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DIRECT AND INDIRECT BOUNDARY ELEMENT METHODS FOR SOLVING THE HEAT CONDUCTION PROBLEM

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The boundary element method is used to solve the stationary heat conduction problem as a Dirichlet, a Neumann or as a mixed boundary value problem. Using singularities which are interpreted physically, a number of Fredholm integral equations of the first or second kind is derived by the indirect method. With the aid of Green's third identity and Kupradze's functional equation further direct integral equations are obtained for the given problem. Finally a numerical method is described for solving the integral equations using Hermitian polynomials for the boundary elements and constant, linear, quadratic or cubic polynomials for the unknown functions.

1. Introduction

When solving the stationary heat conduction problem using the boundary element method, then, depending on the approach used, there are several Fredholm integral equations of the first or second kind available for the Dirichlet, for the Neumann and for the mixed boundary value problem. To make things clearer these can be divided into

(a) indirect, and
(b) direct

integral equations.

In group (a) integral equations are gathered which present the boundary condition of the problem under consideration (here the temperature boundary condition). These equations can be derived with the aid of a fictitious distribution of singularities in an infinite plane along a definite boundary curve congruent to the given boundary. The fictitious distribution of singularities is the unknown function which can usually be interpreted physically and can be determined from the satisfaction of the boundary conditions. In a further step, hence the indirect method, the temperature field in each point of the infinite plane can be calculated. Heat sources and heat dipoles are used here as singularities. Further singularities, such as vortex, vortex-dipole or quadrupole are not considered since they lead formally to the same integral equations. Indirect integral equations describing the temperature field are derived and solved numerically in [1] by calculating the thermal stresses in discs. With the aid of an integral equation of the first kind the Dirichlet problem is solved in [2] using heat sources as singularities. Numerical results confirm there the applicability of this integral equation. The process is then transferred to non-stationary temperature fields. The solution of Dirichlet's and

Neumann's problem with the indirect method is then dealt with in detail in [3] for the St.-Venant's torsion problem.

The direct integral equations in group (b) are those which contain as an unknown function either the function sought, or its derivatives at the boundary. They are derived generally in terms of mathematical relationships of potential theory. The solution of the integral equations thus obtained provides directly the boundary values of the unknown function. The values of the function at internal points can be calculated from the known boundary values. A direct method for solving the heat conduction problem can be found in [4–9]. Starting from Green's third identity an integral equation is formulated there which is of the second kind for the Neumann problem and of the first kind for the Dirichlet problem. For the mixed problem first and second kind integral equations both occur in various parts of the boundary. A review of direct and indirect integral equations is given in [10] for the St.-Venant torsion problem.

The aim of this paper is firstly to gather together known integral equations using a uniform notational system, and secondly to derive further direct and indirect integral equations in order to give as comprehensive a presentation as possible of the heat conduction problem in terms of integral equations. On the basis of this presentation in a further paper [11] the integral equations will be solved numerically and compared. Such a comparison has not previously been possible since most investigations have only solved certain, mainly direct integral equations.

2. The basic equations

A finite, homogeneous and thermal isotropic solid body $\bar{D}$ is given, see Fig. 1. The boundary curve Γ of the body is made up of continuous smooth curves Γ_i, $i = 0, 1, \ldots, m$. The approximately valid assumption is made that the conductivity λ and the specific heat c are independent of temperature. In the stationary case the temperature field arising from a thermal load is sought.

The two-dimensional heat conduction problem is described by the Laplace differential equation

$$\Delta T(x, y) = 0, \quad (x, y) \in D, \tag{1}$$

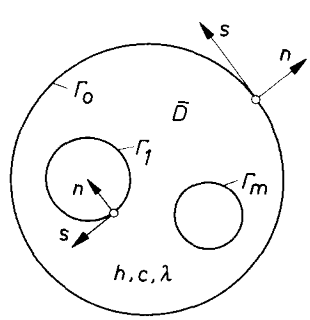


Fig. 1. Geometry of the solid.

and, according to the boundary conditions, can be treated

(a) as a Dirichlet problem,

$$T(x, y) = f_1(x, y), \quad (x, y) \in \Gamma ; \tag{2}$$

(b) as a Neumann problem,

$$\frac{\partial T}{\partial n}(x, y) = f_2(x, y), \quad (x, y) \in \Gamma ; \tag{3}$$

(c) as a mixed boundary value problem, with a boundary temperature $T(x, y)$ for a part of the boundary Γ and a temperature gradient $\partial T(x, y)/\partial n$ for the rest of the boundary as the given boundary conditions.

All these problems are described here using integral equations.

3. The singularities in the infinite elastic plane

As already mentioned in Section 1, in the indirect method the boundary conditions of the given problem are formulated as integral equations with the aid of singularities. It is thus necessary to examine the singularities in the elastic infinite plane.

3.1. The heat source in the infinite plane

A heat source at a source point $Q(x_Q, y_Q)$, see Fig. 2, with intensity W produces a temperature $T(P)$ at a field point $P(x_P, y_P)$. The temperature at the field point P can be calculated from the following relationship, see also [12]:

$$T(P) = -\frac{W}{2h\pi\lambda} \ln r_{PQ} = -\frac{w}{2\pi} \ln r_{PQ}, \tag{4}$$

with

$$r_{PQ}^2 = (x_P - x_Q)^2 + (y_P - y_Q)^2,$$

where λ is the conductivity and h the thickness of the solid. The source intensity W has the same dimension as the temperature.

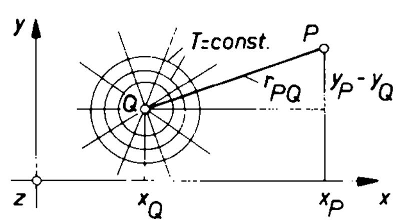


Fig. 2. Heat source and isothermals in the infinite plane.

3.2. The heat dipole in the infinite plane

In a ξ, η coordinate system, see Fig. 3, consider the heat source in the point $Q_1(\varepsilon, 0)$ with the intensity w and in the point $Q_2(-\varepsilon, 0)$ the heat source with the intensity $-w$ (heat sink). The intensity w is related to λh as in (4).

At the field point $P(x_P, y_P)$ the following temperature is obtained from the superposition of the temperature fields from the heat sources in Q_1 and Q_2 with intensities w and $-w$ respectively:

$$T(P) = T_1 + T_2 = \frac{w}{2\pi}(\ln r_2 - \ln r_1), \tag{5}$$

where

$$r_1^2 = (\xi_P - \xi_Q - \varepsilon)^2 + (\eta_P - \eta_Q)^2, \qquad r_2^2 = (\xi_P - \xi_Q + \epsilon)^2 + (\eta_P - \eta_Q)^2. \tag{6}$$

Applying the mean value theorem to the right-hand side of (5) gives

$$T(P) = -\frac{w}{2\pi} 2\varepsilon \frac{\partial}{\partial \xi}\{\ln[(\xi_P - \xi)^2 + (\eta_P - \eta_Q)^2]\}_{\xi_k}, \tag{7}$$

if ξ_k adopts a suitable value between $(\xi_Q - \varepsilon)$ and $(\xi_Q + \varepsilon)$. Now if ε is allowed to tend to zero, and at the same time w increases so that the product $2\varepsilon w$ approaches a finite limit M_ξ, then a heat dipole is formed, see Fig. 3, with 'moment' M_ξ. In this case, since ξ_k tends to ξ_Q, the temperature at a field point P is given by

$$T(P) = \frac{M_\xi}{2\pi} \frac{\xi_P - \xi_Q}{(\xi_P - \xi_Q)^2 + (\eta_P - \eta_Q)^2} = \frac{M_\xi}{2\pi} \frac{(x_P - x_Q)\cos\beta + (y_P - y_Q)\sin\beta}{r_{PQ}^2}. \tag{8}$$

The approach of source and sink is along an arbitrary axis (dipole axis) ξ, determined by $\beta = \sphericalangle(x, \xi)$ in the direction from $-w$ to $+w$.

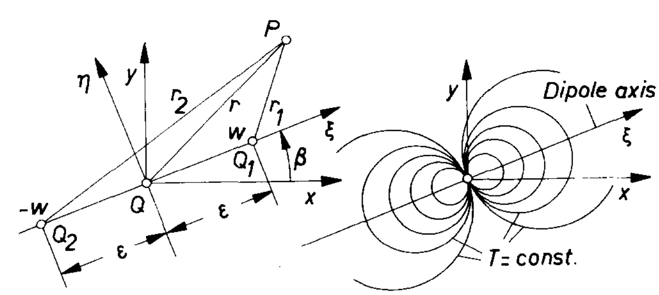


Fig. 3. Heat source and heat sink producing a heat dipole.

4. Indirect integral equations

In the indirect method (or method of singularities) the body given in Fig. 1 is now on an infinitely extended plane with the same elastic and thermal properties, see Fig. 4. The boundary Γ in Fig. 4 is congruent to the given boundary curve and is made up of Γ_0, $\Gamma_1, \ldots, \Gamma_m$. Along it the initially unknown singularities are placed, producing a certain temperature in each point of the infinite elastic body. Consider now the boundary curve Γ^+ of the domain D^+, consisting of Γ_0^+, $\Gamma_1^+, \ldots, \Gamma_m^+$, enclosed by Γ, though sharing no common point. If now let Γ^+ approach Γ, and if on Γ^+, as a consequence of the distribution of singularities on Γ, the boundary conditions of the Dirichlet or Neumann problem are fulfilled, then the temperature field of the area enclosed by Γ coincides with the actual temperature field. Thus a Fredholm integral equation of the first or second kind is obtained for the boundary condition.

From the solution of the integral equation the distribution of singularities is obtained which is necessary along Γ to satisfy the boundary condition all along Γ^+. With this distribution of singularities the state of temperature at all points on the infinite plane can be obtained, and thus also at all points in D^+.

At this point it should be mentioned that the heat singularities produce stresses and displacements at all points in D, in addition to temperature effects. They do not however in general conform with the stresses and displacements arising as a result of the given thermal load. To calculate the actual stresses and displacements further singularities are needed. They must be distributed along Γ so that in Γ^+ ($\Gamma^+ \rightarrow \Gamma$) the stresses or displacements caused by the heat singularities can be compensated. Which singularities should be used in which case and how stresses and displacements resulting from temperature load can be calculated using the indirect Boundary Element Method is described in [1] and in [13].

4.1. A heat source distribution and its integral equations

The temperature field in an infinite plane as a result of a heat source can be calculated from (4). A continuous heat source distribution $q(Q) = w(Q)\,\mathrm{d}s_Q$ along Γ, see Fig. 4, leads at the

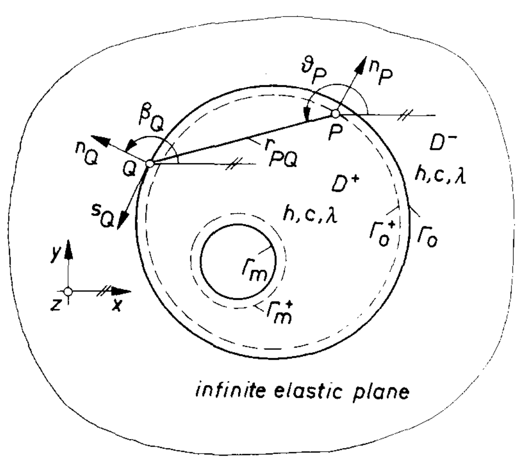


Fig. 4. The solid on an infinite plane.

field point $P(x_P, y_P)$ of the elastic infinite plane to the temperature

$$T(P) = -\frac{1}{2\pi}\oint_\Gamma q(Q) \ln r_{PQ} \, \mathrm{d}s_Q, \quad P \in D \cup \Gamma. \tag{9}$$

The equation is initially valid for all points $P \in D$, $D = D^+ \cup D^-$. For the boundary points non-integral terms can arise from the charge of limits $\Gamma^+ \to \Gamma$. The Cauchy principal value ($\oint$) must then be calculated for the curve integral. The kernels delivered by the non-integral terms can be taken from [3]. There it is shown that (9) is also valid for boundary points. In the following the integral equations are given with the non-integral terms, if any. Equation (9) is a Fredholm integral equation of the first kind for the Dirichlet problem and represents the single layer potential. It is used in [1] and [2] for calculating thermal stresses and the temperature field for two-dimensional problems. In potential theory it is shown (e.g. in [14, 15]) that (9) does not always have a unique solution, which can however be avoided by a simple change in scale. Further details of the numerical solution of (9) can be found in [11] and the literature given there.

If partial derivatives $\partial/\partial n$ and $\partial/\partial s$ are formed from (9) at the point P, keeping the source point Q fixed, this gives

$$\frac{\partial T}{\partial n}(P) = \alpha q(P) - \frac{1}{2\pi}\oint_\Gamma q(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \, \mathrm{d}s_Q, \quad \alpha = \begin{cases} 0 & \text{for } P \in D, \\ \frac{1}{2} & \text{for } P \in \Gamma; \end{cases} \tag{10}$$

$$\frac{\partial T}{\partial s}(P) = -\frac{1}{2\pi}\oint_\Gamma q(Q) \frac{\partial \ln r_{PQ}}{\partial s_P} \, \mathrm{d}s_Q, \quad P \in D \cup \Gamma. \tag{11}$$

Equation (10) is a Fredholm integral equation of the second kind for the boundary condition of the Neumann problem. This problem can only be solved when the condition

$$\oint_\Gamma \frac{\partial T}{\partial n}(Q) \, \mathrm{d}s_Q = 0 \tag{12}$$

is satisfied. However, the derivative $\partial T/\partial n$ represents here, with the exception of one factor, the heat flux. For stationary temperature fields (12) is satisfied and thus (10) is definitely soluble. Solving (10) gives the source distribution $q(Q)$, enabling the calculation from (9) of the temperature $T(P)$, with the exception of a constant, in all points $P \in D \cup \Gamma$. To determine the constant the temperature at some point of the area under consideration must be known. The constant can be determined with the help of (9). The Neumann problem is thus fully solved.

When handling the mixed boundary value problem (9) must be satisfied for a part of the boundary and for the rest (10), i.e., there are segments with integral equations of the first and other of the second kind.

An integral equation of the second kind can be made available for the Dirichlet problem by using other singularities. The mixed problem can thus be solved if, as a result of a suitable combination of singularities, only integral equations of the second kind occur at the boundary.

However, at the transition points where the Neumann problem changes to the Dirichlet or vice versa, the continuity conditions of the functions of the singularities must be satisfied. The path to solving boundary value problems with mixed boundary conditions is described in more detail in [11] and in [16].

4.2. *The integral equations resulting from a heat dipole distribution*

Before the integral equations are formulated some mathematical relationships will be established. They can be easily derived with reference to Fig. 4. The function $\ln r_{PQ}$ and $\vartheta_P(P, Q)$ satisfy the Cauchy–Riemann differential equations

$$\frac{\partial \ln r_{PQ}}{\partial n} = \frac{\partial \vartheta_P(P, Q)}{\partial s} \quad \text{and} \quad \frac{\partial \ln r_{PQ}}{\partial s} = -\frac{\partial \vartheta_P(P, Q)}{\partial n} . \tag{13}$$

The derivative from (13) can be formed for the field point P as well as for the source point Q. The partial derivatives in the direction of n_Q, keeping here the field point P fixed, are given by the relationships

$$\frac{\partial \ln r_{PQ}}{\partial n_Q} = -\frac{(x_P - x_Q)\cos\beta_Q + (y_P - y_Q)\sin\beta_Q}{r_{PQ}^2} , \tag{14}$$

$$\frac{\partial \ln r_{PQ}}{\partial s_Q} = -\frac{(y_P - y_Q)\cos\beta_Q - (x_P - x_Q)\sin\beta_Q}{r_{PQ}^2} . \tag{15}$$

The notation used can be seen in Fig. 4.

In Section 3.2 the equation for the temperature field arising from a dipole with the dipole axis in the ξ direction was derived. Such a dipole distribution $M_\xi(Q) = m_n(Q)\,\mathrm{d}s_Q$ is now introduced on Γ, such that at every point Q the dipole axis coincides with the direction of the normal n_Q. From Fig. 4 it can be seen that in this case $\beta = \sphericalangle(x, \xi) = \beta_Q$. With this dipole distribution, and taking (14) into consideration, the temperature field is obtained from (8),

$$T(P) = \alpha m_n(P) - \frac{1}{2\pi}\oint_\Gamma m_n(Q)\frac{\partial \ln r_{PQ}}{\partial n_Q}\,\mathrm{d}s_Q , \quad \alpha = \begin{cases} 0 & \text{for } P \in D , \\ -\frac{1}{2} & \text{for } P \in \Gamma , \end{cases} \tag{16}$$

and thus a Fredholm integral equation of the second kind for the boundary condition of the Dirichlet problem. This equation was derived in [1] and numerically tested. It represents mathematically the double layer potential and has an unique solution for simple connected areas. For a multiple connected domain the dipole distribution $m_n(Q)$, $Q \in \Gamma$, is not sufficient (cf. [14]). Additional singularities, e.g. heat sources, or heat dipoles, must be introduced. Further details of the solution of the Dirichlet problem with the help of the double layer potential for multiple connnected domains can be found in [11].

In the same way as in Section 4.1, the partial derivatives $\partial/\partial n$ and $\partial/\partial s$ can be formed from

(16) for the field point P while the source point Q is fixed. One obtains the relationships

$$\frac{\partial T}{\partial n}(P) = -\frac{1}{2\pi}\oint_\Gamma m_n(Q)\frac{\partial^2 \ln r_{PQ}}{\partial n_P \partial n_Q}\,\mathrm{d}s_Q, \quad P \in D, \tag{17}$$

$$\frac{\partial T}{\partial s}(P) = -\frac{1}{2\pi}\oint_\Gamma m_n(Q)\frac{\partial^2 \ln r_{PQ}}{\partial s_P \partial n_Q}\,\mathrm{d}s_Q, \quad P \in D, \tag{18}$$

which are initially valid for points P not lying at the boundary. The kernels of the integrals from (17) and (18) are singular with $1/r^2$. In (17) the integral remains unbounded when source point Q and field point P coincide at the boundary. Thus the temperature gradient can only be calculated from (17) for points $P \in D$, since the integral for the boundary points does not exist. Numerical investigations in [16] have shown that integral kernels with poles of the second order also provide useless results in the neighbourhood of the boundary. For this reason such integral equations should be avoided. It follows from this that a solution of the Neumann problem is not directly possible using heat dipoles. However, this difficulty in calculating the boundary values of the temperature gradient can be overcome. From the temperature distribution for points not lying at the boundary, and taking (13) into account, one obtains

$$T(P) = -\frac{1}{2\pi}\oint_\Gamma m_n(Q)\frac{\partial \ln r_{PQ}}{\partial n_Q}\,\mathrm{d}s_Q = -\frac{1}{2\pi}\oint_\Gamma m_n(Q)\frac{\partial \vartheta_P(P, Q)}{\partial s_Q}\,\mathrm{d}s_Q, \quad P \in D, \tag{19}$$

which, after integration by parts, can be rewritten as

$$T(P) = \frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\vartheta_P(P, Q)\,\mathrm{d}s_Q - \frac{1}{2\pi}\oint_\Gamma \frac{\partial}{\partial s_Q}[m_n(Q)\vartheta_P(P, Q)]\,\mathrm{d}s_Q, \quad P \in D. \tag{20}$$

Since the second integral produces a constant, (20) gives

$$T(P) = \frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\vartheta_P(P, Q)\,\mathrm{d}s_Q - m_n(Q_0), \quad P \in D, \tag{21}$$

where Q_0 is an arbitrary boundary point independent of the field point P. For $P \in D^-$, $m_n(Q_0)$ is eliminated. If we now differentiate at the field point P, again taking (13) into account, then we obtain a Fredholm integral equation of the first kind for the temperature gradient,

$$\frac{\partial T}{\partial n}(P) = \frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\frac{\partial \vartheta_P(P, Q)}{\partial n_P}\,\mathrm{d}s_Q = -\frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\frac{\partial \ln r_{PQ}}{\partial s_P}\,\mathrm{d}s_Q, \quad P \in D \cup \Gamma. \tag{22}$$

This is also valid for the boundary points because the kernel gives no non-integral term, when $\Gamma^+ \to \Gamma$ (see [3]). The calculation of the temperature gradient from (22) thus requires a differentiation of the dipole distribution $m_n(Q)$, $Q \in \Gamma$, derived from the solution of (16). This differentiation which reduces the accuracy of the results can be avoided if instead of (16) an integral equation is derived, with the derivative $\partial/\partial s$ of the dipole distribution as the unknown function. This is done below.

The differentiation of (21) in the direction of s_P leads to the following equation:

$$\frac{\partial T}{\partial s}(P) = \frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\frac{\partial \vartheta_P(P,Q)}{\partial s_P}\,\mathrm{d}s_Q = \frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\frac{\partial \ln r_{PQ}}{\partial n_P}\,\mathrm{d}s_Q, \quad P \in D, \tag{23}$$

which at the limit $\Gamma^+ \to \Gamma$, see Fig. 4, takes on the form

$$\frac{\partial T}{\partial s}(P) = -\frac{1}{2}\frac{\partial m_n}{\partial s}(P) + \frac{1}{2\pi}\oint_\Gamma \frac{\partial m_n}{\partial s}(Q)\frac{\partial \ln r_{PQ}}{\partial n_P}\,\mathrm{d}s_Q, \quad P \in \Gamma. \tag{24}$$

Since in the given temperature field $T(P)$, $P \in \Gamma$, $\partial T(P)/\partial s$ is also known, it is advisable, in solving the Dirichlet problem, to solve (24) when seeking to find the temperature gradient at the boundary rather than (16) and then to calculate the boundary values for $\partial T(P)/\partial n$ from (22).

In this section the integral equations have so far been derived from the dipole distribution with the dipole axis in the n_Q direction. In the same way further integral equations can be obtained with the dipole distribution in the s_Q direction. The temperature field of such a distribution can be obtained from (8), if the substitution is made $\beta = \beta_Q + \frac{1}{2}\pi$. The integral equation thus formed,

$$T(P) = -\frac{1}{2\pi}\oint_\Gamma m_s(Q)\frac{\partial \ln r_{PQ}}{\partial s_Q}\,\mathrm{d}s_Q, \quad P \in D \cup \Gamma, \tag{25}$$

which is also valid for boundary points, represents the modified potential of a simple layer, see [14]. The partial derivatives $\partial/\partial n$ and $\partial/\partial s$ for a field point P again keeping the source point Q fixed are

$$\frac{\partial T}{\partial n}(P) = -\frac{1}{2\pi}\oint_\Gamma m_s(Q)\frac{\partial^2 \ln r_{PQ}}{\partial n_P \partial s_Q}\,\mathrm{d}s_Q, \quad P \in D, \tag{26}$$

$$\frac{\partial T}{\partial s}(P) = -\frac{1}{2\pi}\oint_\Gamma m_s(Q)\frac{\partial^2 \ln r_{PQ}}{\partial s_P \partial s_Q}\,\mathrm{d}s_Q, \quad P \in D, \tag{27}$$

the kernels of which, as in (17) and (18) are singular with poles of the second order. It can be shown that (26) is also valid for boundary points. With the dipole distribution obtained by solving (25), the temperature gradient at the boundary can be calculated from (26). The integral equation (25) is however of the first kind and not uniquely soluble. The unique solution of (25) would demand an additional condition. Further details about it can be found in

[11]. The integration by parts of (25) gives the relationship

$$T(P) = -\frac{1}{2\pi}\oint_{\Gamma} \frac{\partial m_s}{\partial s}(Q) \ln r_{PQ}\, \mathrm{d}s_Q, \quad P \in D \cup \Gamma, \tag{28}$$

which on comparison with (9) gives

$$q(Q) = \frac{\partial m_s}{\partial s}(Q). \tag{29}$$

In this case are obtained formally the same integral equations as with a heat source distribution $q(Q)$. Different singularities do not help to gain new integral equations for the stated problems. A vortex distribution, for example, gives the same integral equations already obtained from the distribution $\partial m_n(Q)/\partial s_Q$. It is not therefore necessary to produce further derivations for these integral equations.

5. Direct integral equations

Direct integral equations are understood to be those which have either the boundary temperature $T(Q)$, or the derivatives $\partial T(Q)/\partial n$, $\partial T(Q)/\partial s$ as unknown functions. These can be regarded as a special case of the indirect integral equations, since they can be derived from them, given a suitable combination of singularities, cf. [17] for the torsion problem. In the same way the indirect can also be viewed as a special case of the direct integral equations, cf. [18] and thus the two formulations are fully equivalent.

5.1. *The integral equation for the boundary temperature $T(Q)$*

According to Green's third identity in the plane, the following equations apply for a harmonic function $T(x, y)$:

$$\oint_{\Gamma} \left[T(Q)\frac{\partial \ln r_{PQ}}{\partial n_Q} - \frac{\partial T}{\partial n}(Q) \ln r_{PQ} \right] \mathrm{d}s_Q = \begin{cases} \pi T(P), & P \in \Gamma, \quad (30) \\ 2\pi T(P), & P \in D^+ \quad (31) \\ 0, & P \in D^-. \quad (32) \end{cases}$$

When tackling the heat conduction problem (30) is first solved. For the Neumann problem, i.e. with a given temperature gradient, (30) is a Fredholm integral equation of the second kind, the solution of which gives the boundary temperature except for a constant. To determine the constant, the temperature at any one point must be known. With the boundary values $T(Q)$ and the given temperature gradient the temperature for any point $P \in D^+$ can be then calculated from (31). With a Dirichlet problem, however, where the temperature $T(Q)$, $Q \in \Gamma$, is given, then (30) is an integral equation of the first kind for the function $\partial T/\partial n$. With the boundary value $\partial T/\partial n$ and the known temperature $T(Q)$ at the boundary, then the temperature $T(P)$, $P \in D^+$, can again be calculated from (31). For the mixed problem (30) is an integral equation of the first kind for part of the boundary and of the second kind for the rest. In [4–9] the heat conduction problem is treated according to (30) as a mixed problem.

From Green's formula further direct integral equations can be derived in addition to (30). It thus suggests itself to attempt to derive a direct integral equation of the second kind for the Dirichlet boundary value problem.

5.2. The integral equations for $\partial T/\partial n$ and $\partial T/\partial s$

Green's third identity is again taken as the starting point from which to derive direct integral equations of the second kind for the temperature gradient $\partial T/\partial n$ and for $\partial T/\partial s$. Taking (13) into account, (31) is rewritten as

$$-2\pi T(P) + \oint_\Gamma T(Q) \frac{\partial \vartheta_P(P, Q)}{\partial s_Q} \, \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial n}(Q) \ln r_{PQ} \, \mathrm{d}s_Q, \quad P \in D^+ . \tag{33}$$

After integration by parts the curve integral on the left-hand side of (33) can be expressed as

$$\oint_\Gamma T(Q) \frac{\partial \vartheta_P(P, Q)}{\partial s_Q} \, \mathrm{d}s_Q = -\oint_\Gamma \frac{\partial T}{\partial s}(Q) \vartheta_P(P, Q) \, \mathrm{d}s_Q + \oint_\Gamma \frac{\partial}{\partial s_Q}[T(Q)\vartheta_P(P, Q)] \, \mathrm{d}s_Q, \quad P \in D^+ . \tag{34}$$

Substituting in (33), and recognising that the second integral on the right-hand side of (34) gives the value $2\pi T(Q_0)$, where Q_0 is an arbitrary boundary point independent of P, we obtain

$$-2\pi T(P) - \oint_\Gamma \frac{\partial T}{\partial s}(Q) \vartheta_P(P, Q) \, \mathrm{d}s_Q + 2\pi T(Q_0) = \oint_\Gamma \frac{\partial T}{\partial n}(Q) \ln r_{PQ} \, \mathrm{d}s_Q, \quad P \in D^+ . \tag{35}$$

Differentiating (35) in the direction from n_P and s_P, with Q fixed, gives the equations

$$-2\pi \frac{\partial T}{\partial n}(P) - \oint_\Gamma \frac{\partial T}{\partial s}(Q) \frac{\partial \vartheta_P(P, Q)}{\partial n_P} \, \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial n}(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \, \mathrm{d}s_Q, \quad P \in D^+ , \tag{36}$$

$$-2\pi \frac{\partial T}{\partial s}(P) - \oint_\Gamma \frac{\partial T}{\partial s}(Q) \frac{\partial \vartheta_P(P, Q)}{\partial s_P} \, \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial n}(Q) \frac{\partial \ln r_{PQ}}{\partial s_P} \, \mathrm{d}s_Q, \quad P \in D^+ , \tag{37}$$

which can be rewritten, with (13), as

$$2\pi \frac{\partial T}{\partial n}(P) + \oint_\Gamma \frac{\partial T}{\partial n}(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \, \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial s}(Q) \frac{\partial \ln r_{PQ}}{\partial s_P} \, \mathrm{d}s_Q, \quad P \in D^+ , \tag{38}$$

$$-2\pi \frac{\partial T}{\partial s}(P) - \oint_\Gamma \frac{\partial T}{\partial s}(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \, \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial n}(Q) \frac{\partial \ln r_{PQ}}{\partial s_P} \, \mathrm{d}s_Q, \quad P \in D^+ . \tag{39}$$

To obtain the integral equations for the boundary points the limit must be transferred from $\Gamma^+ \to \Gamma$, see Fig. 4. With this limit transfer only the kernel of the left-hand side of (38) or (39) gives a non-integral term. With the relationship

$$\lim_{\Gamma^+ \to \Gamma} \oint_\Gamma g(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \mathrm{d}s_Q = -\pi g(P) + \oint_\Gamma g(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \mathrm{d}s_Q, \quad P \in \Gamma, \tag{40}$$

where $g(Q)$ represents the functions $\partial T(Q)/\partial n$ and $\partial T(Q)/\partial s$, we obtain the following equations for the boundary points from (38) and (39):

$$\pi \frac{\partial T}{\partial n}(P) + \oint_\Gamma \frac{\partial T}{\partial n}(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial s}(Q) \frac{\partial \ln r_{PQ}}{\partial s_P} \mathrm{d}s_Q, \quad P \in \Gamma, \tag{41}$$

$$\pi \frac{\partial T}{\partial s}(P) + \oint_\Gamma \frac{\partial T}{\partial s}(Q) \frac{\partial \ln r_{PQ}}{\partial n_P} \mathrm{d}s_Q = -\oint_\Gamma \frac{\partial T}{\partial n}(Q) \frac{\partial \ln r_{PQ}}{\partial s_P} \mathrm{d}s_Q, \quad P \in \Gamma. \tag{42}$$

For the Dirichlet problem, i.e. with a given $T(Q)$, $Q \in \Gamma$, and thus also $\partial T(Q)/\partial s$, (41) is a Fredholm integral equation of the second kind. Solving this equation the temperature gradient along the boundary Γ is obtained thus making it possible to calculate the temperature field from (31) for any point $P \in D^+$.

Equation (42) is, in addition to (30), a further direct integral equation of the second kind for the Neumann problem. The solution of this equation however provides values for $\partial T(Q)/\partial s$. The boundary temperature is thus determined except for a constant. Again the complete solution of the problem requires that the temperature be known at some point $P \in D \cup \Gamma$. Further advantages and disadvantages of, as well as applications for (41) and (42) are reported in detail in [11].

5.3. Kupradze's functional equation

Kupradze's functional equation (cf. [19, 20]) for plane problems of potential theory is

$$\oint_\Gamma T(Q) \frac{\partial \ln r_{PQ}}{\partial n_Q} \mathrm{d}s_Q = \oint_\Gamma \frac{\partial T}{\partial n}(Q) \ln r_{PQ} \, \mathrm{d}s_Q, \quad Q \in \Gamma, \quad P \in \hat{\Gamma}, \tag{43}$$

where $\hat{\Gamma}$ is an arbitrary boundary in the outer area which encloses the boundary Γ. It can be derived from Green's third identity when the field point P lies outside the boundary Γ. In the case of a multiple connected domain $\hat{\Gamma}$ is made up of $\hat{\Gamma}_0, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_m$. The solution of (43) as a Neumann problem gives the boundary temperature $T(Q)$ except for a constant which can be determined if the temperature at any point is known. With these boundary values and the given temperature gradient, then the temperature field can be calculated at all points $P \in D^+$ from (31).

Similar to the torsion problem in [17], it is possible to derive (43) with the aid of a combination of heat sources and heat dipoles, since the left-hand side has the form of a double layer potential and the right-hand side has the form of a single layer potential. In the limit, as $\hat{\Gamma}$ approaches Γ, Green's identity for the boundary points is obtained.

The solution of the heat conduction problem from (43) at first seems to be very convenient, because its kernels are regular (as opposed to the previous integral equations) due to the fact that the source point Q and the field point P do not coincide. But it is to be expected that the numerical solubility is dependent to a very high degree both on the proximity of Γ and $\hat{\Gamma}$ and on whether (43) is used to solve a Neumann, Dirichlet or mixed boundary value problem, see [21].

6. Numerical method for solving the integral equations

The integral equations derived here are only approximately soluble for arbitrary bounded areas. In order to be able to compare the computing efforts for solving the equations they are all solved using the same method. A restriction is made for the boundary Γ, see Fig. 1, namely that it is without corners, although some of the integral equations can also be applied when the edges have corners. Where corners occur they are replaced with small arches. This restriction is unimportant for the heat conduction problem under consideration, since it does not lead to any significant change in the temperature field. With the aid of an approximation of the unknown function and of the boundary the integral equation can be reduced to a system of linear equations. For this approximation the same method is used as was used in [22] to calculate potential flow. There Hermitian polynomials of the third order are used for the boundary and Lagrangian polynomials of the second order for the function. This boundary approximation ensures that the boundaries are smooth which, together with a continuous function, is necessary for the existence of the Cauchy principal value ($⨏$). The method is generalised here by using constant, linear, quadratic and cubic approximation polynomials for the functions.

6.1. Boundary approximation

The boundary Γ of the area under consideration is subdivided into N curvilinear elements, see Fig. 5. Each of these boundary elements is mapped onto the interval $[-1, 1]$. With the coordinates of the initial and end point of the individual element and with the Hermitian

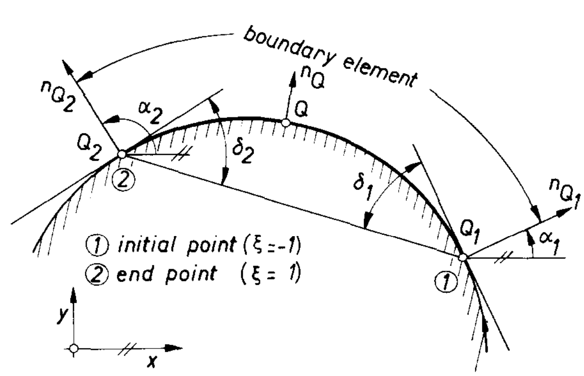


Fig. 5. The geometry of the boundary element.

polynomials

$$\begin{aligned} &H_1 = \tfrac{1}{4}(2 - 3\xi + \xi^3), \qquad H_2 = \tfrac{1}{4}(1 - \xi - \xi^2 + \xi^3), \\ &H_3 = \tfrac{1}{4}(2 + 3\xi - \xi^3), \qquad H_4 = \tfrac{1}{4}(-1 - \xi + \xi^2 + \xi^3), \end{aligned} \qquad \xi \in [-1, 1], \tag{44}$$

the coordinates of a point Q in the element can be calculated from

$$\begin{aligned} x_Q &= H_1 x_1 - H_2 m_1 \sin\alpha_1 + H_3 x_2 - H_4 m_2 \sin\alpha_2 , \\ x_Q &= H_1 y_1 + H_2 m_1 \cos\alpha_1 + H_3 y_2 + H_4 m_2 \cos\alpha_2 . \end{aligned} \tag{45}$$

The coordinates x_i, y_i and the direction of the normal n_i, $i = 1, 2$, are known. From the angle α_i, see Fig. 5, it is possible to calculate δ_i and thus m_i from

$$m_i = \tfrac{1}{2} l \frac{\delta_i}{\sin\delta_i}, \quad i = 1, 2, \quad l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 . \tag{46}$$

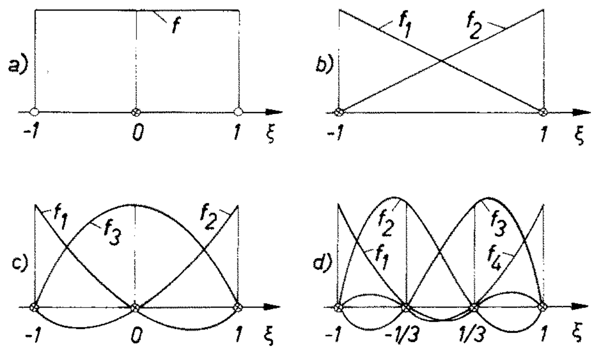
For the transformation of ds it is also necessary to consider the relationship

$$\mathrm{d}s = \sqrt{\dot{x}_Q^2 + \dot{y}_Q^2}\, \mathrm{d}\xi, \quad \dot{x}_Q = \frac{\partial x_Q}{\partial \xi}, \quad \dot{y}_Q = \frac{\partial y_Q}{\partial \xi} . \tag{47}$$

The partial derivatives $\dot{x}_Q$, $\dot{y}_Q$ can be gained from (45). For further details of the boundary approximation see [22] and [3].

6.2. Function approximation

The unknown function of the integral equation under consideration is depicted in such a way that at certain collocation points the function values are introduced as unknowns and between these points is approximated with polynomials. If one now demands that the integral equation is exactly satisfied in all collocation points, then one obtains a system of linear equations. The unknowns of the system are the values of the function at the collocation points and its coefficients are integrals which in general can be numerically calculated. According to the accuracy required it is possible to choose constant, linear, quadratic or cubic approximation polynomials, see Fig. 6. These approximations, with the exception of constant polynomials, secure a continuous function, which is necessary for the existence of the Cauchy principal value at the collocation points. If constant polynomials are used for the function approximation, then the integral equation must be satisfied in the intermediate points of the boundary elements, see Fig. 6(a), in order to satisfy the demand for continuity. The constant approximation has the important advantage that the integrals (i.e. the coefficients of the system matrix) can be computed closed, which implies a short computing time. It has the disadvantage however that for an accurate result very small boundary elements are necessary, which introduces a large number of unknowns and increases the demands on computing capacity. Since however an important disadvantage of the Boundary Element Methods is the non-symmetrical and fully occupied system matrix, one should always try to use as few



⊗ points where the boundary condition must be fulfilled

Fig. 6. Polynomials for the function approximation.

unknowns as possible, even if this requires somewhat longer computing times. For this reason the constant polynomials for the functions should only be used where the number of unknowns remains relatively small, or where the boundary is polygonal, since in these cases it is highly suitable and poses no difficulties.

The linear approximation polynomials, see Fig. 6(b),

$$f_1(\xi) = \tfrac{1}{2}(1-\xi), \qquad f_2(\xi) = \tfrac{1}{2}(1+\xi), \quad \xi \in [-1, 1], \tag{48}$$

have the advantage, as did the constant ones, that the integration is closed for straight or arched boundary elements (which make up the majority of cases in practical applications). In many cases the results are accurate enough. The integral equation must be satisfied at the points $\xi = -1$ and $\xi = 1$, see Fig. 6(b). For higher demands the Lagrange polynomials, see Fig. 6(c),

$$f_1(\xi) = \tfrac{1}{2}(\xi^2 - \xi), \qquad f_2(\xi) = \tfrac{1}{2}(\xi^2 + \xi), \qquad f_3(\xi) = 1 - \xi^2, \quad \xi \in [-1, 1], \tag{49}$$

can be used. Here the integral equations must be satisfied at $\xi = -1$, $\xi = 0$ and at $\xi = 1$. In [3] it has been shown that quadratic polynomials give better results than linear polynomials, where the number of unknowns are retained. In addition in the quadratic case less data are needed, because it is possible to make do with fewer elements. The only disadvantage to be mentioned as against the linear approximation is the complicated programme required.

Whether the cubic polynomials, see Fig. 6(d),

$$\begin{aligned} f_1(\xi) &= -\tfrac{1}{16}(3\xi+1)(3\xi-1)(\xi-1), & f_2(\xi) &= \tfrac{9}{16}(\xi^2-1)(3\xi-1), \\ f_3(\xi) &= -\tfrac{9}{16}(\xi^2-1)(3\xi+1), & f_4(\xi) &= \tfrac{1}{16}(3\xi+1)(3\xi-1)(\xi+1), \end{aligned} \quad \xi \in [-1, 1], \tag{50}$$

will be able to achieve even better results, again naturally retaining the number of unknowns, is not known, and as yet no calculations have been presented which would confirm this. Here the integral equations are satisfied at $\xi = -1$, $\xi = -\tfrac{1}{3}$, $\xi = \tfrac{1}{3}$ and at $\xi = 1$.

The progress of the function $g(Q)(g(Q)$ represents here the unknown function) can be represented with the approximation polynomials for each element by

$$g(Q)=\sum_{k=1}^{n} g_k f_k(\xi)\,, \tag{51}$$

with $n=2$ for linear, $n=3$ for quadratic and $n=4$ for cubic polynomials.

7. Reduction to a linear system of equations

From the approximation in Section 6 the solution of an integral equation can be reduced to the solution of a linear system of equations. This process will be explained here for the case of the linear approximation. The quadratic or cubic approximation is then analogous.

The boundary Γ is divided into N boundary elements L_j, $j=1,\ldots,N$. For the linear approximation the function values at the collocation points P_i are introduced as unknowns. Thus an unknown value must be determined at every point P_i, $i=1,\ldots,N$. If we now require that the integral equation being considered is satisfied exactly at all N collocation points P_i then a system of N equations is obtained with the N unknown function values. An indirect integral equation (e.g. (10)) can then be written for the point P_i,

$$\pi q(P)-\sum_{j=1}^{N}\int_{L_j}\sum_{k=1}^{2} q_k f_k(\xi)\frac{\partial \ln r_{PQ}}{\partial n_P}\,\mathrm{d}s_Q = 2\pi\frac{\partial T}{\partial n}(P),\quad P=P_i\in\Gamma\,. \tag{52}$$

Analogously for a direct integral equation (e.g. for (30)), we obtain

$$\pi T(P)-\sum_{j=1}^{N}\int_{L_j}\sum_{k=1}^{2} T_k f_k(\xi)\frac{\partial \ln r_{PQ}}{\partial n_Q}\,\mathrm{d}s_Q = -\sum_{j=1}^{N}\int_{L_j}\sum_{k=1}^{2}\frac{\partial T}{\partial n}\bigg|_k f_k(\xi)\ln r_{PQ}\,\mathrm{d}s_Q\,,\quad P=P_i\in\Gamma\,. \tag{53}$$

In (52) and (53) the function values q_k and T_k respectively are only introduced into the collocation points and thus for each boundary element L_j are no longer dependent on the integration path. They can then be written for every element outside the integral. The corresponding integrand is then known and the integral can be numerically evaluated for each element.

With the vectors

$$\boldsymbol{q}^{\mathrm{t}}=(q_1, q_2,\ldots,q_N)\,, \tag{54}$$

$$\boldsymbol{b}^{\mathrm{t}}=\left(\frac{\partial T}{\partial n}\bigg|_1,\ldots,\frac{\partial T}{\partial n}\bigg|_N\right), \tag{55}$$

$$\boldsymbol{t}^{\mathrm{t}}=(T_1, T_2,\ldots,T_N)\,, \tag{56}$$

the following linear system of equations is obtained:

$$\boldsymbol{A}\boldsymbol{q} = \boldsymbol{b}\ ; \tag{57}$$

and from (53)

$$\boldsymbol{B}\boldsymbol{t} = \boldsymbol{C}\boldsymbol{b}\ , \tag{58}$$

which can be solved by known algorithms. The coefficients of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are curve integrals from (52) and (53) which are to be calculated for the boundary elements L_j, $j = 1, \ldots, N$.

The solution of the systems (57) or (58) gives the function values at the collocation points from which, as described elsewhere, the values required can be established for every point.

Equations (52) and (53) represent two systems of equations deriving from integral equations of the second kind. In these systems the main diagonal of the relevant matrix is strongly occupied by the presence of non-integral terms, as opposed to the integral equations of the first kind, where non-integral terms are not present, because the unknown function appears only behind the integral sign. Integral equations of the first kind have therefore properties which are quite different from those of the second kind. They deliver in general an ill-conditioned system matrix. Any attempt to solve the system directly is likely to produce a wildly oscillating solution. The numerical solution of equations of the first kind makes special conditions necessary for the stabilisation of the solution, because only a small perturbation on the right-hand side of the system can give rise to an arbitrarily large perturbation in the unknown function. For the more detailed mathematics of the numerical solution of integral equations of the first kind see e.g. [24].

Finally, it should be mentioned that there are integral equations of the first kind which deliver good results without serious extra numerical work. Such equations are investigated for the heat conduction problem in [11].

8. Concluding remarks

In previous works on the solution of the heat conduction problem direct integral equations were used almost exclusively, although others were just as suitable or even better in certain cases. The main aim of this paper was to provide as comprehensive a collection as possible of direct and indirect integral equations for the heat conduction problem. To enable a uniform presentation, necessary if the integral equations are to be compared with each other, all the equations were derived from the start, although many of them were already known. Certainly, not all of the integral equations presented are well suited to solving the problems considered. In order to establish the advantages and disadvantages of the individual integral equations, it will have to be examined which of them can be solved at all, and under which conditions. The experience gained in such investigations can be transferred to other areas, since from a mathematical point of view, many technical problems can be described by the same integral equations.

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