The Quantum Rabi Model

Gabriel O. Alves

Undergraduate studies report

Contents

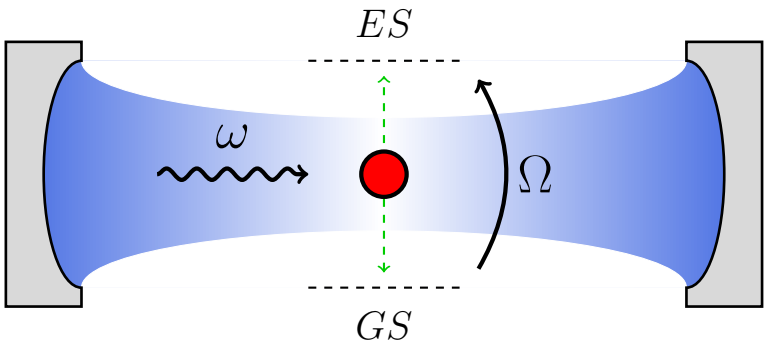
1 Introduction

1.1 An overview of the Rabi Model

The Quantum Rabi model describes the interaction of a quantized field with a two-level atom and is characterized by a Hamiltonian containing three terms:

$$H_{Rabi} = H_f + H_a - \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}} \tag{1.1.1}$$

The first term represents the Hamiltonian for the quantized electromagnetic field, the second term represents the Hamiltonian for a two-level system and the third term takes into account the interaction between the radiation and the atom - i.e the energy of the dipole. The expression for H_a is simple: for a system with energies $\pm\Omega/2$, it's simply given by $\Omega/2\sigma_z$. The construction of the other two terms require a little more attention, therefore we'll study them in the next two sections, albeit briefly, since this is not our focus.



1.1.1 Quantization of the electromagnetic field

For an one-dimensional cavity of length L along the z direction, the expressions for the EM-field that satisfy the boundary conditions (i.e. the vanishing condition at $z = 0$ and $z = L$) and the Maxwell equations are:

$$\begin{aligned} \boldsymbol{E}(z,t) &= \sqrt{\frac{2\omega^2}{V\epsilon_0}} q(t) \sin(kz)\hat{\boldsymbol{x}} \\ \boldsymbol{B}(z,t) &= \frac{\mu_0\epsilon_0}{k}\sqrt{\frac{2\omega^2}{V\epsilon_0}} \dot{q}(t) \cos(kz)\hat{\boldsymbol{x}} \end{aligned} \tag{1.1.2}$$

with $q(t)$ being a time-dependent factor and V the effective volume of the cavity. We already know that the energy density of the EM field is given by $u = \epsilon_0|\boldsymbol{E}|^2 + 1/\mu_0|\boldsymbol{B}|^2$. If we denote $\dot{q}(t)$ by $p(t)$, the Hamiltonian for the single-mode field simplifies to

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{1.1.3}$$

after we integrate. But this is simply the Hamiltonian of the Harmonic oscillator, which can be written in terms of the creation and annihilation operators. Therefore, the Rabi Hamiltonian assumes the form: [1]

$$H_{Rabi} = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z - \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}} \tag{1.1.4}$$

1.1.2 Atom - field interaction

The electric field operator, after we perform the dipole approximation ($\lambda_{photon} \gg r_{atom}$) is given by (in the Schrödinger picture):

$$\hat{\boldsymbol{E}} = \hat{\boldsymbol{e}}\sqrt{\frac{\omega}{\epsilon_0 V}}(a + a^\dagger)\sin kz \tag{1.1.5}$$

here $\hat{\boldsymbol{e}}$ denotes an unit vector in an arbitrary direction. If we write the dipole operator as $\hat{d} = \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{e}}$, the interaction term simplifies to

$$-\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}} = -g\hat{d}(a + a^\dagger)$$

where we have absorbed all the constants in a single parameter g. Since the dipole operator couple the fundamental and excited states, it can be written as $d\,|0\rangle\langle 1| + d^*\,|1\rangle\langle 0|$. If we assume that is entries are real, we get $\hat{\boldsymbol{d}} = d\sigma_x$ and the quantum Rabi Hamiltonian assumes the following form:

$$H_{Rabi} = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z - \lambda(a + a^\dagger)\sigma_x \tag{1.1.6}$$

with $\lambda = dg$.

[1] We dropped the zero point energy term. Moreover, we'll assume that $\hbar = 1$ from now on.

2 Rabi Model

2.1 The parity operator

We can show that the parity operator

$$\Pi = \exp\left[a^\dagger a + \frac{1}{2}(\sigma_z + 1)\right] \tag{2.1.1}$$

commutes with the Hamiltonian. It suffices to show that

$$\Pi H \Pi^\dagger = H$$

Since σ_z and $a^\dagger a$ live in different spaces we can decompose the operator as:

$$\Pi = \exp\left(a^\dagger a\right) \exp\left(\frac{\sigma_z + 1}{2}\right)$$

The operator will obviously commute with the first and second terms of the Hamiltonian. For the third term we may begin by evaluating $\Pi\sigma_x\Pi^\dagger$. Since $\sigma_z^2 = 1$, when we exponentiate the operator we get

$$e^{i\alpha\sigma_z} = \cos\alpha + i\sigma_z \sin\alpha$$

So, $\Pi\sigma_x\Pi^\dagger = \sigma_z\sigma_x\sigma_z = -\sigma_z$. If we use the the BCH formula we can also see that

$$e^{i\alpha a^\dagger a} a e^{-\alpha a^\dagger a} = a + i\alpha[a^\dagger a, a] - \frac{\alpha^2}{2!}[a^\dagger a, [a^\dagger a, a]] + \ldots = a(\cos\alpha - i\sin\alpha)$$

because $[a^\dagger a, a] = -a$. Therefore, since $\alpha = \pi$ we have $\Pi(a + a^\dagger)\Pi^\dagger = -(a + a^\dagger)$, therefore $\Pi(a + a^\dagger)\sigma_x\Pi^\dagger = (a + a^\dagger)\sigma_x$, and the Hamiltonian is invariant under this transformation.

2.2 Low-energy effective Hamiltonian

Since the Pauli matrix σ_x has only off-diagonal elements, the interaction term $\lambda(a + a^\dagger)\sigma_x$ will be an off-diagonal block matrix. So, our first task is to find a method to diagonalize the Hamiltonian. As it was done in [5], the procedure that we'll use is the Schrieffer-Wolff transformation. We can write the Rabi Hamiltonian as:

$$H_{Rabi} = H_0 - \lambda V = \omega a^\dagger a - \lambda \frac{\Omega}{2}\sigma_z + (a + a^\dagger)\sigma_x \tag{2.2.1}$$

this will split the Hamiltonian into two terms: an unperturbed diagonal Hamiltonian H_0, and an interaction off-diagonal Hamiltonian V. Now, we apply a unitary transformation $U = e^S$ on this Hamiltonian:

$$H' = e^{-S} H_{Rabi} e^S = \sum \frac{1}{k!}[H_{Rabi}, S]^{(k)} \tag{2.2.2}$$

with S being an anti-Hermitian and block off-diagonal operator. Our objective is to find a generator S so that this procedure diagonalizes the matrix above. We can split up the Hamiltonian into diagonal terms

$$H'_d = \sum \frac{1}{(2k)!}[H_0, S]^{(2k)} - \sum \frac{1}{(2k+1)!}[\lambda V, S]^{(2k+1)} \tag{2.2.3}$$

and off - diagonal terms

$$H'_{od} = \sum \frac{1}{(2k+1)!}[H_0, S]^{(2k+1)} - \sum \frac{1}{(2k)!}[\lambda V, S]^{(2k)} \tag{2.2.4}$$

since the product between diagonal and off-diagonal matrices will yield off-diagonal matrices and so on. [2]The expanded terms of lower orders will be:

$$H'_{od} = -\lambda V + [H_0, S]^{(1)} - \frac{1}{2!}[\lambda V, S]^{(2)} + \frac{1}{3!}[H_0, S]^{(3)} + ...$$

We can parametrize the generator S as:

$$S \equiv \lambda S_1 + \lambda^3 S_3$$

So, the off-diagonal Hamiltonian becomes

$$\begin{aligned} H'_{od} = &- \lambda V + \lambda[H_0, S_1] + \lambda^3[H_0, S_3] - \frac{1}{2}[[\lambda V, \lambda S_1 + \lambda^3 S_3], \lambda S_1 + \lambda^3 S_3] \\ &+ \frac{1}{6}[[[H_0, \lambda S_1 + \lambda^3 S_3], \lambda S_1 + \lambda^3 S_3], \lambda S_1 + \lambda^3 S_3] + ... \end{aligned}$$

If we expand the commutators we will see that

$$[\lambda V, S]^{(2)} = \lambda^3[[V, S_1], S_1] + O(\lambda^5)$$

and that

[2]the fact that we're dealing with $2x2$ block matrices make it easy to decide whether the product will be diagonal or not

$$[H_0, S]^{(3)} = \lambda^3[[[H_0, S_1], S_1], S_1] + O(\lambda^5)$$

If we require that the first order terms go to zero, the following condition can be found:

$$[H_0, S_1] = V \tag{2.2.5}$$

if we do the same for the terms of third order we will find that:

$$[H_0, S_3] - \frac{1}{2}\lambda^3[[V, S_1], S_1] + \frac{1}{6}[[[\underbrace{H_0, S_1}_{=V}], S_1], S_1] = 0$$

Thus,

$$[H_0, S_3] = \frac{1}{3}[[V, S_1], S_1] \tag{2.2.6}$$

The generator that satisfies the first condition is simply

$$S_1 = \frac{1}{\Omega}(a + a^\dagger)(\sigma_+ - \sigma_-) + O\left(\frac{\omega_0}{\Omega^2}\right) \tag{2.2.7}$$

And for the second one we must have

$$S_3 = -\frac{4}{3\Omega^3}(a + a^\dagger)^3(\sigma_+ - \sigma_-) + O\left(\frac{\omega_0}{\Omega^4}\right) \tag{2.2.8}$$

The diagonal terms up to fourth order will be:

$$H' = H_0 - \frac{\lambda^2}{2}[V, S_1] + \frac{\lambda^4}{24}[[[V, S_1], S_1], S_1] - \frac{\lambda^4}{2}[V, S_3]$$

We can use the identities $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$, $\sigma_+^2 = \sigma_-^2 = 0$ and $\sigma_\pm\sigma_\mp = \sigma_\pm$ to evaluate the commutators. This will give us:

$$H' = \omega_0 a^\dagger a + \frac{\Omega}{2}\sigma_z + \frac{g^2\omega_0^2}{4\Omega}\sigma_0 + \frac{\omega_0 g^4}{4}(a+a^\dagger)^2\sigma_z - \frac{g^4\omega_0^2}{16\Omega}(a+a^\dagger)^4\sigma_z + O\left(\frac{g^2\omega_0^2}{\Omega^2}\right) \tag{2.2.9}$$

With $g = 2\lambda/\sqrt{\omega_0\Omega}$. The last step is to project this Hamiltonian onto the $|\downarrow\rangle$ space. Since $\langle\downarrow|\,\sigma_z\,|\downarrow\rangle = -1$, we get, after the projection

$$H^\Omega_{np} = \langle\downarrow|\,H'\,|\downarrow\rangle = \omega_0 a^\dagger a - \frac{\Omega}{2} - \frac{\omega_0 g^4}{4}(a + a^\dagger)^2 + \frac{g^4\omega_0^2}{16\Omega}(a + a^\dagger)^4 + \frac{g^2\omega_0^2}{4\Omega} \tag{2.2.10}$$

A projection onto the $|\uparrow\rangle$ space would give us a kinectic term $(a + a^\dagger)^2 \propto x^2$,

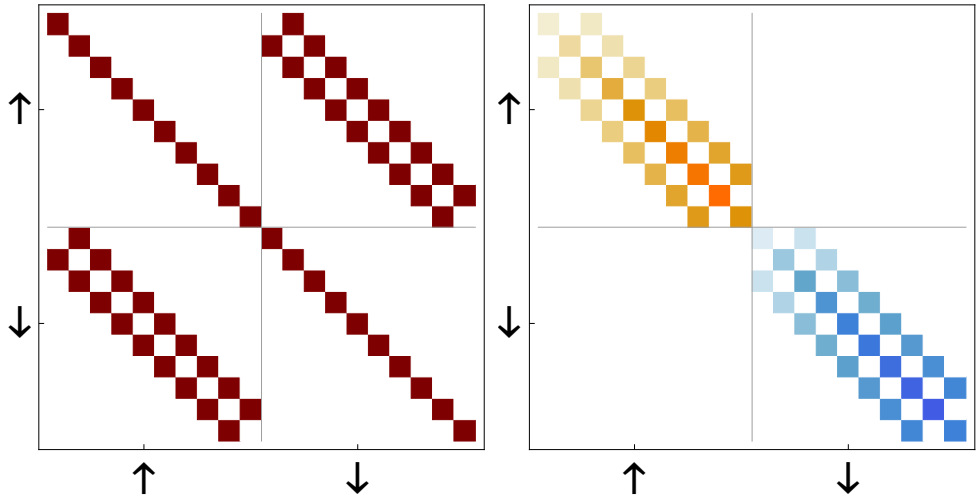


Figure 1: A representation of the Hamiltonian in matrix form. On the left we have the non-diagonalized Hamiltonian. The off-diagonal that couple the spin subspaces term arise from the interaction between the atom and the field. On the right we have the diagonalized Hamiltonian. The blue block matrix represents the low-energy subspace that we obtain after the projection.

with a plus sign. This kind of term will in general *increase* the energy of the harmonic oscillator. Therefore, since we want to capture the low-energy physics, the projection that we chose makes more sense. Lastly, we parametrize the field-atom coupling constant as $\lambda \propto N$, the frequency as $\Omega \propto N^2$ and take $N \to \infty$. By doing so, the coupling constant g will remain *finite* and we'll be able to diagonalize the Hamiltonian exactly. In this thermodynamic limit we finally obtain:

$$H_{np} = \omega_0 a^\dagger a - \frac{\Omega}{2} - \frac{\omega_0 g^2}{4}(a + a^\dagger)^2 \tag{2.2.11}$$

This Hamiltonian is much easier to work with, since the presence of σ_z instead of σ_x in the mixed term assures that the Matrix will remain block diagonal.

2.3 Normal Phase

2.3.1 Diagonalization of the Hamiltonian in the $\Omega/\omega_0 \to \infty$ limit

First of all, we should rewrite the projected Hamiltonian as:

$$\bar{H}_{np} = \omega \left(1 - \frac{g^2}{2}\right) a^\dagger a - \frac{\omega g^2}{4}(aa + a^\dagger a^\dagger) - \frac{\Omega}{2} \tag{2.3.1}$$

By doing so we can relate these parameters to the Hamiltonian (6.3.1) in the appendix through:

$$\xi = \omega\left(1 - \frac{g^2}{2}\right), \quad \text{and} \quad \kappa = -\frac{\omega g^2}{4} \tag{2.3.2}$$

Using the results already found in the appendix, we can see that the solution is simply:

$$r = \frac{1}{2}\tanh^{-1}\left(\frac{\frac{g^2}{2}}{1 - \frac{g^2}{2}}\right)$$

But this can be written as a logarithm. Since

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

The squeezing parameter becomes:

$$r_{np} = -\frac{1}{4}\ln\left(1 - g^2\right) \tag{2.3.3}$$

By making use of the equation (6.3.4) in the appendix, we can finally find the diagonalized Hamiltonian:

$$H_{np} = \omega_0\sqrt{1-g^2}a^\dagger a - \frac{\Omega}{2} + \frac{\epsilon_{np} - \omega_0}{2} \tag{2.3.4}$$

with excitation energy ϵ_{np}:

$$\epsilon_{np} = \tilde{\xi} = \sqrt{1-g^2}$$

and ground state energy $E_{G,np}$:

$$E_{G,np} = \frac{\tilde{\xi} - \xi}{2} - \frac{\Omega}{2} = \frac{\epsilon_{np} - \omega_0}{2} - \frac{\Omega}{2}$$

Note that the gap closes at the critical point $g = g_c = 1$. This suggests that the GS is degenerate and we have a QPT. Therefore, in the next section we will take this criticality into account when diagonalizing the Hamiltonian.

2.3.2 Exact solution in the $\Omega/\omega_0 \to \infty$ limit

Since the eigenvalue if $a^\dagger a$ is m, the eigenvalue of this Hamiltonian will be:

$$E^m_{np} = m\epsilon_{np} + E_{G,np} \tag{2.3.5}$$

And since we squeeze the Hamiltonian, transforming it as $S^\dagger H_{Rabi} S$, the eigenvector in the field space will be $S[r_{np}]\,|m\rangle$, but we are also projecting the Hamil-

tonian onto the $H_\downarrow$ space, therefore the eigenvector will be:

$$|\phi_{np}^m\rangle = S[r_{np}]\,|m\rangle\,|\downarrow\rangle \tag{2.3.6}$$

In the ground state $|\phi_{np}^0\rangle$, the expected value for the rescaled photon number will be (we should normalize it, otherwise since its proportional to Ω, which goes to infinity):

$$n_{c,np} = \frac{\omega_0}{\Omega}\,\langle\phi_{np}^0|\,a^\dagger a\,|\phi_{np}^0\rangle = \langle 0|\,S^\dagger[r_{np}](a^\dagger a)S[r_{np}]\,|0\rangle$$

But $\langle a^\dagger a\rangle = \langle a^2\rangle = \langle (a^\dagger)^2\rangle = 0$. Thus:

$$n_{c,np} = \frac{\omega_0}{\Omega}\sinh^2 r_{np} = \frac{\omega_0}{\Omega}\frac{\cosh(2r_{np})-1}{2} \tag{2.3.7}$$

The variance for the position will be:

$$\Delta x_{np} = e^{rp} = (1-g^2)^{-\frac{1}{4}} \tag{2.3.8}$$

Due to the imaginary term the variance of the moment will be a bit different:

$$\Delta p_{np} = e^{-rp} = (1-g^2)^{\frac{1}{4}} \tag{2.3.9}$$

We've simply used the results for the squeezing operator stated in the appendix. Note the these expression also give us the critical exponents. For Δx_{np} for example, the critical exponent is $-1/4$.

2.4 Superradiant Phase

The procedure to diagonalize the Rabi Hamiltonian in the superradiant case requires an extra step. When $g > 1$ we leave the low-energy regime, and as we'll see, the number of photons gets proportional to Ω. The strategy to bypass this problem apply the Displacement operator on the Hamiltonian to displace the field:

$$\tilde{H}_{Rabi} = D^\dagger[\alpha]H_{Rabi}D[\alpha] = \omega_0(a^\dagger+\alpha)(a+\alpha) - \lambda(a+a^\dagger)\sigma_x + \frac{\Omega}{2}\sigma_z - 2\lambda\alpha\sigma_x \tag{2.4.1}$$

Here we have taken $\alpha \in \mathrm{R}$. This will give us rescaled constants and we'll recover the low-energy physics. But we should first find an appropriate change of basis to lead us to the Hamiltonian of the same form that we had before.

2.4.1 Change of basis

We should aim to find the eigenvalues and eigenvectors of the atomic part of this new Hamiltonian, i.e. $\Omega/2\sigma_z - 2\lambda\alpha\sigma_x$. If we use the formula that we found in the appendix for the eigenvalues and eigenvectors we will see that the Pauli matrices in the old and new basis are related by:

$$\sigma_x = \sin 2\theta\tau_z + \cos 2\theta\tau_x \tag{2.4.2}$$

and that its eigenvalues turn out to be:

$$\eta_{1,2} = \frac{\tilde{\Omega}}{2} = \frac{\sqrt{\Omega^2 + 16\lambda^2\alpha^2}}{2} \tag{2.4.3}$$

With $\tan 2\theta = -4\lambda\alpha/\Omega$ Moreover, we can also use the spectral decomposition to find the value of

$$\frac{\Omega}{2}\sigma_z - 2\lambda\alpha\sigma_x$$

in the new basis. We can simply write it as

$$\frac{\Omega}{2}\sigma_z - 2\lambda\alpha\sigma_x = \sum_i \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| = \frac{\tilde{\Omega}}{2}\left|\tilde{\uparrow}\right\rangle \left\langle\tilde{\uparrow}\right| - \frac{\tilde{\Omega}}{2}\left|\tilde{\downarrow}\right\rangle \left\langle\tilde{\downarrow}\right|$$

or, in terms of the pauli matrix in the new basis,

$$\frac{\Omega}{2}\sigma_z - 2\lambda\alpha\sigma_x = \frac{\tilde{\Omega}}{2}\tau_z \tag{2.4.4}$$

After this change of basis the Hamiltonian will become

$$\tilde{H}_{Rabi} = \omega_0 a^\dagger a - \lambda\cos 2\theta(a + a^\dagger)\tau_x + \omega_0\alpha^2 + (\alpha\omega_0 + \lambda\sin 2\theta\tau_z)(a + a^\dagger)$$

As we've done before, we will also project this Hamiltonian on the $H_\downarrow$ space. If we require that the projected term $(\alpha\omega_0 - \lambda\sin 2\theta)(a + a^\dagger) = 0$, we will recover the same Hamiltonian that we had in the previous section and we can apply the same procedure. This can be achieved if

$$\alpha = \pm\alpha_G = \pm\sqrt{\frac{\Omega}{4g^2\omega_0}}\sqrt{g^4 - 1}$$

and we get:

$$\tilde{H}_{Rabi}(\pm\alpha_G) = \omega_0 a^\dagger a - \tilde{\lambda}(a + a^\dagger)\tau_x + \frac{\tilde{\Omega}}{2}\tau_z + \omega_0\alpha_g^2 \tag{2.4.5}$$

Note that after we apply the Schrieffer-Wolf perturbation and project this Hamiltonian it will assume the same form that (2.2.11), with rescaled constants $\tilde{\lambda} = \lambda\cos 2\theta = \sqrt{\omega_0\Omega}/2g$ and $\tilde{\Omega} = g^2\Omega$.

2.4.2 Exact solution in the $\Omega/\omega_0 \to \infty$ limit

The effective Hamiltonian in this case is:

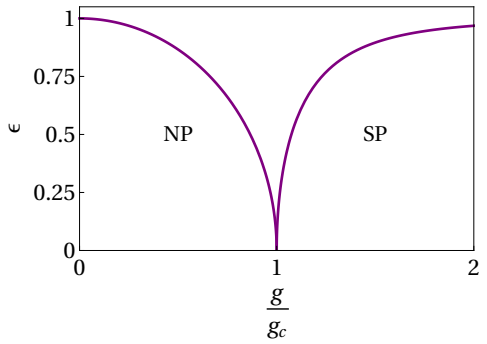
$$H_{sp} = \omega_0 a^\dagger a - \frac{\omega_0}{4g^4}(a + a^\dagger)^2 - \frac{\Omega}{4}(g^2 + g^{-2}) \tag{2.4.6}$$

To see this, we can note that the quadratic term will have a new coefficient given by $\omega_0\tilde{g}^2/4 = \tilde{\lambda}^2/\tilde{\Omega} = \omega_0/4g^2$. The last term on the other hand comes from $\omega_0\alpha_g^2$.

We can notice that H_{sp} and H_{np} differ only by a substitution $g^2 \to 1/g^2$ in the term $(a + a^\dagger)$, therefore we can follow the same procedure that we did before we find:

$$\epsilon_{sp}(g) = \omega_0\sqrt{1 - g^{-4}} \tag{2.4.7}$$

and a ground state energy:



$$E_{G,sp}(g) = \frac{\epsilon_{sp}(g) - \omega_0}{2} - \frac{\Omega}{4}(g^2 - g^{-2}) \tag{2.4.8}$$

with eigenstate

$$|\phi_{sp}^m(g)\rangle_\pm = D[\alpha_g\pm]S[r_{sp}]\,|m\rangle\,|\downarrow^\pm\rangle \tag{2.4.9}$$

with the new states being:

$$|\downarrow^\pm\rangle = \mp\sqrt{\frac{1 - g^{-2}}{2}}\,|\uparrow\rangle + \sqrt{\frac{1 + g^{-2}}{2}}\,|\downarrow\rangle \tag{2.4.10}$$

Critical exponents			
ϵ	Δp	Δx	n_c
$1/2$	$1/4$	$-1/4$	1

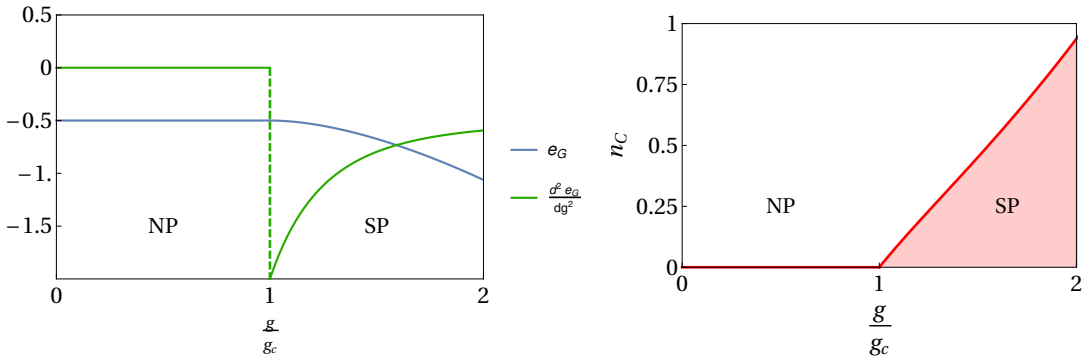


Figure 2: On the left we have a plot for the rescaled ground state energy. We can see that its second derivative is discontinuous at the critical point. On the right we have a plot for the (normalized) number of photons, which increases monotonically in the superradiant phase.

2.5 The no-go theorem

Let's suppose that we have a quadratic term in the Hamiltonian, i.e.

$$H^A_{Rabi} = \omega_0 a^\dagger a + \frac{\Omega}{2}\sigma_z - \lambda(a + a^\dagger)\sigma_x + D(a + a^\dagger)^2 \tag{2.5.1}$$

If we arrange the equation above to write it in the form (6.3.1), we get:

$$H^A_{Rabi} = (\omega_0 + 2D)a^\dagger a + \frac{\Omega}{2}\sigma_z - \lambda(a + a^\dagger)\sigma_x + D(aa + a^\dagger a^\dagger) + D \tag{2.5.2}$$

If we apply the squeezing operator in this Hamiltonian with the following substitutions:

$$\xi = \omega_0 + 2D \tag{2.5.3}$$

and

$$\kappa = D \tag{2.5.4}$$

we get:

$$e^{4r} = \frac{(\omega_0 + 2D) - 2D}{(\omega_0 + 2D) + 2D} \implies e^{-4r} = 1 + \frac{4D}{\omega_0} \tag{2.5.5}$$

This will lead us to the following Hamiltonian:

$$\bar{H}^{A}_{Rabi} = \bar{\omega}_0 a^\dagger a - \bar{\lambda}(a + a^\dagger)\sigma_x + \frac{\Omega}{2}\sigma_z \tag{2.5.6}$$

With the first renormalized frequency being:

$$\bar{\omega}_0 = \xi \cosh 2r + 2\kappa \sinh 2r = \omega_0 \left(1 + \frac{4D}{\omega_0}\right)^{\frac{1}{2}}$$

and

$$\bar{\lambda} = \lambda(\cosh r + \sinh r) = \lambda e^r = \lambda \left(1 + \frac{4D}{\omega_0}\right)^{\frac{1}{4}}$$

2.6 Finite-Frequency Scaling

2.6.1 Ground state

Now we will look back to the Hamiltonian ?? and take into account the fourth order terms. Since we can't apply the same procedure to diagonalize this Hamiltonian, we will make use of the variational method instead to estimate groud state energy and other relevant parameters. A reasonable strategy is to propose the squeezed vacuum $|\psi_0(s)\rangle = S[s]\,|0\rangle$ as a trial wave function. The variational method would then provide a suitable value for s.

To evalute this extra term, one should notice that

$$S^\dagger[s](a + a^\dagger)^4 S[s] = e^{4r}(a + a^\dagger)^4$$

Even though fourth orders terms are cumbersome, this one can be substantially simplified because the only terms that will survive when sandwiched by the vacuum will be:

$$\langle 0|\left[(aa^\dagger)^2 + 1\right]|0\rangle = 3$$

Thus, the extra term in the Hamiltonian yields:

$$\frac{g^4\omega_0^2}{16\Omega}\langle\psi_0(s)|\,(a + a^\dagger)^4\,|\psi_0(s)\rangle = \frac{3\omega_0^2 g^4}{16\Omega}e^{4s} \tag{2.6.1}$$

Therefore, we obtain the following expression for the ground state energy:

$$E_0 = \langle\psi_0(s)|\,H_{np}\,|\psi_0(s)\rangle = \frac{\omega_0}{2}\cosh(2s) - \frac{\omega_0 g^2}{4}e^{2s} + \frac{3\omega_0^2 g^4}{16\Omega}e^{4s} - \frac{\Omega}{2} + \frac{g^2\omega_0^2}{4\Omega} \tag{2.6.2}$$

If we derive this expression we obtain:

$$\frac{\partial E}{\partial s} = \frac{3g^4\omega_0}{2\Omega}e^{6s} + (1-g^2)e^{4s} - 1 = 0$$

At the critical point $g = g_c = 1$, the solution acquires a simple form:

$$s_c = \frac{1}{6}\ln\left(\frac{2\Omega}{3\Omega_0}\right) \tag{2.6.3}$$

If we insert this parameter into (2.6.1), we can obtain an expression for the normalized correction of the ground state energy:

$$e_{G,g_c}(\Omega/\omega_0) = \frac{\omega_0}{\Omega}(E_0(s) - E_{G,np}(g_c)) \tag{2.6.4}$$

This parameter evaluates the difference between $E_{G,np}(g_c)$, i.e. the ground state energy found analytically, when we considered only the second order terms, and the ground state energy $E_0(s)$ found by means of the variational method, where we considered fourth order terms in the bosonic operators. It's interesting to write this in terms of power of the ratio Ω/ω_0, so we just use a bit of algebra here in order to simplify the expression. If we rewrite the normalizartion factor as $(2\Omega/3\omega_0)^{-1}(2/3)$, the correction will be:

$$e_{G,g_c}(\Omega/\omega_0) = \frac{\omega_0}{4}\left(\frac{2\Omega}{3\omega_0}\right)^{-\frac{4}{3}} + O\left(\frac{\omega_0^2}{\Omega^2}\right) \tag{2.6.5}$$

and for the number of photons, we have:

$$n_{c,g_c}(\Omega/\omega_0) = \langle\psi_0(s)|\, a^\dagger a\, |\psi_0(s)\rangle = \frac{\omega_0}{\Omega}\left(\frac{\cosh(2r)-1}{2}\right) \tag{2.6.6}$$

If we compute the leading order¿

$$n_{c,g_c}(\Omega/\omega_0) = \frac{1}{6}\left(\frac{2\Omega}{3\omega_0}\right)^{-\frac{2}{3}} + O\left(\frac{\omega_0}{\Omega}\right) \tag{2.6.7}$$

and for the quadratures we have:

$$\Delta x_{g_c} = \left(\frac{2\Omega}{3\omega_0}\right)^{\frac{1}{6}}, \qquad \Delta p_{g_c} = \left(\frac{2\Omega}{3\omega_0}\right)^{-\frac{1}{3}}$$

2.6.2 Energy gap and finite-frequency scaling

The second order contribution will be:

$$\frac{\omega_0}{2}e^{2r}\langle m|\left(a^\dagger a+\frac{1}{2}\right)|m\rangle=\frac{m}{2}+\frac{1}{4}$$

The fourth order contribution in turn, will be:

$$\frac{\omega_0}{16}\left(\frac{\omega_0}{\Omega}\right)e^{4s}\langle m|\left(a+a^\dagger\right)^4|m\rangle=\frac{\omega_0}{16}\left(\frac{\omega_0}{\Omega}\right)e^{4s}\langle m|\left(2a^\dagger a+aa+a^\dagger a^\dagger+1\right)^2|m\rangle$$

We can use the completeness of the fock states to evaluate this expression:

$$\langle m|\left(a+a^\dagger\right)^4|m\rangle=\sum_n\langle m|\left(2a^\dagger a+aa+a^\dagger a^\dagger+1\right)|n\rangle\langle n|\left(2a^\dagger a+aa+a^\dagger a^\dagger+1\right)|m\rangle$$

If we make use the the orthotornomality of the eigenstates, we can see that the first term is:

$$\langle m|\left(2a^\dagger a+aa+a^\dagger a^\dagger+1\right)|n\rangle=(2n\delta_{n,m}+\sqrt{n(n-1)}\delta_{m,n-2}+\sqrt{(n+1)(n+2)}\delta_{m,n+2}+\delta_{n,m})$$

We find a similar expression for its conjugate. Therefore the contribution of the fourth order term is:

$$\langle m|\left(a+a^\dagger\right)^4|m\rangle=6m^2+6m+3$$

Hence, the energy of the m-th excited state is:

$$E_m(s_c)=\langle\psi_m(s)|H_{np}|\psi_m(s)\rangle=\frac{\omega_0}{4}\left(\frac{m^2+3m}{4}+\frac{3}{8}\right)\left(\frac{2\Omega}{3\omega_0}\right)^{-\frac{1}{3}}\tag{2.6.8}$$

So, the gap between two arbitrary is states is:

$$\epsilon_{n,m}(g_c,\Omega/\omega_0)=\frac{\omega_0}{4}\frac{(m-n)(m+n+3)}{4}\left(\frac{2\Omega}{3\omega_0}\right)^{-\frac{1}{3}}\tag{2.6.9}$$

In particular, we have

$$\epsilon_{1,0}(g_c,\Omega/\omega_0)=\omega_0\left(\frac{2\Omega}{3\omega_0}\right)^{-\frac{1}{3}}$$

for the gap between the ground and the first excited state.

Finite-frequency scaling exponents			
ϵ	Δp	Δx	n_c
$-1/3$	$-1/6$	$1/6$	$-2/3$

Lastly, since the finite-frequency scalin g of the transition amplitude is deifned as:

$$\xi_{n,m}(g,\Omega/\omega_0) = -\langle n|\, S^\dagger[s_c]\frac{\partial}{\partial g}S[s_c]\,|m\rangle \tag{2.6.10}$$

We have

$$\xi_{n,m}(g,\Omega/\omega_0) = -\frac{\partial s_c}{\partial g}\langle n|\,(a^\dagger a^\dagger - aa)\,|m\rangle$$

Since $a\,|m\rangle = \sqrt{m}\,|m-1\rangle$ and $a^\dagger\,|m\rangle = \sqrt{m+1}\,|m+1\rangle$, we have:

$$\xi_{n,m}(g,\Omega/\omega_0) = -\frac{\partial s_c}{\partial g}\left(\sqrt{(m+2)(m+1)}\delta_{n,m+2} - \sqrt{m(m-1)}\delta_{n,m-2}\right) \tag{2.6.11}$$

To find the derivative of the squeezing parameters we simply use implicit dwerivation, which yields

$$-\frac{\partial s}{\partial g} = \frac{-12g^2\omega_0 e2s + 4\Omega^2}{18\omega_0\Omega e^{2s}}$$

If we simplify, we get:

$$\xi_{n,m}(g,\Omega/\omega_0) = \left[\frac{2}{3} - \frac{1}{3}\left(\frac{2\Omega}{3\omega_0}\right)^{\frac{2}{3}}\right]\left(\sqrt{(m+2)(m+1)}\delta_{n,m+2} - \sqrt{m(m-1)}\delta_{n,m-2}\right)$$

3 Dynamics

3.1 Adiabatic Perturbation Theory

The instantaneous eigenstates are given by:

$$|\psi(t)\rangle = \sum_n e^{i\Theta_n} |\psi(0)\rangle = \sum_n \alpha_n(t) e^{-i\Theta_n(t)} |r_{np}(g(t)), n\rangle \tag{3.1.1}$$

the convention here is to define

$$\Theta_n(t) = \int_0^t \epsilon_n(t')dt'$$

The Schrödinger equation tells us that (we are using $\hbar = 1$):

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle = \epsilon_n |\psi(t)\rangle \tag{3.1.2}$$

if we derive the first expression we get

$$\sum_n i\dot{\alpha}_n(t)e^{-i\Theta_n(t)} |r_{np}(g(t)), n\rangle + \dot{\Theta}_n |\psi(t)\rangle + i\alpha_n(t)e^{-i\Theta_n(t)} \frac{d}{dt} |r_{np}(g(t)), n\rangle = \epsilon_n |\psi(t)\rangle$$

since $\dot{\Theta}_n = \epsilon_n$, the second term on the left-hand side and the term in the right-hand side cancel each other, and we arrive at

$$\dot{\alpha} = -\sum_m \alpha_m(t) \langle r_{np}(g(t)), n| \frac{\partial}{\partial t} |r_{np}(g(t)), m\rangle e^{i(\Theta_n(t) - \Theta_m(t))}$$

therefore, if we make the substitution $g = \dot{g}t$, the solutions are given by

$$\alpha_n(g) = -\sum_m \int_0^g dg' \alpha_m(g') \langle r_{np}(g'), n| \frac{\partial}{\partial t} |r_{np}(g'), m\rangle e^{i(\Theta_n(g') - \Theta_m(g'))}$$

know, supposing that our initial state was $|\Phi(0)\rangle = |r_{np}(g), 0\rangle$ (this means that $\alpha_0 = 1$) and that the quench is slow, i.e $g \ll 1$, we can approximate the solution of the previous equation to

$$\alpha_n(g) \approx -\int_0^g \langle r_{np}(g'), n| \partial_{g'} |r_{np}(g'), 0\rangle e^{i(\Theta_n(g') - \Theta_0(g'))} dg' \tag{3.1.3}$$

By using a Taylor expansion, we can evaluate the oscillating integral above, since

$$\int_{x_1}^{x_2} f(x)e^{i\alpha g(x)} = \frac{1}{i\alpha}\frac{f(x)}{g'(x)}e^{i\alpha g(x)}\Big|_{x_1}^{x_2} + O(\alpha^{-2})$$

thus, only the first term in the expansion contributes. We should remember that

$$\Theta_n(g) = \frac{1}{g}\int_0^g \epsilon_n(g')dg'$$

to obtain

$$\alpha_n(g) \approx i\dot{g}\frac{\langle r_{np}(g'), n|\, \partial_{g'}\, |r_{np}(g'), 0\rangle}{\epsilon_n(g) - \epsilon_0(g)} e^{i(\Theta_n(g') - \Theta_0(g'))}\Big|_0^g + O((\dot{g})^2)$$

We know that $r_{np}(g) = -1/4 \ln(1-g^2)$ and that $a^{\dagger 2}|0\rangle = \sqrt{2}|2\rangle$, thus the inner product in the expression above is given by

$$\langle r_{np}(g'), n|\, \partial_{g'}\, |r_{np}(g'), 0\rangle = -\frac{\sqrt{2}g}{4(1-g^2)}\delta_{m,2}$$

and the only correction for the wave function is given by the coefficient

$$\alpha_2(g) \approx \frac{-ig\dot{g}}{4\sqrt{2}(1-g^2)^{z\nu+1}} e^{i(\Theta_n(g') - \Theta_0(g'))} \tag{3.1.4}$$

the expression for the energy gap was written as $\epsilon_n(g) = n\omega_0(1-g^2)^{z\nu}$ aiming generality, so the expression is valid even for a different set of critical exponents. With this expression in hands we can finally find the residual energy at the end of the quench, given by

$$E_r(g_f) = \sum_{n>0} \epsilon_n(g_f)|\alpha_n(g_f)|^2 \approx \tau_q^{-2}\frac{g_f^4}{16\omega_0(1-g_f^2)^{z\nu+2}} \tag{3.1.5}$$

because $\dot{g} = g_f/\tau_q$.

3.2 Kibble - Zurek Mechanism

3.2.1 Classical case

It's known in the scaling theory that both the correlation length and relaxation time of system that undergoes a phase transition diverge as we approach the critical

point. We say that there was a breakdown in the adiabacity of the system, and the phase transition ends incompletely.

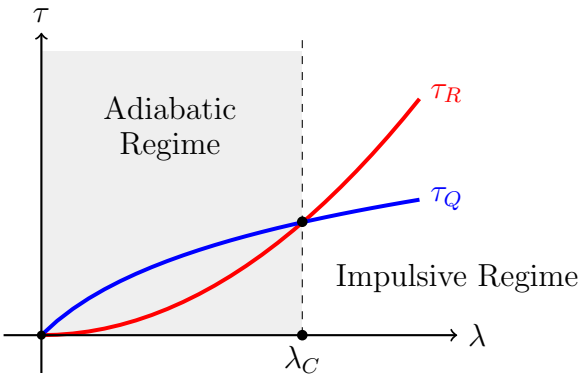
$$\xi(\epsilon) \propto \xi_0 \epsilon^{-\nu} \tag{3.2.1}$$

$$\tau_R \propto \tau_0 \epsilon^{-z\nu} \tag{3.2.2}$$

Since the process is not instantaneous, the system is not able to respond to external stimuli fast enough, i.e. it doesn't relax to equilibrium, and as a consequence the changes in the system occur *locally*, rather than *globally*.

The KZM mechanism is used to describe second-order phase transitions while taking this effect into account. Let's consider that the system is able to relax to equilibrium until a certain critical point in time called $\hat{t}$. We'll divide the behaviour of the system into two different regimes:

- An adiabatic regime, where the typical relaxation time is much lower than the inverse of the quench rate and the changes occur adiabatically
- An impulsive regime, where we observe a breakdown in the adiabacity



If we assume tha the temperature is quenched linearly as

$$T(t) = T_C \left(1 - \frac{t}{\tau}\right)$$

the critical time $\hat{t}$ is the instant when the relaxation time is equal to the *remaining* time until the system reaches the critical point $\tau(T - T_C)$. Using the relation (3.2.2) from the scaling theory, we get

$$\hat{t} \propto \left(\frac{\tau}{\tau_0}\right)^{-\frac{z\nu}{z\nu+1}}$$

3.2.2 Quantum Counterpart

In the case of the QPT, the relevant quantity will be a parameter λ which is driven towards the critical point λ_C. Ww'll also assume in our analysis that the energy gap scales as $\Delta \propto \Delta_0(\lambda - \lambda_c)^{z\nu}$. The relaxation time - or 'reaction time' - can be taken as

$$\tau_R = \frac{\hbar}{\Delta}$$

So, as we can see, as the gap closes the system diverges. On the other range, the timescale related to how fast the Hamiltonian is driven by external changes is given by:

$$\frac{1}{\tau_Q} = \frac{1}{\Delta}\frac{d\Delta}{dt}$$

Thus, if we solve the relation:

$$\frac{d\Delta}{dt} = \Delta^2 \tag{3.2.3}$$

we arrive at the appropriate scaling relation. If we assume that the parameter λ changes linearly (at least close to the critical point), i.e. $d\lambda/dt \propto 1/\tau$, we arrive at the following result:

$$\hat{\lambda} = \tau^{-\frac{1}{1+z\nu}} \tag{3.2.4}$$

and

$$\hat{t} = \tau^{\frac{z\nu}{1+z\nu}} \tag{3.2.5}$$

where $\hat{\lambda}$ and $\hat{\lambda}$ represent the values of the parameter and the time at the crossover between the two regimes, respectively.

3.2.3 KZM and the Rabi Model

In units of $\hbar = 1$ energy is given by $E = f$, thus we can take the relaxation time of the Rabi system to be

$$\eta(g) = 2\epsilon_{np}(g)$$

Since some states are not accessible, making some transitions forbidden due to the parity symmetry, we have the factor of two above. So we can simply use the formalism developed in the previous section taking η as the 'effective' gap. Thus, in a similar spirit , we have to solve:

$$(2\omega_0(1-\dot{g}^2)^{z\nu})^2 = |2\omega_0\dot{g}(-2z\nu\hat{g})(1-\hat{g}^2)^{z\nu-1}|$$

Taking the modulus and breaking the diffence of squares apart on the LHS we get:

$$4\omega_0^2(1+\hat{g})^{2z\nu}(1-\hat{g})^{2z\nu} = \frac{g_f}{\tau_q}4\omega_0 zz\nu\hat{g}(1+\hat{g})^{z\nu-1}(1-\hat{g})^{z\nu-1}$$

which simplifies to

$$(1-\hat{g})^{z\nu+1} = \frac{g_f}{\omega_0\tau_q}z\nu\hat{g}(1+\hat{g})^{-z\nu-1}$$

If we quench the system during a sufficient long time, while ending it at the critical point $g_f = 1$, the crossover point $\hat{g}$ will be close to 1, and will can approximate the expression $\hat{g}/(1+\hat{g})^{-z\nu-1} \approx 2^{-3/2}$, since $z\nu = 1/2$ the present model. The equation above becomes

$$\hat{g} \approx 1 - (4\sqrt{2}\omega_0\tau_q)^{-\frac{1}{z\nu+1}}$$

We can use this value for g to calculate the residual energy at the crossover, which will remain constant at the impulsive regime. Substituting into the (3.1.5):

$$E_r(\hat{g}) \approx \tau_q^{-2}\frac{1}{16\omega_0}(4\sqrt{2}\omega_0\tau_q)^{\frac{z\nu+2}{z\nu+1}} \propto \tau_q^{-\frac{z\nu}{z\nu+1}} \tag{3.2.6}$$

using or set of critical exponents, we get $E_r \propto \tau_q^{-1/3}$, in contrast to $E_r \propto \tau_q^{-2}$ in the adiabatic regime.

3.3 Equation of Motion for Quench Dynamics

3.3.1 Infinite ω_0/Ω

Let's go back to our quadratic Hamiltonian in the normal phase, this time making it time dependent in the quench parameter:

$$H_{np}(g(t)) = \omega_0 a^\dagger a - \frac{\omega g^2(t)}{4}(a+a^\dagger)^2 - \frac{\Omega}{2} \tag{3.3.1}$$

In the Heisenberg picture, the bosonic operator is given by

$$a_H(t) = u(t)a + v^*(t)a^\dagger$$

And in order to preserve the commutation relation the parameters $u(t)$ and $v(t)$ must satisfy $|u(t)|^2 - |v(t)^2| = 1$ at all times. Its equation of motion is:

$$iu(t)a + iv^*(t)a^\dagger = \omega_0(u(t)a + v^*(t)a^\dagger) - \frac{\omega_0 g^2(t)}{4}[(u(t)+v(t))a + (u(t)+v(t))^*a^\dagger]$$

If we commute the expression above with $[..., a^\dagger]$ and $[a, ...]$ respectively, we get the following set of couplet differential equations:

$$\begin{aligned} \frac{i}{\omega_0}\frac{du(t)}{dt} &= \left(1 - \frac{g^2(t)}{2}\right)u(t) - \frac{g^2(t)}{2}v(t) \\ -\frac{i}{\omega_0}\frac{dv(t)}{dt} &= \left(1 - \frac{g^2(t)}{2}\right)v(t) - \frac{g^2(t)}{2}u(t) \end{aligned} \tag{3.3.2}$$

with $u(0) = 1$ and $v(0) = 0$ as initial conditions. The residual energy at the end of the quench will be:

$$E_r = \omega_0|v(\tau_q)|^2 - \frac{\omega_0 g_f^2}{4}|u(\tau_q) + v(\tau_q)|^2 - \frac{\epsilon_{np}(g_f) - \omega_0}{2} \tag{3.3.3}$$

3.3.2 Effect of finite ω_0/Ω

Since in this case we have a fourth order term in the Hamiltonian, we'll arrive at a slightly different set of ODEs. In this case the Hamiltonian is

$$H_{np}(g(t)) = \omega_0 a^\dagger a - \frac{\omega g^2(t)}{4}(a + a^\dagger)^2 + \frac{\omega_0 g^4(t)}{16\Omega}(a + a^\dagger)^4 - \frac{\Omega}{2} + \frac{\omega_0^2 g^2(t)}{4\Omega} \tag{3.3.4}$$

The Heisenberg EOM under this Hamiltonian is

$$\begin{aligned} iu(t)a + iv^*(t)a^\dagger = \omega_0(u(t)a + v^*(t)a^\dagger) - \frac{\omega_0 g^2(t)}{4}[(u(t)+v(t))a + (u(t)+v(t))^*a^\dagger \\ + \frac{3\omega_0 g^4(t)}{4\Omega}|u(t)+v(t)|^2[(u(t)+v(t))a + (u(t)+v(t))^*a^\dagger] \end{aligned}$$

This time we get this following set of ODEs, which will share an extra term, the leading order correction

$$\frac{i}{\omega_0}\frac{du(t)}{dt} = \left(1 - \frac{g^2(t)}{2}\right)u(t) - \frac{g^2(t)}{2}v(t) + \frac{3g^4(t)}{4\Omega}(u(t)+v(t))|u(t)+v(t)|^2$$
$$-\frac{i}{\omega_0}\frac{dv(t)}{dt} = \left(1 - \frac{g^2(t)}{2}\right)v(t) - \frac{g^2(t)}{2}u(t) + \frac{3g^4(t)}{4\Omega}(u(t)+v(t))|u(t)+v(t)|^2$$

In this case, the residual energy will be:

$$E_r^{\Omega} = \omega_0|v(\tau_q)|^2 - \frac{\omega_0 g_f^2}{4}|u(\tau_q)+v(\tau_q)|^2 + \frac{3\omega_0 g_f^4}{4\Omega}|u(\tau_q)+v(\tau_q)|^4 - \frac{\omega_0}{4}\left(\frac{2\Omega}{3\omega_0}\right)^{-1/3}$$

These relations could be confirmed with numerical simulations, as shown below.

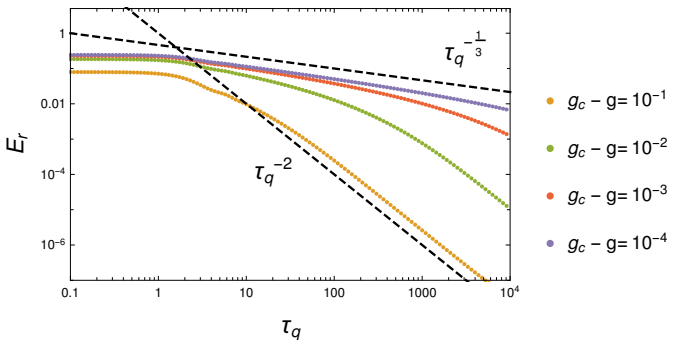


Figure 3: Residual energy as a function of the final quench time. The dashed lines show the expected behavior for both regimes. The colored lines represent the numerical solutions for the system Dynamics.

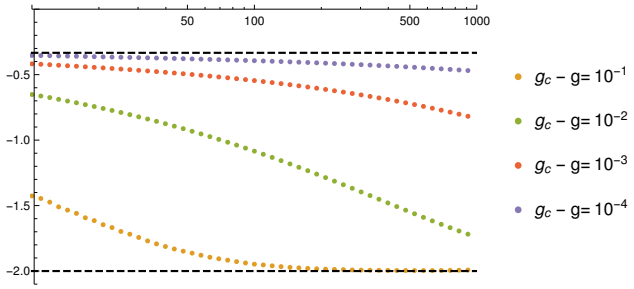


Figure 4: Plot for the critical exponent as a function of time, fitted for different values of final quench parameter g.

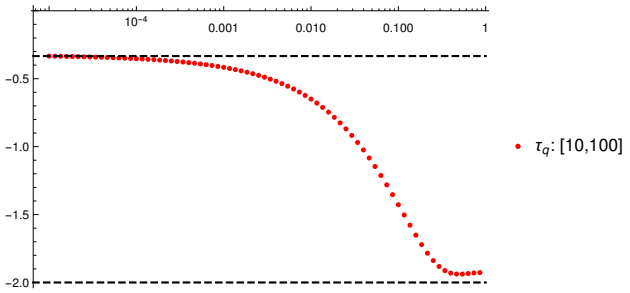


Figure 5: Plot for the critical exponent as a function of the final parameter g, fitted for the interval of time $[0, 100]$.

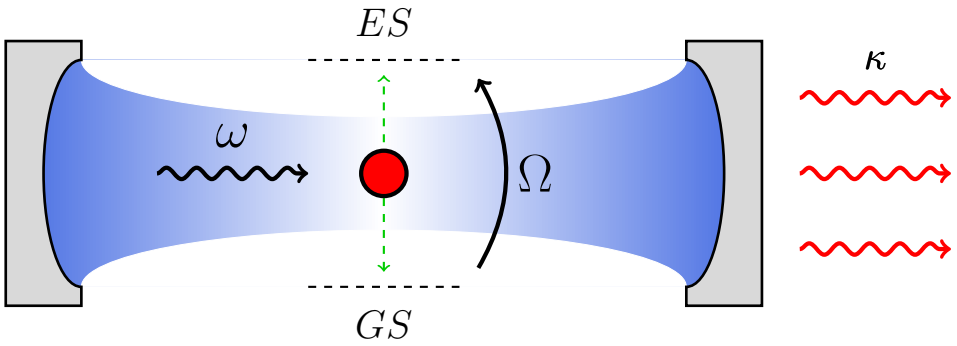
4 Dissipative Phase Transtion

The dissipation in the model, which corresponds to a possible loss of photons in the cavity, can be described through a master equation of the form

$$\dot{\rho} = \mathcal{L}[\rho] = -i[H_{Rabi}, \rho] + 2\kappa\mathcal{D}[a] \tag{4.0.1}$$

with a dissipator of the form

$$\mathcal{D}[a] = a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a \tag{4.0.2}$$



4.1 Semiclassical analysis

Under the presence o a dissipator, the evolution of the mean value of an observable is given by the following expression:

$$\frac{d\langle O\rangle}{dt} = i\langle[H, O]\rangle + \mathrm{Tr}\left\{O\mathcal{D}(\rho)\right\} \tag{4.1.1}$$

If we define a *adjoint dissipator* as:

$$\bar{D}[L](O) = \frac{1}{2}L^\dagger[O, L] + \frac{1}{2}[L^\dagger, O]L \tag{4.1.2}$$

The expected values can simply be calculated through:

$$\frac{d\langle O\rangle}{dt} = i\langle[H, O]\rangle + \langle\bar{D}(O)\rangle \tag{4.1.3}$$

Our work reduces to the calculation of some commutators. Luckly for the open QRM we have $\bar{D} = \kappa\mathcal{D}[a]$. The first EOM for example, is simply:

$$\langle \dot{a} \rangle = i\langle [H, a] \rangle + \langle \bar{D}(a) \rangle$$

So, let's calculate the relevant dissipators. We have $\omega_0[a^\dagger a, a] = -\omega_0 a$, $-\lambda[(a + a^\dagger)\sigma_x, a] = \lambda\sigma_x$, $-\lambda[\sigma_x, \sigma_+] = \lambda\sigma_z$ and $-\lambda[\sigma_x, \sigma_z] = 2\lambda(\sigma_+ - \sigma_-)$. The dissipators on the other hand are:

$$D[a](a) = -\frac{a}{2}$$
$$D[a](a^\dagger) = -\frac{a^\dagger}{2}$$

and we obviously have $D[a](\sigma_+) = D[a](\sigma_-) = 0$, since the photon dissipation should have no direct effect on the qubit part of the system. After doing these calculations we get the first we get the EOM:

$$\begin{aligned} \langle \dot{a} \rangle &= -i(\omega_0 - i\kappa)\langle a \rangle - i\lambda\langle \sigma_x \rangle \\ \langle \dot{\sigma_+} \rangle &= i\Omega\langle \sigma_+ \rangle - i\lambda(\langle a \rangle + \langle a \rangle^*)\langle \sigma_z \rangle \\ \langle \dot{\sigma_z} \rangle &= -2i\lambda(\langle a \rangle + \langle a \rangle^*)(\langle \sigma_+ \rangle - \langle \sigma_- \rangle) \end{aligned} \tag{4.1.4}$$

Now we introduce the same coupling parameter g that we use in the close QRM. Moreover, we write the frequency of the atom Ω in units of ω_0 as η. We will do the same for the dissipation factor, writing $\bar{\kappa} \equiv \kappa/\omega_0$ Defining the averages as $\alpha = \langle a \rangle/\eta$ $s_+ \equiv \langle \sigma_+ \rangle$ and $s_z \equiv \sigma_z$. We finally get the equations for the steady state:

$$\begin{aligned} 0 =& \bar{\kappa})\alpha + g\frac{(s_+ + s_+^*)}{2} \\ 0 =& -s_+ + g\frac{(\alpha + \alpha^*)}{2}s_z \\ 0 =& g(\alpha + \alpha^*)(s_+ - s_+^*) \end{aligned} \tag{4.1.5}$$

Since in the mean field approximation we consider that the state is pure (for example, we do approximations of the form $\langle \sigma_z a \rangle = \langle \sigma_z \rangle \langle a \rangle$, which means that there's no correlation), we use the constant of motion

$$\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1 \tag{4.1.6}$$

this yields two set of stable solutions. For $g < g_c$ we have the *first* one, where $s_+ = \alpha = 0, s_z = \pm 1$. For the *second* stable solution on the other hand we have

$$s_+ = \pm\frac{1}{2}\sqrt{1-\frac{g_c^4}{g^4}}, \quad \alpha = \mp\frac{g}{2(1-i\bar{\kappa})}\sqrt{1-\frac{g_c^4}{g^4}}, \quad s_z = -\frac{g_c^2}{g^2} \tag{4.1.7}$$

There's also a solution with positive s_z but since we're interested in the regime of lower energy we discard it.

4.2 Master Equations

Now, we simply repeat the procedure that we did in the close QRM to derive the master equation for both phases, but this time we apply the unitaries into the Lindblad equation, instead of the Hamiltonian. For the normal phase we'll have:

$$\dot{\rho} = -i[\omega a^\dagger a - \frac{\omega_0 g^2}{4}(a+a^\dagger)^2, \rho] + \kappa\mathcal{D}[a] + O(\eta^{-1/2}) \tag{4.2.1}$$

Since we have $\eta \to \infty$ in the thermodynamic limit, only the zeroth order term of the transformed dissipator survives, all the corrections of higher order are negligible. For the superradiant phase we once again apply the displacemene operator, this time using the parameter α that we got in the last section through semiclassical analysis. In this case the parameter won't coincide with what we found for the closed model, since in this case the dissiapation introduces new physics and we should expect a different displacement in the steady state. In particular, the term has now an imaginary part:

$$\alpha = \frac{g\sqrt{\eta}}{2g_c^2}(1+i\bar{\kappa})\sqrt{1-\frac{g_c^4}{g^4}} \tag{4.2.2}$$

The displacement operator actually affects the dissipator, the net effect is that we'll have new terms on the commutator:

$$\dot{\rho}' = -i[D^\dagger[\alpha]H_{Rabi}D[\alpha] + i\kappa(\alpha^* a - \alpha a^\dagger), \rho'] + \kappa(2a\rho' a^\dagger - a^\dagger a\rho' - \rho' a^\dagger a) \tag{4.2.3}$$

After we apply the displacement operator and the usual procedure, the resulting Hamiltonian for the superradiant phase is:

$$H_{sp} = \omega_0 a^\dagger a - \frac{\omega_0 g_c^6}{4g^4}(a+a^\dagger)^2 - \frac{\Omega}{4}\left(\frac{g^2}{g_c^2}+\frac{g_c^2}{g^2}\right) \tag{4.2.4}$$

with the last term corresponding to the ground state energy. Our master equation is thus:

$$\dot{\rho}' = -i[H_{sp}, \rho'] + \kappa(2a\rho' a^\dagger - a^\dagger a\rho' - \rho' a^\dagger a) \tag{4.2.5}$$

so we can see that in comparison to the normal phase, we simply have a substitution of the form:

$$g \to \frac{g_c^3}{g^2}$$

4.3 Equations of motion

The equation of motion for the bosonic operators will be given by $\boldsymbol{u} = L_{np}\boldsymbol{u}$. With Liouvillian:

$$L_{np} = \omega_0 \begin{pmatrix} -i(1-\frac{g^2}{2}) - \bar{\kappa} & \frac{ig^2}{2} \\ -\frac{ig^2}{2} & i(1-\frac{g^2}{2}) - \bar{\kappa} \end{pmatrix} \tag{4.3.1}$$

We have an analog equation for the superradiant phase. It suffices make the appropriate constant substitution that we saw at the end of the previous section. The eigenvalues of the matrix will be:

$$\ell_{np} = -\kappa \pm i\omega_0\sqrt{1-g^2} \tag{4.3.2}$$

The first point of interest is $g_1 = 1$. In this case the second terms become purely imaginary, and the eigenvalue ℓ becomes purely real, and we begin to have a decrease in the so-called asymptotic decay rate, defined as:

$$\kappa_{ADR} \equiv -Re[\ell_{np}]$$

Which is a quantity related to the "damping" of the system, if we consider that the imaginary part plays a oscillatory role. The second point of interest is the actual critical point of the system. We can see that for

$$g_c = \sqrt{1+\bar{\kappa}^2} \tag{4.3.3}$$

the decay rate actually becomes zero. In the closed QRB, the phase transition is characterized by the closing of the energy gap. But in the dissipative case the notion of an energy gap is pretty much lost, and instead what we observe is the closing of the Liouvillian gap. In this case we can see that the decay rate becomes positive when we reach the critical point, and the solution becomes unstable. This is really similar to what we saw in the closed system. If we do the same for the superradiant phase, we get

$$\ell_{sp} = -\kappa \pm i\omega_0\sqrt{1-\frac{g_c^6}{g^4}} \tag{4.3.4}$$

The point of interest here is $g_2 = g_c^{3/2}$. Between g_c and g_2 the square root is imaginary, and thus, the second term contributes to the decay rate. After we cross this point the term becomes purely imaginary once again, and the decay back returns to its original value.

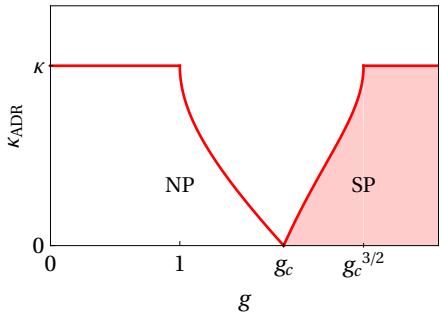


Figure 6: Asymptotic decay rate as a function of g.

Moreover, we can calculate the expected value for the bosonic operators at the steady state. For the normal phase they will be the trivial solution $\langle a \rangle = 0$. This means that there's no field displacement. However, since we displace the Hamiltonian to solve for the superradiant phase we have, at the steady state $\langle a \rangle = \alpha$. This means that we will have displacement in both quadratures. The effect of the dissipation is to introduce a displacement into the p-quadrature, since $\langle p \rangle \propto \mathrm{Im}(\alpha) \propto \kappa$, something that does not happen in the closed system.

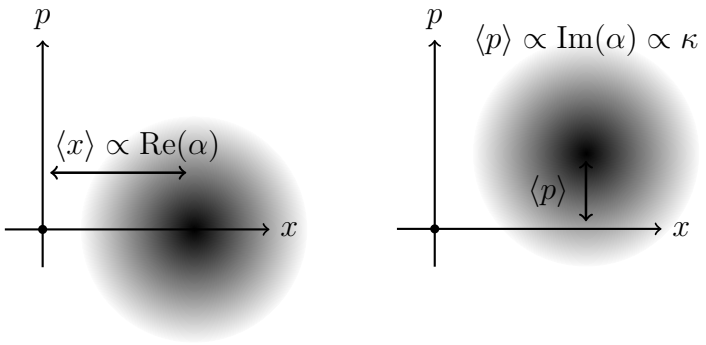


Figure 7: Depiction of the distribution in the quantum phase space at the steady state.

5 The Dicke Model

5.1 Introduction

The Dicke model is a many-body version of the Rabil model, describing the interaction between N atom and an electromagnetic mode:

$$H_{Dicke} = \omega a^\dagger a + \sum_i^N \Omega s_z^i + \frac{\lambda}{\sqrt{N}}(a + a^\dagger)s_x^i \tag{5.1.1}$$

We first write the Hamiltonian in a Rabi-like way defining an overall spin operator:

$$S_z = \frac{1}{2}\sum \sigma_z^i, \;\; S_x = \frac{1}{2}\sum \sigma_x^i \tag{5.1.2}$$

The Hamiltonian assumes the form:

$$H_{Dicke} = \omega a^\dagger a + \Omega S_z + \frac{\lambda}{\sqrt{N}}(a + a^\dagger)S_x \tag{5.1.3}$$

As we'll soon show, this model also presents a quantum phase transition with a superradiant phase in the thermodynamic limit of infinite particles $N \to \infty$, which is different from what we did for the QRM. This also means that the diagonalization procedure that we'll employ is slightly different.

5.2 Finding the eigenstates for the S_z operator

First of all, we must look for a convenient basis to express our eigenstates. We should first note that when we define the overall spin operators in this way, we should have an invariance under a permutation of spins, i.e. the ordering is not important. We can achieve this if we use the basis:

$$|N, n\rangle = \frac{1}{\sqrt{B_{Nn}}}\sum_p |1\rangle^n \otimes |0\rangle^{N-n} \tag{5.2.1}$$

where B_{Nn} represents the binomial coefficient. The index p indicates the sum over all permutations. So, in this notation for example we have:

$$|3, 1\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{5.2.2}$$

We can thus expand the ground-state energy as:

$$|\psi_{GS}\rangle = \sum_n \psi_n |N, n\rangle \tag{5.2.3}$$

There's one further observation that we should make. Let's apply the operator S_z on $|3, 2\rangle$ for example. We can verify that:

$$S_z |3, 2\rangle = \frac{1}{2\sqrt{3}}[\underbrace{(-|011\rangle + |011\rangle + |011\rangle)}_{=2S_z|011\rangle} + (|101\rangle - |101\rangle + |101\rangle) + (|110\rangle + |110\rangle - |110\rangle)]$$

thus

$$S_z |3, 2\rangle = \frac{1}{2} |3, 2\rangle \tag{5.2.4}$$

For a more general state, after we apply S_z on a term $|...\rangle$ of $|N, n\rangle$, we'll create n positive copies of this state and $N - n$ negative copies. Therefore, when S_z acs on this kind of state we have:

$$S_z |...\rangle = \frac{n}{2} |...\rangle - \frac{N-n}{2} |...\rangle = \left(n - \frac{N}{2}\right) |...\rangle \tag{5.2.5}$$

For the state $|3, 2\rangle$ for example, all of its terms have 2 spins up, and $3 - 2 = 1$ spin down. Thus, we can verify that

$$S_z |N, n\rangle = \left(n - \frac{N}{2}\right) |N, n\rangle \tag{5.2.6}$$

In other words, if we generalize this idea we can see that S_z is the eigenstate of $|m\rangle$, with eigenvalue m:

$$S_z |m\rangle = m |m\rangle \tag{5.2.7}$$

with a mapping

$$m \to \frac{N}{2} - n, \ \ |m\rangle \to |N, n\rangle \tag{5.2.8}$$

moreover, since n goes from 0 to N, we have the following bounds for m:

$$m = S, S - 1, ... - S + 1, -S$$

where $s = N/2$ is the maximum "overall" spin, i.e., the maximum eigenvalue of S_z. Therefore, we can see that we have only $2s + 1$ distinct eigenvalues, and only the overall spin has physical relevance. We can then factorize the problem to a

much smaller $\mathcal{H} = \mathcal{H}_{Bosonic} \otimes \mathcal{H}_{2s+1}$ Hilbert state, since we have highly degenerate states to this problem and states with no real physical meaning. We can thus write the Hamiltonian as:

$$H_{Dicke} = \omega a^\dagger a + \Omega S_z + \frac{\lambda}{\sqrt{2S}}(a + a^\dagger)S_x \tag{5.2.9}$$

5.3 Diagonalizing the Hamiltonian

We can know use a similar strategy to diagonalize this Hamiltonian. We first apply a rotation around the y-axis and then we displace the field. This means that our transformed Hamiltonian will be:

$$\tilde{H} = D^\dagger(\alpha)e^{i\theta S_y} H e^{-i\theta S_y} D(\alpha) \tag{5.3.1}$$

this will be:

$$\begin{aligned}\tilde{H} = &\omega[a^\dagger a + \alpha(a^\dagger + a) + \alpha^2] \\ &+ \Omega(S_x \sin\theta + S_z \cos\theta) \\ &+ \frac{2\lambda}{\sqrt{2s}}(S_x \cos\theta - S_y \sin\theta)(a^\dagger + a + 2\alpha)\end{aligned}$$

There spin operators are cumbersome, so we'll apply the **Holsten - Primakoff transformation**, which can be used to describe the spin operators through bosonic modes. We write:

$$\begin{aligned}S_z &= b^\dagger b - s \\ S_+ &= b^\dagger \sqrt{2s - b^\dagger b} \\ S_- &= b\sqrt{2s - b^\dagger b}\end{aligned} \tag{5.3.2}$$

We can verify that this correspondence obeys the appropriate algebras. Since in the thermidynamic we have a infinite number of particles, this corresponds to $s \to \infty$. We can then approximate these relations to

$$\begin{aligned}S_+ &\approx \sqrt{2s} b^\dagger \\ S_- &\approx \sqrt{2s} b\end{aligned} \tag{5.3.3}$$

considering that s is very large. This enable us to reconstruct the Hamiltonian entirely trough bosonic operators. If we make the terms linear in b equal to zero,

in the thermodynamic limit $s \to \infty$, we get the following equation:

$$\frac{\omega}{2}\sqrt{2s}\sin\theta + 2\alpha\lambda\cos\theta = 0 \tag{5.3.4}$$

and for a:

$$\Omega\alpha - \sqrt{2s}\lambda\sin\theta = 0 \tag{5.3.5}$$

These two equations yield:

$$\cos\theta = \frac{\omega\Omega}{4\lambda^2} = \frac{\lambda_c^2}{\lambda^2} \tag{5.3.6}$$

with $\lambda_c = 2\lambda/\sqrt{\Omega\omega}$, which is analogue to the parameter g in the Rabi model. This means that we have two types of solutions: the trivial one, $\theta = 0$ with $\alpha = 0$, and a solution for $\lambda > \lambda_c$ we have a solution that is not trivial, i.e:

$$\text{Phases} : \begin{cases} \theta = 0, \text{ if } \lambda < \lambda_c \\ \cos\theta = \frac{\lambda_c^2}{\lambda^2}, \text{ if } \lambda > \lambda_c \end{cases}$$

this corresponds to:

$$\alpha = \begin{cases} 0, \text{ if } \lambda < \lambda_c \\ -\lambda\sqrt{2s}\sqrt{1 - \frac{\lambda_c^4}{\lambda^4}}, \text{ if } \lambda > \lambda_c \end{cases}$$

These phases correspond to the normal and superradiant phases, respectively.

6 Appendix

6.1 Single-mode Squeezing

6.1.1 Action on the bosonic operators

Definition 1. *We define the squeezing operator as*

$$S(z) = e^{\frac{za^\dagger a^\dagger - z^* aa}{2}} \tag{6.1.1}$$

where $z = re^{i\theta}$, as we'll see, is a complex number related to uncertainity of the quadratures.

These operators are relevant in this context because when we apply them to the creation and annihilation operators, the following Bogoliubov transformation is obtained:

Result 1. *The squeezing operators enable us to consruct a new set of bosonic operators given by*

$$\tilde{a} = S^\dagger(z) a S(z) = a \cosh r + a^\dagger e^{i\theta} \sinh r \tag{6.1.2}$$

and

$$\tilde{a}^\dagger = S^\dagger(z) a^\dagger S(z) = a^\dagger \cosh r + a e^{-i\theta} \sinh r \tag{6.1.3}$$

First of all, we should notice that the operator in the exponential is skew-hermitian, since $A^\dagger = \frac{1}{2}(za^\dagger a^\dagger - z^* aa)^\dagger = \frac{1}{2}(z^* aa - za^\dagger a^\dagger) = -A$. Thus, the expression that we want to evaluate is basically $e^{-A} a e^{A}$. It's possible to use one of the BCH formulae to find:

$$e^{-A} a e^{A} = a - [A, a] + \frac{1}{2!}[A, [A, a]] - \frac{1}{3!}[A, A, [A, a]]] + ... \tag{6.1.4}$$

The first commutator is simply

$$\frac{1}{2}[za^\dagger a^\dagger - z^* aa, a] = \frac{z}{2}[a^\dagger a^\dagger, a] = \frac{z}{2}(a^\dagger [a^\dagger, a] + [a^\dagger, a] a^\dagger) = -za^\dagger = -re^{i\theta} a^\dagger$$

The second one, on the other hand, yields

$$[A,[A,a]] = [z^* aa, za^\dagger] = r^2 a$$

The third commutator results in $-r^3 e^{i\theta} a^\dagger$, and so on. Therefore, the expansion that we are looking for is

$$e^{-A} a e^{A} = S_z^\dagger a S_z = a\left(1 + \frac{r^2}{2} + ...\right) + a^\dagger e^{i\theta}\left(r + \frac{r^3}{3} + ...\right) \tag{6.1.5}$$

which can also be written as:

$$S_z^\dagger a S_z = a\cosh r + a^\dagger e^{i\theta}\sinh r \tag{6.1.6}$$

The proof for the creation operator is analogous.

6.1.2 Variance of the quadrature operators

Suppose we define the quadrature operators as $x = a + a^\dagger$ and $p = i(a^\dagger - a)$. We have the following result:

Result 2. *The variance of the quadrature operators in the squeezed vacuum state are give by:*

$$\Delta x^2 = \cosh 2r + \sinh 2r \cos\theta \tag{6.1.7}$$

and

$$\Delta p^2 = \cosh 2r - \sinh 2r \cos\theta \tag{6.1.8}$$

Since the Fock states are orthogonal, it's easy to see that the first moments $\langle a\rangle$ and $\langle a^\dagger\rangle$ will be null. For the second moment of x on the other hand, we have:

$$\langle x^2\rangle = \langle 0|\, S^\dagger[r](a + a^\dagger)^2 S[r]\,|0\rangle$$

But:

$$S^\dagger[r](a + a^\dagger)S[r] = (\cosh r + e^{-i\theta}\sinh r)a + (\cosh r + e^{i\theta}\sinh r)a^\dagger$$

If we take the square of this expression, only one term will remain, because all the other one will act on the vacuum and go to zero. The surviving term will be:

$$\langle x^2\rangle = (\cosh r + e^{-i\theta}\sinh r)(\cosh r + e^{i\theta}\sinh r)\,\langle 0|\, aa^\dagger\,|0\rangle$$

Since $aa^\dagger = a^\dagger a + 1$. we have

$$\langle x^2 \rangle = \cosh^2 r + \sinh^2 r + 2 \sinh r \cosh r (e^{i\theta} + e^{-i\theta}) \tag{6.1.9}$$

and this simplifies to the expressions written above. Note that when the angle of hte squeezing parameter is real, i.e $\theta = 0$, we have $\Delta x = e^r$ and $\Delta p = e^{-r}$.

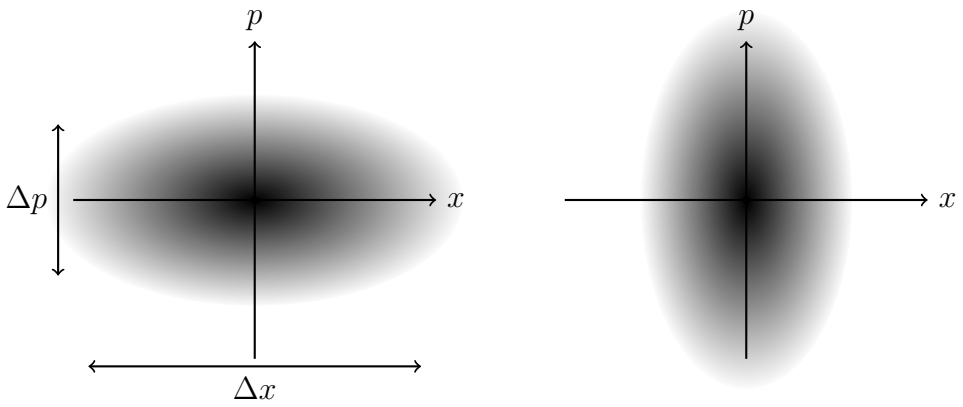


Figure 8: Depiction of the uncertainities of a squeezed vacuum state.

6.2 Displacement Operator

Definition 2. *We define the displacement operator as*

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{6.2.1}$$

Where α is a arbitrary complex number related to the displacement in the phase space.

By making use of the BCH formula, as we did for the Squeezing operator, we can show that the Displacement operator has the following property:

Result 3. *The displacement acts on the annihilation operator according to*

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha \tag{6.2.2}$$

and on the creation operator as

$$D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^* \tag{6.2.3}$$

This operator also has a pretty intuitive effect in the phase space - if we define the *displaced vacuum* as $|\alpha\rangle \equiv D(\alpha)\,|0\rangle$ the expected value for the quadratures are:

$$\langle x\rangle = Re(\alpha), \quad \text{and} \quad \langle p\rangle = Im(\alpha) \tag{6.2.4}$$

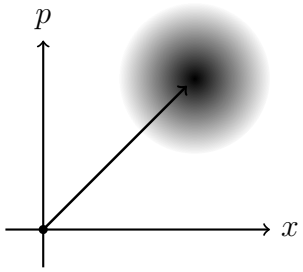


Figure 9: Depiction of the a displaced squeezed vacuum state.

so the displacement operator basically "displaces" the expected value of quadratures of the field but doesn't change the uncertainties.

6.3 Diagonalization of the Hamiltonian

Suppose we have a Hamiltonian of the form:

$$H = \xi a^\dagger a + \kappa a^\dagger a^\dagger + \kappa^* aa \tag{6.3.1}$$

A good strategy to diagonalize this kind of Hamiltonian is the use of the Squeeze operator. If we use the results of the last section for the transformation of the creation and annihilation operators, we will see that the first term transform as:

$$S_z^\dagger a^\dagger a S_z = (S_z^\dagger a^\dagger S_z)(S_z^\dagger a S_z) = (a^\dagger \cosh r + a e^{-i\theta}\sinh r)(a\cosh r + a^\dagger e^{i\theta}\sinh r)$$

In the first equality I made use of the fact that S_z is unitary. Hence, the expression for the diagonalized Hamiltonian can be rewritten as:

$$\begin{aligned} S_z^\dagger H S_z = {} & \xi(a^\dagger\cosh r + e^{-i\theta}a\sinh r)(a\cosh r + e^{i\theta}a^\dagger\sinh r) \\ & + \kappa(a^\dagger\cosh r + a e^{-i\theta}\sinh r)^2 \\ & + \kappa^*(a\cosh r + e^{i\theta}a^\dagger\sinh r)^2 \end{aligned}$$

If we write the constant κ in polar form as $\kappa = |\kappa|e^{i\phi}$, we can conveniently choose the angle θ of the squeezing parameter to be the same as ϕ in order to simplify the expansions, obtaining:

$$\begin{aligned} S_z^\dagger H S_z = &(\xi \cosh 2r + |\kappa| \sinh 2r) a^\dagger a \\ &+ (\frac{\xi}{2} \sinh 2r + |\kappa| \cosh 2r)(e^{i\phi} a^\dagger a^\dagger + e^{-\phi} a a) \\ &+ |\kappa| \sinh 2r + \xi \sinh^2 r \end{aligned}$$

We should choose an r such that makes the expression in the second line go to zero, therefore:

Result 4. *A Hamiltonian of the form* (6.3.1) *can be diagonalized for a real squeezing parameter given by:*

$$\tanh 2r = -\frac{2|\kappa|}{\xi} \tag{6.3.2}$$

or, alternatively

$$e^{4r} = \frac{\xi - 2|\kappa|}{\xi + 2|\kappa|} \tag{6.3.3}$$

This choice of parameters yields the following diagonalized form for the Hamiltonian:

$$H = \tilde{\xi} a^\dagger a + \frac{\tilde{\xi} - \xi}{2} \tag{6.3.4}$$

with

$$\tilde{\xi} = \sqrt{\xi^2 - 4|\kappa|^2} \tag{6.3.5}$$

It's also clear that we should have:

$$|\xi| > |2\kappa| \tag{6.3.6}$$

6.4 Two level systems

Suppose we parametrize an arbitrary 2×2 square hermitian matrix as:

$$A = A_0 + \boldsymbol{a} \cdot \boldsymbol{\sigma} = \begin{pmatrix} a_0 + a_z & a_x - i a_y \\ a_x + i a_y & a_0 - a_z \end{pmatrix} \tag{6.4.1}$$

We may also define $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ and $\boldsymbol{n} = \boldsymbol{a}/a = (n_x, n_y, n_z)$. With these definitions, we can parametrize the matrix as:

$$A = A_0 + a(\boldsymbol{n} \cdot \boldsymbol{\sigma}) \tag{6.4.2}$$

If we express the vector $\boldsymbol{n} = (\sin 2\theta \cos 2\phi, \sin 2\theta \sin 2\phi, \cos 2\theta)$ in spherical coordinates, the matrix assumes a nice form:

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \cos 2\theta & e^{-2i\phi} \sin 2\theta \\ e^{2i\phi} \sin 2\theta & -\cos 2\theta \end{pmatrix} \tag{6.4.3}$$

This method is useful because the matrix of eigenvectors assumes a particularly simple form:

$$G = \begin{pmatrix} e^{-i\phi} \cos\theta & -e^{i\phi} \sin\theta \\ e^{i\phi} \sin\theta & e^{i\phi} \cos\theta \end{pmatrix} \tag{6.4.4}$$

In the absence of a y-component the angle ϕ is null and this matrix assumes a particularly simple form, which is the case of our work:

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{6.4.5}$$

The only relevant transformation will be the effect of the change of basis on σ_x, which will become:

$$\sigma_x = U^\dagger \tau_x U = \sin 2\theta \tau_x + \cos 2\theta \tau_z \tag{6.4.6}$$

because $\tau_x = U \sigma_x U^\dagger$. We should also note that the Eigenvalues are given by:

$$\mathrm{Eig}(A) = a_0 \pm a \tag{6.4.7}$$

In our case $\boldsymbol{a} = (-2\lambda\alpha, 0, \Omega/2)$, therefore:

$$\mathrm{Eig}(A) = \pm \frac{\sqrt{\Omega^2 + 16\lambda^2\alpha^2}}{2} = \pm \frac{\tilde{\Omega}}{2} \tag{6.4.8}$$

and lastly, by identifying the vector $\boldsymbol{a}$ we can also write the relationship between the angle θ and the parameters of the system:

$$\tan 2\theta = -\frac{4\lambda\alpha}{\Omega} \tag{6.4.9}$$

References

[1] How to fix a broken symmetry: quantum dynamics of symmetry restoration in a ferromagnetic Bose-Einstein condensate. 10(4):45023, 2008.

[2] Clive Emary and Tobias Brandes. Chaos and the quantum phase transition in the dicke model. *Phys. Rev. E*, 67:066203, Jun 2003.

[3] Christopher Gerry and Peter Knight. *Introductory Quantum Optics*. Cambridge University Press, 1 edition, 2004.

[4] Jonathan A Gross, Carlton M Caves, Gerard J Milburn, and Joshua Combes. Qubit models of weak continuous measurements: markovian conditional and open-system dynamics. *Quantum Science and Technology*, 3(2):24005, 2018.

[5] Myung-Joong Hwang, Ricardo Puebla, and Martin B. Plenio. Quantum phase transition and universal dynamics in the rabi model. *Physical Review Letters*, 115(18), oct 2015.

[6] Myung-Joong Hwang, Peter Rabl, and Martin B. Plenio. Dissipative phase transition in the open quantum rabi model. *Phys. Rev. A*, 97:013825, Jan 2018.

[7] Sílvio R. A. Salinas. *Introdução à Física Estatística*. Edusp, 2 edition, 1999.

[8] Marlan O. Scully. *Quantum Optics*. Cambridge University Press, 1 edition, 1997.