HIGHER CATEGORY THEORY: EXERCISE SHEET 3

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Exercise 1. Recall that a groupoid is a category in which all morphisms are invertible. A $(2,1)$-*category* is a 2-category $\mathcal{C}$ such that, for all $x, y \in \mathcal{C}$, the category $\mathcal{C}(x, y)$ is a groupoid.

1. Given a groupoid $\mathcal{G}$, let
$$\pi_0(\mathcal{G}) := \mathrm{Obj}(\mathcal{G})/\sim$$
be the set of equivalence classes of objects under the relation $x \sim y$ if there exists $g\colon x \to y$ in $\mathcal{G}$ Show that the assignment
$$\pi_0\colon \mathbf{Grpd} \to \mathbf{Set}; \quad \mathcal{G} \mapsto \pi_0(\mathcal{G})$$
on objects defines a functor.
2. Show that, given a $(2,1)$-category $\mathcal{C}$, there is a 1-category $h\mathcal{C}$ such that:
   - $\mathrm{Obj}(h\mathcal{C}) = \mathrm{Obj}(\mathcal{C})$.
   - for all $x, y \in \mathcal{C}$, $h\mathcal{C}(x, y) = \pi_0(\mathcal{C}(x, y))$.

Solution. 1. To check that π_0 is a functor, first we need to define how it acts on morphisms. Let
$$f\colon \mathcal{G} \to \mathcal{H}$$
be a morphisms of groupoids. Since $\mathcal{G}$ and $\mathcal{H}$ are actually categories, f is a functor $f \in \mathbf{Fun}(\mathcal{G}, \mathcal{H})$. Define $\pi_0(f)$ to be the map
$$\begin{aligned} f^*\colon \pi_0(\mathcal{G}) &\to \pi_0(\mathcal{H}) \\ [x] &\mapsto [f(x)] \end{aligned}$$
where $[x]$ denotes the equivalence class of $x \in \mathrm{Obj}(\mathcal{G})$ in $\pi_0(\mathcal{G})$. This is a well-defined map between sets. Indeed, take $x, y \in \mathrm{Obj}(\mathcal{G})$ such that $[x] = [y]$, that is, $x \sim y$. Then, by construction, there is a morphism $g\colon x \to y$ in $\mathcal{G}$. Consider the morphism
$$h\colon f(x) \to f(y)$$
in $\mathcal{H}$ given by $h = f(g)$. Then $f(x) \sim f(y)$ and therefore $[f(x)] = [f(y)]$ and so the map is well-defined.

Also, given a groupoid $\mathcal{G}$, we have the identity morphism $\mathrm{id}_{\mathcal{G}}\colon G \to G$. Then clearly
$$\mathrm{id}_{\mathcal{G}}^*([x]) = [\mathrm{id}_{\mathcal{G}}(g)] = [g] = \mathrm{id}_{\pi_0(\mathcal{G})}([g]).$$
Finally, given two morphisms of groupoids $f\colon \mathcal{H} \to \mathcal{K}, g\colon \mathcal{K} \to \mathcal{G}$, the morphism
$$(g \circ f)^*\colon \pi_0(\mathcal{H}) \to \pi_0(\mathcal{K})$$
satisfies
$$\begin{aligned} (g \circ f)^*([x]) &= [(g \circ f)(x)] \\ &= [g(f(x))] \\ &= g^*([f(x)]) \\ &= g^*(f^*([x])) \\ &= (g^* \circ f^*)([x]) \end{aligned}$$
Therefore, $\pi_0\colon \mathbf{Grpd} \to \mathbf{Set}$ is a functor.

2. The exercise asks to check that $h\mathcal{C}$ is a category. First of all, it has a set of objects equal to $\mathrm{Obj}(\mathcal{C})$. Secondly, given any two elements $x, y \in \mathrm{Obj}(\mathcal{C})$, there is a set (by construction), $\pi_0(\mathcal{C}(x, y))$. On the other hand, given $x, y \in \mathrm{Obj}(h\mathcal{C}) = \mathrm{Obj}(\mathcal{C})$,
$$h\mathcal{C}(x, y) = \pi_0(\mathcal{C}(x, y)) = \mathrm{Obj}(\mathcal{C}(x, y))/\sim\ .$$
Therefore, each morphism $f \in h\mathcal{C}(x, y)$ is represented by an equivalence class $f = [a]$ for some $a \in \mathrm{Obj}(\mathcal{C}(x, y))$. Now, since $\mathcal{C}$ is a $(2,1)$-category, it comes equipped with a functor
$$\mu\colon \mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$

for every $x, y, z \in \mathrm{Obj}(\mathcal{C})$. In particular maps a pair of objects (a, b), $a \in \mathrm{Obj}(\mathcal{C}(x, y)), b \in \mathrm{Obj}(\mathcal{C}(y, z))$, to an object $\mu(a, b) \in \mathrm{Obj}(\mathcal{C}(x, z))$. Therefore, for every $x, y, z \in \mathrm{Obj}(\mathcal{C})$ and morphisms $f \in h\mathcal{C}(x, y), g \in h\mathcal{C}(y, z)$ represented by $[a]$ and $[b]$ for $a \in \mathrm{Obj}(\mathcal{C}(x, y)), b \in \mathrm{Obj}(\mathcal{C}(y, z))$, we define

$$\begin{aligned} g \circ f \colon h\mathcal{C}(x, y) \times h\mathcal{C}(y, z) &\to h\mathcal{C}(x, z) \\ ([a], [b]) &\mapsto [\mu(a, b)] \end{aligned}$$

This map is well-defined. Given $([a], [a']) = ([b], [b'])$, then $[a] = [b]$ and $[a'] = [b']$, so there are morphisms

$$g \colon a \to b$$

and

$$g' \colon a' \to b' .$$

But then,

$$\mu(g, g) \colon \mu(a, a') \to \mu(b, b')$$

and therefore $[\mu(a, a')] = [\mu(b, b')]$. Also, unitality and associativity follow again by the fact that μ is a functor. We need to check as well that this is well defined and that it satisfies Unitality and Associativity.

$\square$

Exercise 2. Let $f \colon H \to G$ be an arbitrary homomorphism between small groups and $\phi \colon BH \to BG$ the corresponding functor of grupoids. For a small field K and a representation $F \colon BH \to \mathbf{Vect}_K$, describe the right Kan extension $\phi_* F$.

Solution. The category $[BH, \mathbf{Vect}_K] = \mathbf{Vect}_K^{BH}$ is the category whose objects are functors $F \colon BH \to \mathbf{Vect}_K$, and this can be seen as representations of the group H, $\rho \colon H \to \mathrm{GL}(V)$ for $V \in \mathrm{Obj}(\mathbf{Vect}_K)$. Given a particular representation

$$F \colon BH \to \mathbf{Vect}_K$$

a right extension of $\phi \colon BH \to BG_K$ along F consists of

- a representation $\overline{F} \colon BG \to \mathbf{Vect}_K$;
- a natural transformation $\eta \colon \phi^* \colon [BG, \mathbf{Vect}_K] \to [BH, \mathbf{Vect}_K]$.

Therefore, given a representation of H, we are asked to define a representation $\overline{F}$ of G. When H is a subgroup of G, there is this construction called the induced representation that does it. However, I don't know how to do it for arbitrary groups and arbitrary homomorphisms. The construction for H a subgroup of G defines, for $\rho \colon H \to \mathrm{GL}(V)$,

$$\mathrm{Ind}_H^G \rho = K[G] \otimes_{K[H]} V .$$

I believe that we might do something similar for arbitrary homomorphisms $f \colon H \to G$. Now, the natural transformations are already given by equivariant maps and also the universal property follows from the one of tensor products and free groups, therefore making it a right Kan extension. $\square$