Homework 5. Advanced Algebra

Advanced Algebra

Exercise 1. Let Γ be a finite, connected graph with vertices $(e_1, \dots, e_E)$ and edges $(k_1, \dots, k_K)$. We endow the edges wih an orientation and obtain the incidence matrix $I_\Gamma \in \mathrm{Mat}_{E\times K}(\mathbb{Z})$ of Γ as

$$(I_\Gamma)_{ij} := \begin{cases} +1 & \text{if } k_j \text{ starts in } e_i \\ -1 & \text{if } k_j \text{ ends in } e_i \\ 0 & \text{else} \end{cases} .$$

Consider the complex C_* given by

$$C_1 = e_1\mathbb{Z} \oplus \cdots \oplus e_E\mathbb{Z}\,,\ C_0 = k\mathbb{Z} \oplus \cdots \oplus k_K\mathbb{Z}\,,\ C_n = 0$$

for all $n \geq 2$, endowed with the differential $d_1 \colon C_1 \to C_0$ given by multiplication with I_Γ and $d_n = 0$ for $n \neq 1$. Prove the following statements:

a) (C_*, d) is a complex.

b) The only nonzero homology modules are $H_0(C_*)$ and $H_1(C_*)$. We have $\mathrm{rk}(H_0(C_*)) = 0$ and $\mathrm{rk}(H_1(C_*)) = E - K - 1$.

c) The number of closed paths in Γ is $E - K - 1$.

Solution. At the moment I'm writing this there are some typos in the exercise. $(e_1, \dots, e_E)$ should be the **edges** and not the vertices, since we want C_1 to be the free group $\mathbb{Z}$-module generated by the edges and C_0 freely generated by the vertices.

Also, the rank of H_1 is not $E - K - 1$ but rather $E - K + 1$. I'm changing the notation a little bit, so denote by $(e_1, \dots, e_E)$ the edges and by $(v_1, \dots, v_V)$ the vertices, so that C_1 is the free module generated by the edges and C_0 the free module generated by the vertices.

a) Well, it is clearly a complex, since multiplication by I_Γ is linear, and so it is a module homomorphism. Since there is only one non-trivial map, $d = d_1 \colon C_1 \to C_0$, we have that $d^2 = 0$, and so it is a complex.

b) Since the only non-trivial map is d, then the only non-trivial homology groups will be $H_0(C_*)$ and $H_1(C_*)$, which will be given respectively by

$$H_0 = C_0/\operatorname{im} d\,,\ H_1 = \ker d\,.$$

Next, notice that if we have an edge e joining two vertices v_i and v_j, we can define a source and target map as

$$t(e) = v_j\,, s(e) = v_i\,.$$

These induce then source and target maps $t, s \colon E \to V$ from the set of edges to the set of vertices, and we view mulitplication by the incidence matrix I_Γ as $I_\Gamma e = t(e) - s(e)$.

Furthermore, if we are given a path $e_1, \dots, e_L$ from v_i to v_j, we have

$$s(e_1) = v_i\,,\ t(e_L) = v_j \text{ and } t(e_k) = s(e_{k+1})\,,$$

so

$$I_\Gamma \cdot \sum_{k=1}^{L} e_k = \sum_{k=1}^{L} I_\Gamma \cdot e_k = \sum_{k=1}^{L}(t(e_k) - s(e_k)) = v_j - v_i$$

since it is a telescopic sum being $e_1, \dots, e_L$ a path. Therefore, we see that the elements $v_j - v_i$ are on the image of d provided that there exists a path between them. But Γ is path connected, which means that any two vertices have a path, so we can fix some vertex $\tilde{v}$, so that for any v there exists a path from v_j to $\tilde{v}$. This shows that C_0 is generated by

$$\{\tilde{v}, v_1 - \tilde{v}, \dots, v_V - \tilde{v}\}$$

and since the elements $v_1 - \tilde{v}$ belong to the image of d, they get zeroes out in H^0, so we only get that $\tilde{v}$ generates H_0, and so

$$\operatorname{rk}(H_0(C_*)) = 1\,.$$

For H_1 we use the Rank-Nullity Theorem to see that

$$\begin{aligned} E = \operatorname{rk} C_1 &= \operatorname{rk}(\ker d) + \operatorname{rk}(\operatorname{im} d) \\ &= \operatorname{rk}(H_1) + (\operatorname{rk}(C_0) - \operatorname{rk}(H_0)) \\ &= \operatorname{rk}(H_1) + V - 1 \end{aligned}$$

which shows that $\operatorname{rk}(H_1(C_*)) = E - V + 1$.

c) A closed path is a sequence $e_1, \dots, e_L$ such that $s(e_1) = t(e_L)$. Then, we have $I_\Gamma \cdot \sum_{k=1}^{L} e_k = 0$, so these generate the kernel of d, and so there are $E - V + 1$ of those, by the above.

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Exercise 2. a) Verify that

$$\cdots \xrightarrow{d_{n+1}} C_n(K,R) \xrightarrow{d_n} \cdots \xrightarrow{d_2} C_1(K,R) \xrightarrow{d_1} C_0(K,R) \longrightarrow 0$$

is a complex. It is called the *simplicial chain complex.*

b) Determine the simplicial chain complex with coefficients in $\mathbb{R}$ for the simplicial complex K on $\{0,1,2,3\}$ given by all subsets of cardinality ≤ 2. Compute the homology groups $H_n(K,\mathbb{R}) = \ker(d_n)/\operatorname{im}(d_{n+1})$.

c) For any (geometric) simplicial complex one can consruct and abstract simplicial complex by only retaining the set of vertices. Profe that, conversely, for any finite abstract simplicial complex K one can construct a (geometric) simplicial complex K'.

Solution. a) First, we notice that the face maps $\partial_i \colon K_n \to K_{n-1}$ satisfy the identity

$$\partial_j \circ \partial_i = \partial_{i-1}\partial_j$$

whenever $j < i$. Indeed, going by the left hand side we obtain the

$$\{x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}\,.$$

On the other hand, if we first apply ∂_j, then we would have shifted by 1 the index of i so that x_i is now in the position $i-1$, so we get

$$\{x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$$

as well. Then, we compute

$$\begin{aligned}
d \circ d = d\left(\sum_{i=0}^{n}(-1)^i e_{\partial_i \sigma}\right) &= \left(\sum_{j=0}^{n-1}(-1)^j e_{\partial_j}\right) \circ \left(\sum_{i=0}^{n}(-1)^i e_{\partial_i \sigma}\right) \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} e_{\partial_j \partial_i} + \sum_{0\le i\le j\le n-1} (-1)^{i+j} e_{\partial_j \partial_i \sigma} \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} e_{\partial_{i-1} \partial_j} + \sum_{0\le i\le j\le n-1} (-1)^{i+j} e_{\partial_j \partial_i \sigma} \\
&= \sum_{0\le i<j\le n} (-1)^{i+j} e_{\partial_{j-1} \partial_i \sigma} + \sum_{0\le i\le j\le n-1} (-!)^{i+j} e_{\partial_j \partial_i \sigma} \\
&= \sum_{0\le i\le j\le n-1} (-1)^{i+j-1} e_{\partial_j \partial_i \sigma} + \sum_{0\le i\le j\le n-1} (-1)^{i+j} e_{\partial_j \partial_i \sigma} \\
&= 0
\end{aligned}$$

where we have split the sum between $j < i$ and $i \le j$, used the identity $\partial_j \circ \partial_i = \partial_{i-1}\partial_j$ from before and then identify i with j and finally j with $j-1$.

b) We obtain the following complexes:

$$\begin{aligned}
C_0 &= \{\{0\}, \{1\}, \{2\}, \{3\}\} \\
C_1 &= \{\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}\}
\end{aligned}$$

and the rest of them all trivial. For the zero-th homology group $H_0(K, \mathbb{R})$, we need to look at $\mathrm{im}(d_1)$. This is given by

$$\begin{aligned}
d_1(\{0,1\}) &= \{1\} - \{0\} \\
d_1(\{0,2\}) &= \{2\} - \{0\} \\
d_1(\{0,3\}) &= \{3\} - \{0\} \\
d_1(\{1,2\}) &= \{2\} - \{1\} \\
d_1(\{1,3\}) &= \{3\} - \{1\} \\
d_1(\{2,3\}) &= \{3\} - \{2\}
\end{aligned}$$

which we can write in matrix form as

$$\mathrm{im}(d_1) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

which has rank 3. This $\mathrm{im}(d_1) \cong \mathbb{R}^3$ and thus

$$H_0(K,\mathbb{R}) = C_0/\mathrm{im}(d_1) \cong \mathbb{R}^4/\mathbb{R}^3 \cong \mathbb{R}\,.$$

For the other one, however, now we have that $\mathrm{im}(d_2) \cong 0$, and thus

$$H_1(K,\mathbb{R}) \cong \ker(d_1)/\mathrm{im}(d_2) \cong \mathbb{R}^3/0 \cong \mathbb{R}^3\,.$$

The rest of the homology groups are trivial.

c) Given an anstract simplicial complex K, define a category $\mathrm{Face}(K)$ whose objects are the faces of K and morphisms are given by inclusions. Let

$$|\Delta^n| = \left\{ (t_0,\dots,t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \right\}$$

be the standard topological n-simplex. Define a functor

$$\begin{aligned} \mathcal{F}\colon \mathrm{Face}(K) &\to \mathbf{Top} \\ X &\mapsto |\Delta^{\dim X}|\,. \end{aligned}$$

Given another face Y of dimension $n \leq m = \dim X$, there is a map $F(Y) \to F(X)$ consisting of sending $|\Delta^m|$ to the m-dimensional face of $|\Delta^n|$. Then we define a simplicial complex as the colimit of the functor $\mathcal{F}$.

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Exercise 3. Let M, N be smooth manifolds.

(a) Show that the dimension of the 0-th de Tham cohomology group $H^0_{dR}(M)$ euals the number of connected components of M.

(b) Let $f, g\colon M \to N$ be homotopic smooth maps. Prove that hte induced maps $\overline{f}, \overline{g}\colon \Omega^*(N) \to \Omega^*(M)$ of complexes (on $\Omega^p(N)$ they are given by the pullbacks f^*, g^*) are homotopic. In particular, we have $H^p_{dR}(f) = H^p_{dR}(g)$ for all $p \geq 0$.

(c) Compute the de Rham cohomology groups for the n-dimensional sphere S^n.

(d) Let $v, w \in \mathbb{R}^n$. Compute the de Rham cohomology groups for $\mathbb{R}^n \setminus \{v\}$ and $\mathbb{R}^n \setminus \{v, w\}$.

Solution. a) We can decompose M as a sum of its connected components, and since cohomology commutes with finite coproducts, we have that

$$H^0_{dR}(M) \cong \bigoplus_{i=1}^n H^0_{dR}(M_i)$$

where M_i denotes the i-th connected component of M. Now, $H^0_{dR}(M_i)$ is simply the set of closed 0-forms on M_i, that is, smooth functions on M_i such that $df = 0$. These is the group of locally constant functions on M_i, but since M_i is connected, locally constant implies globally constant, so $H^0_{dR}(M_i) \cong \mathbb{R}$. Therefore,

$$H^0_{dR}(M) \cong \mathbb{R}^n$$

so the dimension of the 0-th de Rham cohomology group equals the number of conncted components.

b) Let $H\colon M\times[0,1]\to N$ be a homotopy between f and g, that is, such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$. Consider a cocycle $\omega\in\Omega^k(N)$. The pullback is then along H can be wrriten as

$$H^*\omega=\omega_0+\mathrm{d}\,t\wedge\omega_1$$

where $\omega_0\in\Omega^k(M)$ and $\omega_1\in\Omega^{k-1}(M)$, so we get

$$f^*\omega=\omega_0=g^*\omega\,.$$

Since $F^*\omega$ is a cocylce, then

$$0=\mathrm{d}\,F^*\omega=\mathrm{d}\,t\wedge\left(\frac{\partial\omega_0}{\partial t}-d_M\omega_1\right)+\cdots$$

and thus

$$f^*\omega-g^*\omega=\omega_0(1)-\omega_0(0)=\int_0^1\frac{\partial\omega_0}{\partial t}\,\mathrm{d}\,t=\int_0^1 d_M\omega_1\,\mathrm{d}\,t=d_M\int_0^1\omega_1\,\mathrm{d}\,t\,.$$

This means that $f^*\omega$ and $g^*\omega$ are the same in cohomology.

c) First we need the case base for S^1. We do this by choosing a cover $S^1=U\cup V$ of two semicircles overlapping, so that $U\cap V$ is the union of two disconnected segments. Applying Mayer-Vietoris to this cover, it becomes

$$0\longrightarrow\mathbb{R}\longrightarrow\mathbb{R}^2\longrightarrow\mathbb{R}^2\longrightarrow H^1(S^1)\longrightarrow 0$$

since $H^0(S^1)\cong\mathbb{R}$, and $H^0(U)\cong\mathbb{R}\cong\mathbb{R}^2$. This implies that $H^1(S^1)=\mathbb{R}$. The claim is that

$$H^n(S^m)=\begin{cases}\mathbb{R} & \text{if } n=0,m\\ 0 & \text{otherwise}\end{cases}$$

To see this. choose a cover of S^m by U and V being S^m minus the north and south pole respectively. Using Poincare Lemma we have that

$$H^k(U)=H^k(V)\cong 0$$

for all k. On the other hand, $U\cap V$ deformation retracts onto S^{m-1}, for which the induction hypothesis applies. Hence we know all cohomology groups except $H^n(S^m)$ for $1\leq n\leq m$.

If $n=1$, first notice that the map $H^0(S^m)\to H^0(U)\oplus H^0(V)$ has trivial kernel and its image is isomorphic to $\mathbb{R}$, so $H^0(S^n)\cong\mathbb{R}$. On the other hand, the connecting homomorphism $\delta\colon H^0(S^{m-1})\to H^1(S^m)$ is surjective since the map $H^1(S^2)\to H^1(U)\oplus H^1(V)$ is trivial. Using Poincaré Lemma for $H^k(U)=H^k(V)=0$ for $k>0$, a similar argument as used for $H^1(S^1)$ shows that $H^1(S^m)=0$. If $1<n<m$, then all the mapso going from $H^n(S^m)$ and into it are trivial, so $H^n(S^m)=0$.

Finally, if $m=n$, the connecting homomorphism δ is surjective, and $H^{n-1}(S^{n-1})\cong\mathbb{R}$ by induction hypothesis. Moreover, the image of the substraction map that goes into $H^{n-1}(S^{n-1})$ is zero, so the last connecting homomorphism δ has trivial kernel and is therefore an isomorphism. Hence $H^n(S^n)\cong\mathbb{R}$.

d) We have done the first one. $\mathbb{R} \setminus \{v\}$ deformation retracts onto the sphere S^{n-1}, for which we already know the cohomology groups.

For $\mathbb{R}^n \setminus \{u, v\}$, I'm guessing that this deformation retracts onto the wedge of two S^{n-1}, so we would have

$$H^m(\mathbb{R} \setminus \{u, v\}) \cong H^m(S^{n-1}) \oplus H^m(S^{n-1}) \, .$$

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