Some Advances in Directed (Co)Homology

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OSU Mathematics REU 2019

The Ohio State University

Overview

A directed space vaguely refers to a space with some notion of time flowing on it. Some concrete instances of directed spaces are spacetimes, directed graphs, oriented simplicial complexes, and oriented cube complexes. We motivate and introduce a theory of directed (co)homology, suggested by existing work in homological algebra for monoids [1, 3].

We work on the relationship between regular and directed (co)homology and prove that there exists an injective map from $H^n(X;\mathbb{N})$ to $H^n(X;\mathbb{Z})$ for pre-cubical sets. In addition, we conjecture and attempt to prove an analog of the universal coefficient theorem for cancellative monoids. We also give an algorithm for computing $H_1(X;\mathbb{R}_{\geq 0})$ of an oriented 1-pre-cubical complex (directed graph).

Definitions

We work on a pre-cubical set X, i.e. a cubical set without degeneracy maps, with coefficients in a commutative monoid M. Let $X = \{X_k \mid k = 0, 1, \dots\}$ where X_k is the set of k-dimensional pre-cubes in X. We denote the i-th element in X_k as x_{k_i}. The properties of cubical sets imply that a k-pre-cube has exactly $2k$ faces, and opposite faces can be grouped into k pairs. For a k-pre-cube, we pick one face and all its neighboring faces and define the k faces to be top faces, and the remaining faces to be bottom faces.

Directed Homology

Let the **k-th chain element** $C_k = \{\sum_\alpha m_\alpha x_\alpha \mid m_\alpha \in M,\ x_\alpha \in X_k\}$, i.e. the set of linear combinations of k-cubes with coefficients in M. Define **k-th boundary maps**

$$\partial_k^+, \partial_k^- : C_k \longrightarrow C_{k-1}$$

such that ∂_k^+ can be represented as a $|C_{k-1}| \times |C_k|$ matrix and $\partial_{+k(i,j)=1}$ if $x_{(k-1)_i} \in X_{k-1}$ is a top face of $x_{k_j} \in X_k$. Otherwise, $\partial_{k(i,j)} = 0$ [2]. ∂_k^- is defined similarly with bottom faces.

Let the **k-th equalizer**

$$Z_k = \{z \in C_k \mid \partial_k^+(z) = \partial_k^-(z)\}.$$

For $z_1, z_2 \in Z_k$, define $z_1 \equiv_k z_2$ if there exists $f \in C_{k+1}$ such that

$$z_1 + \partial_k^+(f) = z_2 + \partial_k^-(f)$$

and let the congruence relation $z_1 \sim_k z_2$ be generated by $\equiv_k$. Let the **k-th homology** $H_k = Z_k / \sim_k$. Note that when X is a graph, $C_2 = \varnothing$ and $H_1 = Z_1$.

Directed Cohomology

Let the **cochain element** $C^k = \mathrm{Hom}(C_k; M)$ and define **coboundary maps**

$$d_k^+, d_k^+ : C^k \longrightarrow C^{k-1}$$

where for $\phi \in C^k$, $d_k^+(\phi) = \phi \circ \partial_{k+1}^+$ and $d_k^-(\phi) = \phi \circ \partial_{k+1}^-$, analogous to the regular definition. Same as in regular (co) homology, the matrices of coboundary maps d_k^+, d_k^+ are transposes of boundary maps $\partial_{k-1}^+, \partial_{k-1}^-$ correspondingly.

Let the **equalizer**

$$Z^k = \{z \in C^k \mid d_k^+(z) = d_k^-(z)\}.$$

For $z_1, z_2 \in Z^k$, define $z_1 \equiv^k z_2$ if there exists $f \in C^{k-1}$ such that

$$z_1 + d_k^+(f) = z_2 + d_k^-(f)$$

and let the congruence relation $z_1 \sim^k z_2$ be generated by $\equiv^k$. Let the **k-th cohomology** $H^k = Z^k / \sim^k$. Note that $H^1 = C^1 / \sim^1 = M^{|X_1|} / \sim^1$ because d_1^+ and d_1^- are zero maps whose equalizer is C^1.

Directed and Regular (Co)Homology: A Comparison

Concept	Regular	Directed
chain element	linear combinations of X_k with G or M coefficients	
cochain element	$\mathrm{Hom}(C_k; G)$ or $\mathrm{Hom}(C_k; M)$	
boundary map	∂	∂^+, ∂^- s.t. $\partial^+ - \partial^- = \partial$
coboundary map	d	d^+, d^- s.t. $d^+ - d^- = d$
cycle	$\mathrm{Ker}\,\partial$	equalizer of ∂^+, ∂^-
cocycle	$\mathrm{Ker}\,d$	equalizer of d^+, d^-
$z_1 \sim z_2$	$z_1 = z_2 + \partial(f)$ $z_1 = z_2 + d(f)$	$z_1 \pm \partial^+(f) = z_2 \mp \partial^-(f)$ $z_1 \pm d^+(f) = z_2 \mp d^-(f)$
homology	$Z_n / \mathrm{Im}\,\partial_{n+1}$	$Z_n / \sim_n$
cohomology	$Z^n / \mathrm{Im}\,d_{n-1}$	$Z^n / \sim^n$

An Example

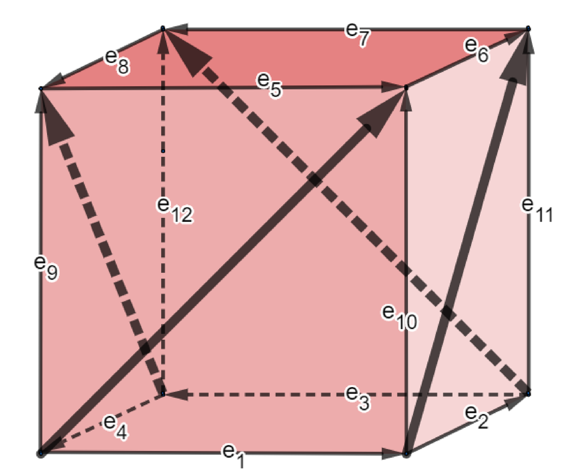


Fig. 1: A directed cube without top and bottom faces. It has four faces $f_1 = (e_1, e_{10}, e_9, e_5)$, $f_2 = (e_2, e_{11}, e_6, e_{10})$, $f_3 = (e_{11}, e_7, e_{12}, e_3)$ and $f_4 = (e_{12}, e_8, e_9, e_4)$.

For each face, let the two bottom left edges be "bottom" and top right edges be "top". For example, $\partial_2^-(f_1) = e_1 + e_9$ and $\partial_2^+(f_1) = e_5 + e_{10}$.

Observe that the cube has two cycles with four edges at the top/bottom. Let $c_1 = e_1 + e_2 + e_3 + e_4$ and $c_2 = e_5 + e_6 + e_7 + e_8$. In fact, $c_1 \sim_2 c_2$ because

$$\begin{aligned} & c_1 + \partial_2^+(f_1 + f_2 + f_3 + f_4) \\ =& \sum_{i=1}^{12} e_i \\ =& c_2 + \partial_2^-(f_1 + f_2 + f_3 + f_4) \end{aligned}$$

Results

After defining directed (co)homology, we want to explore its relation to properties of the space and the regular (co)homology, and try to conjecture and prove analogues of important theorems in regular (co)homology.

Results about Directed (Co)Homology

Proposition $H_1(X, \mathbb{N})$ is non-trivial if and only if X has a directed cycle.

Theorem $H_1(X;\mathbb{N}) \otimes \mathbb{R}_{\geq 0} \cong H_1(X;\mathbb{R}_{\geq 0})$

Definition Consider the equivalence relation on directed cycles where two cycles are equivalent if they pass through the same vertices.

Definition Let $E = \{0, e\}$ be the commutative monoid where e is idempotent.

Conjecture $H^1(X; E)$ counts these equivalence classes.

Relationship between Directed and Regular (Co)Homology

Consider the monoid $M = \mathbb{N}$. Define $c \subset C_1$ as a **d-cycle** if c is a cycle equipped with a direction and $c = \{e_j^+\} \cup \{e_k^-\}$ where e_j^+ (e_k^-) are edges along (against) the direction of the d-cycle. Let $w \in C^1$ and define the **weight** of a d-cycle c as $\hat{w}(c) = \sum_j w(e_j^+) - \sum_k w(e_k^-)$. Let $w_1, w_2 \in C^1$. We prove the following proposition.

Proposition $w_1 \sim^1 w_2 \iff \hat{w}_1(c) = \hat{w}_2(c)$ for any 1-d-cycle c.

The proposition implies that the equivalence classes of 1-cocycles, i.e., the first cohomology, are exactly decided by the ways to assign weights to 1-d-cycles. Using this proposition, we can show that if the d-cycles can be determined by cycles in the regular sense, then there exists a natural injection from the directed to regular cohomology.

Let a d-cycle $c = \{e_p^+\} \cup \{e_q^-\} \subset C^1$ be **directed** if $\{e_q^-\} = \varnothing$. A d-cycle is **undirected** if it is not directed. A d-cycle c is **generated** by $\{c_i\}$ if $c = \sum_i n_i c_i$, where $n_i \in \mathbb{N}$. Let X be a graph where all undirected d-cycles are generated by directed d-cycles. The setup helps us determine the cohomology only by weights of directed d-cycles, which equal the weights of regular cycles, resulting in an injection between directed and regular cohomology.

Define $\phi : H^1(X;\mathbb{N}) \longrightarrow H^1(X;\mathbb{Z})$ and $\phi([w]_\mathbb{N}) = [w]_\mathbb{Z}$, where $w \in C^1(X;\mathbb{N})$, $[w]_\mathbb{N} \in H^1(X;\mathbb{N})$ and $[w]_\mathbb{Z}$ is the corresponding cocycle equivalence class in $H^1(X;\mathbb{Z})$, which constructs a natural map from directed to regular cohomology.

Proposition ϕ is injective.

Notice that the proof only used $\hat{w}(c_\mathbb{N}) = \hat{w}(c_\mathbb{Z})$ for directed d-cycles c, we can see a more general conclusion as the following.

Theorem If $w_1(c_\mathbb{Z}) = w_2(c_\mathbb{Z})$ for any cycle $c_\mathbb{Z} \subset C_1(X, \mathbb{Z})$ where the corresponding d-cycle of $c_\mathbb{Z}$ in $C_1(X, \mathbb{N})$ is directed, then $\hat{w}_1(c_\mathbb{N}) = \hat{w}_2(c_\mathbb{N})$ for any d-cycle $c_\mathbb{N} \subset C_1(X, \mathbb{N})$.

To generalize this result, we notice that the core property being used in the proof is the equal number of top and bottom faces of pre-cubical sets. Thus, the theorem can be applied to H^n for pre-cubical sets and H^1 for any complex, because a 1-complex (an edge) always has one source and one sink vertex. For the same reason, the theorem holds for homology, too. Moreover, we check that the technique used in the proof is applicable to $M = \mathbb{R}_{\geq 0}$, so there exists a natural injection from $H^n(X;\mathbb{R}_{\geq 0})$ to $H^n(X;\mathbb{R})$ as well.

An Analog of Universal Coefficient Theorems

Conjecture For a cancellative monoid M, we have a short exact sequence

$$0 \longrightarrow H_n(X;\mathbb{N}) \otimes M \longrightarrow H_n(X; M) \longrightarrow \mathrm{Tor}(H_{n-1}(X;\mathbb{N}), M) \longrightarrow 0$$

that splits, though not naturally.

To prove of the conjecture, we attempt to develop a homological algebra theory for semimodules, so that we can make sense of semimodule exact sequence, projective semimodule and Tor functor on the category of semimodules, etc.

Assuming cancellativity of M, we can maintain the definition of exactness in regular homological algebra and prove that projectivity is equivalent to right split short exact sequence. However, it is not always true that a right split semimodule exact sequence implies the direct sum statement in the splitting lemma. This inconsistency causes trouble in constructing the desired projective resolution of H_{n-1}

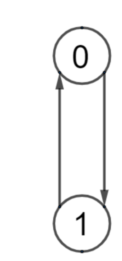
$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

In addition, we have not yet proved that $\mathrm{Tor}_k(A, B)$, defined as the k-th homology of the chain complex obtained from any projective resolution of A tensor B is unique. Nevertheless, the conjecture will be proved if the equalizers are projective and the Tor functor is unique.

Computation

Algorithm for $DiH_1(Graph, \mathbb{R}_{\geq 0})$

Input: $G = (V, E)$
Output: Generators for Directed Cycles
Build the maps d^*_{-1}, d^*_{+1} as $|V| \times |E|$ matrices
Compute the basis for the nullspace of $d^*_{-1} - d^*_{+1}$
Restrict the solutions to $\mathbb{R}^n_{\geq 0}$



$$d^*_{+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$d^*_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Future Research

By varying the monoid coefficients, does directed cohomology capture anything interesting about the directed space?

Does directed homology satisfy the Eilenberg–Steenrod axioms?

When are Z_n and Z_{n-1} projective so that we can have a projective resolution of H_{n-1} involving Z_n, C_n and Z_{n-1}? Can we define Ext for semimodules and prove a universal coefficient theorem for directed cohomology?

Is there an analogue of the Mayer-Vietoris sequence for directed (co)homology?

Acknowledgements

We would like to acknowledge our advisors Dr. Sanjeevi Krishnan and Scott Newton for all their help on this research project as well as the OSU Mathematics Department for funding this program.

References

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