Rational BV-Algebra In String Topology

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Plan for the Talk

1. Goals
2. Motivation
3. Hochschild complexes
4. Gerstenhaber and BV-algebras
5. BV-algebra structure on Hochschild cohomology
6. BV-algebra structure on Loop Homology
7. Proof of Main Theorem

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Main Theorem

The goal of this talk is to prove the following statement:

Theorem

If M is 1-connected and the field of coefficitnets has characteristic zero, then

- there is a BV-structure on $HH^\bullet(C^\bullet(M), C^\bullet(M))$ extending the structure of a Gerstenhaber algebra,
- there is an isomorphism of BV-algebras

$$\mathbb{H}_\bullet(\mathsf{L}M) \cong HH^\bullet(C^\bullet(M), C^\bullet(M)).$$

Notation I

All manifolds are assumed to be smooth orientable, closed, with $\dim(M) = m$.

Denote by

$$\mathsf{L}M = \mathsf{Map}(\mathbb{S}^1, M)$$

the *loop space of* M. Its *shifted homology* will be denoted by

$$\mathbb{H}_\bullet(\mathsf{L}M) = H_{\bullet+m}(\mathsf{L}M)\,.$$

$C_\bullet(M)$ and $C^\bullet(M)$ denote the singular chains and cochains of M.

Convention: complexes will have cohomological degree $V^i = V_{-i}$.

Notation II

A will denote a DG-algebra, sA denotes the "shift by 1", i.e.

$$(sA)^i = A^{i+1},$$

and $T(A)$ denotes the *tensor coalgebra*

$$T(A) = \bigoplus_{i=0} (sA)^{\otimes i}.$$

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Motivation

Assume the coefficients ring is a field. Jones [3] showed there is a linear isomorphism

$$HH_\bullet(C^\bullet(M), C^\bullet(M)) \cong H^\bullet(\mathsf{L}M)\,.$$

By duality, $H_\bullet(\mathsf{L}M) \cong HH^\bullet(C^\bullet(M), C_\bullet(M))$.

Next, the cap product induces an isomorphism of graded vector spaces

$$HH^\bullet(C^\bullet(M), C_\bullet(M)) \cong HH^{\bullet-m}(C^\bullet(M), C^\bullet(M))\,,$$

and therefore an isomorphism

$$\mathbb{H}_\bullet(\mathsf{L}M) \cong HH^\bullet(C^\bullet(M), C^\bullet(M))\,.$$

Motivation (II)

On the other hand, Lambrechts and Stanley ([2]) showed the following:

Theorem

There exists a commutative differential graded algebra A satisfying:

- A is quasi-isomorphic to the DG-algebra $C^\bullet(M)$.
- A is connected, finite dimensional and satisfies Poincaré duality in dimension m (i.e. there is an A-linear isomorphism $\theta : A \to A^\vee$ commuting with the differentials.

A will be referred to as a *Poincaré duality model for M*.

Idea of Proof

Using the result of Lambrechts and Stanley, we first replace $C^\bullet(M)$ by its quasi-isomorphic DG-algebra A.

By a result in [4] there is an isomorphism of Gerstenhaber algebras

$$HH^\bullet(A,A) \cong HH^\bullet(C^\bullet(M),C^\bullet)).$$

Then, show that $HH_\bullet(A,A) \cong H^\bullet(\mathsf{L}M)$ as an isomorphism of graded vector spaces.

Idea of Proof (II)

By the result of Menichi, the dual of Connes' operator $B^\vee$ on $HH^{\bullet+1}(A,A)^\vee$ gets transfered via the duality isomorphism

$$\theta\colon HH_\bullet(A,A)^\vee \cong HH^\bullet(A,A^\vee) \xrightarrow{\simeq} HH^\bullet(A,A)$$

to a BV-algebra structure on $HH^\bullet(A,A)$ extending the Gerstenhaber algebra structure.

Finally, show that the isomorphism $HH_\bullet(A,A) \cong H^\bullet(\mathrm{L}M)$ transfers Connes' operator B to the operator Δ' induced by the action of $\mathbb{S}^1$ on $\mathrm{L}M$

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Hochschild (co)homology: Bar construction I

Fix a DG-algebra A, a right A-module P and a left A-module N.

Denote by $\mathrm{Bar}(P,A,N)$ the following (two-sided) complex,

$$\mathrm{Bar}_k(P,A,N)^l = (P \otimes T^k(A) \otimes N)^l \,.$$

Here, k is the *lower degree*. Given an element $p[a_1 \mid a_2 \mid \cdots \mid a_k]n$

in $\mathrm{Bar}_k(P,A,N)$, we will say it has *upper degree*

$$l = |p| + |n| + \sum_{i=1}^{k} |sa_i| \,.$$

Hochschild (co)homology: Bar construction II

We equip $\mathrm{Bar}(P,A,N)$ with the differential $b_{\mathrm{Bar}} = \partial_{DG} + \partial_{\mathrm{Bar}}$:

$$\begin{aligned}\partial_{DG}(p[a_1 \mid \cdots \mid a_k]n) = d(p)[a_1 \mid \cdots \mid a_k]n \\ - \sum_{i=1}^{k} (-1)^{\mathrm{sign}_i} p[a_1 \mid \cdots \mid d(a_i) \mid \cdots \mid a_k]n \\ + (-1)^{\mathrm{sign}_{k+1}} p[a_1 \mid \cdots \mid a_k]d(n)\,.\end{aligned}$$

$$\begin{aligned}\partial_{\mathrm{Bar}}(p[a_1 \mid \cdots \mid a_k]n) = (-1)^{|p|} pa_1[a_2 \mid \cdots \mid a_k]n \\ + \sum_{i=2}^{k} (-1)^{\mathrm{sign}_i} p[a_1 \mid \cdots \mid |a_{i-1}a_i \mid \cdots \mid a_k]n \\ - (-1)^{\mathrm{sign}_k} p[a_1 \mid \cdots \mid a_{k-1}]a_k n\,.\end{aligned}$$

Here, sign_i denotes $|p| + \sum_{j<i} |sa_j|$.

Hochschild (co)homology

Let $A^e = A \otimes A^{\mathsf{op}}$ and P a DG right A^e-module. Define

$$C_\bullet(P,A) = P \otimes_{A^e} \mathsf{Bar}(A,A,A)\,.$$

The complex $C_\bullet(P,A)$ is called the *Hochschild chain complex of A*

with coefficients in P. Its homology is called the *Hochschild homology of A with coefficients in P*. We denote it by $HH_\bullet(A,P)$.

Similarly, define for a left DG A^e-module N

$$C^\bullet(A,N) = \mathrm{Hom}_{A^e}(\mathsf{Bar}(A,A,A),N)\,.$$

$C^\bullet(A,N)$ is the *Hochschild cochain complex of A with coefficients in N*. Its cohomology is called the *Hochschild cohomology of A with coefficients in N*. We denote it by $HH^\bullet(N,A)$.

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Gerstenhaber algebras

Recall that a *Gerstenhaber algebra structure* on a commutative graded algebra $H = \{H_i\}_{i\in\mathbb{Z}}$ is given by a (Gerstenhaber) bracket

$$[-,-]\colon H_i \otimes H_j \to H_{i+j+1}\,,\; x\otimes y \mapsto [x,y]\ .$$

The bracket $[-,-]$ satisfies, for all $h,h',h'' \in H$:

- $[h,h'] = (-1)^{(|h|-1)(|h'|-1)}[h',h]$,
- $[h,[h',h'']] = [[h,h'],h''] + (-1)^{(|a|-1)(|a'|-1)}[h',[h,h'']]$.

Hochschild cohomology becomes a Gerstenhaber algebra via:

- Cup product (this gives graded commutativity),
- Gerstenhaber bracket: circle product of Hochschild cochains.

BV-algebras

Recall that a *Batalin-Vilkovisky algebra* (BV-algebra) is a commutative graded algebra H together with a linear map of degree -1 (BV-operator)

$$\Delta\colon H^k \to H^{k-1}$$

such that:

- $\Delta^2 = 0$,
- H becomes a Gerstenhaber algebra with bracket

$$[h, h'] = (-1)^{|h|}(\Delta(hh') - (-1)^{|h|}\Delta(h)h' - h\Delta(h') + h\Delta(1)h' .$$

Connes' B operator I

The Hochschild complex carries a natural cyclic action induced by the cyclic bar construction.

Connes' B operator is a degree -1 homogeneous map on Hocshchild chains that takes this action into account.

For a a general element $a_0 \otimes [a_1 \mid \cdots \mid a_n]$, one has

$$B(a_0 \otimes [a_1 \mid \cdots \mid a_n]) = \sum_{i=0}^{n} (-1)^{\text{sign'}_i} 1 \otimes [a_i \mid \cdots \mid a_n \mid a_0 \mid \cdots \mid a_{i-1}], ,$$

where

$$\text{sign'}_i = (|sa_0| + |sa_1| + \cdots + |sa_{i-1}|)(|sa_i| + \cdots + |sa_n|) .$$

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Connes' B operator II

The Hochschild complex $C_\bullet(A)$ together with the operators b and B form a *mixed complex* $(B^2 = 0, B \circ b + b \circ B = 0)$.

Its homology is known as the *cyclic homology of* A, and it is denoted by $HC_\bullet(A)$.

BV-Structure in Hochschild cohomology

Recall that $HH^\bullet(A)$ is a Gerstenhaber algebra where:

- Graded commutativity comes from the cup product,
- Gerstenhaber algebra comes from circle product.

By duality, Connes' operator B induces a map in cohomology

$$\overline{B}\colon HH^{n+1}(A) \to HH^n(A)\,.$$

Here, $HH^\bullet(A) = HH^\bullet(A, A^\vee)$.

$$\overline{B}([f])([a_1 \otimes \cdots \otimes a_n])(a_0) = \sum_{i=0}^{n} \pm f(a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1})(1)\,.$$

BV-algebra structure on Hochschild cohomology

Hochschild cohomology with the dual of Connes' operator, $\overline{B}$, is a BV-algebra.

Remarks

The BV-algebra structure constructed above works more generally in the *ungraded case*.

However, for our purposes, as we want to compare $HH^\bullet(C^\bullet, C^\bullet)$ with the homology of the loop space, we quote the following result by L. Menichi ([1])

Theorem

The dual of Connes' operator B together with the cup product induce a BV-algebra structure on Hochschild cohomology $H^\bullet(C^\bullet, C^\bullet)$.

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Chas-Sullivan Product

Recall from previous talks, we have a loop product

$$- \bullet - : LM \times LM \to LM\,.$$

using the notion of *transversal geometric chains*.

Moreover, the circle action

$$\rho : \mathbb{S}^1 \times \mathsf{L}M \to \mathsf{L}M \;\; \rho(s,\alpha)(t) = \alpha(s+t)$$

defines an operator Δ of degree 1 commuting with the differential. We have seen the following:

BV-structure on $\mathbb{H}_\bullet(\mathsf{LM})$

The loop product $\bullet$ together with the operator Δ define a BV-algebra structure on the loop homology $\mathbb{H}^\bullet(\mathsf{L}M)$.

Chas-Sullivan Product (Dual)

We are interested in the *dual* of the loop product $\bullet$. Consider the diagram

$$\begin{tikzcd}
\mathsf{L}M^{\times 2} \arrow[d, "{(p_0,p_0)}"] & \mathsf{L}M \times_M \mathsf{L}M \arrow[l, "i"'] \arrow[r, "\mathsf{Comp}"] \arrow[d, "p_0"] & \mathsf{L}M \arrow[d, "p_0"] \\
M^{\times 2} & M \arrow[l, "\mathsf{diag}"] \arrow[r, equal] & M
\end{tikzcd}$$

By Poincaré duality, given $f\colon E \to M$, we can construct the *Gysin homomorphism* via the commutative diagram

$$\begin{tikzcd}
H^k(E) \arrow[r, "f_*"] \arrow[d, "\mathsf{P.D.}"] & H^{k-(e-m)}(M) \arrow[d, "\mathsf{P.D.}"] \\
H_{e-k}(E) \arrow[r, "f_*"] & H_{e-k}(M)
\end{tikzcd}$$

where $e = \dim(E), m = \dim(M)$.

Chas-Sullivan Product (Dual) II

Taking $\mathrm{diag}\colon M \to M \times M$ and $i\colon \mathsf{L}M \times_M \mathsf{L}M$, we obtain the Gysin maps

$$\mathrm{diag}^!\colon H^k(M) \to H^{k+m}(M^{\times 2}),\ \ i^!\colon H^k(\mathsf{L}M \times_M \mathsf{L}M) \to H^{k+m}(\mathsf{L}M^{\times 2}).$$

We obtain the following commutative diagram

$$\begin{tikzcd}
H^{k+m}(\mathsf{L}M^{\times 2}) & H^k(\mathsf{L}M \times_M \mathsf{L}M) \arrow[l, "i^!"'] & H^k(\mathsf{L}M) \arrow[l, "H^k(\mathrm{Comp})"'] \\
H^{k+m}(M^{\times 2}) \arrow[u, "H^*(p_0)^{\times 2}"] & H^k(M) \arrow[l, "\mathrm{diag}^!"'] \arrow[u, "H^*(p_0)"] \arrow[r, equal] & H^k(M) \arrow[u, "H^*(p_0)"]
\end{tikzcd}$$

Define the *dual of the loop product* as the composition

$$i^! \circ H^\bullet(\mathrm{Comp})\colon H^*(\mathsf{L}M) \to H^{\bullet+m}(\mathsf{L}M^{\times 2}).$$

Remarks

The construction of the dual loop product is not quite like that.

We need the notion of *Thom spaces* and the *Thom-Pontryagin construction*. More concretely, the square formed by the maps i and diag is a pullback square which allows for a Thom-Pontryagin map

$$\mathsf{L}M \times \mathsf{L}M \to (\mathsf{L}M \times_M \mathsf{L}M)^{TM}.$$

Then, one argues in a similar fashion to define the dual of the loop product as before.

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$C_\bullet(A,A)$ is a model of LM

Proposition

We have an isomorphism of graded vector spaces $HH_\bullet(A,A) \cong H^\bullet(\mathrm{L}M)$. The composite

$$\begin{tikzcd}
C_\bullet(A,A) \arrow[r] \arrow[d, equal] & C_\bullet(A,A) \otimes_A C_\bullet(A,A) \\
A \otimes T(A) \arrow[r, "\mathrm{id}\otimes\phi"] & A \otimes T(A) \otimes T(A) \arrow[u, "\simeq"']
\end{tikzcd}$$

is a model of the composition of free loops.

Sketch of Proof

Consider the diagram

$$
\begin{tikzcd}
 & \mathsf{L}M \times_M \mathsf{L}M \arrow[rr, "j"] \arrow[dl, "\mathsf{Comp}"'] \arrow[dd, dashed, "p_0"'] & & M^I \times_M M^I \arrow[dl, "\mathsf{Comp}^I"] \arrow[dd, "{(p_0,p_0,p_0)}"] \\
\mathsf{L}M \arrow[rr, "j"] \arrow[dd, "p_0"'] & & M^I \arrow[dd] & \\
 & M \arrow[rr, "(\mathsf{id}\times\Delta)\circ\Delta"] \arrow[dl, equal] & & M^{\times 3} \arrow[dl, "\mathsf{pr}"'] \\
M \arrow[rr, "\Delta"'] & & M^{\times 2} &
\end{tikzcd}
\tag{1}
$$

where Comp^I denotes composition of paths, Δ the diagonal and p_0 denotes evaluation at $p_0 \in M$.

Sketch of Proof (II)

Let (A,d) be a commutative DG-algebra quasi-isomorphic to $C^\bullet(M)$. Consider

$$\begin{tikzcd}
\mathrm{Bar}(A,A,A) \arrow[r, "\Psi"] & \mathrm{Bar}(A,A,A) \otimes_A \mathrm{Bar}(A,A,A) \\
A^{\otimes 2} \arrow[u] \arrow[r, "\psi"] & A^{\otimes 3} \arrow[u]
\end{tikzcd} \tag{2}$$

where

$$\Psi(a \otimes [a_1 \mid \cdots \mid a_k]a') = \sum_{i=0}^{k} a \otimes [a_1 \mid \cdots \mid a_i] \otimes 1 \otimes [a_{i+1} \mid \cdots \mid a_k] \otimes a',$$

and

$$\psi(a \otimes a') = a \otimes 1 \otimes a'.$$

Diagram (2) is a cochain model of the right hand square in (1).

Sketch of Proof (III)

Tensoring by A diagram (2), we obtain

$$\begin{tikzcd}
A \otimes_{A^{\otimes 2}} \mathsf{Bar}(A,A,A) \arrow[r, "\mathrm{id}\otimes\Psi"] & A \otimes_{A^{\otimes 3}} (\mathsf{Bar}(A,A,A) \otimes_A \mathsf{Bar}(A,A,A)) \\
A \otimes_{A^{\otimes 2}} A^{\otimes 2} \arrow[u] \arrow[r, "\mathrm{id}\otimes\psi"] & A \otimes_{A^{\otimes 3}} A^{\otimes 3} \arrow[u]
\end{tikzcd} \tag{3}$$

Then, one realises that (3) is actually a cochain model of the left hand square in (1). But we also have

$$\begin{tikzcd}
A \otimes_{A^{\otimes 2}} \mathsf{Bar}(A,A,A) \arrow[r, "\mathrm{id}\otimes\Psi"] & A \otimes_{A^{\otimes 3}} \mathsf{Bar}(A,A,A) \otimes_A \mathsf{Bar}(A,A,A) \\
A \otimes T(A) \arrow[u, "\simeq"] \arrow[r, "\mathrm{id}\otimes\phi"] & A \otimes T(A) \otimes T(A) \arrow[u, "\simeq"]
\end{tikzcd}$$

from which the Proposition follows.

Consider the following diagram

$$\begin{tikzcd} A^\vee \arrow[r, "\mu^\vee"] & (A\otimes A)^\vee = A^\vee \otimes A^\vee \\ A \arrow[u, "\theta"] \arrow[r, dotted, "\mu_A"] & A\otimes A \arrow[u, "\theta\otimes\theta"'] \end{tikzcd} \tag{4}$$

Here, $\mu : A\otimes A \to A$ denotes multiplication of A, θ is the dual isomorphism, and μ_A is defined by the commutative diagram.

We have the following

Lemma

The map μ_A represents the Gysin map $\mathrm{diag}^!$ and the map of degree m

$$C_\bullet(A,A)\otimes_A C_\bullet(A,A) \xrightarrow{\simeq} A\otimes_{A^{\otimes 2}} C_\bullet(A,A)^{\otimes 2} \xrightarrow{\mu_A\otimes\mathrm{id}} C_\bullet(A,A)^{\otimes 2}$$

commutes with the differential and induces $i^!$ in homology.

Define a map Φ via the diagram

$$\begin{tikzcd}
A \otimes T(A) \arrow[r, "\mathrm{id}\otimes\phi"] \arrow[d, dotted, "\Phi"] & A \otimes T(A) \otimes T(A) \cong A \otimes_{A^{\otimes 2}} (A \otimes T(A))^{\otimes 2} \arrow[d, "\mu_A \otimes \mathrm{id}"] \\
(A \otimes T(A))^{\otimes 2} & A^{\otimes 2} \otimes_{A^{\otimes 2}} (A \otimes T(A))^{\otimes 2} \arrow[l, "\simeq"']
\end{tikzcd} \tag{5}$$

We have the following

Theorem

The isomorphism $HH_{\bullet+m}(A,A)^{\vee} \cong HH^{\bullet+m}(A,A^{\vee}) \cong HH^{\bullet}(A,A)$ transfers Φ_* to the Gerstenhaber product on $HH^{\bullet}(A,A)$.

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