Binormal Evolution of Generalized Elastic Curves

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Geometry of Submanifolds. Celebrating Bang-Yen Chen's 80th Anniversary

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Motivation

- Da Rios (1906): Modeled the movement of a thin vortex filament in a viscous fluid by the motion of a curve propagating according to

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  the localized induction equation.
- This evolution represents a binormal flow: $X_t = \kappa B$.
- Da Rios also obtained the so-called Da Rios equations for the vortex filament:

$$\begin{aligned} \kappa_t &= -2\kappa_s \tau - \kappa \tau_s \,, \\ \tau_t &= \left(\frac{\kappa_{ss}}{\kappa} + \frac{\kappa^2}{2} - \tau^2 \right)_s \,. \end{aligned}$$

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- Hasimoto (1972): Discovered that the localized induction equation is equivalent to the non-linear Schrödinger equation (a soliton equation).
- Hasimoto (1971): Found that if the evolution according to the localized induction equation is by isometries the initial vortex filament must be a classical elastic curve.
- Question: What happens if we consider a binormal flow of the type

$$X_t = \mathcal{F}(\kappa) B \,,$$

for arbitrary smooth functions $\mathcal{F}$?

Binormal Evolution Surfaces

Given a smooth map $X : U \subseteq \mathbb{R}^2 \longrightarrow M_r^3(\rho)$, we consider the evolution problem

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- The corresponding immersed surface (U, X) in $M^3(\rho)$ is called a binormal evolution surface with velocity $\dot{P}(\kappa)$.
- We can employ the theory of submanifolds to compute the Gauss-Codazzi equations and extend the classical Da Rios equations.

Initial Filament

For the binormal flow $X_t = \dot{P}(\kappa)B$, we have

Theorem (Garay & P., 2016)

Traveling wave solutions of the Gauss-Codazzi equations correspond with the evolution under isometries and slippage of a general Kirchhoff centerline.

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In particular, if there is no slippage then the initial filament is critical for

$$\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)\, ds\,.$$

It is a generalized elastic curve.

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The vector field along the critical curve γ defined by

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Theorem (Langer & Singer, 1984)

Killing vector fields along curves can uniquely be extended to Killing vector fields in $M^3(\rho)$.

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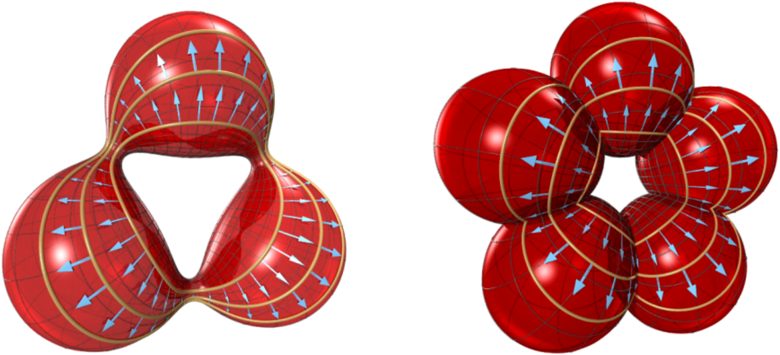
Theorem (Arroyo, Garay & P., 2017)

By construction S_γ is a ξ-invariant surface. If γ is planar ($\tau = 0$), S_γ is either flat isoparametric or a rotational surface.

Illustration



(Arroyo, Garay & A. P., 2019)

Application I: CMC Surfaces

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Theorem (Arroyo, Garay & P., 2018)

Invariant CMC surfaces in $M^3(\rho)$ are binormal evolution surfaces whose initial filaments are critical for

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In particular, if $\gamma \subset \mathbb{S}^2(\rho)$, we have (Arroyo, Garay & P., 2019):

- There exist non-trivial closed critical curves for any value of μ.
- If γ is a simple closed critical curve, then $\mu \neq -\sqrt{\rho/3}$ is negative.

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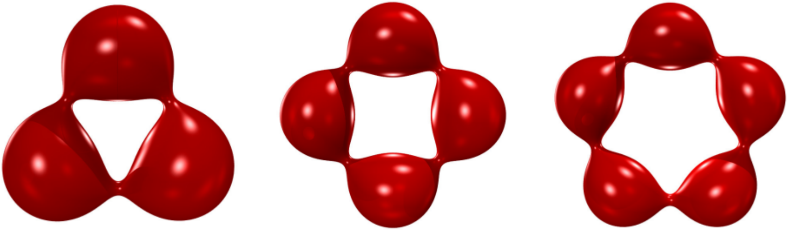

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- Coincides with previous results of Perdomo and Ripoll.
- Verify the Lawson's conjecture (proved by Brendle in 2013).
- After Pinkall-Sterling's conjecture (proved by Andrews-Li in 2015), these are all embedded CMC tori.

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Non-CMC biconservative surfaces in $M^3(\rho)$ are rotational.
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Theorem (Montaldo & P., 2023)

Non-CMC biconservative surfaces in $M^3(\rho)$ are binormal evolution surfaces whose initial filaments are critical for

$$\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^{1/4}\, ds\,.$$

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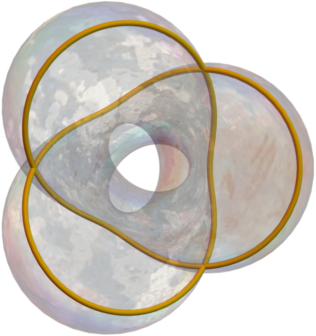

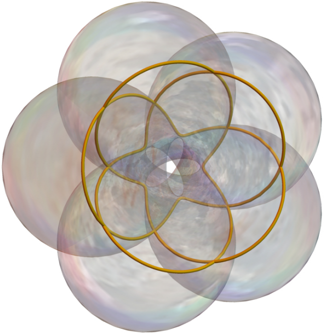

THE END

Thank You!