

LECTURE NOTES

Math 2450, Calculus III with Applications

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1 Vectors in Plane and Space (Chapter 9)

Definition 1.1 The plane can be described as the set,

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\},$$

while the space is the set,

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Both, \mathbb{R}^2 and \mathbb{R}^3 are examples of vector and affine spaces.

Definition 1.2 A vector \vec{v} is an element of a vector space V . Roughly speaking, a vector is a mathematical object which has both:

(i) Magnitude (norm), and

(ii) Direction.

(It can be represented by an arrow).

Remark 1.3 A vector has an initial and final point. For instance, if the initial point is $P = (p_1, p_2, p_3)$ and the final point is $Q = (q_1, q_2, q_3)$, the vector $\vec{v} = \vec{PQ}$ is given by

$$\vec{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle,$$

i.e., “the components of \vec{PQ} are the coordinates of Q minus the coordinates of P ”.

Vectors can be translated, so we can always think about them as fixed in the origin, that is, we may assume $P = (0, 0, 0)$.

Definition 1.4 The norm of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

(This quantity is always non-negative and, it is zero if and only if the vector \vec{v} is identically zero).

Remark 1.5 The norm of a vector comes from the Pythagorean Theorem.

Proposition 1.6 (General Formula) Let \vec{v} be a vector with initial point $P = (p_1, p_2, p_3)$ and final point $Q = (q_1, q_2, q_3)$. Then, \vec{v} can be expressed in terms of its components as

$$\vec{v} = \vec{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

Moreover, the norm (magnitude) of \vec{v} can be computed using

$$\|\vec{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}.$$

Definition 1.7 The distance between two points P and Q is defined as $d(P, Q) = \|\vec{PQ}\| = \|\vec{QP}\|$.

Definition 1.8 A unit vector is a vector of norm one. The standard (or, canonical) unit vectors are:

$$\vec{i} = \langle 1, 0, 0 \rangle, \quad \vec{j} = \langle 0, 1, 0 \rangle, \quad \vec{k} = \langle 0, 0, 1 \rangle.$$

Remark 1.9 Given a vector \vec{v} , there exists a unit vector with the same direction,

$$\frac{\vec{v}}{\|\vec{v}\|}.$$

Proposition 1.10 Operations:

1. Scalar Multiplication. Let $c > 0$ be a real number and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ a vector, then $c\vec{v}$ is a vector with the same direction as \vec{v} and with norm $c\|\vec{v}\|$. If $c < 0$, the direction is the opposite and the norm is $|c|\|\vec{v}\|$. In terms of its components, for any $c \in \mathbb{R}$,

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle.$$

2. Sum of Vectors. Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$. The sum $\vec{v} + \vec{u}$ is the diagonal of the parallelogram with sides \vec{v} and \vec{u} . In terms of its components,

$$\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle.$$

Remark 1.11 The difference between two vectors \vec{v} and \vec{u} can be understood in terms of above two operations. Indeed,

$$\vec{v} - \vec{u} = \vec{v} + (-\vec{u}),$$

where $-\vec{u}$ is, precisely, the scalar multiplication of \vec{u} by -1 .

Notation 1.12 Using above operations and the canonical vectors, a vector \vec{v} can be denoted in the following two equivalent ways:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}.$$

1.1 The Dot Product

Definition 1.13 The dot product (or, also inner or scalar product) of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar (real number) given by

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Proposition 1.14 (Properties of the Dot Product) Let \vec{u} , \vec{v} and \vec{w} be three vectors and $c \in \mathbb{R}$. Then, the following properties are satisfied:

- (i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,
- (ii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$,
- (iii) $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$,
- (iv) $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$, and
- (v) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$, where θ is the angle between \vec{u} and \vec{v} .

Definition 1.15 Two vectors are said to be orthogonal (or, perpendicular) if the angle between them is $\pm\pi/2$.

Proposition 1.16 Two vectors \vec{u} and \vec{v} are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

1.2 The Cross Product

Definition 1.17 The cross product (or, also outer or vector product) between two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the vector given by

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle.$$

Remark 1.18 In order to remember about definition is useful to understand the cross product as the following formal determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Proposition 1.19 (Properties of the Cross Product) Let \vec{u} and \vec{v} be two vectors. Then, the following properties are satisfied:

- (i) The cross product $\vec{u} \times \vec{v}$ is orthogonal to both \vec{u} and \vec{v} .
- (ii) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, and

(iii) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin \theta$, where $\theta \in [0, \pi)$ is the angle between \vec{u} and \vec{v} .

Theorem 1.20 The norm of the cross product $\vec{u} \times \vec{v}$ is the area of the parallelogram with sides \vec{u} and \vec{v} . Similarly, the area of a triangle with sides \vec{u} and \vec{v} is, precisely, $\|\vec{u} \times \vec{v}\|/2$.

Definition 1.21 The triple product of \vec{u} , \vec{v} and \vec{w} is the number $\vec{u} \cdot (\vec{v} \times \vec{w})$.

Theorem 1.22 The absolute value of the triple product (i.e., $|\vec{u} \cdot (\vec{v} \times \vec{w})|$) represents the volume of the parallelepiped with sides \vec{u} , \vec{v} and \vec{w} .

1.3 Lines in \mathbb{R}^3

Definition 1.23 A line is an affine subspace of dimension one.

Proposition 1.24 (The First Postulate of Euclide) For every two points in space there is a line passing through them, and such a line is unique.

Proposition 1.25 Let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ be two points in space and denote by $\vec{v} = \vec{PQ}$ the director vector. Then, the unique line passing through P and Q can be represented as:

1. Vector Equation.

$$\vec{r}(t) = P + t\vec{v} = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle,$$

where t is the parameter of the line.

2. Parametric Equations.

$$\begin{cases} x = p_1 + tv_1, \\ y = p_2 + tv_2, \\ z = p_3 + tv_3. \end{cases}$$

3. Symmetric Equations. Assuming $v_1, v_2, v_3 \neq 0$,

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3}.$$

(If any v_i , $i = 1, 2, 3$ is zero, the line is contained in a plane.)

4. Implicit Equations.

$$\begin{cases} ax + by + cz = d, \\ \tilde{a}x + \tilde{b}y + \tilde{c}z = \tilde{d}. \end{cases}$$

Example 1.26 Write the equations of the line passing through the points $P = (1, -1, 0)$ and $Q = (2, 1, -1)$.

Proposition 1.27 *Two lines in the space can be:*

- (i) *The same.*
- (ii) *Parallel.*
- (iii) *Skew.*
- (iv) *Intersecting (the intersection is a point).*

Let \vec{v}_1 and \vec{v}_2 be the director vectors, respectively. If \vec{v}_1 is a multiple of \vec{v}_2 (i.e., there exists a real number $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$, they are proportional), then the two lines are either the same or parallel. In the other case, the lines are:

- *Skew, if they do not share any points.*
- *Intersecting, if there exists a common point.*

Example 1.28 *Determine the relative position of the following pair of lines:*

1.

$$\begin{cases} \frac{x-1}{2} = y + 1 = \frac{z-2}{4}, \\ \frac{x+2}{4} = \frac{-y}{3} = z + 1. \end{cases}$$

2.

$$\begin{cases} \frac{x-1}{2} = y + 1 = \frac{z-2}{4}, \\ \frac{x+2}{4} = \frac{-y}{3} = \frac{-2z+1}{2}. \end{cases}$$

1.4 Planes in \mathbb{R}^3

Definition 1.29 A plane is an affine subspace of dimension two.

Proposition 1.30 *For every three non-aligned points in space there is a unique plane passing through them. Equivalently, it is enough to know a point and two non proportional vectors, or a point and a normal vector.*

Proposition 1.31 *Let $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$ and $R = (r_1, r_2, r_3)$ be three non-aligned points in space and denote by $\vec{v} = \vec{PQ}$ and by $\vec{u} = \vec{PR}$. Then, the unique plane passing through P , Q and R can be represented as:*

1. Vector Equation.

$$\pi(s, t) = P + t\vec{v} + s\vec{u} = \langle p_1 + tv_1 + su_1, p_2 + tv_2 + su_2, p_3 + tv_3 + su_3 \rangle,$$

where t and s are the parameters of the plane.

2. Parametric Equations.

$$\begin{cases} x = p_1 + tv_1 + su_1, \\ y = p_2 + tv_2 + su_2, \\ z = p_3 + tv_3 + su_3. \end{cases}$$

3. Implicit Equations. Let $\vec{N} = \langle a, b, c \rangle$ be any normal vector to the plane (a normal vector can be computed as the cross product of \vec{v} and \vec{u}), then the implicit equation of the plane is:

$$ax + by + cz = d.$$

Example 1.32 Represent the plane passing through the points $P = (1, 1, 1)$, $Q = (2, 4, 3)$ and $R = (-1, -2, -1)$.

Proposition 1.33 Two planes in the space can be:

- (i) The same.
- (ii) Parallel.
- (iii) Intersecting (the intersection is a line).

Let \vec{N}_1 and \vec{N}_2 be the normal vectors, respectively. If they are proportional, then the two planes are either the same or parallel. In the other case, the planes intersect in a line.

Proposition 1.34 Every line is the intersection of two planes.

Example 1.35 Determine the relative position of the following planes:

$$\begin{cases} 2x + y - 4z = 3, \\ x - y + z = 2. \end{cases}$$

1.5 Distance

Definition 1.36 The distance between two affine subspaces (points, lines, planes) is the minimum of the distances between all the possible pair of points, i.e.,

$$d(A, B) = \min\{d(P, Q) \mid P \in A, Q \in B\}.$$

Example 1.37 Compute the distance between the following subspaces:

1. Distance from the point $P = (0, 3, 6)$ to the line $\vec{r}(t) = \langle 1 - t, 1 + 2t, 5 + 3t \rangle$.
2. Distance between the point $P = (1, 1, 1)$ and the plane of equation $x - y + z = 2$.
3. Distance between the lines

$$\mathcal{L}_1 : \begin{cases} x = 2t + 1 \\ y = t + 1 \\ z = -t + 1 \end{cases}, \quad \text{and} \quad \mathcal{L}_2 : \begin{cases} x - 2z + 1 = 0 \\ x - y + z = 0 \end{cases}.$$

1.6 Exercises

1. Find the equation of the plane passing through $P = (2, 5, 3)$ and perpendicular to the vector $\vec{N} = \langle 1, -4, -3 \rangle$. (Answer: $x - 4y - 3z = -27$.)
2. Find the equation of the plane through the point $P = (2, 5, 3)$ and parallel to the plane $x - 4y - 3z = -8$. (Answer: $x - 4y - 3z = -27$.)
3. Find a nonzero vector parallel to the line of intersection of the two planes $x - 4y - 3z = -4$ and $-3x + 3y - z = 3$. (Answer: $\langle -13, -10, 9 \rangle$.)
4. Find the implicit equation for the plane through $P = (-3, 3, -1)$ and normal to the vector $\langle 1, -4, -3 \rangle$. (Answer: $x - 4y - 3z = -12$.)
5. Find the equation of the plane passing through the points $P = (-4, 3, 0)$, $Q = (-3, -1, -3)$ and $R = (-7, 6, -1)$. (Answer: $13x + 10y - 9z = -22$.)

2 Vector-Valued Functions (Chapter 10)

Definition 2.1 Let $f(t)$, $g(t)$ and $h(t)$ be usual functions in the variable t . A vector-valued function $\vec{R}(t)$ is a function of the type

$$\vec{R}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k},$$

and its image for a fixed t is a vector.

Example 2.2 The vector parametric equation of a line,

$$\vec{r}(t) = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle,$$

is a vector valued-function.

Example 2.3 Compute the inner product $\vec{R} \cdot \vec{F}$ and the cross product $\vec{R} \times \vec{F}$ of the following vector valued functions:

$$\vec{R}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}, \quad \vec{F}(t) = \cos t \vec{i} - t \sin t \vec{k}.$$

Remark 2.4 Vector-valued functions can be understood as the parameterizations of a curve. Consequently, the vector-valued function $\vec{R}(t)$ is usually called the position vector.

Definition 2.5 The velocity vector of a curve is the first derivative of the position vector $\vec{R}(t)$ with respect to the parameter t , i.e., $\vec{R}'(t)$. The speed is the norm of the velocity vector, i.e., $\|\vec{R}'(t)\|$. And, the acceleration is the second derivative of the position vector (or, equivalently, the first derivative of the velocity vector) with respect to the parameter t , i.e., $\vec{R}''(t)$.

Example 2.6 Consider the position vector of a line, i.e.,

$$\vec{r}(t) = (p_1 + tv_1)\vec{i} + (p_2 + tv_2)\vec{j} + (p_3 + tv_3)\vec{k}.$$

The velocity vector is

$$\vec{r}'(t) = \langle v_1, v_2, v_3 \rangle,$$

that is, precisely, the director vector. While, the acceleration is identically zero, i.e., $\vec{r}''(t) = \langle 0, 0, 0 \rangle$.

Example 2.7 Compute the velocity and the acceleration of the curve parameterized by

$$\vec{R}(t) = (1 - 2t)\vec{i} - t^2\vec{j} + e^t\vec{k},$$

at $t_0 = 0$.

Example 2.8 Given the acceleration

$$\vec{A}(t) = \cos t \vec{i} - t \sin t \vec{k}$$

the initial velocity $\vec{V}(0) = \langle 1, -2, 1 \rangle$ and initial position $\vec{R}(0) = \langle 0, 2, 3 \rangle$, find the position vector $\vec{R}(t)$.

Definition 2.9 Given a curve parameterized by a vector-valued function $\vec{R}(t)$, the unit tangent vector is given by

$$T(t) = \frac{\vec{R}'(t)}{\|\vec{R}'(t)\|}.$$

The unit normal vector is given by

$$N(t) = \frac{T'(t)}{\|T'(t)\|},$$

and the unit binormal is $B(t) = T(t) \times N(t)$. We call $\{T(t), N(t), B(t)\}$ the Frenet frame.

Definition 2.10 The curvature $\kappa(t)$ of a curve is given by the following formula

$$\kappa(t) = \frac{\|\vec{R}'(t) \times \vec{R}''(t)\|}{\|\vec{R}'(t)\|^3}.$$

Example 2.11 Compute the curvature of the curve parameterized by

$$\vec{R}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}.$$

2.1 Exercises

1. Find the position vector $\vec{R}(t)$ given the velocity $\vec{V}(t) = e^t \vec{i} + 8 \cos(4t) \vec{j} + 16t^3 \vec{k}$ and the initial position $\vec{R}(0) = \langle 2, -2, 3 \rangle$.
(Answer: $\vec{R}(t) = (e^t + 1) \vec{i} + 2(\sin(4t) - 1) \vec{j} + (4t^4 + 3) \vec{k}$.)
2. Find the curvature of the planar curve $y = 2x + 1/x$ at $x = 1$. (Answer: $\kappa = 1/\sqrt{2}$.)
3. Find the curvature of the planar curve $y = 3e^x$ at $x = 0$. (Answer: $\kappa = 3/(\sqrt{10})^3$.)
4. Find the curvature of the curve parameterized by $\vec{R}(t) = 2 \sin(2t) \vec{i} + 2 \cos(2t) \vec{j} + \vec{k}$.
5. Find the curvature of the curve parameterized by $\vec{R}(t) = t^2 \vec{i} - t e^t \vec{j} + t^2 \cos(t) \vec{k}$ at $t = 0$.

3 Partial Differentiation (Chapter 11)

3.1 Functions of Several Variables

Definition 3.1 A function of several variables $f(x_1, \dots, x_n)$ is a map from a subset \mathcal{D} of \mathbb{R}^n to the real numbers, i.e., it is a scalar-valued function. The subset \mathcal{D} is the domain of the function. The range of f is the set of real numbers that can be written as $f(x_1, \dots, x_n)$.

Remark 3.2 In particular, we will focus on functions of two variables $f(x, y)$.

Example 3.3 Consider the function $f(x, y) = 1/\sqrt{x^2 - y^2}$. Compute the domain and range of f .

Remark 3.4 The graph $z = f(x, y)$ of a function of two variables is a surface in space. Locally, any surface can be described as a graph of a function.

Definition 3.5 A level curve for the constant value $c \in \mathbb{R}$ is the set of points in the plane satisfying $f(x, y) = c$.

Definition 3.6 A quadric is a surface which is described by an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

Proposition 3.7 Quadrics can be classified in the following groups:

(i) Cylinders.

(ii) Ellipsoids:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

In particular, if $a = b = c = R$ we have a sphere of radius R .

(iii) Cones:

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

(iv) Hyperboloids:

- Hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

- Hyperboloid of two sheets:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$

(v) *Paraboloids:*

- *Elliptic paraboloid:*

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

- *Hyperbolic paraboloid:*

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

3.2 Limits of Functions in Two Variables

Definition 3.8 Let $f : \mathcal{D} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of two variables and $(a, b) \in \mathcal{D}$. We say that the limit of $f(x, y)$ when (x, y) tends to (a, b) is L ,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L,$$

if when (x, y) is “close” to (a, b) , the value $f(x, y)$ is “close” to L .

Remark 3.9 For functions of two variables there are more options of being “close” than for functions of one variable, i.e., we can approach the point (a, b) in many ways.

Definition 3.10 The limit $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ exists if the function $f(x, y)$ approaches the same value L along every possible way of approaching (a, b) .

Proposition 3.11 (Iterated Limits) If the iterated limits

$$\lim_{x \rightarrow a} \left(\lim_{y \rightarrow b} f(x, y) \right), \quad \text{and} \quad \lim_{y \rightarrow b} \left(\lim_{x \rightarrow a} f(x, y) \right),$$

are not equal (or one of them is $\pm\infty$), then the limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y),$$

does not exist. (However, if they coincide we need to keep working.)

Proposition 3.12 (Limits Along Lines) Consider the general equation of all lines passing through the point (a, b) , $y = m(x - a) + b$. If the limit

$$\lim_{x \rightarrow a} f(x, m[x - a] + b),$$

depends on the constant m , then the limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y),$$

does not exist. (However, if they coincide we need to keep working.)

Proposition 3.13 (Polar Coordinates) Consider the parameterization of the circles of radii r centered at (a, b) given by the polar coordinates,

$$\begin{cases} x = a + r \cos \theta, \\ y = b + r \sin \theta. \end{cases}$$

Then,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{r \rightarrow 0} f(a + r \cos \theta, b + r \sin \theta).$$

In particular, if above limit depends on θ , then the limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y),$$

does not exist. (Otherwise, polar coordinates give us the answer.)

Example 3.14 Compute the following limits:

1.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2}{x^2 + y^2}.$$

2.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}.$$

3.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2}.$$

4.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}.$$

5.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{2\sqrt{x^2 + y^2 + 9} - 6}.$$

3.3 Continuity of Functions in Two Variables

Definition 3.15 We say that a function of two variables $f(x, y)$ is continuous at a point (a, b) in its domain if:

(i) $f(a, b)$ exists,

(ii) $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ exists, and

(iii) $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$.

We say that $f(x, y)$ is continuous in a domain \mathcal{D} if it is continuous at all $(a, b) \in \mathcal{D}$.

Remark 3.16 All the properties about continuous functions in one variable extend to two variables:

1. The sum of continuous functions is continuous.
2. The product of continuous functions is continuous.
3. The composition of continuous functions is continuous.

Example 3.17 Study the continuity of the following functions:

1.

$$f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

2.

$$f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

3.

$$f(x, y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ A, & (x, y) = (0, 0). \end{cases}$$

4.

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

3.4 Partial and Directional Derivatives

Definition 3.18 Let $f(x, y)$ be a function in two variables. The partial derivative of f with respect to x is given by:

$$f_x = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Similarly, the partial derivative of f with respect to y is:

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Remark 3.19 From the definition, it is clear that for f_x we can differentiate assuming y is constant. And, similarly for f_y .

Example 3.20 Compute the partial derivatives of the following functions:

- (i) $f(x, y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5$
- (ii) $f(x, y) = \cos(x^3 - 3x^2y^2 + 2y^4)$

Definition 3.21 The gradient of f is the vector field given by

$$\nabla f(a, b) = f_x(a, b)\vec{i} + f_y(a, b)\vec{j}.$$

Remark 3.22 Geometrically, f_x represents the slope of the line tangent to the graph $z = f(x, y)$ parallel to the x -axis. Similarly, for f_y .

Remark 3.23 The gradient points in the direction of maximum rate of change.

Definition 3.24 Let $S \subset \mathbb{R}^3$ be a surface and let $P = (a, b, f(a, b)) \in S$ be a point in the surface. The tangent plane at P is given by the implicit equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example 3.25 Find the tangent plane to the surface that is the graph of $f(x, y) = x^3 - x^2y + y^2 - 2x + 3y$ at $(a, b) = (-1, 3)$.

Definition 3.26 Let $\vec{v} = \langle v_1, v_2 \rangle$ be a unit vector in the plane \mathbb{R}^2 . The directional derivative of f with respect to \vec{v} is:

$$D_{\vec{v}}f = \lim_{h \rightarrow 0} \frac{f(x + hv_1, y + hv_2) - f(x, y)}{h}.$$

In general, for an arbitrary vector \vec{u} (unit or not) we compute:

$$D_{\vec{u}}f = \nabla f \cdot \frac{\vec{u}}{\|\vec{u}\|}.$$

Example 3.27 Compute the directional derivatives of the following functions $f(x, y)$:

- (i) $f(x, y) = \ln(x^2 + y^2)$ at $P = (1, -3)$ in the direction of $\vec{u} = \langle 2, -3 \rangle$.
- (ii) $f(x, y) = x^2 - y^2$ at $P = (1, 0)$ in the direction of $\vec{u} = \langle \sqrt{3}/2, 1/2 \rangle$.

3.5 Differentiability

Definition 3.28 A function $f(x, y)$ is differentiable at a point (a, b) if

$$\lim_{(x, y) \rightarrow (a, b)} \frac{f(x, y) - f(a, b) - f_x(a, b)(x - a) - f_y(a, b)(y - b)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

Remark 3.29 Roughly speaking, if the function $f(x, y)$ at (a, b) can be approximated by a linear function, the tangent plane.

Example 3.30 Is the function $f(x, y) = 3x - 4y^2$ differentiable at $(-1, 2)$?

Proposition 3.31 If $f(x, y)$ is differentiable at (a, b) , then $f(x, y)$ is continuous at (a, b) .

Theorem 3.32 Let $f(x, y)$ be a function in two variables and $(a, b) \in \mathcal{D}$ a point in the domain of f . If the partial derivatives of f , f_x and f_y exist and are continuous at (a, b) , then f is differentiable at (a, b) .

Remark 3.33 A function can have partial derivatives, without being differentiable. For instance,

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

3.6 Higher Order Partial Derivatives

Remark 3.34 Given a function $f(x, y)$ its partial derivatives f_x and f_y are also functions in two variables, so we can compute their partial derivatives:

$$\begin{aligned} f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \\ f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right), \\ f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right), \\ f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right), \end{aligned}$$

and we can keep going, i.e., f_{xxx}, \dots

Theorem 3.35 (Clairut-Schwarz) Let $f(x, y)$ be a function of two variables and assume that f has continuous second order partial derivatives at a point (a, b) , then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Remark 3.36 Continuity is essential in above result. If not, the following function serves as a counterexample:

$$f(x, y) = \begin{cases} \frac{x^3 y - y^3 x}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Example 3.37 Compute the second order partial derivatives of the function $f(x, y) = \sin(3x - 2y) + e^{x+4y}$.

3.7 The Chain Rule

Proposition 3.38 Let $x(u, v)$ and $y(u, v)$ be differentiable functions in the variables u and v , and assume that $f(x, y)$ is a differentiable function in x and y . Then,

$$f(x(u, v), y(u, v))$$

is a differentiable function in u and v and

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.\end{aligned}$$

Example 3.39 Let $f(x, y) = 3x^2 - 2xy$, $x(u, v) = e^2 \sin v$ and $y(u, v) = u + 2v + e^{uv}$. Compute f_u and f_v .

Remark 3.40 The chain rule can be used to implicitly differentiate a function.

Example 3.41 Let $z \equiv z(x, y)$ given by the implicit equation $x^2z + yz^3 = x$. Compute the partial derivatives z_x and z_y .

Example 3.42 Find the tangent plane to the surface $4x^2 - 2y^2 + z^2 = 12$ at $(2, 2, 2)$.

3.8 Extrema of Functions of Two Variables

Definition 3.43 Let $f(x, y)$ be a differentiable function of two variables. We say that a point $(a, b) \in \mathcal{D}$ in the domain of f is a critical point if either:

- (i) $\nabla f(a, b) = \langle 0, 0 \rangle = 0$, or,
- (ii) one of the partial derivatives, f_x or f_y , does not exist.

Example 3.44 Find the critical points of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$.

Definition 3.45 Let $f(x, y)$ be a differentiable function of two variables. We say that a point $(a, b) \in \mathcal{D}$ is:

- (i) A local maximum for f , if $f(x, y) \leq f(a, b)$ for all (x, y) “close” to (a, b) .
- (ii) A global maximum for f , if $f(x, y) \leq f(a, b)$ for all (x, y) .
- (iii) A local minimum for f , if $f(x, y) \geq f(a, b)$ for all (x, y) “close” to (a, b) .

(iv) A global minimum for f , if $f(x, y) \geq f(a, b)$ for all (x, y) .

Remark 3.46 Clearly global extrema are also local/relative.

Theorem 3.47 Let $f(x, y)$ be a differentiable function of two variables and assume that $f_x(a, b)$ and $f_y(a, b)$ exist. If $(a, b) \in \mathcal{D}$ is a local maximum (or, minimum), then

$$\nabla f(a, b) = \langle 0, 0 \rangle = 0.$$

Remark 3.48 The converse is not true. For instance, the function $f(x, y) = x^2 - y^2$ has a critical point at $(0, 0)$ which is not a local maximum nor a local minimum.

Definition 3.49 A saddle point is a critical point which is not a local maximum nor a local minimum.

Remark 3.50 To understand the local nature of critical points we need to check, at least, the second order partial derivatives.

Definition 3.51 The Hessian matrix of a function $f(x, y)$ at a point $(a, b) \in \mathcal{D}$ is

$$\mathcal{H}f(a, b) = \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix}.$$

We will denote by $D = \det \mathcal{H}f(a, b)$ to the determinant of the Hessian matrix of f at (a, b) . (That is, $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)f_{yx}(a, b)$.)

Theorem 3.52 (Second Partial Derivative Test) Let $f(x, y)$ be a differentiable function and assume that (a, b) is a critical point where $\nabla f(a, b) = \langle 0, 0 \rangle$. Then:

- (i) If $D > 0$ and $f_{xx}(a, b) > 0$, the point (a, b) is a local minimum.
- (ii) If $D > 0$ and $f_{xx}(a, b) < 0$, the point (a, b) is a local maximum.
- (iii) If $D < 0$, the point (a, b) is a saddle point.
- (iv) If $D = 0$, the test is “inconclusive”.

Example 3.53 Compute the critical points of the following functions and classify them:

1. $f(x, y) = x^3 + 2xy - 6x - 4y^2$
2. $f(x, y) = 2x^2 + 2xy + y^2 - 2x + 5$
3. $f(x, y) = x^2y^4$
4. $f(x, y) = 8x^3 - 24xy + y^3$

Theorem 3.54 (Global Extrema) *A continuous function defined on a compact (bounded and closed) domain attains both its global minimum and global maximum.*

Definition 3.55 *A set is closed if the boundary is included.*

Remark 3.56 *How to find the global extrema?*

1. *Compute the critical points in the interior of the domain. (The interior is the set without the boundary.)*
2. *Determine the maximum and minimum value of the function at the boundary. (One variable problem.)*
3. *Compare the values of the function at the following points: the points of 1), the points of 2) and the points in the boundary of the boundary.*

Example 3.57 *Compute the extrema of the following functions:*

1. $f(x, y) = e^{x^2-y^2}$ defined on $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.
2. $f(x, y) = \frac{9x}{x^2+y^2+1}$ for $x \in [-1, 1]$ and $y \in [-1, 1]$.
3. $f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$, for $(x, y) \in [0, 2] \times [-1, 3]$.
4. $f(x, y) = x^2 + y^2 - 2y + 1$ defined on $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

Remark 3.58 *Step 2 in previous remark is usually hard if the domain has a weird shape (other than a circle, an ellipse or a square). To solve this problem we may need the Lagrange multipliers principle.*

3.9 Lagrange Multipliers

Remark 3.59 *Lagrange multipliers serve to compute maxima and minima of a differentiable function $f(x, y)$ subject to a constraint $g(x, y) = 0$.*

Theorem 3.60 (Lagrange Multipliers Principle) *Let $f(x, y)$ and $g(x, y)$ be functions in two variables with continuous partial derivatives and assume that $f(x, y)$ restricted to the curve $g(x, y) = 0$ has a local extrema (a, b) such that $\nabla g(a, b) \neq \langle 0, 0 \rangle$. Then, there exists a number $\lambda \in \mathbb{R}$ such that*

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

Definition 3.61 *The number $\lambda \in \mathbb{R}$ is referred as to the Lagrange multiplier.*

Remark 3.62 *If we want to find extrema of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$, from Lagrange multipliers principle, we need to solve*

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y), \\ g(x, y) = 0, \end{cases}$$

to obtain the candidates. Observe that this is a system of three equations (possibly, nonlinear) with three unknowns, i.e., x , y and λ .

Example 3.63 *Compute maxima and minima of the following functions with their respective constraints:*

1. $f(x, y) = x + y$ subject to $g(x, y) = x^2 + y^2 - 1 = 0$.
2. $f(x, y) = 1 - x^2 - y^2$ subject to $g(x, y) = x + y - 1 = 0$.
3. $f(x, y) = x^2 + y^2$ subject to $g(x, y) = x^2 + xy + y^2 - 1 = 0$.
4. $f(x, y) = e^{x^2 - y^2}$ subject to $g(x, y) = x^2 + y^2 - 1 = 0$.
5. $f(x, y) = x^2 + y^2 - 2y + 1$ subject to $g(x, y) = x^2 + y^2 - 4 = 0$.

Review Problems

1. Compute the distance from the point $P = (3, -1, 4)$ to the line

$$\mathcal{L} : \begin{cases} x = t, \\ y = 2 + t, \\ z = 1 - t. \end{cases}$$

(Use the family of planes orthogonal to \mathcal{L} to find the closest point in \mathcal{L} to P .)

(Answer: $Q = (-1, 1, 2)$ and $d(P, \mathcal{L}) = \sqrt{24}$.)

2. Compute the distance from the point $P = (1, 1, 2)$ to the line $\mathcal{L}(t) = \langle -2 + 3t, 1 - t, 3 + 2t \rangle$.

(Use the family of planes orthogonal to \mathcal{L} to find the closest point in \mathcal{L} to P .)

(Answer: $Q = (-1/2, 1/2, 4)$ and $d(P, \mathcal{L}) = \sqrt{26}/2$.)

3. Compute the distance from the point $P = (-2, 1, 0)$ to the plane $\pi : x - y + 2z = 7$.

(Use the line perpendicular to π to find the closest point in π to P .)

(Answer: $Q = (-1/3, -2/3, 10/3)$ and $d(P, \mathcal{L}) = \sqrt{50/3}$.)

4. Determine the relative position between the lines

$$\mathcal{L}_1 : \begin{cases} x = 2 - t, \\ y = 1 + t, \\ z = -2 - t, \end{cases} \quad \mathcal{L}_2 : \begin{cases} x = 1 + s, \\ y = s, \\ z = 5 + s. \end{cases}$$

(If they are skew, compute the distance between them finding the closest points P and Q .)

5. Determine the relative position between the lines

$$\mathcal{L}_1 : \begin{cases} x = 2t + 3, \\ y = t + 2, \\ z = -t, \end{cases} \quad \mathcal{L}_2 : \begin{cases} x - y + z = 0, \\ 2x - y - z + 1 = 0. \end{cases}$$

(If they are skew, compute the distance between them finding the closest points P and Q .)

6. Compute the position vector given the acceleration $\vec{A}(t) = -4 \cos(2t)\vec{i} + 4 \sin(2t)\vec{j} - 2t\vec{k}$ and initial velocity and position $\vec{V}(0) = \langle 1, 0, 1 \rangle$ and $\vec{R}(0) = \langle 1, 1, 1 \rangle$, respectively.

(Answer: $\vec{R}(t) = (\cos(2t) + t)\vec{i} + (-\sin(2t) + 2t + 1)\vec{j} + (-t^3/3 + t + 1)\vec{k}$.)

7. Compute the position vector given the acceleration $\vec{A}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + 3t\vec{k}$ and initial velocity and position $\vec{V}(0) = \langle 1, 0, 1 \rangle$ and $\vec{R}(0) = \langle 1, 1, 1 \rangle$, respectively.

(Answer: $\vec{R}(t) = (-\cos t + t + 2)\vec{i} + (-\sin t + t + 1)\vec{j} + (t^3/2 + t + 1)\vec{k}$.)

8. Compute the acceleration and position vector given the velocity $\vec{V}(t) = \sqrt{t}\vec{i} - e^t\vec{j} + \sin t\vec{k}$ and initial position $\vec{R}(0) = \langle 1, -1, 2 \rangle$.

(Answer: $\vec{A}(t) = 1/(2\sqrt{t})\vec{i} - e^t\vec{j} + \cos t\vec{k}$ and $\vec{R}(t) = (2t^{3/2}/3 + 1)\vec{i} - e^t\vec{j} + (3 - \cos t)\vec{k}$.)

9. Compute the acceleration and position vector given the velocity $\vec{V}(t) = t^2\vec{i} - \sin(2t)\vec{j} + 2te^{t^2}\vec{k}$ and initial position $\vec{R}(0) = \langle 1, -1/2, 0 \rangle$.

(Answer: $\vec{A}(t) = 2t\vec{i} - 2\cos(2t)\vec{j} + (2e^{t^2} + 4t^2e^{t^2})\vec{k}$ and $\vec{R}(t) = (t^3/3 + 1)\vec{i} + (\cos(2t)/2 - 1)\vec{j} + (e^{t^2} - 1)\vec{k}$.)

10. Compute the acceleration and position vector given the velocity $\vec{V}(t) = e^{3t}\vec{i} - 2\sin(2t)\vec{j} + \sqrt{t}\vec{k}$ and initial position $\vec{R}(0) = \langle 1/3, -1, 1 \rangle$.

(Answer: $\vec{A}(t) = 3e^{3t}\vec{i} - 4\cos(2t)\vec{j} + 1/(2\sqrt{t})\vec{k}$ and $\vec{R}(t) = e^{3t}/3\vec{i} + (\cos(2t) - 2)\vec{j} + (2t^{3/2}/3 + 1)\vec{k}$.)

11. Study the continuity of the following function:

$$f(x, y) = \begin{cases} \frac{4xy^2}{x^2+3y^4}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

(Answer: The function $f(x, y)$ is continuous everywhere.)

12. Study the continuity of the following function:

$$f(x, y) = \begin{cases} \frac{3xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

(The function $f(x, y)$ is continuous everywhere, but at $(0, 0)$.)

13. Study the continuity of the following function:

$$f(x, y) = \begin{cases} \frac{2(x^2+y^2)}{\sqrt{x^2+y^2+4}-2}, & (x, y) \neq (0, 0), \\ 8, & (x, y) = (0, 0). \end{cases}$$

(Answer: The function $f(x, y)$ is continuous everywhere.)

14. Study the continuity of the following function:

$$f(x, y) = \begin{cases} \frac{x-y}{x^2+4y^4}, & (x, y) \neq (0, 0), \\ A, & (x, y) = (0, 0). \end{cases}$$

(Answer: There is no possible value of $A \in \mathbb{R}$ such that $f(x, y)$ is continuous at $(0, 0)$.)

15. Study the continuity of the following function:

$$f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

(Answer: The function $f(x, y)$ is continuous everywhere.)

16. Let $f(x, y) = x^2 + 2xy + 3y^2 - 4x - 3y$. Obtain critical points and classify them as local maximum, local minimum or saddle points.
17. Let $f(x, y) = x^2 - 3xy + y^2$. Obtain critical points and classify them as local maximum, local minimum or saddle points.
18. Let $f(x, y) = x^2 + 3xy + 5y^3 - 7x - 11y$. Obtain critical points and classify them as local maximum, local minimum or saddle points.
19. Let $f(x, y) = x^3 - 3x + y^3 - 3y^2$. Obtain critical points and classify them as local maximum, local minimum or saddle points.
(Answer: $P_1 = (1, 0)$ is a saddle point, $P_2 = (1, 2)$ is a local minimum, $P_3 = (-1, 0)$ is a local maximum and $P_4 = (-1, 2)$ is a saddle point.)
20. Let $f(x, y) = 2x^2 + y^2 + 6y$ be defined on \mathbb{R}^2 . Obtain global extrema, if possible.
(Answer: $P = (0, -3)$ is the global minimum and there is no global maximum.)
21. Let $f(x, y) = 2x^2 + y^2 + 6y$ be defined on $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 16\}$. Obtain extrema.
(Answer: $P = (0, -3)$ is the global minimum and $P_{\pm} = (\pm\sqrt{7}, 3)$ are the global maxima.)
22. Let $f(x, y) = x^2 + 3y^2$ be defined on \mathbb{R}^2 . Obtain global extrema, if possible.
(Answer: $P = (0, 0)$ is the global minimum and there is no global maximum.)
23. Let $f(x, y) = x^2 + 3y^2$ be defined on $\{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 \leq 4\}$. Obtain extrema.
(Answer: $P = (0, 0)$ is the global minimum and $P_{\pm} = (3/2, \pm\sqrt{15}/2)$ are the global maxima.)
24. Let $f(x, y) = 4x^2 + y^2 - 4x$ be defined on \mathbb{R}^2 . Obtain global extrema, if possible.
(Answer: $P = (1/2, 0)$ is the global minimum and there is no global maximum.)
25. Let $f(x, y) = 4x^2 + y^2 - 4x$ be defined on $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Obtain extrema.
(Answer: $P = (1/2, 0)$ is the global minimum and $P = (-1, 0)$ is the global maximum.)
26. Let $f(x, y) = x^2 + y^2 + 2x - 1$ be defined on \mathbb{R}^2 . Obtain global extrema, if possible.
(Answer: $P = (-1, 0)$ is the global minimum and there is no global maximum.)
27. Let $f(x, y) = x^2 + y^2 + 2x - 1$ be defined on $\{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^2 \leq 8\}$. Obtain extrema.
(Answer: $P = (-1, 0)$ is the global minimum and $P_{\pm} = (1, \pm\sqrt{6})$ are the global maxima.)
28. Let $f(x, y) = -x^2 - y^2 + 2x - 1$ be defined on \mathbb{R}^2 . Obtain global extrema, if possible.
(Answer: $P = (1, 0)$ is the global maximum and there is no global minima.)
29. Let $f(x, y) = -x^2 - y^2 + 2x - 1$ be defined on $\{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^2 \leq 8\}$. Obtain extrema.
(Answer: $P = (1, 0)$ is the global maximum and $P_{\pm} = (-1, \pm\sqrt{6})$ are the global minima.)

4 Multiple Integration (Chapter 12)

4.1 Double Integrals Over Rectangular Regions

Definition 4.1 Let $f(x, y)$ be defined on a closed, bounded rectangular region R in the xy -plane. The double integral of f over R is defined by

$$\int \int_R f(x, y) dA = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^N f(x_k^*, y_k^*) \Delta A_k,$$

where A_k is the area of the k -th representative cell and $\|P\|$ is the norm of the partition, i.e., the length of the longest diagonal of any rectangle in the partition.

Definition 4.2 The sum $\sum_{k=1}^N f(x_k^*, y_k^*) \Delta A_k$ is called the Riemann sum of $f(x, y)$.

Definition 4.3 If above limit exists, f is said to be integrable over R .

Proposition 4.4 Let f and g be integrable functions on a rectangular region R . Then, the following properties of double integrals hold:

(i) *Linearity.* For constants $a, b \in \mathbb{R}$,

$$\int \int_R (af(x, y) + bg(x, y)) dA = a \int \int_R f(x, y) dA + b \int \int_R g(x, y) dA.$$

(ii) If $f(x, y) \geq g(x, y)$ on R ,

$$\int \int_R f(x, y) dA \geq \int \int_R g(x, y) dA.$$

(iii) If R is subdivided on two rectangles R_1 and R_2 whose intersection is just a line,

$$\int \int_R f(x, y) dA = \int \int_{R_1} f(x, y) dA + \int \int_{R_2} f(x, y) dA.$$

(iv) Let $R = [a, b] \times [c, d]$ and assume $f(x, y) = f_1(x)f_2(y)$, then

$$\int \int_R f(x, y) dA = \left(\int_a^b f_1(x) dx \right) \left(\int_c^d f_2(y) dy \right).$$

Remark 4.5 The integral $\int \int_R f(x, y) dA$ represents the volume of the region bounded between the xy -plane and the graph of $f(x, y)$.

Theorem 4.6 (Fubini) Let $f(x, y)$ be an integrable function over a rectangular region $R = [a, b] \times [c, d]$. Then,

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

Example 4.7 Compute the following double integrals:

1.

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) dy dx$$

2. Let $R = [1, 2] \times [0, 1]$,

$$\int \int_R x^2 y^5 dA$$

3. Let $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq \log 5\}$,

$$\int \int_R x e^{xy} dA$$

4.

$$\int_0^1 \int_0^{\pi/2} x \cos(xy) dx dy$$

4.2 Double Integrals Over General Regions

Definition 4.8 Let $\mathcal{D} \subset \mathbb{R}^2$ be an arbitrary bounded domain and consider a rectangle R such that $\mathcal{D} \subset R$. Assume f is continuous on \mathcal{D} and define

$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\int \int_{\mathcal{D}} f(x, y) dA = \int \int_R F(x, y) dA.$$

Remark 4.9 We will consider regions of the form

$$\mathcal{D}_1 = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

and

$$\mathcal{D}_2 = \{(x, y) \in \mathbb{R}^2 \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}.$$

Theorem 4.10 (Fubini) Let $f(x, y)$ be continuous on \mathcal{D}_1 (and \mathcal{D}_2), then

$$\int \int_{\mathcal{D}_1} f(x, y) dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx,$$

and

$$\int \int_{\mathcal{D}_2} f(x, y) dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy.$$

Example 4.11 Compute the following double integrals:

1.

$$\int \int_{\mathcal{D}} x^2 e^{xy} dA,$$

where \mathcal{D} is the triangular region enclosed by the lines $y = 1$, $x = 0$ and $y = x/2$.

2.

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 160xy^2 dydx.$$

3.

$$\int \int_{\mathcal{D}} (x + y) dA,$$

where \mathcal{D} is the triangular region enclosed by the lines $y = 0$, $y = 2x$ and $x = 1$.

4.

$$\int_0^1 \int_x^1 e^{y^2} dydx.$$

5.

$$\int \int_{\mathcal{D}} (x^2 + y^2) dA,$$

where \mathcal{D} is the planar region enclosed by the curves $y = 2x$ and $y = x^2$.

6. Change the order of integration for

$$\int_0^2 \int_1^{e^x} f(x, y) dydx.$$

Definition 4.12 Given a region $\mathcal{D} \subset \mathbb{R}^2$, the area of \mathcal{D} is the integral

$$\mathcal{A}(\mathcal{D}) = \int \int_{\mathcal{D}} 1 dA.$$

Example 4.13 Compute the areas of the following domains \mathcal{D} :

1. Find the area of the region \mathcal{D} bounded by the parabola $y = x^2 - 2$ and the line $y = x$.

2. Area of \mathcal{D} bounded by $y = x^3$ and $y = x^3 + 1$ for $x \in [0, 1]$.

3. Area of \mathcal{D} bounded by $y = 1$, $y = x$, $y = \log x$ and $y = 0$.

4.3 Change of Variable in Double Integrals

Definition 4.14 A transformation $T : \mathcal{D} \subset \mathbb{R}^2 \rightarrow \tilde{\mathcal{D}} \subset \mathbb{R}^2$ defined by $T(u, v) = (x(u, v), y(u, v))$ is bijjective (or, one-to-one) if:

- (i) Every point in $\tilde{\mathcal{D}}$ is in the range of T .
- (ii) Different points of \mathcal{D} map to different points in $\tilde{\mathcal{D}}$.

Theorem 4.15 (Change of Variables) Let $T(u, v) = (x(u, v), y(u, v))$ be a bijective \mathcal{C}^1 transformation such that $T(S) = R$. Let $f(x, y)$ be a continuous and bounded function, then

$$\int \int_R f(x, y) dA = \int \int_S f(x(u, v), y(u, v)) |\det \text{Jac}(T)| du dv ,$$

where

$$\text{Jac}(T) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} ,$$

is the Jacobian matrix associated to the transformation.

Remark 4.16 (Polar Coordinates) We can use the polar coordinates as one possible change of variable:

$$\begin{aligned} x &= r \cos \theta , \\ y &= r \sin \theta . \end{aligned}$$

In this case, $|\det \text{Jac}(T)| = r$.

Example 4.17 Compute the following integrals using a suitable change of variable:

1.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx .$$

2. Let $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, x \geq 0\}$ and

$$\int \int_R (x + y) dA .$$

3.

$$\int_{-\infty}^{\infty} e^{-x^2} dx .$$

4. Let R be the parallelogram with vertices $(1, 2)$, $(3, 4)$, $(4, 3)$ and $(6, 5)$, and

$$\int \int_R (x - y) dA .$$

4.4 Triple Integrals Over General Regions

Theorem 4.18 Let $f(x, y, z)$ be a continuous and bounded function over the region

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \mathcal{D}, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Then,

$$\int \int \int_B f(x, y, z) dV = \int \int_{\mathcal{D}} \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right) dA.$$

Remark 4.19 Once we solve the integral with respect to z , we have a double integral over the region \mathcal{D} , which we know how to deal with.

Example 4.20 Compute the following triple integrals:

1.

$$\int_{-1}^2 \int_1^2 \int_0^1 z^2 y e^x dx dy dz.$$

2. Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

3. Let B be the region defined by $x^2 + y^2 \leq 4$, $z \geq 0$ and $z \leq 4 - 2y$. Compute

$$\int \int \int_B x dV.$$

Remark 4.21 Change of variables also apply for triple integrals in a similar way as in double integrals.

Remark 4.22 (Cylindrical Coordinates) We can use the cylindrical coordinates as one possible change of variable:

$$\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta, \\ z &= z. \end{aligned}$$

In this case, $|\det \text{Jac}(T)| = r$.

Example 4.23 Compute the following integrals using cylindrical coordinates:

1. Let B be the region bounded by $z = x^2 + y^2$ and $z = 4$. Compute

$$\int \int \int_B \sqrt{x^2 + y^2} dV.$$

2. Find the volume of the region B bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 2 - x^2 - y^2$.

Remark 4.24 (Spherical Coordinates) We can use the spherical coordinates as one possible change of variable:

$$\begin{aligned}x &= \rho \cos \theta \sin \phi, \\y &= \rho \sin \theta \sin \phi, \\z &= \rho \cos \phi.\end{aligned}$$

In this case, $|\det \text{Jac}(T)| = \rho^2 \sin \phi$.

Example 4.25 Compute the volume of the region bounded by the cone $z = \sqrt{3(x^2 + y^2)}$ and the half-sphere $z = \sqrt{4 - x^2 - y^2}$

4.5 Exercises

1.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$$

Solution: $\pi/8$.

2. Let $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}$ and

$$\int \int_{\mathcal{D}} (4 - x^2 - y^2) dA.$$

Solution: $5\pi/2$.

3. Let \mathcal{D} be the region bounded by the lines $x+y = 1$, $x+y = 3$ and the curves $x^2 - y^2 = -1$, $x^2 - y^2 = 1$. Compute,

$$\int \int_{\mathcal{D}} (x-y)e^{x^2-y^2} dA.$$

Hint: Use the change of variable $u = x+y$ and $v = x-y$.

Solution: $2/(3e)$.

4. Let B be the solid bounded by the surface $y^2 = x$, $x = 4$, $y = 0$, $z = 0$, and $z = 1$. Compute,

$$\int \int \int_B xyz dV.$$

Solution: $16/3$.

5. Let $B = \{(x, y, z) \in \mathbb{R}^3 \mid -y \leq x \leq y, 0 \leq y \leq 1, 0 \leq z \leq 1 - x^4 - y^4\}$. Compute,

$$\int \int \int_B z dV.$$

Solution: $113/450$.

6. Let $B = \{(x, y, z) \in \mathbb{R}^3 \mid 1 \leq x^2 + y^2 \leq 9, y \leq 0, 0 \leq z \leq 1\}$. Compute,

$$\int \int \int_B \sqrt{x^2 + y^2} \, dV .$$

Solution: $26\pi/3$.

7. Let B be the region bounded by $x^2 + y^2 + z^2 = 16$, $z \geq 0$ and $3z^2 = x^2 + y^2$. Compute,

$$\int \int \int_B z \, dV .$$

Solution: 48π .

8. Write the integral for the volume of the region bounded by $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{16 - x^2 - y^2}$, in Cartesian (x, y and z), cylindrical and spherical coordinates. Compute the volume using cylindrical coordinates.

Solution: $64\pi (2 - \sqrt{2}) / 3$.

5 Vector Analysis (Chapter 13)

5.1 Scalar Line Integrals

Remark 5.1 A scalar line integral is a single integral of a function of several variables along a curve.

Theorem 5.2 Let $f(x, y, z)$ be a continuous function defined on a domain that includes the curve C parameterized by $\vec{r}(t)$ with $t \in [a, b]$. Then,

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt.$$

Remark 5.3 The scalar line integral is independent of the parameterization of the curve.

Example 5.4 Evaluate the following line integrals:

1. Evaluate $\int_C (x^2 + yz) \, ds$ where C is parameterized by $\vec{r}(t) = \langle 2t, 5t, -t \rangle$ with $0 \leq t \leq 10$.
2. Evaluate $\int_C x e^y \, ds$ where C is the curve of equation $x = e^y$ from $(1, 0)$ to $(e, 1)$.

Example 5.5 Find the parameterization of the intersection between $x^2 + y^2 + z^2 = 50$ and the plane $y = 5$.

Remark 5.6 If the curve can be divided into different parts (i.e., $C = C_1 \cup C_2$), then

$$\int_C f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds.$$

Example 5.7 Evaluate $\int_C xy^2 \, ds$ where C is the triangle of vertices $(0, 1, 2)$, $(0, -1, 0)$ and $(1, 0, 3)$.

5.2 Vector Fields

Definition 5.8 A vector field (in space) is an assignment of a three-dimensional vector $\vec{F}(x, y, z)$ to each point $(x, y, z) \in \mathcal{D} \subset \mathbb{R}^3$. The subset \mathcal{D} is the domain of the vector field.

Remark 5.9 A vector field is a vector-valued function in several variables, i.e., $\vec{F} : \mathcal{D} \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

Example 5.10 $\vec{F}(x, y) = x\vec{i} + y\vec{j}$.

Definition 5.11 We say that $\vec{F} : \mathcal{D} \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a unit vector field if $\|\vec{F}(x, y, z)\| = 1$ for every $(x, y, z) \in \mathcal{D}$.

Remark 5.12 For convenience, we will often use the following notation

$$\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k},$$

where $f_i(x, y, z)$ for $i = 1, 2, 3$, are real-valued functions in several variables.

Definition 5.13 We say that a vector field on \mathbb{R}^3 is smooth if all derivatives (of all orders) of $f_i(x, y, z)$, $i = 1, 2, 3$, exist and are continuous in the domain of the vector field.

Definition 5.14 Let $\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$ be a smooth vector field. The divergence of \vec{F} is the function

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

Remark 5.15 We may think ∇ as a formal “vector field”

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k},$$

so the divergence $\nabla \cdot \vec{F}$ is, roughly speaking, the dot product (or scalar/inner) of ∇ and \vec{F} .

Remark 5.16 (Physical Interpretation) If a vector field \vec{F} is the velocity of a fluid moving in space, the divergence at a point measures the flow of the fluid.

Definition 5.17 If the divergence of a vector field is zero, we say that the vector field is incompressible.

Example 5.18 Compute the divergence of the vector field $\vec{F}(x, y) = x\vec{i} + y\vec{j}$.

Definition 5.19 Let $\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$ be a smooth vector field. The curl of \vec{F} is the vector field

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix},$$

that is,

$$\nabla \times \vec{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \vec{k}.$$

Remark 5.20 As above, understanding ∇ as a “vector field”, the curl $\nabla \times \vec{F}$ is the cross product (or vector/outer) of ∇ with \vec{F} .

Remark 5.21 In the case that the vector field \vec{F} is contained in the plane \mathbb{R}^2 , then we can consider $f_3 = 0$ and f_1 and f_2 do not depend on z . Therefore,

$$\nabla \times \vec{F} = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \vec{k}.$$

Remark 5.22 (Physical Interpretation) The curl of a vector field measures the spin of the vector field, i.e., if \vec{F} is the velocity of a fluid, at each point, the curl measures the tendency of the fluid to rotate. The direction of $\nabla \times \vec{F}$ is parallel to the axis of rotation, while the magnitude represents the speed of rotation.

Definition 5.23 We say that a vector field \vec{F} is irrotational if the curl is the vector zero, i.e., if

$$\nabla \times \vec{F} = \langle 0, 0, 0 \rangle.$$

Example 5.24 Determine if the following vector fields are irrotational or not:

1. $\vec{F}(x, y) = y\vec{i}$.
2. The gravitational potential energy:

$$\vec{F}(x, y, z) = \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle.$$

Proposition 5.25 Let $\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$ be a smooth vector field. Then, the divergence of the curl of \vec{F} is zero, i.e.,

$$\nabla \cdot (\nabla \times \vec{F}) = 0.$$

Definition 5.26 Let $f(x, y, z)$ be a smooth function. The gradient vector field of f is

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

Proposition 5.27 Let $f(x, y, z)$ be a smooth function. Then, the curl of the gradient of f is the vector zero, i.e.,

$$\nabla \times (\nabla f) = \langle 0, 0, 0 \rangle.$$

Remark 5.28 The other implication is true under an additional assumption.

Definition 5.29 We say that a domain \mathcal{D} is connected if any pair of points can be joined by a path contained in \mathcal{D} . We say that \mathcal{D} is simply-connected if it is connected and any simple loop in \mathcal{D} can be shrunk to a point.

Remark 5.30 In two dimensions, a domain is simply-connected if it is connected and has no holes.

Proposition 5.31 Let \vec{F} be a smooth vector field defined on a simply-connected domain. If \vec{F} is irrotational (i.e., $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$), then \vec{F} is a gradient vector field. That is, it exists a function f such that

$$\vec{F} = \nabla f.$$

Definition 5.32 The scalar-valued function f is called a (scalar) potential for \vec{F} .

Example 5.33 Determine if the following vector fields are gradient vector fields and, if yes, find the scalar potentials:

1. $\vec{F}(x, y) = 2xy^3\vec{i} + (3x^2y^2 + \cos y)\vec{j}$.
2. $\vec{F}(x, y, z) = 2xy\vec{i} + (x^2 + 2yz^3)\vec{j} + (3y^2z^2 + 2z)\vec{k}$.

Definition 5.34 Let $f(x, y, z)$ be a smooth function. We define the Laplacian as the function given by the divergence of the gradient of f , i.e.,

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}.$$

5.3 Vector Line Integrals

Definition 5.35 An orientation on a curve C is a choice of direction on C . For a closed curve the positive orientation is counter-clockwise, i.e., enclosing a domain on the left.

Remark 5.36 Unless the opposite stated, we will always assume that our curves are positively oriented.

Definition 5.37 Let \vec{F} be a smooth vector field defined on a domain containing a curve C and assume that C is parameterized by $\vec{r}(t)$ with $t \in [a, b]$. The vector line integral of \vec{F} along C is

$$\int_C \vec{F} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt.$$

Remark 5.38 Sometimes, the vector line integral can also be denoted as

$$\int_C \vec{F} d\vec{r}.$$

Example 5.39 Evaluate the following vector line integrals:

1. $\int_C \vec{F} ds$, where $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$ and C is the semicircle parameterized by $\vec{r}(t) = \langle \cos t, \sin t \rangle$, with $0 \leq t \leq \pi$.
2. $\int_C \vec{F} ds$, where $\vec{F}(x, y, z) = 4x\vec{i} + 2y\vec{j} + 4y^2\vec{k}$ and C is parameterized by $\vec{r}(t) = \langle 4 \cos 2t, 2 \sin 2t, 3 \rangle$, with $t \in [0, \pi/4]$.
3. The vector line integral of the vector field $\vec{F}(x, y) = y^2\vec{i} + (2xy + 1)\vec{j}$ along the triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 5)$, positively oriented.

Theorem 5.40 (Fundamental Theorem of Line Integrals) Let \vec{F} be a smooth gradient vector field (i.e., $\vec{F} = \nabla f$, for some function f) and let C be a curve parameterized by $\vec{r}(t)$ with $t \in [a, b]$. Then,

$$\int_C \vec{F} ds = \int_C \nabla f ds = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Definition 5.41 We say that a vector field is conservative if the line integral $\int_C \vec{F} ds$ does not depend on the path C , but only on the initial and final points.

Remark 5.42 If C is closed and \vec{F} is conservative, then

$$\int_C \vec{F} ds = 0,$$

since the initial and final points of a closed curve are the same.

Proposition 5.43 A vector field is conservative if and only if it is a gradient vector field. In other words, if and only if there exists a scalar potential.

Remark 5.44 To sum up, we have that a vector field \vec{F} is conservative (i.e., line integrals only depend on the endpoints, not on the path) if and only if \vec{F} is a gradient vector field (i.e., $\vec{F} = \nabla f$, for some function f). Moreover, gradient vector fields (and so, conservative vector fields) are always irrotational (that is, $\nabla \times \vec{F} = \nabla \times (\nabla f) = \langle 0, 0, 0 \rangle$). However, for the converse of this last statement, we need a simply-connected domain.

Example 5.45 Consider the vector field

$$\vec{F}(x, y) = \frac{-y}{x^2 + y^2} \vec{i} + \frac{x}{x^2 + y^2} \vec{j}.$$

Is this vector field irrotational? Can we conclude that it is a gradient vector field? Compute the line integral $\int_C \vec{F} ds$ along the closed circle of radius one centered at $(0, 0)$.

5.4 Green's Theorem

Theorem 5.46 (Green's Theorem) Let \mathcal{D} be a region in the plane with boundary $\partial\mathcal{D}$ together with the induced positive orientation, and consider a smooth vector field $\vec{F}(x, y) = f_1(x, y)\vec{i} + f_2(x, y)\vec{j}$ defined on \mathcal{D} . Then,

$$\oint_{\partial\mathcal{D}} \vec{F} ds = \int \int_{\mathcal{D}} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA.$$

Remark 5.47 The notation \oint just means that we are computing the line integral \int along closed curves.

Remark 5.48 Observe that the integrand on the right-hand side of Green's Theorem is just the curl of \vec{F} , $\nabla \times \vec{F}$.

Example 5.49 Use Green's Theorem to compute the followings:

1. The line integral of the vector field $\vec{F}(x, y) = y^2\vec{i} + (2xy + 1)\vec{j}$ along the triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 5)$.
2. The line integral of the vector field $\vec{F}(x, y) = \sin(x^2)\vec{i} + (3x - y)\vec{j}$ along the triangle with vertices $(-1, 2)$, $(4, 2)$ and $(4, 5)$.
3. The area of the region enclosed by the curve $\vec{r}(t) = \langle \sin t \cos t, \sin t \rangle$, $0 \leq t \leq \pi$.
4. The line integral of the vector field $\vec{F}(x, y) = x^3\vec{i} + (5x + e^y \sin y)\vec{j}$ along $\partial\mathcal{D}$ where $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 + y^2 \leq 25\}$.

Remark 5.50 The area of a region \mathcal{D} can be computed, for instance, as

$$\mathcal{A}(\mathcal{D}) = \frac{1}{2} \oint_{\partial\mathcal{D}} \langle -y, x \rangle ds.$$

There are more ways to compute the area using Green's Theorem.

5.5 Surface Integrals

Remark 5.51 A surface integral is similar to a line integral, but the integration is over a surface not a curve.

Definition 5.52 A parameterized surface is a map $\vec{X}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ from a domain $\mathcal{D} \subset \mathbb{R}^2$ to \mathbb{R}^3 . The image of \mathcal{D} is the surface $S = \vec{X}(\mathcal{D})$.

Definition 5.53 We say that a surface $\vec{X}(u, v)$ is regular if the vector $\vec{X}_u \times \vec{X}_v$ never vanishes. The vector field $\vec{X}_u \times \vec{X}_v$ is the normal vector field to the surface.

Definition 5.54 The surface integral of a function f over a surface S parameterized by $\vec{X}(u, v)$ is

$$\int \int_S f \, dS = \int \int_{\mathcal{D}} f(\vec{X}(u, v)) \|\vec{X}_u \times \vec{X}_v\| \, dA,$$

where \mathcal{D} is the domain of the parameterization.

Definition 5.55 Let $\vec{X}(u, v)$ be a smooth parameterization of a surface S with $(u, v) \in \mathcal{D}$. The surface area of S is

$$\mathcal{A}(S) = \int \int_{\mathcal{D}} \|\vec{X}_u \times \vec{X}_v\| \, dA.$$

Example 5.56 Compute the surface area of a sphere of radius ρ .

Example 5.57 Find the area of the surface generated by rotating the curve $y = f(x)$, $x \in (a, b)$, around the x -axis. Answer:

$$\mathcal{A}(S) = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx.$$

Example 5.58 Calculate the surface integral $\int \int_S (x+y^2) \, dS$ where S is the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 3$.

Definition 5.59 An orientation on a surface S is the choice of a unit normal vector \vec{N} on S . We say that S has the positive orientation if \vec{N} points outward (of any convex domain).

Definition 5.60 Let \vec{F} be a vector field in \mathbb{R}^3 defined on a domain that contains a surface S oriented with a unit normal vector \vec{N} . The flux of \vec{F} (vector surface integral) along S is defined as the surface integral

$$\int \int_S \vec{F} \, dS = \int \int_S \vec{F} \cdot \vec{N} \, dS = \int \int_{\mathcal{D}} \vec{F}(\vec{X}(u, v)) \cdot \vec{X}_u \times \vec{X}_v \, dA,$$

where $\vec{X}(u, v)$ is the parameterization of S .

Example 5.61 Calculate the flux of the vector field $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j}$ over $\vec{X}(u, v) = \langle u, v^2 - u, u + v \rangle$, $0 \leq u \leq 3$ and $0 \leq v \leq 4$.

5.6 Divergence Theorem

Theorem 5.62 (Divergence Theorem) Let B be a solid in space whose boundary is a surface S and assume S is positively oriented (i.e., outward unit normal). Then, for a smooth vector field \vec{F} defined on B ,

$$\int \int_S \vec{F} \, dS = \int \int \int_B \nabla \cdot \vec{F} \, dV.$$

Example 5.63 Let $\vec{F}(x, y, z) = (x - y)\vec{i} + (x + z)\vec{j} + (z - y)\vec{k}$ and S be the cone $x^2 + y^2 = z^2$, $0 \leq z \leq 1$ with the top disc. Verify the Divergence Theorem.

Remark 5.64 The Divergence Theorem allows us to give an interpretation of the divergence vector field. Let B_r be a ball of radius r (very small) centered at P and S_r be the boundary of B_r . The flux of a vector field \vec{F} across S_r can be approximated as

$$\int \int_{S_r} \vec{F} dS = \int \int \int_{B_r} \nabla \cdot \vec{F} dV \simeq \nabla \cdot \vec{F}(P) \mathcal{V}(B_r),$$

and we can conclude that

$$\nabla \cdot \vec{F}(P) = \lim_{r \rightarrow 0} \frac{1}{\mathcal{V}(B_r)} \int \int_{S_r} \vec{F} dS.$$

If $\nabla \cdot \vec{F}(P)$ is positive (respectively, negative) it means that the vector field goes outside (resp., inside) the ball.

Example 5.65 Compute $\int \int_S \vec{F} dS$ where S is the cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$ and

$$\vec{F}(x, y, z) = \left(\frac{x^3}{3} + yz \right) \vec{i} + \left(\frac{y^3}{3} - \sin(xz) \right) \vec{j} + (z - x - y) \vec{k}.$$

5.7 Stokes' Theorem

Theorem 5.66 (Stokes' Theorem) Let S be a surface with boundary C , which is simple and closed, positively oriented. Then, for a smooth vector field \vec{F} defined on a domain containing S ,

$$\oint_C \vec{F} ds = \int \int_S \nabla \times \vec{F} dS.$$

Example 5.67 Let $\vec{F}(x, y, z) = y\vec{i} + 2z\vec{j} + x^2\vec{k}$ and S be $z = 4 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' Theorem.

Remark 5.68 If the surface S is a planar region \mathcal{D} , Stokes' Theorem is just Green's Theorem.

Example 5.69 Compute $\int \int_S \nabla \times \vec{F} dS$ where $\vec{F}(x, y, z) = z\vec{i} + x\vec{j} + y\vec{k}$ and for any surface S whose boundary is the circle of radius 1 centered in the origin of the xz -plane.

5.8 Exercises

1. Compute $\int \int_S z^2 dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$ and the disc enclosed by the intersection of the plane and the sphere.

Solution: $37\pi/3$.

2. Let $\vec{F}(x, y, z) = xy\vec{i} + (x^2 + y^2 + z^2)\vec{j} + yz\vec{k}$ be a vector field and C be the boundary of the parallelogram with vertices $(0, 0, 1)$, $(0, 1, 0)$, $(2, 0, -1)$ and $(2, 1, -2)$. Compute $\int_C \vec{F} ds$.
Hint: Use Stokes' Theorem.

Solution: 3.

3. Consider the vector field $\vec{F}(x, y, z) = x^2y^3\vec{i} + \vec{j} + 2\vec{k}$ and let C be the intersection of the cylinder $x^2 + y^2 = 4$ and the half-sphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$. Compute $\int_C \vec{F} ds$.

Solution: -8π .

Review Problems

- Let \mathcal{D} be the region bounded by the triangle of vertices $(-3, 0)$, $(0, 0)$ and $(0, 3)$ and consider the vector field $\vec{F}(x, y) = (xy^2 + x^2)\vec{i} + (4x - 1)\vec{j}$.
 - Compute the (vector) line integral of \vec{F} along the triangle $\partial\mathcal{D}$, using the definition of line integrals. (Answer: $99/4$.)
 - Compute the (vector) line integral of \vec{F} along the triangle $\partial\mathcal{D}$, using Green's Theorem. (Answer: $99/4$.)
- Let \mathcal{D} be the region below $x^2 + y^2 = 2$ and above $y = x^2$.
 - Compute the area of \mathcal{D} using double integrals. (Answer: $\pi/2 + 1/3$.)
 - Compute the area of \mathcal{D} applying Green's Theorem. (Answer: $\pi/2 + 1/3$.)
 - Compute the line integral of $\vec{F}(x, y) = (x^2 - y)\vec{i} + (x - \sin^2 y)\vec{j}$ along the positively oriented boundary of \mathcal{D} . (Answer: $\pi + 2/3$.)
- Let \mathcal{D} be the region enclosed by $x^2 + y^2 = 25$ and such that $x \leq 0$. Consider the vector field $\vec{F}(x, y) = yx^2\vec{i} - x^2\vec{j}$.
 - Compute the (vector) line integral of \vec{F} along the boundary $\partial\mathcal{D}$, using the definition of line integrals. (Answer: $500/3 - 625/8\pi$.)
 - Compute the (vector) line integral of \vec{F} along the boundary $\partial\mathcal{D}$, using Green's Theorem. (Answer: $500/3 - 625/8\pi$.)
- Consider the vector field $\vec{F}(x, y, z) = (-4x + 4 - 4y^2 + 2z^3)\vec{i} + (-8xy + 2z)\vec{j} + (6xz^2 + 2y)\vec{k}$.
 - Is the vector field \vec{F} conservative? (Answer: Yes.)
 - If possible, find all the scalar potentials. (Answer: $f(x, y, z) = -2x^2 + 4x - 4xy^2 + 2xz^3 + 2yz + c$.)
 - Compute the (vector) line integral of \vec{F} over the closed curve C which is the intersection of the cylinder $x^2 + y^2 = 2x$ and the sphere $x^2 + y^2 + z^2 = 4$. (Answer: 0.)
 - Compute the (vector) line integral of \vec{F} over the curve C parameterized by $\vec{r}(t) = \langle t^3 - t^2, \sin^2(\pi t) + t, 1 - \cos(\pi t) \rangle$, for $t \in [0, 1]$. (Answer: 4.)
- Consider the vector field $\vec{F}(x, y, z) = (2x + \frac{1}{x} + ye^{xy})\vec{i} + (z + xe^{xy})\vec{j} + y\vec{k}$.
 - Is the vector field \vec{F} conservative? (Answer: Yes, if $x > 0$.)
 - If possible, find all the scalar potentials. (Answer: $f(x, y, z) = x^2 + \log x + e^{xy} + yz + c$.)
 - Compute the (vector) line integral of \vec{F} along every possible closed curve in the space. (Answer: 0.)

- (d) Compute the (vector) line integral of \vec{F} along every possible curve joining the points $(1, 0, 0)$ and $(2, 0, 1)$. (Answer: $3 + \log 2$.)
6. Let B be the region enclosed by $z = x^2 + y^2$ and the plane $z = 9$.
- Compute the volume of B , using triple integrals. (Answer: $81\pi/2$.)
 - Compute the volume of B , using the Divergence Theorem. (Answer: $81\pi/2$.)
 - Compute the flux of the vector field $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + z\vec{k}$ over the boundary ∂B . (Answer: $81\pi/2$.)
7. Compute the flux of the vector field $\vec{F}(x, y, z) = yx^2\vec{i} + (xy^2 - 3z^4)\vec{j} + (x^3 + y^2)\vec{k}$ over the boundary of the region enclosed by the sphere of radius 4, $z \leq 0$ and $y \leq 0$. (Answer: 0.)
8. Compute the flux of the vector field $\vec{F}(x, y, z) = \vec{i} + z\vec{j} + 6x\vec{k}$ over the sphere of radius 3. (Answer: 0.)