Lista III

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1.1 Q1

We are concerned with the following Hamiltonian:

$$H = \frac{\omega}{2}\sigma_z + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k(\sigma_+ b_k + \sigma_- b_k^\dagger) \tag{1.1.1}$$

We can identify H_0 as

$$H_0 = H_s + H_e = \frac{\omega}{2}\sigma_z + \Omega b_k^\dagger b_k \tag{1.1.2}$$

and the operator V that connects the two spaces as:

$$V = \lambda_k(\sigma_+ b_k + \sigma_- b_k^\dagger) \tag{1.1.3}$$

We know that the commutation relations for the Pauli Matrices are:

$$[\sigma_z, \sigma_+] = 2\sigma_+ \quad \text{and} \quad [\sigma_z, \sigma_-] = -2\sigma_- \tag{1.1.4}$$

And that $[b^\dagger b, b] = -b$ and $[b^\dagger b, b^\dagger] = b^\dagger$. With these relations in hand we can compute the effective Hamiltonian $V(t)$ connecting the two spaces, and it can be found by means of the BCH formula, as we've seen before:

$$V(t) = e^{iH_0 t} V e^{-iH_0 t} = V - t[H_0, V] + \frac{t^2}{2!}[H_0, [H_0, V]] - ... \tag{1.1.5}$$

Computing the first commutator, we find:

$$[H_0, V] = \lambda_k(\omega - \Omega)(\sigma_+ b_k - \sigma_- b_k^\dagger)$$

For the second commutator we find $\lambda_k(\omega - \Omega)^2(\sigma_+ b_k + \sigma_- b_k^\dagger)$, and so on. Thus, the operator $V(t)$ can be written as:

$$V(t) = \lambda_k \left(e^{i\Delta_k t}\sigma_+ b_k + e^{-i\Delta_k t}\sigma_- b_k^\dagger \right) \tag{1.1.6}$$

Therefore, the first term in the Nakajima - Zwanzig equation is:

$$V(t)\rho_S\rho_E V(t-s) = \lambda_k\lambda_q \left(e^{i\Delta_k t}\sigma_+ b_k + e^{-i\Delta_k t}\sigma_- b_k^\dagger \right) \rho_S\rho_E \left(e^{i\Delta_q (t-s)}\sigma_+ b_q + e^{-i\Delta_q (t-s)}\sigma_- b_q^\dagger \right)$$

which we can expanded as:

$$
\begin{aligned}
V(t)\rho_S\rho_E V(t-s) =& \lambda_k\lambda_q\Big[e^{i(\Delta_k+\Delta_q)t}e^{-i\Delta_q s}\sigma_+ b_k\rho_S\rho_E\sigma_+ b_q \\
&+ e^{i(\Delta_k-\Delta_q)t}e^{-i\Delta_q s}\sigma_+ b_k\rho_S\rho_E\sigma_- b_q^\dagger \\
&+ e^{-i(\Delta_k-\Delta_q)t}e^{-i\Delta_q s}\sigma_- b_k^\dagger\rho_S\rho_E\sigma_+ b_q \\
&+ e^{-i(\Delta_k+\Delta_q)t}e^{i\Delta_q s}\sigma_- b_k^\dagger\rho_S\rho_E\sigma_- b_q^\dagger\Big]
\end{aligned}
$$

We can see that this equation is quite similar to the one found in the lecture notes. Tracing it out in the environment space:

$$
\begin{aligned}
\mathrm{tr}_E[V(t)\rho_S\rho_E V(t-s)] =& \lambda_k\lambda_q\Big[e^{i(\Delta_k+\Delta_q)t}e^{-i\Delta_q s}\sigma_+\rho_S\sigma_+\langle b_q b_k\rangle \\
&+ e^{i(\Delta_k-\Delta_q)t}e^{-i\Delta_q s}\sigma_+\rho_S\sigma_-\langle b_q^\dagger b_k\rangle \\
&+ e^{-i(\Delta_k-\Delta_q)t}e^{-i\Delta_q s}\sigma_-\rho_S\sigma_+\langle b_q b_k^\dagger\rangle \\
&+ e^{-i(\Delta_k+\Delta_q)t}e^{i\Delta_q s}\sigma_-\rho_S\sigma_-\langle b_q^\dagger b_k^\dagger\rangle\Big]
\end{aligned}
$$

We should calculate the expected values now. If we assume that the density matrix for the environment is simply the one of a thermal state, we can evaluate the expected value of the number operator as:

$$
\langle b_k^\dagger b_k\rangle = \frac{1}{\mathrm{tr}(e^{-\beta\Omega b_k^\dagger b_k})}\mathrm{tr}(e^{-\beta\Omega b_k^\dagger b_k}b_k^\dagger b_k) = (1-e^{-\beta\Omega})\mathrm{tr}(e^{-\beta\Omega b_k^\dagger b_k}b_k^\dagger b_k)
$$

This can be done in the Fock basis:

$$
\mathrm{tr}(e^{-\beta\Omega b_k^\dagger b_k}b_k^\dagger b_k) = \sum_{n=0}^{\infty}\langle n|\, e^{-\beta\Omega b_k^\dagger b_k}b_k^\dagger b_k\,|n\rangle = \sum_{n=0}^{\infty} n\,\langle n|\, e^{-\beta\Omega b_k^\dagger b_k}\,|n\rangle
$$

But since the Fock states are orthonormal and $e^{-\beta\Omega b_k^\dagger b_k}|n\rangle = e^{-\beta\Omega n}|n\rangle$, we get:

$$
\mathrm{tr}(e^{-\beta\Omega b_k^\dagger b_k}b_k^\dagger b_k) = \sum_{n=0}^{\infty} n e^{-\beta\Omega n}
$$

We can use a little trick now - we know that the closed formula for the geometric series is:

$$\sum_{n=0}^{\infty} e^{-\beta\Omega n} = \frac{1}{1-e^{-\beta\Omega n}}$$

If we derive the previous equation with respect to Ω we arrive at the desired series:

$$-\frac{1}{\beta}\frac{d}{d\Omega}\left(\sum_{n=0}^{\infty} e^{-\beta\Omega n}\right) = \sum_{n=0}^{\infty} ne^{-\beta\Omega n} = \frac{e^{-\beta\Omega}}{(1-e^{-\beta\Omega})^2} \tag{1.1.7}$$

Therefore, the expected value for the number operator is simply:

$$\langle b_k^\dagger b_k\rangle = \frac{e^{-\beta\Omega}}{1-e^{-\beta\Omega}} = \frac{1}{e^{\beta\Omega}-1} = \bar{n}(\Omega) \tag{1.1.8}$$

If we calculate $\langle b_k b_k\rangle$ or $\langle b_k^\dagger b_q\rangle$, we'll find out that they are both null, because we will end up with an expression proportional to the inner product between orthogonal elements of the Fock basis. This is quite analogous to what was done in order to evaluate the expected value of the number operator. Lastly, we have $\langle b_k b_k^\dagger\rangle = \bar{n}(\Omega)+1$, because of the relation $[b_k^\dagger, b_k] = 1$. With these results we are able to continue:

$$\mathrm{tr}_E[V(t)\rho_S\rho_E V(t-s)] = \lambda_k^2(e^{i\Delta_k s}\bar{n}(\Omega_k)\sigma_+\rho_S\sigma_- + e^{-i\Delta_k s}[\bar{n}(\Omega)+1]\sigma_-\rho_S\sigma_+) \tag{1.1.9}$$

This expression is a lot cleaner, because all the terms with $q \neq k$ disappear, as well as the terms containing $\langle b_k b_k\rangle$ and $\langle b_k^\dagger b_k^\dagger\rangle$.

If we follow the procedure of the spectral density of the bath, as exposed in the lecture notes, we get the following expression for the first term in the Nakajima-Zwanzig equation:

$$\int_0^\infty ds \int_0^\infty \frac{d\Omega}{2\pi} J(\Omega)\left(e^{i\Delta_k s}\bar{n}(\Omega_k)\sigma_+\rho_S\sigma_- + e^{-i\Delta_k s}[\bar{n}(\Omega)+1]\sigma_-\rho_S\sigma_+\right)$$

which yields

$$\frac{J(\omega)}{2}\left(\bar{n}(\omega)\sigma_+\rho_S\sigma_- + [\bar{n}(\omega)+1]\sigma_-\rho_S\sigma_+\right) \tag{1.1.10}$$

and for the second term we get:

$$\frac{J(\omega)}{2}\left(\bar{n}(\omega)\rho_S\sigma_-\sigma_+ + [\bar{n}(\omega)+1]\rho_S\sigma_+\sigma_-\right) \tag{1.1.11}$$

Thus, we finally arrive at:

$$\frac{d\rho_S}{dt} = \frac{\gamma}{2}\bar{n}\left(\sigma_+\rho_S\sigma_- - \sigma_-\sigma_+\rho_S\right) + \frac{\gamma}{2}(\bar{n}+1)\left(\sigma_-\rho_S\sigma_+ - \sigma_+\sigma_-\rho_S\right) + \text{h.c.} \quad (1.1.12)$$

Since $a^\dagger\rho_S a$ and $a\rho_S a^\dagger$ are hermitian, we finally get the master equation (in the interaction picture):

$$\frac{d\rho_S}{dt} = \gamma\bar{n}\left(\sigma_+\rho_S\sigma_- - \frac{1}{2}[\sigma_-\sigma_+, \rho_S]\right) + \gamma(\bar{n}+1)\left(\sigma_-\rho_S\sigma_+ - \frac{1}{2}[\sigma_+\sigma_-, \rho_S]\right) \quad (1.1.13)$$

1.2 Q2

1.2.1 Item a)

We can begin by finding the first moments. For the creation operator a we get

$$\langle a\rangle = \mathrm{tr}(\rho a) = \mathrm{tr}(S_z|0\rangle\langle 0|S_z^\dagger a) = \langle 0|S_z^\dagger a S_z|0\rangle \quad (1.2.1)$$

after making use of the a cyclic permutation. We already know how the squeezing operator acts on a, so

$$\langle a\rangle = \langle 0|(a\cosh r + b^\dagger e^{i\theta}\sinh r)|0\rangle = 0$$

because $a|0\rangle = 0$ and $\langle 0|b^\dagger = 0$. With this same reasoning we find out that the other first moments are also null. Due to this fact the covariance matrix assumes a much simpler form:

$$\Theta = \begin{pmatrix} \langle a^\dagger a\rangle + \frac{1}{2} & \langle aa\rangle & \langle ab^\dagger\rangle & \langle ab\rangle \\ \langle a^\dagger a^\dagger\rangle & \langle a^\dagger a\rangle + \frac{1}{2} & \langle a^\dagger b^\dagger\rangle & \langle a^\dagger b\rangle \\ \langle a^\dagger b\rangle & \langle ab\rangle & \langle b^\dagger b\rangle + \frac{1}{2} & \langle bb\rangle \\ \langle a^\dagger b^\dagger\rangle & \langle ab^\dagger\rangle & \langle b^\dagger b^\dagger\rangle & \langle b^\dagger b\rangle + \frac{1}{2} \end{pmatrix} \quad (1.2.2)$$

The second moment $\langle a^\dagger a\rangle$ is (we can just follow the previous steps and use the fact that $S_z^\dagger a^\dagger a S_z = S_z^\dagger a^\dagger S_z S_z^\dagger a S_z$):

$$\langle a^\dagger a\rangle = \langle 0|\left(a^\dagger a\cosh^2 r + \sinh r\cosh r[e^{-i\theta}a^\dagger b^\dagger + e^{i\theta}ba] + bb^\dagger\sinh^2 r\right)|0\rangle \quad (1.2.3)$$

Since $bb^\dagger = 1 + b^\dagger b$ a new term will arise and we'll get a single nonzero coefficient, thus:

$$\langle a^\dagger a\rangle + \frac{1}{2} = \sinh^2 r + \frac{1}{2} = \frac{1}{2}\cosh 2r \tag{1.2.4}$$

Naturally, we should find the same for $b^\dagger b$. For $\langle aa\rangle$ we get:

$$\langle aa\rangle = \langle 0|\,(aa\cosh^2 r + \sinh r\cosh r[e^{-i\theta}a^\dagger a + e^{i\theta}aa^\dagger] + a^\dagger a^\dagger \sinh^2 r)\,|0\rangle = 0$$

and $\langle a^\dagger a^\dagger\rangle = 0$. If we do the same for the operators $ab^\dagger$ and $a^\dagger b$ we'll see that they're also zero. The equation for the expected value of ab on the other hand, yields:

$$\langle 0|\,(ab\cosh^2 r + \sinh r\cosh r[e^{i\theta}aa^\dagger + e^{i\theta}b^\dagger b] + e^{2i\theta}b^\dagger a^\dagger \sinh^2 r)\,|0\rangle = \frac{1}{2}e^{i\theta}\sinh 2r$$

Similarly:

$$\langle a^\dagger b^\dagger\rangle = \frac{1}{2}e^{-i\theta}\sinh 2r$$

We can then fill the covariance matrix as follows:

$$\Theta = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & 0 & e^{i\theta}\sinh 2r \\ 0 & \cosh 2r & e^{-i\theta}\sinh 2r & 0 \\ 0 & e^{i\theta}\sinh 2r & \cosh 2r & 0 \\ e^{-i\theta}\sinh 2r & 0 & 0 & \cosh 2r \end{pmatrix} \tag{1.2.5}$$

The upper-left block corresponds to the CM of the subsystem A, thus:

$$\Theta_A = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 \\ 0 & \cosh 2r \end{pmatrix} \tag{1.2.6}$$

It was shown in the lecture notes that the covariance matrix of a squeezed thermal state is

$$\Theta_{th} = (\tilde{n} + \frac{1}{2})\begin{pmatrix} \cosh 2r & e^{i\theta}\sinh 2r \\ e^{-i\theta}\sinh 2r & \cosh 2r \end{pmatrix} \tag{1.2.7}$$

When there's no squeezing ($r \to 0$) this CM reads:

$$\Theta_{th} = (\tilde{n} + \frac{1}{2})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.8}$$

If we compare Θ_A with Θ_{th} we'll finally see that:

$$\tilde{n} = \sinh^2 r \tag{1.2.9}$$

1.2.2 Item b)

The determinant of the CM Θ_A is:

$$|\Theta_A| = \frac{1}{4}\cosh^2 2r \tag{1.2.10}$$

The purity is then given by,

$$\mathcal{P}_A = \frac{1}{2\sqrt{|\Theta_A|}} = \frac{1}{\cosh 2r} \tag{1.2.11}$$

Threfore, we get the following expression for the entropy of the system A:

$$S_A = -\ln \mathcal{P}_A = \ln\cosh 2r \tag{1.2.12}$$

In the limit $r \gg 1$ the entropy can be approximated to $S_A \approx 2r$.

1.3 Q3

1.3.1 Item a)

First of all, in order to build the matrices for the Lyapunov equation, we should find the equation that describes the time evolution of the vector of averages $x = \langle \boldsymbol{X} \rangle$ and then identify the matrix W.

$$\frac{dx}{dt} = Wx - f \tag{1.3.1}$$

Since we know that the expectation value of an observable evolves according to

$$\frac{d\langle \mathcal{O} \rangle}{dt} = i\langle [H, \mathcal{O}] \rangle + \langle \mathcal{D}(\mathcal{O}) \rangle \tag{1.3.2}$$

for a given dissipator $\mathcal{D}$, we can just insert the desired operators on this equation. In our current system, the relationship between the dissipator and the *adjoint* dissipator is:

$$\mathcal{D}(O) = 2\kappa\tilde{D}[a](\mathcal{O}) + 2\kappa\tilde{D}[b](\mathcal{O}) \tag{1.3.3}$$

Thus, we can begin by figuring out how the adjoint dissipators act on the creation and annihilation operators. For a and $a^\dagger$ we have:

$$\tilde{D}[a](a) = \frac{1}{2}a^\dagger[a, a] + \frac{1}{2}[a^\dagger, a]a = -\frac{a}{2} \tag{1.3.4}$$

and also,

$$\tilde{D}[a](a^\dagger) = \frac{1}{2}a^\dagger[a^\dagger, a] + \frac{1}{2}[a^\dagger, a^\dagger]a = -\frac{a^\dagger}{2} \tag{1.3.5}$$

now, the commutator between the Hamiltonian and the operators should be evaluated:

$$\begin{aligned} [H, a] &= \omega[a^\dagger a, a] + i\lambda[a^\dagger, a]b^\dagger = -\omega a - i\lambda b^\dagger \\ [H, a^\dagger] &= \omega[a^\dagger a, a^\dagger] - i\lambda[a, a^\dagger]b = \omega a^\dagger - i\lambda b \end{aligned} \tag{1.3.6}$$

Since both the dissipator and the Hamiltonian are symmetric, we find similar expressions for b and $b^\dagger$, with a and b swapped. Now, we can write

$$\begin{aligned} \frac{d\langle a\rangle}{dt} &= -(i\omega + \kappa)\langle a\rangle + \lambda\langle b^\dagger\rangle \\ \frac{d\langle a^\dagger\rangle}{dt} &= (i\omega - \kappa)\langle a^\dagger\rangle + \lambda\langle b\rangle \end{aligned} \tag{1.3.7}$$

Therefore for $\boldsymbol{X} = (a, a^\dagger, b, b^\dagger)$ we have $f = 0$, since there are no independent terms, and:

$$W = \begin{pmatrix} -i\omega - \kappa & 0 & 0 & \lambda \\ 0 & i\omega - \kappa & \lambda & 0 \\ 0 & \lambda & -i\omega - \kappa & 0 \\ \lambda & 0 & 0 & i\omega - \kappa \end{pmatrix} \tag{1.3.8}$$

We can proceed by doing the same for the second moments. For the adjoint dissipators we have

$$\tilde{D}[a](aa) = \frac{1}{2}a^\dagger[aa, a] + \frac{1}{2}[a^\dagger, aa] = -aa \tag{1.3.9}$$

and

$$\tilde{D}[a](a^\dagger a) = \frac{1}{2}a^\dagger[a^\dagger a, a] + \frac{1}{2}[a^\dagger, a^\dagger a] = -a^\dagger a \tag{1.3.10}$$

The commutator on the other hand, yields

$$[H, a^\dagger a] = \omega[a^\dagger a, aa] + i\lambda[a^\dagger, aa]b = -2(\omega aa - i\lambda ab^\dagger) \tag{1.3.11}$$

and

$$[H, a^\dagger a] = i\lambda([a^\dagger, a^\dagger a]b^\dagger - [a, a^\dagger a]b) = -i\lambda(a^\dagger b^\dagger + ab) \tag{1.3.12}$$

Thus:

$$\frac{d\langle aa\rangle}{dt} = -2(i\omega + \kappa)\langle aa\rangle + 2\lambda\langle ab^\dagger\rangle \tag{1.3.13}$$

and

$$\frac{d\langle a^\dagger a\rangle}{dt} = -2\kappa\langle a^\dagger a\rangle + \lambda\langle a^\dagger b^\dagger\rangle + \lambda\langle ab\rangle \tag{1.3.14}$$

Now, with these informations in hand we should be able to build the matrix F. We may write the first of entry of the matrix as

$$\frac{d\Theta_{11}}{dt} = \frac{d\langle a^\dagger a\rangle}{dt} - \langle a\rangle\frac{d\langle a^\dagger\rangle}{dt} - \frac{d\langle a\rangle}{dt}\langle a^\dagger\rangle \tag{1.3.15}$$

which reads,

$$\begin{aligned}
\frac{d\Theta_{11}}{dt} =& -2\kappa\langle a^\dagger a\rangle + \lambda\langle a^\dagger b^\dagger\rangle + \lambda\langle ab\rangle \\
& - (i\omega - \kappa)\langle a^\dagger\rangle\langle a\rangle + (i\omega + \kappa)\langle a^\dagger\rangle\langle a\rangle \\
& - \lambda\langle a\rangle\langle b\rangle + \lambda\langle a^\dagger\rangle\langle b^\dagger\rangle \\
& = -2\kappa(\langle a^\dagger a\rangle + \langle a^\dagger\rangle\langle a\rangle) + \lambda(\langle a^\dagger b^\dagger\rangle + \langle a\rangle\langle b\rangle) + \lambda(\langle ab\rangle + \langle a^\dagger\rangle\langle b^\dagger\rangle) \\
& = -2\kappa\langle \delta a^\dagger \delta a\rangle + \lambda\langle \delta a^\dagger \delta b^\dagger\rangle + \lambda\langle \delta a \delta b\rangle
\end{aligned}$$

Since $\langle \delta a^\dagger \delta a\rangle = \Theta_{11} - 1/2$, the equation above can be written as

$$\frac{d\Theta_{11}}{dt} = \frac{d\Theta_{22}}{dt} = -2\kappa\Theta_{11} + \lambda\Theta_{14} + \lambda\Theta_{23} + \kappa \tag{1.3.16}$$

If we do the same for Θ_{12} we get

$$\frac{d\Theta_{12}}{dt} = \frac{d\Theta_{21}^\dagger}{dt} = -2(i\omega + \kappa)\Theta_{12} + 2\lambda\Theta_{13} \tag{1.3.17}$$

Since the dissipators acting on a and b are independent, the resulting F matrix will me block diagonal. [1] Due to the symmetry between a and b, we should also find a similar expression for Θ_{33}. Therefore, we may identify the matrix F as:

[1] If we take a look at the expression that we found for Θ_{11} and Θ_{12} we'll be able to see that the independent term κ arises from the $1/2$ term in the relation $\langle \delta a^\dagger \delta a\rangle = \Theta_{11} - 1/2$. Since this only holds for the diagonal entries of the covariance matrix, the off-diagonal elements won't have any independent terms. Thus, we should expect the F matrix to be diagonal.

$$F = \kappa \mathbb{I}_4 \tag{1.3.18}$$

The following steady-state solution was found by *Mathematica*:

$$\Theta_0 = \begin{pmatrix} \frac{k^2+w^2}{2(k^2+w^2-\lambda^2)} & 0 & 0 & \frac{(k-iw)\lambda}{2(k^2+w^2-\lambda^2)} \\ 0 & \frac{k^2+w^2}{2(k^2+w^2-\lambda^2)} & \frac{(k+iw)\lambda}{2(k^2+w^2-\lambda^2)} & 0 \\ 0 & \frac{(k-iw)\lambda}{2(k^2+w^2-\lambda^2)} & \frac{k^2+w^2}{2(k^2+w^2-\lambda^2)} & 0 \\ \frac{(k+iw)\lambda}{2(k^2+w^2-\lambda^2)} & 0 & 0 & \frac{k^2+w^2}{2(k^2+w^2-\lambda^2)} \end{pmatrix} \tag{1.3.19}$$

1.3.2 Item b)

The determinant of the covariance matrix in the steady-state is:

$$|\Theta_0| = \frac{\left(\kappa^2 + w^2\right)^2}{16\left(\kappa^2 - \lambda^2 + w^2\right)^2} \tag{1.3.20}$$

We saw in the notes that the purity is related to the CM according to $\mathcal{P} = \frac{1}{2^N |\Theta|}$, so we find:

$$\mathcal{P}_0 = \frac{1}{2^2\sqrt{|\Theta_0|}} = \frac{\kappa^2 - \lambda^2 + w^2}{\kappa^2 + w^2} = 1 - \frac{\lambda^2}{\kappa^2 + \omega^2} \tag{1.3.21}$$

which is, in general, mixed. In the limit $\lambda \to 0$, on the other hand, we have a pure state:

$$\mathcal{P}_{\lambda\to 0} = 1 \tag{1.3.22}$$

This result is expected, since the Hamiltonian $H_{\lambda\to 0} = \omega a^\dagger a + \omega b^\dagger b$ becomes decoupled in this limit. When there's no dissipation present we get:

$$\mathcal{P}_{\kappa\to 0} = 1 - \frac{\lambda^2}{\omega^2} \tag{1.3.23}$$

So we'll have a state a bit more mixed when dissipation is present. Now, we have the following results concerning the CM: when λ goes to zero we get

$$\Theta_{\lambda\to 0} = \frac{1}{2}\mathbb{I}_4 \tag{1.3.24}$$

which is the CM of a thermal state with $\tilde{n} = 0$. If we so the same for κ we arrive at:

$$\Theta_{\kappa\to 0} = \frac{1}{2(\omega^2-\lambda^2)}\begin{pmatrix} \omega^2 & 0 & 0 & -i\omega\lambda \\ 0 & \omega^2 & i\omega\lambda & 0 \\ 0 & -i\omega\lambda & \omega^2 & 0 \\ i\omega\lambda & 0 & 0 & \omega^2 \end{pmatrix} \tag{1.3.25}$$

1.3.3 Item c)

Since we already know the purity of the system, it's quite easy to evaluate the entropy. The mutual informaiton is given by:

$$I_{AB} = S_A + S_B - S_{AB} \tag{1.3.26}$$

So we should find the reduced CMs of the system in order to calculate their individual entropies. By taking the upper-left block and the lower-right block CM we get:

$$\Theta_0^A = \Theta_0^B = \frac{\kappa^2+\omega^2}{2(\kappa^2+\omega^2-\lambda^2)}\mathbb{I}_2 \tag{1.3.27}$$

Their purity is:

$$\mathcal{P}_A = \mathcal{P}_B = \frac{\kappa^2+\omega^2+\lambda^2}{\kappa^2+\omega^2} \tag{1.3.28}$$

Thus, we have: [2]

$$I_{AB} = -\ln\left(\frac{\mathcal{P}_A\mathcal{P}_B}{\mathcal{P}_0}\right) = \ln\left(\frac{\kappa^2+\omega^2}{\kappa^2+\omega^2-\lambda^2}\right) \tag{1.3.29}$$

As special cases we have both $I_{AB,\lambda\to 0} = 0$ and:

$$I_{AB,\kappa\to 0} = \ln\left(\frac{\omega^2}{\omega^2-\lambda^2}\right) \tag{1.3.30}$$

Thus, as we increase the coupling between the modes, mutual entropy also goes up.

[2] When $\lambda^2 > \omega^2+\kappa^2$, we get negative values for the purity and the entropy value blows up. I'm not entirely sure of what this means. Mathematically, we get positive eigenvalues for the matrix W (They are $\lambda_\pm = -k \pm \sqrt{\lambda^2-\omega^2}$). Thus, it ceases to be negative-definite. Physically I'd argue that this means there isn't any steady state solution for the system, since the system $\dot{x} = Wx$ is not asymptotically stable anymore .

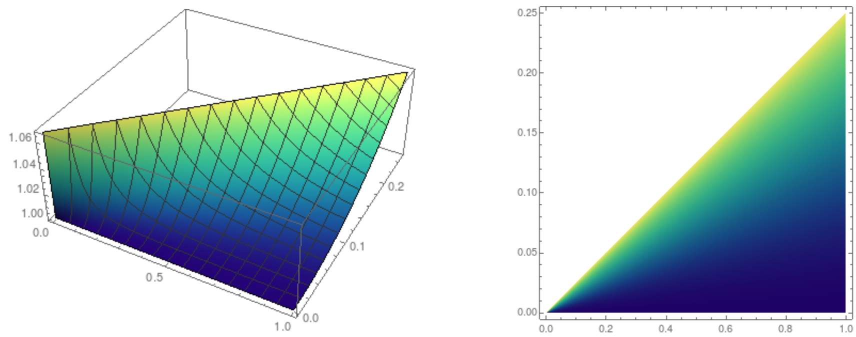

Figura 1: Density plot of the mutual information as a function of μ and λ.

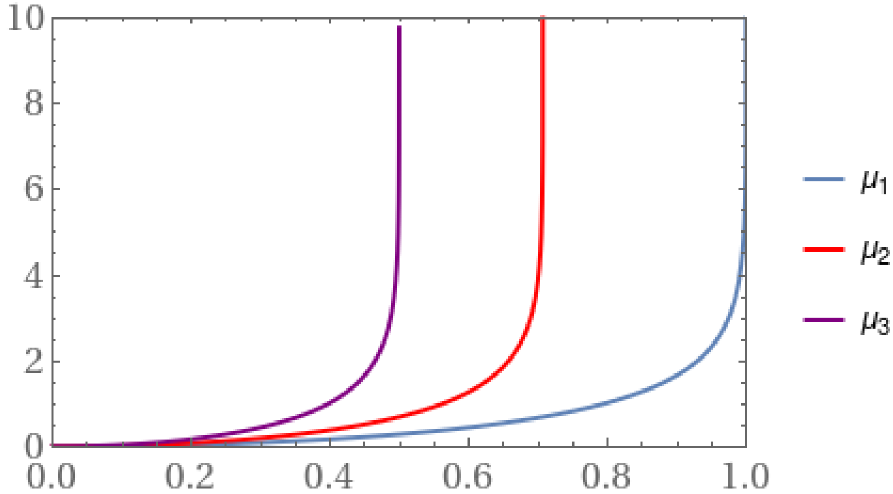

Figura 2: Mutual information for different values of μ. Here we have $\mu_1 > \mu_2 > \mu_3$. μ is a new variable that I defined as $\mu^2 = \omega^2 + \kappa^2$

1.3.4 Item d)

According to the lecture notes we have:

$$\langle Q_+^2 \rangle + \langle P_-^2 \rangle = \langle a^\dagger a \rangle + \frac{1}{2} + \langle b^\dagger b \rangle + e^{i\theta} \langle a^\dagger b^\dagger \rangle + e^{-i\theta} \langle ab \rangle \tag{1.3.31}$$

Thus, we can use the CM to calculate this quantity:

$$\langle Q_+^2 \rangle + \langle P_-^2 \rangle = \Theta_{11} + \Theta_{33} + e^{i\theta} \Theta_{23} + e^{-i\theta} \Theta_{32}$$

Making the appropriate substitutions:

$$\langle Q_+^2 \rangle + \langle P_-^2 \rangle = \frac{\kappa^2 + \omega^2}{\kappa^2 + \omega^2 - \lambda^2} + \frac{e^{i\theta}(\kappa + i\omega) + e^{-i\theta}(\kappa - i\omega)}{\kappa^2 + \omega^2 - \lambda^2}$$

All the imaginary term cancel out and we get:

$$\langle Q_+^2\rangle + \langle P_-^2\rangle = \frac{\kappa^2+\omega^2+\lambda\kappa\cos\theta-\lambda\omega\sin\theta}{\kappa^2+\omega^2-\lambda^2} \tag{1.3.32}$$

The minimum occurs when:

$$\frac{d(\kappa\cos\theta-\omega\sin\theta)}{d\theta}=0 \implies \tan\theta=-\frac{\omega}{\kappa}$$

hence,

$$\cos\theta=\pm\frac{\kappa}{\sqrt{\omega^2+\kappa^2}}, \qquad \text{and} \qquad \sin\theta=\mp\frac{\omega}{\sqrt{\omega^2+\kappa^2}}$$

Which implies

$$\langle Q_+^2\rangle + \langle P_-^2\rangle\Big|_{min} = \frac{1\pm\frac{\lambda}{\sqrt{\omega^2+\kappa^2}}}{1-\frac{\lambda^2}{\omega^2+\kappa^2}} = \frac{1}{1\pm\frac{\lambda}{\sqrt{\kappa^2+\omega^2}}} \tag{1.3.33}$$

or, for the + solution (We can't conclude anything with the other solution):

$$\langle Q_+^2\rangle + \langle P_-^2\rangle\Big|_{min} = \frac{\mu}{\mu+\lambda}$$

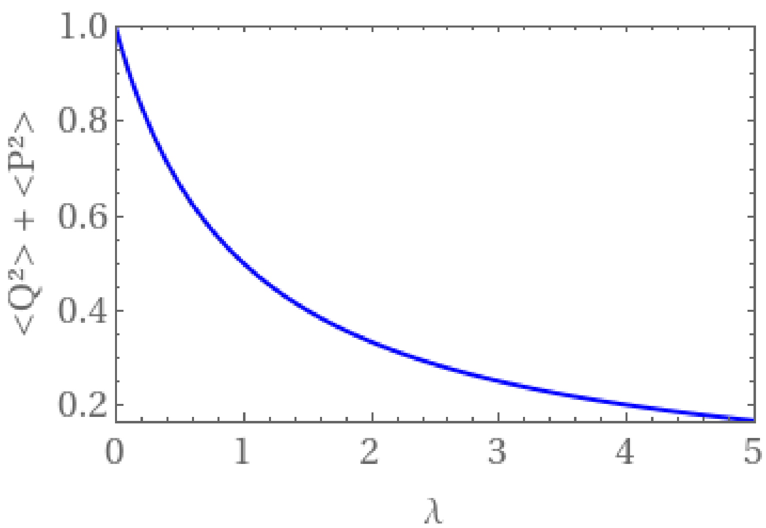

Figura 3: Density plot of $\langle Q_+^2\rangle + \langle P_-^2\rangle$ for a fixed μ.

This quantity is always smaller than one, so the system is always entangled.

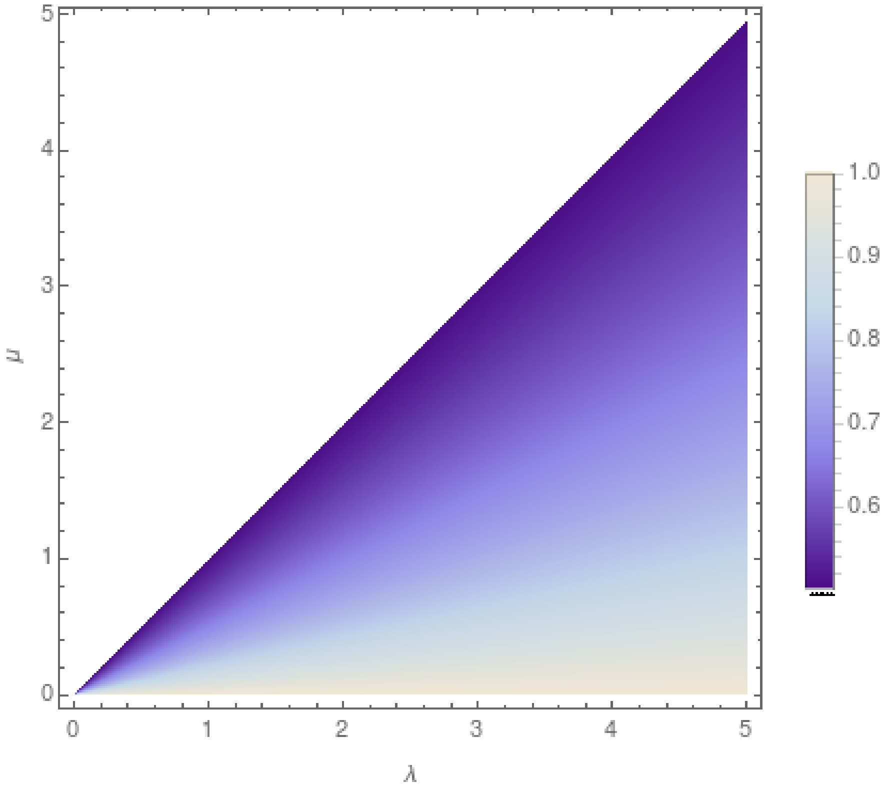

Figura 4: Density plot of $\langle Q_+^2\rangle + \langle P_-^2\rangle$ versus μ and λ.

1.4 Q4

Since the displacemente and Squeezing operators are both unitary, the purity is simply given by:[3]

$$\mathrm{tr}(\rho^2) = \mathrm{tr}(\rho_{th}^2) = (1 - e^{-\beta\omega})^2 \mathrm{tr}(\mathrm{Exp}(-2\beta\omega a^\dagger a)) \tag{1.4.1}$$

We know that the eigenvalues λ_i of the number operator are simply the natural numbers $0, 1, 2, ...$ Thus, we can easily find the eigenvalues of its exponential. The i-th eigenvalue is given by (The result from the Problem Set 1 was used here):

$$\mathrm{Eig}(\mathrm{Exp}(a^\dagger a))_i = \sum_{i=0}^{\infty} \frac{\lambda_i^n}{n!} = e^{\lambda_i} \tag{1.4.2}$$

Then, the eigenvalues of $\mathrm{Exp}(-2\beta\omega a^\dagger a)$ are simply $1, e^{-2\beta\omega}, e^{-4\beta\omega}, ...$ The last result from Linear Algebra that we should use is the one which tells us that the

[3]We can write $Exp(-\beta\omega a^\dagger a)Exp(-\beta\omega a^\dagger a) = Exp(-2\beta\omega a^\dagger a)$ because $a^\dagger a$ obviously commutes with itself.

trace of a Matrix equals the sum of its eigenvalues, therefore:

$$\mathrm{tr}(\rho_{th}^2) = (1-e^{-\beta\omega})^2(1+e^{-2\beta\omega}+e^{-4\beta\omega}+....) = (1-e^{-\beta\omega})^2\frac{1}{1-e^{-2\beta\omega}} \tag{1.4.3}$$

We can write $1-e^{-2\beta\omega}$ as $(1-e^{-\beta\omega})(1+e^{-\beta\omega})$, and after a little bit of algebra we find the desired result:

$$\mathrm{tr}(\rho^2) = \frac{1}{\frac{2}{e^{\beta\omega}+1}-1} = \frac{1}{2\tilde{n}+1} \tag{1.4.4}$$

If we try to calculate from the CM we'll find:

$$\mathcal{P}_{th} = \frac{1}{2\sqrt{(\tilde{n}+\frac{1}{2})^2}} = \frac{1}{2\tilde{n}+1}$$

1.5 Q5

1.5.1 Item a)

First of all, we should move the Hamiltonian to the rotating frame by applying a unitary $S(\omega_p) = e^{i\omega_p a^\dagger a}$, as we did in section 3.3. This will remove is time dependence, since:

$$SaS^\dagger = e^{-i\omega_p t}a, \quad \text{and} \quad Sa^\dagger S^\dagger = e^{i\omega_p t}a^\dagger$$

So the exponentials in the Hamiltonian cancel out. Besides, the unitaries have no effect on the quadratic term, since $Sa^\dagger aS^\dagger = Sa^\dagger S^\dagger SaS^\dagger = (ei\omega_p ta^\dagger)(e{-}i\omega_p ta) = a^\dagger a$. The same argument applies to the U-term. Moreover, we already know that

$$\frac{dS}{dt}S^\dagger = i\frac{d(\omega_p a^\dagger a)}{dt} = i\omega_p a^\dagger a$$

Thus, we finally find an equivalent Hamiltonian with detuned frequency $\Delta = \omega_c - \omega_p$ given by:

$$H = \Delta a^\dagger a + \frac{U}{2}a^\dagger a^\dagger aa + i\epsilon(a^\dagger - a) \tag{1.5.1}$$

Now, for the commutator

$$[H,a] = \Delta[a^\dagger a, a] + \frac{U}{2}[a^\dagger a^\dagger aa, a] + i\epsilon[a^\dagger - a, a]$$

The U-term is a bit cumbersome but the commutator isn't hard to calculate because

$$[a^\dagger a^\dagger aa, a] = a^\dagger a^\dagger [aa, a] + [a^\dagger a^\dagger, a]aa = -2a^\dagger aa$$

this leads us to:

$$[H, a] = -\Delta a - \frac{U}{2} a^\dagger aa - i\epsilon$$

We already found the contribution of the dissipator in the problem 3, it's simply:

$$\tilde{D}[a](a) = -\kappa a$$

Consequently, the equation of motion for a will be:

$$\frac{d\langle a\rangle}{dt} = \epsilon - (\kappa + i\Delta)\langle a\rangle - iU\langle a^\dagger aa\rangle \tag{1.5.2}$$

1.5.2 Item b)

If we ignore the fluctuations we'll arrive at a simple expression for $\langle a^\dagger aa\rangle$, which will be written as a function of the first momoments. We may express the annihilation operator as $a = \delta a + \langle a\rangle$, therefore, if we assume that the fluctations are small we get:

$$\langle a^2\rangle = \langle a\rangle^2 + 2\langle a\rangle\delta a + \delta a^2 \approx \langle a\rangle^2$$

and,

$$\langle a^\dagger aa\rangle = \langle(\delta a^\dagger + \langle a^\dagger\rangle)a^2\rangle \approx \langle\langle a^\dagger\rangle a^2\rangle = \langle a^\dagger\rangle\langle a^2\rangle \approx \langle a^\dagger\rangle\langle a\rangle^2$$

This results in the following ODE:

$$\frac{d\langle a\rangle}{dt} = \epsilon - (\kappa + i\Delta)\langle a\rangle - iU\langle a^\dagger\rangle\langle a\rangle^2$$

We can also denote $\langle a\rangle$ as α and write $\langle a^\dagger\rangle\langle a\rangle^2 = \langle a\rangle^\dagger\langle a\rangle^2 = |\alpha|^2\alpha$ to obtain:

$$\frac{d\alpha}{dt} = \epsilon - (\kappa + i\Delta)\alpha - iU\alpha|\alpha|^2 \tag{1.5.3}$$

1.5.3 Item c)

In the steady state we must have:

$$\frac{d\alpha}{dt} \implies \epsilon - (\kappa + i\Delta)\alpha - iU\alpha|\alpha|^2 = 0 \tag{1.5.4}$$

For a positive detuning, we obtain the following plot:

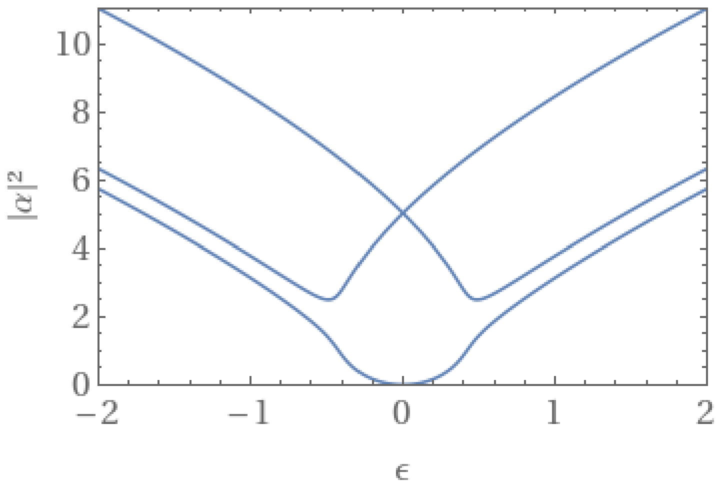

Figura 5: Positive detuning frequency. Here the parameters were fixed at $\kappa = 0.05$, $U = 0.1$ and $\Delta = 0.5$

The plot for the negative detuning on the other hand, is:

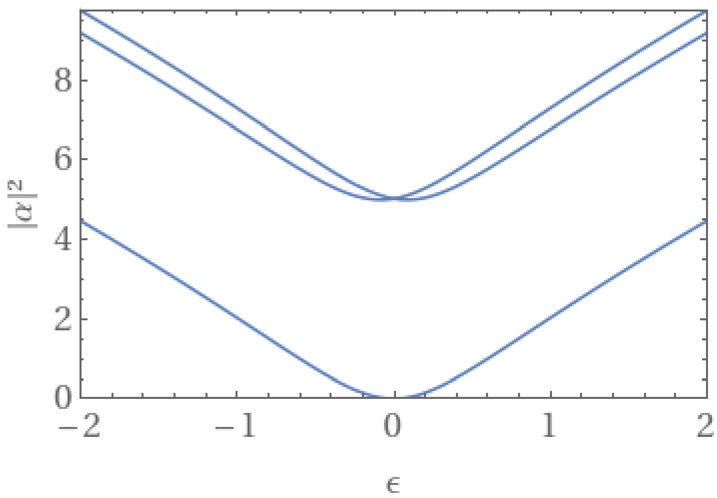

Figura 6: Negative detuning frequency $\Delta = -0.5$.