New classes of efficiently solvable generalized Traveling Salesman Problems

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We consider the n-city traveling salesman problem (TSP), symmetric or asymmetric, with the following attributes. In one case, a positive integer k and an ordering $(1,\dots,n)$ of the cities is given, and an optimal tour is sought subject to the condition that for any pair i, j $(1,\dots,n)$, if $j \geq i + k$, then i precedes j in the tour. In another case, position i in the tour has to be assigned to some city within k positions from i in the above ordering. This case is closely related to the TSP with time windows. In a third case, an optimal tour visiting m out of n cities is sought subject to constraints of the above two types. This is a special case of the Prize Collecting TSP (PCTSP). In any of the three cases, k may be replaced by city-specific integers $k(i)$, $i = 1,\dots,n$. These problems arise in practice. For each class, we reduce the problem to that of finding a shortest source–sink path in a layered network with a number of arcs linear in n and exponential in the parameter k (which is independent of the problem size). Besides providing linear time algorithms for the solution of these problems, the reduction to a shortest path problem also provides a compact linear programming formulation. Finally, for TSPs or PCTSPs that do not have the required attributes, these algorithms can be used as heuristics that find in linear time a local optimum over an exponential-size neighborhood.

1. Introduction

The Traveling Salesman Problem (TSP) is NP-complete, but some special cases of it are polynomially solvable. There is a considerable literature on this subject (see [8–15,20–31,33–35]), starting with a 1957 paper by Supnick [33]. The paper by Gilmore et al. [26] is a comprehensive survey of this literature up to 1985, along with some original contributions.

Most of the polynomially solvable cases of the TSP owe this property to some special attribute of the cost matrix. In particular, all of the sizeable Russian literature (known to us) on the subject (see [9,15,21–23,28]) belongs to this class, as well as

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most of the Western papers surveyed in [26] (see [8,20,24,25,27,29,33]). Furthermore, most of the literature after 1985 consists of cases of this type (see [10–13,30, 34,35]).

A second type of polynomially solvable TSPs are those restricted to some special class of (typically sparse) graphs. Thus, Cornuéjols et al. [14] showed that the TSP is polynomially solvable on Halin graphs; Ratliff and Rosenthal [31] identified a practical problem that can be formulated as a TSP on a graph whose sparse structure allows a polynomial time solution procedure; and Gilmore et al. in their survey [26] describe a class of sparse graphs, termed bandwidth-limited, which they show to be efficiently solvable.

In this paper, we introduce a class that belongs to a third type of polynomially solvable cases. These are generalized TSPs, where the generalization goes in two directions. First the solutions have to satisfy a restriction on the positions that each city can take in the tour, or a restriction on the set of candidates for each position in the tour. These restrictions may be given indirectly in the form of a certain type of precedence constraints, but may also be specified directly. Precedence-constrained TSPs have been studied in their own right (see [1,5,16,18]), but the precedence constraints involved here are of a special type. Nevertheless, these special types of precedence constraints frequently arise in practice. When the “positional” constraints are given directly, we have a class of problems closely related to the TSP with time windows, a model known to have many applications. The other direction in which the TSP is generalized is that the tour to be constructed need not contain all n cities, but may instead be required to contain a certain number $m < n$ of cities. This is a special case of the Prize Collecting TSP, again a problem that has been studied in its own right (see [2–4,19]). The PCTSP with the type of precedence constraints examined in this paper arises in the scheduling of steel rolling mills [6].

The contribution of this paper is two-fold: (i) it gives highly efficient procedures for solving several new classes of generalized TSPs that arise in practice, and (ii) the procedures it develops can be used as heuristics on general TSPs that do not have the properties assumed in the paper, and for certain situations these heuristics look promising. Preliminary computational experience with this approach is described in [32], and a more extensive computational study is in preparation [7].

All the classes of TSPs and their generalizations discussed in this paper can be symmetric or asymmetric; the results apply to both cases.

The structure of the paper is as follows. Section 2 describes the results: it defines three new problem classes (two versions of each) that are solvable in time linear in the number of cities and exponential in some parameters that are independent of problem size. Section 3 discusses applications of the models introduced. Section 4 describes the auxiliary graph whose construction provides the proof of the main theorem as well as the solution method for the basic model. Section 5 describes the auxiliary graphs needed for proving the other theorems and solving the remaining models. Finally, section 6 discusses the connection between our models and the bandwidth-limited TSPs.

2. The results

Consider the n-city traveling salesman problem (TSP) defined on a complete directed or undirected graph, and fix city 1 as the home city, where all tours start and end. Suppose now that we are given an integer k, $1 \leq k < n$, and an ordering $(1,\ldots,n)$ of the set N of cities, and we want to find a minimum cost permutation π of $(1,\ldots,n)$ (and associated tour) subject to the condition

(i) for all $i, j \quad (1,\ldots,n)$, $j \geq i + k$ implies $\pi(i) < \pi(j)$.

Then we have the following

Theorem 2.1. Any TSP with condition (i) can be solved in time $O(k^2 2^{k-2} n)$.

Proof outline. One can construct an arc-weighted acyclic digraph $G^* = (V^*, A^*)$, to be called the *auxiliary digraph*, with the following properties:

(a) V^* consists of $n + 1$ layers V_i^*, such that for $i = 1,\ldots,n$, if $j \quad V_i^*$ and $(j, \ell) \quad A^*$, then $\ell \quad V_{i+1}^*$;

(b) $V_1^* = \{s\}$, $V_{n+1}^* = \{t\}$, where s is a source and t is a sink, and for $i = 2,\ldots,n$, $|V_i^*| \leq (k + 1)2^{k-2}$;

(c) for each $j \quad V_i^*$, $i = 2,\ldots,n$, $\deg^-(j) \leq k$;

(d) there is a 1:1 correspondence between optimal tours in G satisfying (i), (ii), (iii), and shortest $s-t$ paths in G^*.

The construction of G^* is the subject of section 3. The theorem then follows from the fact that a shortest $s-t$ path in the layered digraph G^* can be found in $O(|A^*|)$ time. □

The condition defining this class of TSPs solvable in time linear in n is a special type of precedence constraint: it states that every city i has to precede in the tour every city j such that $j \geq i + k$. Note that the case $k = 1$ corresponds to the situation where the ordering $(1,\ldots,n)$ is itself optimal. More generally, in order for property (i) to hold with a relatively small k, it is necessary that the ordering $(1,\ldots,n)$ be relatively close to an optimal one; and, roughly speaking, the further removed the ordering $(1,\ldots,n)$ from an optimal one, the larger k has to be for property (i) to hold. Viewed this way, theorem 2.1 essentially says that if we have an ordering whose “divergence” from an optimal ordering can be bounded by (i), then an optimal ordering can be found in time linear in n (but exponential in the bounding constant k). On the other hand, it also follows from theorem 2.1 that, unless $\mathcal{P} = \mathcal{NP}$, for an arbitrary TSP one cannot find in polynomial time an ordering whose “divergence” from an optimal tour is bounded by (i).

The property underlying theorem 2.1 can be generalized: it is not necessary that the constant k be the same for each pair of cities; it can be replaced with positive constants $k(i)$, $1 \leq k(i) \leq n - i + 1$, $i \quad N$, specific to each city i. In that case, condition (i) becomes

(ia) for all $i, j \quad (1,\ldots,n)$, $j \geq i + k(i)$ implies $\pi(i) < \pi(j)$.

The result corresponding to theorem 2.1 is then

Theorem 2.2. Any TSP with condition (ia) can be solved in time $O(\sum_{i=2}^{n} k^*(i)^2 2^{k^*(i)-2})$, where

$$k^*(i) := \max\{k(j) : j + k(j) \geq i + 1 \text{ and } j \leq i\}.$$

Proof outline. As in the case of theorem 2.1, one can construct an auxiliary digraph $G^{*(a)} = (V^{*(a)}, A^{*(a)})$ such that the optimal tours in G satisfying (ia) are in a 1:1 correspondence with the shortest $s-t$ paths in G^*. This time, $|V_i^{*(a)}| \leq (k^*(i) + 1)2^{k^*(i)-2}$ and $\deg^-(j) \leq k^*(i)$ for $j \quad V_i^{*(a)}$, $i \quad \{2,\ldots,n-1\}$, with $k^*(i)$ as defined in the theorem. The construction of $G^{*(a)}$ is discussed in section 5. □

Obviously, when $k(i) = k$ for all $i \quad N$, then $k^*(i)$ becomes k and thus theorem 2.1 is a special case of theorem 2.2.

A class of problems closely related to the above ones is the one in which condition (i) is replaced by

(ib) for all $i \quad (1,\ldots,n)$, $i - k + 1 \leq \pi^{-1}(i) \leq i + k - 1$.

This condition says that the candidates for position i in an optimal tour must come from the interval of integers $\{i - k + 1,\ldots, i + k - 1\}$. In this case, we have

Theorem 2.3. Any TSP with condition (ib) can be solved in time $O(k^{1.5} 2^{2(k-1)} n)$.

Proof outline. One can construct an auxiliary digraph $G^{*(b)}$ with the same properties as in the case of theorem 2.1 but with more nodes and arcs. This construction is discussed in section 5. □

As in the case of theorem 2.1, the conditions for which theorem 2.3 holds can also be generalized by replacing the integer k with position-specific integers $k(i)_L$, $k(i)_U$. This leads to a "position-window" condition that requires the candidates for position i to come from the interval of integers $\{i - k(i)_L + 1,\ldots, i + k(i)_U - 1\}$. The corresponding problem can then be solved in time linear in n, but exponential in $\max_i \max\{k(i)_L, k(i)_U\}$.

The above property of TSPs generalizes to an important class of Prize Collecting TSPs (PCTSPs). In the PCTSP (directed or undirected), a salesman has to visit a subset of the cities, and gets a prize for every city that he visits. The objective is a minimum-cost tour that visits enough cities to collect a required amount of prize money. Consider now the special case when all the prizes are equal; then the requirement to collect a specified amount of prize money can be expressed as a requirement to include into the PC tour a specified number of cities. We call this class of PCTSPs *uniform*. In other words, a uniform PCTSP is one whose "prize collection constraint" is of the form

$$\sum_{i \in N} y_i \geq m,$$

where each y_i is a 0–1 variable taking the value 1 if a city i is included, 0 if city i is not included in the PC tour, and m is the number of cities that have to be included. When the travel costs are nonnegative, which is the usual case, the above inequality can be treated as an equation.

Consider now a uniform PCTSP in which a minimum-cost PC tour of m cities from N is sought, starting and ending at city 1, and suppose that we are given an integer k, $\lceil n/m \rceil \leq k < n$, and an ordering $(1,\ldots,n)$ of N. Although we do not put an explicit lower bround on m, the relevant cases are those in which $\lceil n/m \rceil$ is a small integer, say 2, 3 or 4. We wish to find an optimal subset $M \subseteq N$ with $\{1\} \in M$, and the associated optimal PC tour, i.e. permutation π of the elements of M, with $\pi(1) = 1$ and with the property that

(ic) for all $j, \ell \in M$, $\ell \geq j + k$ implies $\pi(j) < \pi(\ell)$; and

(iic) for all $j \in N - \{1\}$ and $i \in M - \{1\}$, city j can be assigned to position i in the tour if and only if

$$\left\lceil \frac{n}{m}(i-1) \right\rceil - k + 1 \leq j \leq \left\lceil \frac{n}{m}(i-1) \right\rceil + k - 1.$$

We then have

Theorem 2.4. Any uniform PCTSP with conditions (ic), (iic) can be solved in time $O(k^2 2^{2k-1} m)$.

Proof outline. An auxiliary digraph $G^{*(c)} = (V^{*(c)}, A^{*(c)})$ can be constructed, with the same properties as in the case of theorem 2.1, but with m layers instead of n and more nodes in every layer (see section 5). □

As in the case of the TSP, the property of a uniform PCTSP defined by (ic), (iic) can be generalized by replacing the constant k with constants $k(j)$ specific to each city j, satisfying $k(j)$, $1 \leq k(j) \leq n - j + 1$, $j \in N$, with $k(1) \geq \lceil n/m \rceil$. Conditions (ic), (iic) then become

(id) for all $j, \ell \in M$, $\ell \geq j + k(j)$ implies $\pi(j) < \pi(\ell)$; and

(iid) for all $j \in N - \{1\}$ and $i \in M - \{1\}$, city j can be assigned to position i in the tour if and only if

$$\left\lceil \frac{n}{m}(i-1) \right\rceil - k(j) + 1 \leq j \leq \left\lceil \frac{n}{m}(i-1) \right\rceil + k(i) - 1.$$

The corresponding result is then

Theorem 2.5. Any uniform PCTSP with conditions (id), (iid) can be solved in time $O(\sum_{i=2}^{m} k^*(i)^2 2^{2k^*(i)-1})$.

Proof outline. An auxiliary digraph $G^{*(d)}$ like the one constructed for the proof of theorem 2.2 applies here, with n replaced by m as the number of layers (see section 5). □

Note that the time bounds of theorems 2.4 and 2.5 involve only k and m.

All of the above results can be generalized by allowing the constants k or $k(i)$ to be replaced with affine functions of $\log_2 n$. This will cause the complexity bounds given in the above theorems to cease to be linear, but they will still remain polynomial in n. For instance, if k is replaced by $\alpha \log_2 n$ for some positive constant α, the complexity bound given in theorem 2.1 becomes polynomial of degree $\alpha + 1$ in n. A similar transformation applies to each of the other bounds.

Finally, it should be mentioned that reducing the TSP with condition (i) or its analogues to a shortest $s-t$ path problem in G^* or its analogues also provides a compact linear programming formulation for these problems:

$$\min \left\{ cy : y \in \mathbb{R}^{A^*},\ y \geq 0,\ y(\delta^+(i)) - y(\delta^-(i)) = \begin{cases} 1 & i = s, \\ 0 & i \in V^* - \{s,t\} \end{cases} \right\},$$

where $\delta^+(i) := \{(k,\ell) \in A^* : k = i\}$, $\delta^-(i) := \{(k,\ell) \in A^* : \ell = i\}$, and where the number of variables and constraints is linear in n and exponential in the parameter k (which does not depend on problem size).

An interesting open problem is how to use an extended formulation of the TSP with condition (i) involving the above variables and constraints as well as the variables x_{ij} of the standard formulation, with constraints of the form

$$\sum_{(k,\ell) \in A_{ij}^*} y_{k\ell} - x_{ij} = 0, \quad (i,j) \in E,$$

where A_{ij}^* is the set of those arcs of G^* corresponding to edge (i,j) of G, to obtain a polyhedral characterization in the x-space by projecting out the y-variables.

3. Applications

The primary inspiration for this paper came from a class of Prize Collecting TSPs that arise in practice and that have a structure similar to the one expressed by conditions (id)–(iid). In scheduling a steel rolling mill that processes slabs into sheet, a subset of slabs is selected from an inventory and ordered into a “round” (a sequence between two general roll changes) in such a way as to minimize the cost of the round while satisfying a lower bound on the total weight of the slabs rolled [6]. The “cost of the round” can be expressed as a linear function of the arc variables, which assigns a transition cost c_{ij} to having item j processed right after item i. Thus, the problem can be formulated as a Prize Collecting TSP, where the prizes are the weights of the slabs and the tour to be found is the round to be generated (plus a dummy node 1). However, besides the conditions that can be captured in the cost function, there are also some precedence constraints: if the width of slab i exceeds that of slab j by at least Δ units, then i must precede j in the round. Suppose now that the slabs in the inventory are ordered into a sequence $(1,\dots,n)$ according to nonincreasing width. Then if w_i is the width of slab i and ℓ is the smallest integer such that $w_\ell \leq w_i - \Delta$, we can define $k(i) := \ell - i$ and formulate the above described precedence constraint as the requirement that for any pair i, j N, if $j \geq i + k(i)$, then $\pi(i) < \pi(j)$, i.e. i must precede j in the round. But this is precisely condition (id).

Another class of real-world problems where the approach discussed in this paper may have wide applicability is scheduling or routing with time-windows. Consider a scheduling problem with sequence-dependent setup times and a time window for each job, defined by a release time and a deadline. Or consider a delivery problem, where a vehicle has to deliver goods to clients within a time-window for each client. Whenever there is a way to translate the time windows into "position-windows", i.e. into a lower and upper bound on the position (rank) of each job or client in the tour to be constructed, the TSP with condition (ib), possibly with position-windows of different sizes for the different items, comes into its own. One way of translating time windows into position-windows is the following: suppose one is able to generate a feasible sequence $\sigma := (1,\dots,n)$, i.e. one that satisfies the time-windows. Then for each position i in the tour, one can define its position-window, i.e. the interval $\{i - k(i)_L + 1,\dots, i + k(i)_U - 1\}$, as the segment of σ covering the time-window, plus possibly a few more positions to the right and/or the left. Admittedly, this is only an approximation of the time-windows, but (a) the approach can be refined in many ways, and (b) the shortest path calculation executed during the procedure can be amended so as to check at every step (at a modest extra cost) the actual time-windows, and stop building a path as soon as it violates one of them.

While inspired by some real-world problems, the class of models discussed here has much wider applicability. We suspect that a condition of the type considered here is often present in real-world problems, though not explicitly stated and possibly hard to recognize. To illustrate what we have in mind, consider a TSP in the two-dimensional plane, defined on a complete undirected graph G whose vertex set is shown in figure 1(a), where vertex 0 represents the home city. Suppose the cost of traveling from or to the home city is 0 for all the other cities, while the remaining costs are the Euclidean distances between the cities, the unit being the smallest distance between any two cities. It is easy to check that any tour starting at city 0, traversing the remaining 24 cities by always choosing as the next city one that is at unit distance, has length 23 and is optimal. One such tour is shown in figure 1(b).

Suppose now that the horizontal distances are twice as large as the vertical distances, as shown in figure 2(a). In this case, the tour corresponding to the one in figure 1(b) has length 37 and is no longer optimal: an optimal one is shown in figure 2(b), of

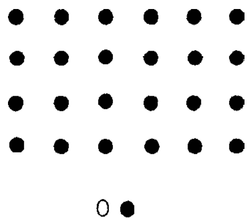

Figure 1(a)

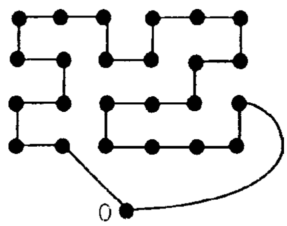

Figure 1(b)

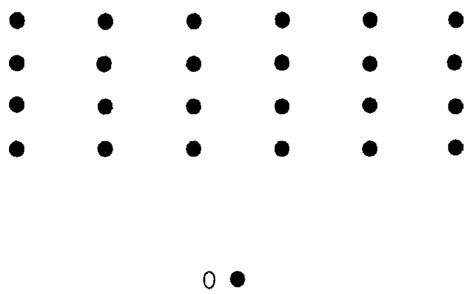

Figure 2(a)

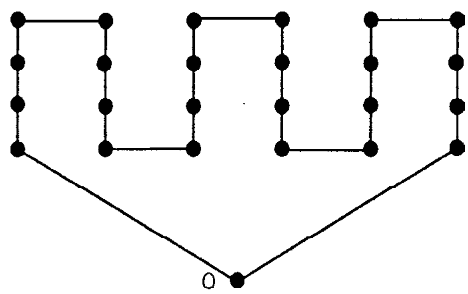

Figure 2(b)

length 28. Clearly, with the horizontal distances twice as large as the vertical ones, it pays to traverse all the cities at a given "latitude" before moving horizontally to the next latitude. Taking any ordering that respects this rule to be $(1,\ldots,n)$, we can state that there exists an optimal tour such that for any pair of cities i, j, if $j \geq i + k$, then i has to precede j. Here, k is the number of cities on "the same latitude", in our example 4.

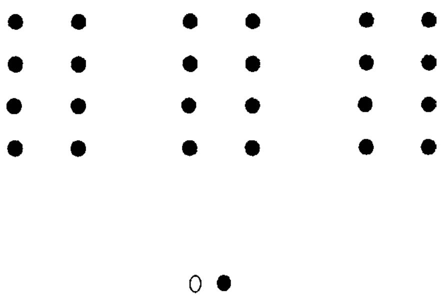

Figure 3(a)

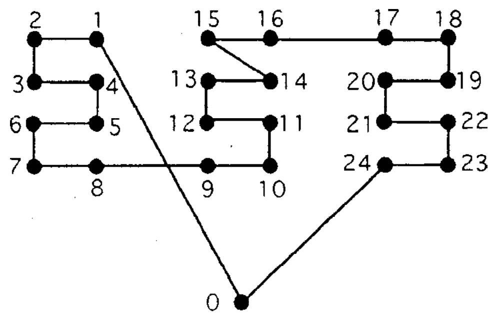

Figure 3(b)

To take this example one step further, suppose now that the 24 non-home cities are clustered into three groups of size 8 each, as shown in figure 3(a). It is not hard to see that no tour can be optimal if it visits the middle cluster first or last. In other words, if we number the clusters from left to right as 1, 2, 3, there is always an optimal tour that visits the clusters in that order. If we now choose any ordering of the cities such that the first cluster gets the numbers from 1 to 8 (in whatever order), the second the numbers from 9 to 16, and the third those from 17 to 24 (figure 3(b) shows such an ordering with associated nonoptimal tour), there is always an optimal tour which satisfies the following condition:

- for any pair i, j of cities, if $j \geq i + k(i)$, then i has to precede j.

Here, the numbers $k(i)$ can be defined as

$$k(i) = \begin{cases} 9 - i & \text{if } i \in \{1, \ldots, 8\}, \\ 17 - i & \text{if } i \in \{9, \ldots, 16\}, \\ 25 - i & \text{if } i \in \{17, \ldots, 24\}. \end{cases}$$

This property has nothing to do with the regularity of intra-cluster distances: it just captures the pattern of inter-cluster distances and the fact that the latter are larger than the intra-cluster distances.

Finally, to take another example, suppose again that we have a symmetric cost function representing inter-city Euclidean distances, and suppose there is a line or curve along which the distances are larger than those in the orthogonal directions. We wish to find a minimum-cost Hamilton path. Taking as the ordering $(1, \ldots, n)$ the one

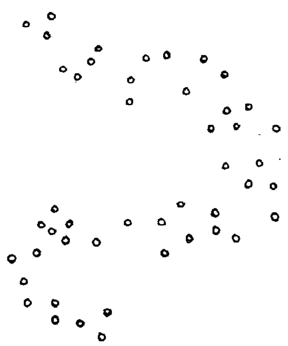

Figure 4(a). Layout of cities.

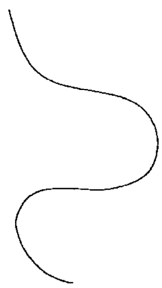

Figure 4(b) Associated curve.

implied by any Hamilton path that roughly follows the direction of the curve, as produced for instance by a "sweep" heuristic which does not move ahead "longitudinally" (i.e. along the curve) until it has visited all the cities reachable "laterally" (i.e. by moving in directions orthogonal to the curve), we can then state the condition that in an optimal Hamilton path,

- for any pair of cities i, j, if $j \geq i + k$, then i is visited before j.

The number k itself depends on the ratio between distances along the curve and distances along perpendiculars to the curve, as well as those distances themselves; but clearly, for large enough k the above condition will be satisfied.

The examples could be continued, but the problem of detecting when a condition of the type discussed here applies to some given problem is likely to be very hard. What the examples suggest, however, is that given an ordering $(1,\dots,n)$ obtained by a "good" TSP heuristic, and an integer k either adopted by educated guessing or derived from looking at the maximum deviation from the above ordering in a number of "good" but different heuristic tours, solving exactly the TSP with condition (ia) is likely to yield an improved solution.

4. The auxiliary digraph G^*

Consider a TSP with condition (i) of section 1. A tour will be called *feasible* if it satisfies the condition. This condition implies that the assignment of a given city to some position in the tour is restricted to a certain segment of the list of those positions; and vice versa: the candidates for a given position in the tour must be drawn from a segment of the list $(1,\dots,n)$ of cities. More specifically, we have

Proposition 4.1. Let π define a feasible tour. Then

(a) for any city j N,

$$j - k + 1 \leq \pi(j) \leq j + k - 1; \quad \text{and} \tag{4.1}$$

(b) for any position i in the tour,

$$i - k + 1 \leq \pi^{-1}(i) \leq i + k - 1. \tag{4.2}$$

Proof. (a) Suppose city j is assigned to a position $\pi(j)$ for which the first inequality of (4.1) does not hold, i.e. $\pi(j) \leq j - k$. Then some city $h \leq j - k$ must be assigned to a position $\pi(h) > \pi(j)$, contrary to the requirement (i) that h precede j. Similarly, if the second inequality of (4.1) is violated, i.e. $\pi(j) \geq j + k$, then some city $\ell \geq j + k$ must be assigned to a position $\pi(\ell) < \pi(j)$, contrary to the requirement that j precede ℓ.

(b) Condition (4.2) can be written as

$$\pi(j) - k + 1 \leq j \leq \pi(j) + k - 1, \tag{4.3}$$

where we put $j := \pi^{-1}(i)$. Now the first (second) inequality of (4.3) is the second (first) inequality of (4.1). Thus, (4.2) follows from (4.1). □

Proposition 4.1 says that (a) every city j can be assigned to only one of the $2k - 1$ positions of the integer interval $\{j - k + 1,\dots, j + k - 1\}$, and that (b) every position i can be filled by only one of the $2k - 1$ cities of the integer interval $\{i - k + 1,\dots, i + k - 1\}$. Note the crucial fact that the number of cities assignable to a position, and the number of positions assignable to a city, do not depend on n.

Corollary 4.2. The number of feasible tours is bounded by $(2k - 1)^{n-1}$.

While this is a substantial reduction with respect to the number of tours in the complete graph or digraph G, which is $(n-1)!/2$ and $(n-1)!$, respectively, it is still a number exponential in n.

Since the number of candidates for a given position in the tour depends only on k and not on n, an optimal tour can be determined through a dynamic programming recursion that depends only on k. In general, the cost of an optimal tour segment starting at city 1, traversing the cities of a subset $W \subset N$ in positions $2,\ldots,i-1$, and visiting city j in position i, can be calculated recursively as

$$C(W, i, j) = \min_{\ell \in W}\{C(W\backslash\{\ell\}, i-1, \ell) + c_{\ell j}.$$

The problem with this is that there are $\binom{n}{i-2}$ different subsets W of N of size $i-2$, and as i grows, the number of such subsets becomes exponential in n. It is the precedence constraints involving k that make it possible to eliminate this exponential growth: they render most of the subsets inadmissible; and make it possible to describe those that are admissible by specifying a pair of sub-subsets whose joint size is bounded by k. In particular, every feasible ordering $\pi : N \to N$ defines for every position i in the tour the unique pair of subsets

$$S^{-}(\pi, i) := \{\ell \in (1, \ldots, n) : \ell \geq i, \pi(\ell) \leq i-1\}$$

and

$$S^{+}(\pi, i) := \{h \in (1, \ldots, n) : h \leq i-1, \pi(h) \geq i\}.$$

Proposition 4.3. For all feasible π and all $i \in N$,

$$|S^{-}(\pi, i)| = |S^{+}(\pi, i)| \leq \lfloor k/2 \rfloor.$$

Proof. From the definitions, $S^{-}(\pi, i) \cap S^{+}(\pi, i) = \emptyset$ and $|S^{-}(\pi, i)| = |S^{+}(\pi, i)|$. Since $\pi(\ell) < \pi(h)$ for all $\ell \in S^{-}(\pi, i)$ and $h \in S^{+}(\pi, i)$, condition (i) implies $\ell \leq h + k - 1$ for all such pairs ℓ, h; hence,

$$\max\{\ell : \ell \in S^{-}(\pi, i)\} - \min\{h : h \in S^{+}(\pi, i)\} \leq k-1.$$

We then have $|S^{-}(\pi, i) \cup S^{+}(\pi, i)| \leq (k-1) + 1 = k$, and the claimed result follows. □

Specifying the pair of subsets $S^{-}(\pi, i)$, $S^{+}(\pi, i)$ is equivalent to fully specifying the subset of cities encountered by the tour corresponding to π between positions 1 and i. Indeed, if for $i \in N$ we denote by N_i the set of the first i elements of $(1,\ldots,n)$, then the subset W referred to above is

$$(N_{i-1} \backslash S^{+}(\pi, i)) \cup S^{-}(\pi, i).$$

Thus, we construct the layered digraph $G^{*} = (V^{*}, A^{*})$ as follows. We put $V^{*} := \bigcup_{i=1}^{n+1} V_i^{*}$, where $V_1^{*} := \{s\}$, the source node, and $V_{n+1}^{*} := \{t\}$, the sink node. The nodes

s and t, the only ones in layer 1 and $n+1$, correspond to the first and last node of the tour, a role assigned by definition to city 1. For every other layer i, we create a node $(i, j, S_{ij}^-, S_{ij}^+)$ for every city j that can be a candidate for position i, and every pair of sets S_{ij}^-, S_{ij}^+ (corresponding to $S^-(\pi, i)$, $S^+(\pi, i)$ in the earlier notation) compatible with the feasibility conditions of a tour. This means, in particular, that j has to belong to the integer interval

$$I(i) := \{\max\{2, i-k+1\}, \quad , \min\{n, i+k-1\}\},$$

and that the sets S_{ij}^-, S_{ij}^+ have to be drawn from $\{i, i+1, \ldots, \min\{n, i+k-1\}\} \setminus \{j\}$ and $\{i-1, i-2, \ldots, \max\{2, i-k+1\}\}$, respectively, in such a way as to satisfy condition (i). We will use the notation $V_i^* = \bigcup_{j \quad I(i)} V_{ij}^*$, where V_{ij}^* is the set of nodes associated with the pair $(i, j := \pi^{-1}(i))$. The nodes in this set – as well as the pairs S_{ij}^-, S_{ij}^+ by which the nodes differ among themselves – will be numbered from 1 to $|V_{ij}^*|$, i.e. V_{ij}^* consists of the nodes

$$(i, j, r) := (i, j, S_{ijr}^-, S_{ijr}^+), \quad r = 1, \quad , |V_{ij}^*|.$$

To put it slightly differently, each node (i, j, r) of G^* represents a family of paths (tour segments) in G from city 1 in position 1 to city j in position i, with the cities in the set

$$N(i, j, r) := (N_{i-1} \backslash S_{ijr}^+) \cup S_{ijr}^-$$

in the positions between 1 and i (including 1 but not i). For the sake of uniformity, we will use the same notation for the source and the sink, too, i.e.

$$s := (1,1,1) := (1,1, \quad , \quad),$$
$$t := (n+1,1,1) := (n+1,1, \quad , \quad).$$

The arcs of G^* are defined as follows. All arcs join nodes in consecutive layers. Two nodes in consecutive layers, $(i-1, \ell, t) \quad V_{i-1}^*$ and $(i, j, r) \quad V_i^*$, will be called *compatible* if each path (tour segment) T_{ij} in G that belongs to the family corresponding to (i, j, r) can be obtained from a path (tour segment) $T_{i-1,\ell}$ that belongs to the family corresponding to $(i-1, \ell, t)$, by adjoining to $T_{i-1,\ell}$ the city j. Since $T_{i-1,\ell}$ uses the cities (nodes of G) in $N(i-1, \ell, t) \cup \{\ell\}$ and T_{ij} uses the cities in $N(i, j, r) \cup \{j\}$, clearly nodes $(i-1, \ell, t)$ and (i, j, r) of G^* are compatible if and only if

$$N(i, j, r) = N(i-1, \ell, t) \cup \{\ell\}. \tag{4.4}$$

We define an arc of G^* for every pair of nodes in consecutive layers that are compatible with each other. To ascertain compatibility in an efficient manner, i.e. without having to deal directly with the sets $N(i, j, r)$ and $N(i-1, \ell, t)$ whose size increases with i, we have to express condition (4.4) in terms of the sets S_{ijr}^-, S_{ijr}^+ and $S_{i-1,\ell,t}^-, S_{i-1,\ell,t}^+$. This is done in the next proposition. We have to consider five different situations and in each situation a slightly different condition has to be satisfied.

Proposition 4.4. Nodes $(i-1, \ell, t)$ and (i, j, r) of G^* are compatible if and only if $j \neq \ell$ and in each of the situations (a), (b), (c), (d), (e), the condition specified below holds:

(a) $\ell < i-1$ and $i-1 \quad S^-_{i-1,\ell,t}$:

$$S^-_{i,j,r} = S^-_{i-1,\ell,t}, \quad S^+_{ijr} = S^+_{i-1,\ell,t} \setminus \{\ell\};$$

(b) $\ell < i-1$ and $i-1 \quad S^-_{i-1,\ell,t}$:

$$S^-_{i,j,r} = S^-_{i-1,\ell,t}, \quad S^+_{ijr} = (S^+_{i-1,\ell,t} \setminus \{\ell\}) \cup \{i-1\};$$

(c) $\ell = i-1$:

$$S^-_{i,j,r} = S^-_{i-1,\ell,t}, \quad S^+_{ijr} = S^+_{i-1,\ell,t};$$

(d) $\ell > i-1$ and $i-1 \quad S^-_{i-1,\ell,t}$:

$$S^-_{i,j,r} = S^-_{i-1,\ell,t} \setminus \{i-1\}) \cup \{\ell\}, \quad S^+_{ijr} = S^+_{i-1,\ell,t};$$

(e) $\ell > i-1$ and $i-1 \quad S^-_{i-1,\ell,t}$:

$$S^-_{i,j,r} = S^-_{i-1,\ell,t} \cup \{\ell\}, \quad S^+_{ijr} = S^+_{i-1,\ell,t} \cup \{i-1\}.$$

Proof. If $j = \ell$, the two nodes are obviously incompatible, so $j \neq \ell$ is a necessary condition for compatibility. We now consider the different cases, and show for each one that the stated condition (along with $j \neq \ell$) is necessary and sufficient for (4.4) to hold. To make the argument easier to follow, we restate (4.4) as

$$(N_{i-2} \setminus S^+_{i-1,\ell,t}) \cup S^-_{i-1,\ell,t} \cup \{\ell\} = (N_{i-1} \setminus S^+_{ijr}) \cup S^-_{ijr}, \tag{4.4$'$}$$

where $N_{i-1} = N_{i-2} \cup \{i-1\}$.

(a) If $\ell < i-1$ and $i-1 \quad S^-_{i-1,\ell,t}$, then $\ell \quad S^+_{i-1,\ell,t}$, but since $\pi(\ell) = i-1$, ℓ cannot belong to S^+_{ijr}. On the other hand, compatibility requires that S^+_{ijr} contain the rest of $S^+_{i-1,\ell,t}$; hence the condition $S^+_{ijr} = S^+_{i-1,\ell,t} \setminus \{\ell\}$. Further, $i-1 \quad S^-_{i-1,\ell,t}$, but according to the definitions, $i-1$ cannot belong to S^{-1}_{ijr}; on the other hand, compatibility requires that S^{-1}_{ijr} contain the rest of $S^-_{i-1,\ell,t}$.

(b) If $\ell < i-1$ but $i-1 \quad S^-_{i-1,\ell,t}$, then for the reason stated above we must have $S^-_{ijr} = S^-_{i-1,\ell,t}$. Further, again for the reason stated above, S^+_{ijr} cannot contain ℓ (which belongs to $S^+_{i-1,\ell,t}$). On the other hand, since $\pi(\ell) = i-1$, $\pi(j) = i$ and $i-1 \quad S^-_{i-1,\ell,t}$, which together imply that $\pi(i-1) > i$, we must have $i-1 \quad S^+_{ijr}$.

(c) If $\ell = i-1$, then neither ℓ nor $i-1$ belong to either $S^-_{i-1,\ell,t}$ or $S^+_{i-1,\ell,t}$, and there is no change in the two sets as we move from $(i-1, \ell, t)$ to (i, j, r).

(d) If $\ell > i-1$ and $i-1 \quad S^-_{i-1,\ell,t}$, then again S^-_{ijr} cannot contain $i-1$ according to its definition, but must contain ℓ (since $\pi(\ell) = i-1$ and $\ell \geq i$), which yields $S^-_{ijr} = (S^-_{i-1,\ell,t} \setminus \{i-1\}) \cup \{\ell\}$. On the other hand, neither $i-1$ nor ℓ belongs to $S^+_{i-1,\ell,t}$, hence we need $S^+_{ijr} = S^+_{i-1,\ell,t}$.

(e) Finally, when $\ell > i - 1$ and $i - 1$ $S^-_{i-1,\ell,t}$, ℓ must belong to S^-_{ijr} for the same reason as under (d); and since $i - 1$ $S^-_{i-1,\ell,t}$ together with $\pi(j) = i$ implies $\pi(i-1) > i$, $i - 1$ must belong to S^+_{ijr}. □

Note that given a node (i, j, r) of G^* and a city ℓ assigned to position $i - 1$, there is at most one index t such that the node $(i - 1, \ell, t)$ is compatible with (i, j, r). Similarly, given a node $(i - 1, \ell, t)$ and a city j assigned to position i, there is at most one index r such that (i, j, r) is compatible with $(i - 1, \ell, t)$. Thus, (i, j, r) has at most one predecessor in $V^*_{i-1,\ell}$ for every ℓ, and $(i - 1, \ell, t)$ has at most one successor in V^*_{ij} for every j.

Finally, we have to specify the arc weights or lengths for G^*. For any arc $((i - 1, \ell, t), (i, j, r))$ A^*, we define its length as $c((i - 1, \ell, t), (i, j, r)) := c_{\ell j}$, where $c_{\ell j}$ is the length (cost) of the arc (ℓ, j) of G, the graph or digraph on which the TSP is defined.

Theorem 4.5. There is a 1:1 correspondence between (optimal) feasible tours in G and (shortest) $s-t$ paths in G^*.

Proof. We will show that there is a 1:1 correspondence between $s-t$ paths in G^* and feasible tours in G, with the cost of a feasible tour in G the same as the cost of the corresponding $s-t$ path in G^*. It then follows that optimal feasible tours in G correspond to shortest $s-t$ paths in G^*.

Let P be an $s-t$ path in G^* involving the following sequence of arcs:

$$((i_1, j_1, r_1), (i_2, j_2, r_2)), ((i_2, j_2, r_2), (i_3, j_3, r_3)), \quad , ((i_n, j_n, r_n), (i_{n+1}, j_{n+1}, r_{n+1})),$$

where $(i_1, j_1, r_1) = (1, 1, 1)$ and $(i_{n+1}, j_{n+1}, r_{n+1}) = (n + 1, 1, 1)$. Then the sequence of arcs $(j_1, j_2), (j_2, j_3), \ldots, (j_n, j_{n+1})$ of G, where $j_1 = j_{n+1} = 1$, is a feasible tour in G, say T. To see this, notice that from the definition of the nodes and arcs of G^*, $((i_p, j_p, r_p), (i_q, j_q, r_q))$ A^* if and only if (i) $q = p + 1$, (ii) j_p and j_q are valid candidates for positions i_p and i_q, respectively, in a tour in G, and (iii) $N(i_q, j_q, r_q) = N(i_p, j_p, r_p) \cup \{j\}$; in other words, if and only if the arc (j_p, j_q) of G is joining city j_p in position i_p to city j_q in position i_q of some feasible tour in G. Furthermore, since the length of arc $((i_p, j_p, r_p), (i_q, j_q, r_q))$ of G^* is $c_{j_p j_q}$, the same as the cost of arc (j_p, j_q) of G, it follows that the length of P in G^* is the same as the cost of T in G. □

Example 1. For $k = 3$, the digraph G^* is shown in figures 5(a), 5(b). The first label of each node, marking its position in the tour, is omitted and shown instead at the top of the column of nodes as the index of the corresponding layer. So the notation in these figures is $(j, S^-_{ijr}, S^+_{ijr})$.

Example 2. For $k = 4$, the nodes in V^*_i, for any i satisfying $k \le i \le n - k + 1$, are shown in table 1, and the arcs incident into nodes in $V^*_{i,i+1}$ are shown in table 2.

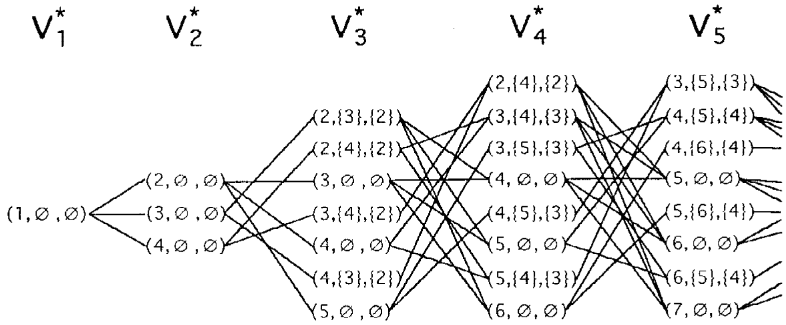

Figure 5(a).

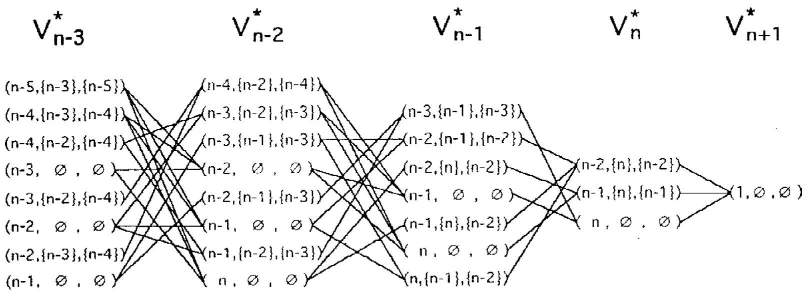

Figure 5(b).

Theorem 4.5 implies that an optimal feasible tour in G can be found by identifying a shortest path in G^*, a task of $O(|A^*|)$. Next we evaluate the size of V^* and A^*. To do this, we need some auxiliary results which we proceed to derive. In dealing with binomial coefficients $\binom{r}{s}$, we define $\binom{r}{s} = 1$ for $s = 0$ and $\binom{r}{s} = 0$ for $r < s$.

Lemma 4.6. For any positive integer k,

$$\sum_{m=0}^{\lfloor k/2 \rfloor} \sum_{\ell=m}^{k-m-1} \binom{k-2-\ell}{m-1}\binom{\ell}{m} = 2^{k-2}.$$

Proof. We claim that

$$\sum_{\ell=m}^{k-m-1} \binom{k-2-\ell}{m-1}\binom{\ell}{m} = \binom{k-1}{2m}.$$

To show this, we set $h := k - 2m - 1$, $j := \ell - m$, and write the left-hand side of the equation as

Table 1

Nodes in V_{ij}^* for $k = 4$.

Subset V_{ij}^* of V_i^*	Node (i, j, r)	S_{ijr}^-	S_{ijr}^+
$V_{i,i-3}^*$	$(i, i-3, 1)$	$\{i\}$	$\{i-3\}$
$V_{i,i-2}^*$	$(i, i-2, 1)$	$\{i\}$	$\{i-3\}$
	$(i, i-2, 2)$	$\{i+1\}$	$\{i-2\}$
	$(i, i-2, 3\}$	$\{i, i+1\}$	$\{i-2, i-1\}$
$V_{i,i-1}^*$	$(i, i-1, 1)$	$\{i\}$	$\{i-1\}$
	$(i, i-1, 2)$	$\{i+1\}$	$\{i-1\}$
	$(i, i-1, 3)$	$\{i+2\}$	$\{i-1\}$
	$(i, i-1, 4)$	$\{i, i+1\}$	$\{i-2, i-1\}$
$V_{i,i}^*$	$(i, i, 1)$		
	$(i, i, 2)$	$\{i+1\}$	$\{i-2\}$
	$(i, i, 3)$	$\{i+1\}$	$\{i-1\}$
	$(i, i, 4)$	$\{i+2\}$	$\{i-1\}$
$V_{i,i+1}^*$	$(i, i+1, 1)$		
	$(i, i+1, 2)$	$\{i\}$	$\{i-2\}$
	$(i, i+1, 3)$	$\{i\}$	$\{i-1\}$
	$(i, i+1, 4)$	$\{i+2\}$	$\{i-1\}$
$V_{i,i+2}^*$	$(i, i+2, 1)$		
	$(i, i+2, 2)$	$\{i\}$	$\{i-1\}$
	$(i, i+2, 3)$	$\{i+1\}$	$\{i-1\}$
$V_{i,i+3}^*$	$(i, i+3, 1)$		

Table 2

Arcs of G^* incident to nodes in $V_{i,i+1}^*$ for $k = 4$.

Tail of arc			Head of arc			Length of arc
$(i-1, \ell, r)$	$S_{i-1,\ell,t}^-$	$S_{i-1,\ell,t}^+$	$(i, i+1, r)$	$S_{i,i+1,r}^-$	$S_{i,i+1,r}^+$	
$(i-1, i-4, 1)$	$\{i-1\}$	$\{i-4\}$	$(i, i+1, 1)$			$c_{i-4,\,i+1}$
$(i-1, i-3, 1)$	$\{i-1\}$	$\{i-3\}$				$c_{i-3,\,i+1}$
$(i-1, i-2, 1)$	$\{i-1\}$	$\{i-2\}$				$c_{i-2,\,i+1}$
$(i-1, i-1, 1)$						$c_{i-1,\,i+1}$
$(i-1, i-3, 3)$	$\{i-1, i\}$	$\{i-3, i-2\}$	$(i, i+1, 2)$	$\{i\}$	$\{i-2\}$	$c_{i-3,\,i+1}$
$(i-1, i-1, 3)$	$\{i\}$	$\{i-2\}$				$c_{i-1,\,i+1}$
$(i-1, i, 3)$	$\{i-1\}$	$\{i-2\}$				$c_{i,\,i+1}$
$(i-1, i-3, 2)$	$\{i\}$	$\{i-3\}$	$(i, i+1, 3)$	$\{i\}$	$\{i-1\}$	$c_{i-3,\,i+1}$
$(i-1, i-2, 2)$	$\{i\}$	$\{i-2\}$				$c_{i-2,\,i+1}$
$(i-1, i+2, 1)$			$(i, i+1, 4)$	$\{i+2\}$	$\{i-1\}$	$c_{i+2,\,i+1}$

$$\sum_{j=0}^{h}\binom{m+h-j-1}{h-j}\binom{m+j}{j}=\binom{m+(m+1)+h-1}{h}$$
$$=\binom{k-1}{2m},$$

where the first equation follows from a well-known identity for binomial coefficients (see (12.16) in [17, p. 62]).

Now the equation of the lemma becomes

$$\sum_{m=0}^{\lfloor k/2\rfloor}\binom{k-1}{2m}=\frac{1}{2}\sum_{p=0}^{2\lfloor k/2\rfloor}\binom{k-1}{p}$$
$$=\frac{1}{2}2^{k-1}.$$

□

Lemma 4.7. For any positive integers k and h, $k \geq h$,

$$\sum_{m=0}^{\lfloor k/2\rfloor}\binom{k-h-1}{m}\sum_{\ell=m-1}^{h-1}\binom{\ell}{m-1}=\binom{k-2}{h}+\binom{k-2}{h-1}.$$

Proof.

$$\sum_{\ell=m-1}^{h-1}\binom{\ell}{m-1}=\sum_{j=0}^{h-1}\binom{h-1-j}{m-1}$$
$$=\binom{h}{m}$$

(see (12.6) in [17, p. 61] for the last equation). Thus, the left-hand side of the equation in the lemma becomes

$$\sum_{m=0}^{\lfloor k/2\rfloor}\binom{k-h-1}{m}\binom{h}{m}=\sum_{m=0}^{h}\binom{k-h-1}{h-m}\binom{h}{h-m}$$

(since for $m > \min\{\lfloor k/2\rfloor, h\}$, the expression is 0

$$=\binom{k-1}{h}\text{ (see (12.9) in [17, p.62])}$$

$$=\binom{k-2}{h}+\binom{k-2}{h-1}.$$

□

Lemma 4.8. For any positive integers k and h, $k \geq h$,

$$\sum_{m=1}^{\lfloor k/2\rfloor}\sum_{\ell=h+1}^{k-1}\binom{\ell-1}{m-1}\binom{k-\ell-1}{m}=2^{k-2}-\sum_{\ell=0}^{h}\binom{k-2}{\ell}.$$

Proof. For any positive integer $\ell \le k-1$,

$$\sum_{m=1}^{\lfloor k/2 \rfloor} \binom{\ell-1}{m-1}\binom{k-\ell-1}{m} = \sum_{m=1}^{\ell} \binom{k-\ell-1}{m}\binom{\ell-1}{\ell-m} = \binom{k-2}{\ell}$$

(since for $m > \min\{\lfloor k/2 \rfloor, \ell\}$, the expression is 0). Thus, the equation of the lemma becomes

$$\begin{aligned} \sum_{\ell=h+1}^{k-1} \binom{k-2}{\ell} &= \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} - \sum_{\ell=0}^{h} \binom{k-2}{\ell} \\ &= 2^{k-2} - \sum_{\ell=0}^{h} \binom{k-2}{\ell}. \end{aligned}$$

□

Lemma 4.9. For any positive integer k,

$$\sum_{h=2}^{k-1} \sum_{\ell=0}^{h-2} \binom{k-2}{\ell} = (k-2)2^{k-3}.$$

Proof.

$$\begin{aligned} \sum_{h=2}^{k-1} \sum_{\ell=0}^{h-2} \binom{k-2}{\ell} &= \sum_{\ell=0}^{0} \binom{k-2}{\ell} + \sum_{\ell=0}^{1} \binom{k-2}{\ell} + \cdots + \sum_{\ell=0}^{k-3} \binom{k-2}{\ell} \\ &= (k-2)\binom{k-2}{0} + (k-3)\binom{k-2}{1} + \cdots + 2\binom{k-2}{k-4} + 1\binom{k-2}{k-3} \\ &= 1\binom{k-2}{1} + 2\binom{k-2}{2} + \cdots + (k-3)\binom{k-2}{k-3} + (k-2)\binom{k-2}{k-2} \\ &= (k-2)2^{k+3}, \end{aligned}$$

where the last equation is the well-known identity (12.1) in [17, p. 61]. □

Theorem 4.10. For $k+1 \le i \le n-k+1$, $|V_i^*| = (k+1)2^{k-2}$.

Proof. For given i, $k+1 \le i \le n-k+1$, $V_i^* = \cup V_{ij}^*$, where the union is taken over all j such that $i-k+1 \le j \le i+k-1$.

To count the nodes in a set V_{ij}^*, we will count all the sets $S_{ij}^- \subseteq \{i+1,\ldots, i+k-1\}$ of elements that can be placed in positions belonging to $\{i-1,\ldots, i-k+1\}$ without violating the precedence constraints. Each such S_{ij}^- forms the first member of a pair of feasible sets, and each set of positions $S_{ij}^+ \subseteq \{i-1,\ldots, i-k+1\}$, corresponding to the elements dislocated by those in S_{ij}^-, forms the second member of a pair.

(1) We first consider the case $j = i$. For $m = 0, 1,\ldots, \lfloor k/2 \rfloor$, we will count the distinct pairs S_{ij}^-, S_{ij}^+ such that $|S_{ij}^-| = |S_{ij}^+| = m$.

$|S_{ij}^-| = |S_{ij}^+| = 0$. Clearly, there is only one pair satisfying this condition, with $S_{ij}^- = S_{ij}^+ = \emptyset$.

$|S_{ij}^-| = |S_{ij}^+| = 1$. City $i+1$ can be assigned to any of the positions $\{i-1,\ldots, i-k+2\}$, city $i+2$, to any of the positions $\{i-1,\ldots, i-k+3\}$, etc., and city $i+k-2$ can be assigned only to position $i-1$. Thus, the number of pairs S_{ij}^-, S_{ij}^+ of size 1 is $(k-2)+(k-3)+\cdots+1 = (k-1)(k-2)/2$, a number which (for reasons soon to become clear) we choose to write as

$$\Sigma(1) = \sum_{\ell=1}^{k-2} \binom{k-2-\ell}{0}\binom{\ell}{1}.$$

$|S_{ij}^-| = |S_{ij}^+| = m$ for $2 \le m \le \lfloor k/2 \rfloor$. Consider the collection $M(\ell)$ of m-subsets of $\{i+1,\ldots, i+k-1\}$ whose largest element is $i+k-1-\ell$ for some $\ell \in \{1,\ldots, k-m-1\}$. For a given ℓ,

$$|M(\ell)| = \binom{k-2-\ell}{m-1},$$

since fixing the largest of the $k-1-\ell$ elements to choose from leaves $k-2-\ell$ elements to be grouped into subsets of size $m-1$. The elements of each m-set of the collection $M(\ell)$ can be placed into any m positions of the set $\{i-1,\ldots, i-\ell\}$ without violating the precedence constraints, and the number of such m-subsets of positions is $\binom{\ell}{m}$. Thus, for a given ℓ there are $\binom{k-2-\ell}{m-1}\binom{\ell}{m}$ distinct pairs S_{ij}^-, S_{ij}^+ whose members are of size m. Therefore, the number of such pairs for all $\ell \in \{1,\ldots, k-m-1\}$ is

$$\Sigma(m) = \sum_{\ell=m}^{k-m-1} \binom{k-2-\ell}{m-1}\binom{\ell}{m},$$

where we have started the summation with $\ell = m$, since the terms corresponding to $\ell < m$ are all zero; and from the proof of lemma 4.6, the above expression can be restated as $\Sigma(m) = \binom{k-1}{2m}$.

Taking now the sum of the last expression over all $m \in \{0, 1,\ldots, \lfloor k/2 \rfloor\}$, we obtain the number of nodes in V_{ii}^*. The reason for stopping at $m = \lfloor k/2 \rfloor$ is that this is the largest possible size for a set S_{ij}^- (see proposition 4.3):

$$\sum_{m=0}^{\lfloor k/2 \rfloor} \Sigma(m) = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k-1}{2m} = \frac{1}{2}\sum_{p=0}^{2\lfloor k/2 \rfloor} \binom{k-1}{p} = \frac{1}{2} 2^{k-1}.$$

Thus, $|V_{ii}^*| = 2^{k-2}$.

(2) Next we consider the case of V_{ij}^* such that $i+1 \le j \le i+k-1$. We will write $j = i+h$, with $1 \le h \le k-1$.

There is only one pair of feasible sets S_{ij}^-, S_{ij}^+ such that $|S_{ij}^-| = |S_{ij}^+| = 0$. Next, we proceed directly to count the feasible pairs such that $|S_{ij}^-| = |S_{ij}^+| = m$ for some $1 \le m \le \lfloor k/2 \rfloor$. We first note that since all $\ell \in S_{ij}^+$ must satisfy $\ell \ge j - k + 1$, it follows that a set S_{ij}^+ of cardinality m exists only if its smallest element, say $\ell \le i - m$, satisfies the above inequality, i.e. only if $i - m \ge j - k + 1 = i + h - k + 1$, or $h \le k - m - 1$. So we assume this property.

There are now two types of feasible pairs of sets, depending on whether S_{ij}^- does or does not contain elements larger than $j (= i + h)$, and they have to be counted differently.

Type 1. $S_{ij}^- \subseteq \{i, i+1, \ldots, i+h-1\}$. For $h \le m - 1$, there are no sets of type 1. So assume $m \le h \le k - m - 1$. For every $\ell \in \{m-1, m, \ldots, j-i-1\}$, every m-subset of $\{i, i+1, \ldots, i+h-1\}$ whose largest element is $i + \ell$ can be the first member S_{ij}^- of a feasible pair. The number of such m-sets for a given ℓ is $\binom{\ell}{m-1}$, and for all ℓ it is $\sum_{\ell=m-1}^{h-1} \binom{\ell}{m-1}$. The elements of each such m-set can be assigned to any m positions in the set $\{i-1, i-2, \ldots, i-k+h+1\}$, and the number of such m-sets S_{ij}^+ of positions is $\binom{k-h-1}{m}$. Thus, for all $\ell \in \{m-1, \ldots, h-1\}$, there are

$$\binom{k-h-1}{m} \sum_{\ell=m-1}^{h-1} \binom{\ell}{m-1} \tag{4.5}$$

pairs of type 1.

Type 2. $S_{ij}^- \cap \{i+h+1, \ldots, i+k-1\} \ne \emptyset$. For every $\ell \in \{h - 1, \ldots, k-1\}$, every m-subset of $\{i, i+1, \ldots, i+h-1, i+h+1, \ldots, i+\ell\}$ whose largest element is $i + \ell$ can be the first member S_{ij}^- of a feasible pair. The number of such m-sets for a given ℓ is $\binom{\ell-1}{m-1}$. The elements of each such m-set can be assigned to any m positions in the set $\{i-1, i-2, \ldots, i-k+\ell+1\}$, and the number of such m-subsets of positions is $\binom{k-\ell-1}{m}$. Thus, for a given ℓ, there are $\binom{\ell-1}{m-1}\binom{k-\ell-1}{m}$ feasible pairs of sets of type 2, and for all $\ell \in \{h+1, \ldots, k-1\}$, there are

$$\sum_{\ell=h+1}^{k-1} \binom{\ell-1}{m-1} \binom{k-\ell-1}{m} \tag{4.6}$$

such pairs.

Thus, the total number of feasible pairs (of both types) of cardinality m, for $1 \le m \le \lfloor k/2 \rfloor$, is (4.5) + (4.6), and the total number for all $m \in \{0, 1, \ldots, \lfloor k/2 \rfloor\}$, for a given $j = i + h \ge i$, is

$$\sum[i+h] = 1 + \sum_{m=1}^{\lfloor k/2 \rfloor} \binom{k-h-1}{m} \sum_{\ell=m-1}^{h-1} \binom{\ell}{m-1} + \sum_{m=1}^{\lfloor k/2 \rfloor} \sum_{\ell=h+1}^{k-1} \binom{\ell-1}{m-1} \binom{k-\ell-1}{m}.$$

Now from lemma 4.7, we have

$$1 + \sum_{m=1}^{\lfloor k/2 \rfloor} \binom{k-h-1}{m} \sum_{\ell=m-1}^{h-1} \binom{\ell}{m-1} = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k-h-1}{m} \sum_{\ell=m-1}^{h-1} \binom{\ell}{m-1}$$

$$= \binom{k-2}{h} + \binom{k-2}{h-1}$$

and from lemma 4.8,

$$\sum_{m=1}^{\lfloor k/2 \rfloor} \sum_{\ell=h+1}^{k-1} \binom{\ell-1}{m-1} \binom{k-\ell-1}{m} = 2^{k-2} - \sum_{\ell=0}^{h} \binom{k-2}{\ell}.$$

Thus,

$$\sum [i+h] = \binom{k-2}{h} + \binom{k-2}{h-1} + 2^{k-2} - \sum_{\ell=0}^{h} \binom{k-2}{\ell}$$

$$= 2^{k-2} - \sum_{\ell=0}^{h-2} \binom{k-2}{\ell}.$$

Thus, we have proved that for $j = i + h$, $1 \le h \le k-1$, $|V_{ij}^*| = 2^{k-2} - \sum_{\ell=0}^{h-2} \binom{k-2}{\ell}$. In particular, for $h = 1$ and $h = 2$, we have $|V_{i,i+1}^*| = 2^{k-2}$ and $|V_{i,i+2}^*| = 2^{k-2} - 1$, respectively.

Next we add up the numbers of nodes in all layers V_{ij}^*, $j = i + h$, $1 \le h \le k-1$ to calculate $|\bigcup_{h=1}^{k-1} V_{i,i+h}^*|$:

$$\sum_{h=1}^{k-1} \sum [i+h] = (k-1)2^{k-2} - \sum_{h=1}^{k-1} \sum_{\ell=0}^{h-2} \binom{k-2}{\ell}$$

$$= (k-1)2^{k-2} - (k-2)2^{k-3},$$

where the last equation follows from lemma 4.9 and the fact that for $h = 1$, $\sum_{\ell=0}^{h-2} \binom{k-2}{\ell} = 0$.

(3) We now turn to the node sets V_{ij}^* for $j < i$. We claim that $|\bigcup_{h=1}^{k-1} V_{i,i-h}| = |\bigcup_{h=1}^{k-1} V_{i,i+h}|$, and prove the claim from symmetry considerations.

Suppose we replace the directed graph G^* by its reverse, $G^{**} = (V^{**}, A^{**})$, which is based on the ordering $\{n, n-1, \dots, 1\}$ of N. Clearly, there is a 1:1 correspondence between V_1^* and V_{n+1}^{**}, V_{n+1}^* and V_1^{**}, and in general between V_i^* and V_{n+2-i}^{**}. Consider now the subset V_{ij}^* of V_i^*, for some $j \ne i$. There is then a 1:1 correspondence between the nodes of V_{ij}^* and those of $V_{n+2-i,n+2-j}^*$. Thus, we can count these nodes in either set and the result will be the same. But if $j > i$, then $n + 2 - j < n + 2 - i$ and vice versa. This proves the claim.

We have now come to the point where we can add up the numbers of nodes in V_{ii}^*, $\bigcup_{h=1}^{k-1} V_{i,i+h}^*$ and $\bigcup_{h=1}^{k-1} V_{i,i-h}^*$. The total number is

$$\begin{aligned}|V^*_{ii}| + 2|\textstyle\bigcup_{h=1}^{k-1} V^*_{i,i+h}| &= 2^{k-2} + 2(k-1)2^{k-2} - 2(k-2)2^{k-3}\\ &= 2^{k-2}(1 + 2(k-1) - (k-2))\\ &= (k+1)2^{k-2}.\end{aligned}$$ □

We have counted the nodes in a typical layer i, $k+1 \le i \le n-k+1$. Since the number of nodes in the other layers is smaller, we have

$$|V^*| \le (k+1)2^{k-2}n.$$

Next we bound the number of arcs of G^*. We do this by deriving a tight bound on the indegree of a node.

Proposition 4.11. For any node $(i, j, r) \in V^*$, $\deg^-(i, j, r) \le k$.

Proof. If $S^-_{ijr} = \emptyset$, then for any predecessor $(i-1, \ell, t)$ of (i, j, r) we must have $\ell \le i-1$; and since $\ell \ge (i-1)-k+1 = i-k$, ℓ is restricted to the k integer values in $\{i-1,\dots, i-k\}$.

Now let $S^-_{i,j,r} \ne \emptyset$, and let h be the largest element of $S^-_{i,j,r}$. Then for any predecessor $(i-1, \ell, t)$ of (i, j, r) we must have $h \le \ell + k - 1$, since $\pi(h) < i-1 = \pi(\ell)$. On the other hand, $\ell \le h$ by the choice of h. This restricts ℓ to the k integers in $\{h-k+1,\dots, h\}$.

Since we have shown before that a node (i, j, r) has at most one predecessor in $V^*_{i-1,\ell}$ for every ℓ, this completes the proof. □

Combining the bound on the indegree of a node with that on the number of nodes gives a bound on the number of arcs of G^*:

$$|A^*| \le k(k+1)2^{k-2}n.$$

Since finding a shortest $s-t$ path in G^* takes $O(|A^*|)$ time, this is the time it takes to find an optimal feasible tour in G. Thus, theorem 2.1 follows from the above results.

5. The other auxiliary digraphs

(1) We now consider TSPs with condition (ia), which allows for different integers $k(j)$ for different cities j. A tour will be called feasible if it conforms to these conditions. Again, the conditions imply that the position in the tour to which a given city may be assigned is restricted to a certain segment of the tour, and vice versa: the candidates for a given position in the tour must be drawn from a certain segment of the list of cities. However, the corresponding segments now have somewhat different characteristics. The following result corresponds to proposition 4.1.

For $j \quad N$, define

$$P_j := \{\ell \quad N : \ell < j \text{ and } \ell + k(\ell) \geq j + 1\}.$$

Proposition 5.1. For any city j, the assignment $\pi(j) = i$ is feasible if and only if

$$j - |P_j| \leq i \leq j + k(j) - 1. \tag{5.1}$$

Proof. *Necessity*. Suppose $\pi(j) = i$ is a feasible assignment. If $i \geq j$, the first inequality of (5.1) is clearly satisfied, and if $i \leq j$, the second inequality is satisfied. We have to show that the first inequality is also satisfied when $i < j$ and the second inequality is also satisfied when $i > j$.

Let $i > j$. Then $\pi(j) > \pi(\ell)$ for some $\ell \geq i$, which implies

$$\begin{aligned} i &\leq \ell \\ &\leq j + k(j) - 1 \quad \text{(from (i a))}, \end{aligned}$$

i.e. the second inequality of (5.1) is satisfied.

Let $i < j$. Then $\pi(\ell) > \pi(j)$ for $j - i$ cities ℓ such that $\ell < j$. But $\pi(\ell) > \pi(j)$ implies (from (ia)) $j \leq \ell + k(\ell) - 1$, and thus the set of cities $\ell < j$ such that $\pi(\ell) > \pi(j)$ is precisely P_j. Therefore, $|P_j| \geq j - i$, or

$$i \geq j - |P_j|,$$

which is the first inequality of (5.1).

Sufficiency. Suppose (5.1) is satisfied for some $i, j \quad N$. We show that $\pi(j) = i$ is a feasible assignment by exhibiting a feasible tour that includes it. If $i = j$, then $\pi(j) = j$ for all $j \quad N$ defines a feasible tour.

Let $i > j$, and consider the tour defined by $\pi(\ell) = \ell$ for $\ell = 1,\ldots, j - 1$ and $\ell = i + 1,\ldots, n$, $\pi(j) = i$, and $\pi(\ell) = \ell - 1$ for $\ell = j + 1,\ldots, i$. This tour is feasible, since whenever $\pi(\ell) < \pi(j)$ for some $\ell > j$, which occurs only for $\ell \quad \{j + 1,\ldots, i\}$, we have $\ell \leq j + k(j) - 1$ from (5.1).

Now let $i < j$. Let P_j be ordered increasingly, and let $P_j^i := \ell_1,\ldots, \ell_{j-i}\}$ be the subsequence of P_j consisting of the last $j - i$ elements. Clearly, $\ell_1 \leq i$. Consider the tour constructed as follows. Let

$$\begin{aligned} \pi(\ell) &= \ell \quad \text{for } \ell = 1, \quad , \ell_1 \text{ and } \ell = j + 1, \quad , n, \\ \pi(j) &= i \quad \text{and } \pi(\ell_r) = i + r \text{ for } \ell_r \quad P_j^i,\ r = 1, \quad , j - i. \end{aligned}$$

This assigns positions in the tour to all cities except those in $L := \{\ell \quad N \setminus P_j^i : \ell_1 \leq \ell \leq j - 1\}$ and fills all the positions in the tour except for those in $H := \{h \quad N : \ell_1 \leq h \leq i - 1\}$. From the definition of P_j^i, $|L| = |H|$. Now assign the pth city in L to the pth position in H. The resulting tour violates no precedence constraint, since from the definition of P_j^i as a subsequence of P_j, if $\pi(\ell) > \pi(j)$, then $j \leq \ell + k(\ell) - 1$. This construction is possible because the first inequality of (5.1) is satisfied, i.e. $|P_j| \geq j - i$. □

From (5.1), by using $j = \pi^{-1}(i)$, we have the following constraints satisfied by every candidate city $\pi^{-1}(i)$ for position i in the tour:

$$i - k(\pi^{-1}(i)) + 1 \leq \pi^{-1}(i) \leq i + |P_{\pi^{-1}(i)}|.$$

These relations are not very useful, since $\pi^{-1}(i)$ appears on both sides of each inequality. Bounds that do not have this shortcoming can be obtained as follows. For $i, j \in N$, define

$$j_{(i)} := \min\{j : j + k(j) \geq i + 1\},$$
$$j^{(i)} := \max\{j : j - i \leq |P_j|\}.$$

Proposition 5.2. For any feasible tour and any position i in that tour,

$$j_{(i)} \leq \pi^{-1}(i) \leq j^{(i)}. \tag{5.2}$$

Proof. When considering candidates j for position i, by (5.1) we have $\pi(j) \leq j + k(j) - 1$, which can be written as $j + k(j) \geq i + 1$. Since $j_{(i)}$ is defined to be the smallest j satisfying $j + k(j) \geq i + 1$, this proves $j_{(i)} \leq \pi^{-1}(i)$. We also have $j - |P_j| \leq \pi(j)$, which we can write as $j - i \leq |P_j|$. Since $j^{(i)}$ is defined to be the largest j satisfying $j - k \leq |P_j|$, this proves $\pi^{-1}(i) \leq j^{(i)}$. □

While the bounds on $\pi(j)$ defined by (5.1) are necessary and sufficient conditions for an assignment (of city j to position $\pi(j)$) to be feasible, the bounds on $\pi^{-1}(i)$ defined by (5.2) are necessary but not sufficient. Consider, for instance, the case where $k(3) = 3$ and $k(i) = 1$ for $i \neq 3$. The only feasible ways to start a tour are: 1–2–3–4–5–6, 1–2–4–3–5–6, and 1–2–4–5–3–6, and thus the only candidates for position 5 are cities 3 and 5, not city 4. This means that, while the interval defined by (5.1) is contiguous, the interval defined by (5.2) is not. However, it is not hard to show that the following holds.

Proposition 5.3. Assume the numbers $k(i)$ satisfy the condition

$$k(i) - k(i+1) \leq 1, \quad i = 1, \ldots, n.$$

Then for every position i and every city $j \in \{j_{(i)}, \ldots, j^{(i)}\}$, there exists a feasible tour in which $\pi^{-1}(i) = j$.

We now turn to the layered acyclic digraph $G^{*(a)} = V^{*(a)}, A^{*(a)})$. The nodes and arcs of $G^{*(a)}$ are constructed in the same way as those of G^*, but the sets $V_i^{*(a)}$ (layers) are no longer of the same size for all i other than the first few and the last few. For a typical i, $V_i^{*(a)} = \cup V_{ij}^{*(a)}$, where each $V_{ij}^{*(a)}$ contains the nodes associated with city j being assigned to position i in the tour, as in the case of G^*, and the union is taken over the set of all such cities j. However, in this case the number of distinct cities j

that are candidates for a given position i is bounded by $2k^*(i)$, where

$$k^*(i) := \max\{k(j) : i - k(j) + 1 \le j \le i\}.$$

The number of nodes in each $V_i^{*(a)}$ can be bounded by essentially the same technique as in computing the cardinality of V_i^*, with the parameter k replaced by $k^*(i)$. This yields the bound $|V_i^{*(a)}| \le (k^*(i)+1)2^{k^*(i)-2}$, and hence $|V^{*(a)}| \le \sum_{i=2}^{n}(k^*(i)+1)2^{k^*(i)-2}$. Similarly, counting the arcs yields the bound

$$|A^*| \le \sum_{i=2}^{n} k^*(i)(k^*(i)+1)2^{k^*(i)-2}.$$

This proves theorem 2.2.

(2) Next we consider TSPs with condition (ib). These differ from the TSPs with condition (i) in that the integer interval from which the candidate cities for a given position i can be drawn is given directly, rather than through precedence constraints. Note that although the precedence constraint (i) implies the intervals of proposition 5.1, the converse is not true: although the interval constraint

$$i - k + 1 \le \pi^{-1}(i) \le i + k - 1 \tag{5.3}$$

implies a precedence constraint, this is weaker than the precedence constraint of (i); if $j_1 = i + k - 1$ and $j_2 = i - k + 2$, then $j_1 \ge j_2 + 2k - 3$, but $\pi(j_1) < \pi(j_2)$ is still feasible, with $\pi(j_1) = i$, $\pi(j_2) = i + 1$. In fact, we have

Remark 5.4. If condition 5.3 is satisfied, then for all $j, \ell \in (1,\dots,n)$, $\ell \ge j + 2(k-1)$ implies $\pi(j) < \pi(\ell)$.

Proof. Let $\pi(j) = i$, $\pi(\ell) = h$. From (5.3) $j \ge i - k + 1$ and $\ell \le h + k - 1$. Now let $\ell \ge j + 2(k-1)$; then

$$\begin{aligned} i &\le j + k - 1 \\ &\le \ell - 2(k-1) + k - 1 \\ &\le (h + k - 1) - k + 1 = h. \end{aligned}$$

□

Next we turn to the digraph $G^{*(b)} = (V^{*(b)}, A^{*(b)})$. This is very similar to G^*: $V^{*(b)}$ has $n+1$ layers $V_i^{*(b)}$, $i = 1,\dots,n+1$, and for $i \in \{k+1,\dots,n-k+1\}$, $V_i^{*(b)} = \cup V_{ij}^{*(b)}$, with the union taken over all $j \in \{i-k+1,\dots,i+k-1\}$. The only difference with respect to G^* is that in the case of $G^{*(b)}$, the subsets S_{ij}^-, S_{ij}^+ are not restricted to size $\lfloor k/2 \rfloor$, but to size $k-1$, and thus the number of nodes in each set $V_{ij}^{*(b)}$ is larger than in the corresponding set V_{ij}^*.

Proposition 5.5. $|V^{*(b)}| \le (2k-1)(k-2)^{-1/2}2^{2(k-1)}n$, and $|A^*| \le (2k-1)^2(k-1)^{-1/2}2^{2(k-1)}n$.

Proof. If $j \in \{i,\ldots, i+k-1\}$, there are $\binom{k-1}{m}^2$ feasible pairs S_{ij}^-, S_{ij}^+ with $|S_{ij}^-| = |S_{ij}^+| = m$ for each $m \in \{0,\ldots, k-1\}$; namely, the $k-1$ elements of $\{i,\ldots, i+k-1\}\setminus\{j\}$ can form $\binom{k-1}{m}$ different subsets S_{ij}^- of size m, each of which can replace any of the subsets S_{ij}^+ of size m of $\{i-1,\ldots, i-k+1\}$, of which there are also $\binom{k-1}{m}$. This yields $\binom{k-1}{m}^2$ pairs S_{ij}^-, S_{ij}^+ of size m. If $j \in \{i-1,\ldots, i-k+1\}$, the corresponding number is $\binom{k}{m}\binom{k-2}{m} < \binom{k-1}{m}^2$. So in either case, the number of feasible pairs S_{ij}^-, S_{ij}^+ of sets of size m is bounded by $\binom{k-1}{m}^2$; hence, for all $m \in \{0,\ldots, k-1\}$ this number is bounded by

$$\sum_{m=0}^{k-1} \binom{k-1}{m}^2 = \binom{2(k-1)}{k-1}$$

(see (12.11) in [17, p. 62]), which is bounded above – from Stirling's formula – by $(k-1)^{-1/2}2^{2(k-1)}$. Thus, for any pair i, j, $|V_{ij}^{*(b)}| \le (k-1)^{-1/2}2^{2(k-1)}$. Since for a given position i there are $2k-1$ candidates j, the number of vertices of $G^{*(b)}$ is bounded by $(2k-1)(k-1)^{-1/2}2^{2(k-1)}n$,; and since the indegree of any node is bounded by $2k-1$, the number of arcs of $G^{*(b)}$ is bounded by $(2k-1)^2(k-1)^{-1/2}2^{2(k-1)}n$. □

This proves theorem 2.3.

(3) We now address the digraph $G^{*(c)}$ associated with a uniform Prize Collecting TSP with conditions (ic), (iic). Here, the difference with respect to the TSP with condition (i) is that (a) the tour (a PC tour) is to have length m rather than n, and (b) the interval from which the candidates for a given position i in the PC tour can be drawn is prescribed explicitly as $\{t(i)-k+1,\ldots, t(i)+k-1\}$, where $t(i) := \lceil (n/m)(i-1)\rceil$. Now, in the case of the TSP with condition (i), the condition implies a similar interval, namely $\{i-k+1,\ldots, i+k-1\}$. In both cases, the interval is of cardinality $2k-1$, but in one case it is centered around i, in the other around $\lceil (n/m)(i-1)\rceil$. In order to construct the auxiliary graph $G^{*(c)}$, we need to specify for every node of the layer V_i^* corresponding to position i in the tour the city $\pi^{-1}(i) = j \in N$ assigned to position i, and the sets

$$S^-(\pi, i) := \{h \in (1, \ldots, n) : h \ge t(i) \text{ and } h \in M \text{ with } \pi(h) \le i-1\},$$
$$S^+(\pi, i) := \{h \in \{t(i)-k+1, \ldots, t(i)-1\} : h \notin M \text{ or } \pi(h) \ge i\}.$$

As in the case of G^*, specifying the subsets $S^-(\pi, i)$ and $S^+(\pi, i)$ amounts to fully specifying the subset of cities available for assignment to position $i+1$ after having assigned city j to position i; namely, this subset is $\{t(i),\ldots, t(i)+k\} \cup S^+(\pi, i)\setminus S^-(\pi, i) \cup \{j\})$.

It is easy to see that, just as in the case of G^*, $|S^-(\pi, i)| \le \lfloor k/2\rfloor$ for any tour π and any position i in that tour. However, unlike in the case of G^*, the sets $S^-(\pi, i)$ and $S^+(\pi, i)$ are not of equal size. In fact, it is easy to see that, assuming $m < n$, if $S^-(\pi, i) = \emptyset$ we can still have $S^+(\pi, i) \ne \emptyset$, since not all $k-1$ nodes in the interval $\{t(i)-k+1,\ldots, t(i)-1\}$ need to (or even can) belong to M. Rather than count explicitly the number of pairs S_{ij}^-, S_{ij}^+ that can be associated with a given position $i \in M$ and city

$j \quad N$ assigned to i, we will give a (non-tight) bound on this number. Since $|S_{ij}^{-}| \leq \lfloor k/2 \rfloor$, $|S_{ij}^{+}| \leq k$, and each of the sets S_{ij}^{-}, S_{ij}^{+} is drawn from an interval of size k, evidently the number of pairs (S_{ij}^{-}, S_{ij}^{+}) is bounded by 2^{2k-1}. Further, since for each position $i \quad M$ there are $2k - 1$ candidate cities $j \quad N$, we have $|V_i^{*(c)}| \leq (2k-1)2^{2k-1}$ for all $i \quad M$ and $|V^{*(c)}| \leq (2k-1)2^{2k-1}m$. Also, the number of arcs of $G^{*(c)}$ satisfies

$$|A^{*(c)}| \leq (2k-1)^2 2^{2k-1} m.$$

This proves theorem 2.4.

(4) Finally, we turn to the uniform Prize Collecting TSP with conditions (id), (iid) in which the constant k is replaced with position-specific constants $k(i)$, $i \quad M$. The relationship between this problem and the last one is the same as the relationship between the TSP with condition (ia) and the one with condition (i). The digraph $G^{*(d)} = (V^{*(d)}, A^{*(d)})$ associated with this problem can thus be constructed in the same way as $G^{*(a)}$. The number of candidate cities for a position i in the PC tour is bounded by $2(k^*(i) - 1)$, where $k^*(i) := \max\{k(j) : t(i) - k(j) + 1 \leq j \leq t(i)\}$. In bounding the nodes in a set $V_i^{*(d)}$, we can use the same technique as in the case of $V_i^{*(c)}$. This yields $V_i^{*(d)} \leq (2k^*(i) - 1)2^{2k^*(i)-1}$ and $|V^{*(d)}| \leq \sum_{i=2}^{m}(2k^*(i) - 1)2^{2k^*(i)-1}$. Similarly, counting the arcs yields $|A^*| \leq \sum_{i=2}^{m}(2k^*(i) - 1)^2 2^{2k^*(i)-1}$.

This proves theorem 2.5.

6. Relation to the TSP on bandwidth-limited graphs

Among the classes of polynomially solvable TSPs, the one on bandwidth-limited graphs (see [26]) has some affinity to the classes introduced in this paper.

A graph $G = (V, E)$ is said to have bandwidth b if $(i, j) \quad E$ implies $|i - j| \leq b$. In other words, all the edges joining vertices i, j such that $|i - j| > b$ are missing. For fixed b, the TSP on graphs with bandwidth b can be solved efficiently by a dynamic programming recursion. Gilmore et al. [26] formulate this recursion for the case of an undirected graph and give its complexity as $O((b + 1)!\,n)$, where $n = |V|$ and b is the bandwidth.

Consider now our basic model, the TSP with condition (i). It is not hard to see that this condition implies a limit on the bandwidth of G. Indeed, we have

Proposition 6.1. In a TSP with condition (i), if cities i and j are adjacent in an optimal tour, then $|i - j| \leq 2k - 1$ and this bound is tight.

Proof. To prove the first statement by contradiction, suppose i is the predecessor of j in the tour and $j - i \geq 2k$ (the same argument can be made if j precedes i and $i - j \geq 2k$). For any node h that comes before i in the tour, $h \leq i + k - 1$; and for any node ℓ that comes after j in the tour, $h \geq j - k + 1 \geq i + k + 1$. It follows that the tour cannot contain node $i + k$, a contradiction.

To see that the bound is tight, notice that assigning positions p and $p+1$ in the tour to nodes $i = p - k + 1$ and $j = (p+1) + k - 1$, respectively, satisfies condition (i), and yields $j - i = 2k - 1$. □

It then follows that an optimal tour remains unchanged if all edges (i, j) such that $|i - j| > 2k - 1$ are removed from G, i.e. if G is replaced with a graph of bandwidth $b = 2k - 1$. This is true for the directed as well as the undirected case.

On the other hand, a TSP with bandwidth b does not necessarily satisfy the precedence requirements of (i) with $k := (b+1)/2$. Consider a graph on n nodes, with bandwidth $b = 3$. The tour $(1, 3, 5, \ldots, n, n-1, n-3, \ldots, 4, 2, 1)$ if n is odd or $(1, 3, 5, \ldots, n-1, n, n-2, n-4, \ldots, 4, 2, 1)$ if n is even is entirely "within the bandwidth", but clearly violates the precedence constraints for $k = (b+1)/2 = 2$.

The TSP on bandwidth-limited graphs is solvable by a dynamic programming recursion in time linear in n, just like the TSPs satisfying condition (i) of this paper. However, the two recursions are different, and the complexity of our procedure in terms of the parameter k is orders of magnitude lower. To be more specific, a comparison of the two complexities, $O((b+1)!\,n) = O((2k)!\,n)$ for the TSP on graphs with bandwidth $b = 2k + 1$ and $O(k^2 2^{k-2} n)$ for the TSP with condition (i), yields the following:

k	$(2k)!\,n$	$k^2 2^{k-2} n$
3	$720n$	$18n$
4	$40{,}320n$	$64n$
5	$3{,}628{,}800n$	$200n$
6	$479{,}011{,}600n$	$576n$
7	$87{,}178{,}291{,}200n$	$1{,}568n$

To look at it differently, if we let k grow with the logarithm of n, i.e. put $k := \alpha \log n$, then the TSP with condition (i) is solvable in time polynomial in n, namely, $O((\alpha \log n)^2 n^{\alpha+1})$; but this does not seem to be true for the TSP on bandwidth-limited graphs.

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