

Calculating the Efficient Frontier: Part 3

June 21, 2011

Posted by calcrv at 9:52 pm

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Calculating the Efficient Frontier with a Risk-Free Asset

So, I've been feeling a little bit bogged down on this efficient frontier topic, and I'm anxious to wrap things up with this post and move on to something more original!

In the previous two posts, I have demonstrated how to [calculate the efficient frontier for any number of risky assets](#), and I've shown how to [calculate the portfolio weights for each point on the efficient frontier](#).

In this post, I will conclude this series on the efficient frontier (finally!) by demonstrating how the frontier changes with the addition of a risk-free asset.

An example Octave script is provided for calculating and plotting the efficient frontier, capital allocation line, and tangency portfolio.

Tangency Portfolio with a Risk-Free Asset w/return R

The tangency portfolio is the intercept point if we draw a tangent line from the risk-free rate of return (on the y-axis) to the efficient frontier for risky assets.

The equations used to calculate the portfolio weights, expected return, and variance for the tangency portfolio are shown in this section.

The variables used in these equations are described in more detail in the previous posts. However, to review briefly, $\bar{\mathbf{z}}$ is the vector of expected returns for the risky assets, and \mathbf{S} is the covariance matrix for the returns of these assets. The values for A, B, C, and Δ are intermediate values calculated using the portfolio statistics and the equations for these values are shown in my [initial post](#) on this topic. The value R is the return on the risk-free asset. The value \mathbf{w}_t is a vector showing the weight of each asset (weights sum to 1) in the tangency portfolio, and \bar{Z}_t and σ_t^2 are the expected return and variance of the tangency portfolio.

$$\mathbf{w}_t = \frac{\mathbf{S}^{-1}(\bar{\mathbf{z}} - R\mathbf{1})}{B - AR}$$

$$\bar{Z}_t = \mathbf{w}_t' \bar{\mathbf{z}} = \frac{C - BR}{B - AR}$$

$$\sigma_t^2 = \mathbf{w}_t' \mathbf{S} \mathbf{w}_t = \frac{C - 2RB + R^2 A}{(B - AR)^2}$$

Calculating a Point on the Capital Allocation Line

The Capital Allocation Line or CAL is the tangent line from the risk-free rate of return (on the y-axis) to the efficient frontier for risky assets. The points on this line represent the highest expected return we can get at a given level of risk when we have access to a risk-free asset.

All points on this line are a combination of the tangency portfolio and the risk-free asset. If the weight of the portfolio in the tangency portfolio is given as y then the weight in the risk-free asset must be $1 - y$. The value of y can be calculated for any desired expected return, \bar{Z}_c on the CAL using this equation:

$$y = \frac{\bar{Z}_c - R}{Z_t - R}$$

Using this y -value we can calculate the standard deviation at this point on the CAL:

$$\sigma_c = y\sigma_t$$

By combining these equations, we get the general equation for the CAL:

$$\bar{Z}_c = R + \left[\frac{Z_t - R}{\sigma_t} \right] \sigma_c$$

Example Calculation using Octave

As an example, let's consider the four risky assets used in the efficient frontier examples in the previous posts. The expected return vector and covariance matrix for these assets is given here:

$$\bar{\mathbf{z}} = \begin{bmatrix} 14 \\ 12 \\ 15 \\ 7 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 185 & 86.5 & 80 & 20 \\ 86.5 & 196 & 76 & 13.5 \\ 80 & 76 & 411 & -19 \\ 20 & 13.5 & -19 & 25 \end{bmatrix}$$

The risk-free rate will be assumed to be 3%, and the target return will be set to 14%.

If we implement the equations for the tangency portfolio and CAL in Octave, we can calculate the portfolio weights for the tangency portfolio (\mathbf{w}_t), and the weight of the total portfolio which should be in the tangency portfolio (y) and the risk-free asset ($1 - y$) to achieve the target expected return.

The portfolio weights for the tangency portfolio are shown here:

$$\mathbf{w}_t = \begin{bmatrix} 0.1063 \\ 0.0600 \\ 0.1319 \\ 0.7019 \end{bmatrix}$$

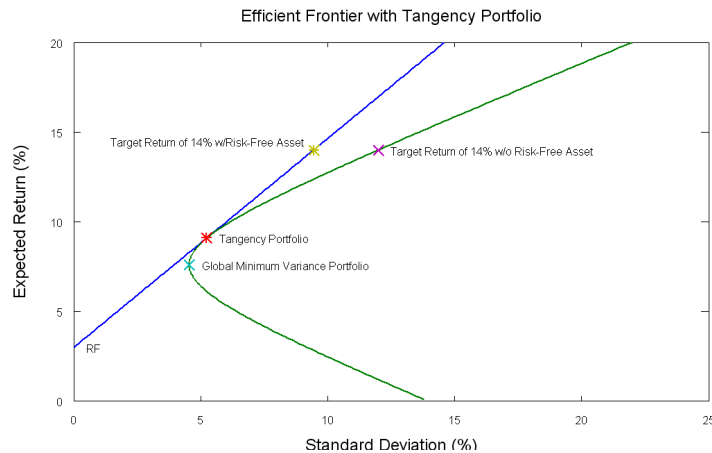
The weight in the tangency portfolio is:

$$y = 1.804$$

Since this value is greater than 1, that means we have a short position in the risk-free asset with a weight of $1 - y$ or -0.804.

Note that this is a somewhat unrealistic scenario since we cannot borrow at the true risk-free rate. In practice our borrowing rate would be higher, and the actual risk-free rate would be used only for portfolios where we had a positive weight in the risk-free asset.

The plot of the efficient frontier, tangency portfolio, and the CAL are shown here:



Octave Code:

This Octave code will calculate and plot the efficient frontier, tangency portfolio and the CAL. The script can be modified for a different set of assets by updating the expected returns vector, the covariance matrix, the target return, and the risk free rate. The script will also work in Matlab.

```
1 clear all;
2 close all;
3
4 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5 % Mean Variance Optimizer Inputs
6 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
7
8 % S is matrix of security covariances
9 S = [185 86.5 80 20; 86.5 196 76 13.5; 80 76 411 -19; 20 13.5 -1
10
11 % Vector of security expected returns
12 zbar = [14; 12; 15; 7]
13
14 % Risk Free Asset Return
15 R = 3
16
17 % Target Return
18 mu_tar = 14
19
```

```

20 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
21 % Calculating Variables
22 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
23
24 % Unity vector..must have same length as zbar
25 unity = ones(length(zbar),1)
26
27 % Vector of security standard deviations
28 stdevs = sqrt(diag(S))
29
30 A = unity'*S^-1*unity
31 B = unity'*S^-1*zbar
32 C = zbar'*S^-1*zbar
33 D = A*C-B^2
34
35 % Calculate Lambda and Gamma
36 lambda_target = (C - mu_tar*B)/D;
37 gamma_target = (mu_tar*A-B)/D;
38
39 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
40 % Efficient Frontier
41 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
42 mu = 1:200;
43 mu = mu/10;
44
45 minvar = ((A*mu.^2)-2*B*mu+C)/D;
46 minstd = sqrt(minvar);
47
48 plot(minstd,mu,stdevs,zbar,'*')
49 title('Efficient Frontier with Individual Securities','fontsize'
50 ylabel('Expected Return (%)','fontsize',18)
51 xlabel('Standard Deviation (%)','fontsize',18)
52
53 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
54 % Global Minimum Variance Portfolio
55 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
56
57 % Mean and Variance of Global Minimum Variance Portfolio
58 mu_g = B/A
59 var_g = 1/A
60 std_g = sqrt(var_g)
61
62 % Minimum Variance Portfolio Weights
63 w_g = (S^-1*unity)/A
64
65 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
66 % Tangency Portfolio with a Risk Free Asset
67 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
68
69 % Expected Return of Tangency Portfolio
70 ztan = (C-B*R)/(B-A*R);
71
72 % Variance and Standard Deviation of Tangency Portfolio
73 vartan = (C-2*R*B + R^2*A)/((B-A*R)^2);
74 stdtan = sqrt(vartan);
75
76 % Weights for Tangency Portfolio
77 w_tan = (S^-1*(zbar - R*unity))/(B-A*R)
78
79 % Tangency Line
80 mu_tan = mu(mu >= R);
81 minvar_rf = (mu_tan-R).^2/(C-2*R*B+A*R^2);
82 minstd_rf = sqrt(minvar_rf);
83
84 % Weights for w_d (tangency when R=0)
85 w_d = (S^-1*zbar)/B;
86
87 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
88 % Target Return Portfolio w/and w/o Risk Free Asset

```

```

89 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
90 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
91 % Weights for portfolio with target return = 14%, w/o risk-free
92
93 w_s = (lambda_target*A)*w_g + (gamma_target*B)*w_d;
94
95 % Expected Return of Target Portfolio (should match target)
96 mu_s = w_s'*zbar;
97
98 % Variance and Standard Deviation of target portfolio
99 var_s = w_s'*S*w_s;
100 std_s = sqrt(var_s);
101
102 % Weights for portfolio with target return = 14%, w/risk free as
103
104 y = (mu_tar - R)/(ztan-R);
105 stdtar = stdtan*y;
106
107 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
108 % Tangency Plot
109 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
110
111 figure
112 plot(minstd_rf,'linewidth',2,mu_tan,minstd,'linewidth',2,mu,stdt
113 text(0.5,R,'RF','fontsize',12);
114 text(0.5+std_g,mu_g,'Global Minimum Variance Portfolio','fontsiz
115 text(0.5+stdtan,ztan,'Tangency Portfolio','fontsize',12);
116 text(0.5+std_s,mu_s,'Target Return of 14% w/o Risk-Free Asset','
117 text(stdtar-8,mu_tar+0.5,'Target Return of 14% w/Risk-Free Asset
118 title('Efficient Frontier with Tangency Portfolio','fontsize',18
119 ylabel('Expected Return (%)','fontsize',18)
120 xlabel('Standard Deviation (%)','fontsize',18)

```