Calculating the Efficient Frontier: Part 2

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A Matrix Based Approach for Calculating Portfolio Weights

In a previous post, I showed how to calculate and plot the efficient frontier for a set of risky assets. In this post, I extend the previous example and show how to use Octave or Matlab to calculate the portfolio weights for each of the various risky assets for any point on the efficient frontier.

In my next post, I'll conclude this series on the efficient frontier by adding a risk free asset, showing the calculation for the tangency portfolio, and demonstrating how this creates a new frontier.

Calculating the Weights for Key Points on the Efficient Frontier

We can calculate the weights for any point on the efficient frontier once we know the weights for any two points on the efficient frontier.

Two easy points to calculate are the global minimum variance portfolio and the tangency portfolio for the case where the risk free rate is assumed to be zero.

Global Minimum Variance Portfolio

A point of particular interest on the efficient frontier is the "global minimum variance portfolio". This portfolio is the point on the efficient frontier which has the minimum variance or standard deviation.

We can solve for the mean and variance of the global minimum variance portfolio by setting the derivative of the equation for the variance to zero and solving for μ. The equations for calculating the A,B,C and Δ values are given in the previous post.

$$\sigma^2 = \frac{A\mu^2 - 2B\mu + C}{\Delta}$$

Setting the derivative with respect to μ to zero gives:

$$0 = \frac{2A\mu - 2B}{\Delta}$$

Running through all the algebra gives:

$$\mu_g = \frac{B}{A}$$

$$\sigma_g^2 = \frac{1}{A}$$

Finally, the portfolio weights for the global minimum variance portfolio are given by:

$$\mathbf{w_g} = \frac{\mathbf{S^{-1}1}}{A}$$

Calculating a Second Point on Efficient Frontier (Tangency Portfolio with R=0%)

We need two points on the efficient frontier to calculate any other point. We have the global minimum variance portfolio as a first point, and a second easy point to calculate is the tangency portfolio for the case where the risk-free rate is set to zero.

The equations for the expected return, standard deviations, and weights for this portfolio are given below:

$$\mathbf{w_d} = \frac{\mathbf{S^{-1}\bar{z}}}{B}$$

$$\mu_d = \mathbf{w_d'\bar{z}}$$

$$\sigma_d^2 = \mathbf{w_d'Sw_d}$$

The Two-Fund Separation Theorem

The two fund separation theorem states that all minimum variance portfolios on the efficient frontier are combinations of only two distinct portfolios. So, any two points on the mean variance frontier will span the set.

To calculate any point on the frontier, we can use the minimum variance portfolio and the tangency portfolio when R=0, but as a first step we need to calculate two intermediate values using the target expected return or μ:

$$\lambda = \frac{C-\mu B}{\Delta}$$

$$\gamma = \frac{\mu A-B}{\Delta}$$

With these values, and the portfolio weights for the two portfolios calculated above, we can find the weights for any point on the efficient frontier.

$$\mathbf{w}^* = (\lambda A)\,\mathbf{w_g} + (\gamma B)\,\mathbf{w_d}$$

Portfolio Weights Example

As an example, we can use the four assets which I used for the example in the previous post. As a reminder, the expected returns and covariance matrix for these assets are shown below:

$$\mathbf{\bar{z}} = \begin{bmatrix} 14 \\ 12 \\ 15 \\ 7 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 185 & 86.5 & 80 & 20 \\ 86.5 & 196 & 76 & 13.5 \\ 80 & 76 & 411 & -19 \\ 20 & 13.5 & -19 & 25 \end{bmatrix}$$

Using these assets, we can use the equations shown above to calculate the portfolio weights for the two reference portfolios, and we can then use these two portfolios to calculate portfolio weights for any point on the efficient frontier.

Global Minimum Variance Portfolio:

The equations presented above are implemented in an Octave script, and the portfolio weights, standard deviation, and expected return for the Global Minimum Variance Portfolio are shown below. Note that a negative portfolio weight indicates a short position in that security.

$$\mathbf{w_g} = \begin{bmatrix} -0.0399 \\ 0.0223 \\ 0.0966 \\ 0.9210 \end{bmatrix}$$

$$\mu_g = 7.60\%$$

$$\sigma_g^2 = 20.69$$

$$\sigma_g = 4.55\%$$

Note that this standard deviation is the minimum standard deviation for any point on the efficient frontier.

Tangency Portfolio when R=0:

The portfolio weights, standard deviation, and expected return for the tangency portfolio when the risk free rate (R) is assumed to be zero are shown here.

$$\mathbf{w_d} = \begin{bmatrix} 0.0486 \\ 0.0451 \\ 0.1180 \\ 0.7883 \end{bmatrix}$$

$$\mu_d = 8.51\%$$

$$\sigma_d^2 = 23.16$$

$$\sigma_d = 4.81\%$$

Calculating Portfolio Weights for an Arbitrary Expected Return:

For this example, we will assume that our target portfolio return is 14%.

Using this target return, the target portfolio weights can be calculated. Again, remember that a negative portfolio weight indicates a short position in that security.

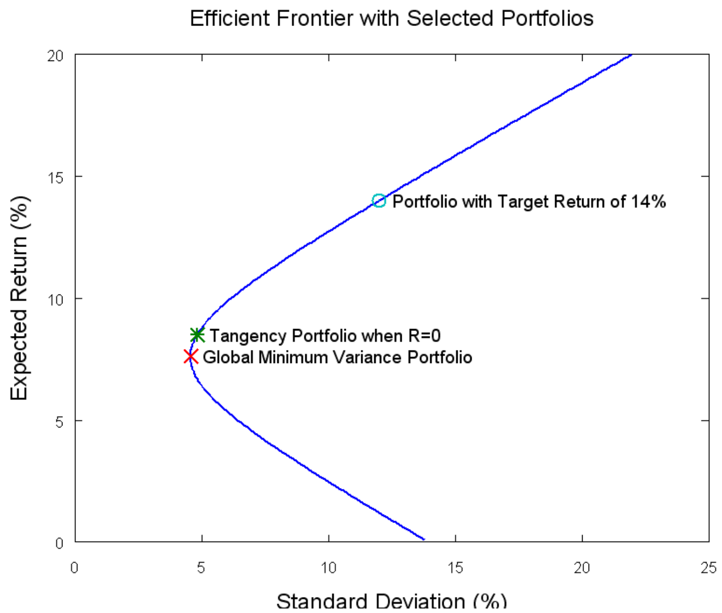
$$\mathbf{w}^* = \begin{bmatrix} 0.5857 \\ 0.1836 \\ 0.2477 \\ -0.0171 \end{bmatrix}$$

$$\mu^* = 14.00\%$$

$$\sigma_*^2 = 143.72$$

$$\sigma^* = 11.99\%$$

Plot of Efficient Frontier with Key Points:



Octave Code:

The script below can be run in Matlab or Octave to calculate, plot, and determine portfolio weights for any point on the efficient frontier.

By updating the expected returns ('zbar') and covariance matrix ('S"), the script can be used to compute and plot the efficient frontier for any desired set of assets. It will also calculate the portfolio weights for a point on the efficient frontier with a specified target return ('mu_tar'). The portfolio weights will be returned in the variable 'w_s'.

```
clear all;
close all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Mean Variance Optimizer Inputs (Update this section with your
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% S is matrix of security covariances
S = [185 86.5 80 20; 86.5 196 76 13.5; 80 76 411 -19; 20 13.5 -1

% Vector of security expected returns
zbar = [14; 12; 15; 7]

% Target Return
mu_tar = 14

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculating Variables
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Unity vector..must have same length as zbar
unity = ones(length(zbar),1)

% Vector of security standard deviations
stdevs = sqrt(diag(S))

A = unity'*S^-1*unity
B = unity'*S^-1*zbar
C = zbar'*S^-1*zbar
D = A*C-B^2

% Calculate Lambda and Gamma
lambda_target = (C - mu_tar*B)/D;
gamma_target =  (mu_tar*A-B)/D;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Efficient Frontier
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
mu = 1:200;
mu = mu/10;

minvar = ((A*mu.^2)-2*B*mu+C)/D;
minstd = sqrt(minvar);

plot(minstd,mu,stdevs,zbar,'*')
title('Efficient Frontier with Individual Securities','fontsize'
ylabel('Expected Return (%)','fontsize',18)
xlabel('Standard Deviation (%)','fontsize',18)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Global Minimum Variance Portfolio
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Mean and Variance of Global Minimum Variance Portfolio
mu_g = B/A
var_g = 1/A
std_g = sqrt(var_g)

% Minimum Variance Portfolio Weights
w_g = (S^-1*unity)/A

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Tangency Portfolio with a Risk Free Rate = 0
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Weights for Tangency Portfolio, R=0
w_d = (S^-1*zbar)/B;

% Expected Return of Tangency Portfolio
mu_d = w_d'*zbar;

% Variance and Standard Deviation of Tangency Portfolio
var_d = w_d'*S*w_d;
std_d = sqrt(var_d);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Portfolio with Expected Return = 14%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Weights for portfolio with target return = 14%

w_s = (lambda_target*A)*w_g + (gamma_target*B)*w_d;

% Expected Return of Target Portfolio (should match target)
mu_s = w_s'*zbar;

% Variance and Standard Deviation of target portfolio
var_s = w_s'*S*w_s;
std_s = sqrt(var_s);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plot Efficient Frontier and Key Points
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure
plot(minstd,mu,'linewidth',2,std_d,mu_d,'*','linewidth',2,std_g,
text(0.5+std_g,mu_g,'Global Minimum Variance Portfolio','fontsiz
text(0.5+std_d,mu_d,'Tangency Portfolio when R=0','fontsize',14)
text(0.5+std_s,mu_s,'Portfolio with Target Return of 14%','fonts
title('Efficient Frontier with Selected Portfolios','fontsize',1
ylabel('Expected Return (%)','fontsize',18)
xlabel('Standard Deviation (%)','fontsize',18)
```