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Ratio-Like and Recurrence Relation Tests for Convergence of Series

Y. F. CHANG AND G. CORLISS

Department of Computer Science, University of Nebraska, Lincoln, Nebraska 68588, *U.S.A.*

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Three tests for convergence of series are given which are well suited for the automatic determination of the radius of convergence of a series. The first test is similar to the standard ratio test, while the other two tests use a recurrence relation for the terms of the series. Together, these tests determine the radius of convergence of many series which arise as solutions to ordinary differential equations. Numerical examples using these tests are given.

1. Introduction

THE PAST SEVERAL YEARS have seen a renewed interest in the numerical solution of ordinary differential equations by methods based on finding a Taylor series expansion for the solution. The essential features of such methods are given in most standard textbooks in ordinary differential equations (for example, Kreysig, 1972, chapter 3). In 1946, Miller used recurrence schemes for the terms of a Taylor series to compute the Airy Integral for the British Association Mathematical Tables (Miller, 1946). The Taylor series approach has been successfully applied to a wide range of non-linear initial and boundary value problems in ordinary and partial differential equations. See reference in Barton, Willers & Zahar (1970) and Chang (1974).

We consider the solution of systems of ordinary differential equations

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{1}$$

We wish to approximate y by its truncated Taylor series and to extend the approximation by analytic continuation. A full discussion of the method may be found in Moore (1966).

In order to overcome the difficulty of finding analytic expressions for arbitrarily high derivatives, several authors (Barton *et al.*, 1970; Chang, 1968; Gibbons, 1960; Leavitt, 1966; Moore, 1966) have written compiler-like programs which accept the differential equations as input. Leibniz' rule for differentiating a product is used to generate long Taylor series using recurrence relations for higher derivatives. The compiler produces a computer program which can be run later. For many test problems, these compiler-generated programs achieve a high degree of accuracy and proved to be competitive with standard methods (Barton *et al.*, 1970; Chang, 1974).

One problem with the Taylor series approach is the difficulty of testing the convergence of the resulting series. The stepsize should be produced as large as

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possible consistent with error control demands. The radius of convergence of the series is not usually known *a priori*, so analysis to recognize it is very important. Both Barton *et al.* (1970) and Gibbons (1960) estimate the truncation error from the magnitude of the first neglected term. Chang (1974) gave an algorithm which recognized the radius of convergence of many test series. This allowed him to adjust the stepsize under program control to maintain an acceptable truncation error.

This paper reports on three tests which are suitable for the automatic computer determination of the radius of convergence of a general series. Two of these tests, the three-term test considered in Section 3 and the five-term test considered in Section 5, are especially attractive because they are successful even when the order of the primary singularity is not an integer. Section 2 reviews some well-known methods for recognizing the radius of convergence of a series. The model problems we wish to consider are also defined. In Section 3, the three-term test which is most useful for functions with a single singularity on the circle of convergence is discussed. It is shown that the usual ratio test sometimes dangerously over-estimates the radius of convergence of the series for these problems. In Section 4 we prove the principle result of this paper, a test for the convergence of a series based on the form of a recurrence relation for its terms. In Sections 5 and 6, we consider two special cases which are of interest because of their applications to Chang's Automatic Taylor Series (ATS) method. The five-term test considered in Section 5 is most useful for recognizing the convergence of the series for a function with a pair of conjugate singularities. In Section 7, we present three numerical examples which illustrate the ability of these tests to recognize the radius of convergence in non-linear problems in ordinary differential equations. In Section 8, our results are summarized and a direction for further investigations of convergence tests of this type is suggested.

2. Estimates for Radius of Convergence

If f is a meromorphic function, many estimates for the radius of convergence (R_c) of its power series are known, for this is a very old problem. Hadamard's classical thesis on the subject (Hadamard, 1892) included estimates for the radius of convergence, for all of the poles on the circle of convergence, and for the location and order of all of the poles of highest order on the circle of convergence in terms of lim sup's of ratios of certain determinants. Golomb (1943) considered hypotheses under which Hadamard's lim sup could be replaced by lim. Although the singularities of a function may occur in any pattern, many functions which arise as solutions to (1) are characterized by real poles or pairs of complex conjugate poles in the function or in its derivatives. The simplest form of Golomb's tests are given in the following Theorem.

THEOREM 1 (Golomb, 1943; p. 590). *If $f(z) = \sum_0^\infty a_n z^n$ is meromorphic and has no singularities on the circle of convergence $|z| = R_c$ except poles, and if $z_1 = \lim (a_n/a_{n+1})$ exists, then f has order*

$$m = 1 + \lim_{n\to\infty} (\log |a_n| + n \log R_c)/\log n \tag{2}$$

on $|z| = R_c$, and z_1 is the only pole of order m on $|z| = R_c$.

A closely related method using Hankel determinants is discussed at some length by

Henrici (1974). He observes that in principle, the problem of determining the poles of a meromorphic function from the coefficients of its Taylor series is completely solved. However, this method is computationally expensive, especially if the number of poles on the circle of convergence is not known *a priori*. Perhaps the most widely used algorithm for locating poles of a meromorphic function involves the quotient-difference (qd) table (Henrici, 1974).

If there is a single pole on the circle of convergence, each of these tests reduces to the usual ratio test. Further, their proofs do not hold for functions like $\log(z)$ or $z^{2/3}$ where the singularities are branch points, not poles. The three- and five-term tests considered in this paper are able to recognize the radius of convergence of such functions. In Section 3, we compare the estimates of the ratio test and of (2) with those provided by the three-term test and examine the effects of secondary singularities.

Many functions which arise as solutions to (1) are characterized by real singularities or pairs of complex conjugate singularities. Accordingly, we define the two model problems

$$v(a, \alpha, z) = \begin{cases} (a-z)^{-\alpha} & \alpha \neq 0, -1, \dots \\ v(z) \text{ such that } v^{(1-\alpha)}(z) = v(a, 1, z) & \alpha = 0, -1, \dots \end{cases} \tag{3}$$

and

$$w(b, c, \alpha, z) = \begin{cases} (z^2+bz+c^2)^{-\alpha} & \alpha \neq 0, -1, \dots \\ w(z) \text{ such that } \mathrm{w}^{(1-\alpha)}(z) = w(b, c, 1, z) & \alpha = 0, -1, \dots \end{cases} \tag{4}$$

where α, a, b, and $c \in R$. A function with only one primary singularity at $z = a$ has a series which is asymptotically like that for $v(a, \alpha, z)$, while a function with a conjugate pair of primary singularities has a series which is asymptotically like that for $w(b, c, \alpha, z)$.

Define the reduced derivatives of a function g by

$$G_{i+1} = g^{(i)}(a)h^i/i!.$$

Then the series for the model problems (3) and (4) expanded at $z = 0$ satisfy the recurrence relations

$$V_{i+1} = V_i(1+(\alpha-1)/i)h/a \quad \begin{cases} i = 1, 2, \dots & \text{for } \alpha \neq 0, -1, \dots \\ i = 2-\alpha, 3-\alpha, \dots & \text{for } \alpha = 0, -1, \dots \end{cases}$$

$$iW_{i+1} = (2-2\alpha-i)(h/c)^2 W_{i-1} + (1-\alpha-i)bhW_i/c^2 \qquad i = 2, 3, \dots \qquad \alpha \neq 0, -1, \dots$$

$$iW_{i+1} = (1-i)^{-1}(i+\alpha-2)(i+\alpha-1)(h/c)^2 W_{i-1} + (1-\alpha-i)bhW_i/c^2 \qquad i = 3-\alpha, 4-\alpha, \dots, \qquad \alpha = 0, -1, \dots$$

The initial terms of each series are determined by the initial value problem.

3. Three-Term Test

We first consider the case in which $y(x)$, the solution to (1), has a single singularity on the circle of convergence so that the power series for y is asymptotic to the power series for $v(a, \alpha, z)$. In order to determine the radius of convergence a, we let $R_i = V_{i+1}/V_i$ and solve the equations

$$R_i = (1+(\alpha-1)/i)h/a \quad \text{and} \quad R_{i-1} = (1+(\alpha-1)/(i-1))h/a$$

to yield

$$h/a = iR_i-(i-1)R_{i-1} \quad \text{and} \quad \alpha = \frac{i(i-2)R_i-(i-1)^2R_{i-1}}{-iR_i+(i-1)R_{i-1}}. \tag{5}$$

This suggests the following ratio-like test which we call the three-term test.

THEOREM 2. *Let $\sum_1^\infty a_i$ be a non-zero series of real numbers such that*

$$\lim_{i\to\infty}\left[i\left(\frac{a_{i+1}}{a_i}\right)-(i-1)\left(\frac{a_i}{a_{i-1}}\right)\right] \tag{6}$$

exists and equals L. Then

(*i*) *if $|L| < 1$, $\sum a_i$ is absolutely convergent,*
(*ii*) *if $|L| = 1$, the test fails, and*
(*iii*) *if $|L| > 1$, $\sum a_i$ diverges.*

Theorem 2 is proved by the following lemma which shows that the three-term test is weaker than the usual ratio test.

LEMMA 1. *Under the hypotheses of Theorem* 2, lim (a_{i+1}/a_i) *exists and equals L.*

To prove the lemma, let d_1 and d_2 be arbitrary numbers satisfying $-\infty < d_1 < L < d_2 < \infty$ and define a sequence of constants $b_i = a_{i+1}/a_i$. For i sufficiently large,

$$d_1 < ib_i-(i-1)b_{i-1} < d_2,$$

or

$$((i-1)b_{i-1}+d_1)/i < b_i < ((i-1)b_{i-1}+d_2)/i.$$

Proof by induction on p shows that

$$((I-1)b_{I-1}+(p+1)d_1)/(I+p) < b_{I+p} < ((I-1)b_{I-1}+(p+1)d_2)/(I+p). \tag{7}$$

Taking lim inf of the left hand and lim sup of the right hand inequalities in (7) gives $d_1 \leqslant \liminf b_i \leqslant \limsup b_i \leqslant d_2$. d_1 and d_2 were arbitrary lower and upper bounds, respectively, for L, so the lemma follows.

We remark that the sequence $\{a_i\}_{i=2}^\infty$ given by

$$a_{2j} = 2jb_j, \qquad b_1 = 1,$$
$$a_{2j+1} = (2j+1)b_j, \qquad b_j = b_{j-1}\left(\frac{2j-1}{2j}\right)^2,$$

shows that the converse of Lemma 1 is false.

Although the three-term test is weaker than the ratio test, it is more appropriate for recognizing R_c under program control. For the model problem $v(a, \alpha, z)$, each of the

TABLE 1
Estimates from 30 term series for $(1-z)^{-\alpha}$

Order α	Ratio test R_c	3-Term test R_c	Hadamard's order	3-Term order
10·0	0·7632	1·0000	6·5596	9·9995
5·5	0·8657	1·0000	4·4103†	5·5002
2·0	0·9667	1·0000	2·0000	2·0008
1·0	1·0000	1·0000	1·0000	1·0001
0·333	1·0235	1·0000	0·0485†	0·3333
0·0	1·0357	1·0000	0·0100†	0·0000
−1·0	1·0741	1·0000	−0·9697†	−1·0001
−5·5	1·2889	1·0000	−3·9117†	−5·5000
−10·0	1·6111	1·0000	−9·2498†	−9·9997

† Hadamard's order estimate does not apply in these cases.

methods given in Section 2 reduces to the standard ratio test, so we shall use this problem to compare the ratio test with the three-term test.

The series $\sum_1^\infty V_i$ is asymptotic to the geometric series $a^{-\alpha}\sum_1^\infty (h/a)^{i-1}$, so the standard ratio test correctly determines its convergence.

$$|V_{i+1}/V_i| = |1+(\alpha-1)/i|\cdot|h/a|,$$

so the relative error in the value for h/a computed by the ratio test is $|1+(\alpha-1)/i|$. Although this relative error approaches 1 as $i\to\infty$, the error made in estimating R_c from a truncated series may not be negligible. Table 1 shows the effect of applying the ratio and three-term tests to the series for $(1-z)^{-\alpha}$ truncated to 30 terms using IBM360 single precision arithmetic. Table 1 also compares Hadamard's estimate for the order of the singularity (2) with the three-term estimate (5).

If $\alpha > 1$, the ratio test underestimates R_c, and the algorithm for solving the differential equation takes a smaller step than is necessary, but its accuracy is maintained. However, if $\alpha < 1$, R_c is over-estimated, and the algorithm may attempt to sum a divergent series with disastrous consequences.

TABLE 2
Estimates with a neighbouring secondary singularity

Order α	Ratio test R_c	3-Term test R_c	Hadamard's order	3-Term order
10·0	0·7650	0·9939	6·5667	9·6785
5·5	0·8688	0·9925	4·4211†	5·1303
2·0	0·9714	0·9916	2·0149	1·6010
1·0	1·0054	0·9914	1·0164	0·5946
0·333	1·0294	0·9912	0·0659†	−0·0758
0·0	1·0419	0·9912	0·0279†	−0·4106
−1·0	1·0810	0·9911	−0·9500†	−1·4135
−5·5	1·3013	0·9914	−3·8820†	−5·9049
−10·0	1·6337	0·9942	−9·2052†	−10·3514

† Hadamard's order estimate does not apply in these cases.

Both the ratio test and the three-term test estimate the distance to the closest singularity, and both may be affected by the presence of secondary singularities. Repeating the computations shown in Table 1 for 30 terms of the series for the function $(1-z)^{-\alpha}+(1{\cdot}1-z)^{-\alpha}$ yields results shown in Table 2. The presence of the secondary singularity caused the performance of each test to deteriorate.

The series $\sum W_i$ is the sum of two series, each of which is asymptotic to a geometric series. Both the ratio test and the three-term test usually fail because the primary singularities occur at a pair of conjugate points. In Sections 5 and 6, we will consider tests which can successfully recognize R_c for this problem.

4. Recurrence Relation Convergence Test

We have seen in Section 2 that the power series for the solution to (1) is often asymptotically like the power series for the model problems (3) and (4). The difference equations satisfied by V_i and W_i are known, so we consider the convergence of a general series $\sum a_i$ whose terms satisfy a certain linear recurrence relation. Before stating these tests, we pause to state some results from the theory of difference equations. These results follow from results on products of matrices in Ostrowski (1966, chapters 20 and 21).

THEOREM 3 (Ostrowski, 1966). *Consider the linear system of difference equations*

$$d_{i+1} = (A+B_i)d_i,$$

where A and B_i are $n\times n$ matrices and $\lim B_i = 0$.

(*i*) *If the spectral radius $\rho(A)$ satisfies* $0<\rho(A)<1$, *then $\|d_i\|$ is bounded above by the terms of a convergent geometric series so* $\lim \|d_i\| = 0$.

(*ii*) *If all of the eigenvalues of the matrix A exceed* 1 *in modulus, then there exists an integer I and a number $\tau>0$ such that*

$$\|d_i\|^2 \geqslant (i-I)\tau, \quad \textit{for } i>I.$$

Now that we have the necessary results from the theory of stability for difference equations, we can proceed to consider a general convergence test.

THEOREM 4. *Let $\sum_1^\infty a_i$ be a series of real numbers whose terms satisfy a linear recurrence relation of the form*

$$a_i = \sum_{j=1}^{k} (a_j + b_{i,j})a_{i-j}, \tag{8}$$

where $\lim\limits_{i\to\infty} b_{i,j} = 0$. *Let the matrix*

$$A = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & 0 \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & 0 & 1 \\ \alpha_k & \cdot & \cdot & \cdot & \alpha_2 & \alpha_1 \end{pmatrix}$$

have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ listed in order of increasing modulus. Then

(i) *if $|\lambda_k| < 1$, $\sum a_i$ converges absolutely,*
(ii) *if $|\lambda_1| > 1$, $\sum a_i$ diverges, and*
(iii) *if $|\lambda_1| \leqslant 1 \leqslant |\lambda_k|$ the test fails.*

The proof of Theorem 4(i) and (ii) follow from Theorem 3. To prove part (iii), observe that the series for $w(0, 1, 1, 1)$ diverges, while the series for $w(0, 1, 0, 1)$ converges.

A convergence test of this type is appropriate to apply to a series whose terms are produced by a recursion relation. The two remaining tests in this paper are special cases of Theorem 4 which are motivated by the form of the recurrence relation for the model problem (4).

5. Five-Term Test

If the singularities of y closest to the point of expansion occur with the same order at a pair of conjugate points, the power series for y is asymptotic to the power series for the model problem $w(b, c, \alpha, z)$. In order to determine $R_c = c$ for the series $\sum W_i$, we fix an integer N and consider the three equations

$$iW_{i+1} = (2-2\alpha-i)(h/c)^2 W_{i-1} + (1-\alpha-i)bhW_i/c^2, \qquad i = N-3, N-2, N-1$$

in the three unknowns $x_1 = h/c$, $x_2 = b/c$, and $x_3 = \alpha$. These equations involve W_{N-4}, W_{N-3}, W_{N-2}, W_{N-1}, and W_N, hence this approach is called the five-term test.

THEOREM 5. *Let $\sum_1^\infty a_i$ be a series of real numbers. Assume that there exist real numbers x_1, x_2, x_3 with $|x_2| \leqslant 2$ such that*

$$(1+\varepsilon_i)ia_{i+1} = (2-2x_3-i)x_1^2 a_{i-1} + (1-x_3-i)x_1 x_2 a_i, \tag{9}$$

where $\lim \varepsilon_i = 0$. *Then*

(i) *if $|x_1| < 1$, then $\sum a_i$ converges absolutely,*
(ii) *if $|x_1| > 1$, then $\sum a_i$ diverges, and*
(iii) *if $|x_1| = 1$, the test fails.*

Equation (9) defines a system of difference equations $d_{i+1} = (A+B_i)d_i$, where $d_i = (a_{i-1}, a_i)^T$,

$$A = \begin{pmatrix} 0 & 1 \\ -x_1^2 & -x_1 x_2 \end{pmatrix}$$

and

$$B_i = \begin{pmatrix} 0 & 0 \\ \left(1 - \dfrac{-2+i+2x_3}{i(1+\varepsilon_i)}\right)x_1^2 & \left(1 - \dfrac{-1+i+x_3}{i(1+\varepsilon_i)}\right)x_1 x_2 \end{pmatrix}.$$

Clearly $\lim B_i = 0$. A has eigenvalues

$$\lambda_1 = (-x_1 x_2 + (x_1^2 x_2^2 - 4x_1^2)^{\frac{1}{2}})/2,$$
$$\lambda_2 = (-x_1 x_2 - (x_1^2 x_2^2 - 4x_1^2)^{\frac{1}{2}})/2,$$

so that $|\lambda_1| = |\lambda_2| = |x_1|$. Our result now follows from Theorem 4.

The five-term test recognizes the singularities of $w(b, c, \alpha, z)$ for $|b/c| < 2$, while if $|b/c| \geqslant 2$, $w(b, c, \alpha, z)$ has two real singularities which can be recognized by the simpler three-term test.

If $a_i \neq 0$, equation (9) may be replaced by the assumption that

$$\lim_{i \to \infty} \left[\frac{(2-2x_3-i)x_1^2 a_{i-1} + (1-x_3-i)x_1 x_2 a_i}{i a_{i+1}} \right]$$

exists and equals 1.

6. Four-Term Test

We can state another test for convergence similar to the five-term test. The four-term test is motivated by considering the recurrence formula for $w(b, c, 1, z)$,

$$W_{i+1} = -(h/c)^2 W_{i-1} - bhW_i/c^2. \tag{10}$$

Two copies of equation (10) can be used to solve for the two unknowns $x_1 = h/c$ and $x_2 = b/c$ in terms of four consecutive W_i's.

THEOREM 6. *Let* $\sum_1^\infty a_i$ *be a series of real numbers. Assume that there exist real numbers* x_1 *and* x_2 *with* $|x_2| \leqslant 2$ *such that*

$$(1+\varepsilon_i)a_{i+1} = -(x_1^2 a_{i-1} + x_1 x_2 a_i), \tag{11}$$

where $\lim \varepsilon_i = 0$. *Then*

(*i*) *if* $|x_1| < 1$, $\sum a_i$ *converges absolutely,*
(*ii*) *if* $|x_1| > 1$, $\sum a_i$ *diverges, and*
(*iii*) *if* $|x_1| = 1$, *the test fails.*

The proof of Theorem 6 is very similar to the proof of Theorem 5, so it is omitted.

If $a_i \neq 0$, Equation (11) may be replaced by the assumption that $\lim\, (x_1^2 a_{i-1} + x_1 x_2 a_i)/(-a_{i+1})$ exists and equals 1.

7. Numerical Examples

Three numerical examples illustrate the ability of our tests to determine correctly the radius of convergence for solutions to non-linear differential equations. For each example, a 30-term series was generated recursively using the program produced by Chang's ATS compiler. Computations were done using single precision arithmetic on an IBM 360/65.

These examples are intended to illustrate the ability of the tests discussed in this paper to recognize the location and order of the singularity(ies) on the circle of convergence. Work is continuing to improve the programming efficiency of these tests, to incorporate them fully into the ATS program, and to use the information they provide to develop a variable order, variable step method. The details and the performance of that method will be the subject of a later paper.

Example 1. First Painlevé transcendent.

$$y'' = 6y^2 + x; \quad y(0) = 1,\ y'(0) = 0.$$

TABLE 3
Pole location and order for $y'' = 6y^2 + x$

x_0	Three-term test R_c	Pole	Order	Ratio test R_c	Pole	Order
0·0	1·2230†	1·2230	—	1·3636	1·3636	3·1835
0·1	1·0962	1·1962	2·4030	1·0514	1·1514	1·0785
0·2	0·9969	1·1969	2·0033	0·9636	1·1636	1·0837
0·3	0·9037	1·2037	2·0001	0·8736	1·1736	1·0904
0·4	0·8083	1·2083	2·0000	0·7814	1·1813	1·0981

† Three-term test failed due to the proximity of the secondary pole. This estimate is from the five-term test.

Poles at $x = 1{\cdot}2068$ and $x = -1{\cdot}256$ are part of a sequence of poles of order two on the real axis. Table 3 contains a comparison of the estimates for the radius of convergence (R_c) and the order of the pole given by the three-term test with those given by the ratio test and Hadamard's order estimate.

Example 2. Singularity with fractional order.

$$y'' = (y')^4 + y; \quad y(0) = 1,\ y'(0) = 0.$$

This equation has a singularity of order $-2/3$ at $x = 1{\cdot}01953$. At each step, R_c was estimated by the three-term test. The series was then summed using a step of approximately 0·6 times the estimated radius of convergence. The first four steps are shown in Table 4.

Example 3. Conjugate pairs.

$$y'' = y - (y')^2; \quad y(0) = 0{\cdot}284,\ y'(0) = -0{\cdot}1.$$

y' has a sequence of simple poles in conjugate pairs. The first pair occurs at $x = 3{\cdot}05 \pm 3{\cdot}05i$. At each step, R_c was estimated by the four-term test and a step of approximately 0·6 times this estimate was taken. Figure 1 shows the circles of convergence estimated at each of the six steps from 0 to 10.

TABLE 4
Pole location and order for $y'' = (y')^4 + y$

x_0	Three-term test R_c	Pole	Order	Ratio test R_c	Pole	Order
0·00	0·9859†	0·9859	—	—	—	—
0·57	0·4495	1·0195	−0·6688	0·4837	1·0537	0·8012
0·83	0·1895	1·0195	−0·6670	0·4492	1·2792	0·7853
0·94	0·0795	1·0195	−0·6667	0·4453	1·3853	0·7617

† The three-term and ratio tests fail because all even terms of the series are 0. This estimate is from the five-term test.

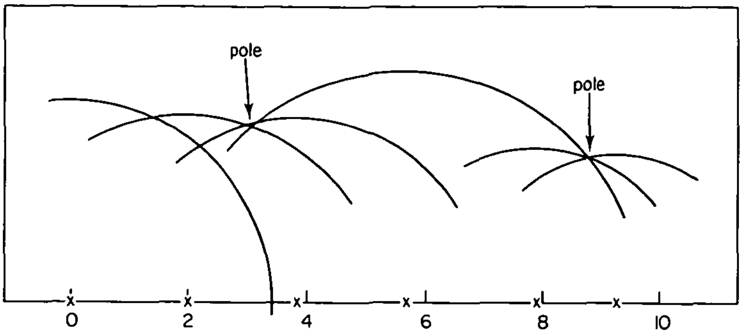

FIG. 1. Four-term test applied to $y'' = y - (y')^2$.

8. Summary

We have given three tests for convergence of a series. These tests are well suited for recognizing the radius of convergence of a series under program control. These tests do not work if the closest singularities do not occur at a single point or at a pair of conjugate points. More advanced convergence tests may be developed by considering model problems with three or more singularities. Such advanced tests are unnecessary, however, since a small step away from the original point of expansion yields a series with one or two closest singularities. The tests given in this paper also fail for functions whose primary singularity is essential. We are continuing our study of methods for determining the radius of convergence of the power series for functions of this type.

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REFERENCES

BARTON, D., WILLERS, I. M. & ZAHAR, R. V. M. 1970 Taylor series methods for ordinary differential equations—an evaluation. In *Proceedings of the Mathematical Software Symposium*, pp. 369–390. Lafayette, Indiania: Purdue University.

CHANG, Y. F. 1968 Convergence properties of Taylor series, and applications in numerical integration, unpublished class notes.

CHANG, Y. F. 1974 Automatic solution of differential equations. In *Lecture Notes in Mathematics, Number 430, Constructive and Computational Methods for Differential and Integral Equations*, D. L. Colton and R. P. Gilbert, Ed., pp. 61–94. New York: Springer-Verlag.

GIBBONS, A. 1960 A program for the automatic integration of differential equations using the method of Taylor series. *Comput. J.*, **3**, 108–111.

GOLOMB, M. 1943 Zeros and poles of functions defined by Taylor series. *Bull. AMS* **49**, 581–592.

HADAMARD, J. 1892 Essai sur l'étude des functions données par leur développement de Taylor. *J. Math. Pures Appl.* **8**, 101–186.

HENRICI, P. 1974 *Applied and Computational Complex Analysis*. Vol. 1. New York: Wiley.

KREYSIG, E. 1972 *Advanced Engineering Mathematics*. 3rd Edit. New York: Wiley.

LEAVITT, J. A. 1966 Methods and applications of power series. *Math. Comput.* **20**, 46–52.
MILLER, J. C. P. 1946 *The Airy Integral*. Part-Vol. B, British Association for the Advancement of Science. London: Cambridge University Press.
MOORE, R. E. 1966 *Interval Analysis*. Englewood Cliffs, New Jersey: Prentice-Hall.
OSTROWSKI, A. M. 1966 *Solutions of Equations and Systems of Equations*. 2nd Edit. New York: Academic Press.