SOME QUESTIONS ABOUT ZFC

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Remark 0.1. The formulation of the ZFC axioms being used are those given in Jech [3]. The difference between their presentation in Jech and Kunen [1] is that in Jech's formulation, the sets that are taken to have more stringent constraints. For example, Pairset in Jech states that for any two elements there is a set which contains *only* those two elements. In Kunen, the set just has to contain the two fixed elements, but can contain others as well. One might guess that some authors might not pay attention much to such details because Separation—formulated appropriately— can show the existence of the sets with more stringent contraints

1. Which axioms of ZFC are true in $\mathbb{N}_{set} = (\omega; \in)$?

Some important definitions and properties of ω. The proofs of the properties can be found in the appendix.

Definition 1.1. α is a *natural number (or finite ordinal)* iff $\forall \beta \leq \alpha(\beta = 0 \vee \beta$ is a successor ordinal).

Definition 1.2. ω is the class of all natural numbers (finite ordinals).

Definition 1.3. For any finite ordinal, x, we define the *successor ordinal* $S(X)$ given by

$$Y = S(x) \text{ iff } \mathbb{N}_{set} \models (x < Y \wedge \forall z(x < z \implies Y \leq z))$$

Remark 1.4. This definition of successor ordinal given picks out the same elements in the model as the usual one of $S(x) := (x \cup \{x\})$, but they are not logically equivalent.

Lemma 1.5. *ω is well-founded.*

Proposition 1.6. $2 = \langle 0, 0 \rangle = \{0, \{0\}\}$ *is the only ordered pair.*

Corollary 1.7. *There are no relations in $\mathbb{N}_{set}$. Consequently, there are no functions or well-orderings in $\mathbb{N}_{set}$.*

1.1. $\mathbb{N}_{set} \models$ **Axiom of Extensionality.**

Proof. This follows easily because ω is transitive and any transitive class satisfies Extensionality. □

1.2. $\mathbb{N}_{set} \models$ **Axiom of Foundation.**

Proof. Since ω is well-founded under $\in$, then Foundation holds. □

1.3. $\mathbb{N}_{set} \nvDash$ **Axiom of Separation.**

Proof. Let $\varphi(x)$ be the wff (where x is free) that is true in $\mathbb{N}_{set}$ iff x is 0 or 2. Define $\varphi(x)$ by

$$\varphi(x) \coloneqq (x = 0 \vee x = 2)$$

The set $|\varphi(x)| = \{x \in 10 : \varphi(x)\} = \{0, 2\}$ does not exist in $\mathbb{N}_{set}$ because it is not a finite ordinal. □

1.4. $\mathbb{N}_{set} \nvDash$ **Axiom of Pairing.**

Proof. As mentioned previously, the set $\{0, 2\}$ does not exist in $\mathbb{N}_{set}$ because it is not a finite ordinal. □

Remark 1.8. Pairing is true in $\mathbb{N}_{set}$ using the less restrictive definition where the set can contain other elements besides the two in consideration.

1.5. $\mathbb{N}_{set} \vDash$ **Axiom of Union.**

Proof. Suppose $\mathscr{F}$ is a family of sets. Since $\mathscr{F}$ is a finite ordinal, then for every element $x \in \mathscr{F}$, $x \subset \mathscr{F}$. Thus $\mathscr{F} = \bigcup \mathscr{F}$. □

1.6. $\mathbb{N}_{set} \nvDash$ **Axiom of Replacement.**

Proof. Let φ be the wff that defines the function $f(x) = x + 1$ given by

$$\varphi(x, y) \coloneqq (y = S(x))$$

Let $A = 1 = \{0\}$ and note that the hypothesis of Replacement holds for $f\big|_A$, that is $\mathbb{N}_{set} \vDash \varphi(0, 1)$. But $Ran(f) = \{1\}$ is not a finite ordinal, thus it does not exist. Therefore Replacement fails. □

1.7. $\mathbb{N}_{set} \nvDash$ **Axiom of Infinity.**

Proof. Suppose to the contrary that

$$\mathbb{N}_{set} \vDash \exists x \Big(0 \in x \wedge \forall y \in x \big(S(y) \in x\big)\Big).$$

Let Ω be a witness to the statement above. Since Ω is a finite ordinal, then for every $\beta \leq \Omega(\beta = 0 \vee \beta$ is a successor ordinal). Choose $\beta = \Omega \leq \Omega$. Therefore Ω is either 0 or a successor ordinal. Ω is clearly not 0. If Ω is a successor ordinal, then $\Omega = S(\gamma)$ for some γ. Since $\gamma \in \Omega$ and Ω satisfies Infinity, then $S(\gamma) = \Omega \in \Omega$. This is a contradiction. □

1.8. $\mathbb{N}_{set} \vDash$ **Axiom of Powerset.** Recall the definition of the subset relation and $\mathscr{P}(X)$,

$$A \subseteq B \text{ iff } \mathbb{N}_{set} \vDash (\forall x (x \in A \implies x \in B))$$
$$Y = \mathscr{P}(X) \text{ iff } \mathbb{N}_{set} \vDash (\forall z (z \subseteq X \iff z \in Y))$$

Proof. Let X be a finite ordinal. We claim that $S(X) = \mathscr{P}(X)$. Observe that a finite ordinal z is a subset of X iff $z < X$, and $z < X$ iff $z \in X$. If $z = X$, then $z \in S(X)$. Thus, $S(X) = \mathscr{P}(X)$. □

1.9. $\mathbb{N}_{set} \nvDash$ **Axiom of Choice (Well-Ordering Theorem).**

Proof. This follows from Corollary 1.5. Since no relation exists, then well-orders don't exist. □

2. Which axioms of ZFC are true in $\mathbb{N}_< = (\mathbb{N}; <)$?

We will first rewrite and interpret the axioms of ZFC where "$\in$" is replaced with "$<$" which means strictly less than. Then we will analyze which axioms hold in $\mathbb{N}_<$. Lastly, we will see if we can construct a model for ZFC under this interpretation of "$\in$". The author makes no promises as to whether this is a fruitful endeavor.

Definition 2.1. An *ordered structure* is a structure $(M; R)$, where R is a strict-partial order. That is, R is irreflexive, transitive, and anti-symmetric.

Definition 2.2. Let x be an element in an ordered structure. The *downward closure of* x, denoted "$dwcl(x)$", is the subset of the structure that contains all the elements less than x.

Remark 2.3. The $dwcl(x)$ is definable in an ordered structure by $dwcl(x) \coloneqq (y < x)$.

Definition 2.4. Let x be an element in an ordered structure M. The *successor of* x, denoted "$S(x)$", is the unique element, if it exists, given by

$$Y = S(x) \text{ iff } M \models (x < Y \wedge \forall z(x < z \implies Y \leq z))$$

Axiom of Extensionality:

$$\forall x \forall y \Big(\forall z (z < x \iff z < y) \implies x = y \Big)$$

Informally this says that two elements are equal if they are bigger than precisely the same elements. This will hold in a strict total order or a poset in the shape of a tree.

Axiom of Foundation:

$$\forall x [\exists y (y < x) \implies \exists y (y < x \wedge \neg \exists z (z < x \wedge z < y))]$$

This still seems to capture the spirit of Foundation under "$\in$". It says that in a strict ordering (total or partial), if some element x is not minimal, then there is a minimal element in a chain that contains x. In other words, every chain bottoms out.

Axiom of Comprehension: For each formula φ with free variables among $x, z, w_1, ..., w_n$,

$$\forall z \forall w_1, ..., w_n \exists y \forall x (x < y \iff x < z \wedge \varphi)$$

Informally this says that for every property φ and every element z, there is some y whose downward closure is *precisely* the set of elements in $dwcl(z)$ that satisfy φ. This seems to be false in any structure where you can define an interval that isn't bounded below by some minimal element. The counterexample would be that $\varphi(x)$ expresses that x is in the interval and the interval is bounded below by something less than y.

Axiom of Pairing:

$$\forall x \forall y \exists z \forall w (w < z \iff w = x \vee w = y)$$

This is false, if x and y are not minimal. This seems like it's almost always false. The less restricted version states that for any pair of elements there is always a greater element. This requires any model to be infinite.

Axiom of Union:

$$\forall \mathscr{F} \exists A \forall Y \forall x (x < Y \wedge Y < \mathscr{F} \implies x < A)$$

Informally this says that for every element $\mathscr{F}$ there is an element A such that for every starting point Y less than $\mathscr{F}$, anything less than Y will also be less than A. This is trivially true in any ordered-structure because of transitivity. Let $A = \mathscr{F}$ or $A > \mathscr{F}$.

Axiom of Replacement: For each formula $\varphi(x, y, p)$,

$$\forall x \forall y \forall z \Big[(\varphi(x,y,p) \wedge \varphi(x,z,p) \implies y = z) \implies \forall X \exists Y \forall y (y \in Y \iff (\exists x < X)\varphi(x,y,p)\Big]$$

This states that for any class function and any domain $dwcl(X)$, $R(f|_{dwcl(X)}) = dwcl(Y)$ for some Y. This is false in any strict total order with a minimal element for the function $f(x) = S(X)$. The $dwcl(Y)$ for any Y will contain the least element which won't be in the range.

Remark 2.5. The wff that defines 0 will now define the minimal element in the ordered structure which may not be unique.

Axiom of Infinity:

$$\exists x \Big(0 < x \wedge \forall y < x (S(y) < x) \Big)$$

This says that there is an inaccessible maximal element to some chain with a minimal element. An element is inaccessible if it has no immediate predecessor.

Axiom of Powerset: To write this axiom out, we need to reinterpret the subset wff.

$$A \subseteq B \text{ iff } \forall x (x < A \implies x < B)$$

Which is now just stating that the $dwcl(A) \subseteq dwcl(B)$. Note that it need not be the case that $A < B$.

$$\forall x \exists y \forall z (z \subset x \implies z < y)$$

Informally this says that for any element x there is an upperbound to all of its subchains. This seems trivially true in a total order by letting the upperbound be x itself. This is not true for posets.

Axiom of Choice: It would take a lot of work to rewrite Choice as the Well-Ordering Theorem so I'll rewrite the existence of a choice function.

$$\forall X [0 \nless x \implies \exists f \forall A < X (f(A) < A)]$$

I haven't the slightest idea what this says. Nonetheless, if there is a least element 0, then it is vacuously true. If there isn't a least element, then I am not sure yet.

Lemma 2.6. *Every natural number is definable as the nth successor of 0, $S^n(0)$, where 0 is defined as the least element.*

2.1. $\mathbb{N}_< \vDash$ **Axiom of Extensionality.**

Proof. It is not hard to see that this axiom holds in any strict total order because of comparability. But it can fail in a strict partial order. $\square$

2.2. $\mathbb{N}_< \vDash$ **Axiom of Foundation.**

Proof. This statement holds in any strict total order with a least element. $\square$

2.3. $\mathbb{N}_< \nvDash$ **Axiom of Comprehension.**

Proof. As a counterexample, let $z = 10$ and $\varphi(x) \coloneqq (5 < x < 10)$. There does not exist a y such that $x < y$ iff $5 < x < 10$. $\square$

2.4. $\mathbb{N}_< \nvDash$ **Axiom of Pairing.**

Proof. As a counterexample, take $x = 2$ and $y = 3$, there is no element z such that for anything less than z it is either 2 or 3. $\square$

Remark 2.7. The less restrictive version of Pairing holds in $\mathbb{N}_<$.

2.5. $\mathbb{N}_< \vDash$ **Axiom of Union.**

Proof. Let x be an element in $\mathbb{N}_<$ and let $x = A$. This statement is trivially true. $\square$

2.6. $\mathbb{N}_< \nvDash$ **Axiom of Replacement.**

Proof. Let $f(x) = S(x)$ and restrict f to $dwcl(5)$. Then the range of f is not the downward closure of any element because any downward closure contains 0, but the range of $f\big|_{dwcl(5)}$ does not contain 0. $\square$

2.7. $\mathbb{N}_< \nvDash$ **Axiom of Infinity.**

Proof. This is false since every element in $\mathbb{N}_<$ has an immediate predecessor. $\square$

2.8. $\mathbb{N}_< \vDash$ **Axiom of Powerset.**

Proof. Let x be an ordered element and let $y = x$. This makes the statement trivially true since $\square$

2.9. $\mathbb{N}_<$ **and Axiom of Choice.** This is left as an exercise to the reader.

3. Model Existence of ZFC(<)

Axiom of Pairing

$$\forall x \forall y \exists z \forall w (w < z \iff w = x \vee w = y)$$

Proposition 3.1. *There are no models of the Axiom of Pairing where the binary relation is a strict poset on the ordered-structure.*

Proof. Let M be an ordered-structure. Let $x = y$ be some element x_1 in an ordered-structure and suppose there exists an element $z = z_1$ such that Pairing holds. Because of irreflexivity z_1 must be different from x_1. Now let $x = x_1$ and $y = z_1$. Again, the irreflexivity of $<$ forces the witness z to be different from x_1 and z_1. Say it's some element z_2. Now issues arises for $x = z_2$ and $y = z_3$. There does not exist a z such that for anything, it is less than it iff it is either z_2 or z_3. Any z will be such that $z_1 < z$.
Note: If there is an arrow from x *to* z, then $x < z$.

$$x_1 \longrightarrow z_1 \longrightarrow z_2 \longrightarrow z_3$$

□

Corollary 3.2. *There are no models of ZFC where the binary relation is a strict poset.*

4. Membership Relation as Directed-Edge Relation

In this section we will interpret the $\in$-relation as the directed-edge relation on nodes of a directed graph.

Definition 4.1. xEy iff there is a edge *from* x *to* y

Remark 4.2. We will tend to say "*node x points to node y*" whenever xEy.

Definition 4.3. An *isolated node* is a node that has no nodes pointing to it.

Proposition 4.4. *$I(x)$ is definable*

$$I(x) \text{ iff } M \vDash (\neg \exists y (yEx))$$

Axiom of Extensionality:

$$\forall x \forall y \Big(\forall z (zEx \iff zEy) \implies x = y \Big)$$

This says that if two nodes have the same nodes pointing to them, then they are equal.

Axiom of Foundation:

$$\forall x [\exists y (yEx) \implies \exists y (yEx \wedge \neg \exists z (zEx \wedge zEy))]$$

If x is not an isolated node, then there is a node y that points to x that has a node pointing to it that does not point to x.

Axiom of Comprehension: For each formula φ with free variables among $x, z, w_1, ..., w_n$,

$$\forall z \forall w_1, ..., w_n \exists y \forall x (xEy \iff xEz \wedge \varphi)$$

For every node x, there is a node y where the nodes that point to it are precisely those that point to x and satisfy φ.

Axiom of Pairing:

$$\forall x \forall y \exists z \forall w (wEz \iff w = x \vee w = y)$$

For every pair of nodes x, y there is a node where only x and y point to it. This will require models to be infinite.

Axiom of Union:

$$\forall \mathscr{F} \exists A \forall Y \forall x (xEY \wedge YE\mathscr{F} \implies xEA)$$

For every node $\mathscr{F}$, there is a node A, where if x points to something that points to $\mathscr{F}$, then x points to A.

Axiom of Replacement: For each formula φ with free variables among $x, y, A, w_1, ..., w_n$,

$$\forall A \forall w_1, ..., w_n [\forall xEA \exists ! y \varphi \implies \exists Y \forall xEA \exists yEY \varphi]$$

Let φ be some class function. For every node A, if φ is a function on everything that points to A, then there is a node where everything that points to it is the range.

Axiom of Infinity: We have to see what the wff that defines $\varnothing$ defines under the directed-edge relation.

Definition 4.5. 0 is the unique set y such that $\forall x(\neg xEy)$

In other words, 0 is the only isolated node. So any model must have only one isolated node for this axiom to make sense.

Remark 4.6. The usual definition of $S(x) := x \cup \{x\}$ doesn't make much sense immediately. First, $\{x\}$ becomes the unique node that x points towards. Second, the element that has only x and x pointing to it (the "union") would be a node for which only x and $\{x\}$ point towards. With the restrictive definition of Pairing, this won't work as $\{x\}$ is supposed to be the only thing that x points towards. With the looser definition, we can make sense of $x \cup x$.

$$\exists x \Big(0Ex \wedge \forall yEx (S(y)Ex) \Big)$$

There is some node that is pointed to by the isolated node. Furthermore, if something points to x, then so does it's successor.

Axiom of Powerset: To write this axiom out, we need to reinterpret the subset wff. The new wff is $A \subset B$ iff

$$\forall x (xEA \implies xEB)$$

This states that $A \subset B$ iff everything that points to A also points to B.

$$\forall x \exists y \forall z (z \subset x \implies zEy)$$

This can be read as saying that for any node x, there is a node y such that if a node is a "subset" of x, then it points to y.

Axiom of Choice: It would take a lot of work to rewrite Choice as the Well-Ordering Theorem so I'll rewrite the existence of a choice function.

$$\forall X [0 \not{E}\, x \implies \exists f \forall AEX (f(A)EA)]$$

I haven't the slightest idea what this says.

5. Appendix

(1) **Proposition 1.6.** 2 is the only ordered pair in $\mathbb{N}_{set}$.

Proof. An ordered pair $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ has only two elements. 2 is an order pair. $2 = \langle 0, 0\rangle = \{0, \{0, 1\}\} = \{0, 1\} = 2$. We claim that 2 is the only set with two-elements and is therefore the only ordered pair.

Suppose that there exists a finite ordinal x with 2 elements such that $x \neq 2 = \{0, 1\}$. So then either $0 \notin x$ or $1 \notin x$. If $0 \notin x$, then x is not a finite ordinal. This is a contradiction. If $1 \notin x$, then $x = \{0, y\}$. Now $y \in x$, but $y \not\subseteq x$ so x is not transitive. This is a contradiction. □

(2) **Corollary 1.7.** There are no relations in $\mathbb{N}_{set}$. Consequently, there are no functions nor well-orderings in $\mathbb{N}_{set}$.

Proof. By **Lemma 1**, 2 is the only ordered pair. Since a relation is a set of ordered pairs, the only possible relation is $R = \{2\}$. This is not a set in $\mathbb{N}_{set}$ since it is not transitive. □

References

1. K. Kunen, *Set Theory: An Introduction to Independence Proofs*
2. H. B. Enderton, *A Mathematical Introduction to Logic*
3. T. Jech, *Set Theory*