1. Notation

In the rest of the document, we shall analyze a power distribution network represented as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of power lines connecting these nodes. An edge (k, t) indicates a power line that connects node k to node t. We define $\mathcal{N}_k$ as the set of nodes directly connected to node k, and $\mathcal{A}_k$ as the set of edges that either originate from or terminate at node k.

2. Second Order Conic Programming Power Flow Formulation

This work considers the power flow formulation for distribution networks introduced in [2]. In that research, the following variables are adopted: $c_{k,k} = e_k^2 + f_k^2$, $c_{k,t} = e_k e_t + f_k f_t$ and $s_{k,t} = e_k f_t - f_k e_t$ where e_k and f_k represents the real and imaginary components of voltage at node k, v_k, respectively. Note that $c_{k,t} = c_{t,k}$ and $s_{k,t} = -s_{t,k}$ for every pair of connected nodes.

Using the notation described above and considering the single-line equivalent circuit depicted in Figure 1, the power flowing from node k to node t can be expressed in a linear form as follows:

$$
\begin{aligned}
P_{k,t} &= G_{k,t} c_{k,k} - G_{k,t} c_{k,t} - B_{k,t} s_{k,t}, \\
Q_{k,t} &= B_{k,t} c_{k,k} - B_{k,t} c_{k,t} + G_{k,t} s_{k,t},
\end{aligned}
$$

where $c_{k,t}^2 + s_{k,t}^2 = c_{k,k} c_{t,t}$ constraints the new problem variables.

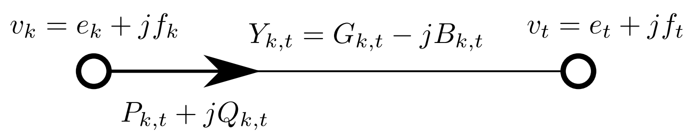

Figure 1: Distribution line model.

Thus, the power flow problem can be reformulated as a second order conic program as:

$$
\max \sum_{\forall (k,t) \in \mathcal{E}} c_{k,t}, \tag{1}
$$

$$
\begin{aligned}
s.t. P_{g,k} - P_{l,k} &= \sum_{\forall t \in \mathcal{N}_k} P_{k,t}, \quad \forall k \in \mathcal{V}, \qquad (2) \\
Q_{g,k} - Q_{l,k} &= \sum_{\forall t \in \mathcal{N}_k} Q_{k,t}, \quad \forall k \in \mathcal{V}, \\
c_{k,t}^2 + s_{k,t}^2 &\leq c_{k,k} c_{t,t}, \quad \forall (k,t) \in \mathcal{E},
\end{aligned}
$$

where $P_{g,k}$ and $Q_{g,k}$ are the active and reactive power injections at node k; and $P_{l,k}$ and $Q_{l,k}$ are, respectively, the active and reactive power demanded at that node. As evident from the analysis, there are two linear equations corresponding to each node, excluding the slack node, and one nonlinear equation for each line in the network that corresponds to a second order rotated conic equality constraint.

In sight of problem formulation (1), the set of searched variables can be stacked in column vector $x = [\text{col}(c_{k,k})_{k \in \mathcal{V}}^\top, \text{col}(c_{k,t})_{(k,t) \in \mathcal{E}}^\top, \text{col}(s_{k,t})_{(k,t) \in \mathcal{E}}^\top]^\top \in \mathbb{R}^n$ in such a way that the problem can be rewritten in a compact form as:

$$
\begin{aligned}
Ax &= b, \\
g_{(r,t)}(x) &\leq 0, \quad \forall (r,t) \in \mathcal{E},
\end{aligned} \tag{3}
$$

where linear power flow equations are embedded in $Ax = b$; and $g_{(r,t)}(x)$ are a set of inequalities that constitutes the rotated quadratic cone constraints [1].

3. SOCP optimization algorithm

The original problem:

$$\begin{aligned} \min \quad & f^\top x, \\ s.t. \quad & Ax = b, \\ & g_{(r,t)}(x) \leq 0, \quad \forall (k,t) \in \mathcal{E}. \end{aligned}$$

We introduce new vectors of variables $z_{(k,t)}$ to divide the problem:

$$\begin{aligned} \min \quad & f^\top x, \\ s.t. \quad & Ax = b, \\ & z_{(k,t)} = S_{(k,t)} x, \quad \forall (k,t) \in \mathcal{E}, \\ & g_{(r,t)}(z_{(k,t)}) \leq 0, \quad \forall (r,t) \in \mathcal{E}. \end{aligned}$$

where $S_{(k,t)}$ is a selection matrix that takes from x the variables that constitute the cone associated to line (r,t).

Based on above formulation, the augmented Lagrangian is constructed:

$$\begin{aligned} \mathcal{L}(x,z,y,u) \quad = \quad & f^\top x + I_{g_{(k,t)}}(z_{(k,t)}) + y^\top (Ax - b) + \frac{\rho}{2} ||Ax - b||_2^2 \\ + \quad & \sum_{\forall (k,t) \in \mathcal{E}} u_{(k,t)}^\top (S_{(k,t)} x - z_{(k,t)}) + \sum_{\forall (k,t) \in]} \frac{\rho}{2} ||S_{(k,t)} x - z_{(k,t)}||_2^2, \end{aligned}$$

where I is the indicator function that returns 0 if the constraints are fulfilled and ∞ otherwise.

Next, the problem is solved relying in the Alternating Direction Method of the Multipliers (ADMM) method:

1. x-update: deducir
2. $z_{(k,t)}$-update:
$$z_{(k,t)}^{k+1} \quad = \quad arg \min_{z_{(k,t)} \in \mathcal{K}(k,t)} expresion = P_{\mathcal{K}}(k,t)\,(algomas)$$
3. dual-update:
$$y^{k+1} = y^k + \rho(Ax^{k+1} - b)$$
$$u_{(k,t)}^{k+1} = u_{(k,t)}^k + \rho(S_{(k,t)} x - z_{(k,t)})$$

The projection of $z_{(k,t)}$ in the area defined by the intersection of all the considered needs to be studied. Consider again the standard form of the rotated quadratic cone $2uv \geq \alpha^2$ where we have denoted $\alpha = ||w||_2$. Then:

- If $u, v \geq 0$ and $2uv \geq \alpha^2$, the solution is inside the cone and, therefore, no projection is needed.
- If $u, v \leq 0$, set all the variable to zero.
- If only $u \leq 0$ or $v \leq 0$, set that variable and w to zero.
- If $u, v \geq 0$ and $\mathbf{2uv < \boldsymbol{\alpha}^2}$:
  - Solve the problem:

$$\begin{aligned} \min \quad & ||(u', v', w') - (u, v, w)||_2^2 \\ s.t. \quad & (u', v', w') \in \mathbf{K} \end{aligned}$$

– We assume that the new point is in the limit of the rotated cone, thus, $2u'v' = ||w'||_2^2$.

– defining the Lagragnian of the problem it yields:

$$\mathcal{L}(u, v, w, \mu) = ||(u', v', w') - (u, v, w)||_2^2 + \mu \left(2u'v' - ||w'||_2^2\right)$$

– Solving the problem it yields:

$$\begin{aligned} u' &= \frac{u - \mu v}{1 - \mu^2}, \\ v' &= \frac{v - \mu u}{1 - \mu^2}, \\ w' &= \frac{1}{1 - \mu} w, \end{aligned}$$

where the value of μ can be obtained from the second order equation $a\mu^2 + b\mu + c = 0$ being:

$$\begin{aligned} a &= 2uv - ||w||_2^2, \\ b &= -2 \left(u^2 + v^2 + ||w||_2^2\right), \\ c &= 2uv - ||w||_2^2. \end{aligned}$$

References

[1] E. D. Andersen, C. Roos, and T. Terlaky. On implementing a primal-dual interior-point method for conic quadratic optimization. *Mathematical Programming*, 95:249–277, 2003.

[2] R. A. Jabr. Radial distribution load flow using conic programming. *IEEE transactions on power systems*, 21(3):1458–1459, 2006.