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Introduction

Sometimes in mathematics it is useful to understand a certain object by studying the properties it shares with other objects. For example, to study the set of polynomials in one variable with coefficients in the reals $\mathbb{R}[x]$, one might want to study as well structures such has $\mathbb{R}[x, y]$ or $\mathbb{C}[x]$. Therefore it is helpful to somehow relate those constructions by the properties they share. In particular, $\mathbb{R}[x], \mathbb{R}[x, y]$ and $\mathbb{C}[x]$ are rings and therefore we can talk about them in terms of two binary operations that satisfy a certain set of axioms. This realisation of spaces of polynomials as a certain algebraic structure helps us to translate properties to one object to another. The purpose of defining a group, a ring or a topological space is not just to mess up with the minds of undergraduate students, but to help us understand the underlying relations between objects. And it is not the only one.

Try to picture now what would the next step be if we wish to continue studying the relations between different objects. Or first, *what exactly would our new objects be?* We already have structures that captures certain sets into groups, rings, topological spaces, vector spaces, ... what is left to study? As it turns out, we have not exhausted yet all possible relations between objects.

Take a group G, for example. Is it not somehow similar to a ring? Both structures are given by a set, and even a ring is an abelian group for one of its binary operations. When speaking about groups, we can talk about *group homomorphisms* $f : G \to H$. Isn't the definition of a group homomorphism similar to the one given for a ring homomorphism? In a more topological set up, given a topological space (X, x_0) with a base point $x_0 \in X$, we can relate it to a group via the *fundamental group of* (X, x), $\pi_1(X; x_0)$. This maps $\pi_1(X, x_0)$ gives us a relation between groups and topological spaces with a base point, a somewhat strange relation.

The goal of Category Theory is to capture those inherent notions in certain types of structures and study the connections between them. Groups and rings have all a set of objects: its elements are elements of a set. So does a topological space, or a spaces of matrices. We can also map groups to groups and rings to rings and topological spaces to topological space. Isn't that a hint? Instead of considering groups, sets, rings, R-modules, etc... we would like to talk about the "set"[1] of all groups, the "set" of all sets and so on an so fourth, and study the relations between those "sets". Therefore we end up with the definition of a category.

[1]The meaning of the quotes "set" will be clear on the next section.

Chapter 1

Categories, Functors and Natural Transformations

1.1 Axioms for Categories

Definition 1.1.1. A category $\mathcal{C}$[1] consists of:

- a class Obj of objects.
- for every pair (x, y) of objects, a set $\mathrm{Hom}_{\mathcal{C}}(x, y) = \mathcal{C}(x, y) = \{x \to y\}$ of morphisms from x to y.

equipped with, for every triple $(x, y, z) \in \mathrm{Obj}$, a composition law:

$$\begin{aligned} \mathcal{C}(x, y) \times \mathcal{C}(y, z) &\to \mathcal{C}(x, z) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

and, for any object x, an identity morphism

$$\mathrm{id}_x \in \mathcal{C}(x, x)$$

such that

(a) Associativity: $(f \circ g) \circ h = f \circ (g \circ h)$.

(b) Unitality: $f \circ \mathrm{id}_x \circ f = f$.

Example 1.1.2. Let us picture some very basic categories:

(1) The **0** category is the empty category, where both the set of objects and the set of morphisms are the empty set.

(2) The **1** category is the category that has exactly one object and one arrow, the identity.

(3) The **2** category is the category that has two objects and just one non identity arrow,

$$\bullet \longrightarrow \bullet$$

[1]During the rest of the book, we will use calligraphic letter to denote categories

(4) The **3** category is the category that has three objects and whose morphisms are depicted in the following commutative triangle,

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

(5) We can construct other categories with the same number of objects with different morphisms. We only have to be careful that we define the composition maps correctly. For example, this is the picture of a category with three elements

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

and this would be an example of a category with four elements

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[rr, bend right] & \bullet \arrow[l, shift left] \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Example 1.1.3. Given a preorder $(P, \leq)$[2], we can consider P has a category where the arrows are given by $x \to x'$ if and only if $x \leq x'$. This indeed defines a category that has the property that there is at most one morphism between to given elements $x, x' \in P$. A poset is a preorder in which the relation $\leq$ is also antisymmetric. Both of this are categories.

Example 1.1.4. Let G be a group. We can see G as a category with one object, BG, that has as set of morphisms $\mathrm{Hom}_{BH}(G, G)$ the elements of G. The composition is given by the binary operation $\circ$ in the group G. Notice that since every element $g \in G$ has an inverse, every morphism is an isomorphism in BG. We used the fact that G was a group to assert that each morphism is an isomorphism, but if we only care about the categorical structure, we do not need a group at all. We just need a set X with a binary closed operation on X which is associative and has an identity. This particular sort of objects deserve a name of their own, monoids.

Definition 1.1.5. We say that a category is small if the underlying class of objects is a set.

Example 1.1.6. The following are examples of categories:

(1) **Set**, where objects are all small sets and the arrows are given by functions between them.

(2) $\mathbf{Set}_*$, where objects are small sets each with a selected base point and arrows are base point preserving functions.

[2] $\leq$ is a transitive and reflexive relation.

(3) **Cat**, where the objects are all small categories and the morphisms are functors between them[3].

(4) **Grp**, where the objects are all small groups and the arrows are all group homomorphisms.

(5) **Ab**, where the objects are all small abelian groups and again the morphisms are group homomorphism.

(6) **Rng**, where the objects are all small rings and the morphisms are ring homomorphisms.

(7) **CRng**, where as before, morphisms are ring homomorphisms but objects are commutative rings.

(8) **Top**, where the objects are topological spaces and the morphisms are continuous map between them.

(9) **Top**$_*$, where the objects are topological spaces with selected base point, and morphisms are base point preserving continuous maps between them.

1.2 Functors

Definition 1.2.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ consists of:

- a map $F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$.
- for every pair (x, y) of objects of $\mathcal{C}$, a map

$$F\colon \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$$

subject to the conditions

(a) for every $x \in \mathcal{C}$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

(b) for every pair $g\colon x \to y$ and $g\colon y \to z$, of compatible morphisms, we have

$$F(f \circ g) = F(f) \circ F(g)\,.$$

and such that $F(\mathrm{id}_x) = \mathrm{id}_y$.

Example 1.2.2. A simple example of a functor $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ is given by the power set: for every set X, $\mathcal{P}$ maps it to $\mathcal{P}(X)$, and for every map $f\colon X \to Y$, $\mathcal{P}$ maps it to $\mathcal{P}(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ sending each subset $A \subset X$ to its image $f(A) \subset Y$.

Example 1.2.3. The fundamental group π_1 is a functor $\pi_1\colon \mathbf{Top}_* \to \mathbf{Grp}$ taking each topological space with a base point (X, x_0) to the group $\pi_1(X, x_0)$, and each continuous map $f\colon (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$ to a group homomorphism

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, x_0)\,.$$

Similarly, higher homotopy groups $\pi_n(X, x_0)$ for $n > 1$ are all functors.

[3] We give the proper definition of a functor in the next section.

Definition 1.2.4. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ that drops of forgets part of the underlying structure of the objects in $\mathcal{C}$ is called a forgetful functor.

Example 1.2.5. Every structure built from a set induces a forgetful functor:

(1) The functor **Grp** $\to$ **Set** that forgets the group structure is forgetful.

(2) The functor **Rng** $\to$ **Set** that forgets the ring structure is forgetgul as well.

(3) The functor **CRng** $\to$ **Rng** that gorgets the commutativity axiom is again forgetful.

Just like in the case for morphisms, we can compose functors. Given three categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ and functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$, we can compose them to form a functor $\mathcal{A} \to \mathcal{C}$. This functor is called the composite functor; it is denoted by $G \circ F$ or GF and it is defined in the most obvious way,

$$A \mapsto F(A) \mapsto G(F(A)) \quad \text{and} \quad f \mapsto F(f) \mapsto G(F(f)) \,.$$

For each category $\mathcal{C}$, there is an identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ which acts as an identity for this composition. An isomorphisms $F\colon \mathcal{C} \to \mathcal{D}$ of categories is a functor F from $\mathcal{C}$ to $\mathcal{D}$ which is a bijection both on objects and on arrows. Equivalently we can define an isomorphism as a functor F that has a two-sided inverse $G = F^{-1}$, that is, if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that the composits $G \circ F$ and $F \circ G$ are the identity functors.

Definition 1.2.6. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is

(a) full when to every pair x, y of objects of $\mathcal{C}$ and to every arrow $g\colon F(x) \to F(y)$ of $\mathcal{D}$, there is an arrow $f\colon x \to y$ of $\mathcal{C}$ such that $g = F(f)$. In other words, a full functor is a surjective functor on arrows;

(b) faithful when to every pair x, y of objects of $\mathcal{C}$ and to every pair $f, g\colon x \to y$ of parallel arrows of $\mathcal{C}$, the equality $Tf = Fg\colon Tc \to Tc'$[4]. In other words, a faithful functor is an injective functor on arrows;

(c) fully faithful if it is full and faithful[5].

Example 1.2.7. The forgetful functor **Grp** $\to$ **Set** is clearly faithful, but it is not full and also not a bijection on objects. Not every map between sets if a group homomorphism, and it cannot be a bijection since, for example, the underlying set of the groups $\mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ are isomorphic, but the groups are not.

Definition 1.2.8. A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is a collection of some of the objects and some of the arrows of $\mathcal{C}$, which includes

- with each arrow $f\colon x \to y$ both x and y as objects,
- with each object x its identity morphism id_x,
- with each pair $f\colon x \to y$ and $g\colon y \to z$ of composible arrows, their composite.

[4] We are dropping the parenthesis on both the expressions $F(x)$ for objects and $F(f)$ for arrows, replacing those notations simply by Fx or Ff.

[5] A fully faithful functor $F\colon \mathcal{B} \to \mathcal{C}$ does not mean that it is a bijection between $\mathcal{D}$ and $\mathcal{C}$. It is a bijection between the arrows but **not** a bijection between the objects, since there could be objects on $\mathcal{C}$ that do not correspond via F to objects on $\mathcal{D}$.

This three conditions ensure that the collection $\mathcal{B}$ is a category.

Given a category $\mathcal{C}$ and a subcategory $\mathcal{B}$ of $\mathcal{C}$, we have an inclusion functor

$$\iota \colon \mathcal{B} \to \mathcal{C}$$

sending each object in $\mathcal{B}$ to the same one in $\mathcal{C}$, and every arrow $f \colon x \to y$ in $\mathcal{B}$ to the same arrow in $\mathcal{C}$. By construction, the inclusion functor is faithful. We say that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$ when the inclusion functor ι is full.

Example 1.2.9. The following are examples of subcategories:

(1) The category $\mathbf{Set}_f$ of finite sets is a full subcategory of $\mathbf{Set}$.

(2) The category $\mathbf{Ab}$ is a full subcategory of $\mathbf{Grp}$.

(3) The category of sets where the morphisms are given my bijections is a non-full subcategory of $\mathbf{Set}$, since not every map between sets is a bijection.

1.3 Natural Transformations

Definition 1.3.1. Given two functors $F, G \colon \mathcal{C} \to \mathcal{D}$, a natural transformations $\tau \colon F \Rightarrow G$ is a function which assigns to each object $x \in \mathcal{C}$ an arrow

$$\eta_x = \eta x \colon Fx \to Gx$$

of $\mathcal{D}$ in such a way that every arrow $f \colon x \to y$ in $\mathcal{C}$ yields in a commutative diagram

$$\begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta y"] & Gy \end{tikzcd} \tag{1.1}$$

We can view natural transformations $\tau \colon F \Rightarrow G$ as a set of functions $\{\tau_x \colon Fx \to Gx\}$ for each $x \in \mathrm{Obj}(\mathcal{C})$ translating commutative diagrams. By the latter we mean that if the diagram

$$\begin{tikzcd} & x \arrow[dl, "h"'] \arrow[dr, "f"] & \\ z & & y \arrow[ll, "g"] \end{tikzcd} \tag{1.2}$$

is commutative, a natural transformation maps translates the diagram 1.2 to the diagram

$$\begin{tikzcd} Fx \arrow[rr, "\eta_x"] \arrow[dr, "Ff"] \arrow[dd, "Fh"] & & Gx \arrow[dr, "Gf"] \arrow[dd] & \\ & Fy \arrow[rr, "\eta_y"] \arrow[dl, "Fg"] & & Gy \arrow[dl, "Gf"] \\ Fz \arrow[rr, "\eta_z"] & & Gz & \end{tikzcd}$$

where everything is commutative. The same thing happens with commutative squares. Consider the commutative square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h"'] & y \arrow[d, "g"] \\ z \arrow[r, "i"] & w \end{tikzcd} \tag{1.3}$$

Then the a natural transformation $\tau\colon F \Rightarrow G$ would translate into a "commutative cube"

$$\begin{tikzcd} & Fy \arrow[rr, "\eta_y"] \arrow[dd] & & Gy \arrow[dd, "Gg"] \\ Fx \arrow[ur, "Ff"] \arrow[rr] \arrow[dd, "Fh"'] & & Gx \arrow[ur, "Gf"] \arrow[dd] & \\ & Fw \arrow[rr, "\eta_w"] & & Gw \\ Fz \arrow[ur, "Fi"] \arrow[rr, "\eta_z"] & & Gz \arrow[ur, "Gi"] & \end{tikzcd}$$

A more natural way to represent pictorically natural transformations is the following:

$$\begin{tikzcd} A \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & B \end{tikzcd}$$

Example 1.3.2. In this example, we will see that the determinant is a natural transformation. Let M be a $n \times n$ matrix with entries in a commutative ring K and let $\det_K M$ denote the determinant. Let K^* denote the group of units[6] of K. Thus, a matrix is non-singular when $\det_K M$ is a unit, and $\det_K$ is a morphism

$$\det_K\colon \mathrm{GL}_n K \to K^*$$

of groups (an arrow in **Grp**). Because the determinant is defined by the same formula for all rings K, each morphism $f\colon K \to K'$ of commutative rings leads to a commutative diagram

$$\begin{tikzcd} \mathrm{GL}_n K \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL}_n f"'] & K^* \arrow[d, "f'"] \\ \mathrm{GL}_n K' \arrow[r, "\det_{K'}"] & K'^* \end{tikzcd}$$

This states that the transformation $\det\colon \mathrm{GL}_n \to (\quad)^*$ is a natural transformation between two functors **CRng** $\to$ **Grp**.

Example 1.3.3. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form a category whose objects consist of all functors $F\colon \mathcal{C} \to \mathcal{D}$, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Morphism are then natural transformations $\eta\colon F \Rightarrow$, where composition is done component-wise:

[6]Recall that a unit of a commutative ring R is an invertible element.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=2ex, "\Downarrow\gamma"] & \mathcal{D} \arrow[r, mapsto] & \mathcal{C} \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\gamma\star\eta"] & \mathcal{D} \end{tikzcd}$$

where $\gamma \star \eta = \{\gamma_x \circ \eta_x\}_{x \in \mathrm{Obj}(\mathcal{C})}$.

There are further operations between natural transformations and functors. For example, given a functor $F \colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta \colon G \Rightarrow H$ between two functors $G, H\mathcal{D} \to \mathcal{E}$, we can compose the functor F with the natural transformation η, $\eta \circ F = \{\eta_{F(x)}\}_{x \in \mathrm{Obj}(\mathcal{C})}$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{E} \arrow[r, mapsto] & \mathcal{C} \arrow[r, bend left=50, "G\circ F"] \arrow[r, bend right=50, "H\circ F"'] \arrow[r, phantom, "\Downarrow\eta\circ F"] & \mathcal{E} \end{tikzcd}$$

The same operation can be done the other way around. Given a natural transformation $\eta \colon F \Rightarrow G$ between two functors $F, G \colon \mathcal{C} \to \mathcal{D}$, and a functor $H \colon \mathcal{D} \to \mathcal{E}$, we can com'ose $H \circ \eta = \{H(\eta_x)\}_{x \in \mathcal{C}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \arrow[r, mapsto] & \mathcal{C} \arrow[r, bend left=50, "H\circ F"] \arrow[r, bend right=50, "G\circ F"'] \arrow[r, phantom, "\Downarrow\eta\circ F"] & \mathcal{E} \end{tikzcd}$$

We can also compose two natural transformations "horizontally". Given functors $F, G \colon \mathcal{C} \to \mathcal{D}$ and $H, I \colon \mathcal{D} \to \mathcal{E}$, and natural transformations $\eta \colon F \to G$ and $\gamma \colon H \Rightarrow I$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "I"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{E} \arrow[r, mapsto] & \mathcal{C} \arrow[r, bend left=50, "H\circ F"] \arrow[r, bend right=50, "I\circ G"'] \arrow[r, phantom, "\Downarrow\gamma\circ\eta"] & \mathcal{D} \end{tikzcd}$$

From the naturality of γ, we can see that the diagram

$$\begin{array}{ccc} H(G(x)) & \xrightarrow{\gamma_{F(x)}} & I(F(x)) \\ \Big\downarrow{\scriptstyle H(\eta_s)} & & \Big\downarrow{\scriptstyle I(\eta_x)} \\ H(G(x)) & \xrightarrow{\gamma_{G(x)}} & I(G(y)) \end{array}$$

commutes, and thus,

$$\gamma \circ \eta = \left\{I(\eta_x) \circ \gamma_{F(x)}\right\}_{x \in \mathcal{C}} = \left\{\gamma_{G(x)} \circ H(\eta_x)\right\}_{x \in \mathcal{C}} .$$

Definition 1.3.4. An equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is defined to be a pair of functors $F \colon \mathcal{C} \to \mathcal{D}, G \colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\eta \mathrm{id}_{\mathcal{C}} \Rightarrow \colon G \circ F$ and $\epsilon \colon F \circ G \to \mathrm{id}_{\mathcal{D}}$.

1.4 Monic, Epis and Zeros

Definition 1.4.1 (Isomorphisms)**.** We say that an arrow $e\colon x \to y$ is invertible in $\mathcal{C}$ if there is an arrow $e'\colon y \to x$ in $\mathcal{C}$ such that $e'e = \mathrm{id}_x$ and $ee' = \mathrm{id}_y$. If such e' exists, it is unique and we shall write it as $e' = e^{-1}$. We say that two objects x, y are isomorphic in the category $\mathcal{C}$ if there is an invertible arrow (isomorphism) $e\colon x \to y$. If it is such case, we will write $a \cong b$.

Definition 1.4.2 (Monics and Epis)**.** An arrow $m\colon x \to y$ is monic in $\mathcal{C}$ when for any two parallel arrows $f_1, f_2, \colon z \to x$, the equiality $m \circ f_1 = m \circ f_2$ implies $f_1 = f_2$. In other words, m is monic if it can always be cancelled on the left. In categories such as **Set** and **Grp**, monic arrows are precisely the injections (monomorphisms) in the usual sense.

An arrow $h\colon x \to y$ is epi in $\mathcal{C}$ when for any two arrows $g_1, g_2\colon y \to w$, the equality $g_1 \circ h = g_2 \circ h$ implies $g_1 = g_2$. In other words, h is epi when it can always be cancelled on the right. In **Set**, the epi arrrows are precisely the surjections (epimorphisms) in the usual sense.

Definition 1.4.3 (Right and Left Inverses)**.** For an arrow $h\colon x \to y$, a right inverse is an arrow $r\colon y \to x$ such that $hr = \mathrm{id}_y$. A right inverse is also called a section of h. If h has a right inverse, it is epi[7]. Similarly, a left inverse for h is called a retraction for h, and any arrow with a left inverse is necessarily monic.

Definition 1.4.4 (Terminal and Initial Objects)**.** An object t is terminal in $\mathcal{C}$ if to each object $x \in \mathcal{C}$ there is exactly one arrow $x \to t$. If t is terminal, the only arrow $t \to t$ is the identity, and any two terminal objects of $\mathcal{C}$ are isomorphic in $\mathcal{C}$. An object s is initial if $\mathcal{C}$ if to each object x there is exactly one arrow $s \to x$. A null object $\mathbf{0}$ in $\mathcal{C}$ is an object which is both initial and terminal. If $\mathbf{0}$ has a null object, that object is unique up to isomorphism, while for any two objects x, y in $\mathcal{C}$, there is a unique arrow

$$x \to \mathbf{0} \xrightarrow{y}$$

called the zero arrow from x to y.

Example 1.4.5. In **Set**, the empty set $\emptyset$ is an initial object and any one point set is a terminal object. In **Grp**, the group with one element is both initial and terminal, making it a null object. This happens as well in categories such as **Ab** or **R-Mod**.

A grupoid is a category in which every arrow is invertible. A typical grupoid is the fundamental grupoid $\pi(X)$ of a topological space X. An object of $\pi(X)$ is a point $x \in X$, and an arrow $x \to y$ of $\pi(X)$ is a homotopy class of paths f from $x \to y$.

[7]Tbe converse holds in **Set** by fails in categories like **Grp**.

Chapter 2

Constructions on Categories

2.1 Contravariance and Opposites

Categorical duality can be phrased in terms of the following mantra: "reverse all arrows". Categorically, we can do that:

Definition 2.1.1 (Opposite Category)**.** To each category $\mathcal{C}$, we can associate its corresponding opposite category $\mathcal{C}^{\text{op}}$, where the objects are the same as those in $\mathcal{C}$, but the arrows are exactly the reverse arrows of $\mathcal{C}$. That is, given $x, y \in \text{Obj}(\mathcal{C})$ and an arrow $f \in \text{Hom}_{\mathcal{C}}(x, y)$, we get a corresponding opposite arrow $f \in y \to x$ on $\text{Hom}_{\mathcal{C}^{\text{op}}}(x, y)$. In other words,

$$\text{Hom}_{\mathcal{C}}(x, y) = \text{Hom}_{\mathcal{C}^{\text{op}}}(y, x)\,.$$

Definition 2.1.2. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called as define previously is called a covariant functor. A contravarian functor will be consider to be a covariant functor $G\colon \mathcal{C}^{\text{op}} \to \mathcal{B}$.

Hom-sets provide an important example of co- and contravariant functors:

Definition 2.1.3 (Hom-functors)**.** Suppose that $\mathcal{C}$ is a category with small hom-sets [1], so that each $\text{Hom}_{\mathcal{C}}(x, y)$ is a small set. For each object $x \in \mathcal{C}$ we can define the covariant hom-functor

$$\mathcal{C}(x, -) = \text{Hom}_{\mathcal{C}}(x, -)\colon \mathcal{C} \to \mathbf{Set}$$

that sends each object $y \in \text{Obj}(\mathcal{C})$ to the set $\text{Hom}_{\mathcal{C}}(x, y)$ and sends an arrow $f\colon y \to z$ to the function in

$$\text{Hom}_{\mathcal{C}}(x, f)\colon \text{Hom}_{\mathcal{C}}(x, y) \to \text{Hom}_{\mathcal{C}}(x, z)$$

defined by the assignment $g \mapsto f \circ g$ for every $g\colon x \to y$.

[1] These hom-functors have been defined only for a category $\mathcal{C}$ with small hom-sets. The familiar categories $\mathbf{Grp}, \mathbf{Set}, \mathbf{Top}$, etc. do have this property. To include categories without this property, we can proceed as follows: given a category $\mathcal{C}$, take a set V large enough to incude all subsets of the set of arrows of $\mathcal{C}$. Let $\mathbf{Ens} = \mathbf{Set}_V$ be the category with objects all sets $X \in V$, arrows all functions $f\colon X \to Y$ between two such sets and composistion the usual composition of functions. Then each hom-set $\text{Hom}(x, y) = \mathcal{C}(x, y)$ is an object of this category $\mathbf{Ens}$, so the above procedure defines two hom-functors

$$\mathcal{C}(x, -)\colon \mathcal{C} \to \mathbf{Ens}\,, \quad \mathcal{C}(-, y)\colon \mathcal{C}^{\text{op}} \to \mathbf{Ens}\,.$$

In particular, when V is the universe of all small sets, $\mathbf{Ens} = \mathbf{Set}$. In general, $\mathbf{Ens}$ is a category of sets which acts as a receiving category for the hom-functors of a category or categories of interests.

We have a dual notion of the covariant hom-functor, the contravariant hom-functor, defined for each object $y \in \mathcal{C}$ as

$$\mathcal{C}(-,y) = \mathrm{Hom}_{\mathcal{C}}(x,y) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

and sends each object x to the set $\mathrm{Hom}_{\mathcal{C}}(x,y)$[2] and each arrow $f \colon x \to z$ on $\mathcal{C}$ to the function

$$\mathrm{Hom}(f,y) \colon \mathrm{Hom}(z,y) \to \mathrm{Hom}(x,y)$$

defined by $g \mapsto g \circ f$.

Example 2.1.4. Let X be a topological space. Then set $\mathbf{Open}(X)$ to be the category of all open subsets U of X ordered by inclusion. There is an arrow $V \to U$ precisely when $V \subset U$. Let $C(U)$ be the set of all continuous real-valued functions $f \colon U \to \mathbb{R}$; the assignment $f \mapsto h|_V$ restricting each f to the subset V is a function $C(U) \to C(V)$. This makes C a contravariant functor on $\mathbf{Open}(X)$ to $\mathbf{Set}$.

Example 2.1.5. Examples are above works not only for continuous real valued functions, but for differential and smooth functions as well. This examples are related to each other and correspond with the notion of a presheaf. A presheaf is defined to be a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

2.2 Products of Categories

Definition 2.2.1. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can construct a new category $\mathcal{C} \times \mathcal{D}$ called the product of $\mathcal{C}$ and $\mathcal{D}$. An object of $\mathcal{C} \times \mathcal{D}$ is a pair (x,y) of objects where $x \in \mathrm{Obj}(\mathcal{C})$ and $y \in \mathrm{Obj}(\mathcal{D})$, and an arrow $(x,y) \to (x',y')$ is a pair (f,g) of arrows $f \colon x \to x'$ and $g \colon y \to y'$, where the composite of two such arrows

$$(x,y) \xrightarrow{(f,g)} (x',y') \xrightarrow{(f',g')} (x'',y'')$$

is defined in terms of the composites in $\mathcal{C}$ and $\mathcal{D}$, that is

$$(f',g') \circ (f,g) = (f' \circ f, g' \circ g)\,.$$

A product category $\mathcal{C} \times \mathcal{D}$ comes equipped with natural functors $F \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $G \colon \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ called the projections of the product. These are defined on objects and arrows as

$$F(x,y) = x\,, \quad G(x,y) = y$$

and

$$F(f,g) = f \quad G(f,g) = g\,.$$

They have the following property: given any category $\mathcal{E}$ and two functions $T \colon \mathcal{E} \to \mathcal{C}, R \colon \mathcal{E} \to \mathcal{D}$, there is a unique functor

$$U \colon \mathcal{E} \to \mathcal{C} \times \mathcal{D}$$

such that $FU = R, GU = T$. The construction of U is better visualized by the following vommutative diagram of functors:

$$\begin{tikzcd}
& \mathcal{E} \arrow[dl, "R"'] \arrow[d, dotted, "U"] \arrow[dr, "T"] & \\
\mathcal{C} & \mathcal{C} \times \mathcal{D} \arrow[l, "F"] \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

[2]When it is clear the category $\mathcal{C}$ over which we are working, we will usually drop the subscript in $\mathrm{Hom}_{\mathcal{C}}$ and write only Hom.

2.3 Functor Categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, we can consider all functors from $\mathcal{C}$ to $\mathcal{D}$. If $\eta\colon F \to G$ and $\tau\colon G \to H$ are two natural transformations, their components for each $x \in \mathcal{C}$ define as we have seen composite arroes which are the components of a transformation $\tau \circ \eta\colon F \to H$. Naturality follows inmediately from the fact the the two small square in the diagram

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"] \arrow[dd, bend right=60, "(\tau\circ\eta)_x"'] & Fy \arrow[d, "\eta_y"] \arrow[dd, bend left=60, "(\tau\circ\eta)_y"] \\ Gx \arrow[r, "Gf"] \arrow[d, "\tau_x"] & Gy \arrow[d, "\tau_y"] \\ Hx \arrow[r, "Hf"] & Hy \end{tikzcd}$$

are commutative. This composition of transformation is associative; moreover it has for each functor F an identity, the natural transformation $\mathrm{id}_F\colon F \to F$ with components $\mathrm{id}_F x = \mathrm{id}_{Fx}$. Hence, given categories $\mathcal{C}$ and $\mathcal{D}$, we may construct formally a functor category

$$\mathcal{D}^{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \mathcal{D}) = [\mathcal{C}, \mathcal{D}]$$

with objects the functors $F\colon \mathcal{C} \to \mathcal{D}$ and morphisms the natural transformations between two such functors. It is often suggestive to write

$$\mathrm{Nat}(F, G) = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) = \{\eta \mid \eta\colon F \Rightarrow G \text{ natural transformation}\}\ .$$

Example 2.3.1. If K is a commutative ring and G is a group, then the functor category $(K{-}\mathbf{Mod})^G$ is the category of K-linear representations of G. Specificaly, each functor $F\colon G \to K{-}\mathbf{Mod}$ is determined by a K-module V (the image of the single object of the category G) and morphism $F\colon G \to \mathrm{Aut}(V)$ of groups. If F' is a second such representations, a natural transformation $\eta\colon F \to F'$ is given by a single arrow $\eta\colon V \to V'$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\eta"] \arrow[d, "Fg"'] & V' \arrow[d, "F'g"] \\ V \arrow[r, "\eta"] & V' \end{tikzcd}$$

commutes for every $g \in G$. Such a η is called an intertwining operator.

2.4 Comma Categories

There is another general construction of a category whose objects are certain arrows, as in the following several special cases. If x is an object of the category $\mathcal{C}$, the category of objects under x is the category $(x \downarrow \mathcal{C})$ with objects all pairs (f, y) where y is an object of $\mathcal{C}$ and $f\colon x \to y$ an arrow of $\mathcal{C}$, and with arrows $h\colon (f, y) \to (g, z)$ those arrows $h\colon y \to z$ of $\mathcal{C}$ for which $h \circ f = g$. Thus, an object of $(x \downarrow \mathcal{C})$ is a commutative triangle with top vertex x. In displayed form, the objects are

$$\begin{tikzcd} x \arrow[d, "f"] \\ y \end{tikzcd}$$

and arrows $h\colon (f,x) \to (f,y)$ are

$$
\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
y \arrow[rr, "h"] & & z
\end{tikzcd}
$$

The composition of arrows in $(x \downarrow \mathcal{C})$ is given by the composition in $\mathcal{C}$.

Example 2.4.1. If we consider $\bullet$ to be a one point set and X any set, we can consider the category $(\bullet \downarrow \mathbf{Set})$. An object in $(\bullet \downarrow \mathbf{Set})$ is just a pair (f, X) where $f\colon \bullet \to X$. But since $\bullet$ was a one point set, f is nothing but a choice of a base point of X. Therefore $(\bullet \downarrow \mathbf{Set})$ is the category of pointed sets.

Example 2.4.2. In a similar fashion, we could consider $\mathbb{Z}$ and the category **Ab** of abelian groups. Then $(\mathbb{Z} \downarrow \mathbf{Ab})$ would be the category of all abelian groups with a distinguished element.

Similarly, if x is an object of $\mathcal{C}$, we could consider as well the category $(\mathcal{C} \downarrow x)$ of objects over x. It has objects arrows $f\colon y \to x$ for $y \in \mathrm{Obj}(\mathcal{C})$ and arrows are commutative triangles:

$$
\begin{tikzcd}
y \arrow[d, "f"] \\
x
\end{tikzcd}
\quad ; \quad
\begin{tikzcd}
y \arrow[rr, "h"] \arrow[dr, "f"'] & & z \arrow[dl, "f'"] \\
 & x &
\end{tikzcd}
$$

Example 2.4.3. Since $\bullet$ is a terminal object in **Set**, there is always a unique $X \to \bullet$ and therefore $(\mathbf{Set} \downarrow \bullet)$ is isomorphic to *sets*.

If x is an object of $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, the category $(x \downarrow F)$ of objects F-under x has as objects all pairs (f, y) with $y \in \mathrm{Obj}(\mathcal{D})$ and $f\colon x \to Fy$ and as arrows $h\colon (f,y) \to (f', z)$ all those arrows $h\colon y \to z$ in $\mathcal{D}$ for which $f' = Fh \circ f$. In pictures, the objects and arrows are respectively

$$
\begin{tikzcd}
x \arrow[d, "f"] \\
Fy
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
Fy \arrow[rr, "Fh"] & & Fy'
\end{tikzcd}
\qquad \text{(commutative)}
$$

Again, composition is given by composition of the arrows h in $\mathcal{D}$.

Example 2.4.4. Let $U\colon \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor. Then for each set x, an object of $(x \downarrow U)$ is a function $x \to Ug$ from x into the underlying set of some group g; for example, the function mapping x into the underlying set of the free group generated by the elements of the set x is one such object.

Again, if $x \in \mathcal{C}$ and $G\colon \mathcal{E} \to \mathcal{D}$ is a functor, one may construct a category $(G \downarrow x)$ of objects G-over x.

Definition 2.4.5 (Comma Category)**.** Given categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ and functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{E} \to \mathcal{D}$, the comma category $(F \downarrow G)$, also written as (F, G) has objects all triples (x, y, f) with $x \in \mathcal{E}, y \in dcal$ and $f\colon Fx \to Sy$, and arrows $(x, y, f) \to (x', y', f')$ all pairs (h, h') of arrows $h\colon x \to x', h'\colon y \to y'$ such that $g \circ Th' = Gh \circ f$. In pictures, it has

$$\begin{tikzcd} Fx \arrow[d, "f"] \\ Gy \end{tikzcd}$$

as arrows (x, y, f), and

$$\begin{tikzcd} Fx \arrow[r, "Fh'"] \arrow[d, "f"] & Fx' \arrow[d, "f'"] \\ Gy \arrow[r, "Gh"] & Gy' \end{tikzcd} \qquad \text{(commutative)}$$

as arrows (h, h').

Chapter 3

Universals and Limits

3.1 Universal Arrows