

# Quantum computing for computational finance

## overview, challenges and opportunities

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# Motivation

- Quantum computers could bring unparalleled competitive advantage to financial companies in areas like portfolio optimization, option pricing, quantitative risk management or Machine Learning models.
- Quantum computers are able to handle exponentially growing (in qubits) Hilbert spaces.
- Thus, quantum computing becomes an attractive framework for calculations over large multi-dimensional domains.
- Quantum algorithms could potentially overcome their classical counterparts in dealing with combinatorial explosions and the curse of dimensionality.
- However, bringing this to practice encounters several bottlenecks, especially with the current or near-term quantum technologies (NISQ).

# Disclaimer

- Quantum computing literature is experiencing an explosion: This presentation incorporates only a few of the current trends.
- The selection of the addressed topics reflects only my view (interests) within the vast scope of the computational finance field.
- Then, many important topics are not addressed here: optimization, time series, blockchain, cryptography, etc.
- There might be inconsistencies or certain abuse in the (mathematical and/or quantum) notation. In some cases, that is intentional, for the sake of clarity. In others...sorry in advance!

# Outline

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Quantum Computing basics

Promising Quantum Algorithms for pricing (and risk measures)

Quantum Monte Carlo

Quantum PDE solvers

Quantum Machine/Deep Learning

Challenges

Opportunities

# Quantum Computing basics

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# Quantum Computing basics (I)

- The basic unit of information is the *qubit* (alternatively to the *bit*).
- A qubit is represented by a (column) vector:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with the *amplitudes*  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ .

- Basis states:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $\{|0\rangle, |1\rangle\}$  is a computational basis for a quantum state:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

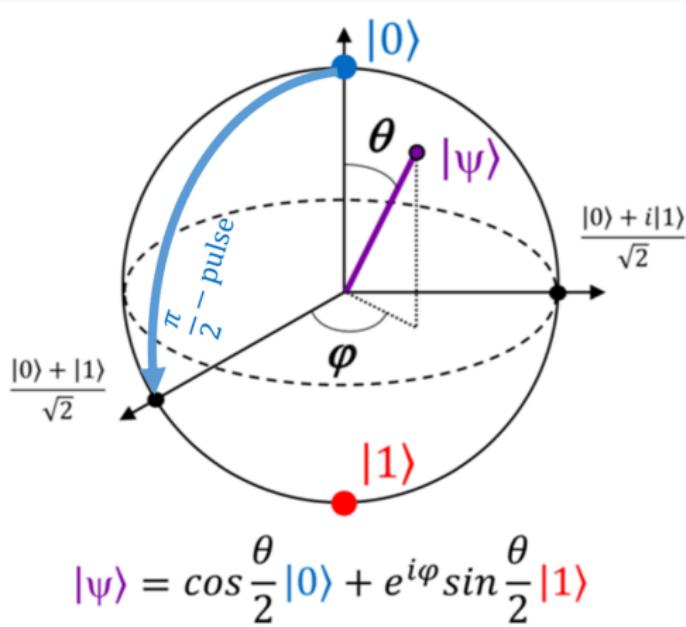
- When *measuring* the state:

- get 0 with probability  $|\alpha|^2$
- get 1 with probability  $|\beta|^2$

- So, *measurement*  $\equiv$  *distribution sampling*

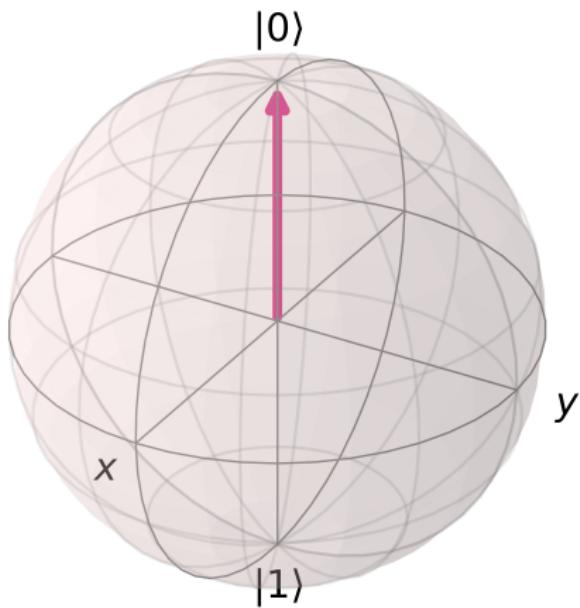
## Quantum Computing basics (II)

- The Bloch sphere provides a representation of qubit state
- Measuring a qubit occurs along the Z axis, so it is irreversible and will collapse to either 0 or 1



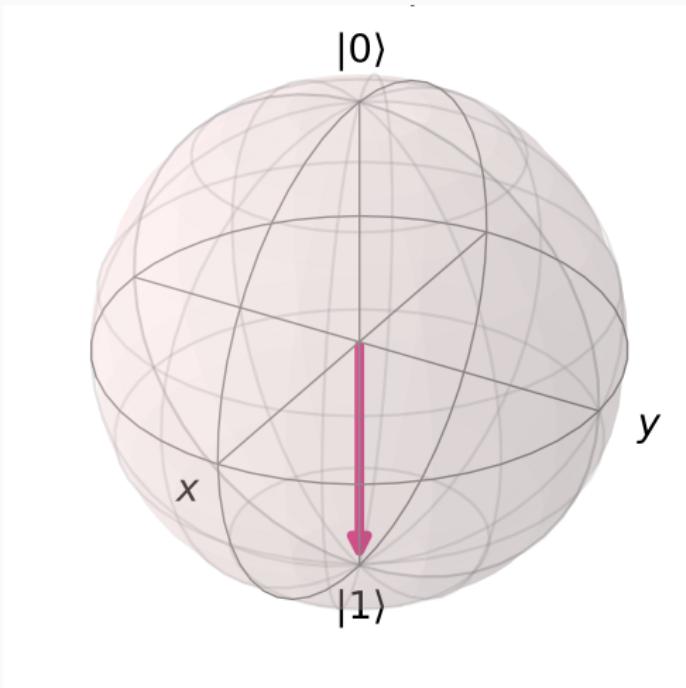
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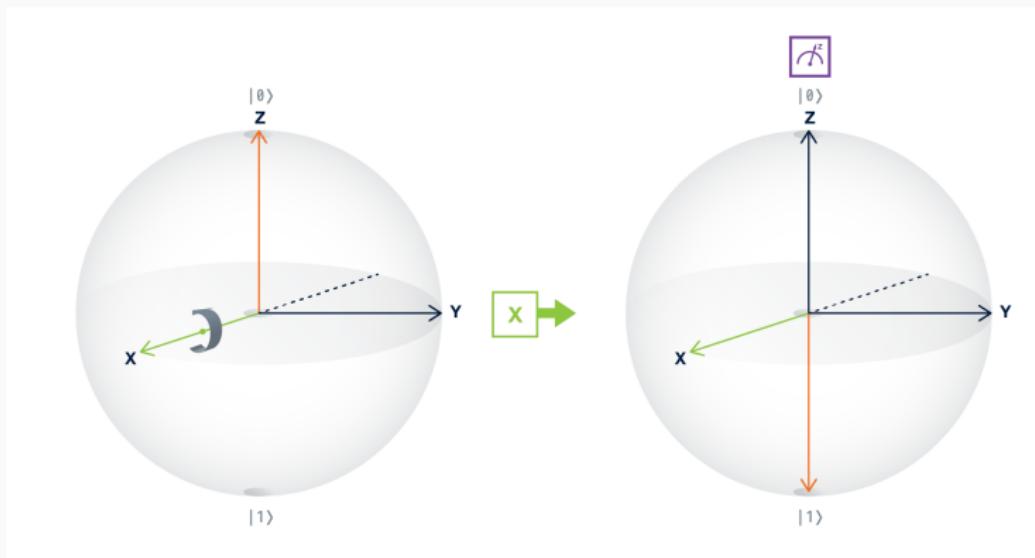
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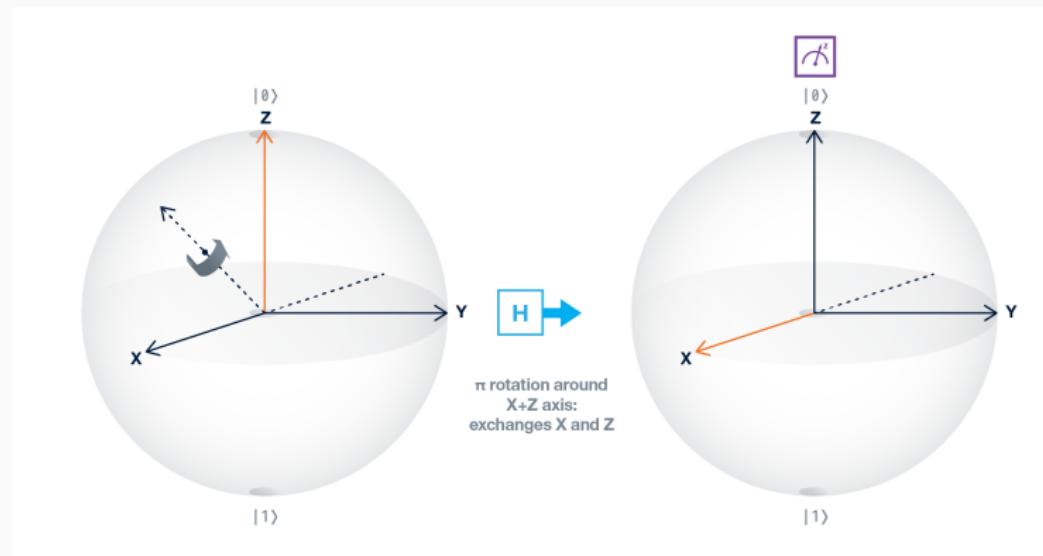
# Quantum Computing basics (III)

- Quantum gates to perform operations on qubits
- Gates are reversible and can be represented as unitary matrices acting on the qubit vectors.



# Quantum Computing basics (III)

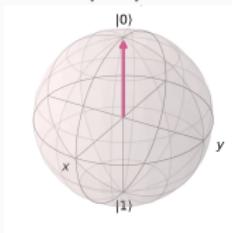
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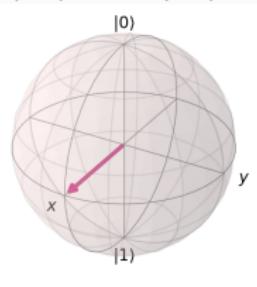
# Quantum Computing basics (IV)

- Superposition: Identically prepared qubits can still behave randomly
- The randomness is inherent in the quantum nature

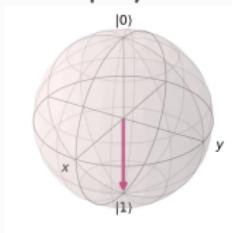
$|0\rangle$



$|0\rangle + |1\rangle$



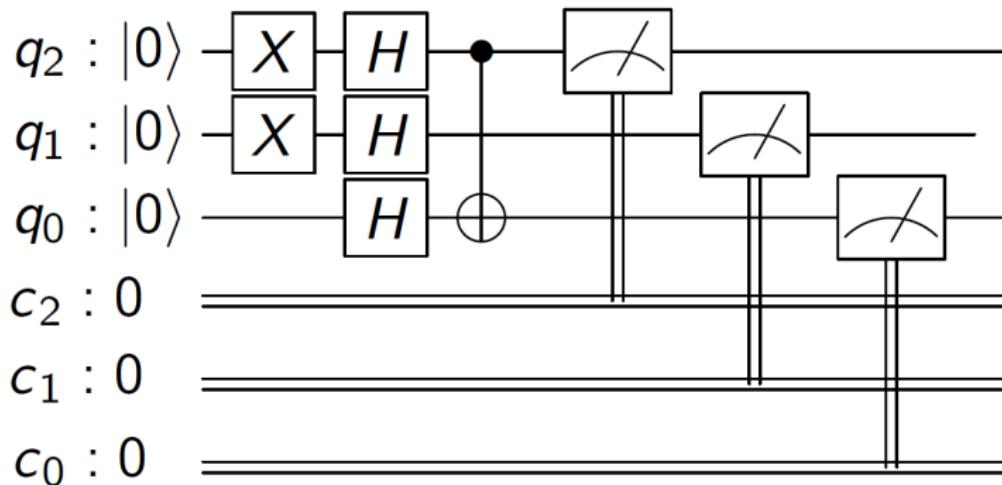
$|1\rangle$



$\sim 50/50$  chance of  
being  $|0\rangle$  or  $|1\rangle$

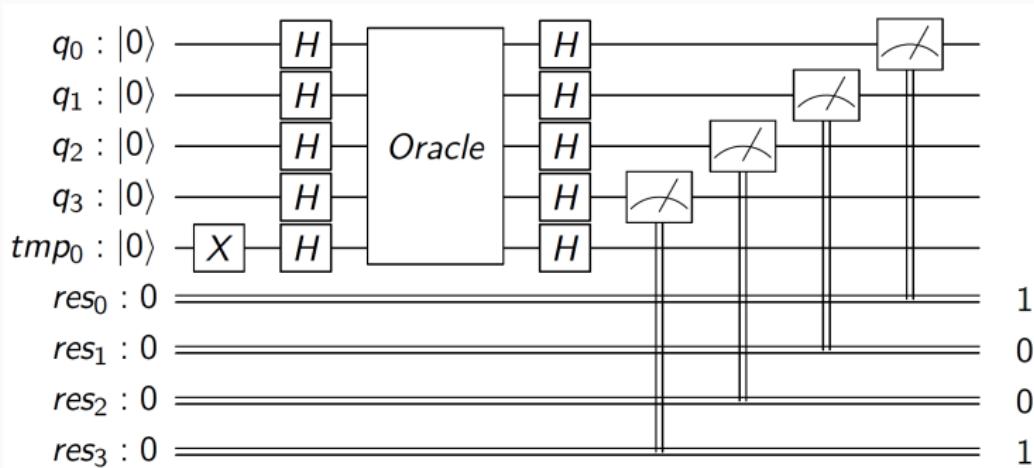
# Quantum Computing basics (and V)

- Each row represents a bit, either quantum or classical
- The operations are performed each qubit from left to right
- Measurement to extract the information



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# Promising Quantum Algorithms for pricing (and risk measures)

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# Quantum Monte Carlo

# Quantum Monte Carlo (QMC)

The quantum-accelerated Monte Carlo could potentially/theoretically provide a quadratic speedup for option pricing and risk measures calculation [Gómez et al., 2022].

How? Quantum Amplitude Estimation.

Monte Carlo methods in finance can be informally defined as

$$\frac{1}{M} \sum_{i=0}^{M-1} f(X_i) \approx \mathbb{E}[f(X)] = \int f(x)p(x)dx \approx \sum_{j=0}^{N-1} f(x_j)p(x_j)$$

where  $p(x)$  is a density function.

Analogously, Quantum Monte Carlo (QMC) assumes a state of the form

$$|\psi\rangle = |0\rangle \otimes \sum_{j=0}^{N-1} f(x_j)p(x_j) |j\rangle + |1\rangle \otimes \sum_{j=0}^{N-1} \sqrt{1 - f^2(x_j)p(x_j)} |j\rangle$$

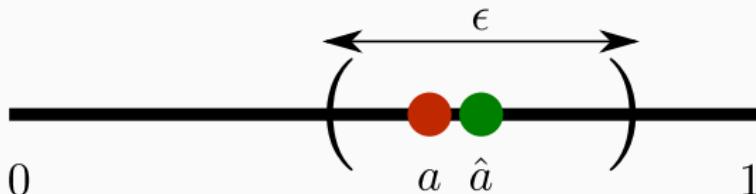
where the quantity of interest is encoded in the state's amplitude.

# QMC: Quantum Amplitude Estimation

Given a state:

$$|\psi\rangle = a |\phi\rangle + \sqrt{1 - a^2} |\phi^\perp\rangle,$$

Quantum Amplitude Estimation (QAE) is an algorithm which gives an estimation  $\hat{a} \pm \frac{\epsilon}{2}$  of the amplitude  $a$ .



This technique promises to obtain a **quadratic speedup over its classical counterpart**.

To achieve so, it relies on two main subroutines:

- Grover (search) amplification.
- Quantum Phase Estimation.

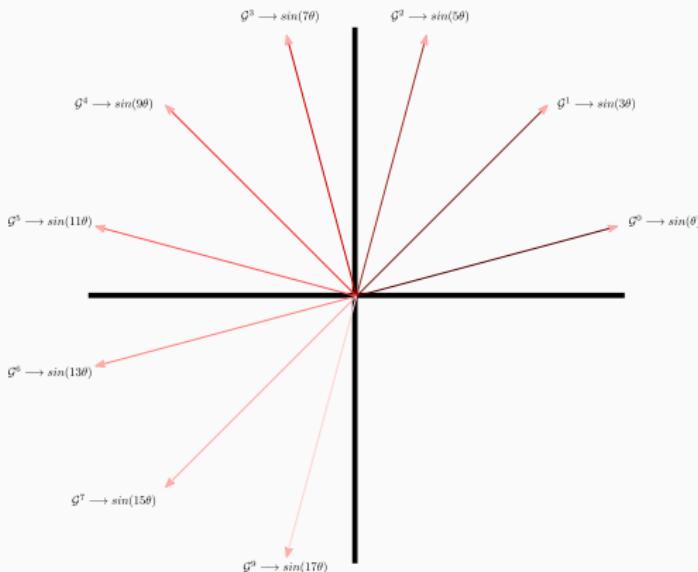
# QMC: Grover Amplification

Given a state:

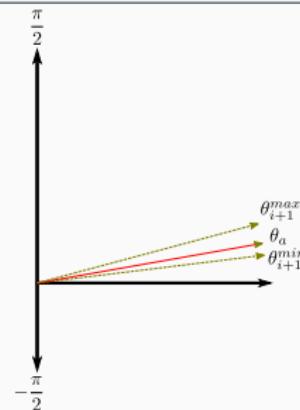
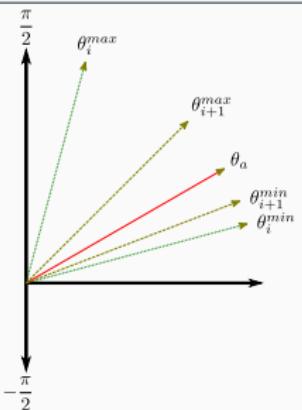
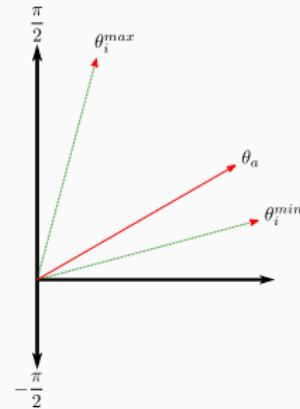
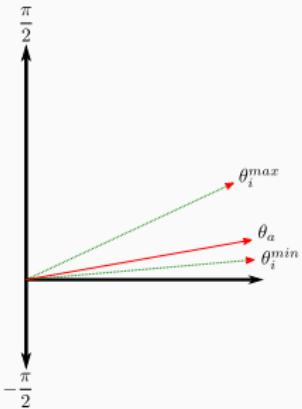
$$|\psi\rangle = \sin(\theta) |\phi\rangle + \cos(\theta) |\phi^\perp\rangle,$$

Grover operator performs the following transformation:

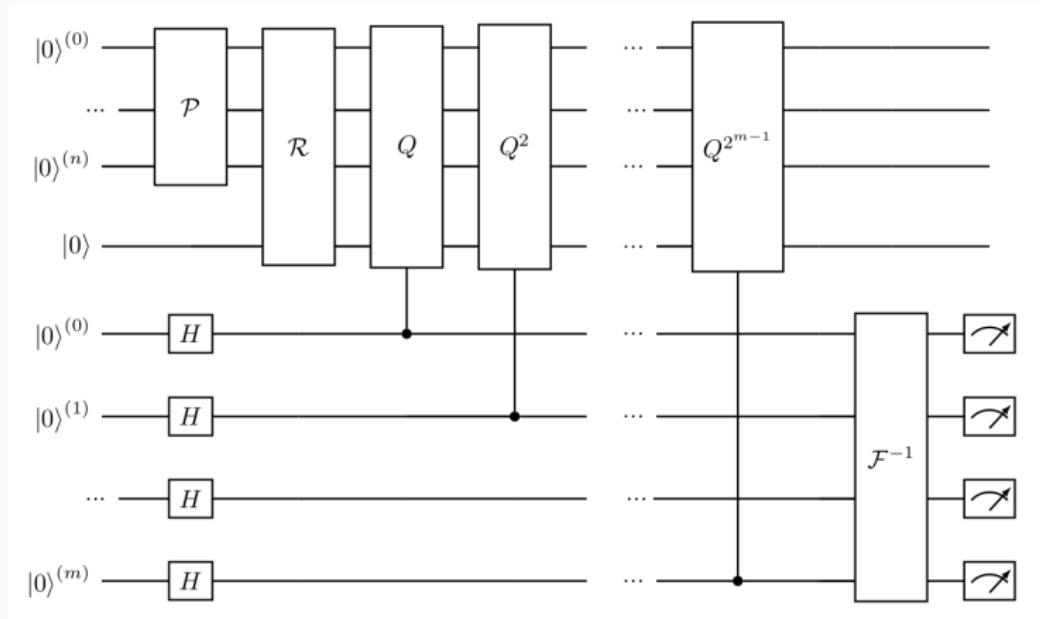
$$\mathcal{Q}^k |\psi\rangle = \sin((2k+1)\theta) |\phi\rangle + \cos((2k+1)\theta) |\phi^\perp\rangle.$$



# QMC: Quantum Amplitude Estimation (graphically)



# QMC: Quantum Amplitude Estimation (circuit)



**Figure 1:** Quantum Amplitude Estimation

# QMC: Quantum Amplitude Estimation (convergence)

**Theorem (Mean estimation for  $[0, 1]$  bounded functions**

**[Montanaro, 2015]**

Let there be given a quantum circuit  $\mathcal{P}$  on  $n$  qubits. Let  $v(\mathcal{P})$  be the random variable that maps to  $v(x) \in [0, 1]$  when the bit string  $x$  is measured as the output of  $\mathcal{P}$ . Let  $\mathcal{R}$  be defined as

$$\mathcal{R}|x\rangle|0\rangle = |x\rangle \left( \sqrt{1 - v(x)}|0\rangle - \sqrt{v(x)}|1\rangle \right).$$

Let  $|\mathcal{X}\rangle$  be defined as  $|\mathcal{X}\rangle = \mathcal{R}(\mathcal{P} \otimes \mathcal{I}_2)|0^{n+1}\rangle$ . Set  $\mathcal{U} = \mathcal{I}_{2^{n+1}} - 2|\mathcal{X}\rangle\langle\mathcal{X}|$ .

There exists a quantum algorithm that uses  $\mathcal{O}(\log 1/\delta)$  copies of the state  $\mathcal{X}$ , uses  $\mathcal{U}$  for a number of times proportional to  $\mathcal{O}(m \log 1/\delta)$  and outputs an estimate  $\hat{\mu}$  such that

$$|\hat{\mu} - \mathbb{E}[v(\mathcal{P})]| \leq C \left( \frac{\sqrt{\mathbb{E}[v(\mathcal{P})]}}{m} + \frac{1}{m^2} \right),$$

with probability at least  $1 - \delta$ , where  $C$  is a universal constant. In particular, for any fixed  $\delta > 0$  and any  $\epsilon$  such that  $0 < \epsilon \leq 1$ , to produce an estimate  $\hat{\mu}$  such that, with probability at least  $1 - \delta$ ,  $|\hat{\mu} - \mathbb{E}[v(\mathcal{P})]| \leq \epsilon \mathbb{E}[v(\mathcal{P})]$ , it suffices to take  $m = \mathcal{O}((\epsilon \mathbb{E}[v(\mathcal{P})])^{-1})$ . To achieve  $|\hat{\mu} - \mathbb{E}[v(\mathcal{P})]| \leq \epsilon$  with probability at least  $1 - \delta$ , it suffices to take  $m = \mathcal{O}(\epsilon^{-1})$ .

# QMC: variations on QAE

Plain QAE is not feasible in NISQ era, due to the use of a Quantum Fourier Transform (QFT).

Algorithm	Performance
Monte Carlo	$N_{\mathcal{A}}^{MC} \sim \mathcal{O}\left(\frac{1}{\epsilon_p^2}\right)$
QPE[Brassard et al., 2002]	$N_{\mathcal{A}}^{QPE} \sim \mathcal{O}\left(\frac{1}{\epsilon_p}\right)$
MLAE-LIS[Suzuki et al., 2020]	$N_{\mathcal{A}}^{LIS} \sim \mathcal{O}\left(\epsilon_p^{-4/3}\right)$
MLAE-EIS[Suzuki et al., 2020]	$N_{\mathcal{A}}^{EIS} \sim \mathcal{O}\left(\frac{1}{\epsilon_p}\right)$
PLAE[Giurgica-Tiron et al., 2020]	$N_{\mathcal{A}}^{PLAE} \sim \mathcal{O}\left(\frac{1}{\epsilon_p^{1+\beta}}\right), d \sim \mathcal{O}\left(\frac{1}{\epsilon_p^{1-\beta}}\right)$
Improved MLAE[Callison and Browne, 2022]	$N_{\mathcal{A}}^{imp EIS} \sim \mathcal{O}\left(\frac{1}{\epsilon_p} \frac{1}{d} \log\left(\frac{1}{\gamma}\right)\right), d = 2^{q-2}$
IQAE [Grinko et al., 2021]	$N_{\mathcal{A}}^{IQAE} < \frac{50}{\epsilon_p} \log\left(\frac{2}{\gamma} \log_2 \frac{\pi}{4\epsilon_p}\right)$
mIQAE[Fukuzawa et al., 2023]	$N_{\mathcal{A}}^{mIQAE} < \frac{123}{\epsilon_p} \log \frac{6}{\gamma}$
QCoin [Abrams and Williams, 1999]	$N_{\mathcal{A}}^{QCoin} \sim \mathcal{O}\left(\frac{1}{a} \frac{1}{\epsilon_p} \log \frac{1}{\gamma}\right), k \geq 2, 1 \geq q \geq (k-1)$
QoPrime [Giurgica-Tiron et al., 2020]	$N_{\mathcal{A}}^{QoPrime} < C \lceil \frac{k}{q} \rceil \frac{1}{\epsilon_p^{1+q/k}} \log\left(\frac{4}{\gamma} \lceil \frac{k}{q} \rceil\right), d \sim \mathcal{O}\left(\frac{1}{\epsilon_p^{1-q/k}}\right)$
FasterAE [Nakaji, 2020]	$N_{\mathcal{A}}^{fasterAE} < \frac{4.1 \cdot 10^3}{\epsilon_p} \log\left(\frac{4}{\gamma} \log_2\left(\frac{2\pi}{3\epsilon_p}\right)\right)$
AdaptiveAE [Zhao et al., 2022]	$N_{\mathcal{A}}^{adaptiveAE} < \mathcal{O}\left(\frac{1}{\epsilon_p} \log\left(\frac{\pi^2(T+1)}{3\gamma}\right)\right), T = \lceil \frac{\log \frac{\pi}{K\epsilon_p}}{\log K} \rceil$
RQAE [Manzano et al., 2023b]	$N_{\mathcal{A}}^{RQAE} < \frac{C_1(q)}{\epsilon_a} \log\left(\frac{3.3}{\gamma} \log_q\left(\frac{C_2(q)}{\epsilon_a}\right)\right)$
mRQAE [Ferro and Manzano, 2024]	$N_{\mathcal{A}}^{mRQAE} < \frac{C_1(q)}{\epsilon_a} \log\left(\frac{C_2(q)}{\gamma}\right)$

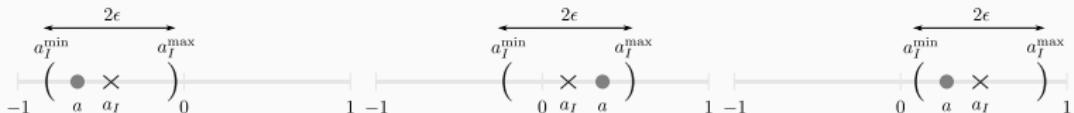
# RQAE: Settings

- Consider a one-parameter family of oracles  $\mathcal{A}_b$  that, acting on the state  $|0\rangle$ , yield

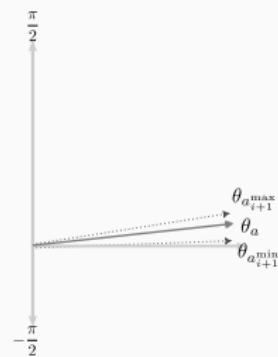
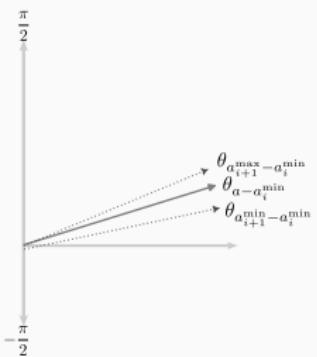
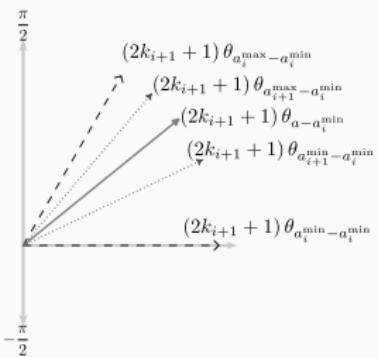
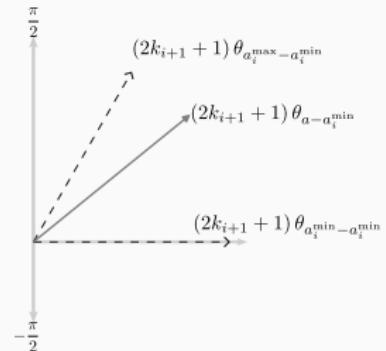
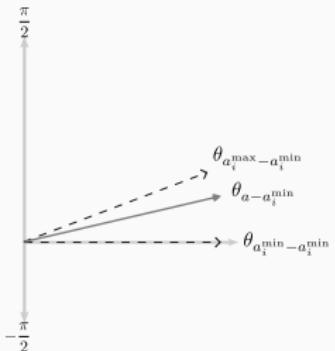
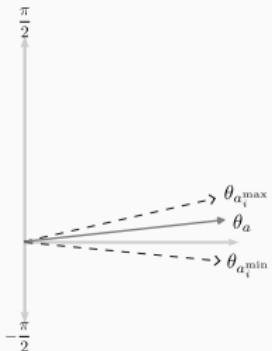
$$\mathcal{A}_b |0\rangle = |\psi\rangle = (a + b) |\phi\rangle + c_b |\phi^\perp\rangle_b ,$$

where  $a$  is a real number,  $b$  is an auxiliary, continuous and real parameter that we call “shift”.

- Given a precision level  $\epsilon$  and a confidence level  $1 - \gamma$ , the goal of the RQAE algorithm is to compute an interval  $(a_I^{\min}, a_I^{\max}) \subset [-1, 1]$  of width smaller than  $2\epsilon$  which contains the value of  $a$  with probability greater or equal to  $1 - \gamma$ .



# RQAE: Algorithm (graphically)



## RQAE: Main properties

Given  $\epsilon, \gamma$  and an amplification policy  $(k_i, i = 0, \dots, I)$ :

- Circuit depth bounded by:

$$k_I \leq \left\lceil \frac{\arcsin(\sqrt{2\epsilon^p})}{\arcsin(2\epsilon)} - \frac{1}{2} \right\rceil = k^{\max}.$$

- Precision  $\epsilon$  with confidence  $1 - \gamma$  (Proof of Correctness):

$$\mathbb{P}\left[a \notin (a_I^{\min}, a_I^{\max})\right] \leq \gamma.$$

- The total number of calls to the oracle is bounded by:

$$N_{\text{oracle}} < C_1 \frac{1}{\epsilon} \log\left(\frac{C_2}{\gamma}\right) ,$$

where the constants  $C_1$  and  $C_2$  depend on the amplification policy.

# RQAE: Numerical results (I)

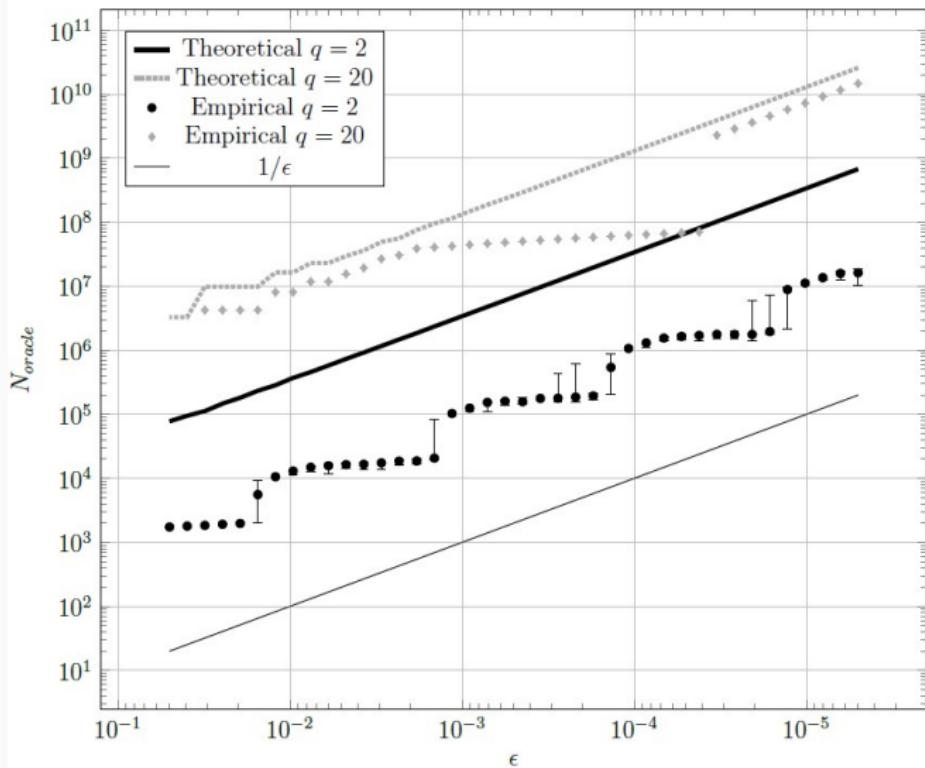
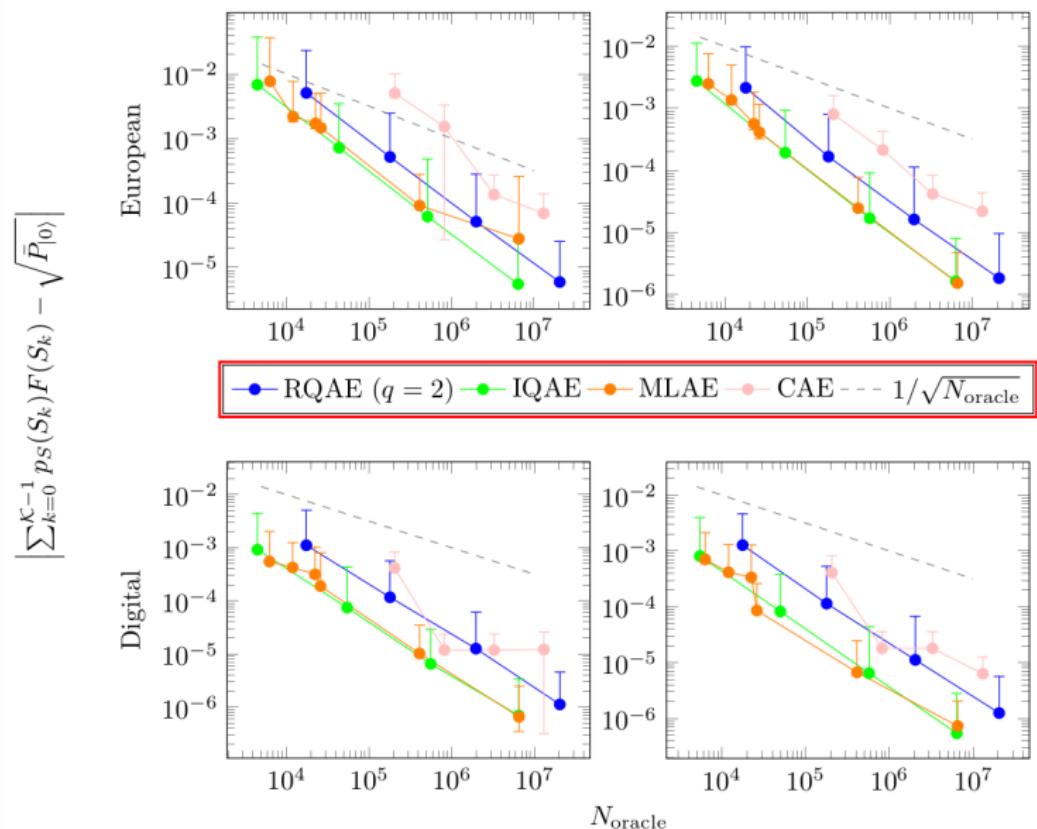


Figure 2: Number of calls to the oracle  $N_{\text{oracle}}$  versus the required precision  $\epsilon$ .

# RQAE: Numerical results (and II)

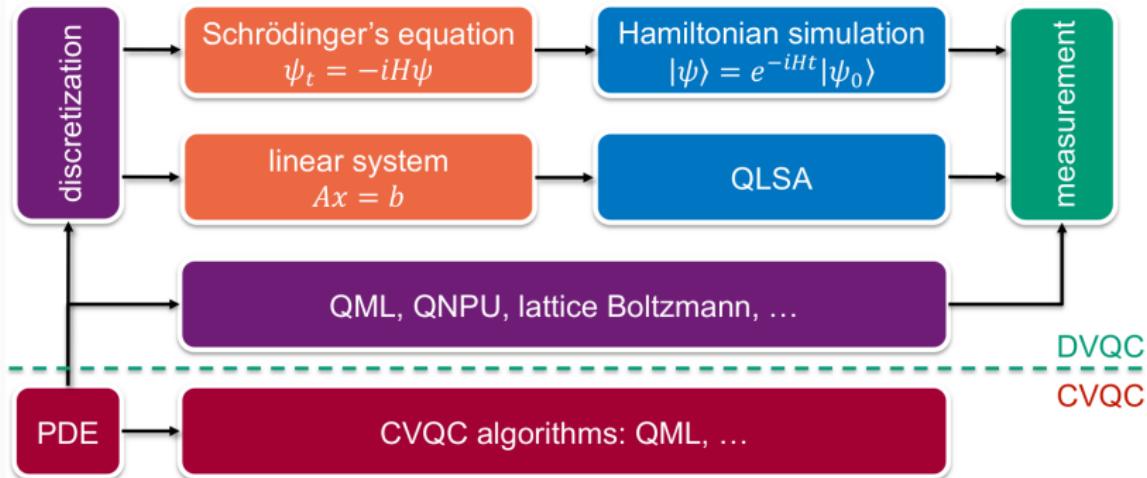


## RQAE: Main Conclusions

- More information from the quantum circuit than just the module of the amplitude, i.e., the sign of the quantity of interest, increasing the applicability.
- RQAE is an iterative algorithm which offers explicit control over the amplification policy through adjustable parameters.
- Control (also via the free parameters) the depth of the circuit, a crucial feature in the current NISQ era.
- A rigorous (and clean) theoretical analysis of the RQAE performance is provided, proving that it achieves a quadratic speedup (w.r.t. unamplified sampling), modulo logarithmic corrections.

# Quantum PDE solvers

# Quantum PDE solvers



**Figure 3:** Classification of quantum PDE solvers. Extracted from the presentation “A survey of quantum computing for PDEs” by Matthias Möller (TU Delft).

# Quantum approaches for Black-Scholes PDE

Some financial PDEs can be mapped into the propagation governed by a Hamiltonian [Gonzalez-Conde et al., 2021, Fontanella et al., 2021].

Applying the change of variable  $S = e^x$  on the Black-Scholes eq.,

$$\frac{\partial V}{\partial t} + \left( \mu - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} - \mu V = 0 ,$$

which can be written as a Schrödinger-like equation,

$$\frac{\partial V}{\partial t} = -i\hat{H}_{\text{BS}} V ,$$

where

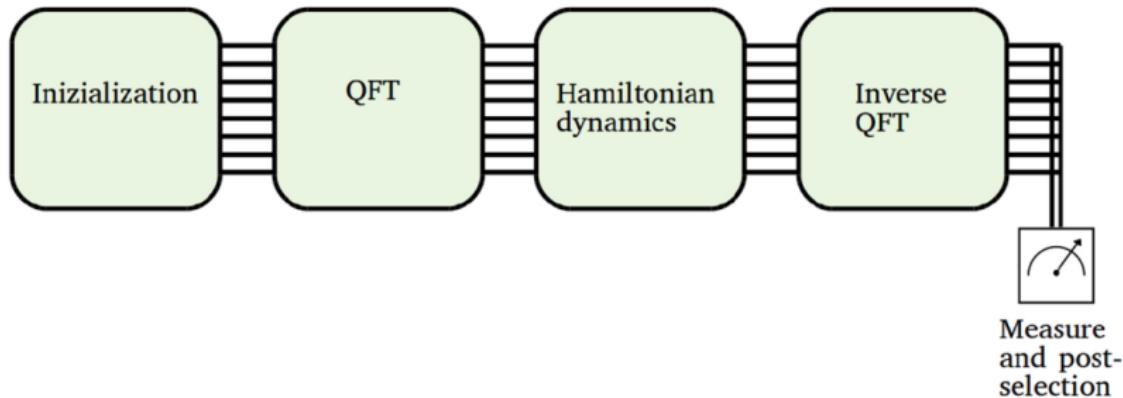
$$\hat{H}_{\text{BS}} = i \frac{\sigma^2}{2} \hat{p}^2 - \left( \frac{\sigma^2}{2} - \mu \right) \hat{p} + i\mu \mathbb{I} , \quad \text{with} \quad \hat{p} = -i \frac{\partial}{\partial x} .$$

The Hamiltonian  $\hat{H}_{\text{BS}}$  is *not* Hermitian.

Therefore, the associated evolution operator  $\hat{U}(t, t_0) = e^{-i\hat{H}_{\text{BS}}(t-t_0)}$  is not unitary.

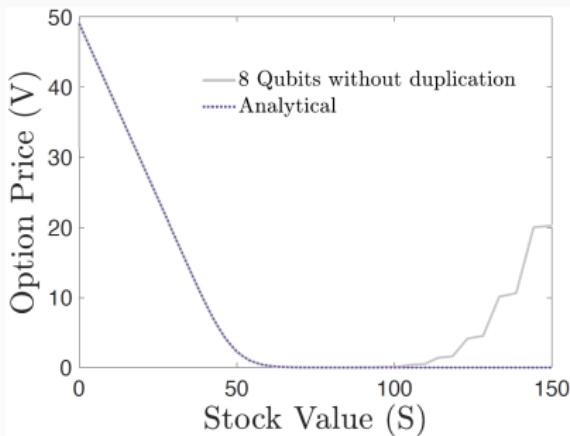
## PDEs: Real-time propagation

- To implement  $\hat{U}(t, t_0)$ , consider an enlarged system, i.e. a doubled unitary operator [Gonzalez-Conde et al., 2021].
- Require of adding an auxiliary qubit.
- $\hat{H}_{\text{BS}}$  is diagonal in momentum space  $\rightarrow$  diagonal operator  $\rightarrow$  QFT (and Inverse QFT)  $\rightarrow$  exponential speedup.
- But, an overall exponential speedup requires efficient loading of the model and payoff function.
- Again, QFT (IQFT) is gate-wise demanding (incompatible NISQ).

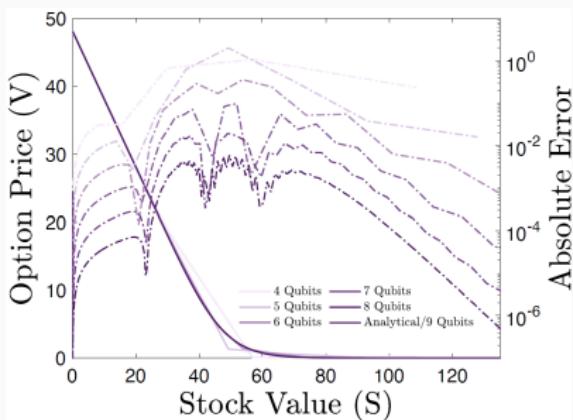


# PDEs: Real-time propagation (solution)

- The algorithm achieves a high degree of agreement in a fault-tolerant quantum computer...
- ...but with a 60% success probability in the measurement and post-selection (depending on the financial parameters).
- Not tested in a real NISQ quantum system.



(a) Boundary error without duplication.



(b) Convergence in qubits (point).

## PDEs: Imaginary-time propagation

- Additional change of variable  $\tau = \sigma^2(T - t)$  and transformation  $v(x, \tau) = \exp(-ax - b\tau)V(t, s)$ , with suitable constants  $a$  and  $b$ ,

$$\frac{\partial v}{\partial \tau} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2}.$$

- Using the Wick rotation  $\tilde{\tau} = -i\tau$  (real time to imaginary time), the heat equation turns into a Schrödinger-like equation,

$$\frac{\partial v}{\partial \tilde{\tau}} = -\hat{H}_{\text{HE}} v,$$

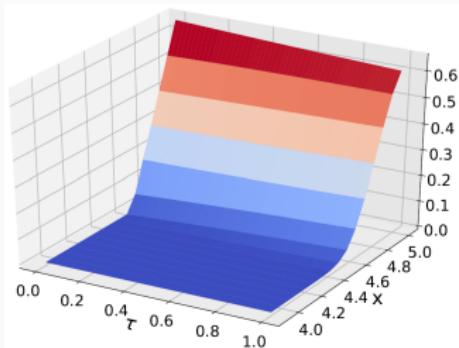
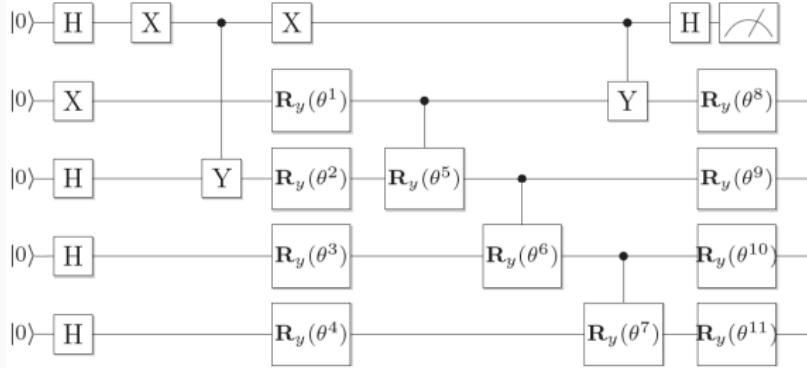
where

$$\hat{H}_{\text{HE}} = -\frac{i}{2} \hat{q}^2, \quad \text{with} \quad \hat{q} = -i \frac{\partial}{\partial x}.$$

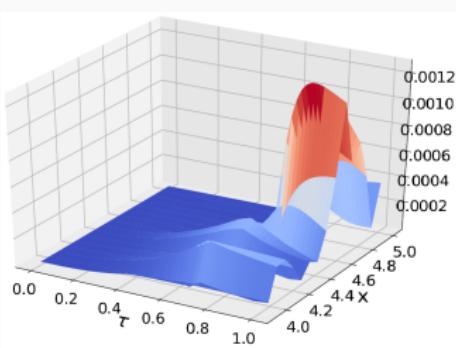
- This leads to a purely anti-Hermitian Hamiltonian operator.
- Imaginary-time propagation transforms oscillations into dampings.
- Problem of finding the ground state of quantum systems, well investigated in condensed matter physics and chemistry.
- The imaginary time evolution operator is approximated by an ansatz circuit in [Fontanella et al., 2021].

# PDEs: Imaginary-time propagation (solution)

- The solution is retrieved by a hybrid quantum-classical algorithm:



(c) Prices of European option.



(d) Errors of European option.

# **Quantum Machine/Deep Learning**

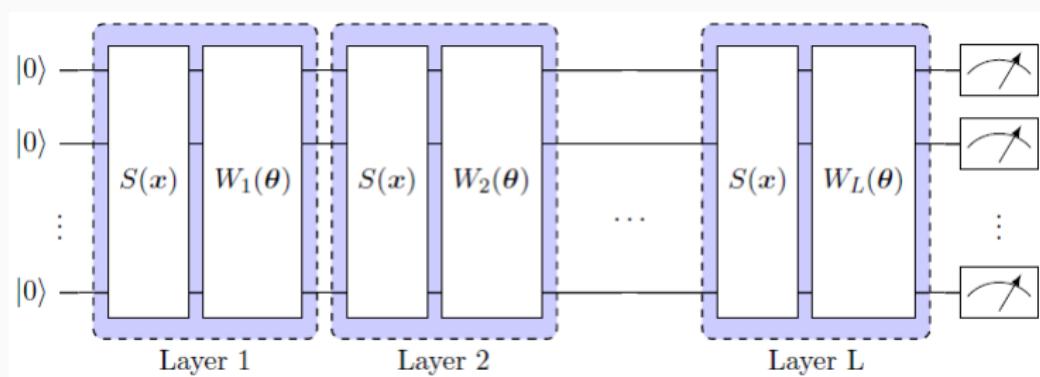
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# Quantum Machine/Deep Learning

- (Quantum) Principal Component Analysis:
  - Eigenvalues by QPE [Nielsen and Chuang, 2001].
  - Covariance matrix → density matrix (QPCA)  
[Lloyd et al., 2014, Abhijith et al., 2020].
- (Quantum) Regression:
  - Solving linear systems by the HHL algorithm [Wiebe et al., 2012].
  - Quantum Kernel Estimation [Egger et al., 2020].
  - Quantum regression with Gaussian processes [Zhao et al., 2019].
- Hybrid classical-quantum deep learning:
  - Training in QC (quantum annealing) [Adachi and Henderson, 2015].
  - Quantum-enhanced reinforcement learning [Saggio et al., 2021].
  - Quantum GANs [Nakaji et al., 2021, Assouel et al., 2022].
  - Boltzman machines → Born machines  
[Vinci et al., 2020, Alcazar et al., 2020].
- Full Quantum Neural Network (QNN):
  - ANNs based on the principles of quantum mechanics [Kak, 1995].
  - How to train QNN? See [Beer et al., 2020, Coyle et al., 2021].
- Promising approach: *Parametrized Quantum Circuits*

# Parametrized Quantum Circuits (PQCs)

- Also known as *variational circuits* or *quantum circuit learning*.
- First theoretical results on *accessibility*, *expressivity* and *universality*.
- Circuits with both fixed and adjustable (“parametrized”) gates.
- The training is carried out by a classical optimiser.
- Each layer composed by a trainable circuit block  $W_i(\theta)$  and a data-encoding block  $S(x)$ :



**Figure 4:** Parametrized Quantum Circuit.

- A PQC model can be written as a generalized trigonometric series:

$$\mathbb{E}[M] = \langle 0 | U^\dagger(\mathbf{x}; \boldsymbol{\theta}) M U(\mathbf{x}; \boldsymbol{\theta}) | 0 \rangle = f(\mathbf{x}; \boldsymbol{\theta}) = \sum_{\omega \in \Omega} c_\omega(\boldsymbol{\theta}) e^{i\omega \mathbf{x}},$$

where  $M$  is an observable,  $U(\mathbf{x}; \boldsymbol{\theta})$  is a quantum circuit that depends on inputs  $\mathbf{x} = (x_0, x_1, \dots, x_N)$  and the parameters  $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_T)$ .

- Accessibility: with  $\Omega \subset \mathbb{Z}^N \rightarrow$  (partial) Fourier series!
- The coefficients  $c_\omega$  determine the expressivity (how the accessible functions can be combined).
- But the expressivity is also limited by the data encoding strategy.
- Universality: the Fourier series formalism allows to study quantum models using the results in Fourier analysis (see [Schuld et al., 2021] and [Manzano et al., 2023a]).

# PQCs: Universality results (I)

## Definition

Let  $U(\mathbf{x}; \boldsymbol{\theta})$  be modelled as a unitary such that (1 layer):

$$U(\boldsymbol{\theta}, \mathbf{x}) = W^{(2)}(\boldsymbol{\theta}^{(2)}) S(\mathbf{x}) W^{(1)}(\boldsymbol{\theta}^{(1)}),$$

and

$$S(\mathbf{x}) = e^{-x_1 H} \otimes \cdots \otimes e^{-x_N H} =: S_H(\mathbf{x})$$

where  $H$  is a particular Hamiltonian.

## Definition

Let  $\{H_m | m \in \mathbb{N}\}$  be a Hamiltonian family where  $H_m$  acts on  $m$  subsystems of dimension  $d$ . Such a Hamiltonian family gives rise to a family of models  $\{f_m\}$  in the following way:

$$f_m(\mathbf{x}) = \langle \Gamma | S_{H_m}^\dagger(\mathbf{x}) M S_{H_m}(\mathbf{x}) | \Gamma \rangle . \quad (1)$$

with  $|\Gamma\rangle := W^{(1)}(\boldsymbol{\theta}^{(1)})|0\rangle$ .

## PQCs: Universality results (II)

**Theorem (Convergence in  $L^2$  [Schuld et al., 2021])**

Let  $\{H_m\}$  be a universal Hamiltonian family, and  $\{f_m\}$  the associated quantum model family, defined via (1). For all functions

$f^* \in L^2([0, 2\pi]^N)$ , and for all  $\epsilon > 0$ , there exists some  $m' \in \mathbb{N}$ , some state  $|\Gamma\rangle \in \mathbb{C}^{m'}$  and some observable  $M$  such that

$$\|f_{m'} - f^*\|_{L^2} < \epsilon.$$

**Theorem (Convergence in  $L^p$  [Manzano et al., 2023a])**

Let  $\{H_m\}$  be a universal Hamiltonian family, and  $\{f_m\}$  the associated quantum model family, defined via (1). For all functions

$f^* \in L^p([0, 2\pi]^N)$  where  $1 \leq p < \infty$ , and for all  $\epsilon > 0$ , there exists some  $m' \in \mathbb{N}$ , some state  $|\Gamma\rangle \in \mathbb{C}^{m'}$ , and some observable  $M$  such that:

$$\|f_{m'} - f^*\|_{L^p} < \epsilon.$$

## PQCs: Universality results (and III)

**Theorem (Convergence in  $C^0$  [Manzano et al., 2023a])**

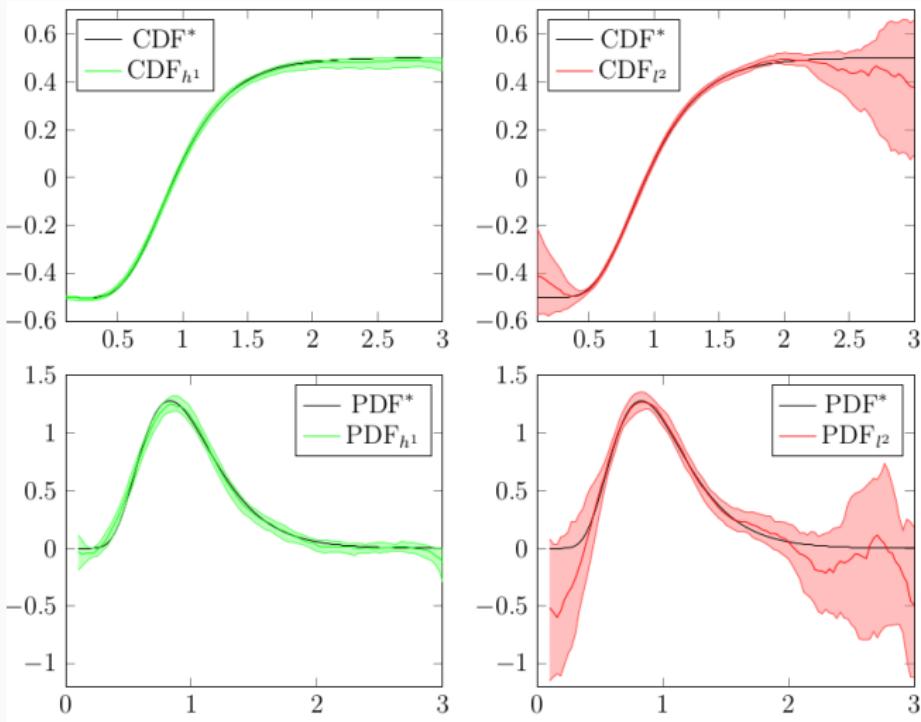
Let  $\{H_m\}$  be a universal Hamiltonian family, and  $\{f_m\}$  the associated quantum model family, defined via (1). For all functions  $f^* \in C^0(U)$  where  $U$  is compactly contained in the closed cube  $[0, 2\pi]^N$ , and for all  $\epsilon > 0$ , there exists some  $m' \in \mathbb{N}$ , some state  $|\Gamma\rangle \in \mathbb{C}^{m'}$ , and some observable  $M$  such that  $f_{m'}$  converges uniformly to  $f^*$ :

$$\|f_{m'} - f^*\|_{C^0} < \epsilon,$$

with

$$\|f_{m'} - f^*\|_{C^0} := \sup_{\mathbf{x} \in [0, 2\pi]^N} \|f_{m'}(\mathbf{x}) - f^*(\mathbf{x})\|.$$

# PQCs for Black-Scholes distribution



**Figure 5:** PQC approximating the Black-Scholes distribution, using the two different empirical risk functions associated to  $L^2$  and  $C^0$  convergence results, respectively.

## Challenges

---

# Discussion on Quantum Monte Carlo

Is the Quantum Monte Carlo what we (computational finance community) expect?

In [Stamatopoulos et al., 2020] they divide the routine for computing the price of a plain vanilla in three steps:



They **promise a quadratic speedup over classical Monte Carlo**:

*"This represents a theoretical quadratic speed-up compared to classical Monte Carlo methods."*

# Classical Monte Carlo vs Quantum Monte Carlo

When claiming a “quadratic” speedup of the QMC over the Classical,  
**what are they comparing?**

Steps involved in Classical and Quantum Monte Carlo :

Quantum Monte Carlo	Classical Monte Carlo
Load Distribution	Load parameters... Simulate the paths
Load Payoff	Compute payoff
Amplitude Estimation	Sum over paths Print the results

*“In most of the existing literature on option pricing for equities using quantum computers... an SDE is tacitly solved... Once this SDE is solved... the pricing of a particular security begins by applying QAE.[Alghassi et al., 2021]”*

# Bottleneck

The bottleneck in Classical Monte Carlo is in simulating paths.

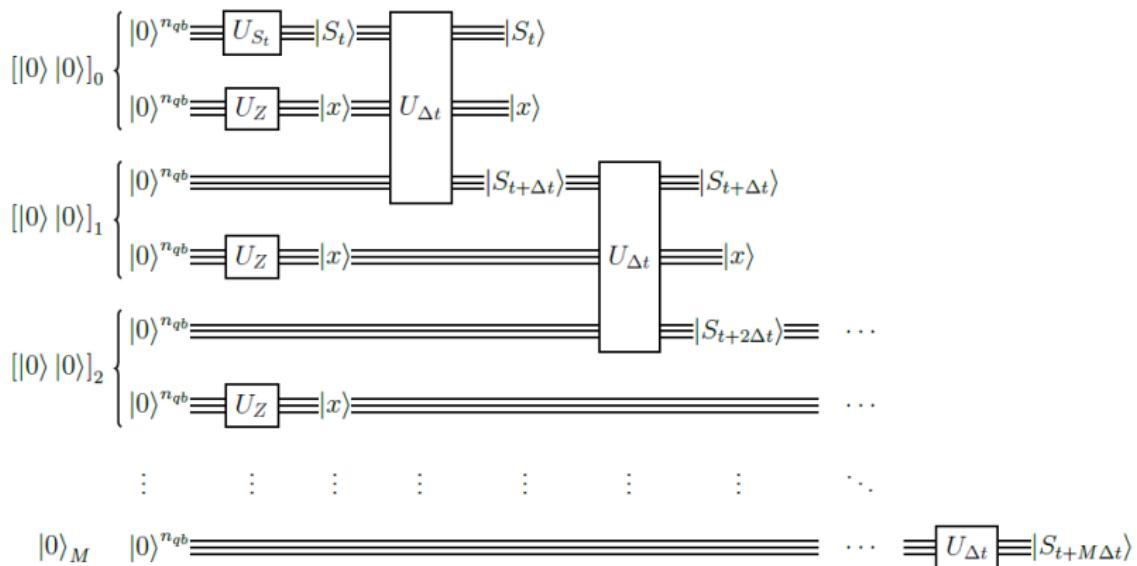
**Analogously**, the bottleneck in the quantum algorithm is in the loading/simulation/computation of the distribution.



The quantum advantage might disappear when taking into account the cost of simulation:

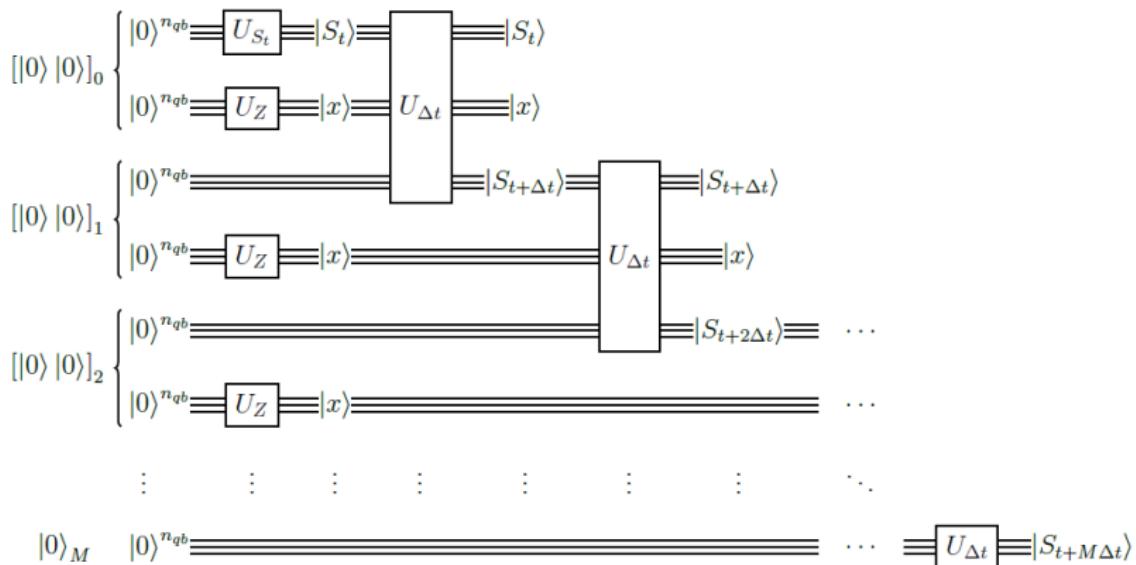
*“Although preparing such states is in principle always possible for reasonable stochastic processes, efficient realization of this method demands a careful analysis and may not always result in a practical quantum advantage.” (see [Alghassi et al., 2021])*

# Quantum Monte Carlo simulation



**Figure 6:** Quantum Monte Carlo simulation.

# Quantum Monte Carlo simulation



**Figure 6:** Quantum Monte Carlo simulation.

In single precision and  $M = 12$ :  $\sim 800$  (logical) qubits!

# Ranking of quantum computers in number of qubits

## Circuit-based quantum processors [edit]

These QPUs are based on the [quantum circuit](#) and [quantum logic gate-based model of computing](#).

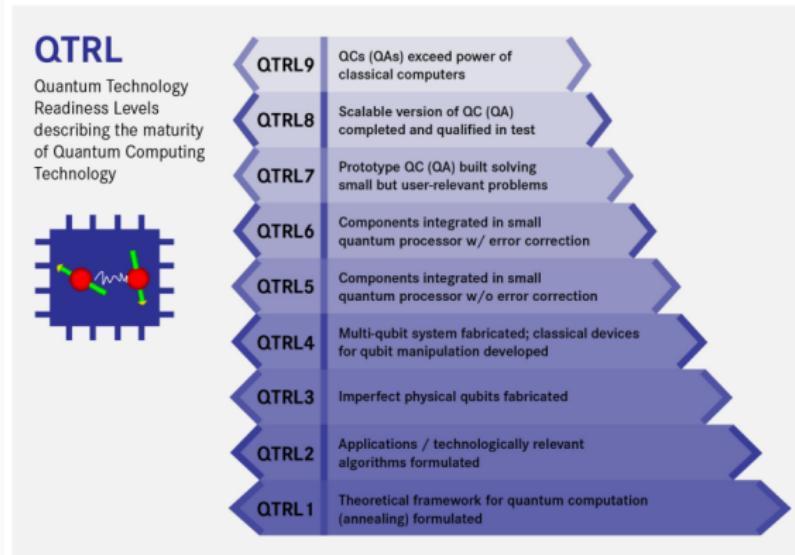
Manufacturer	Name/codename designation	Architecture	Layout	Fidelity (%)	Qubits (physical)	Release date	Quantum volume
Atom Computing	N/A	Neutral atoms in optical lattices			1180 <sup>[6][7]</sup>	October 2023	
IBM	IBM Condor <sup>[16][6]</sup>	Superconducting	N/A	N/A	1121 <sup>[15]</sup>	December 2023	
CAS	Xiaohong <sup>[64]</sup>	Superconducting	N/A	N/A	504 <sup>[64]</sup>	2024	
IBM	IBM Osprey <sup>[6][7]</sup>	Superconducting	N/A	N/A	433 <sup>[15]</sup>	November 2022	
Xanadu	Borealis <sup>[62]</sup>	Photonics (Continuous-variable)	N/A	N/A	216 <sup>[62]</sup>	2022 <sup>[62]</sup>	
M Squared Lasers	Maxwell	Neutral atoms in optical lattices		99.5 (3-qubit gate), 99.1 (4-qubit gate) <sup>[31]</sup>	200 <sup>[32]</sup>	November 2022	
IBM	IBM Heron <sup>[16][6]</sup>	Superconducting	N/A	N/A	133	December 2023	
IBM	IBM Eagle	Superconducting	N/A	N/A	127 <sup>[15]</sup>	November 2021	
Atom Computing	Phoenix	Neutral atoms in optical lattices			100 <sup>[5]</sup>	August 10, 2021	

**Figure 7:** Quantum processors with more than 100 (physical) qubits (21/11/2024).

# Other challenges in algorithms for quantitative finance

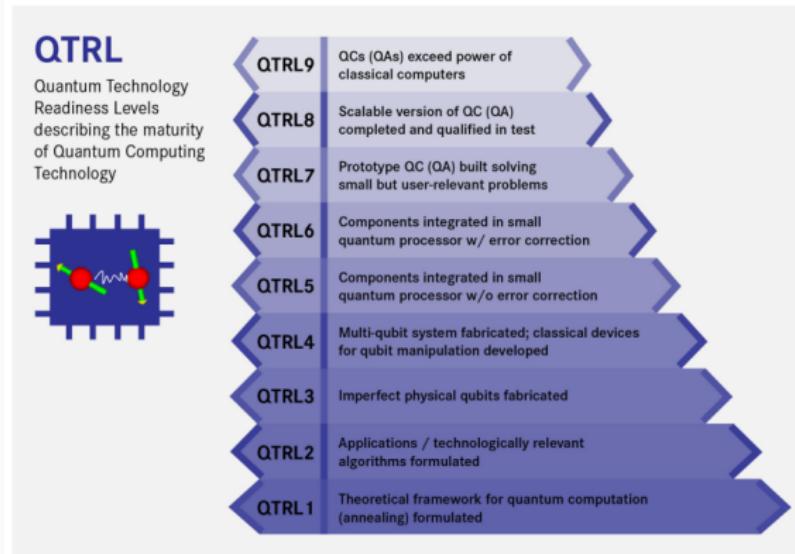
- Data accessibility for Quantum Machine Learning models.
- Quantum-native function implementations (using unitary transforms).
- Information extraction from a quantum state:
  - QAE can be seen as an efficient information extraction routine
  - Post selection in PDE-Hamiltonian simulation algorithms?
- Rigorous proofs for:
  - Speedups (quantum advantage)
  - Estimation convergence
  - Circuits complexity (depth)
- Quantum volume (NISQ):
  - Intrinsic noise of the current quantum systems (the shallower the better)
  - Limited number of qubits (i.e. to represent floating-point numbers)
  - Others: coherence time, measurement errors, circuit compiler efficiency, etc.

# Quantum Technology Readiness Levels



**Figure 8:** Copyright: Kristel Michielsen, Thomas Lippert – Forschungszentrum Jülich. [https://www.fz-juelich.de/ias/jsc/EN/Research/ModellingSimulation/QIP/QTRL/\\_node.html](https://www.fz-juelich.de/ias/jsc/EN/Research/ModellingSimulation/QIP/QTRL/_node.html)

# Quantum Technology Readiness Levels



**Figure 8:** Copyright: Kristel Michielsen, Thomas Lippert – Forschungszentrum Jülich. [https://www.fz-juelich.de/ias/jsc/EN/Research/ModellingSimulation/QIP/QTRL/\\_node.html](https://www.fz-juelich.de/ias/jsc/EN/Research/ModellingSimulation/QIP/QTRL/_node.html)

QTRL 9 expected in 2035!

## Opportunities

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# Opportunities

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- Quantum Monte Carlo: All the described above!
- PDEs:
  - Mapping to linear systems (classical numerical methods)
  - Proofs of algorithm complexity (close to exponential speedup?)
- Proofs for the universality/expressivity of quantum DL models:
  - Theory behind the general reproducing kernels
- Adapt/use QML algorithms for financial applications.
- Efficient quantum versions of successful classical algorithms:
  - Series expansions
  - PINNs
  - Differential Machine Learning
- Other:
  - Alternatives to Harrow-Hassidim-Lloyd (HHL) for linear systems.
  - Similar opportunities in optimization algorithms.
- At practical level: NISQ era!

# Conclusions

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- In recent years we have seen significant advances in quantum algorithms with application to financial mathematical problems.
- While this progress is very encouraging, further work will be required to prove that Quantum Computing can deliver real-world advantage.
- Especially if this advantage is to be delivered on NISQ technology with limitations to both the number of logical qubits and the depth of quantum circuits.
- Research into financial applications of quantum computing is accelerating with new ideas emerging at rapid pace...
- ...but important breakthroughs across the technology stack will be needed to make the approaches viable.
- Theory/software is ahead of practice/hardware!
- Plenty of room for contributions!

# Thanks!!

More: [alvaro.leitao@udc.gal](mailto:alvaro.leitao@udc.gal) and [alvaroleitao.github.io](https://github.com/alvaroleitao)

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## QMC: Risk measures

Find  $\text{VaR}_\alpha(X) = \inf\{x : \mathbb{P}[X \leq x] \geq 1 - \alpha\} = \inf\{x : F_X(x) \geq 1 - \alpha\}$ :

$$f_J(x) = \begin{cases} 1 & \text{if } x \leq x_J \\ 0 & \text{otherwise} \end{cases}$$

Thus, the original QMC state becomes

$$|\psi\rangle = |0\rangle \otimes \sum_{j=J+1}^{N-1} p(x_j) |j\rangle + |1\rangle \otimes \sum_{j=0}^J \sqrt{p(x_j)} |j\rangle$$

A bisection search over  $J$  and measuring  $|1\rangle$  gives the  $x_{J_\alpha} \approx \text{VaR}_\alpha(X)$

To estimate  $\text{CVaR}_\alpha(X)$ , take  $f(x) = \frac{x}{x_{J_\alpha}} f_{J_\alpha}(x)$ , so

$$|\psi\rangle = |0\rangle \otimes \left( \sum_{j=J_\alpha+1}^{N-1} p(x_j) |j\rangle + \sum_{j=0}^{J_\alpha} \left(1 - \frac{x_j}{x_{J_\alpha}}\right) p(x_j) |j\rangle \right) + |1\rangle \otimes \sum_{j=0}^{J_\alpha} \sqrt{\frac{x_j}{x_{J_\alpha}} p(x_j)} |j\rangle$$

and measure  $|1\rangle$ . Then,  $\text{CVaR}_\alpha(X) \approx \frac{x_{J_\alpha}}{1-\alpha} \sum_{j=0}^{J_\alpha} \frac{x_j}{x_{J_\alpha}} p(x_j)$

# QMC: Pricing

Using Y-rotations and a comparator (in  $K$ ), we can construct:

$$\begin{aligned} |\psi\rangle &= |0\rangle \otimes \sum_{x_j < K} \sqrt{p(x_j)} |j\rangle [\cos(g_0) |0\rangle + \sin(g_0) |1\rangle] \\ &\quad + |1\rangle \otimes \sum_{x_j \geq K} \sqrt{p(x_j)} |j\rangle [\cos(g_0 + g(x_j)) |0\rangle + \sin(g_0 + g(x_j)) |1\rangle] \end{aligned}$$

The probability of measuring the second *ancilla* (auxiliary) state  $|1\rangle$  is:

$$P = \sum_{x_j < K} p(x_j) \sin^2(g_0) + \sum_{x_j \geq K} p(x_j) \sin^2(g_0 + g(x_j))$$

For a European call ( $\max(0, x_j - K)$ ), set  $g(x) = \frac{2c(x-K)}{x_{\max}-K}$ ,  $g_0 = \frac{\pi}{4} - c$ .

Thus, using that  $\sin^2(cf(x) + \frac{\pi}{4}) = cf(x) + \frac{1}{2} + \mathcal{O}(c^3 f^3(x))$ , we have

$$\begin{aligned} P &\approx \sum_{x_j < K} p(x_j) \left( \frac{1}{2} - c \right) + \sum_{x_j \geq K} p(x_j) \left( \frac{2c(x_j - K)}{x_{\max} - K} + \frac{1}{2} - c \right) \\ &= \frac{1}{2} - c + \frac{2c}{x_{\max} - K} \sum_{x_j \geq K} p(x_j)(x_j - K) \end{aligned}$$

