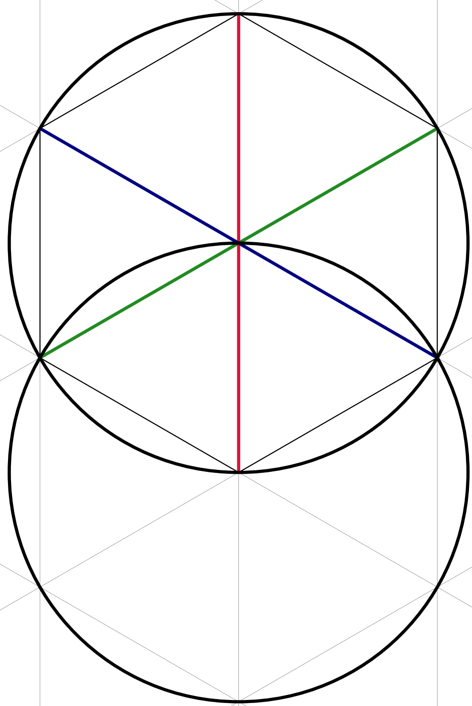
Semiclassical Gravity

Níckolas de Aguiar Alves



Nel mezzo del cammin di nostra vida
mi ritrovai per una selva oscura
ché la diritta via era smaritta.

DANTE ALIGHIERI, *Inferno*, I, vv. 1–3.

Preface

I've been writing these notes as a way to organize my studies for my MSc project at the Federal University of ABC (UFABC), Brazil. They are still far from being complete, assuming they ever will be.

Currently, I'm interested in learning how to describe quantum fields in the presence of a classical gravitation field, which is a framework commonly known as Quantum Field Theory in Curved Spacetime or Semiclassical Gravity. This is the main subject of these notes, which assume reasonable familiarity with both General Relativity and Quantum Field Theory in flat spacetime. The main reference throughout most of this text is [19]. Chapter 1 on page 1 is an exception and relies mainly on the references there listed. Supplementary references are used and cited throughout the text. I don't claim originality on the results presented here.

I appreciate the interest in my work and I would be extremely pleased to receive comments, critics, compliments and etc through my e-mail (alves.nickolas@ufabc.edu.br). If you wish to check some more works, please check my personal website `https://alves-nickolas.github.io`. In case you are reading this document in a distant future in which I've already concluded my MSc project, you might want to check my ORCID iD (`https://orcid.org/0000-0002-0309-735X`) for updated contact information.

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Níckolas de Aguiar Alves
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Contents

One

How Do Elementary Particles Fall?

> L'amor che move il sole e l'altre stelle.
>
> Dante Alighieri, *Paradiso*, XXXIII, v. 145.

SOME problems in Physics are simple to understand, but hard to answer. Why do things fall? What is everything made of? Humankind has been trying to find an appropriate answer to these questions for millennia. Some pre-socratic philosophers had their preferred explanations, such as Heraclitus' belief on fire as a primordial element, with similar thoughts by Thales with water or Anaximenes with air[12, p. 71]. Some time later Aristotle proposed an explanation for both questions in terms of five elements. As Russell puts it,[12, pp. 241–242]

> The treatise *On the Heavens* sets forth a pleasant and simple theory. Things below the moon are subject to generation and decay; from the moon upwards, everything is ungenerated and indestructible. The earth, which is spherical, is at the centre of the universe. In the sublunary sphere, everything is composed of the four elements, earth, water, air, and fire; but there is a fifth element, of which the heavenly bodies are composed. The natural movement of the terrestrial elements is rectilinear, but that of the fifth element is circular. The heavens are perfectly spherical, and the upper regions are more divine than the lower. The stars and planets are not composed of fire, but of the fifth element; their motion is due to that of spheres to which they are attached. (All this appears in poetical form in Dante's *Paradiso*).
>
> The four terrestrial elements are not eternal, but are generated out of each other—fire is absolutely light, in the sense that its natural motion is upward; earth is absolutely heavy. Air is relatively light, and water is relatively heavy.

Our view of Fundamental Physics* changed throughout the years. Eventually we learned matter could be described in terms of atoms, atoms in terms of smaller particles, and so on. Our view of gravity eventually gained a new description in terms of an

*A term we use instead of "elementary particle physics and relativistic theories", in the spirit of [16].

inverse-square law. Electric and magnetic interactions were studied and eventually unified. These provided improvements to Aristotle's view, which can then be thought of as a simplified description which works for a more restricted collection of phenomena.

Fundamental Physics in the XI century involves two remarkable theories which are able to describe a myriad of phenomena: the Standard Model of Particle Physics (SM), a quantum field theory (QFT) endowed with $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ gauge symmetry in a Minkowski spacetime[14], and General Relativity (GR), a classical field theory describing spacetime as a pseudo-Riemannian manifold and gravity as a geometric consequence[18].

While centuries of work were able to provide more accurate descriptions of reality regarding both questions, they also took the answers along two separate ways. Aristotle's theory might have worked only up to some low energy scale, but it unified all interactions in a few simple concepts. Our current theories, on the other hand, are quite different. For example, the SM lives on a flat spacetime where probabilities are essential to understanding everything. GR, on the other hand, describes a curved spacetime as a smooth manifold where deterministic phenomena take place.

In principle, this is not a problem, in the sense that solving it is not a matter of trying to fix a measurement we are not able to describe. Instead, it is a matter of consistency. GR is unaware of quantum effects and the SM is unaware of gravitational effects. Maybe one could dare to say that, as far as we know, elementary particles don't fall. But it is expected that at sufficiently high energies these descriptions will fail, just as Aristotle's has, and a new, more profound theory should become necessary.

A way of perceiving this is by realizing that, when brought together, quantum mechanics (QM), special relativity (SR) and GR would imply one can't determine the value of a given field on an arbitrarily small location. To see this, we follow the argument on [10, pp. 6–7].

Suppose we want to measure the value of a field on some position x. In particular, this requires us to have a measurement locating such position. Will assume we want it to be of precision at least L, *id est*, we want to have $\Delta x < L$. Let us assume we'll measure such position by placing a particle there. Heisenberg's uncertainty principle implies then that the momentum of the particle will respect $\Delta p \gtrsim \frac{\hbar}{\Delta x} > \frac{\hbar}{L}$. However, we know that $\langle p^2 \rangle = \Delta p^2 + \langle p \rangle^2 \geq \Delta p^2$, and as a consequence we see that $p^2 \gtrsim \left(\frac{\hbar}{L}\right)^2$. For high energies (*id est*, $E \gg m$, where m is the mass of the particle we are considering), we can take $E \sim pc$ and we see that the energy of the particle respects $E \gtrsim \frac{\hbar c}{L}$.

This already incorporates SR and QM, but we have not considered gravitational effects so far. The stress-energy tensor is the source of gravitational effects and an amount of energy E will induce gravitational effects analogous to those of a mass M respecting $E \sim Mc^2$. Furthermore, a mass M is related to a Schwarzschild radius $R \sim \frac{GM}{c^2}$. If the energy of the particle we are using to probe the field is too large, it will form a black hole with radius $R > L$, which forbids us of localizing the point we want to measure. Hence, we are interested in the limiting case $L \sim \frac{GM}{c^2} \sim \frac{GE}{c^4}$. This will allows us to compute the maximum precision we would be able to measure without creating a black hole that forbids us from measuring. Substituting this approximate expression in our previous estimate for L leads to

$$E \gtrsim \frac{\hbar}{cL}, \tag{1.1a}$$

$$\frac{Lc^4}{G} \gtrsim \frac{\hbar}{cL}, \tag{1.1b}$$

$$L^2 \gtrsim \frac{\hbar G}{c^3}, \tag{1.1c}$$

which leads us to the definition of the so-called Planck length,

$$L_P = \sqrt{\frac{\hbar G}{c^3}}. \tag{1.2}$$

Of course, to state black holes would form if we try to look too close need not to be true, for this is a result that occurred from extrapolating GR to incredibly high energies. Nevertheless, it allows us to see that interesting Physics should occur at this scale, known as Planck scale. In particular, it puts a shadow of doubt on the validity of the predictions of GR in extreme conditions, such as in black hole singularities or the Big Bang and invites us to look further for a new theory.

While we currently do not know how gravity behaves near the Planck scale, there are candidate theories, such as String Theory[2], Loop Quantum Gravity[10], Asymptotically Safe Quantum Gravity[4], and much more[7]. Nevertheless, we need not, and in these notes we shall not, dive so deep to start studying quantum effects in the presence of gravitational fields. In the same way one can study the interaction of atomic systems with a classical electromagnetic field (as discussed in [13, 20], for example), one can consider what happens to quantum fields in classical, but curved, spacetime. Naively, we could say we are about to investigate how elementary particles fall, though this silly view should be taken with care since, as we shall realize, the notion of a particle will stop making sense once things start to fall.

Two

Quantum Mechanics Revisited

> Nessun maggior dolore che ricordarsi del tempo
> felice ne la miseria
>
> Dante Alighieri, *Inferno*, V, vv. 121–123.

2.1 Classical Mechanics

Geometric Formulation of Hamiltonian Mechanics

A wide range of physical systems can be described in terms of Hamiltonian mechanics. This framework, explained for example in [6], consists in describing the state of a system with N degrees of freedom in terms of some generalized coordinates, $q^1, \ldots, q^N$, and their canonically conjugate momenta, $p_1, \ldots, p_N$. Dynamics is then provided by a Hamiltonian function $H = H(q^1, \ldots, q^N, p_1, \ldots, p_N)$ which determines the equations of motion through the so-called canonical equations or Hamilton's equations, given by

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i}. \end{cases} \tag{2.1}$$

This can be expressed in a more compact notation if we define a $2N$-component object z^i such that $z^i = q_i$ and $z^{i+N} = p_i$ for $1 \leq i \leq N$ and a $2N \times 2N$ object Ω^{ij} with components given by

$$\Omega^{ij} = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}, \tag{2.2}$$

where $\mathbb{1}_N$ denotes the $N \times N$ identity matrix. With this notation, we may write

$$\dot{z}^i = \Omega^{ij} \frac{\partial H}{\partial z^j}. \tag{2.3}$$

This expression resembles another possible formulation of Hamiltonian dynamics, described, for example, in [1]. Let us still consider the same physical system with N

degrees of freedom we've been thinking about. Given a N-dimensional configuration manifold $\mathcal{Q}$, which naively consists of the space locally described by the generalized coordinates $q^1, \ldots, q^N$, we can construct the theory's phase space by considering the cotangent bundle to the manifold, $T^*\mathcal{Q}$. This is a $2N$-dimensional manifold and, given the coordinate system $q^1, \ldots, q^N$ on the configuration manifold, we can obtain a local coordinate system on $T^*\mathcal{Q}$ by employing the generalized coordinates $q^1, \ldots, q^N$ and the components of cotagent vectors, $p_1, \ldots, p_N$. We can equip the phase space with a non-degenerate, closed, 2-form given in terms of the local (q^i, p_i) coordinates by

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i \,, \tag{2.4}$$

where we are summing over repeated indices.

We say it is non-degenerate because $\omega_{ab}v^b = 0$ if, and only if, v^b vanishes. Indeed, if we let

$$v^b = f_i \frac{\partial}{\partial p_i} + g^i \frac{\partial}{\partial q^i}, \tag{2.5}$$

then

$$\omega_{ab}v^b = (\mathrm{d}p_i \otimes \mathrm{d}q^i)(v, \cdot) - (\mathrm{d}q^i \otimes \mathrm{d}p_i)(v, \cdot), \tag{2.6a}$$

$$= f_i \, \mathrm{d}q^i - g^i \, \mathrm{d}p_i \,. \tag{2.6b}$$

We say it is closed* because $\mathrm{d}\omega = 0$. In fact, notice that

$$\omega = \mathrm{d}\big(p_i \, \mathrm{d}q^i\big) \,, \tag{2.7a}$$

$$= \mathrm{d}p_i \wedge \mathrm{d}q^i \,, \tag{2.7b}$$

which means ω is locally exact and, *a fortiori*, it is closed.

We could also build this form ω in a coordinate invariant way. While our present construction does not depend on our choice of coordinates on $\mathcal{Q}$, it depends on our choice of coordinates on $T^*\mathcal{Q}$, but of the great strengths of the Hamiltonian formulation of Mechanics is the possibility of performing canonical transformations, which are nothing but coordinate changes in $T^*\mathcal{Q}$.

Our discussion follows [1, p. 202]. Our goal is to obtain a closed, non-degenerate, 2-form ω on $T^*\mathcal{Q}$. Our discussion with coordinates already showed we can get a locally exact form, and hence we'll look for the 1-form $\theta \in T_z^*(T^*\mathcal{Q})$ such that $\mathrm{d}\theta = \omega$. This is a map $\theta \colon T_z(T^*\mathcal{Q}) \to \mathbb{R}$.

There is a natural projection map $f \colon T^*\mathcal{Q} \to \mathcal{Q}$ which takes a $1 - form$ at $q \in \mathcal{Q}$ to the point q itself. Notice that $f^{-1}(q) = T_q^*\mathcal{Q}$. We want to obtain a function θ with domain $T_z(T^*\mathcal{Q})$. However, we notice that the derivative f_* of f is a map $f_* \colon T(T^*\mathcal{Q}) \to T\mathcal{Q}$. Hence, we have a natural function from $T(T^*\mathcal{Q})$ into $T\mathcal{Q}$, since it is a mere projection.

*We recall a p-form ω is called exact if, and only if, there is a $p-1$-form θ such that $\omega = \mathrm{d}\theta$. While all exact forms are closed, the non-trivial problem of determining which closed forms are globally exact is the issue dealt with by de Rham cohomology. In the particular case of $\mathbb{R}^n$, and hence locally on any manifold, a p-form is closed if, and only if, it is exact.

$$\begin{tikzcd} T_{(q,p)}(T^*\mathcal{Q}) \arrow[r, "f_*"] \arrow[dr, "\Theta"'] & T_q\mathcal{Q} \arrow[d, "p"] \\ & \mathbb{R} \end{tikzcd}$$

Figure 2.1: *To get a 1-form θ on $T^*\mathcal{Q}$, we can exploit the projection map from $T^*\mathcal{Q}$ into $\mathcal{Q}$ and the very point at which the tangent vector to $T^*\mathcal{Q}$ we are considering is given.*

We'll be able to obtain a candidate for θ if we can find a candidate map between an arbitrary tangent space $T_q\mathcal{Q}$ and $\mathbb{R}$.

Given $\xi \in T_z(T^*\mathcal{Q})$, for $z = (q, p) \in T^*\mathcal{Q}$, we want to obtain a real number. We are already able to obtain a tangent vector to $\mathcal{Q}$ at q. This is done by employing f_*, for $f_*(\xi) \in T_q\mathcal{Q}$. However, we started with a 1-form at $z = (q, p)$, which naturally includes an element of $T_q^*\mathcal{Q}$: p. We might then map $f_*(\xi)$ to a real number by simply computing $p(f_*(\xi))$.

Hence, we define $\theta \colon T_{(q,p)}(T^*\mathcal{Q}) \to \mathbb{R}$ according to $\theta(\xi) = p(f_*(\xi))$, as illustrated on Fig. 2.1.

Suppose now we are given a system of coordinates $q^1, \ldots, q^N$ on $\mathcal{Q}$. Notice then that if $\xi \in T_{(q,p)}(T^*\mathcal{Q})$ is given by $\xi = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$ we get

$$f_*(\xi) = a^i \frac{\partial}{\partial q^i}. \tag{2.8}$$

Furthermore, the 1-form $p \in T_q^*\mathcal{Q}$ is of the form $p = p_i \, \mathrm{d}q^i$, which means

$$p(f_*(\xi)) = p_i a^i, \tag{2.9}$$

which means $\theta(\xi) = p_i \, \mathrm{d}q^i \, (\xi)$, *id est*, $\theta = p_i \, \mathrm{d}q^i$ and we recover the local expression $\omega = \mathrm{d}\theta = \mathrm{d}p_i \wedge \mathrm{d}q^i$.

A $2N$-dimensional manifold $\mathcal{M}$ endowed with a non-degenerate, closed, 2-form ω is known as a symplectic manifold, and ω is known as a symplectic form. We state, without proof, the so-called Darboux Theorem

Theorem 2.1 [Darboux]**:**
Let $(\mathcal{M}, \omega)$ be a $2N$-dimensional symplectic manifold. One can always choose local coordinates $(q^1, \ldots, q^N, p_1, \ldots, p_N)$ such that ω reads

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i \,. \qquad \square$$

An atlas in which every chart leaves ω in this form, and hence every transition function preserves the symplectic structure, is known as a symplectic atlas.

For a proof, one can refer to [1, pp. 229–232]. Notice this result means we'll be able to start from the manifold constructions and recover "standard" Hamiltonian mechanics.

The presence of a symplectic structure allows one to obtain an isomorphism between $T_z(T^*\mathcal{Q})$ and $T_z^*(T^*\mathcal{Q})$. If we are given a vector $\xi \in T_z(T^*\mathcal{Q})$, the structure is simple. We can obtain a one-form ω_ξ by defining

$$\omega_\xi(\eta) = \omega(\eta, \xi) \tag{2.10}$$

for every $\eta \in T_z(T^*\mathcal{Q})$.

Since ω is a one-form, it follows that $\xi \mapsto \omega(\cdot, \xi)$ is indeed linear. Since ω is non-degenerate, it also follows that it is injective (its kernel is just the zero vector). Finally, the Rank-Nullity Theorem ensures it is an isomorphism. Indeed, if we denote this linear map by Ω (as we shall from now on), we then see that $\dim(\operatorname{Ran}\Omega) = \dim(T_z^*(T^*\mathcal{Q}))$, even though $\operatorname{Ran}\Omega \subseteq T_z^*(T^*\mathcal{Q})$. The only way this can happen is if $\operatorname{Ran}\Omega = T_z^*(T^*\mathcal{Q})$.

Notice that Ω can also be considered as a $(2,0)$-type tensor with $\Omega^{ab}\omega_{bc} = \delta^a{}_c$.

With this at hand, we are able to start talking about dynamics. If we are given a Hamiltonian $H\colon T^*\mathcal{Q} \to \mathbb{R}$, then we can obtain a 1-form by considering $\mathrm{d}H$. To this 1-form we can associate a vector field $h^a \equiv \Omega^{ab}(\mathrm{d}H)_b$, known as the Hamiltonian vector field associated to H. The one-parameter group of diffeomorphisms $g_H^t\colon T^*\mathcal{Q} \to T^*\mathcal{Q}$ associated to h^a through

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} g_H^t(z)\right|_{t=0} = h(z) \tag{2.11}$$

is known as the Hamiltonian phase flow associated to the Hamiltonian H. It can be shown that this flow preserves the symplectic structure[1, p. 204], and as a consequence it preserves the volume form $\epsilon = \frac{1}{N!}\bigwedge_{i=1}^N \omega$.

If we write Eq. (2.11) in a particular coordinate basis $z = (q, p)$ in which $\omega = \mathrm{d}p_i \wedge \mathrm{d}q_i$, we get

$$\dot{z}^i = \Omega^{ij}\frac{\partial H}{\partial z^j}. \tag{2.12}$$

Since $\omega = \mathrm{d}p_i \wedge \mathrm{d}q_i$, we can arrange it in a matrix

$$\omega = \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \tag{2.13}$$

with entries given by (q, p). Similarly,

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix} \tag{2.14}$$

and hence Eq. (2.12) simply recovers Eq. (2.3) on page 5 locally when we are in an appropriate coordinate system. Hence, the Hamiltonian flow corresponds precisely to the solutions of the equations of motion.

As a consequence, given a point in $T^*\mathcal{Q}$, we have sufficient initial data to obtain the evolution of the physical system. As a second consequence, we can then identify the phase space $T^*\mathcal{Q}$ with $\mathcal{S}$, the manifold comprised of the solutions of the equations of motion.

Within the context of Classical Mechanics, observables are smooth functions defined on phase space. We'll write $\mathcal{O} \equiv \mathcal{C}^\infty(T^*\mathcal{Q}) = \mathcal{C}^\infty(T^*\mathcal{S})$. Naturally, we can add smooth functions to obtain a new smooth function as we can multiply them by a real number to obtain another smooth function, which means $\mathcal{O}$ is a vector space. Due to the symplectic structure on the phase space, we are able to also equip $\mathcal{O}$ with a so-called Poisson bracket, given by

$$\{F, H\}(z) = \frac{\mathrm{d}}{\mathrm{d}t} F(g_H^t(z))\bigg|_{t=0} = \dot{z}^i \frac{\partial F}{\partial z^i}, \tag{2.15}$$

where the dot denotes differentiation with respect to the parameter of the Hamiltonian flow associated to H. Notice the Poisson bracket is the derivative of F along the flow. In particular, F is conserved along the flow if, and only if, $\{F, H\} = 0$.

If we employ Eq. (2.11) on the preceding page, we can rewrite the Poisson bracket as

$$\{F, H\} = (\mathrm{d}F)_a \Omega^{ab} (\mathrm{d}H)_b, \tag{2.16}$$

which can also be rewritten as

$$= \omega_{ab} \Omega^{bc} (\mathrm{d}F)_c \Omega^{ad} (\mathrm{d}H)_d, \tag{2.17}$$

$$= \omega_{ab} h^a f^b, \tag{2.18}$$

where h^a and f^a are the Hamiltonian vector fields associated to H and F, respectively.

Since $\{F, H\} = \omega_{ab} h^a f^b$, the Poisson bracket is antisymmetric. It also holds that

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \tag{2.19}$$

which is known as the Jacobi identity. See, *exempli gratia*, [1, pp. 216–217].

Of particular relevance is the Poisson bracket of the (q, p) coordinates of a symplectic chart. By employing Eqs. (2.14) and (2.16) on the facing page and on the current page we see that

$$\{q^i, q^j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta^i{}_j. \tag{2.20}$$

Linear Dynamical Systems

We are going to be concerned with a particular case of interest within Classical Mechanics. Namely, that of linear dynamical systems.

A dynamical system is a tuple (G, M, φ) in which G is a semigroup, M is a set and $\varphi \colon G \times M \to M$ is a function that maps $(g, x) \mapsto \varphi_g(x)$ with the properties that $\varphi_g \circ \varphi_h = \varphi_{g \circ h}$ and $\varphi_e = \mathrm{id}_M$, e being the identity of the semigroup[17, p. 187]. In our present situation, we have the real numbers as our semigroup, the phase space as our set (we might also project onto the configuration space later on), and the Hamiltonian flow as the function. Given a point in the phase space and some time t, the Hamiltonian flow tells us where that point is headed.

The linearity we mean in "linear dynamical system" comes in by some extra impositions we'll make on the configuration space and on the Hamiltonian. Namely,

i. we'll demand $\mathcal{Q}$ to have a vector space structure. This implies that given a vector basis we can build "linear coordinates" $q^1, \ldots, q^N$ on $\mathcal{Q}$. Furthermore, such properties will lead to a linear structure on $T^*\mathcal{Q}$ and, *a fortiori*, to linear coordinates $q^1, \ldots, q^N, p_1, \ldots, p_N$ on phase space;

ii. with linear coordinates at hand, we'll also impose that the Hamiltonian is quadratic on the phase space variables, so that the equations of motion will be linear when expressed in terms of the linear coordinates.

This is a mathematical way of saying "we'll consider a system comprised of harmonic oscillators", with the difference we shall allow them to have time-dependent spring constants, negative masses, and so on.

The linear coordinates on $\mathcal{Q}$ arise from simply taking the vector space coordinates as the manifold coordinates. A vector $(3, 2, 1)^\intercal$ will have coordinates $q^1 = 3$, $q^2 = 2$ and $q^1 = 1$, for example.

This structure allows us to find an identification between $\mathcal{Q}$ and $T_q\mathcal{Q}$, where $q \in \mathcal{Q}$ is any particular point. Indeed, we can obtain an isomorphism by mapping the basis vector in $\mathcal{Q}$ associated to the q^1 coordinate to the tangent vector $\left(\frac{\partial}{\partial q^1}\right)_q$. Naturally, this shall also hold for $T^*\mathcal{Q}$ and $T_z(T^*\mathcal{Q})$ once we are able to equip it with a linear structure. We already knew how to sum momenta, since $T_q^*\mathcal{Q}$ already has a vector space structure. The fact we now also know how to add positions together due to the newly acquired linear structure of $\mathcal{Q}$ allows us to treat $T^*\mathcal{Q}$ linearly.

Since $T^*\mathcal{Q}$ and $T_z(T^*\mathcal{Q})$ are isomorphic, we can now see the symplectic form ω from a different perspective. Now we are able to consider the bilinear function $\omega\colon T^*\mathcal{Q}\times T^*\mathcal{Q} \to \mathbb{R}$, which is independent of the particular point z we choose for the isomorphism, since its components in (linear) symplectic coordinates are constant. By employing Eq. (2.13) on page 8 we see that

$$\omega(z_1, z_2) = {z_1}^a \omega_{ab} {z_2}^b, \tag{2.21a}$$

$$= \begin{pmatrix} q_1 & p_1 \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}, \tag{2.21b}$$

$$= p_1 q_2 - q_1 p_2, \tag{2.21c}$$

where we are omitting indices for simplicity.

With such structures at hand, we can trade the symplectic manifold $(T^*\mathcal{Q}, \omega)$ for a symplectic vector space, which we shall denote by $(\mathcal{M}, \omega)$.

Be it in a manifold or in a vector space, we are still making a lot of references to a particular choice of basis, which could be dangerous and hard to generalize. In particular, it would be interesting for us to look for an expression of Eq. (2.20) on the previous page, the fundamental Poisson brackets from which we can obtain the ones for any other observable, in a basis-independent form. By that we mean an expression which reflects Eq. (2.20) on the preceding page without making explicit reference to a specific choice of linear coordinates.

To do so, we begin by noticing that given $z \in \mathcal{M}$, we can use ω to obtain a linear function $\omega(z, \cdot)\colon \mathcal{M} \to \mathbb{R}$. Let us denote by y^k the particular vector with entries δ^{ik}, where i is an index labeling the entry and k is the constant associated to y^k. For example, y^1 is given by $(1, 0, \ldots, 0)^\intercal$, y^{2N-1} is given by $(0, \ldots, 0, 1, 0)^\intercal$. Let then $z = (q, p)$ be some point in $\mathcal{M}$. Notice that for $1 \leq k \leq N$ we get

$$\omega(y^k, z) = \begin{pmatrix} \cdots & 1 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{2.22a}$$

$$= \begin{pmatrix} \cdots & 1 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} -p \\ q \end{pmatrix}, \tag{2.22b}$$

$$= -p_k, \tag{2.22c}$$

while $N < k \leq 2N$ leads to

$$\omega(y^k, z) = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 1 & \cdots \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{2.23a}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & \cdots & 1 & \cdots \end{pmatrix} \begin{pmatrix} -p \\ q \end{pmatrix}, \tag{2.23b}$$

$$= q^k. \tag{2.23c}$$

Hence, by employing linear combinations of the vectors y^k (*id est*, by picking an appropriate vector $z \in \mathcal{M}$), we can write any liner combination of the linear coordinates. As a consequence, we can write any expression involving the linear coordinates in terms of functions $\omega(z, \cdot)$ for $z \in \mathcal{M}$. By employing this we can rewrite Eq. (2.20) on page 9 as

$$\{\omega(y_1, \cdot), \omega(y_2, \cdot)\} = -\omega(y_1, y_2). \tag{2.24}$$

Let us pick an example to make sense of this equation. Say $y_1 = y^j$ and $y_2 = y^{i+N}$ for $1 \leq i, j \leq N$. Then according to Eqs. (2.22) and (2.23) we have $\omega(y_1, \cdot) = -p_j$ and $\omega(y_2, \cdot) = q^i$, meaning $\omega(y_1, \cdot)$ is the function that receives a point $z \in \mathcal{M}$ and returns, up to a sign, its j-th "momentum" component, with a similar remark for $\omega(y_2, \cdot)$. With this in mind, we see Eq. (2.24) becomes

$$-\{p_j, q^i\} = -\omega(y_1, y_2), \tag{2.25a}$$

$$= -\begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.25b}$$

$$= \delta^i{}_j, \tag{2.25c}$$

$$\{q^i, p_j\} = \delta^i{}_j, \tag{2.25d}$$

as we would expect. The difference is that now we have an expression that holds for any linear combination of the linear coordinates, and hence we can freely change coordinate basis. While the seemingly harmless Eq. (2.20) on page 9 won't generalize so easily when we want to deal with problems in infinite dimensions (fields), Eq. (2.24) on the previous page will.

Let us then pay attention to the consequences of the Hamiltonian being quadratic on the phase space linear coordinates. From the start we already know the most general Hamiltonian has the form

$$H(t;z) = \frac{1}{2}K_{ij}(t)z^i z^j, \tag{2.26}$$

$z \in \mathcal{M}$. Without loss of generality we may assume $K_{ij}(t) = K_{ji}(t)$, since any antisymmetric components would vanish once contracted with $z^i z^j$ and, as a consequence, can't affect the Hamiltonian.

We can use Eq. (2.12) on page 8 to compute the equations of motion, which are

$$\dot{z}^i = \Omega^{ij}K_{jk}(t)z^k. \tag{2.27}$$

Let us then consider two solutions to the equations of motion, denoted $z_1(t)$ and $z_2(t)$, and pick their "symplectic product" $s(t)$, which we shall define as

$$s(t) = \omega(z_1(t), z_2(t)) = \omega_{ij}z_1^i(t)z_2^j(t). \tag{2.28}$$

This is a candidate for a symplectic structure on $\mathcal{S}$, the space of solutions of the equations of motion, given the symplectic structure on $\mathcal{M}$. However, for this to be consistent we would like the symplectic structure not to depend on the particular instant which we call $t = 0$. Hence, we must have $\dot{s}(t) = 0$ if we really want to use this correspondence. To check whether this really is the case, let us compute $\dot{s}(t)$ explicitly. We get

$$\begin{aligned}
\dot{s}(t) &= \omega_{ij}\dot{z}_1^i(t)z_2^j(t) + \omega_{ij}z_1^i(t)\dot{z}_2^j(t), && \text{(2.29a)}\\
&= \omega_{ij}\Omega^{ik}K_{kl}(t)z_1^l(t)z_2^j(t) + \omega_{ij}z_1^i(t)\Omega^{jk}K_{kl}(t)z_2^l(t), && \text{(2.29b)}\\
&= -K_{jl}(t)z_1^l(t)z_2^j(t) + z_1^i(t)K_{il}(t)z_2^l(t), && \text{(2.29c)}\\
&= -K_{lj}(t)z_1^l(t)z_2^j(t) + K_{il}(t)z_1^i(t)z_2^l(t), && \text{(2.29d)}\\
&= 0, && \text{(2.29e)}
\end{aligned}$$

as expected. Hence, the symplectic structure $\omega\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ does induce a symplectic structure on $\mathcal{S}$.

To actually obtain the explicit expression of ω on $\mathcal{S}$, we would need to explicitly solve the equations of motion so we can compute the expression in Eq. (2.28) and substitute for the momenta in terms of the generalized coordinates.

The fundamental structure of Classical Mechanics we shall use to build QFT in Curved Spacetime is the symplectic structure on the space of solutions to the equations of motion, $(\mathcal{S}, \omega)$.

2.2 Weyl Quantization

The description of classical observables takes place in the phase space $T^*\mathcal{Q}$, which is the cotangent bundle to a configuration manifold $\mathcal{Q}$. They are smooth functions defined on such phase space and can be operated upon with a symplectic structure ω. Time evolution is given by a one-parameter family of canonical transformations given by the integral curves of a Hamiltonian vector field.

Quantum theory relies on a completely different structure. Quantum observables are self-adjoint operators acting on a Hilbert space* $\mathcal{F}$. The closest we can get to a symplectic structure in this setting is then a commutator. Time evolution is now a one-parameter family of unitary transformations generated by a self-adjoint Hamiltonian operator.

The difficult question is: given the classical theory, how to we obtain the correct quantum description? How do we find the appropriate Hilbert space $\mathcal{F}$ in which we should describe the theory and which operators should describe the observables?

Since quantum mechanics should recover classical mechanics in an adequate limit, the standard procedure is to use the classical Poisson brackets to impose commutation relations on the quantum observables, a procedure currently known as canonical quantization (see, *exempli gratia*, [3, Sec. 21]). More rigorously, we are searching for a map $\hat{}: \mathcal{O} \to \hat{\mathcal{O}}$ between the space of classical and quantum observables such that

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}}, \tag{2.30}$$

where we choose units with $\hbar = 1$.

For the most general phase spaces, determined simply by the symplectic structure without mention to a configuration space†, one not always can obtain a quantization procedure. Nevertheless, for phase spaces of the form $T^*\mathcal{Q}$ it is always possible to get a Hilbert space and a map $\hat{}$ such that the fundamental commutators,

$$\left[\hat{q}^i, \hat{q}^j\right] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad \left[\hat{q}^i, \hat{p}_j\right] = i\delta^i{}_j\mathbb{1} \tag{2.31}$$

hold for any choice of generalized coordinates $q^1, \ldots, q^N$ on $\mathcal{Q}$[19, p. 18].

As mentioned earlier, our particular interest is in linear dynamical systems, and hence we can exploit the linearity of the configuration space in here as well. Instead of writing Eq. (2.31), we may instead deal with

$$[\hat{\omega}(y_1, \cdot), \hat{\omega}(y_2, \cdot)] = -i\omega(y_1, y_2)\mathbb{1}, \tag{2.32}$$

which is the quantum version of Eq. (2.24) on page 11. To avoid cumbersome notation, we write simply $\hat{\omega}(y_1, y_2)$ instead of the more appropriate $\omega\widehat{(y_1, y_2)}$.

This would be fine, if it were not for the fact that position and momentum operators are usually unbounded operators and, as such, are defined only on a dense subspace of the

*We are already employing the notation $\mathcal{F}$, which shall stand for the Fock space once we get to field theory.

†We could have taken this approach, but I preferred to stick to the case in which the phase space is of the form $T^*\mathcal{Q}$.

Hilbert space due to the Hellinger–Toeplitz Theorem (see [9, p. 84]). As a consequence, compositions, hence the commutator, are not well-defined. Can we circumvent this issue? While self-adjoint unbounded operators can't be defined on the entire Hilbert space, it is possible to exponentiate them and obtain unitary operators, which are defined everywhere (see the note following [11, Theorem 13.38] or [9, Theorem VIII.7]). To *formally* exponentiate Eq. (2.32) on the preceding page, we may use a particular case of the Baker–Campbell-Hausdorff formula for *bounded* operators which commute with their commutator. If $[\hat{A},[\hat{A},\hat{B}]] = 0 = [\hat{B},[\hat{A},\hat{B}]]$, then [5, Theorem 14.1]

$$e^{\hat{A}}e^{\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}+\hat{B}}. \tag{2.33}$$

If view of Eq. (2.32) on the previous page, we can *formally* obtain

$$\begin{aligned} e^{i\hat{\omega}(y_1,\cdot)}e^{i\hat{\omega}(y_2,\cdot)} &= e^{-\frac{1}{2}[\hat{\omega}(y_1,\cdot),\hat{\omega}(y_2,\cdot)]}e^{i(\hat{\omega}(y_1,\cdot)+\hat{\omega}(y_2,\cdot))}, &\text{(2.34a)}\\ &= e^{\frac{i\omega(y_1,y_2)}{2}}e^{i(\hat{\omega}(y_1,\cdot)+\hat{\omega}(y_2,\cdot))}. &\text{(2.34b)}\end{aligned}$$

This does not follow rigorously from Eq. (2.32) on the preceding page, since Eq. (2.33) won't hold in general for unbounded operators. Nevertheless, this eliminates problematic operators which were allowed when we imposed conditions directly on the commutator, as discussed, for example, in [5, pp. 279–287].

Therefore, we'll rephrase the problem. We now start with the expression

$$W(y) \equiv e^{i\omega(y,\cdot)} \tag{2.35}$$

and search for a quantization map $\hat{}$ such that $\hat{W}(y)$ is unitary, is continuous on y on the strong operator topology*, satisfies

$$\hat{W}(y_1)\hat{W}(y_2) = e^{\frac{i\omega(y_1,y_2)}{2}}\hat{W}(y_1+y_2), \tag{2.36}$$

and satisfies

$$\hat{W}(y)^{\dagger} = \hat{W}(-y). \tag{2.37}$$

Eqs. (2.36) and (2.37) are known as the Weyl relations, and we shall use them to fix the way in which we'll quantize our theory. For a finite number of degrees of freedom, they allow us to uniquely determine the quantum description of the classical theory we started with. The rigorous statement of this uniqueness will come through the so-called Stone–von Neumann Theorem.

To be able to state it, we'll first begin by splitting a general problem in smaller pieces. Suppose you start with a Hilbert space $\mathcal{F}$ and a collection of unitary operators $\hat{W}(y)$, continuous on y in the strong operator topology, which satisfy the Weyl relations. It is possible that there are subspaces of $\mathcal{F}$ which are invariant under the action of the operators $\hat{W}(y)$, in the sense that there might be some Hilbert space $\{0\} \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$ such

*Given two Banach spaces X and Y, the strong operator topology on $\mathcal{B}(X,Y)$, the space of bounded linear operators from X to Y, is the topology in which a net of operators $T_\alpha \in \mathcal{B}(X,Y)$ converges to $T \in \mathcal{B}(X,Y)$ if, and only if, $\|T_\alpha x - Tx\| \to 0$ for every $x \in X$[9, pp. 182–183].

that $\Psi \in \mathcal{F}' \Rightarrow \hat{W}(y)\Psi \in \mathcal{F}'$, for every $y \in \mathcal{M}$. In this case, we can might split $\mathcal{F}$ in the form $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}'^{\perp}$ and analyze each subspace separately. If this is the case, $(\mathcal{F}, \hat{W}(y))$ is said to be a reducible representation of the Weyl relations. On the other hand, if $\mathcal{F}$ and $\{0\}$ are the only subspaces of $\mathcal{F}$ invariant under $\hat{W}(y)$ for every $y \in \mathcal{M}$, $(\mathcal{F}, \hat{W}(y))$ is called an irreducible representation of the Weyl relations.

Equivalently, $(\mathcal{F}, \hat{W}(y))$ is an irreducible representation of the Weyl relations if, and only if, given any non-null $\Psi \in \mathcal{F}$ it holds that the span of the vectors of the form $\hat{W}(y)\Psi$ for $y \in \mathcal{M}$ is dense in $\mathcal{F}$[19, p. 19].

We still need to give meaning to "uniqueness". In a quantum theory, what we are really interested in are the expectation values, not the particular operators themselves. If we change the Hilbert space, change the operators, but keep the expected values fixed, the physical description is still the same. Hence, what we mean by uniqueness is that any two irreducible representations of the Weyl relations should be related by a unitary transformation: an isomorphism between Hilbert spaces. More specifically, if we have two Hilbert spaces $\mathcal{F}$ and $\mathcal{F}'$ each with its own collection of operators V_α and V'_α — which will be $\hat{W}(y)$ in our case of interest —, then $(\mathcal{F}, V_\alpha)$ is said to be unitarily equivalent to $(\mathcal{F}', V'_\alpha)$ if, and only if, there is some unitary map $U\colon \mathcal{F} \to \mathcal{F}'$ such that $UV_\alpha U^{-1} = V'_\alpha$. If this is the case, notice that the expected values have (we denote $\Psi' = U\Psi \in \mathcal{F}'$ for any $\Psi \in \mathcal{F}$)

$$\left\langle \Psi', V'_\alpha \Psi' \right\rangle = \left\langle U\Psi, UV_\alpha U^{-1} U\Psi \right\rangle, \tag{2.38a}$$

$$= \left\langle \Psi, U^\dagger V_\alpha \Psi \right\rangle, \tag{2.38b}$$

$$= \left\langle \Psi, V_\alpha \Psi \right\rangle. \tag{2.38c}$$

Having these notions in mind, we state the Stone–von Neumann Theorem.

Theorem 2.2 [Stone–von Neumann]**:**

Suppose $(\mathcal{M}, \omega)$ is a finite-dimensional symplectic vector space and that both $(\mathcal{F}, \hat{W}(y))$ and $(\mathcal{F}', \hat{W}'(y))$ are strongly continuous, unitary, irreducible representations of the Weyl relations, given in Eqs. (2.36) and (2.37) on the preceding page. Under these conditions, it must hold that $(\mathcal{F}, \hat{W}(y))$ and $(\mathcal{F}', \hat{W}'(y))$ are unitarily equivalent. □

For a proof, one can refer to [5, Theorem 14.8], [8, Theorem XI.84], or [15, Theorem 7.5]. See also [9, Theorem VIII.14].

As a first example, we should select the standard Hilbert space we use when doing Quantum Mechanics à la Schrödinger: $\mathcal{F} = L^2(\mathbb{R}^N)$. We'll pick $N = 1$ for simplicity and omit temporal dependence on the formulae for the same reason. The position operator is given by $\hat{q}\Psi(q) = q\Psi(q)$ and the momentum operator if given by $\hat{p}\Psi(q) = -i\Psi'(q)$, where the prime denotes differentiation with respect to q. It is known from Quantum Mechanics that $e^{ir\hat{q}}\Psi(q) = e^{irq}\Psi(q)$ and $e^{is\hat{p}}\Psi(q) = \Psi(q+s)$. We see then that

$$\left\langle \Phi, e^{ir\hat{q}}\Psi \right\rangle = \int \Phi^*(q) e^{irq} \Psi(q) \, \mathrm{d}q\,, \tag{2.39a}$$

$$= \int \left[e^{-irq}\Phi(q)\right]^* \Psi(q) \, \mathrm{d}q\,, \tag{2.39b}$$

$$= \left\langle e^{-ir\hat{q}}\Phi, \Psi \right\rangle \tag{2.39c}$$

and

$$\left\langle \Phi, e^{is\hat{p}}\Psi \right\rangle = \int \Phi^*(q)\Psi(q+s)\,\mathrm{d}q\,, \tag{2.40a}$$

$$= \int \Phi^*(q'-s)\Psi(q')\,\mathrm{d}q'\,, \tag{2.40b}$$

$$= \left\langle e^{-is\hat{p}}\Phi, \Psi \right\rangle, \tag{2.40c}$$

which prove Eq. (2.37) on page 14 for the cases $y = (0, s)^\intercal$ (yielding $\omega(y, \cdot) = q$) and $y = (-r, 0)^\intercal$ ($\omega(y, \cdot) = p$), which form a basis of $\mathcal{M}$ and hence are sufficient for every case.

As for Eq. (2.36) on page 14, we'll rewrite it differently. Notice that it implies

$$e^{-\frac{i\omega(y_1,y_2)}{2}}\hat{W}(y_1)\hat{W}(y_2) = \hat{W}(y_1 + y_2). \tag{2.41}$$

Hence,

$$\hat{W}(y_1)\hat{W}(y_2) = e^{\frac{i\omega(y_1,y_2)}{2}}\hat{W}(y_1 + y_2), \tag{2.42a}$$

$$= e^{\frac{i\omega(y_1,y_2)}{2}}\hat{W}(y_2 + y_1), \tag{2.42b}$$

$$= e^{i\omega(y_1,y_2)}\hat{W}(y_2)\hat{W}(y_1), \tag{2.42c}$$

where $\omega(y_1, y_2) = -\omega(y_2, y_1)$ was employed. This in fact is the form of the Weyl relations that the provided references use to prove the Stone–von Neumann Theorem.

For $y_1 = y_2$, the relation holds trivially. For $y_1 = (0, r)^\intercal$ and $y_2 = (-s, 0)^\intercal$ it becomes

$$e^{ir\hat{q}}e^{is\hat{p}} = e^{-irs}e^{is\hat{p}}e^{ir\hat{q}}. \tag{2.43}$$

To check this, we notice that

$$e^{ir\hat{q}}e^{is\hat{p}}\Psi(q) = e^{ir\hat{q}}\Psi(q+s), \tag{2.44a}$$

$$= e^{irq}\Psi(q+s). \tag{2.44b}$$

On the other hand,

$$e^{-irs}e^{is\hat{p}}e^{ir\hat{q}}\Psi(q) = e^{irs}e^{is\hat{p}}e^{irq}\Psi(q), \tag{2.45a}$$

$$= e^{-irs}e^{ir(q+s)}\Psi(q+s), \tag{2.45b}$$

$$= e^{irq}\Psi(q+s), \tag{2.45c}$$

and hence

$$e^{ir\hat{q}}e^{is\hat{p}} = e^{irq}\Psi(q+s) = e^{-irs}e^{is\hat{p}}e^{ir\hat{q}} \tag{2.46}$$

as expected.

As a consequence, the Stone–von Neumann Theorem justifies this choice for the position and momentum operators, as any other choice would lead to the same theory.

Nevertheless, we still have no information on how to represent observables beyond position and momentum. If we don't restrain ourselves to the linear case, we might run into issues with factor ordering choices, for example. Our interest, though, will be only on Hamiltonians which are quadratic in the linear coordinates, and no factor ordering issues arise for them, since back in Eq. (2.26) on page 12 we chose the matrix K_{ij} to be symmetric.

Notice the hypothesis on the Stone–von Neumann Theorem that the phase space $\mathcal{M}$ is finite dimensional, *id est*, there are finitely many degrees of freedom. In field theory, this, and hence the Stone–von Neumann Theorem, will no longer hold and the choice of a representation, or vacuum, is no longer a trivial matter. When working on Minkowski spacetime, Poincaré invariance will allow us to choose a particular vacuum. Nevertheless, in general curved spacetime this luxurious symmetry won't be available. Instead, we'll use the algebraic formalism to bypass the choice of a preferred representation.

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