DIRECTED HOMOLOGY

1. Overview

A *directed space* vaguely refers to some space with some notion of time flowing on it. Some concrete instances of directed spaces are spacetimes, directed graphs, oriented simplicial complexes, or oriented cube complexes. For concreteness, we will take a directed space to mean a *(pre) cubical set*. Here we give a definition for *directed homology*, the homology of a directed space, and prove some results. While ordinary homology can be seen as representing the various n-dimensional holes in a topological space, similarly can directed homology be seen as representing the various n-dimensional holes in a directed space where the holes have a consistent direction. We shall term these directed holes as $n-cycles$.

1.0.1. ***Precubical Set.*** A *precubical set* X consists of a family $(X_n)_{n\in\mathbb{N}}$ of sets, whose elements are called $n - cubes$ together with for all indices $n, i \in \mathbb{N}$ with $1 \leq i \leq n$, source and target maps

$$d^-_{n,i} : X_n \to X_{n-1} \text{ and } d^+_{n,i} : X_n \to X_{n-1}$$

respectively associating to an n-cube its *back* and *front face* in the ith direction, such that

$$d^\beta_{n,j} d^\alpha_{n+1,i} = d^\alpha_{n,i} d^\beta_{n+1,j+1}$$

[1].

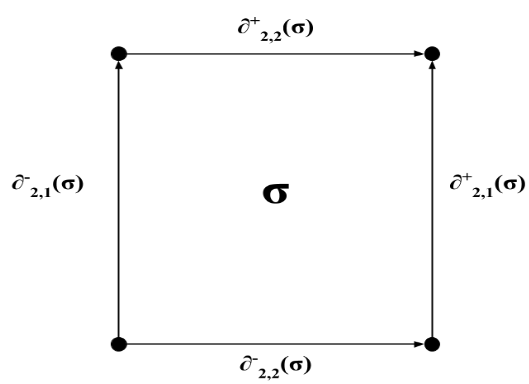

Figure 1. A 2-cube with the action of the source and target maps.

We build up the definitions by first stating them for the 1-dimensional case and then extending them to higher dimensions.

1.1. **Directed 1-Cycles.** Consider a directed space X. Fix a *commutative monoid*, a set M equipped with a commutative, associative, and unital operation $+ : M^2 \to M$.

1.1.1. ***Chain Monoid***. Define $C_n(X; M)$ to be the commutative monoid $\mathbb{N}[X_n] \otimes M$, the $\#X$-fold direct sum of M.

Concretely, $C_n(X; M)$ is the set of linear combinations $\sum_i \lambda_i \sigma_i$, where the σ_i's are abstract n-cubes and λ_i's are elements of M, with a natural addition operation on it. The source and target maps $d_1^-, d_1^+ : X_1 \to X_0$ uniquely extend to homomorphisms $\partial_1^-, \partial_1^+ : C_1(X; M) \to C_0(X; M)$.

1.1.2. *1-D Equalizer*. Define $Z_1(X; M)$ to be the *submonoid*

$$Z_1(X; M) = \text{equalizer}(\partial_1^-, \partial_1^+) = \{x \in C_1(X; M) \mid \partial_1^-(x) = \partial_1^+(x)\}$$

Note: If X is a 1-precubical set, i.e. X has no higher dimensional cubes, than we can take $H_1(X; M)$ to be $Z_1(X; M)$.

1.2. **Higher Directed Homology.** For a precubical set X, define the source and target maps $\partial_n^-, \partial_n^+ : C_n(X; M) \to C_{n-1}(X; M)$ by uniquely extending the equations

$$\partial_n^-(\sigma) = \sum_{i=1}^n d_{n,i}^-(\sigma) \quad \partial_n^+(\sigma) = \sum_{i=1}^n d_{n,i}^+(\sigma)$$

to homomorphisms as before. Here σ denotes an abstract n-cube and $d_{n,i}^-(\sigma)$ refers to the ith back face and $d_{n,i}^+(\sigma)$ refers to the ith front face of σ.

1.2.1. ***Equalizer***. Define $Z_n(X; M) = \text{equalizer}(\partial_n^-, \partial_n^+)$. Define $\equiv_n$ to be the congruence relation on $Z_n(X; M)$ generated by the relations $z_1 \equiv_n z_2$ if there exists $\sigma \in C_{n+1}(X; M)$ such that $z_1 + \partial_{n+1}^-(\sigma) = z_2 + \partial_{n+1}^+(\sigma)$.

1.2.2. ***Directed Homology***. Define

$$H_n(X; M) = Z_n(M)/ \equiv_n .$$

2. Some Results

Note: There is a natural interpretation of a 1-precubical set as a directed graph. The 1-cubes are the directed edges and the 0-cubes are the vertices. We believe that discussing them as such will make the following results and proofs more intuitive for the reader. Thus we will often talk about 1-cubical sets as directed graphs.

2.1. **Proposition.** Let $G = (V, E)$ be a directed graph i.e. a 1-precubical set. $H_1(G; \mathbb{R}_{\geq 0})$ is non-trivial iff G contains a cycle.

Proof. Let G be a directed graph.
($\Longrightarrow$) Suppose $H_1(G; \mathbb{R}_{\geq 0})$ is non-trivial. Then $\exists x \in Z_1(G; \mathbb{R}_{\geq 0})$ a non-trivial element such that $\partial_1^-(x) = \partial_1^+(x)$ where $x = \sum \lambda_i \sigma_i$, interpreted as a dipath. This means,

$$\sum_{i=1}^{n} \lambda_i d_1^-(\sigma_i) = \sum_{i=1}^{n} \lambda_i d_1^+(\sigma_i)$$

Since the two sums are equal, every vertex on the L.H.S appears on the R.H.S, i.e. every vertex must be source and a target. Now we will show that this a sufficient condition for graph to contain a directed cycle.

We build the cycle inductively. Let the first vertex v_1 in the cycle be $v_1 = d_1^-(\sigma_1)$ and the second vertex $v_2 = d_1^+(\sigma_1)$, then since every vertex is a source and target we can find a σ_j such that $d_1^-(\sigma_j) = v_2$. So far the dipath is $\rho = (v_1, v_2, d_1^+(\sigma_j))$. If $d_1^+(\sigma_j) = v_1$, then we are done and have formed a directed cycle. If not, then repeat the process of adding a vertex and checking if it forms a cycle.

This process will terminate since there are finitely many vertices. To see that at some step there must be a cycle formed, suppose that this process reaches the last vertex v_l not already in the dipath. Since every vertex is a source then $v_l = d_1^-(\sigma_i)$ and $v_k = d_1^+(\sigma_i)$. Observe that since v_k is already in the dipath by supposition, then this forms a cycle $\rho = (v_k, ..., v_l, v_k)$.

($\Longleftarrow$) Suppose G has a cycle, now given in terms of edges instead of vertices, $\rho = (\sigma_1, ..., \sigma_n)$. Then $x = \sum_i \sigma_i \in C_1(G; \mathbb{R}_{\geq 0})$. Observe that for $\sigma_i \in \rho$ we have that $d_1^-(\sigma_i) = d_1^+(\sigma_{i-1})$ for $i > 1$ and $d_1^-(\sigma_1) = d_1^+(\sigma_n)$. Now we have that,

$$\partial_1^-(x) = \sum_{i=1}^{n} d_1^-(\sigma_i) = \sum_{i=2}^{n} d_1^+(\sigma_{i-1}) + d_1^+(\sigma_n) = \sum_{i=1}^{n} d_1^+(\sigma_i) = \partial_1^+(x)$$

Thus, $x \in Z_1(G; \mathbb{R}_{\geq 0})$.

□

2.2. **Def.** Tensor Product of Commutative Monoids: $N \otimes M$

$$N \otimes M = F(N \times M)/\approx$$

where $F(N \times M)$ is the free commutative monoid on the cartesian product of N and M, and $\approx$ is the congruence generated by the bilinearity relation $\sim$. The binary operation on $N \otimes M$ is simply $[n][m] = [nm]$.

2.2.1. *Bilinearity Relation.* Define $\sim$ on $F(N \times M)$ given by,

$$(n_1 n_2, m) \sim (n_1, m)(n_2, m) \qquad (n, m_1 m_2) \sim (n, m_1)(n, m_2)$$

$$(0_N, m) \sim (0_N, 0_M) \sim (n, 0_M)$$

Lastly, $\approx$ is generated by taking the intersection of all the relations that subsume $\sim$.

2.3. **Conjecture.** $H_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0} \cong H_1(X; \mathbb{R}_{\geq 0})$

$$\begin{tikzcd}
Z_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0} \arrow[r, "\varphi"] \arrow[d, "(f, id_{\mathbb{R}_{\geq 0}})"'] & Z_1(X;\mathbb{R}_{\geq 0}) \arrow[d, "g"] \\
H_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0} \arrow[r, "h"] & H_1(X;\mathbb{R}_{\geq 0})
\end{tikzcd}$$

The diagram above is for illustrating how we will prove this theorem. To prove the theorem, we will show that φ is an isomorphism, φ preserves the equivalence relation $\equiv_1$, and that $h([x]) = g(\varphi(x))$ is an isomorphism.

2.3.1. ***Lemma***. $Z_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0} \cong Z_1(X; \mathbb{R}_{\geq 0})$

We remind the reader that the elements of $Z_1(X; \mathbb{N}) \times \mathbb{R}_{\geq 0}$ are ordered pairs (x, α) where $x = \sum_i \lambda_i \sigma_i \in Z_1(X; \mathbb{N})$ and $\alpha \in \mathbb{R}_{\geq 0}$, and that elements of the tensor product $Z_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ are equivalence classes of these ordered pairs, $[(x, \alpha)]_{\otimes}$. The authors believe it is hard to keep track of all the notation that will follow.

Proof. Define the function $\phi : F\Big(Z_1(X; \mathbb{N}) \times \mathbb{R}_{\geq 0}\Big) \to Z_1(X; \mathbb{R}_{\geq 0})$ given by

$$\phi((x, \alpha)) = \alpha x = \alpha \sum_{i=1}^{n} \lambda_i \sigma_i$$

Where $x = \sum_i \lambda_i \sigma_i \in Z_1(X; \mathbb{N})$ and $\alpha \in \mathbb{R}_{\geq 0}$. For longer words, extend ϕ linearly.

We claim that ϕ defines an isomorphism φ between $Z_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ and $Z_1(X; \mathbb{R}_{\geq 0})$ where $\varphi([z]) \coloneqq \phi(z)$.

Well-defined. To make sure φ is well-defined we must check that ϕ respects the bilinearity relation defined on the tensor product. That is, if $z_1 \sim z_2$, then $\phi(z_1) = \phi(z_2)$. Suppose $x, y \in Z_1(X; \mathbb{N})$ and that $\alpha, \beta \in \mathbb{R}_{\geq 0}$.

(i) $(x + y, \alpha) \sim (x, \alpha)(y, \alpha)$. Observe that

$$\phi(x + y, \alpha) = \alpha(x + y) = \alpha x + \alpha y = \phi(x, \alpha) + \phi(y, \alpha) = \phi\Big((x, \alpha)(y, \alpha)\Big).$$

(ii) $(x, \alpha + \beta) \sim (x, \alpha)(x, \beta)$. Observe that

$$\phi(x, \alpha + \beta) = (\alpha + \beta)x = \alpha x + \beta x = \phi(x, \alpha) + \phi(x, \beta) = \phi\Big((x, \alpha)(x, \beta)\Big).$$

(iii) $(0, \alpha) \sim (0, 0) \sim (x, 0)$. Observe that

$$\phi(0, \alpha) = \alpha(0) = 0 = \phi(0, 0) = 0(x) = \phi(x, 0).$$

Since ϕ is bilinear, then ϕ will preserve the congruence generated by the bilinearity relation and φ will be well-defined. For example, suppose you have some equivalence class $[H_1]_\otimes$ in $Z_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0}$. Let z and w be arbitrary representatives, then z and w can be "factored" into non-trivial elements of the cartesian product, $Z_1(X;\mathbb{N}) \times \mathbb{R}_{\geq 0}$, say $z = z_1z_2...z_n$ and $w = w_1w_2...w_n$. By the tensor product construction, $z \approx_\otimes w$ because $z_i \approx_\otimes w_i$ for all i. Note that since $z_1 \approx_\otimes w_1$ and $z_2 \approx_\otimes w_2$, then $z_1z_2 \approx_\otimes w_1w_2$. Now since $\phi(z_1) = \phi(w_1)$, $\phi(z_2) = \phi(w_2)$, and z_1 then $\phi(z_1z_2) = \phi(z_1) + \phi(z_2) = \phi(w_1) + \phi(w_2) = \phi(w_1w_2)$. This shows that ϕ is well-defined for equivalence classes of words that have two non-trivial factors. This same argument works for equivalence classes where representatives are of arbitrary length.

Homomorphism. That φ is a homomorphism follows easily from its definition.

$$\varphi([x][y]) = \varphi([xy]) = \phi(xy) = \phi(x) + \phi(y) = \varphi([x]) + \varphi([y])$$

Surjective. Let $z \in Z_1(X;\mathbb{R}_{\geq 0})$, then $z = \sum_{i=1}^{n} \lambda_i\sigma_i$ and consider the simple case in graph theory terms where z represents a directed cycle $c = (\sigma_0, ..., \sigma_n)$ and σ_i is an edge. More precisely, z cannot be partitioned into $z_1 + z_2$ such that both $z_1, z_2 \in Z_1(X;\mathbb{R}_{\geq 0})$, let us call such a sum a *simple sum*. We claim that all λ_is are equal. To see this, look at some $\sigma_i \in c$. In order for $\partial_-(z) = \partial_+(z)$, it is necessary that $\lambda_i = \lambda_{i+1}$ for all i.

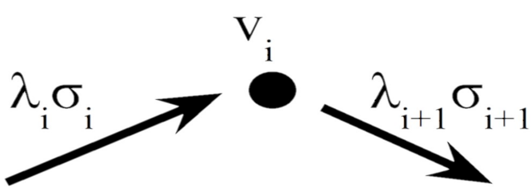

FIGURE 2. A directed cycle needs the same "flow" (coefficient) going in and out of every vertex to be in the equalizer $Z_1(X;\mathbb{R}_{\geq 0})$.

Since $\partial_-(z) = \lambda_{i+1}v_i + ...$ and $\partial_+(z) = \lambda_i v_i + ...$, if $\lambda_i \neq \lambda_{i+1}$, then $\lambda_{i+1}v_i \neq \lambda_i v_i$. Moreover $\partial_-(z) \neq \partial_+(z)$. Hence, it must be the case that $\lambda_i = \lambda_{i+1}$. Due to this fact, for a simple sum we will now use the notation $\sum_i \lambda\sigma_i$, eliminating the index from the λ.

2.3.1.1 Lemma. Any element of $Z_1(X;\mathbb{R}_{\geq 0})$ is a sum of simple sums.

Proof. Incomplete. Let $z \in Z_1(X;\mathbb{R}_{\geq 0})$, and not a simple sum. We know from Proposition 2.1 that if an element in $Z_1(X;\mathbb{R}_{\geq 0})$ is non-trivial, then it contains a simple sum. Let z_1 denote the simple sum contained in z. Since z is in $Z_1(X;\mathbb{R}_{\geq 0})$ we know that

$$\partial_-(z) = \partial_+(z)$$

Since the boundary maps are linear this implies that

$$\partial_-(z_1) + \partial_-(z_2) = \partial_+(z_1) + \partial_+(z_2)$$

With coefficients in $\mathbb{R}_{\geq 0}$ (or $\mathbb{N}$) the equalizer $Z_1(-;M)$ is a cancellative monoid. Thus we have that,

$$\partial_-(z_2) = \partial_+(z_2)$$

We are left with another non-trivial element which again by Proposition 2.1 contains a simple sum. This process can be repeated until the sum is entirely decomposed into a sum of simple sums. □

With the claim in hand, we now show φ is surjective. If $z = \sum_{i=1}^{n} \lambda\sigma_i$ is a simple sum, then simply $\varphi([(z', \lambda)]) = z$ where $z' = \sum_{i=1}^{n} \sigma_i$, the same σ_i's of z, and $[(z', \lambda)] \in Z_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0}$. We can construct (z', λ) because we know all λ_is equal each other, namely they equal λ and $z' \in Z_1(X; \mathbb{N})$. Similarly, if z is a sum of sum of simple sums, then by Lemma 2.3.1.1, z can be decomposed into its component simple sums $z = z_1 + ... + z_k$, and $[(z_1', \lambda_1)]...[(z_k', \lambda_k)]$ constructed in the same way as before will map to z under φ.

Injective. Suppose $[x], [y] \in Z_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ such that $x \not\approx y$ (*i.e* $[x] \neq [y]$) where $x = (x', \alpha)$ and $y = (y', \beta)$. Let us first consider the simple case where $x' = \sum a\sigma_i$ and $y' = \sum b\sigma_i$ are simple sums. We claim that $\varphi([x]) \neq \varphi([y])$. To see this, suppose to the contrary that $\varphi([x]) = \varphi([y])$. Then,

$$\varphi([x]) = \varphi([y])$$

$$\varphi([(x', \alpha)]) = \varphi([(y', \beta)])$$

$$\alpha x' = \beta y'$$

$$\alpha \sum_{i=0}^{n} a\sigma_i = \beta \sum_{i=0}^{n} b\sigma_i$$

$$\sum_{i=0}^{n} (\alpha \cdot a)\sigma_i = \sum_{i=0}^{n} (\beta \cdot b)\sigma_i \tag{1}$$

This means that $\alpha \cdot a = \beta \cdot b$ and moreover in $Z_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ we know that $[(\sum \sigma_i, \alpha \cdot a)] = [(\sum \sigma_i, \beta \cdot b)]$ where the sum of σ_i's are the same ones from (1). From the congruence relation of the tensor product,

$$(\sum \sigma_i, \alpha \cdot a) \approx (\sum \sigma_i, \beta \cdot b)$$

$$a(\sum \sigma_i, \alpha) \approx b(\sum \sigma_i, \beta)$$

$$(a \sum \sigma_i, \alpha) \approx (b \sum \sigma_i, \beta)$$

$$(x', \alpha) \approx (y', \beta)$$

$$x \approx y$$

This is contrary to assumption. Thus, φ is injective for simple sums.

Now we must consider the case where $x = x_1 + x_2 + ... + x_n$ and $y = y_1 + y + 2 + ... + y_n$ are sums of simple sums where $\beta < \alpha$. Suppose that $\varphi(x, \alpha) = \varphi(y, \beta)$. Then we have that

$$\alpha(x_1 + x_2 + ... + x_n) = \beta(y_1 + y_2 + ... + y_n)$$

$$x_1 + x_2 + ... + x_n = \frac{\beta}{\alpha}(y_1 + y_2 + ... + y_n)$$

[Missing remainder of this argument for the general case case]

□

2.3.2. *Lemma.* φ **preserves** $\equiv_1$**.**

This is to say that if $[x], [y] \in Z_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0}$, $x = (x', \alpha)$ and $y = (y', \beta)$ such that $x' \equiv_1 y'$ and $\alpha = \beta$, then $\varphi([x]) \equiv_1 \varphi([y])$. This means that φ induces the map h in the diagram below.

Let f be the quotient map $f : Z_1(X;\mathbb{N}) \mapsto Z_1(X;\mathbb{N})/\equiv_1$ and g be the quotient map $g : Z_1(X;\mathbb{R}_{\geq 0}) \mapsto Z_1(X;\mathbb{R}_{\geq 0})/\equiv_1$.

$$\begin{tikzcd}
Z_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0} \arrow[r, "\varphi"] \arrow[d, "(f, id_{\mathbb{R}_{\geq 0}})"'] & Z_1(X;\mathbb{R}_{\geq 0}) \arrow[d, "g"] \\
H_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0} \arrow[r, "h"] & H_1(X;\mathbb{R}_{\geq 0})
\end{tikzcd}$$

Proof. Let $[x], [y] \in Z_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0}$, $x = (x', \alpha)$ and $y = (y', \alpha)$ such that $x' \equiv_1 y'$. This means $\exists c \in C_2(X;\mathbb{N})$ such that,

$$x' + \partial_-(c) = y' + \partial_+(c) \tag{2}$$

We claim that the existence of c guarantees the existence of a $c' \in C_2(X;\mathbb{R}_{\geq 0})$ that bears witness to the statement $\varphi([x]) \equiv_1 \varphi([y])$. This is quite simple to see, simply multiply α to both sides in (2)

$$\begin{gathered}
\alpha x' + \alpha\partial_-(c) = \alpha y' + \alpha\partial_+(c) \\
\alpha x' + \partial_-(\alpha c) = \alpha y' + \partial_+(\alpha c) \\
\varphi([x]) + \partial_-(\alpha c) = \varphi([y]) + \partial_+(\alpha c) \\
\varphi([x]) \equiv_1 \varphi([y])
\end{gathered}$$

As seen above, $c' = \alpha c \in C_2(X;\mathbb{R}_{\geq 0})$. ∎

2.3.3. *Conjecture.* $h : H_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0} \mapsto H_1(X;\mathbb{R}_{\geq 0})$ is an isomorphism.

Proof. Incomplete. It might be helpful to mention at this point what objects in $H_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ look like as they are a symbolic mess. $H_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ is composed of equivalence classes from the tensor product. The classes are composed of words where each letter is an ordered pair $([x]_{\equiv_1}, \lambda)$. The first coordinate is an equivalence class of formal sums in $Z_1(X;\mathbb{N})$ under the equivalence relation $\equiv_1$. The second coordinate is a non-negative real number.

h is given by the equation, $h\Big([([x]_{\equiv_1}, \alpha)]_{\otimes}\Big) := g(\varphi((x, \alpha)))$. For equivalence classes of longer words, extend this equation linearly. Implicitly, the well-definedness of h is in large part due to φ preserving $\equiv_1$.

Well-Defined To make sure h is well-defined we must ensure that $g \circ \varphi$ preserves the bilinear relation $\sim$. That is, if $z_1 \sim z_2$, then $(g \circ \varphi)(z_1) = (g \circ \varphi)(z_2)$.

(i) $([x + y]_{\equiv_1}, \alpha) \sim ([x]_{\equiv_1}, \alpha)([y]_{\equiv_1}, \alpha)$. Observe that

$$\begin{gathered}
(g \circ \varphi)(x + y, \alpha) = [\alpha(x + y)]_{\equiv_1} = [\alpha x]_{\equiv_1} + [\alpha y]_{\equiv_1} \\
= (g \circ \varphi)(x, \alpha) + (g \circ \varphi)(y, \alpha) = (g \circ \varphi)\Big((x, \alpha)(y, \alpha)\Big).
\end{gathered}$$

(ii) $([x]_{\equiv_1}, \alpha + \beta) \sim ([x]_{\equiv_1}, \alpha)([x]_{\equiv_1}, \beta)$. Observe that

$$(g\circ\varphi)(x, \alpha+\beta) = [(\alpha+\beta)x]_{\equiv_1} = [\alpha x]_{\equiv_1} + [\beta x]_{\equiv_1} = (g\circ\varphi)(x,\alpha) + (g\circ\varphi)(x,\beta) = (g\circ\varphi)\Big((x,\alpha)(x,\beta)\Big).$$

(iii) $(0, \alpha) \sim (0, 0) \sim ([x]_{\equiv_1}, 0)$. Observe that

$$(g\circ\varphi)(0,\alpha) = \alpha(0) = 0 = (g\circ\varphi)(0,0) = [0(x)]_{\equiv_1} = (g\circ\varphi)(x,0).$$

Since $g\circ\varphi$ is bilinear, then $g\circ\varphi$ will preserve the congruence generated by the bilinearity relation and h will be well-defined. For example, suppose you have some equivalence class $[H_1]_\otimes$ in $H_1(X;\mathbb{N})\otimes\mathbb{R}_{\geq 0}$. Let z and w be arbitrary representatives, then z and w can be factored into non-trivial elements of the cartesian product, $H_1(X;\mathbb{N})\times\mathbb{R}_{\geq 0}$, say $z = z_1 z_2 ... z_n$ and $w = w_1 w_2 ... w_n$ where each z_i and w_i is some ordered pair. By the tensor product construction, $z \approx_\otimes w$ because $z_i \approx_\otimes w_i$ for all i. Note that since $z_1 \approx_\otimes w_1$ and $z_2 \approx_\otimes w_2$, then $z_1 z_2 \approx_\otimes w_1 w_2$. Now since

$$(g\circ\varphi)(z_1) = (g\circ\varphi)(w_1) \text{ and } (g\circ\varphi)(z_2) = (g\circ\varphi)(w_2)$$

we have that

$$(g\circ\varphi)(z_1 z_2) = (g\circ\varphi)(z_1) + (g\circ\varphi)(z_2) = (g\circ\varphi)(w_1) + (g\circ\varphi)(w_2) = (g\circ\varphi)(w_1 w_2).$$

This shows that $g\circ\varphi$ is well-defined for equivalence classes of words that have two non-trivial factors. This same argument works for equivalence classes where representatives are of arbitrary length.

Injective Suppose $x \not\approx_\otimes y$ where $x = ([x' = \sum a\sigma_i]_{\equiv_1}, \alpha)$ and $y = ([y' = \sum b\sigma_i]_{\equiv_1}, \beta)$.
Case : $[\mathbf{x}']_{\equiv_1} \neq [\mathbf{y}']_{\equiv_1}$ Here x' and y' are simple sums. Under h, the images are $[\alpha x']_{\equiv_1}$ and $[\beta y']_{\equiv_1}$. We claim that these are not equal. Suppose to the contrary that they are, then $\alpha x' \equiv_1 \beta y'$ which means $\exists c \in C_2(X;\mathbb{R}_{\geq 0})$ such that,

$$\alpha x' + \partial_-(c) = \beta y' + \partial_+(c) \tag{3}$$

We can reconstruct what c would have to be in order for this equation to be true. This c will show there exists a $c' \in C_2(X;\mathbb{N})$ that would make $x' \equiv_1 y'$— contrary to assumption. We can think of c, under the boundary maps, as being some kind of compensation of σ_i's, adding to x' the missing σ_i's of y' with the appropriate coefficients and vice-versa so that the sums in (3) are equal.

To begin this combinatorial game, we know that both sides have to contain the same σ_i's (1-cubes), the union of σ_i's in x' and y'. So $c' = \sum \lambda_i \sigma_i^2$ where the σ_i^2's (2-cubes) are those in c and for some λ_i's. We know that these σ_i^2's, under the boundary maps, are precisely those needed in order to have the same σ_i's on L.H.S and R.H.S of (3) We abuse some notation here, $\sigma_i \in x'$ simply means that σ_i is a term in the sum. The λ_i's are as follows (assume wlog that $a > b$),

$$\lambda_i = \begin{cases} a-b, & \partial_{-j}(\sigma_i^2) \in x' \ \& \ \partial_{+j}(\sigma_i^2) \in y' \\ a, & \partial_{-j}(\sigma_i^2) \in x' \ \& \ \partial_{+j}(\sigma_i^2) \notin y' \\ b, & \partial_{-j}(\sigma_i^2) \notin x' \ \& \ \partial_{+j}(\sigma_i) \in y') \end{cases}$$

It is not hard to see that c' constructed in this way, bears witness to $x' \equiv_1 y'$ which is our contradiction. Note that the important part here is that we can find the right combination of elements of X_2, the σ_i^2's, that will be the compensation of edges under the boundary maps, making sure we can pick the right coefficients in $\mathbb{N}$ is simply this combinatorial game above.

For the general case...[Missing remainder of argument for the general case]

Case : $\alpha \neq \beta$. Under h, the images are $[\alpha x']_{\equiv_1}$ and $[\beta x']_{\equiv_1}$. We claim that these are not equal. Suppose to the contrary that they are, then $\alpha x' \equiv_1 \beta x'$ which means $\exists c \in C_2(X; \mathbb{R}_{\geq 0})$ such that (assume wlog that $\alpha > \beta$),

$$\alpha x' + \partial_-(c) = \beta x' + \partial_+(c)$$
$$(\alpha - \beta)x' + \partial_-(c) = 0 + \partial_+(c)$$
$$x' + \partial_-(\frac{1}{(\alpha-\beta)}c) = 0 + \partial_+(\frac{1}{(\alpha-\beta)}c)$$

These formal operations are justified because Z_1 with coefficients in $\mathbb{R}_{\geq 0}$ or $\mathbb{N}$ is cancellative. Additionally, since $(\alpha - \beta) > 0$, then we know $\frac{1}{(\alpha-\beta)} \in \mathbb{R}_{\geq 0}$. This means $x' \equiv_1 0$, furthermore, in $H_1(X; \mathbb{N}) \otimes \mathbb{R}_{\geq 0}$ we have that $([x']_{\equiv_1}, \alpha) \approx ([x']_{\equiv_1}, \beta)$ contrary to assumption.

Surjective Let $z \in H_1(X; \mathbb{R}_{\geq 0})$, a simple sum, then $z = [\lambda z']_{\otimes}$. Simply, $h\Big([([z']_{\equiv_1}, \lambda)]\Big) = z$. From an earlier result (Appendix item 1), we know that any element in the equalizer is composed of a sum of simple sums. So in the case when $z = [\lambda_1 z_1'][\lambda_2 z_2']...[\lambda_n z_n']$, then we have that $[([z_1']_{\equiv_1}, \lambda_1)][([z_2']_{\equiv_1}, \lambda_2)]...[([z_n']_{\equiv_1}, \lambda_n)]$ maps to z under h.

□

2.4. Future Research.

2.4.1. *Def.* Consider the equivalence relation $\sim$ on the directed cycles of a directed graph G where two cycles are equivalent if they pass through the same vertices.

2.4.2. *Def.* Let $E = \{0, e\}$ be the commutative monoid where e is idempotent.

2.4.3. *Conjecture.* $H^1(X; E)$ counts these equivalence classes.

2.4.4. *Conjecture.* $DiH_1(X; G) \cong H_1(X; G)$ where the LHS is directed homology (as a group), the RHS is regular homology, and G is a group.

References.

(1) *Directed Algebraic Topology and Concurrency* - Lisbeth Fajstrup, Eric Goubault, Emmanuel Haucourt, Samuel Mimram, Martin Raussen, Pg. 50, 2016