we will study to invariant sets associated with equilibrium points. If we let $\mathbf{x}_0$ be such a point, we define the stable and unstable sets, W^s and W^u, associated with $\mathbf{x}_0$ as follows

$$W^s = \{ \mathbf{x} : \phi_{\mathrm{t}}(\mathbf{x}) \to \mathbf{x}_0 \text{ as } \mathrm{t} \to \infty\} \tag{3.1}$$
$$W^u = \{ \mathbf{x} : \phi_{\mathrm{t}}(\mathbf{x}) \to \mathbf{x}_0 \text{ as } \mathrm{t} \to -\infty\} \tag{3.2}$$

The proof that these are indeed invariant sets is trivial so we leave it to the reader. We now turn to computing the stable and unstable sets in the case where the system is linear.

4 Linear Systems

Consider a system given by

$$\dot{\mathbf{x}} = \mathrm{A}\mathbf{x} \tag{4.1}$$

Where A is an n-by-n real matrix. Such a system can be solved directly using a matrix exponential defined as

$$\exp(A) \;=\; \sum_{i=0}^{\infty} \frac{1}{i!} A^i \tag{4.2}$$

We will prove this using the method outlined in Teschl (2012). First, define the matrix norm by

$$\|A\| = \sup_{x:|x|=1} |Ax| \tag{4.3}$$

and notice that the following inequalities hold

$$\max_{j,k} |A_{j,k}| \leq \|A\| \leq n \max_{j,k} |A_{j,k}| \tag{4.4}$$

So that the formal sum of the exponential converges in the matrix norm iff every entry converges. Now, we show that

$$\|A^i\| \leq \|A\|^i \tag{4.5}$$

Proof Notice that

$$\|A^i\| = \sup_{x:|x|=1} |A^i x| = \sup_{x:|x|=1} |A^{i-1}(Ax)| \leq \sup_{x:|x|=1} |Ax| \sup_{x:|x|=1} |A^{i-1}x| \leq \|A\|\|A^{i-1}\|$$

so the conclusion follows by induction.
Thus we have the convergence of the matrix exponential from the convergence of

$$\sum_{i=0}^{\infty} \frac{1}{i!} \|A\|^i$$

Then direct differentiation shows that

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}(0) = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!}\mathbf{x}(0) \tag{4.6}$$

is a solution to the (4.1) with initial condition $\mathbf{x}(0)$. Now, notice that the origin is a fixed point of any such system, and so we will study its stable and unstable sets, which turn out to be linear subspaces of $\mathbb{R}^n$. Although this could be studied for general real matrices, we will restrict ourselves to the case n = 2. Furthermore, we assume the eigenvalues of A have non-zero real part. Any linear system that satisfies the last condition is called **hyperbolic** and the origin would then be a **hyperbolic fixed point**. Again we follow the method given by Teschl(2012) to prove that the invariant sets are the generalized eigenspaces of the matrix A.
First, note that the following holds

$$U^{-1}A^jU = (U^{-1}AU)^j \implies U^{-1}\exp(A)U = \exp(U^{-1}AU)$$

Now, given the system in (4.1) we can perform a change of coordinates so that A is in Jordan form. This is of course done in $\mathbb{C}^2$ rather than $\mathbb{R}^n$ so some care must be taken in checking that the solutions we get are constrained to vectors with zero imaginary part. Assume first that A has two distinct eigenvalues, then it has Jordan form

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \implies \exp(A) = \begin{pmatrix} e^{\alpha_1} & 0 \\ 0 & e^{\alpha_2} \end{pmatrix}$$

In the new coordinates then

$$\mathbf{x}(0) = x_{0,1}u_1 + x_{0,2}u_2 \implies \mathbf{x}(t) = e^{t\alpha_1}x_{0,1}u_1 + e^{t\alpha_2}x_{0,2}u_2 \tag{4.7}$$

Where u_1 and u_2 are the eigenvectors of α_1 and α_2 respectively. If both eigenvalues are real then the eigenvectors can also be chosen to be real so that this solution is indeed an acceptable one for real initial conditions. If not, then we have $\alpha_2 = \alpha_1*$ which then allows us to assume $u_2 = u_1*$ and since $\mathbf{x}(0)* = \mathbf{x}(0)$ for real initial conditions we get $x_{0,2}{}^* = x_{0,1}$ and (4.7) becomes

$$x(t) = 2Re(e^{t\alpha_1}x_{0,1}u_1)$$

In both cases, it is a simple calculation to check that

$$W^s = E^s \tag{4.8}$$
$$W^u = E^u \tag{4.9}$$

Where E^s (respectively E^u) is the direct sum of the generalized eigenspaces of the eigenvalues with negative (respectively positive) real part. For the other case, when there is only one eigenvalue, we have

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \implies \exp(tA) = \begin{pmatrix} e^{t\alpha} & te^{t\alpha} \\ 0 & e^{t\alpha} \end{pmatrix}$$

Proof First, we prove by induction that

$$A^j = \begin{pmatrix} \alpha^j & j\alpha^{j-1} \\ 0 & \alpha^j \end{pmatrix} \tag{4.10}$$

since it clearly holds for j = 1, we move to the inductive step

$$A^{j+1} = \begin{pmatrix} \alpha^j & j\alpha^{j-1} \\ 0 & \alpha^j \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^{j+1} & (j+1)\alpha^j \\ 0 & \alpha^{j+1} \end{pmatrix}$$

so the conclusion follows readily from the definition of the matrix exponential.
Then we get that

$$\mathbf{x}(0) = x_{0,1}u_1 + x_{0,2}u_2 \implies \mathbf{x}(t) = (e^{t\alpha}x_{0,1} + te^{t\alpha}x_{0,2})u_1 + e^{t\alpha}x_{0,2}u_2 \tag{4.11}$$

Now, since α is real, we can choose u_1 and u_2 to be real, so that the solution is valid for real initial conditions. Now, clearly $W^s = \mathbb{R}^n$ if $\alpha < 0$ and $W^u = \mathbb{R}^n$ if $\alpha > 0$ thus we have the same conclusion as in the first case. This is indeed true in general but we will not go through the proof. The main idea is again to use the Jordan form and compute the exponential of a general Jordan block so that a computation similar to the one here can be done. We will however state the theorem as it will be relevant later,

Theorem Given a system of the form (4.1) were the eigenvalues of A have non-zero real parts, then we have

$$W^s = E^s \tag{4.12}$$

$$W^u = E^u \tag{4.13}$$

Where E^s (respectively E^u) is the direct sum of the generalized eigenspaces of the eigenvalues with negative (respectively positive) real part. In particular

$$\mathbb{R}^n = W^s \bigoplus W^u \tag{4.14}$$

5 Non-linear Systems

To apply the results from last section to more general systems, we will need what is known as the stable manifold theorem. The statement is taken from Proctor (2010), the proof is omitted as it's beyond the scope of this paper.

Stable (Invariant) Manifold Theorem Suppose the origin is a fixed point of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Let E^s and E^u be the stable and unstable subspaces of the linearization $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ where A = Df(0) is the Jacobian of f at the origin. If | f($\mathbf{x}$) - A$\mathbf{x}$| = O(|$\mathbf{x}^2$|) then $\exists$ **local stable and unstable manifolds** $W^s_{loc}(0)$ $W^u_{loc}(0)$, which have the same dimension as E^s and E^u and are tangent to them at 0, such that for $\mathbf{x} \in U$ but $\mathbf{x} \neq 0$ for some neighborhood U of 0

$$W^s_{loc}(0) = \{\boldsymbol{x} : \phi_t(\boldsymbol{x}) \to 0 \text{ as t } \to \infty\}$$
$$W^u_{loc}(0) = \{\boldsymbol{x} : \phi_t(\boldsymbol{x}) \to 0 \text{ as t } \to -\infty\}$$

These manifolds can be extended to what are called global manifolds by letting

$$W^s(0) = \{\boldsymbol{x} : \exists t \in \mathbb{R} \text{ s.t } \phi_t(\boldsymbol{x}) \in W^s_{loc}(0)\}$$
$$W^u(0) = \{\boldsymbol{x} : \exists t \in \mathbb{R} \text{ s.t } \phi_t(\boldsymbol{x}) \in W^u_{loc}(0)\}$$

We will apply this theorem to an example taken from Proctor (2010). Consider the following system

$$\dot{x} = -x + y^2, \qquad \dot{y} = y - x^2$$

The origin is a fixed point of this system, and the linearization is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

And we have

$$|f(\mathbf{x}) - A\mathbf{x}| = \begin{vmatrix} y^2 \\ -x^2 \end{vmatrix} = |\mathbf{x}^2|$$

So we can apply the theorem. First, notice that

$$E^u = \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$E^s = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We use the method outlined in Proctor (2010) to get an approximate solution for the manifolds. First, we look at the unstable manifold. It is tangent to $x = 0$ so we let $x = p(y)$ and get

$$-p(y) + y^2 = \dot{x} = p'(y)\dot{y} = p'(y)(y - (p(y))^2) \tag{5.1}$$

Which we solve approximately using a power series. Notice that since we need $p(y)$ to be tangent to $x = 0$ so the first two terms vanish.

$$p(y) = a_2 y^2 + a_3 y^3 + ...$$

Plugging this back into (5.1) we can match coefficients and get a series for $p(y)$. For this problem we will only use a third order approximation, which gives $a_2 = \frac{1}{3}$ and $a_3 = 0$ so our approximate solution is

$$W^u_{loc}(0) = \{\boldsymbol{x} : \boldsymbol{x} = \begin{pmatrix} \frac{1}{3}y^2 \\ y \end{pmatrix}\} \tag{5.2}$$

We can use the same method to for the stable manifold, which gives

$$W^s_{loc}(0) = \{\boldsymbol{x} : \boldsymbol{x} = \begin{pmatrix} x \\ \frac{1}{3}x^2 \end{pmatrix}\} \tag{5.3}$$

This example was used to illustrate how difficulties can arise in computing the manifolds of even simple two-dimensional systems. It is therefore not surprising that the computation of global invariant manifolds for general systems has proven to be extremely difficult. However, some techniques do exist and they have made applications like the one in the next section possible.

6 Applictation to Gravitational Systems

Now we turn our attention to an application of the concepts studied so far to celestial mechanics. Specifically, we will try to use the stable and unstable manifolds of gravitational systems to find an efficient trajectory to put a satellite in lunar orbit. Here we mean fuel efficient when compared to the traditional Hohmann transfer. We use the method described in Cain (2003). The idea is to consider the 4 body system of Earth-Moon-Sun-spacecraft as two 3 body systems, the Earth-Sun-spacecraft and Earth-Moon-spacecraft. These we will consider as two three body systems where the two massive objects describe circular trajectories around their center of mass and the mass of the third object is negligible. Furthermore we will assume both systems are coplanar. These approximations are justified since the time it takes for the spacecraft to perform the orbit transfer is relatively short (about 40 days [3]). Now recall from our study of the Hill Problem that the equations of motion of such a system can be written as

$$\begin{aligned} \ddot{x} &= 2\dot{y} + x + F_x \\ \ddot{y} &= -2\dot{x} + y + F_Y \end{aligned} \tag{6.1}$$

Where F is given as

$$F = \frac{1-\mu}{r_{23}} + \frac{\mu}{r_{13}} \tag{6.2}$$

In keeping with the notation of Cain (2003) we will rewrite these as

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y \end{aligned} \tag{6.3}$$

Where Ω is given as

$$\Omega = \frac{1}{2}(x^2 + y^2) + F \tag{6.4}$$

This system has an integral of motion known as the Jacobi constant C given by

$$C = 2\Omega - (\dot{x}^2 + \dot{y}^2) \tag{6.5}$$

Proof it is a straightforward calculation.

$$\begin{aligned} \dot{C} &= 2\Omega_x\dot{x} + 2\Omega_y\dot{y} - 2(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \\ &= 2(\dot{x}(\Omega_x - \ddot{x}) + \dot{y}(\Omega_y - \ddot{y})) \\ &= 2(-\dot{x}\dot{y} + \dot{x}\dot{y}) \equiv 0 \end{aligned}$$

We will therefore look to use orbits of equal Jacobi constants to conduct the transfer. We will not be able to calculate the trajectories explicitly but the main idea is fairly straightforward.

First, recall that the Lagrange points L_1 L_2 of a planar reduced three body system are the two stable points collinear with the two massive bodies that lie between them and past the less massive one respectively. In the case of the Earth-Sun-spacecraft system they are the Lagrange points closest to Earth. The idea is to use the L_2 point in the Earth-Sun system and the L_2 point in the Earth-Moon system (called EL_2 and LL_2 respectively) to get a spacecraft from the Earth to a Moon orbit. We saw during seminar that in an appropriate rotating coordinate system the Lagrange points are fixed points, so we can define their stable and unstable manifolds. Since the sphere of influence (the region were the gravitational pull from a body dominates the motion of the spacecraft) of the Earth (Moon) lies between L_1 and L_2 a small perturbation can get the spacecraft from L_2 to a capture orbit [3]. For this problem we will use what are called Lyapunov orbits around EL_2 and LL_2 respectively. These are unstable planar orbits[1] and so are ideal to perform the transfer since small maneuvers can get a spacecraft in and out of such an orbit. Even though we haven't proved this, analogous manifolds to the stable and unstable manifolds defined for fixed points exist for periodic orbits[1] which we will use to construct the trajectory of the spacecraft. Points on these invariant manifolds (which are four-dimensional since we need not only position but velocity to determine the state of the system) have the same Jacobi constant as those on the periodic orbits[1]. Therefore, given any two orbits O_1 and O_2 around EL_2 and LL_2 respectively with the same Jacobi constant, we look for an intersection between the unstable manifold of O_1, $W^u(O_1)$, and the stable manifold of O_2, $W^s(O_2)$. Suppose we have found such a point, and call it $\mathbf{x}_0$, then the following hold

$$\phi_t(\mathbf{x}_0) \to O_1 \text{ as } t \to -\infty$$
$$\phi_t(\mathbf{x}_0) \to O_2 \text{ as } t \to \infty$$

So that only two small maneuvers are needed, one to get the spacecraft out of O_1 and into the trajectory of $\mathbf{x}_0$ and another to get it out of this trajectory and into O_2 [1]. Once this is done, another maneuver can be performed to lower the energy and transfer from 0_2 to a lunar orbit [3]. The problem of finding such an intersection is solved computationally using a Poincar section between LL_2 and EL_2 and then looking for common points in the Poincar maps of trajectories in both manifolds [1]. Again, notice that the manifolds in question are four dimensional so the Poincar section will be three dimensional.

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