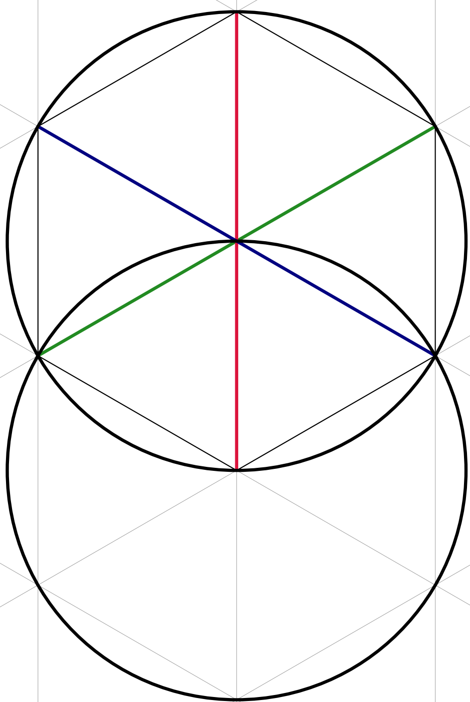
University of São Paulo

# Hyperbolic Equations

Níckolas de Aguiar Alves



If I knew something about it, I wouldn't lecture on it!

Arnold Sommerfeld

# Preface

These are some study notes I've been developing while working on my undergraduate research project. This is a project still being developed, and pretty far from being finished (assuming it will ever be).

The project's goal is to study Hyperbolic Equations, but this requires a lot of previous knowledge. As a consequence, most of the material currently covered in here involves not Hyperbolic Equations, but Topology, Functional Analysis and even some Differential Geometry.

Despite the mess, I appreciate the interest in my work and I would be extremely pleased to receive comments, critics, compliments and etc through my e-mail (nickolas@fma.if.usp.br). If you wish to check some more works, please check my personal website `http://fma.if.usp.br/~nickolas`.

I am supported by grant 2019/12158-4, São Paulo Research Foundation (FAPESP).

Níckolas de Aguiar Alves
February 1, 2021

# Contents

# One

# Elementary Concepts on PDE Theory

You have to do the cruel things in the beginning.

FREDERIC P. SCHULLER quoting NICCOLÒ MACHIAVELLI'S *The Prince*.

## 1.1 What is a Partial Differential Equation?

THIS work is aimed towards a better understanding of physical reality, and this certainly suggests a strong connection between Partial Differential Equations and Physics. Indeed, it is extremely common, as any Physics student should know, for physical problems to be described through PDEs. However, without the knowledge of what is a PDE this affirmations are pointless. Thus, the first question that should be answered is: what is a Partial Differential Equation? Not surprisingly, the second is why we should bother about them at all.

Equations arise on Mathematics from the most elementary High School themes through the most advanced open problems. Some of the most simple equations someone can deal with are algebraic equations, which usually involve finding the roots of a polynomial. As an example, one might want to solve

$$ax^2 + bx + c = 0, \tag{1.1}$$

which can be done by applying the worldly famous quadratic formula.

One can then go a step further and start talking about Ordinary Differential Equations, or ODEs, which relate a function to its derivatives with respect to a single variable. As an example, a Physics student might want to solve a damped harmonic oscillator described by

$$\ddot{x} + \gamma\dot{x} + \omega_0 x = 0. \tag{1.2}$$

Finally, there are the equations we are interested in: Partial Differential Equations, also known as PDEs. These equations are still Differential Equations, and as the name might suggest they still relate a function to its derivatives. Nevertheless, the word "Partial" suggests there is something more: these are now partial derivatives of multi-variable functions. As a simple example, one might want to solve the problem of how small oscillations affect a tensioned string, which is described by the wave equation in one dimension:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{1.3}$$

Of course, this is only an example of what a PDE is, not a definition. In order to give it a proper definition, let us first introduce appropriate notation.

*Notation* [Multiindex Notation]*:*

Let $U \subseteq \mathbb{R}^n$ be an open set, $u\colon U \to \mathbb{R}$ and $\mathbf{x} \in U$. A *multiindex* of order $|\alpha|$ is an n-uple[*] $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ satisfying $|\alpha| = \sum_{i=1}^n \alpha_i$. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write

$$D^\alpha u(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{1.4}$$

Given $k \in \mathbb{N}$, we also write

$$D^k u(\mathbf{x}) \equiv \{D^\alpha u(\mathbf{x}); \alpha \in \mathbb{N}^n, |\alpha| = k\} \tag{1.5}$$

and

$$\left|D^k u\right| := \sqrt{\sum_{|\alpha|=k} |D^\alpha u|^2}. \tag{1.6}$$

Notice that if we assign an order to the set $D^k u(\mathbf{x})$, it is possible to regard it as an element of $\mathbb{R}^{n^k}$. As examples, we might mention the gradient vector,

$$\boldsymbol{\nabla} u \equiv Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \tag{1.7}$$

and the Hessian matrix

$$D^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}. \tag{1.8}$$

The Laplacian of $u$ is defined as the trace of the Hessian matrix: $\nabla^2 u \equiv \Delta u \equiv \tau\big(D^2 u\big)$.

Finally, we might attach a subscript to $D^\alpha u$ in order to specify the variables we are differentiating with respect to. If we are considering a function $u\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and write $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$, $D^\alpha_{\mathbf{x}} u$ denotes differentiation with respect to the first $n$ variables and $D^\alpha_{\mathbf{y}} u$ denotes differentiation with respect to the remaining $m$ variables. As an example, we have

$$D_{\mathbf{x}} u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \quad D_{\mathbf{y}} u = \left(\frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_m}\right). \tag{1.9}$$

◆

[*] $\mathbb{N}$ denotes the set of the natural numbers ($0 \in \mathbb{N}$).

To practice with this notation, it is useful to prove some auxiliary results.

**Proposition 1.1** [Multinomial Theorem]**:**
*Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $k \in \mathbb{N}$. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha, \tag{1.10}$$

*where we denote $\binom{|\alpha|}{\alpha} \equiv \frac{|\alpha|!}{\alpha!}$, $\alpha! \equiv \prod_{i=1}^n \alpha_i$ and $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.* □

*Proof:*

We'll proceed through induction over $n$. The case, $n = 1$, is the binomial theorem, which can be proven through induction over $k$. It is immediate for $k = 1$. If it holds for $k - 1$, then

$$(x_1 + x_2)^{k-1} = \sum_{l=0}^{k-1} \binom{k-1}{l} x_1^l x_2^{k-1-l}, \tag{1.11}$$

$$\begin{aligned}
(x_1 + x_2)^k &= \sum_{l=0}^{k-1} \binom{k-1}{l} \left(x_1^{l+1} x_2^{k-1-l} + x_1^l x_2^{k-l}\right), \\
&= \sum_{l=1}^{k} \binom{k-1}{l-1} x_1^l x_2^{k-l} + \sum_{l=0}^{k-1} \binom{k-1}{l} x_1^l x_2^{k-l}, \\
&= x_2^k + \sum_{l=1}^{k-1} \left(\binom{k-1}{l-1} + \binom{k-1}{l}\right) x_1^l x_2^{k-l} + x_1^k, \\
&= \sum_{l=0}^{k} \binom{k}{l} x_1^l x_2^{k-l},
\end{aligned} \tag{1.12}$$

proving the inductive step for the induction over $k$. Hence, the case $n = 2$ holds for the induction over $n$.

Let us now proceed to the inductive step over $n$. In the following calculation, we denote $\gamma \in \mathbb{N}^{n-1}$ and $\alpha \in \mathbb{N}$. We have

$$\begin{aligned}
\left(\sum_{i=1}^{n} x_i\right)^k &= \left(\sum_{i=1}^{n-1} x_i + x_n\right)^k, \\
&= \sum_{l=0}^{k} \binom{k}{l} \left(\sum_{i=1}^{n-1} x_i\right)^l x_n^{k-l}, \\
&= \sum_{l=0}^{k} \binom{k}{l} \left(\sum_{|\gamma|=l} \binom{|\gamma|}{\gamma} x^\gamma\right) x_n^{k-l}, \\
&= \sum_{l=0}^{k} \sum_{|\gamma|=l} \binom{k}{l} \binom{|\gamma|}{\gamma} x^\gamma x_n^{k-l},
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{|\gamma| \le k} \binom{k}{|\gamma|} \binom{|\gamma|}{\gamma} x^\gamma x_n^{k-|\gamma|}, \\
&= \sum_{|\gamma| \le k} \frac{k!}{|\gamma|!(k-|\gamma|)!} \frac{|\gamma|!}{\gamma!} x^\gamma x_n^{k-|\gamma|}, \\
&= \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha, \qquad (1.13)
\end{aligned}
$$

as claimed. ■

**Proposition 1.2** [Leibniz Rule]**:**

*Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Let $\alpha$ be a multiindex. Then one has*

$$
D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v, \qquad (1.14)
$$

*where we denote $\binom{\alpha}{\beta} \equiv \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\alpha! \equiv \prod_{i=1}^n \alpha^i$ and $\beta \le \alpha \Leftrightarrow \beta^i \le \alpha^i, 1 \le i \le n$.* □

*Proof:*

Let us prove by induction on $n$. We begin with the one-dimensional case. We want to prove that, $\forall\, p \in \mathbb{N}$,

$$
D^p(uv) = \sum_{k=0}^{p} \binom{p}{k} D^k u D^{p-k} v. \qquad (1.15)
$$

This can also be proven by induction. The case $p = 0$ holds because $uv = uv$. The inductive step can then be shown by employing the product rule (which corresponds to the case $p = 1$) and with the binomial coefficients. We have

$$
\begin{aligned}
D^{p-1}(uv) &= \sum_{k=0}^{p-1} \binom{p-1}{k} D^k u D^{p-1-k} v, \\
D^p(uv) &= \sum_{k=0}^{p-1} \binom{p-1}{k} \left( D^{k+1} u D^{p-1-k} v + D^k u D^{p-k} v \right), \\
&= \sum_{k=1}^{p} \binom{p-1}{k-1} D^k u D^{p-k} v + \sum_{k=0}^{p-1} \binom{p-1}{k} D^k u D^{p-k} v, \\
&= u D^p v + \sum_{k=1}^{p-1} \left( \binom{p-1}{k-1} + \binom{p-1}{k} \right) D^k u D^{p-k} v + D^p u v, \\
&= \sum_{k=0}^{p} \binom{p}{k} D^k u D^{p-k} v. \qquad (1.16)
\end{aligned}
$$

Hence, the expression holds in the base case $n = 1$. We now proceed to show the inductive step. Eq. (1.14) holding in $n-1$ dimensions is equivalent to imposing that $\alpha^n = 0$. Given this, we can get the general expression in $n$ dimensions by differentiating

Eq. (1.14) on the preceding page with respect to $x_n$. Since the functions are smooth, all the derivatives commute and we end up with the situation in a single dimension, which we have already proven. All things considered, Eq. (1.14) on the facing page holds in $n$ dimensions. ■

As one might have noticed, sometimes we shall write a derivative of a function $u$ with respect to a real variable $x$ as $\frac{\partial u}{\partial x}$ and sometimes as $u_x$, according to the convention that seems more appropriate at each moment.

With multiindex notation at our hand, it is easier to give a proper definition of what is a PDE.

**Definition 1.3** [Partial Differential Equation]**:**

Let $k \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be an open set. A partial differential equation of order $k$ is an equation of the form

$$F\Big(D^k u(\mathbf{x}), D^{k-1}u(\mathbf{x}), \ldots, Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}\Big) = 0, \tag{1.17}$$

where $F\colon \bigtimes_{m=0}^{k} \mathbb{R}^{n^m} \times U \to \mathbb{R}$ is a given function and $u\colon U \to \mathbb{R}$ is the unknown. ♠

We say a PDE has been solved when we have found every $u$ verifying Eq. (1.17), occasionally satisfying boundary conditions on a set $\Gamma \subseteq \partial U$, where $\partial U$ denotes the boundary of $U$. "Finding every solution" means, in some cases, obtaining a closed expression describing $u$ as a function, while sometimes it means proving $u$ exists and obtaining general properties $u$ must satisfy.

While the search for solutions of an ODE might lead one to change the roles of dependent and independent variables (*exempli gratia*when solving $u'(x) = x$ by separating variables on each side of the equation and integrating). However, this does not occur in the process of solving a PDE: a dependent variable is always a dependent variable and an independent variable is always an independent variable.[70]

It is also common to classify PDEs by their linearity, since linear PDEs are in general easier to solve than non-linear PDEs. Also, if the nonlinearity affects terms high in order, the equation is harder to solve than if it affects only terms low in order[17]. This motivates us to create the following classification:

**Definition 1.4** [Linearity of a PDE]**:**

Let $k \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be an open set and $F\colon \bigtimes_{m=0}^{k} \mathbb{R}^{n^m} \times U \to \mathbb{R}$ be a function. Consider the partial differential equation given by

$$F\Big(D^k u(\mathbf{x}), D^{k-1}u(\mathbf{x}), \ldots, Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}\Big) = 0. \tag{1.18}$$

i. The PDE is said to be linear if it holds that

$$F\Big(D^k u(\mathbf{x}), \ldots, \mathbf{x}\Big) = \sum_{|\alpha| \leq k} f_\alpha(\mathbf{x}) D^\alpha u(\mathbf{x}) - f(\mathbf{x}). \tag{1.19}$$

where $f_\alpha, f\colon U \to \mathbb{R}$ are given functions. If $f$ vanishes everywhere in $U$, the PDE is said to be homogeneous (on the contrary, it is said to be inhomogeneous).

ii. If the PDE is not linear, it is said to be semilinear if it holds that

$$F\Big(D^k u(\mathbf{x}), \dots, \mathbf{x}\Big) = \sum_{|\alpha|=k} f_\alpha(\mathbf{x}) D^\alpha u(\mathbf{x}) + G\Big(D^{k-1}u(\mathbf{x}), \dots, \mathbf{x}\Big), \tag{1.20}$$

where $f_\alpha \colon U \to \mathbb{R}$ and $G \colon \bigtimes_{m=0}^{k-1} \mathbb{R}^{n^m} \times U \to \mathbb{R}$ are given functions.

iii. If the PDE is not linear nor semilinear, it is said to be quasilinear if it holds that

$$F\Big(D^k u(\mathbf{x}), \dots, \mathbf{x}\Big) = \sum_{|\alpha|=k} G_\alpha\Big(D^{k-1}u(\mathbf{x}), \dots, \mathbf{x}\Big) D^\alpha u(\mathbf{x}) + G\Big(D^{k-1}u(\mathbf{x}), \dots, \mathbf{x}\Big), \tag{1.21}$$

where $G_\alpha, G \colon \bigtimes_{m=0}^{k-1} \mathbb{R}^{n^m} \times U \to \mathbb{R}$ are given functions.

iv. If the PDE depends nonlinearly upon the highest order derivatives, it is said to be fully nonlinear. ♠

In Physics it is also extremely common to talk about a PDE involving a vector function (such as electric or magnetic fields, velocities, etc...). This is actually a system of PDEs, which can be defined formally.

**Definition 1.5** [System of PDEs]**:**
Let $k \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be an open set. A system of partial differential equations of order $k$ is an equation of the form

$$\mathbf{F}\Big(D^k \mathbf{u}(\mathbf{x}), D^{k-1}\mathbf{u}(\mathbf{x}), \dots, D\mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x}), \mathbf{x}\Big) = 0, \tag{1.22}$$

where $\mathbf{F} \colon \bigtimes_{l=0}^{k} \mathbb{R}^{mn^l} \times U \to \mathbb{R}^m$ is a given function and $\mathbf{u} \colon U \to \mathbb{R}^m$ is the unknown. ♠

Specially when dealing with PDE systems in three spatial dimensions, we shall write the divergent and the curl of a vector field as $\boldsymbol{\nabla} \cdot \mathbf{u}$ and $\boldsymbol{\nabla} \times \mathbf{u}$.

**Examples:**
Finally, we might list some examples of PDEs and systems of PDEs.

**Transport Equation** a linear, first-order PDE ($c > 0$)

$$\frac{1}{c}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \tag{1.23}$$

**Wave Equation** a linear, second-order PDE ($c > 0$)

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0; \tag{1.24}$$

**Laplace's Equation** a linear, second-order PDE

$$\nabla^2 u = 0; \tag{1.25}$$

**Heat Equation** a second-order PDE ($D > 0$)

$$\frac{\partial u}{\partial t} - D\nabla^2 u = \Phi(u, \mathbf{x}, t); \tag{1.26}$$

**Damped Wave Equation** a linear, second-order PDE ($c, \gamma > 0$)

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} - \nabla^2 u = 0; \tag{1.27}$$

**Schrödinger's Equation** a linear, second-order PDE ($\hbar, m > 0$)

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x}, t)\psi; \tag{1.28}$$

**Telegraph Equation** a linear, second-order PDE ($c, \gamma, \eta > 0$)

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial u}{\partial t} + \eta u = 0; \tag{1.29}$$

**Klein-Gordon Equation** a linear, second-order PDE ($c, m > 0$)

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - m^2\psi = 0; \tag{1.30}$$

**Scalar Conservation Law** a second-order PDE

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F}(u) = 0; \tag{1.31}$$

**Burgers Equation** a semilinear, second-order PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \eta\frac{\partial^2 u}{\partial x^2} = 0; \tag{1.32}$$

**Korteweg-de Vries (KdV) Equation** a semilinear, third-order PDE

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0; \tag{1.33}$$

**Dirac Equation (Free Particle)** a linear, first-order PDE system* ($m > 0$)

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0; \tag{1.34}$$

*$\gamma^\mu$ are four $4 \times 4$ matrices known as Dirac matrices. $\partial_\mu$ denotes differentiation with respect to the respective coordinate (0 stands for time, while 1 through 3 denote the spatial coordinates). Einstein's summation convention (omission of summation symbol for repeated indexes) is being used.

**Maxwell's Equations** a linear, first-order PDE system ($\mu_0, \epsilon_0 > 0$, $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ are given functions)

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}, \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \\ \boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}; \end{cases} \tag{1.35}$$

**Navier-Stokes Equations** a semilinear, second-order PDE system

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\right) + \boldsymbol{\nabla} p - \eta \nabla^2 \mathbf{v} - \left(\zeta + \frac{1}{3}\eta\right) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) = \mathbf{f}. \tag{1.36}$$

♥

## 1.2 Well-Posedness

However, we can't expect that a PDE will have a unique solution in general. When dealing with ODEs, we don't get a unique solution to our problem unless we complement the equation with some extra information (usually called initial or boundary conditions, depending on the independent variables).

The same happens in PDE theory. The process of solving the equation involves integrating the equation (through different methods), and thus some integration constants are expected to appear, and must be fixed somehow if we want the solution to be unique.

Moreover, it is not interesting for our problem to depend arbitrarily on the data we provide. As an example, suppose we are interested in the problem of heat conduction on a homogeneous rod, whose initial temperature is described by a function $u(0, x) = f(x)$ defined on some real interval. If the rod was slightly hotter or colder in a certain region, we do not expect for the solution to change too dramatically. Instead, it would also be only slightly different.

Of course, such properties might be desirable, but, as said before, PDEs might be pretty cumbersome, and thus we can't expect for everything to work so beautifully in every situation. Sometimes, a solution might not even exist.

**Example:**

As an example of a PDE with no solution whatsoever, consider the following PDE for a real function $u$:

$$F(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) = u^2 + \frac{\partial u}{\partial x_1}^2 + \frac{\partial u}{\partial x_2}^2 + 1 = 0. \tag{1.37}$$

Since a real number squared is always non-negative, we are asking for a positive number to be null, which is clearly impossible. Therefore, the equation does not admit any solutions. ♥

These issues give us a reason to give a name for the problems that accept "good solutions". Thus, we are going to say that a problem for a PDE is *well-posed* (in the sense of Hadamard) when

i. it admits a solution;

ii. the solution is unique;

iii. the solution depends continuously on the data provided (this might depend on a certain metric or norm considered).

The data provided are the initial conditions, the boundary conditions, etc.

*Remark* [Solution of a PDE]*:*
Curiously, we defined well-posedness in terms of a concept we do not fully understand: solution.

What we mean by a solution to a PDE can change very much depending on the problem we are dealing with. The idealistic case would be to define a solution to a PDE such as the one in Eq. (1.17) on page 5 as a function $u\colon U \to \mathbb{R}$ which makes Eq. (1.17) on page 5 true in every point $\mathbf{x} \in U$, respects the conditions imposed on the problem (*exempli gratia*, $u\big|_\Gamma = \varphi$, where $\Gamma \subseteq \partial U$ is a given set and $\varphi\colon \Gamma \to \mathbb{R}$ is a given function) and $u$ is $k$ times continuously differentiable. Such a definition seems natural, for it is essentially saying that a solution should solve the problem and behave accordingly (if it solves an equation with derivatives of order $k$, it is expected that this function can be differentiated $k$ times, at least in $U$).

We define solutions in this manner, but call them classical solutions. The adjective is due to the fact that sometimes a classical solution can't be found, but it doesn't mean we should give up. It is possible that if we don't ask for so many properties we might be able to find some $u$ that maintains at least some of them. In some cases, this could be even more interesting than a classical solution would be (the discontinuities of shock waves, for example). If we accept a solution which respects less conditions than a classical solution would, it is possible that a previously ill-posed problem will turn into a well-posed one.

This is barely an example of a simple principle: the more you ask of a solution, the less solutions you can get. The less you ask, the more solutions you get. If you ask only for it to solve the equation everywhere in $U$, it is possible that you will end up with many possible solutions. On the other hand, asking too much of a solution might leave you with none.

As a consequence, it is often more interesting to separate the problems of existence and smoothness (or regularity) of solutions. Given a PDE, we may first prove the problem is well-posed for some wide notion of a weak solution. Sometimes that will be everything we can do, but other times it might turn out that the weak solution we got was in fact a classical solution all along. ♣

# Two

# First-Order Equations: The Method of Characteristics

> Young man, in mathematics you don't understand things. You just get used to them.
>
> JOHN VON NEUMANN in response to a physicist friend saying he did not understand the method of characteristics.

## 2.1 Linear Equations

IN general, nonlinear PDEs are harder to solve than linear PDEs, just as high-order PDEs are harder to solve than low-order problems. Thus, it is quite natural for us to deal firstly with linear first-order problems. The same reason motivates us to work out problems in two dimensions before anything else.

Under these assumptions, the general appearance of a PDE is going to be

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y). \tag{2.1}$$

For now, we are going to assume $c(x,y) = 0$, making the equation even simpler. The case with $c(x,y) \neq 0$ can be treated when dealing with semilinear problems, as we shall see soon.

Therefore, we wish to solve the problem

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = d(x,y). \tag{2.2}$$

Our aim is to find a function $u$ depending on two variables, $x$ and $y$, such that Eq. (2.2) is respected. Geometrically, this is nothing but a surface on 3D space. This surface is described by the equation $z = u(x,y)$, and thus is a level surface of the function

$f(x,y,z) = u(x,y) - z$. The gradient of $f$ is normal to this surface in every point, and is given by

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\ &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right). \end{aligned} \tag{2.3}$$

However, we can write Eq. (2.2) on the previous page as

$$(a(x,y), b(x,y), d(x,y)) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right) = 0, \tag{2.4}$$

which tells us that $(a(x,y), b(x,y), d(x,y))$ is a vector tangent to the surface we are looking for in every point we pick (because it is always perpendicular to $\nabla f$).

This gives us a good geometrical view of our problem: find a surface which is always tangent to $(a(x,y), b(x,y), d(x,y))$, no matter the point you pick. That is certainly very interesting. Now how can we do it? We can simply pick a point on the $xy$-plane, find the vector $(a(x,y), b(x,y), d(x,y))$ and follow it. Repeating the procedure shall infinitely many times shall yield us a curve on the surface we are looking for. Pick enough points and there you go: a nice surface build upon lots of auxiliary curves.

Naively, this is the idea behind the method of characteristics. Let us find the curves which are always tangent to $(a(x,y), b(x,y), d(x,y))$ and build our surface with them.

Such a curve is going to parametrized, as any other curve would as well be, in terms of a real variable $s$ and thus will be written $(x(s), y(s), z(s))$. It also is such that its derivative is given by $(a(x(s), y(s)), b(x(s), y(s)), d(x(s), y(s)))$, and therefore respects the following system of ODEs:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = a(x(s), y(s)), \\ \dfrac{\mathrm{d}y}{\mathrm{d}s} = b(x(s), y(s)), \\ \dfrac{\mathrm{d}z}{\mathrm{d}s} = d(x(s), y(s)). \end{cases} \tag{2.5}$$

We call such a curve an integral curve for the vector field $(a(x,y), b(x,y), d(x,y))$. When trying to solve a PDE such as Eq. (2.2) on the preceding page, we look for the integral curves for the vector field $(a(x,y), b(x,y), d(x,y))$ associated with the PDE, which are the so called characteristic curves. This is done through solving the system of characteristic equations, given by Eq. (2.5). It is thus not surprising to learn that this method is called method of characteristics.

Having the characteristic curves, all one should need to do is join them in the surface we were looking for. Such a surface, which always has $(a(x,y), b(x,y), d(x,y))$ as a tangent vector, is said to be an integral surface for the vector field $(a(x,y), b(x,y), d(x,y))$.

Essentially, the method of characteristic consists in breaking a PDE problem into some ODE problems, which can be solved by ODE theory.

You know, I'm something of a mathematical physicist myself. Therefore, I believe it would be interesting to treat an example of how this method can be applied, and also where the equation we are going to work with, the transport equations, comes from.

**Example** [Transport Equation]**:**
Assume that some fluid is flowing linearly at a constant speed $c$, performing a one-dimensional motion. There is some sort of material in it, such as dirt or a pollutant, suspended in the water. Let's denote its concentration as a function of time and position: $u(t,x)$. We are also assuming the amount of pollutant in the water can be described only in terms of a one-dimensional position, along the flow. This should do it for a thin flow, but perhaps it would be wiser to describe a lake with another model.

The amount of pollutant at a certain instant $t$ contained in the interval $[0,x]$ is given by the integral $M = \int_0^x u(x',t)\,\mathrm{d}x'$. As time passes, the pollutant moves a distance proportional to the velocity of the fluid (we assume there is no relevant diffusion effects), and thus we know that, at the instant $t+\tau$, the same amount of pollutant will be found at the interval $[\tau, x+c\tau]$, *id est*,

$$M = \int_0^x u(x',t)\,\mathrm{d}x' = \int_\tau^{x+c\tau} u(x',t+\tau)\,\mathrm{d}x'\,. \tag{2.6}$$

Differentiating with respect to $x$, we get

$$u(x,t) = u(x+c\tau, t+\tau). \tag{2.7}$$

We then differentiate with respect to $\tau$ and set $\tau = 0$:

$$\begin{aligned} 0 &= cu_x(x+c\tau,t+\tau) + u_t(x+c\tau,t+\tau), \\ 0 &= cu_x(x,t) + u_t(x,t). \end{aligned} \tag{2.8}$$

Given the differential equation, we wish now to work backwards. If we are dealing with a problem concerning the movement of a pollutant in a fluid, how could we find the concentration as a function of time and position?

Our characteristic equations are going to be

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = c, \\ \dfrac{\mathrm{d}t}{\mathrm{d}s} = 1, \\ \dfrac{\mathrm{d}z}{\mathrm{d}s} = 0. \end{cases} \tag{2.9}$$

The solution to this system of ODEs is given by integrating directly each of them:

$$\begin{cases} x(s) = cs + k_1, \\ t(s) = s + k_2, \\ z(s) = k_3, \end{cases} \tag{2.10}$$

with $k_i$ representing integration constants. Of course these constants are equal through each characteristic curve (they are in the expression for the curve!), but not necessarily through the whole integral surface. They can change from a characteristic curve to another.

If we eliminate the parameter $s$ from the solutions to the characteristic equations, we see that

$$x = c \cdot (t - k_2) + k_1, \tag{2.11}$$

$$x - ct = x_0, \tag{2.12}$$

which implies that the characteristic curves are lines in $\mathbb{R}^3$ with $x - ct = x_0$ and $z = k_3$ for each pair of constants $x_0$ and $k_3$.

Notice that if we take the union of all these curves we are going to end with $\mathbb{R}^3$ as a whole! Shouldn't we get only the integral surface to the problem? Of course not! The problem we stated, which was simply a PDE with no conditions whatsoever, is definitely ill-posed! Unless we state some conditions on the solutions, such as how the function should be at $t = 0$, we can't expect for the characteristic curves to provide a single integral surface.

Nevertheless, there is still a good amount of information in this reasoning. As an example, any integral surface we pick (remember: we are looking for a function, as thus this surface cannot be multi-valued when though as $u(x,t)$) is going to assume constant values along the lines $x_0 = x - ct$ and therefore we might regard the solution $u(x,t)$ as a function of a single variable: $u(x,t) = f(x - ct)$.

Actually, the inverse holds as well: not only every solution to this PDE is a function with the form $f(x - ct)$, but also every (differentiable) function $f(x - ct)$ is a solution to the PDE. Indeed, the chain rule allows us to notice that

$$\begin{cases} c\dfrac{\partial u}{\partial x} = cf'(x - ct), \\ \dfrac{\partial u}{\partial t} = -cf'(x - ct). \end{cases} \tag{2.13}$$

$$\therefore \frac{\partial u}{\partial t} c \cdot \frac{\partial u}{\partial x} = 0.$$

The expression of the solution in terms of a function of a single variable also shows an interesting feature of the equation: the information propagates along the lines $x_0 = x - ct$*, and this feature is responsible for the name we give to this particular PDE: the transport equation.

As an example, let us give an initial condition to the problem. Suppose that at a time $t = 0$ the solution we desire is equal to a function $\varphi(x)$, *id est*, $u(x,0) = \varphi(x)$. Since the solution to the transport equation is $u(x,t) = f(x - ct)$ and we need $f(x) = u(x,0) = \varphi(x)$, it is clear that a solution which satisfies both the PDE and the initial condition is

*By the way, we call these lines, which are the projection of the characteristic curves onto the $xy$-plane, projected characteristic curves, which, to be fair, is not very surprising.

$u(x,t) = \varphi(x - ct)$, which is nothing but a translation of the function. Physically, we see the concentration of pollutant moving as a whole without any diffusion or dispersion.

However, not every equation admits such a simple formula, and therefore it would be interesting for us to obtain the same conclusion without using our previous result. Thus, we are going to return to the characteristic equations and solve them with the knowledge of the initial condition.

We are providing data on a curve $\Gamma = (x, 0)$. We might write $(\Gamma, \varphi) \equiv (x, 0, \varphi(x))$ to denote the initial set of data on the integral surface. For each $x = r$, we might then pick a characteristic curve that passes through the point $(r, 0, \varphi(r)$. Essentially, this will allow us to pick the characteristic curves that respect the initial conditions and build the integral surface with them.

In terms of the characteristic equations, this is equivalent to solving Eq. (2.9) on page 13:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = c, \\ \dfrac{\mathrm{d}t}{\mathrm{d}s} = 1, \\ \dfrac{\mathrm{d}z}{\mathrm{d}s} = 0, \end{cases}$$

under the conditions

$$\begin{cases} x(r,0) = r, \\ t(r,0) = 0, \\ z(r,0) = \varphi(r), \end{cases} \tag{2.14}$$

where we have chosen that these values occur at $s = 0$ for simplicity (since $s$ is nothing but a parameter in terms of which we write the curves, we could pick any other value, but 0 should be easy to work with).

If before we had to solve a PDE under a initial condition, now we have simply a system of ODEs, which is far easier. We already knew the solution was given by Eq. (2.10) on page 13, but now that we have the initial conditions we can find the values of the constants $k_i$ (actually, since now our functions depend on a variable $r$ as well, $k_i$ doesn't represent a constant, but a function of $r$). By setting $s = 0$ in Eq. (2.10) on page 13 and comparing it to Eq. (2.14), we see the solution to the system of ODEs is

$$\begin{cases} x(r,s) = cs + r, \\ t(r,s) = s, \\ z(r,s) = \varphi(r). \end{cases} \tag{2.15}$$

Since $z$ is given in terms of $r$ and $s$, we should express these two variables in terms of $x$ and $t$. By doing that, we get

$$\begin{cases} s(x,t) = t, \\ r(x,t) = x - ct. \end{cases} \tag{2.16}$$

Thus, the solution $u(x,t) = z(r(x,t), s(x,t))$ is given by

$$u(x,t) = \varphi(x - ct), \tag{2.17}$$

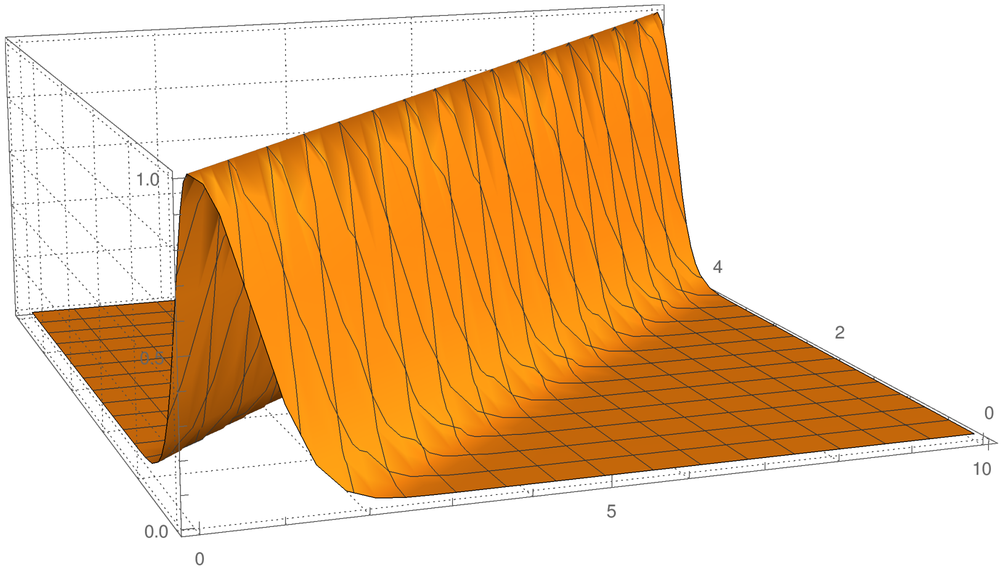



**Figure 2.1:** *Solution to the transport equation with* $c = 2$ *and initial condition* $u(x, 0) = e^{-x^2}$.

as expected.

The solution to the transport equation with $c = 2$ and initial condition $u(x, 0) = e^{-x^2}$ (a gaussian packet) is plotted in Fig. 2.1. ♥

However, a question arises: we provided data on a specific curve $\Gamma$, given by $(x, 0)$. Would this reasoning also hold for any curve $\Gamma$?

Mathematical intuition tells us no. The data we provided was used to choose the relevant characteristic curves to our solution, and thus the curve we provided should pass through at least two characteristic curves (actually more, but *at least* two is certainly guaranteed). If we had data over one characteristic curve, we would not be able to evaluate what happens in many points in the domain. Therefore, we expect that not any curve will do the service, as we shall see in some time.

In general, the problem we are interested in is named a Cauchy Problem. For first-order problems, a Cauchy Problem is a problem of the form

$$\begin{cases} F(\mathbf{x}, u, Du) = 0, \\ u\Big|_{\Gamma} = \varphi, \end{cases} \tag{2.18}$$

with $\mathbf{x} \in \mathbb{R}^n$ and $\Gamma$ being a $(n-1)$-dimensional manifold in $\mathbb{R}^n$.

In order to understand this statement, we must first learn some concepts about Topology.

# Three

# A Compact Introduction to Topology

There is no royal road to geometry.

EUCLID OF ALEXANDRIA, after being asked by Ptolemy I Soter whether there was a faster way to learn geometry than through Euclid's *Elements*.

## 3.1 Metric Spaces

MATHEMATICIANS have a unique view of how things work. As an example, a mathematician would never say he or she has a unique view of how things work before proving that his or her view indeed exists, and only them would proceed to prove that such a view is unique. As a second example, suppose you wanted to discuss the way you measure the distance between two points in $\mathbb{R}^2$. Perhaps a physicist would simply pick a ruler and measure it. Or draw a right triangle and find the length of the hypotenuse with some elementary geometry. A mathematician would first ask "What do you mean by distance?"*.

In Mathematics, everything should be defined with absolutely no possibility of misinterpretation, and the concept of distance is no exception. What are the essential things about distance that really make it a distance? What if I wanted to measure distances in a different way? What should I never, ever, change?

One could simply say that the distance between two points is the radius of the circle that has one of them as the center and the other as a point on the circumference. But is this always what we mean by distance? If you thing about it for a minute, you will realize that there are indeed other meanings.

The easiest example can be given by thinking about a passenger and a driver on a taxicab. They might argue about whether the distance between the departure and arrival points should be calculated by using the definition above or some other idea. If the path was simply a straight line, there should be no argument whatsoever. However, if they

*Perhaps a mathematician would disagree with me, but as I said, they have a unique view of how things work.

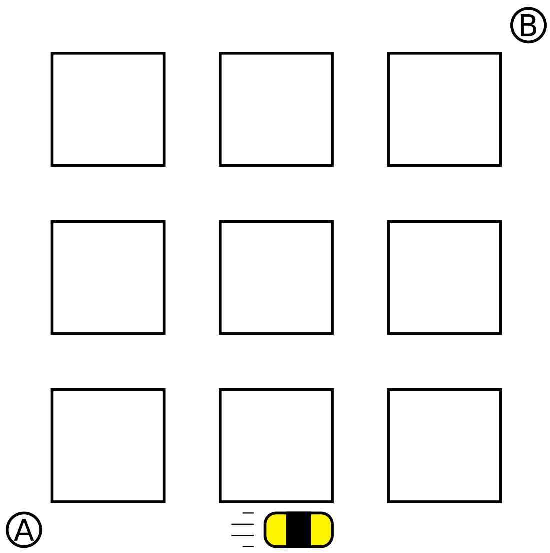



**Figure 3.1:** *What is the distance between the departure point, A, and the arrival point, B? The passenger might wish to argue that it is something like* 4.24 *blocks away, but the taxicab driver certainly will prefer to say it is* 6 *blocks away.*

had to pass through some blocks in order to arrive at their destination, as illustrated in Fig. 3.1, things could get complicated.

This example shows us that distance doesn't always means the same thing. However, not everything is lost! We might still search for the basic properties we expect a distance should have and *define* a distance to be such a thing.

One of the things we could ask is that if you are measuring the distance between two points on a certain space, this distance should never be negative. After all, would it even make sense to speak of a negative distance? Usually, the answer is no, and thus we are going to impose that no distance can be negative.

Another natural thing to require is that the distance between two points can only be zero if they are the same point (it would be weird if the distance between different points was zero, what would that even mean?). Besides, the distance between a point and itself should always be zero.

It is also natural for us to ask that the distance between point $A$ and point $B$ is the same as the distance between point $B$ and point $A$. Perhaps transit officers would disagree with this idea, but we are going to cover this up in a minute. Mathematically, it makes sense for us to ask that distances are symmetrical.

Finally, the last requirement might seem a bit odd at a first glance, but give it a chance. We call it the triangle inequality. Pick two points $A$ and $B$. For any third point $C$ you choose, the distance between point $A$ and point $B$ can never be greater than the sum of the distances between point $A$ and point $C$ and between point $C$ and point $B$. Intuitively, if you make a detour, the distance can never go down (it can stand still though).

Under these assumptions, we can finally give a proper definition to what a distance is. However, we usually don't call it a distance, but a metric.

**Definition 3.1** [Metric Space]**:**
Let $M$ be a non-empty set and $d : M \times M \to \mathbb{R}_+$ be a function such that, $\forall\, x, y, z \in M$:

i. $d(x, y) = 0 \Leftrightarrow x = y$;

ii. $d(x, y) = d(y, x)$ (symmetry);

iii. $d(x, y) \leq d(x, z) + d(y, z)$ (triangle inequality).

Under these conditions, we say that $(M, d)$ is a *metric space* and $d$ is said to be the *metric* defined on $M$. ♠

You might disagree with the hypotheses I made about what should be considered a distance. Perhaps you believe I should also require something else. Or you are a traffic officer and you believe I should not have asked that $d(x, y) = d(y, x)$*. And here comes an interesting part of Mathematics: you can simply use your own definition and work out its properties! Of course, I beg you not to call it a metric for the sake of clarity, but you can still find new results using your different hypotheses. I am going to use the definition I provided because it is the usual one and it shall yield the results I'm looking for, but I can't forbid you to work with something else.

Before we move on, it might be useful for us to give some examples of metric spaces.

**Examples** [$\mathbb{R}^n$]**:**
The Euclidean metric is the most usual metric in $\mathbb{R}^n$, and is defined by

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{3.1}$$

The taxicab metric we mentioned earlier without much rigour can be defined properly as

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|. \tag{3.2}$$

A third interesting metric can be defined as

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} |x_i - y_i|. \tag{3.3}$$

The notation is not random, though it might seem: in fact,

$$d_p(\mathbf{x}, \mathbf{y}) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

*It is alright, you might not ask this one. If you use the triangle inequality as I stated ($d(x, y) \leq d(x, z) + d(y, z)$) and the condition that $d(x, y) = 0 \Leftrightarrow x = y$, you can prove that $d(x, y) = d(y, x)$ and that $d(x, y) \geq 0$. However, if you state the triangle inequality as $d(x, y) \leq d(x, z) + d(z, y)$, I can't guarantee that symmetry will still hold.

defines a metric on $\mathbb{R}^n$ for every $p \geq 1$, though we are not going to prove this fact. $d_\infty$ is simply the limit with $p \to \infty$. For the special cases we've picked, most properties are pretty straightforward: we are taking sums of non-negative numbers (and a positive square root), these numbers can only sum to zero if $x_i = y_i$ for every $i$ and the expression doesn't change if we exchange $\mathbf{x}$ and $\mathbf{y}$. The tricky property is the triangle inequality.

Let us begin by proving the triangle inequality in the case $n = 1$. You should notice that, under this assumption, $d_1(x, y) = d_2(x, y) = d_\infty(x, y) = |x - y|$.

We wish to prove that, given $x, y, z \in \mathbb{R}$, $|x - y| \leq |x - z| + |y - z|$. Without any loss of generality, we might assume $x \geq y$, and thus $|x - y| = x - y$. There are now three possibilities: $z \leq y \leq x$, $y \leq z \leq x$ and $y \leq x \leq z$.

If the first possibility holds, then

$$\begin{aligned} |x - z| + |y - z| &= x - z + y - z, \\ &= x - y + 2y - 2z, \\ &= |x - y| + 2|y - z|, \\ &\geq |x - y|. \end{aligned} \tag{3.4}$$

If the second possibility holds, we have

$$\begin{aligned} |x - z| + |y - z| &= x - z + z - y, \\ &= x - y, \\ &= |x - y|. \end{aligned} \tag{3.5}$$

For the third possibility,

$$\begin{aligned} |x - z| + |y - z| &= -x + z - y + z, \\ &= x - y + 2z - 2x, \\ &= |x - y| + 2|x - z|, \\ &\geq |x - y|. \end{aligned} \tag{3.6}$$

Thus, the triangle inequality holds for all the three metrics we gave when $n = 1$.

For arbitrary $n$, we might simply use this result for each $i$ and realize that

$$\begin{aligned} |x_i - y_i| &\leq |x_i - z_i| + |y_i - z_i|, \\ \sum_{i=1}^{n} |x_i - y_i| &\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |y_i - z_i|, \\ \therefore d_1(\mathbf{x}, \mathbf{y}) &\leq d_1(\mathbf{x}, \mathbf{z}) + d_1(\mathbf{y}, \mathbf{z}). \end{aligned} \tag{3.7}$$

As for $d_\infty$, let $1 \leq i, j, k \leq n$ be such that $|x_i - y_i| = \max_{1 \leq \alpha \leq n} |x_\alpha - y_\alpha|$, $|x_j - z_j| = \max_{1 \leq \beta \leq n} |x_\beta - z_\beta|$ and $|y_k - z_k| = \max_{1 \leq \gamma \leq n} |y_\gamma - z_\gamma|$. Notice that, by definition of $j$ and $k$, it holds that $|x_j - z_j| \geq |x_i - z_i|$ and $|y_k - z_k| \geq |y_i - z_i|$ Using the result we have for $n = 1$, we may write

$$|x_i - y_i| \leq |x_i - z_i| + |y_i - z_i|,$$

$$\begin{aligned}
&\leq |x_j - z_j| + |y_k - z_k|, \\
\max_{1\leq\alpha\leq n} |x_\alpha - y_\alpha| &\leq \max_{1\leq\beta\leq n} |x_\beta - z_\beta| + \max_{1\leq\gamma\leq n} |y_\gamma - z_\gamma|, \\
\therefore d_\infty(\mathbf{x}, \mathbf{y}) &\leq d_\infty(\mathbf{x}, \mathbf{z}) + d_\infty(\mathbf{y}, \mathbf{z}).
\end{aligned} \tag{3.8}$$

Finally, we need to prove the same inequality for $d_2$. You might recall from linear algebra that the usual norm on $\mathbb{R}^n$ looks pretty much like $d_2$*. If we define an inner product on $\mathbb{R}^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i, \tag{3.9}$$

then we know that the norm induced by this inner product is going to be

$$\|\mathbf{x}\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{3.10}$$

Notice then that $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

The triangle inequality can then be proved by making use of either the triangle inequality for the norm or Cauchy-Schwartz inequality (which also can be used to proved that inner products do induce a norm). We are going to use the latter.

The Cauchy-Schwartz inequality can be written (for a vector space over the real numbers) as

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|. \tag{3.11}$$

We might write

$$\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle, \\
\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle, \\
&= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2, \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \cdot \|\mathbf{y}\| + \|\mathbf{y}\|^2, \\
&\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.
\end{aligned} \tag{3.12}$$

Eq. (3.12) proves the triangle inequality for norms induced by an inner product. By using $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ and $(\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y})$, one can prove the triangle inequality for $d_2$. ♥

**Examples** [Function Spaces]**:**

We might as well give some examples concerning function spaces. Let $\mathcal{C}^0([0,1])$ be the set of all continuous functions $f\colon [0,1] \to [0,1]$. Then $(\mathcal{C}^0([0,1], d_p)$ with $p \geq 1$ or $p = \infty$ is a metric space, with $d_p$ defined by

$$d_p(f, g) :- \left[ \int_0^1 |f(x) - g(x)|^p \, \mathrm{d}x \right]^{\frac{1}{p}}, \tag{3.13}$$

*This is an example of the fact that, if $\|\cdot\|$ is a norm on a vector space $V$, then $d(u, v) :- \|u - v\|$ is a metric on $V$. I invite you to prove this theorem.

for finite $p$ or

$$d_\infty(f,g) := \lim_{p\to\infty} d_p(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|, \tag{3.14}$$

for infinite $p$.

Since the functions are continuous, the existence of a point $x_0$ such that $f(x_0) \neq g(x_0)$ implies the existence of an open interval $(a_0, b_0) \subseteq \mathbb{R}$ such that $x_0 \in (a_0, b_0)$ and $f(x) \neq g(x), \forall\, x \in (a_0, b_0)$.

Indeed, since $f$ and $g$ are continuous, so is $h(x) \equiv f(x) - g(x)$. Supposing, without any loss of generality, that $f(x_0) > g(x_0)$, we know that

$$\forall\, \epsilon > 0, \exists\, \delta > 0; |x - x_0| < \delta \Rightarrow |h(x) - h(x_0)| < \epsilon. \tag{3.15}$$

If we pick $\epsilon = \frac{1}{2}h(x_0)$, then there is $\delta > 0$ such that, if $x \in (x_0 - \delta, x_0 + \delta)$, then

$$-\frac{1}{2}h(x_0) < h(x) - h(x_0) < \frac{1}{2}h(x_0). \tag{3.16}$$

Therefore, we found an open interval such that

$$\frac{1}{2}h(x_0) < h(x) < \frac{3}{2}h(x_0) \tag{3.17}$$

for every $x \in (a_0, b_0)$, with $a_0 = x_0 - \delta$ and $b_0 = x_0 + \delta$.

Notice then that the integral of $|f(x) - g(x)|$ over this open interval is certainly positive (never zero), while the integral over the rest of the domain is certainly not negative (since we are integrating a non-negative function). Thus, $d_p(f,g) > 0$ if $f(x) \neq g(x)$. The case $p \to \infty$ is even easier: if there is a point in which $|f(x_0) - g(x_0)| \neq 0$, then $d_\infty(f,g) \geq |f(x_0) - g(x_0)|$, since 0 is the smallest value $|f(x) - g(x)|$ can assume.

The reverse implication and the symmetry property are left as exercises. The proof of the triangle inequality is omitted. ♥

The same process that allowed us to find a satisfying definition to what is a metric allows us to generalize the concept of an open set. Of course, you might be wondering: why would I ever care about open sets so much that I would want to make this concept more general?

As we shall see in a moment, open sets are extremely related to the notions of limits and continuity of a function. Indeed, we are able to define what is a continuous function without ever needing to speak about what is a limit, and this definition would still be consistent with the usual definition based on limits (actually, it would extend this notion!).

For us to speak of continuity, we will need for the domain and range to have some *topological* structure. This is just mathematical slang to say that they should obey some conditions in order for the terms “limit”, “continuity” and everything else to make sense.

That being said, let’s move on to work out some properties of open sets in metric spaces.

Perhaps you remember from advanced calculus courses what is an open ball in $\mathbb{R}^n$ and so on. In terms of metric spaces, these ideas can be expressed in a more general form.

**Definition 3.2** [Open Ball]**:**
Let $(M, d)$ be a metric space. We define the *open ball with center* $x \in M$ *and radius* $r \in \mathbb{R}_+$ as the set

$$\mathcal{B}_r(x) \equiv \mathcal{B}(x, r) :- \{y \in M; d(x, y) < r\}. \qquad \spadesuit$$

As you should notice, this definition is essentially the same you will find in calculus courses, the only difference being it is written in terms of a general metric. In calculus, one would use usually the Euclidean metric.

The notions of interior point of a set and open sets are still the same.

**Definition 3.3** [Interior Point]**:**
Let $(M, d)$ be a metric space and let $X \subseteq M$ be a set. We say a point $x \in X$ is an *interior point* of $X$ if, and only if, $\exists\, r > 0; \mathcal{B}_r(x) \subseteq X$. ♠

**Definition 3.4** [Open Set]**:**
Let $(M, d)$ be a metric space and let $X \subseteq M$ be a set. We say $X$ is *open* if, and only if, every point $x \in X$ is an interior point of $X$. ♠

**Lemma 3.5:**
*Given any metric space* $(M, d)$*, every open ball is an open set.* □

*Proof:*
Let $x_0 \in M$ and $r \in \mathbb{R}_+$. We want to prove that $\mathcal{B}_r(x_0)$ is an open set.

Let $x \in \mathcal{B}_r(x_0)$ and let $s \equiv r - d(x_0, x)$. Since $x \in \mathcal{B}_r(x_0)$, $d(x, x_0) < r$ and thus $s > 0$. I claim $\mathcal{B}_s(x) \subseteq \mathcal{B}_r(x_0)$.

Let $x' \in \mathcal{B}_s(x)$. Due to the triangle inequality, we have that

$$\begin{aligned} d(x_0, x') &\leq d(x_0, x) + d(x, x'), \\ &< d(x_0, x) + s, \\ &= d(x_0, x) + r - d(x, x_0), \\ &= r. \end{aligned} \tag{3.18}$$

Since $d(x_0, x') < r$, we know that $x' \in \mathcal{B}_r(x_0)$. Therefore, $\mathcal{B}_s(x) \subseteq \mathcal{B}_r(x_0)$ and thus every point of $\mathcal{B}_r(x_0)$ is an interior point, *id est*, $\mathcal{B}_r(x_0)$ is an open set. ■

As said before, open sets are extremely interesting due to their relation to continuity. In order to show this, we must of course know what do we mean by continuity.

**Definition 3.6** [Continuous Function]**:**
Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f \colon X \to Y$ be a function. $f$ is said to be *continuous at a point* $x_0$ if, and only if, $\forall\, \epsilon > 0, \exists\, \delta > 0; d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$. If $f$ is continuous at every $x_0 \in X$, then we say $f$ is a *continuous function*. ♠

**Theorem 3.7:**
*Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be metric spaces and* $f \colon X \to Y$ *be a function. Then* $f$ *is a continuous function if, and only if, the preimage* $f^{-1}(A)$ *of* $A$ *is an open set for every open set* $A \subseteq Y$. □

*Proof:*

Let $f$ be a continuous function and let $A \subseteq Y$ be an open set. Suppose $x_0 \in X$ is such that $f(x_0) \in A$. Since $A$ is an open set, we know that there exists $\epsilon > 0$ such that $\mathcal{B}_\epsilon(f(x_0)) \subseteq A$.

However, since $f$ is continuous, we know that there is $\delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$, *id est*, $x \in \mathcal{B}_\delta(x_0) \Rightarrow f(x) \in \mathcal{B}_\epsilon(f(x_0)) \subseteq A$. Thus, we know that the image of $\mathcal{B}_\delta(x_0)$ under $f$, $f(\mathcal{B}_\delta(x_0))$, is contained in $A$. Therefore, $\mathcal{B}_\delta(x_0) \subseteq f^{-1}(A)$, which means that $x_0$ is an interior point of $f^{-1}(A)$. Since the same argument applies to any point with image in $A$, it follows that $f^{-1}(A)$ is open for every open set $A \subseteq Y$, since every point of $f^{-1}(A)$ is an interior point.

If $A$ happened to be empty, then $f^{-1}(A)$ would also be empty and, therefore, trivially open.

Suppose now that, for every open set $A \subseteq Y$, $f^{-1}(A)$ is an open set as well. We wish to prove that $f$ is continuous.

Let $x_0 \in X$ and $\epsilon > 0$. Due to Lemma 3.5 on the preceding page, we know that $\mathcal{B}_\epsilon(f(x_0))$ is an open set, and thus its preimage is an open set as well by hypothesis. Since $x_0 \in f^{-1}(\mathcal{B}_\epsilon(f(x_0)))$, it has to be an interior point and thus there is $\delta > 0$ such that $\mathcal{B}_\delta(x_0) \subseteq f^{-1}(\mathcal{B}_\epsilon(f(x_0)))$. Thus, $\forall\, \epsilon, \exists\, \delta$ such that

$$\begin{aligned} d_X(x, x_0) < \delta \Rightarrow\ & x \in \mathcal{B}_\delta(x_0), \\ \Rightarrow\ & x \in f^{-1}(\mathcal{B}_\epsilon(f(x_0))), \\ \Rightarrow\ & f(x) \in \mathcal{B}_\epsilon(f(x_0)), \\ \Rightarrow\ & d_Y(f(x), f(x_0)) < \epsilon. \end{aligned} \tag{3.19}$$

Therefore, $f$ is continuous. ■

Theorem 3.7 on the previous page shows that studying open sets allows us to understand continuous functions better. Furthermore, if we are able to generalize the concept of an open set, we will be able to extend the definition of continuity beyond metric spaces.

In order to do so, we can prove some more properties about the collection of all open sets in a given metric space $(M, d)$. Such a collection is called a *topology* in $M$.

**Theorem 3.8:**

*Let $(M, d)$ be a metric space, $\tau \subseteq \mathbb{P}(M)$, where $\mathbb{P}(M)$ denotes the powerset of $M$, and $\Lambda$ be an arbitrary set of indexes. Then it holds that*

i. $\varnothing, M \in \tau$;

ii. $X, Y \in \tau \Rightarrow X \cap Y \in \tau$;

iii. $X_\lambda \in \tau, \forall\, \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} X_\lambda \in \tau$. □

*Proof:*

i. since there are no elements in $\varnothing$, it is vacuously true that every point in $\varnothing$ is an interior point. Let $x_0 \in M$ and let $r > 0$. Since $\mathcal{B}_r(x_0)$ is defined as a subset of $M$, of course $\mathcal{B}_r(x_0) \subseteq M$, and thus $x_0$ is an interior point of $M$. Therefore, every point of $M$ is an interior point, which means that $M$ is an open set;

ii. let $x_0 \in X \cap Y$. Since $x_0 \in X$, there exists $r_1 > 0$ such that $\mathcal{B}_{r_1}(x_0) \subseteq X$. Similarly, since $x_0 \in Y$, there exists $r_2 > 0$ such that $\mathcal{B}_{r_2}(x_0) \subseteq Y$. Let $r = \min\{r_1, r_2\}$. Then $\mathcal{B}_r(x_0) \subseteq \mathcal{B}_{r_1}(x_0) \subseteq X$ and $\mathcal{B}_r(x_0) \subseteq \mathcal{B}_{r_2}(x_0) \subseteq Y$. Therefore, $\mathcal{B}_r(x_0) \subseteq X \cap Y$, and thus $x_0$ is an interior point of $X \cap Y$. Since the argument holds for any $x_0 \in X \cap Y$, the set must be open. Of course, if $X \cap Y = \varnothing$, then $X \cap Y$ is an open set due to the previous item;

iii. let $x_0 \in \bigcup_{\lambda \in \Lambda} X_\lambda$. Then $x_0 \in X_\lambda$ for some $\lambda \in \Lambda$. Since $X_\lambda$ is open, there exists $r > 0$ such that $\mathcal{B}_r(x_0) \subseteq X_\lambda \subseteq \bigcup_{\lambda \in \Lambda} X_\lambda$. Therefore, $x_0$ is an interior point of $\bigcup_{\lambda \in \Lambda} X_\lambda$. Of course, if either $\Lambda = \varnothing$ or $X_\lambda = \varnothing, \forall\, \lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} X_\lambda = \varnothing$ and the set is open due to the first item. ■

This structure suggests a way of extending the definition of what is an open set.

## 3.2 Topological Spaces

Previously, we had to enumerate the basic properties we believed to be necessary to define what is distance. Now, in order to extend the definition of what is an open set, we might simply build upon the conclusions of Theorem 3.8 on the facing page.

**Definition 3.9** [Topological Space]**:**
Let $X$ be a non-empty set, let $\tau \subseteq \mathbb{P}(X)$ be a set and let $\Lambda$ be an arbitrary set of indexes. $\tau$ is said to be a *topology* on $X$ and $(X, \tau)$ is said to be a *topological space* if, and only if, the following axioms hold:

i. $\varnothing, X \in \tau$;

ii. $A, B \in \tau \Rightarrow A \cap B \in \tau$;

iii. $A_\lambda \in \tau, \forall\, \lambda \in \Lambda \Rightarrow \displaystyle\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$. ♠

Of course, now that we don't have any metric we need to update our definition of what is an open set. As our definition of topological space might suggest, the open sets are simply the elements of the topology we are considering.

**Definition 3.10** [Open Set]**:**
Let $(X, \tau)$ be a topological space. We say a set $A \subseteq X$ is an *open set* if, and only if, $A \in \tau$. ♠

Of course, given a fixed set $X$, the open sets on $(X, \tau)$ might be very different depending on the topology we choose. As the most elementary example, we proved in the previous section that every metric induces a topology in a metric space (we call it a *metric topology*), and the same set might admit many different metrics.

**Examples** [Some Topological Spaces]**:**

The first example of a topological space one might pick is defined by "What happens if I let every set be an open set?". Given a non-empty set $X$, just set $\tau = \mathbb{P}(X)$. All axioms hold and you have at your hands the *discrete topology*. Curiously, this is also a metric topology, and is induced by the trivial metric:

$$d_T(x,y) = \begin{cases} 0, \text{ if } x = y, \\ 1, \text{ if } x \neq y. \end{cases} \tag{3.20}$$

The next thing that might come to your mind is "If I can make every set be an open set, can I choose a topology so that no set is open?". The answer is no, since one of the requirements for $\tau$ to be a topology is that $\varnothing, X \in \tau$. Nevertheless, $\tau = \{\varnothing, X\}$ does constitute a topology, and is called the *trivial topology* or the *indiscrete topology*.

Given a set $X$ and a set $A \subseteq X$, $\tau \equiv \{B \subseteq X; A \subseteq B\} \cup \{\varnothing\}$ defines a topology in $X$. Indeed, $\varnothing, X \in \tau$ trivially. Given any two sets $B, C \in \tau$, $B \cap C \subseteq X$ and $A \subseteq B \cap C$ (for $A \subseteq B \subseteq X$ and $A \subseteq C \subseteq X$). Finally, given a set of indexes $\Lambda$, a collection of sets $B_\lambda \in \tau$ and any particular $\lambda_0 \in \Lambda$, we have that $A \subseteq B_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} B_\lambda \subseteq X$, the last relation being valid due to the fact that $B_\lambda \subseteq X$ for every $\lambda \in \Lambda$.

Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $\tau' \equiv \{A \cap O; O \in \tau\}$ is a topology on $A$, the so called *subspace* (also called *induced* or *relative*) *topology*. Indeed, since $\varnothing, X \in \tau$, $\varnothing = A \cap \varnothing \subseteq \tau'$ and $A = A \cap X \subseteq \tau'$. Given $B, C \in \tau'$, we know by the definition of $\tau'$ that there are $O_1, O_2 \in \tau$ such that $B = A \cap O_1$ and $C = A \cap O_2$, and thus $B \cap C = (A \cap O_1) \cap (A \cap O_2) = A \cap (O_1 \cap O_2)$. Since $\tau$ is a topology, $O_1 \cap O_2 \in \tau$ and it follows that $B \cap C \in \tau'$. Finally, given a set of arbitrary indexes $\Lambda$ and a collection of sets $B_\lambda \in \tau', \forall \lambda \in \Lambda$, we know there is a collection $O_\lambda \in \tau; B_\lambda = A \cap O_\lambda, \forall \lambda \in \Lambda$. Therefore, $\bigcup_{\lambda \in \Lambda} B_\lambda = \bigcup_{\lambda \in \Lambda} A \cap O_\lambda = A \cap \bigcup_{\lambda \in \Lambda} O_\lambda$. Since $\tau$ is a topology, $\bigcup_{\lambda \in \Lambda} O_\lambda \in \tau$ and thus $\bigcup_{\lambda \in \Lambda} B_\lambda \in \tau'$, proving our claim. ♥

As you see, different topologies might have more or less open sets than others, and such comparisons lead us to the following definition:

**Definition 3.11** [Finer, Coarser and Comparable]**:**

Let $X$ be a set and let $\tau$ and $\tau'$ be topologies on $X$. If $\tau \subseteq \tau'$, we say that $\tau'$ is *finer* than $\tau$. If $\tau \subset \tau'$, we say $\tau'$ is *strictly finer* than $\tau$. Under the same assumptions, we say $\tau$ is *coarser*, or *strictly coarser*, than $\tau'$. Whenever $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$ we say $\tau$ and $\tau'$ are *comparable*. ♠

In fact, it is even possible to define the finest topology containing a given set or the coarsest topology contained within a given set.

**Theorem 3.12:**

*Let $X$ be a set, $\Lambda$ be an arbitrary set of indexes and $(\tau_\lambda)_{\lambda \in \Lambda}$ be a family of topologies on $X$. Then $\bigcap_{\lambda \in \Lambda} \tau_\lambda$ is a topology on $X$.* □

*Proof:*

Since $\varnothing, X \in \tau_\lambda, \forall \lambda \in \Lambda$, we know that $\varnothing, X \in \bigcap_{\lambda\in\Lambda} \tau_\lambda$.

Suppose that $O_1, O_2 \in \bigcap_{\lambda\in\Lambda} \tau_\lambda$. Then we know that $O_1, O_2 \in \tau_\lambda, \forall \lambda \in \Lambda$. Since every $\tau_\lambda$ is a topology, we know that $O_1 \cap O_2 \in \tau_\lambda, \forall \lambda \in \Lambda$. Therefore, it follows that $O_1 \cap O_2 \in \bigcap_{\lambda\in\Lambda} \tau_\lambda$.

Let $\Omega$ be an arbitrary set of indexes and $O_\omega \in \bigcap_{\lambda\in\Lambda} \tau_\lambda, \forall \omega \in \Omega$. Then we know that $O_\omega \in \tau_\lambda, \forall \omega \in \Omega, \forall \lambda \in \Lambda$. Since every $\tau_\lambda$ is a topology, it follows that $\bigcup_{\omega\in\Omega} O_\omega \in \tau_\lambda, \forall \lambda \in \Lambda$. Therefore, we conclude that $\bigcup_{\omega\in\Omega} O_\omega \in \bigcap_{\lambda\in\Lambda} \tau_\lambda$. This concludes the proof. ■

*Remark:*

If the intersection of topologies is still a topology, a natural question one could ask is whether the union of topologies is a topology.

Let $X = \{a, b, c\}$ be a set. Consider the topologies

$$\tau_1 = \{\varnothing, X, \{a\}, \{a, b\}\}, \quad \tau_2 = \{\varnothing, X, \{a\}, \{b, c\}\}. \tag{3.21}$$

I will leave to you the pleasure of proving that $\tau_1$ and $\tau_2$ are indeed topologies. Notice that $\tau \equiv \tau_1 \cup \tau_2 = \{\varnothing, X, \{a\}, \{a, b\}, \{b, c\}\}$ is *not* a topology. Indeed, topologies are always closed under finite intersections, and $\{a, b\} \cap \{b, c\} = \{b\} \notin \tau$, even though $\{a, b\}, \{b, c\} \in \tau$. Thus, $\tau$ is not a topology. ♣

**Proposition 3.13:**

*Let $X$ be a set, $\Lambda$ be an arbitrary set of indexes and $\{\tau_\lambda\}_{\lambda\in\Lambda}$ be a family of topologies on $X$. Then it holds that:*

i. *there exists the coarsest topology in the family of topologies finer than $\tau_\lambda, \forall \lambda \in \Lambda$,* id est, *there is a topology which is the least upper bound of $\{\tau_\lambda\}_{\lambda\in\Lambda}$;*

ii. *there exists the finest topology in the family of topologies coarser than $\tau_\lambda, \forall \lambda \in \Lambda$,* id est, *there is a topology which is the greatest lower bound of $\{\tau_\lambda\}_{\lambda\in\Lambda}$.*

*Least upper bound and greatest lower bound should be understood, in both cases, with respect to the inclusion order.* □

*Proof:*

i. Let $\mathcal{F}$ be the family of all topologies finer than $\tau_\lambda, \forall \lambda \in \Lambda$. Due to the Axiom Schema of Separation this is indeed a set*, since

$$\mathcal{F} = \{\tau \in \mathbb{P}(\mathbb{P}(X)); \tau \text{ is a topology finer than } \tau_\lambda, \forall \lambda \in \Lambda\}. \tag{3.22}$$

Due to Theorem 3.12 on the preceding page, we know that $\tau = \bigcap \mathcal{F}$ is a topology as well. Notice that $\tau \subseteq \tau_\mathcal{F}, \forall \tau_\mathcal{F} \in \mathcal{F}$. Furthermore, since $\tau_\lambda \subseteq \tau_\mathcal{F}, \forall \lambda \in \Lambda, \forall \tau_\mathcal{F} \in \mathcal{F}$,

*Perhaps you are not bored about whether this is or not a set, and in this case you probably are not interested on this footnote. Otherwise, you might be interested in Axiomatic Set Theory and might want to have a look at references [30, 63, 71].

it follows that $\tau_\lambda \subseteq \tau, \forall \lambda \in \Lambda$. Thus, $\tau \in \mathcal{F}$. Given that we already know that $\tau \subseteq \tau_\mathcal{F}, \forall \tau_\mathcal{F} \in \mathcal{F}$, we conclude $\tau$ is the coarsest topology in the family of topologies finer than $\tau_\lambda, \forall \lambda \in \Lambda$.

ii. Let $\mathcal{C}$ be the family of all topologies coarser than $\tau_\lambda, \forall \lambda \in \Lambda$. An argument similar to the one used in the previous item proves the fact that $\mathcal{C}$ is a set. However, this time let us define $\tau = \bigcap_{\lambda \in \Lambda} \tau_\lambda$. Notice that $\tau \subseteq \tau_\lambda, \forall \lambda \in \Lambda$. Therefore, $\tau \in \mathcal{C}$.

Let $\tau' \in \mathcal{C}$. By definition of $\mathcal{C}$, $\tau' \subseteq \tau_\lambda, \forall \lambda \in \Lambda$. Therefore, $\tau' \subseteq \bigcap_{\lambda \in \Lambda} \tau_\lambda = \tau$. Since $\tau' \subseteq \tau, \forall \tau' \in \mathcal{C}$, it is proved that $\tau$ is the finest topology in the family of topologies coarser than $\tau_\lambda, \forall \lambda \in \Lambda$. ■

*Scholium:*

You might realize that nowhere in the proof of Proposition 3.13 on the preceding page i. we used the hypothesis that $\{\tau_\lambda\}_{\lambda \in \Lambda}$ is a family of topologies. We could simply say that a topology $\tau$ is finer than an arbitrary set $\tau_\lambda$ whenever $\tau_\lambda \subseteq \tau$ and the argument would still hold. Thus, we *can* define the coarsest topology that makes a predefined collection of sets be a collection of open sets. ♣

This is our first example of how we can specify a topology by defining a smaller set, instead of the topology as a whole. You might have noticed that every topology we have shown could be explicitly written (except for those which were intersections of families of topologies). However, it is not exactly trivial to write the explicit form of a metric topology, for example. This is only one example of a case in which it is easier for us to specify a small set that can be used to describe the topology as a whole (for metric spaces, it is enough to specify the metric, and therefore which are the open balls). How could we obtain a wider sense of the collection of open balls in a metric sense? Can we define a "basis" for a topology?

When we were dealing with metric spaces, we defined an open set to be a such that had all points as interior points, *id est*, every point of that set could be "covered" by an open ball that was contained within said set.

In order to make things more clear, suppose $(X, \tau)$ is a topological space and we want to describe $\tau$ as if it was generated by some weird collection of "open balls" which is not necessarily associated to a metric. Let $\mathfrak{B}$ denote this collection. Since $X \in \tau$ and we want every point of $X$ to be covered by some "open ball", we must require that $\forall x \in X, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

As a second requirement, we are going to ask that $\forall \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \exists \mathcal{B}_3 \in \mathfrak{B}; \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, which does hold for open balls in metric spaces, though the reason it is so important is not so clear right now, but it should be more evident within some time.

Let us then define a basis for a topology:

**Definition 3.14** [Basis for a Topology on a Set]**:**

Let $X$ be a set. We say $\mathfrak{B} \subseteq \mathbb{P}(X)$ is a *basis* for a topology on $X$ whenever the following conditions hold:

i. $\forall x \in X, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$;

ii. $\forall \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. ♠

We must then check whether we can or not generate a topology using this concept. When dealing with open balls, a set would be open whenever all of its points were interior points. If we denote the collection of open balls as $\mathfrak{B}$, it means the metric topology is

$$\tau :- \{O \subseteq X | \forall x \in O, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O\}. \tag{3.23}$$

Indeed, the same is going to be true in our wider context.

**Theorem 3.15:**

*Let X be a set and $\mathfrak{B}$ be a basis for a topology on X. Then the collection $\tau$ defined as*

$$\tau := \{O \subseteq X | \forall x \in O, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O\} \tag{3.24}$$

*defines a topology on X.* □

*Proof:*

The fact that $\varnothing \in \tau$ is vacuously true. $X \in \tau$ as well, since $\mathcal{B} \subseteq X, \forall \mathcal{B} \in \mathfrak{B}$ and $\forall x \in X, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

Given $O_1, O_2 \in \tau$, $O_1 \cap O_2 \in \tau$. Indeed, $\forall x \in O_1 \cap O_2$ we know there are $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ such that $x \in \mathcal{B}_1 \subseteq O_1$ and $x \in \mathcal{B}_2 \subseteq O_2$, since $O_1, O_2 \in \tau$. Therefore, $x \in \mathcal{B}_1 \cap \mathcal{B}_2$ and we know that $\exists \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, since $\mathfrak{B}$ is a basis. Since $\mathcal{B}_1 \subseteq O_1$ and $\mathcal{B}_2 \subseteq O_2$, it follows that $x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \subseteq O_1 \cap O_2$ and thus $\exists \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq O_1 \cap O_2$, which proves that $O_1 \cap O_2 \in \tau$.

Finally, let $\Lambda$ be an arbitrary set of indexes and $O_\lambda \in \tau, \forall \lambda \in \Lambda$. Let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Then, for any $x \in O$, there must be $\mathcal{B} \in \mathfrak{B}$ such that $x \in \mathcal{B} \subseteq O$, because $x \in O_\lambda$ for some $\lambda \in \Lambda$, and, since $O_\lambda \in \tau$, there is $\mathcal{B} \in \mathfrak{B}$ such that $x \in \mathcal{B} \subseteq O_\lambda$. We know that $O_\lambda \subseteq O$, and thus the theorem is proved. ■

Notice that the proof of Theorem 3.15 exhibits the importance of the second requirement made when defining a basis.

There is still another way of describing the topology generated by a basis. Perhaps you recall from real analysis that a set on $\mathbb{R}$ is open if, and only if, it can be written as unions of open intervals (which are nothing but the open balls in $\mathbb{R}$ with the usual metric). A similar result holds in this much more general context.

**Lemma 3.16:**

*Let X be a set and let $\mathfrak{B}$ be a basis for a topology on X. Then the topology $\tau$ generated by $\mathfrak{B}$ is the collection of all the sets of $\mathbb{P}(X)$ that can be written as unions of elements of $\mathfrak{B}$.* □

*Proof:*

Let $v$ be the collection of all the sets of $\mathbb{P}(X)$ that can be written as unions of elements of $\mathfrak{B}$. We what to prove that $\tau = v$. As usual, we will prove that $v \subseteq \tau$ and $\tau \subseteq v$.

The first inclusion is easy. Let $\mathcal{B} \in \mathfrak{B}$. $\forall x \in \mathcal{B}, \exists \mathcal{B}' = \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}' = \mathcal{B} \subseteq \mathcal{B}$. Therefore, $\mathfrak{B} \subseteq \tau$. Since $\tau$ is closed under arbitrary unions, of course $v \subseteq \tau$.

Let $O \in \tau$. We know that $\forall x \in O, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$. Let $\mathfrak{B}_O$ be the collection of such sets $\mathcal{B}$. Then notice that, $\forall x \in O$, $x \in \bigcup \mathfrak{B}_O$, and thus $O \subseteq \bigcup \mathfrak{B}_O$. Since

$\mathcal{B} \subseteq O, \forall\, \mathcal{B} \in \mathfrak{B}_O$, we know that $x \in O, \forall\, x \in \bigcup \mathfrak{B}_O$. Thus, $\bigcup \mathfrak{B}_O \subseteq O$. It follows that $O = \bigcup \mathfrak{B}_O$. Since the latter is nothing but a union of elements in $\mathfrak{B}$, we have proved that $\tau \subseteq \upsilon$, and the lemma follows. ■

**Example** [Product Topology]**:**

Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Then $X \times Y$ can be turned into a topological space the topology generated by the basis $\mathfrak{B} = \{O_X \times O_Y; O_X \in \tau_X, O_Y \in \tau_Y\}$. This is one of the ways of defining the so called *product topology* (we shall see another one in Definition 3.68 on page 57).

We must proceed to check that $\mathfrak{B}$ is indeed a basis. Let $(x, y) \in X \times Y$. Since $\tau_X$ and $\tau_Y$ are topologies, we know that $X \in \tau_X$ and $Y \in \tau_Y$, and thus $X \times Y \in \mathfrak{B}$. Therefore, $\forall\, (x, y) \in X \times Y, \exists\, \mathcal{B} \in \mathfrak{B}; (x, y) \in \mathcal{B}$.

Let then $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ and $(x, y) \in \mathcal{B}_1 \cap \mathcal{B}_2$. Since $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, we know there are $O_i^X \in \tau_X$ and $O_i^Y \in \tau_Y, i = 1, 2$, such that $\mathcal{B}_1 = O_1^X \times O_1^Y$, with a similar relation for $\mathcal{B}_2$. Thus, $(x, y) \in \left(O_1^X \times O_1^Y\right) \cap \left(O_2^X \times O_2^Y\right) = \left(O_1^X \cap O_2^X\right) \times \left(O_1^Y \cap O_2^Y\right)$. Since $\tau_X$ is a topology, $O_1^X \cap O_2^X \in \tau_X$, with a similar result for $Y$. Therefore, $\mathcal{B} = \left(O_1^X \cap O_2^X\right) \times \left(O_1^Y \cap O_2^Y\right) \in \mathfrak{B}$. It follows that $(x, y) \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. This finishes the proof that $\mathfrak{B}$ is indeed a basis for a topology on $X \times Y$. ♥

We might yet wonder whether we can start with a topological space and find a basis for the topology we are dealing with. In other words, we can use a basis to obtain a topology, but what about going the other way around? Does every topology admit a basis?

**Proposition 3.17:**

*Let $(X, \tau)$ be a topological space and $\mathfrak{B} \subseteq \tau$ be a collection of sets such that $\forall\, O \in \tau, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$. Then $\mathfrak{B}$ is a basis for a topology on $X$, with $\tau$ being generated by $\mathfrak{B}$.* □

*Proof:*

Firstly, we want to prove that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$. This follows directly from the fact that $X \in \tau$, for $\tau$ is a topology. Since $\forall\, O \in \tau, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$, we simply set $O = X$ and we get that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

We must then prove that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$. Since $\mathfrak{B} \subseteq \tau$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are open sets and so is their intersection. Thus, since $\forall\, O \in \tau, \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O$, we might set $O = \mathcal{B}_1 \cap \mathcal{B}_2$ and obtain $\forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, as desired. ■

As we noted earlier, it might be easier to deal with bases instead of topologies. It is a natural conclusion that it would be interesting if we could compare two comparable topologies by simply taking a look at the bases that generate them.

**Proposition 3.18:**

*Let $X$ be a set, $\mathfrak{B}$ and $\mathfrak{B}'$ be each a basis for a topology on $X$ and let $\tau$ and $\tau'$ be the topologies generated by $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Then the following affirmations are equivalent:*

i. *$\tau'$ is finer than $\tau$;*

ii. $\forall x \in X, \forall \mathcal{B} \in \mathfrak{B}$ *with* $x \in \mathcal{B}, \exists \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$. □

*Proof:*

i ⇒ ii: If $\tau'$ is finer than $\tau$, then $\tau \subseteq \tau'$. Therefore, ii surely holds, because $\forall \mathcal{B} \in \mathfrak{B}, \mathcal{B} \in \mathfrak{B}'$.

ii ⇒ i: If $\forall x \in X, \forall \mathcal{B} \in \mathfrak{B}$ with $x \in \mathcal{B}, \exists \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$, then $\mathcal{B}$ is in the topology generated by $\mathfrak{B}'$ (as you can see from Theorem 3.15 on page 29). Therefore, $\mathfrak{B} \subseteq \tau'$. Since a topology is closed under arbitrary unions and $\tau$ is nothing but the collection of all sets that can be written as unions of elements of $\mathfrak{B}$ (Lemma 3.16 on page 29), it follows that $\tau \subseteq \tau'$, *id est*, $\tau'$ is finer than $\tau$. ■

One might wonder why would we even care about which topology is finer or coarser. Is this nomenclature really useful? Well, the usefulness of a name is certainly always questionable. As Shakespeare would put, "What's in a name? That which we call a rose/By any other name would smell as sweet." (Romeo and Juliet (Act II, Scene ii, 45-46)). Nevertheless, the existence of a name for this concept will be useful when we deal with some more topological properties on metric spaces.

Since I've been saying "the usual topology on $\mathbb{R}$" in a sloppy way for some time, perhaps we should define some names for the main topologies we use on $\mathbb{R}$.

**Definition 3.19** [Common Topologies on $\mathbb{R}$]**:**

Consider the real line $\mathbb{R}$.

We define the *standard topology* on $\mathbb{R}$ as the topology generated by the open intervals ($\mathfrak{B} = \{(a,b); a,b \in \mathbb{R}\}$). Unless specified otherwise, every time we mention $\mathbb{R}$ it should be understood that $\mathbb{R}$ comes along with the standard topology. This is the topology I meant (and still mean) by *usual topology*.

We define the *Sorgenfrey topology*, also known as the *lower-limit topology*, on $\mathbb{R}$ as the topology generated by the set $\mathfrak{B}' = \{[a,b)\,; a,b \in \mathbb{R}\}$. The topological space formed by $\mathbb{R}$ with the Sorgenfrey topology is eventually called Sorgenfrey line.

Finally, we let $K = \{\frac{1}{n}; n \in \mathbb{N}^*\}$. The *K-topology* on $\mathbb{R}$ is the topology generated by the basis $\mathfrak{B}'' = \{(a,b)\,; a,b \in \mathbb{R}\} \cup \{(a,b) \setminus K; a,b \in \mathbb{R}\}$. ♠

This time, I will leave you the joy of proving the claim that $\mathfrak{B}, \mathfrak{B}'$ and $\mathfrak{B}''$ are indeed bases.

**Examples** [A Countable Basis for the Standard Topology on $\mathbb{R}$]**:**

Consider $\mathbb{R}$ along with its standard topology. It in fact admits a countable basis, namely*, the collection of open intervals centered at rational numbers with radius of the form $\frac{1}{n}, n \in \mathbb{N}$. We write $\mathfrak{B} = \left\{\mathcal{B}_{\frac{1}{n}}(x); x \in \mathbb{Q}, n \in \mathbb{N}\right\}$.

Firstly, we want to prove that $\forall x \in \mathbb{R}, \exists \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$. If $x \in \mathbb{Q}$, we might simply pick $n = 1$ and have $\mathcal{B}_1(x)$, which clearly contains $x$. On the other hand, if $x \in \mathbb{R} \setminus \mathbb{Q}$, then pick an arbitrary natural number $n$. Consider the interval $\mathcal{B}_{\frac{1}{n}}(x)$. Since $\mathbb{Q}$ is dense in

*Another possibility would be picking intervals with rational extreme points.

$\mathbb{R}$, there is at least one rational number $r \in \mathcal{B}_{\frac{1}{n}}(x)$. Since $|r - x| < \frac{1}{n}$, because $r \in \mathcal{B}_{\frac{1}{n}}(x)$, it follows that $x \in \mathcal{B}_{\frac{1}{n}}(r)$.

Secondly, let $x \in \mathcal{B}_{\frac{1}{n}}(x_1) \cap \mathcal{B}_{\frac{1}{m}}(x_2)$, for some given $x_1, x_2 \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Since $x \in \mathcal{B}_{\frac{1}{n}}(x_1)$, $\frac{1}{n} > |x - x_1|$, with a similar result involving $m$ and $x_2$. Let

$$r = \max\left\{\frac{1}{n} - |x - x_1|, \frac{1}{m} - |x - x_2|\right\}.$$

Due to the Archimedean property, we know that there is a natural number $p \in \mathbb{N}$ such that $\frac{1}{p} < r$. I leave to you the task of proving that $\mathcal{B}_r(x) \subseteq \mathcal{B}_{\frac{1}{n}}(x_1) \cap \mathcal{B}_{\frac{1}{m}}(x_2)$ (suggestion: use the triangle inequality). Notice then that $\mathcal{B}_{\frac{1}{p}}(x) \subseteq \mathcal{B}_r(x) \subseteq \mathcal{B}_{\frac{1}{n}}(x_1) \cap \mathcal{B}_{\frac{1}{m}}(x_2)$, proving that $\mathfrak{B}$ is a basis.

Finally, we must yet prove that $\mathfrak{B}$ is countable. Let $f$ be the function $f\colon \mathbb{Q} \times \mathbb{N} \to \tau$ that maps $(r, n) \to f(r, n) = \mathcal{B}_{\frac{1}{n}}(r)$. $\tau$ denotes the usual topology on $\mathbb{R}$. If we restrict the range of $f$ to its image, which is precisely $\mathfrak{B}$, we obtain a surjective function from $\mathbb{Q} \times \mathbb{N}$ to $\mathfrak{B}$.

$f$ is also a injective function. Let $(r_1, n_1) \neq (r_2, n_2)$, both being elements of $\mathbb{Q} \times \mathbb{N}$. If $r_1 = r_2$, suppose without any loss of generality that $n_2 > n_1$. Since $r_1 + \frac{\frac{1}{n_1} - \frac{1}{n_2}}{2}$ is in $\mathcal{B}_{\frac{1}{n_1}}(r_1)$, but not in $\mathcal{B}_{\frac{1}{n_2}}(r_2)$. Thus, $f(r_1, n_1) \neq f(r_2, n_2)$.

Suppose now that $r_1 \neq r_2$, and suppose for contradiction that there are $n_1, n_2 \in \mathbb{N}$ such that $\mathcal{B}_{\frac{1}{n_1}}(r_1) = \mathcal{B}_{\frac{1}{n_2}}(r_2)$. Notice that $r_i \pm n_i, i = 1, 2$ is an upper (lower) bound for $\mathcal{B}_{\frac{1}{n_i}}(r_i)$, which is thus limited. The least upper bound (greatest lower bound) property then guarantees that there are $a, b \in \mathbb{R}$ such that $\mathcal{B}_{\frac{1}{n_i}}(r_i) = (a, b)$. I leave as an exercise to prove that in fact $b = r_i + n_i$ and $a = r_i - n_i$. It follows then that $r_1 = \frac{a+b}{2} = r_2$, contradicting our initial hypotheses that $r_1 \neq r_2$. It is then necessary that $f(r_1, n_1) \neq f(r_2, n_2)$, proving that $f$ is injective.

We have proven that $f$ is injective and surjective, and thus bijective. Since $\mathbb{Q} \times \mathbb{N}$ is the Cartesian product of two countable sets, it is countable itself. Since there is a bijection between $\mathfrak{B}$ and an countable set, $\mathfrak{B}$ is countable, *quo erat demonstrandum*. ♥

**Proposition 3.20:**

*Consider the real line, $\mathbb{R}$. Sorgenfrey's topology and the K-topology are strictly finer than the standard topology, though they are not comparable to one another.* □

*Proof:*

It is easy to see that the $K$-topology is finer than the standard topology: notice that $\mathfrak{B} \subseteq \mathfrak{B}$. Thus, of course $\forall\, x \in \mathbb{R}, \forall\, \mathcal{B} \in \mathfrak{B}$ with $x \in \mathcal{B}, \exists\, \mathcal{B}'' \in \mathfrak{B}''; x \in \mathfrak{B}'' \subseteq \mathfrak{B}$. Namely, $\mathcal{B}'' = \mathcal{B}$.

As for the Sorgenfrey line, let $x \in \mathbb{R}$ and let $a, b \in \mathbb{R}; a < x < b$, so that $x \in (a, b)$. Let $c = \frac{a+x}{2}$. Then $a < c < x$ and $x \in [c, b) \subseteq (a, b)$. Therefore, we've shown that $\forall\, x \in \mathbb{R}, \forall\, \mathcal{B} \in \mathfrak{B}$ with $x \in \mathfrak{B}, \exists\, \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$.

Proposition 3.18 on page 30 ensures that both the Sorgenfrey topology and the $K$-topology are finer than the standard topology. We still have to prove that they are *strictly* finer than the standard topology and not comparable with each other.

Let $a, b \in \mathbb{R}; a < 0 < b$. Then $0 \in (a, b) \setminus K$. Notice that this is an open set for the $K$-topology. Assume there are $c, d \in \mathbb{R}$ such that $0 \in (c, d) \subseteq (a, b) \setminus K$. Then there are no elements in $(c, d)$ with the form $\frac{1}{n}$. Since $0 \in (c, d)$, this means that $\exists\, d \in \mathbb{R}; \forall\, n \in \mathbb{N}^*, \frac{1}{n} \notin [0, d)$. However, since the Archimedean property holds in $\mathbb{R}$, $\forall\, d \in \mathbb{R}^*_+, \exists\, n \in \mathbb{N}^*; 0 < \frac{1}{n} < d$. Thus, we have reached a contradiction. Therefore, and due to Proposition 3.18 on page 30, the standard topology cannot be finer than the $K$-topology, and thus the $K$-topology is strictly finer than the standard topology.

Let $x, a \in \mathbb{R}, a > x$. Then $x \in [x, a)$, which is an open set for the Sorgenfrey topology. Suppose that there are $b, c \in \mathbb{R}$ such that $x \in (b, c) \subseteq [x, a)$. If $x \in (b, c)$, then $b < x$. However, since $(b, c) \subseteq [x, a)$, $x \leq b$. Since it is impossible for $b < x$ and $x \leq b$ to be true simultaneously, we have reached a contradiction. Due to Proposition 3.18 on page 30, it is not possible for the standard topology to be finer than the Sorgenfrey topology, and thus the latter is strictly finer than the former.

We can show that the Sorgenfrey topology is not finer than the $K$-topology in an analogous way to how we have shown that the standard topology is not finer than the $K$-topology. We can show that the $K$-topology is not finer than the Sorgenfrey topology in analogous way to how we have shown that the standard topology is not finer than the Sorgenfrey topology. Thus, since neither the Sorgenfrey topology nor the $K$-topology is finer than the other, they are not comparable. ■

We have been using only one of the axioms of a topology so far to specify a whole topology from a smaller set (the basis). Namely, the property that topologies are closed under arbitrary unions and we might write any open set as unions of elements from the base. One might then think whether we can generate a topology using the property that the topology is closed under finite intersections?

**Definition 3.21** [Subbasis]**:**

Let $X$ be a set. We say $\mathfrak{S}$ is a *subbasis* for a topology on $X$ if the union of all the elements of $\mathfrak{S}$ equals $X$. The *topology generated by the subbasis* $\mathfrak{S}$ is the collection of unions of finite intersections of elements of $\mathfrak{S}$, *id est*, the topology generated by the basis $\mathfrak{B} = \{\bigcap_{i=1}^{n} \mathcal{B}_i; \mathcal{B}_i \in \mathfrak{S}\}$. ♠

**Proposition 3.22:**

*The topology generated by a subbasis is indeed a topology.* □

*Proof:*

Let $X$ be a set and $\mathfrak{S}$ be a subbasis for a topology on $X$. Notice that our claim is equivalent to proving that $\mathfrak{B} = \{\bigcap_{i=1}^{n} \mathcal{B}_i; \mathcal{B}_i \in \mathfrak{S}\}$ is indeed a basis for a topology on $X$.

Firstly, we need to prove that $\forall\, x \in X, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}, n \in \mathbb{N}^*$. Let $x \in X$. Since $\bigcup \mathfrak{S} = X$, we know that $\exists\, \mathcal{B} \in \mathfrak{S}; x \in \mathcal{B}$. Let $\mathcal{B}_x$ be such a set. Since $\bigcap_{i=1} 1\mathcal{B}_x = \mathcal{B}_x$, we know that $\mathcal{B}_x \in \mathfrak{B}$. Therefore, we have found an element of the basis $\mathfrak{B}$ which contains $x$.

We then proceed to prove that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq$

$\mathcal{B}_1 \cap \mathcal{B}_2$. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ and let $x \in \mathcal{B}_1 \cap \mathcal{B}_2$. We know that there are sets $\mathcal{B}_1^i$ and $\mathcal{B}_2^i$ which are elements of $\mathfrak{S}$ such that $\mathcal{B}_1 = \bigcap_{i=1}^{n} \mathcal{B}_1^i$ and $\mathcal{B}_2 = \bigcap_{i=1}^{m} \mathcal{B}_2^i$. We know that $\left(\bigcap_{i=1}^{n} \mathcal{B}_1^i\right) \cap \left(\bigcap_{i=1}^{m} \mathcal{B}_2^i\right) = \mathcal{B} \in \mathfrak{B}$, since it is composed of finite intersections of elements of the subbasis. Notice now that $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$ and thus it holds that $x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Therefore, $\mathfrak{B}$ is indeed a basis for a topology on $X$ and the topology generated by a subbasis is indeed a topology. ■

Finally, it is common within Mathematics to consider a substructure within a larger set. For example, one might speak about a linear subspace within a linear space. Therefore, it is natural to wonder if there is any topological structure a topological space $(X, \tau)$ could induce on a subset $Y \subseteq X$.

**Definition 3.23** [Subspace of a Topological Space]**:**

Let $(X, \tau)$ be a topological space and let $Y \subseteq X$. We define the *subspace topology on* $Y$, also known as the *relative topology*, as the collection

$$\tau_Y :- \{O \cap Y; O \in \tau\}. \tag{3.25}$$

$(Y, \tau_Y)$ is then said to be a *subspace* of $(X, \tau)$. ♠

Perhaps you remember we have already proven that $\tau_Y$ is indeed a topology on $Y$. Have a look at the examples of topological spaces at page 26.

We already know that working with bases is easier than with the topologies themselves, and thus the following result might prove itself useful.

**Lemma 3.24:**

*Let $(X, \tau)$ be a topological space, $\mathfrak{B}$ be a basis for the topology $\tau$ and $Y \subseteq X$. Then $\mathfrak{B}_Y \equiv \{\mathcal{B} \cap Y; \mathcal{B} \in \mathfrak{B}\}$ is a basis for the subspace topology in $Y$.* □

*Proof:*

Let $\tau_Y$ denote the subspace topology on $Y$.

We know that $\mathcal{B} \in \tau, \forall\, \mathcal{B} \in \mathfrak{B}$ (this follows from Lemma 3.16 on page 29). Thus, given $\mathcal{B} \in \mathfrak{B}$, we know that $\mathcal{B} \cap Y \in \tau_Y$, by the very definition of $\tau_Y$. This implies $\mathfrak{B}_Y \subseteq \tau_Y$.

Due to Proposition 3.17 on page 30, we only have to prove that $\forall\, U \in \tau_Y, \forall\, y \in U, \exists\, \mathcal{B}_Y \in \mathfrak{B}_Y; y \in \mathcal{B}_Y \subseteq U$. Therefore, let $U \in \tau_Y$ and consider some set $O \in \tau; O \cap Y = U$ (the existence of such a set is guaranteed by the definition of $\tau_Y$). Let $y \in U \subseteq O$. Since $\mathfrak{B}$ is a basis for $\tau$, we know that $\exists\, \mathcal{B} \in \mathfrak{B}; y \in \mathcal{B} \subseteq O$. Since $y \in U \subseteq Y$, we know that $y \in \mathcal{B} \cap Y = \mathcal{B}_Y$. Furthermore, notice that $\mathcal{B}_Y = \mathcal{B} \cap Y \subseteq O \cap Y = U$. Thus, $y \in \mathcal{B}_Y \subseteq U$. Notice that $\mathcal{B}_Y = \mathcal{B} \cap Y \in \mathfrak{B}_Y$ by the definition of $\mathfrak{B}_Y$.

Since the argument holds for every $U \in \tau_Y$ and for every $y \in U$, we have proven that $\mathfrak{B}_Y$ is indeed a basis for the subspace topology. ■

*Remark:*

Notice that I avoided saying $U$ is open or $O$ is open in the previous proof. Instead, I preferred saying $U \in \tau_Y$ or $O \in \tau$. When dealing with subspaces, we find a glitch with our current nomenclature, for saying $O$ is open is too ambiguous. Therefore, it is usual for us to say that $U$ is open in $Y$ or $O$ is open in $X$. Or even $U$ is open *relative* to $Y$. Beware: not every set that is open in $Y$ is also open in $X$. ♣

**Example** [A Set That is Open in a Subspace Only]**:**

Let $X = \{a, b, c\}$ and consider the topology $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$. Consider $Y = \{a, b\} \subseteq X$ as a subspace of $(X, \tau)$. Notice that the subspace topology $\tau_Y$ is given by

$$\begin{aligned} \tau_Y &= \{\varnothing \cap Y, X \cap Y, \{a\} \cap Y, \{b, c\} \cap Y\}, \\ &= \{\varnothing, Y, \{a\}, \{b\}\}. \end{aligned} \tag{3.26}$$

The sets $Y = \{a, b\}$ and $\{b\}$ are open in $Y$, but not in $X$. ♥

Okay, that is kind of a bummer. It would certainly be interesting if we could have $\tau_Y \subseteq \tau$. This might not be the general situation, but it is possible if we admit another assumption: $Y \in \tau$.

**Lemma 3.25:**

*Let $(X, \tau)$ be a topological space. Let $Y \subseteq X$ and let $\tau_Y$ be the subspace topology on $Y$. If $Y$ is open in $X$, then $\tau_Y \subseteq \tau$.* □

*Proof:*

Since $\tau_Y = \{O \cap Y; O \in \tau\}$, we simply want to prove that $O \cap Y \in \tau, \forall O \in \tau$. Since $Y \in \tau$ and topologies are closed under finite intersection, it is guaranteed that $O \cap Y \in \tau$. ■

## 3.3 The Road to Limits: Closures and Closed Sets

As suggests the name of this section, the next step we must take in our journey is understanding what is a closed set. It will become clear that there is a strong connection between the concept of a limit point and of a closed set, just as seen, for example, in Real Analysis and Metric Spaces. Besides, if some sets are open, I guess it makes some sense for closed sets to exist.

**Definition 3.26** [Closed Set]**:**

Let $(X, \tau)$ be a topological space. Se say a set $A \subseteq X$ is a *closed set* if, and only if, $A^c \in \tau$, *id est*, whenever the complement of $A$ is an open set. ♠

However, there is a huge difference between sets and doors: a door must be either open or closed, but that is not true for sets. Sets might be open, closed, both or neither.

**Example:**

Let $(X, \tau)$ be a topological space. $X$ and $\varnothing$ are both open and closed sets, because $\varnothing^c = X \in \tau$ and $X^c = \varnothing \in \tau$.

Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$. $\{a\}$ and $\{b\}$ are both open and closed (because one is the complement of the other and vice-versa). Nevertheless, $\{a, b\} \subseteq X$ isn't open nor closed, for $\{a, b\} \notin \tau$ and $\{a, b\}^c = \{c\} \notin \tau$. ♥

Once more, notice that we have a problem with nomenclature when dealing with subspaces. Thus, we are also going to say closed in $Y$ or closed relative to $Y$, and so on.

**Proposition 3.27:**
*Let $(X, \tau)$ be a topological space. Let $Y \subseteq X$ and let $\tau_Y$ be the subspace topology on $Y$. $A \subseteq Y$ is closed in $Y$ if, and only if, $A = F \cap Y$ for some closed set $F \subseteq X$.* $\square$

*Proof:*

$\Leftarrow$: Assume $A = F \cap Y$ for some closed set $F \subseteq X$. Notice that

$$\begin{aligned} Y \setminus A &= Y \cap A^c, \\ &= Y \cap (F \cap Y)^c, \\ &= Y \cap (F^c \cup Y^c), \\ &= Y \cap F^c. \end{aligned} \tag{3.27}$$

Since $F$ is closed in $X$, $F^c$ is open in $X$, and it follows that $Y \setminus A$ is open in $Y$. Therefore, $A$ is closed in $Y$.

$\Rightarrow$: Assume $A$ is closed in $Y$. Then there is some set $O \in \tau$ such that $Y \setminus A = Y \cap O$. Therefore,

$$\begin{aligned} A &= Y \cap (Y \cap O)^c, \\ &= Y \cap (Y^c \cup O^c), \\ &= Y \cap O^c. \end{aligned} \tag{3.28}$$

As $O$ is closed in $X$, the result is proved. ■

**Corollary 3.28:**
*Let $(X, \tau)$ be a topological space. Let $Y \subseteq X$ be a closed set and let $\tau_Y$ be the subspace topology on $Y$. $A \subseteq Y$ is closed in $Y$ if, and only if, it is closed in $X$.* $\square$

*Proof:*
Due to Proposition 3.27 we know that $A$ is closed in $Y$ if, and only if, $A = F \cap Y$ for some closed set in $X$. Since $Y$ is closed in $X$, $F \cap Y$ is closed in $X$. Thus, $A$ is closed in $Y$ if, and only if, it is closed in $X$. ■

**Definition 3.29** [Clopen Sets]**:**
Let $(X, \tau)$ be a topological space. We say a set $A \subseteq X$ is a *clopen set* whenever both $A$ and $A^c$ are open sets, *id est*, whenever $A$ is both closed and open. ♠

An interesting reason for us to study closed sets is because we could also choose to define topological spaces through the structure of the closed sets, instead of the structure of open sets.

**Theorem 3.30:**
*Let $(X, \tau)$ be a topological space. Let $\varphi = \{F \in \mathbb{P}(X); F^c \in \tau\}$. Notice that $\varphi$ is the set of all closed sets in $X$. Finally, let $\Lambda$ be an arbitrary set of indexes. $\varphi$ satisfies the following properties:*

i. $\varnothing, X \in \varphi$;

ii. $A, B \in \varphi \Rightarrow A \cup B \in \varphi$;

iii. $A_\lambda \in \varphi, \forall \lambda \in \Lambda \Rightarrow \bigcap_{\lambda \in \Lambda} A_\lambda \in \varphi$. □

*Proof:*

i. Since $\varnothing, X \in \tau$ and they are the complements of each other, $\varnothing, X \in \varphi$;

ii. Given that $A^c, B^c \in \tau$, we know that $(A^c \cap B^c) \in \tau$. Thus, $(A^c \cap B^c)^c = A \cup B \in \varphi$.

iii. Given that ${A_\lambda}^c \in \tau, \forall \lambda \in \Lambda$, we know that $\bigcup_{\lambda \in \Lambda} {A_\lambda}^c \in \tau$. Therefore,

$$\left(\bigcup_{\lambda \in \Lambda} {A_\lambda}^c\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda \in \varphi. \tag{3.29}$$

This concludes the proof. ■

Although unusual, one could define a topological space as a set $X$ with a collection $\varphi$ of closed sets obeying the properties of Theorem 3.30 on the preceding page, define open sets as complements of closed sets and reobtain the results we have already found.

We shall soon see as well that it is interesting that, given a set, we may obtain an open set, or a closed set related to that set. It is natural to define such generated sets through the properties we already know about intersections of closed sets and unions of open sets.

**Definition 3.31** [Interior, Closure and Boundary]**:**

Let $(X, \tau)$ be a topological space and let $A \subseteq X$. We define the *interior of* $A$, denoted $\mathring{A}$ (or, equivalently, int $A$), as the union of all open sets contained within $A$. We define the *closure of* $A$, denoted $\overline{A}$, as the intersection of all closed sets containing $A$. Finally, we define the boundary of $A$, $\partial A$, as $\partial A = \overline{A} \setminus \mathring{A}$. ♠

**Proposition 3.32:**

*Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then the following results hold:*

i. $\mathring{A} \subseteq A \subseteq \overline{A}$;

ii. $\mathring{A} = A$ *if, and only if,* $A$ *is open;*

iii. $\overline{A} = A$ *if, and only if,* $A$ *is closed.*

□

*Proof:*

i. since $\overset{\circ}{A}$ is the union of every open set contained within $A$, it surely is contained within $A$, *id est*, $\overset{\circ}{A} \subseteq A$. Similarly, provided that $\overline{A}$ is the intersection of all closed sets containing $A$, it surely contains $A$, and thus $A \subseteq \overline{A}$.

ii. $\overset{\circ}{A}$ is an arbitrary union of open sets, and thus is open. Therefore, if $\overset{\circ}{A} = A$, $A$ certainly is open. On the other hand, we know that $\overset{\circ}{A} \subseteq A$. If $A$ is open, then every point of $A$ belongs in the union of every open set contained within $A$, and thus $A \subseteq \overset{\circ}{A}$. Therefore, if $A$ is open, then $\overset{\circ}{A} = A$.

iii. $\overline{A}$ is an arbitrary intersection of closed sets, and thus is closed. Therefore, if $\overline{A} = A$, $A$ certainly is closed. On the other hand, we know that $A \subseteq \overline{A}$. If $A$ is closed, then every point of the intersection of all closed sets containing $A$ is a point of $A$, *id est*, $\overline{A} \subseteq A$. Thus, if $A$ is closed, then $\overline{A} = A$. ■

**Lemma 3.33:**

*Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Then, on the inclusion order, $\overset{\circ}{A}$ is the greatest open set contained within $A$ and $\overline{A}$ is the smallest closed set containing $A$.* □

*Proof:*

Since $\overset{\circ}{A}$ is the union of every open set contained in $A$, if any open set, say $B$, is larger than (or not comparable to) $\overset{\circ}{A}$ and still contained within $A$, then $B$ would also be on the family whose union we are taking, and thus $B \subseteq \overset{\circ}{A}$. Since we have reached a contradiction, it follows that there is no such set $B$.

A similar reasoning holds for $\overline{A}$. Since $\overline{A}$ is the intersection of every closed set containing $A$, the existence of any closed set $B$ containing $A$ implies that $B$ is on the family whose union we are taking. Therefore, $\overline{A} \subseteq B$. ■

We might then examine some properties of the operations that take some set to its closure, interior and/or boundary.

**Proposition 3.34:**

*Let $(X, \tau)$ be a topological space. Let $A, B \subseteq X$. Let $\Lambda$ be an arbitrary family of indexes and $A_\lambda \subseteq A, \forall \lambda \in \Lambda$. Then the following properties hold:*

i. $\overline{\overline{A}} = \overline{A}$;

ii. $B \subseteq A \Rightarrow \overline{B} \subseteq \overline{A}$;

iii. $\overline{B \cup A} = \overline{B} \cup \overline{A}$;

iv. $\overline{\bigcap_{\lambda \in \Lambda} A_\lambda} \subseteq \bigcap_{\lambda \in \Lambda} \overline{A_\lambda}$;

v. $\overline{\varnothing} = \varnothing, \overline{X} = X$. □

*Proof:*

i. We know $\overline{A}$ is a closed set (Lemma 3.33 on the facing page), and thus Proposition 3.32 on page 37 implies $\overline{A} = \overline{\overline{A}}$;

ii. Proposition 3.32 on page 37 guarantees that $B \subseteq A \subseteq \overline{A}$. Provided that $\overline{B}$ is the smallest closed set containing $B$ and $\overline{A}$ is a closed set containing $B$, it follows that $\overline{B} \subseteq \overline{A}$;

iii. We know $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, and therefore $A \cup B \subseteq \overline{A} \cup \overline{B}$. Both $\overline{A}$ and $\overline{B}$ are closed sets and the finite union of closed sets is closed, and thus it follows that $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, and therefore $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

   Furthermore, $A \subseteq A \cup B \subseteq \overline{A \cup B}$. We know $\overline{A}$ is the smallest closed set containing $A$, and therefore $\overline{A} \subseteq \overline{A \cup B}$. An analogous arguments applies to $B$. $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$, and hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. It has already been proved that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, and thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

iv. $A_\lambda \subseteq \overline{A_\lambda}, \forall\, \lambda \in \Lambda$. Therefore, $\bigcap_{\lambda \in \Lambda} A_\lambda \subseteq \bigcap_{\lambda \in \Lambda} \overline{A_\lambda}$. However, $\overline{\bigcap_{\lambda \in \Lambda} A_\lambda}$ is the smallest closed set containing $\bigcap_{\lambda \in \Lambda} A_\lambda$. It follows that $\overline{\bigcap_{\lambda \in \Lambda} A_\lambda} \subseteq \bigcap_{\lambda \in \Lambda} \overline{A_\lambda}$.

v. From Theorem 3.30 on page 36 we know that $\varnothing$ and $X$ are closed sets. The result is then a consequence of Proposition 3.32 on page 37. ■

A very similar set of properties holds for the interior of a set.

**Proposition 3.35:**
*Let $(X, \tau)$ be a topological space. Let $A, B \subseteq X$. Let $\Lambda$ be an arbitrary family of indexes and $A_\lambda \subseteq A, \forall\, \lambda \in \Lambda$. Then the following properties hold:*

i. $\mathring{\mathring{A}} = \mathring{A}$;

ii. $B \subseteq A \Rightarrow \mathring{B} \subseteq \mathring{A}$;

iii. $\operatorname{int}(B \cap A) = \mathring{B} \cap \mathring{A}$;

iv. $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda \subseteq \operatorname{int}(\bigcup_{\lambda \in \Lambda} A_\lambda)$;

v. $\mathring{\varnothing} = \varnothing, \mathring{X} = X$. □

*Proof:*

i. Just as when dealing with closure, we know $\mathring{A}$ is open and thus equals its interior, *id est*, $\mathring{A} = \mathring{\mathring{A}}$;

ii. If $B \subseteq A$, we know $\mathring{B} \subseteq B \subseteq A$. As $\mathring{A}$ is the largest open set contained in $A$ and $\mathring{B}$ is an open set contained in $A$, it follows that $\mathring{B} \subseteq \mathring{A}$;

iii. We know that $A \cap B \subseteq A$, and thus $\operatorname{int}(A \cap B) \subseteq \mathring{A}$, with a similar result for $B$. Therefore, $\operatorname{int}(A \cap B) \subseteq \mathring{A} \cap \mathring{B}$.

On the other hand, $\mathring{A} \subseteq A$ and $\mathring{B} \subseteq B$. Therefore, $\mathring{A} \cap \mathring{B} \subseteq A \cap B$. The finite intersection of open sets is open, and thus $\mathring{A} \cap \mathring{B}$ is an open set contained within $A \cap B$. As $\operatorname{int}(A \cap B)$ is the largest open set contained within $A \cap B$, it follows that $\mathring{A} \cap \mathring{B} \subseteq \operatorname{int}(A \cap B)$. Hence, $\mathring{A} \cap \mathring{B} = \operatorname{int}(A \cap B)$.

iv. As $A_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda, \forall \lambda \in \Lambda$, it holds that $\mathring{A}_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda, \forall \lambda \in \Lambda$. Therefore, one has that $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$. Since the arbitrary union of open sets is open, $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda$ is an open set contained within $\bigcup_{\lambda \in \Lambda} A_\lambda$. However, the largest open set contained within $\bigcup_{\lambda \in \Lambda} A_\lambda$ is $\operatorname{int}(\bigcup_{\lambda \in \Lambda} A_\lambda)$, and therefore $\bigcup_{\lambda \in \Lambda} \mathring{A}_\lambda \subseteq \operatorname{int}(\bigcup_{\lambda \in \Lambda} A_\lambda)$;

v. Both $\varnothing$ and $X$ are open sets and thus are equal to their interiors. ■

We might then relate the concepts of closure and interior.

**Lemma 3.36:**
*Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Then $\mathring{A} = \left(\overline{(A^c)}\right)^c$, or, equivalently, $\overline{A} = \left((\mathring{A^c})\right)^c$.* □

*Proof:*
Firstly, notice that both expressions are indeed equivalent: simply exchange $A \leftrightarrow A^c$ and use the fact that $(A^c)^c = A$.

Let us now focus on the actual result. $\left(\overline{(A^c)}\right)^c$ is the complement of a closed set, and therefore it is an open set. As $A^c \subseteq \overline{(A^c)}$, it holds that $\left(\overline{(A^c)}\right)^c \subseteq A$. We know, however, that $\mathring{A}$ is the largest open set contained in $A$, and thus $\left(\overline{(A^c)}\right)^c \subseteq \mathring{A}$.

Next, we want to prove that $\mathring{A} \subseteq \left(\overline{(A^c)}\right)^c$, which is the same as proving that $\overline{(A^c)} \subseteq \left(\mathring{A}\right)^c$.

Since $\mathring{A} \subseteq A$, $A^c \subseteq \left(\mathring{A}\right)^c$. Furthermore, since it is the complement of an open set, $\left(\mathring{A}\right)^c$ is closed, making it a closed set containing $A^c$. The smallest closed set containing $A^c$ is $\overline{(A^c)}$, and thus $\overline{(A^c)} \subseteq \left(\mathring{A}\right)^c$, as desired. ■

We might as well find properties pertinent to the boundary of a set. However, it is going to be useful to prove a small lemma before we can actually study those properties.

**Lemma 3.37:**
*Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $\partial A = \overline{A} \cap \overline{(A^c)}$.* □

*Proof:*

$$\begin{aligned}\partial A &= \overline{A} \setminus \mathring{A},\\ &= \overline{A} \cap \left(\mathring{A}\right)^{\mathsf{c}},\\ &= \overline{A} \cap \overline{(A^{\mathsf{c}})}.\end{aligned} \tag{3.30}$$

The last step is justified by Lemma 3.36 on the preceding page. ■

**Proposition 3.38:**

*Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then the following hold:*

i. $\overline{\partial A} = \partial A$;

ii. $\partial A = \partial(A^{\mathsf{c}})$;

iii. $\partial(\partial A) \subseteq \partial A$;

iv. $\partial(\partial A) = \partial A \Leftrightarrow \operatorname{int}(\partial A) = \varnothing$;

v. $\partial(\partial(\partial A)) = \partial(\partial A)$;

vi. $\partial\varnothing = \varnothing$ *and* $\partial X = X$. □

*Proof:*

i. Due to Lemma 3.37 on the facing page we know $\partial A$ is the intersection of two closed sets, and thus is closed as well. Therefore, it coincides with its closure;

ii. The definition of boundary presented on Lemma 3.37 on the preceding page is symmetric in $A$ and $A^{\mathsf{c}}$, *id est*, we might interchange them with no difference. Thus, $\partial A = \partial(A^{\mathsf{c}})$;

iii. If $B$ is a closed set, then $\partial B \subseteq B$, for $\partial B = \overline{B} \setminus \mathring{B} \subseteq \overline{B} = B$. Given that $\partial A$ is always closed, the result follows;

iv. If $\operatorname{int}(\partial A) = \varnothing$, then $\partial(\partial A) = \partial A \setminus \varnothing$, where we already used that $\partial A$ is closed and has empty interior.

   On the other hand, if $\partial(\partial A) = \partial A$, then $\partial A = \partial A \setminus \operatorname{int}(\partial A)$, *id est*, no element of $\partial A$ is in $\operatorname{int}\{\partial A\}$, and therefore the latter is the empty set;

v. We know that $\partial(\partial A) = \partial A \cap \overline{((\partial A)^{\mathsf{c}})}$, which may be rewritten as $\partial(\partial A) = \partial A \cap \left(\mathring{(\partial A)}\right)^{\mathsf{c}}$, due to Lemma 3.36 on the facing page. Then Proposition 3.35 on page 39 guarantees that $\operatorname{int}(\partial(\partial A)) = \mathring{\partial A} \cap \operatorname{int}\left(\left(\mathring{(\partial A)}\right)^{\mathsf{c}}\right)$. As $\operatorname{int}\left(\left(\mathring{(\partial A)}\right)^{\mathsf{c}}\right) \subseteq \left(\mathring{(\partial A)}\right)^{\mathsf{c}}$, it follows that $\operatorname{int}(\partial(\partial A)) \subseteq \mathring{\partial A} \cap \left(\mathring{(\partial A)}\right)^{\mathsf{c}} = \varnothing$. Thus, $\operatorname{int}(\partial(\partial A)) = \varnothing$ and the result follows. ■

Something remarkably interesting about the concepts of interior and closure of a set is the possibility of giving yet another definition of topological space. Instead of providing a set with the collection of open (or closed) subsets, we may as well equip it with an unary operation with some properties satisfied by the closure operation.

**Definition 3.39** [Kuratowski Operator]**:**

Let $X$ be a non-empty set and let $\kappa\colon \mathbb{P}(X) \to \mathbb{P}(X)$ be a function. $\kappa$ is said to be a *Kuratowski operator* if, and only if, it satisfies the *Kuratowski axioms*:

i. $\kappa(\varnothing) = \varnothing$;

ii. $A \subseteq \kappa(A), \forall\, A \in \mathbb{P}(X)$;

iii. $\kappa(\kappa(A)) = \kappa(A), \forall\, A \in \mathbb{P}(X)$;

iv. $\kappa(A \cup B) = \kappa(A) \cup \kappa(B), \forall\, A, B \in \mathbb{P}(X)$. ♠

**Theorem 3.40:**

*Let $X$ be a non-empty set and let $\kappa\colon \mathbb{P}(X) \to \mathbb{P}(X)$ be a Kuratowski operator. Let $\tau_\kappa$ be defined as*

$$\tau_\kappa :- \{O \in \mathbb{P}(X); \kappa(O^c) = O^c\}. \tag{3.31}$$

*Under these assumptions, it holds that $(X, \tau_\kappa)$ is a topological space. Furthermore, the closure $\overline{A}$ of a set $A$ according to $\tau_\kappa$ respects $\overline{A} = \kappa(A)$.* □

*Proof:*

Let us first define $\varphi_\kappa :- \{F \in \mathbb{P}(X); \kappa(F) = F\}$. Let $\Lambda$ be an arbitrary set of indexes. I claim that

i. $\varnothing, X \in \varphi_\kappa$;

ii. $A, B \in \varphi_\kappa \Rightarrow A \cup B \in \varphi_\kappa$;

iii. $A_\lambda \in \varphi_\kappa, \forall\, \lambda \in \Lambda \Rightarrow \bigcap_{\lambda\in\Lambda} A_\lambda \in \varphi_\kappa$.

Since $\kappa$ is a Kuratowski operator, it holds that $\kappa(\varnothing) = \varnothing$ by hypothesis, and therefore $\varnothing \in \varphi_\kappa$.

We also know that $A \subseteq \kappa(A), \forall\, A \in \mathbb{P}(X)$. Thus, $X \subseteq \kappa(X)$. However, $\kappa(X) \in \mathbb{P}(X)$, and therefore $\kappa(X) \subseteq X$. It follows that $\kappa(X) = X$, implying that $X \in \varphi_\kappa$.

Let now $A, B \in \varphi_\kappa$, *id est*, $\kappa(A) = A$ and $\kappa(B) = B$. As $\kappa$ is a Kuratowski operator, $\kappa(A \cup B) = \kappa(A) \cup \kappa(B), \forall\, A, B \in \mathbb{P}(X)$. Hence,

$$\begin{aligned} \kappa(A \cup B) &= \kappa(A) \cup \kappa(B), \\ &= A \cup B, \end{aligned} \tag{3.32}$$

proving that $A \cup B \in \varphi_\kappa$.

Finally, we want to prove that

$$A_\lambda \in \varphi_\kappa, \forall\, \lambda \in \Lambda \Rightarrow \bigcap_{\lambda\in\Lambda} A_\lambda \in \varphi_\kappa,$$

*id est*,

$$\kappa(A_\lambda) = A_\lambda, \forall\, \lambda \in \Lambda \Rightarrow \kappa\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} A_\lambda.$$

We know that $\kappa$ is a Kuratowski operator, and therefore $A \subseteq \kappa(A), \forall\, A \in \mathbb{P}(X)$ is a given. We only need to prove that $\kappa(\bigcap_{\lambda\in\Lambda} A_\lambda) \subseteq \bigcap_{\lambda\in\Lambda} A_\lambda$.

$\forall\, A, B \in \mathbb{P}(X)$, if holds that* $A = (A \cap B) \sqcup (A \cap B^{\mathsf{c}})$. Thus, if $B \subseteq A$, it holds that $A = B \sqcup (A \setminus B)$. As $\kappa$ is a Kuratowski operator, it follows that $\kappa(A) = \kappa(B) \cup \kappa(A \setminus B) \supseteq \kappa(B)$. Thus, whenever $B \subseteq A$, it follows that $\kappa(B) \subseteq \kappa(A)$.

Let us assume now that $\lambda_0 \in \Lambda$. We know that $\bigcap_{\lambda\in\Lambda} A_\lambda \subseteq A_{\lambda_0}$, and thus it follows that $\kappa(\bigcap_{\lambda\in\Lambda} A_\lambda) \subseteq \kappa(A_{\lambda_0}), \forall\, \lambda_0 \in \Lambda$. Hence,

$$\kappa\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \subseteq \bigcap_{\lambda_0\in\Lambda} \kappa(A_{\lambda_0}) = \bigcap_{\lambda\in\Lambda} A_\lambda, \tag{3.33}$$

for $\kappa(A_\lambda) = A_\lambda, \forall\, \lambda \in \Lambda$. Notice that as $\lambda_0$ is a dummy index, it can be changed to $\lambda$.

Thus, we already know that, given $(X, \kappa)$, we may equip $X$ with a collection of sets $\varphi_\kappa$ satisfying the same properties closed sets have in topological spaces (Theorem 3.30 on page 36). I claimed previously we could use such a space $(X, \varphi_\kappa)$ to define a topological space and reobtain our usual definition of topology. Let us prove it.

Given $\varphi_\kappa$ defined as before, we want to prove that the collection

$$\tau_\kappa :- \{O \in \mathbb{P}(X); O^{\mathsf{c}} \in \varphi_\kappa\} \tag{3.34}$$

is a topology in $X$. Notice that, currently, the $\kappa$ indexes are merely aesthetic and the proof that a space with the notion of a closed set is a topological space is still completely general and independent of the Kuratowski operator.

Firstly, we want to prove that $\varnothing, X \in \tau_\kappa$. As $X, \varnothing \in \varphi_\kappa$, $X^{\mathsf{c}} = \varnothing \in \tau_\kappa$ and $\varnothing^{\mathsf{c}} = X \in \tau_\kappa$.

Next, we want to prove that given $A, B \in \tau_\kappa$, $A \cap B \in \tau_\kappa$. As $A, B \in \tau_\kappa$, we know that $A^{\mathsf{c}}, B^{\mathsf{c}} \in \varphi_\kappa$. Thus, $A^{\mathsf{c}} \cup B^{\mathsf{c}} \in \varphi_\kappa$. Finally, $(A^{\mathsf{c}} \cup B^{\mathsf{c}})^{\mathsf{c}} = A \cap B \in \tau_\kappa$.

Finally, let $\Lambda$ be an arbitrary set of indexes. Let $A_\lambda \in \tau_\kappa, \forall\, \lambda \in \Lambda$. We want to prove that $\bigcup_{\lambda\in\Lambda} A_\lambda \in \tau_\kappa$.

We have

$$\begin{gathered} {A_\lambda}^{\mathsf{c}} \in \varphi_\kappa, \\ \bigcap_{\lambda\in\Lambda} {A_\lambda}^{\mathsf{c}} \in \varphi_\kappa, \\ \left(\bigcap_{\lambda\in\Lambda} {A_\lambda}^{\mathsf{c}}\right)^{\mathsf{c}} \in \tau_\kappa, \\ \bigcup_{\lambda\in\Lambda} A_\lambda \in \tau_\kappa. \end{gathered} \tag{3.35}$$

*We write $A \sqcup B$ to denote $A \cup B$, with $A \cap B = \varnothing$.

Thus, it is proven that $(X, \tau_\kappa)$ is a topological space. We also proved that given a space with a "closed topology", $(X, \varphi)$, it can be regarded as a topological space.

Finally, we have to prove that $\overline{A} = \kappa(A), \forall A \in \mathbb{P}(X)$, with the closure considered with respect to the topology $\tau_\kappa$.

Let $A \in \mathbb{P}(X)$. Then $\kappa(A)$ is a closed set. Indeed, $\kappa(A) = \kappa(\kappa(A))$ by hypothesis (for $\kappa$ is a Kuratowski operator) and thus $\kappa(A) \in \varphi_\kappa$. Let now $F$ be a closed set such that $A \subseteq F$. Then we have that $F^c \in \tau_\kappa$ (for $F$ is closed) and, thus, $F \in \varphi_\kappa$. Therefore, it holds, from the definition of $\varphi_\kappa$, that $F = \kappa(F)$.

As $A \subseteq F$, it follows that $\kappa(A) \subseteq \kappa(F) = F$, *id est*, $\kappa(A) = F$. Hence, $\kappa(A)$ is the smallest closed set containing $A$, which by definition is the closure of $A$. Therefore, we have proved that $\overline{A} = \kappa(A)$. ■

Once more, we might have problems regarding what happens in subspaces. The following lemma exhibits how the closure in a subspace relates to the the closure on the topological space.

**Lemma 3.41:**

*Let $(X, \tau)$ be a topological space, $(Y, \tau_Y)$ be a subspace, $A \subseteq Y$ and $\overline{A}$ be the closure of $A$ in $X$. Then the closure of $A$ in $Y$ is given by $\overline{A} \cap Y$.* □

*Proof:*

Due to Proposition 3.27 on page 36, we know that $\overline{A} \cap Y$ is a closed set in Y. Suppose $B$ is a closed set in $Y$ such that $A \subseteq B \subseteq \overline{A} \cap Y$. Proposition 3.27 on page 36 guarantees that there is a closed set $D$ in $X$ such that $B = D \cap Y$. Thus, $A \subseteq D \cap Y \subseteq \overline{A} \cap Y$. It follows that $A \subseteq D$ for some closed set $D$. However, since $\overline{A}$ is the intersection of every closed set containing $A$, we know that $\overline{A} \subseteq D$. Finally, we conclude that $\overline{A} \cap Y \subseteq B$, and therefore $\overline{A} \cap Y$ is the smallest closed set containing $A$, which coincides with the intersection of every closed set containing $A$ (Lemma 3.33 on page 38). ■

Even though we have proven a number of results on closures, interiors and boundaries, our current definition of closure is no good for making calculations. Given a set, our present knowledge concerning closures won't allow us to find the closure of such set in an easy way. We can, though, establish a definition of closure based on intersections of the given set with elements of a basis for the topology.

**Definition 3.42** [Intersects]**:**

Let $X$ be a set and $A, B \subseteq X$. We say $A$ *intersects* $B$ if, and only if, $A \cap B \neq \varnothing$. ♠

**Theorem 3.43:**

*Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Then it holds that*

i. *$x \in \overline{A}$ if, and only if, $O$ intersects $A, \forall O \in \tau; x \in O$;*

ii. *if $\mathfrak{B}$ is a basis for the topology $\tau$, then $x \in \overline{A}$ if, and only if, $B$ intersects $A, \forall B \in \mathfrak{B}; x \in \mathfrak{B}$.* □

*Proof:*

We shall do the proof by contrapositive, *id est*, we want to prove that $x \notin \overline{A} \Leftrightarrow \exists\, O \in \tau; x \in O, O \cap A = \varnothing$.

Suppose $x \notin \overline{A}$. Clearly $x \in \overline{A}^{\mathsf{c}}$. By Lemma 3.36 on page 40, we have that $x \in \operatorname{int}(A^{\mathsf{c}})$. We know that $\operatorname{int}(A^{\mathsf{c}})$ is an open set, by the very definition of interior. As $\operatorname{int}(A^{\mathsf{c}}) \subseteq A^{\mathsf{c}}$ and $A^{\mathsf{c}} \cap A = \varnothing$, we have that $\operatorname{int}(A^{\mathsf{c}}) \cap A = \varnothing$, proving the existence of a set $O \in \tau$ such that $x \in O, O \cap A = \varnothing$.

On the other hand, suppose $\exists\, O \in \tau; x \in O, O \cap A = \varnothing$. Since $O \cap A = \varnothing$, we know that $O \subseteq A^{\mathsf{c}}$. Notice then that $A \subseteq O^{\mathsf{c}}$ and that $O^{\mathsf{c}}$ is a closed set, for $O$ is open by hypothesis. Therefore, $\overline{A} \subseteq O^{\mathsf{c}}$, as $\overline{A}$ is the smallest closed set containing $A$. We have then that $\overline{A} \cap O = \varnothing$. As $x \in O$, by hypothesis, we have that $x \notin \overline{A}$, proving the result we wanted.

This proves the first item. We now must prove the second.

The first implication ($x \in \overline{A} \Rightarrow B \cap A \neq \varnothing, \forall\, B \in \mathfrak{B}; x \in B$) is simple: we already know the result is valid in general for any open set. As every basis element is an open set, of course the implication holds.

The second implication is slightly trickier. Suppose $B$ intersects $A$ for every basis element containing $x$. Well, we know any open set is made of unions of basis elements, and by consequence it means that any open set containing $x$ must have one of such basis elements as a subset. As every basis elements with $x \in B$ intersects $A$, it follows that every open set containing $x$ must intersect $A$. Thus, we know from the first item that $x \in \overline{A}$. ■

You might have noticed that we mentioned an "open set containing $x$" sometimes. In fact, this concept is quite common with Topology, and therefore it is handy for us to give a special name for such sets.

**Definition 3.44** [Neighborhood]**:**

Let $(X, \tau)$ be a topological space. Let $x \in X$. We say a set $O \in \tau$ is a *neighborhood* of $x$ if, and only if, $x \in O$. ♠

**Theorem 3.43** [Another Possible Statement]**:**

*Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Then it holds that*

i. *$x \in \overline{A}$ if, and only if, every neighborhood of $x$ intersects $A$;*

ii. *if $\mathfrak{B}$ is a basis for the topology $\tau$, then $x \in \overline{A}$ if, and only if, $B$ intersects $A, \forall\, B \in \mathfrak{B}; x \in \mathfrak{B}$.* □

The second item still depends on some cumbersome notation, but the first one is certainly cleaner.

While we are here, we might as well define the concept of a neighborhood base for future reference.

**Definition 3.45** [Neighborhood Basis]**:**

Let $(X, \tau)$ be a topological space. Let $x \in X$. A *neighborhood basis* for $\tau$ at $x$ is a collection $\mathfrak{N} \subseteq \tau$ satisfying the following conditions:

i. $x \in \mathcal{B}, \forall \mathcal{B} \in \mathfrak{N}$;

ii. $O \in \tau, x \in O \Rightarrow \exists \mathcal{B} \in \mathfrak{N}; \mathcal{B} \subseteq O$. ♠

**Examples** [Calculating Closures]**:**

Consider the real line with the standard topology. The closure of any open interval $I$ is the closed interval with the same extremes.

Let us write $I = (a, b)$. Of course $(a, b) \subseteq \overline{I}$, for $A \subseteq \overline{A}, \forall A \in \mathbb{P}(\mathbb{R})$.

We know the open intervals are a basis for the standard topology on $\mathbb{R}$. Thus, due to Theorem 3.43 on the previous page, we know that $x \in \overline{I}$ if, and only if, every open interval containing $x$ intersects $I$.

Let $a \in (c, d)$. Then, by definition, $c < a < d$, and thus either $d \in I$ or $c < \frac{a+b}{2} < d$ and thus $\frac{a+b}{2} \in (c, d) \cap (a, b)$. A similar argument holds for $b$.

Suppose now that any other number is contained in $\overline{I}$. Let us call it $x_0$ and suppose, without any loss of generality, that $x_0 > b$. Then we pick the open interval $\left(\frac{x_0+b}{2}, x_0 + 1\right)$, for example. As $\frac{x_0+b}{2} > b$, every element of such interval is outside of $I$, and it follows from Theorem 3.43 on the preceding page that $x_0 \notin \overline{I}$.

Still on the same topological space, let us consider a larger set. Namely, $\mathbb{Q}$. What is $\overline{\mathbb{Q}}$?

Let $x \in \mathbb{R}$. We know $\mathbb{R}$ admits as a basis the set of open intervals centered at rational numbers with radius of the form $\frac{1}{n}, n \in \mathbb{N}^*$. As this is a basis, certainly there are intervals of this form that contain $x$ (for $\mathbb{R}$ can be written as an arbitrary union of such intervals). However, by construction, such intervals always contain a rational number (namely, the center). Thus, by Theorem 3.43 on the previous page, every real number is an element of $\overline{\mathbb{Q}}$. As $\overline{\mathbb{Q}} \subseteq \mathbb{R}$, we conclude $\overline{\mathbb{Q}} = \mathbb{R}$. ♥

You might have noticed that we could have simply said that $\mathbb{Q}$ is dense in $\mathbb{R}$ and thus every interval of real numbers contains a rational number. I avoided such nomenclature for a simple reason: we also have a definition of a dense set within topology.

**Definition 3.46** [Dense Set]**:**

Let $(X, \tau)$ be a topological space. We say a set $A \subseteq X$ is *dense* whenever it holds that $\overline{A} = X$. ♠

As usual, this is merely an extension of the similar concept known from metric spaces.

## 3.4 Limits of Sequences

As previous experiences with Real Analysis and Metric Spaces might suggest, the study of limits depends heavily on the notion of a sequence (which, of course, will also receive a more general formulation in terms of nets). Naturally, we should start this section by defining what is a sequence.

**Definition 3.47** [Sequence]**:**

Let $X$ be a non-empty set. A function $x \colon \mathbb{N} \to X$ is commonly called a *sequence* in $X$. ♠

*Notation:*

Instead of writing $x(n)$ for the image of $n \in \mathbb{N}$ through a sequence $x$, it is usual to write simply $x_n$. It is also customary to write $(x_n)_{n\geq 0}$, $(x_n)_{n\geq 1}$, $(x_n)_{n\in\mathbb{N}}$, *et cetera* for the sequence, instead of $x$. Some other notations can also be found (for example, referring to the sequence itself, not a the image of a natural $n$ through the sequence, as $x_n$). ♦

You might already know the definition for the limit of a sequence in a metric space, for it is indeed very similar to the notion of limit introduced at Section 3.1.

**Definition 3.48** [Convergence of a Sequence in a Metric Space]**:**

Let $(M, d)$ be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $M$. We say the sequence *converges* to a point $x \in M$ whenever it holds that

$$\forall\, \epsilon > 0, \exists\, N \in \mathbb{N}; n > N \Rightarrow d(x_n, x) < \epsilon. \tag{3.36}$$

Notice that such $N \in \mathbb{N}$ may, and in general will, depend on $\epsilon$. ♠

We want to make this definition more general and drop the dependence on a metric. After all, no notion of distance in available in topological spaces and we must work solely with notions about sets and open sets.

As the distance between the elements of the sequence decrease, we know that, in terms of sets, the terms of the sequence are contained in smaller and smaller sets. After all, the open balls are nested within each other. This suggests a definition of convergence based on the notion of neighborhoods: for every neighborhood $O$ of the limit value (which we previously denoted as $x$) there should exist a natural number $N$ such that $x_n \in O, \forall\, n > N$.

Nevertheless, not every sequence is convergent, but we could as well be interested in some other cases. For an example, the sequence $x_n = (-1)^n$ in the real line does not converge to any point, but it admits subsequences* that converge to either $+1$ or $-1$. Therefore, it is interesting for us to also develop the theory in the direction of understanding properties of subsequences of a sequence, even if the sequence itself does not converge to any value whatsoever.

Such ideas naturally bring us towards the definitions of points that are frequently and eventually at a set, which do not depend on the topological structure of the space.

**Definition 3.49** [Frequently and Eventually]**:**

Let $X$ be a non-empty set, $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $X$ and $A \subseteq X$. We say $(x_n)_{n\in\mathbb{N}}$ is *frequently* in $A$ if, and only if, there is an infinite amount of indices $m$ such that $x_m \in A$. We say $(x_n)_{n\in\mathbb{N}}$ is *eventually* in $A$ if, and only if, there is a natural number $N$ such that $x_m \in A, \forall\, m > N$. ♠

*Remark:*

Notice that if a sequence is eventually in $A$, then it is frequently in $A$. However, the inverse does not hold: $x_n = (-1)^n$ is frequently in $(0, 2)$, but is is not eventually in such set. ♣

*If $(x_n)_{n\in\mathbb{N}}$ is a sequence and $m \mapsto n_m$ is a crescent function from $\mathbb{N}$ to $\mathbb{N}$, $(x_{n_m})_{m\in\mathbb{N}}$ is said to be a *subsequence* of $(x_n)_{n\in\mathbb{N}}$.

Well, I've said before that topology is all about limits and continuity, and therefore it would be weird if we did not need a topological structure to define limits. If we can talk about points that are eventually in a set or frequently in a set without any need for a topological structure, why did we prove so many results on open sets, bases, closures, and so on?

Suppose we were going to despise any topological structure and try to go for a definition based solely on set theory, no topology allowed. We would lose any and every notion of how close a point is to another. We discussed before an idea of trying to define a limit point by demanding that for every neighborhood there would be a "cutoff point" at our sequence such that every point from there onward would be inside that neighborhood. Well, now that any set is valid, even simple sequences lose their convergence properties. For example, is the sequence $\frac{1}{n}$ in the real line still convergent? If so, does it go to zero? In fact, without any notion of topology, $\{0, 1\}$ would become a neighborhood of 0 (we do not care about the set being open anymore, we don't even know what that means!). But so is $\{0, 2\}$, and the requirements for the sequence to be convergent would require that the sequence is eventually constant, with value 0. This does not happen, and thus the sequence does not converge. Therefore, such a theory is simply not interesting at all, for is does not make our results more general. It restricts them, in fact. Hence, topology.

Therefore, we are motivated to define the concepts of cluster and limit points.

**Definition 3.50** [Cluster Point]**:**

Let $(X, \tau)$ be a topological space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $X$. A point $x \in X$ is said to be a *cluster point* of the sequence $(x_n)_{n\in\mathbb{N}}$ with respect to the topology $\tau$ if, and only if, $(x_n)_{n\in\mathbb{N}}$ is frequently in every neighborhood of $x$. ♠

**Definition 3.51** [Limit Point]**:**

Let $(X, \tau)$ be a topological space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $X$. A point $x \in X$ is said to be a *limit point* (sometimes called simply *limit*) of the sequence $(x_n)_{n\in\mathbb{N}}$ with respect to the topology $\tau$ if, and only if, $(x_n)_{n\in\mathbb{N}}$ is eventually in every neighborhood of $x$. ♠

*Remark:*

As being eventually in a set implies being frequently in the same set, it holds that, giving a sequence, every limit point of such sequence is a cluster point of the sequence. ♣

**Example** [Not Every Cluster Point is a Limit]**:**

There is no reason for a cluster point to be a limit point, but sometimes we tend to believe that such a property would be "likely", and pretend to do Mathematics on a probabilistic manner. Needless to say, this is likely to fail, but it is quite amusing to prove our intuition wrong through some counterexamples. After all, their existence justifies all the effort we have been putting into a theory. Thus, we shall prove that for quite a large family of sequences we can have the whole real line as the set of cluster points, albeit no point at all is a limit point.

Let $q \colon \mathbb{N} \to \mathbb{R}$ be a sequence such that $\operatorname{Ran} q = \mathbb{Q}$. Such a sequence does exist, for $\mathbb{Q}$ is a countable set. I claim that every real number is a cluster point of $q$ with respect to the standard topology, but no real number is a limit point of $q$.

Let $x \in \mathbb{R}$. Let $O_x$ be a neighborhood of $x$. As the open intervals with rational extremes are a basis for the standard topology in $\mathbb{R}$, we may simply prove that every such interval containing $x$ has infinitely many rational numbers (and thus there are infinitely many terms of $q$ in that interval).

Let $a, b \in \mathbb{Q}$ and such that $x \in (a, b)$. Let $m = \min\{x - a, b - x\}$. We know, from the Archimedean property of the real numbers, that there is a natural number $n_0$ such that $\frac{1}{n_0} < m$, and thus $x \in \left(a + \frac{1}{n_0}, b - \frac{1}{n_0}\right)$. It follows that $x \in \left(a + \frac{1}{n}, b - \frac{1}{n}\right), \forall\, n \in \mathbb{N}^*; n > n_0$. As $a + \frac{1}{n} \in \mathbb{Q}, \forall\, n \in \mathbb{N}^*$, we have found infinitely many rational numbers (*id est*, terms of $q$) in an arbitrary neighborhood of $x$. As the argument holds for every $x \in \mathbb{R}$, we have proven that every real number is a cluster point of $\mathbb{R}$.

We now want to prove that no real number is a limit point of $q$. In order to do so, suppose $x \in \mathbb{R}$ is a limit of $q$. Let me write $m$ for the largest integer smaller than or equal to $x$. As $x$ is a limit of $q$, it holds that there is a natural number $n_0 \in \mathbb{N}$ such that $q_n \in (m, m+1), \forall\, n \in \mathbb{N}; n > n_0$. However, this implies that there are no more than $n_0$ terms of $q$ outside of $(m, m+1)$. Given that we have already proved that $(m+2, m+3)$, for example, has infinitely many terms of $q$, we have reached a contradiction and it is impossible for $x$ to be a limit point of $q$. As the argument holds for every point $x \in \mathbb{R}$, $q$ has no limit points in $\mathbb{R}$. ♥

**Example** [Not Every Sequence Has a Single Limit]**:**

As a second example, let us show that there are sequences in topological spaces that admit more than one limit (as opposed to what happens in metric spaces, when limits are always unique). In this example, we are going to consider the line with two origins: we add another element to the real line and introduce a topology in this space.

If we want to add another element to $\mathbb{R}$, we must pick some set which we already know to exist. As any set will do, let ✠ denote the Poor Fellow-Soldiers of Christ and of the Temple of Solomon (yes, the Templars). We write $X = \mathbb{R} \cup \{✠\}$.

As we are interested in topological properties, we still need to define a topology in $X$. Let $\mathfrak{B}_{\mathbb{R}}$ be the basis of open intervals for the standard topology in $\mathbb{R}$. Let $\mathfrak{B}_{✠} \equiv \{\{✠\} \cup B \setminus \{0\}; B \in \mathfrak{B}_{\mathbb{R}}\}$. We define $\mathfrak{B} \equiv \mathfrak{B}_{\mathbb{R}} \cup \mathfrak{B}_{✠}$. I claim $\mathfrak{B}$ is a basis for a topology in $X$. The details of the proof shall be left as an exercise, but I present a sketch.

By construction, $\mathfrak{B}$ covers $X$. In order to prove that $\mathfrak{B}$ is a basis, we still need to prove that $\forall\, B_1, B_2 \in \mathfrak{B}, \forall\, x \in B_1 \cap B_2, \exists\, B_3 \in \mathfrak{B}; x \in B_3 \subseteq B_1 \cap B_2$.

If we have either $B_1, B_2 \in \mathfrak{B}_{\mathbb{R}}$ or $B_1, B_2 \in \mathfrak{B}_{✠}$, the proof is trivial. We must then consider only the case in which each set is in a different collection. Let us suppose, without any loss of generality, that $B_1 = (a, b) \in \mathfrak{B}_{\mathbb{R}} \setminus \mathfrak{B}_{✠}$ and $B_2 = (c, d) \in \mathfrak{B}_{✠} \setminus \mathfrak{B}_{\mathbb{R}}$. We can simply split the intersection of both sets in negative and positive sides and reduce the problem to elements in $\mathfrak{B}_{\mathbb{R}}$.

Let us consider the sequence of elements of $X$ given by $x_n = \frac{1}{n}, \forall\, n \in \mathbb{N}^*$. This sequence admits two limit points: 0 and ✠.

Indeed, let $y$ denote either $0$ or $\maltese$. Consider an arbitrary neighborhood $O$ of $y$ such that $O \in \mathfrak{B}$, for simplicity. As any open set can be written as an union of elements of $\mathfrak{B}$, if we proof that $x_n$ is eventually within any such neighborhood $O$, the argument holds for arbitrary neighborhoods and will follow that $y$ is a limit point of $x_n$.

As $O \in \mathfrak{B}$, we can write it as $O = (a, b)_y$, denoting the interval starting at $a$, ending at $b$ and containing only the origin $y$, but not the other one. For example, if $y = 0$, $0 \in (a, b)_y, \maltese \notin (a, b)_y$, with an analogous relation for $y = \maltese$.

Due to the Archimedean property of the real numbers, we know that, $\forall\, b > 0, \exists\, n_0 > \frac{1}{b}$, and therefore $b > \frac{1}{n_0}$. Thus, $\forall\, n > n_0, x_n < b$ and it follows that $x_n \in (a, b)_y, \forall\, n > n_0$. Hence, $x_n$ is eventually in $O$, for any neighborhood $O$ of $y$ satisfying $O \in \mathfrak{B}$. As every neighborhood of $y$ can be written as unions of such sets, it follows that $x_n$ is eventually in any neighborhood of $y$, *id est*, $y$ is a limit of $x_n$. As $y$ is either $0$ or $\maltese$ and the argument holds for both, we conclude $x_n$ admits two limits: $0$ and $\maltese$. ♥

The line with two origins presents a result which certainly seems odd, considering the usual properties limits respect in metric spaces. Namely, limits in topological spaces need not to be unique, albeit limits in metric spaces are always unique. There are topological spaces whose structure does not allow us to separate some points from others, and sequences in such spaces might admit more than one limit for a simple reason: both points are indistinguishable, from the topological point of view.

If we are interested in studying limits, this is actually quite a bummer. Eventually we could be interested in taking derivatives of functions (we are going to need more complicated spaces in order to do so) and it would be uninteresting for us to have a function with two derivatives at the same point, for we would want to give a general definition for the inclination of a function somewhere and to the procedure of finding tangent lines and planes and *et cetera*. Thus, if we have any dreams of making Calculus more general, we must first be sure the limits we taking are unique.

Motivated by the notion of trying to separate points, we are going to define the *Hausdorff property*.

**Definition 3.52** [Hausdorff Spaces or $T_2$-Spaces]**:**

Let $(X, \tau)$ be a topological space. We say $(X, \tau)$ is a *Hausdorff space*, or a *$T_2$-Space*, or simply that $(X, \tau)$ satisfies the *Hausdorff property*, if, and only if, it holds that given two arbitrary points $x, y \in X$ there are disjoint open sets $O_x, O_y \in \tau$ such that $O_x$ is a neighborhood of $x$ and $O_y$ is a neighborhood of $y$. ♠

*Remark:*

The name *$T_2$-Space* might seem a bit odd. If we consider it alone, it is indeed, but in fact the Hausdorff property is just one of the so called *Separation Axioms* (and yes, there is a definition for a $\mathrm{T}_0$-Space, a $\mathrm{T}_1$-Space, a $\mathrm{T}_{3\frac{1}{2}}$-Space and actually quite a lot of options). Different axioms denote different separation properties which a space might or not obey. As an example, a $\mathrm{T}_0$-Space satisfies the property that, given two points $x, y \in X$, at least one of them admits a neighborhood that does not contain the other. For now, we are interested exclusively in the Hausdorff property (which is often the most interesting), but more details concerning other separation axioms are available at Section 3.6. ♣

Naturally, the next step we should give is proving that such a property does solve the problem we had.

**Theorem 3.53:**
*Let $(X, \tau)$ be a Hausdorff space. Then every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $X$ admits at most one limit point. Furthermore, if it exists, such limit point is the only cluster point of $(x_n)_{n\in\mathbb{N}}$.* □

*Proof:*
Suppose there is $x \in X$ such that $x$ is a limit of $(x_n)_{n\in\mathbb{N}}$. Let $y \in X$ such that $x \neq y$. As $(X, \tau)$ is a Hausdorff space, there are disjoint open sets $O_x$ and $O_y$ such that $x \in O_x$ and $y \in O_y$. Given that $x$ is a limit of $(x_n)_{n\in\mathbb{N}}$, we know that $\exists\, n_0 \in \mathbb{N}; x_n \in O_x, \forall\, n > n_0$. As $O_x$ and $O_y$ are disjoint, it follows that $x_n \notin O_y, \forall\, n > n_0$, and thus there are no more than $n_0$ terms of $(x_n)_{n\in\mathbb{N}}$. Therefore, we have found a neighborhood of $y$ with a finite numbers of terms of $(x_n)_{n\in\mathbb{N}}$, which guarantees that $y$ is not a cluster point of $(x_n)_{n\in\mathbb{N}}$. As every limit point is a cluster point and $y$ is not a cluster point of $(x_n)_{n\in\mathbb{N}}$, it follows by contrapositive that $y$ is not a limit of $(x_n)_{n\in\mathbb{N}}$. As the argument holds for every point $y \in X, y \neq x$, we conclude that $x$ is the only limit point of the sequence.

If $(x_n)_{n\in\mathbb{N}}$ admits no limits points, the result holds trivially. ■

Now that points are being distinguished from a topological point of view (since they can be separated by open sets), limits of sequences are finally unique.

Notice that the Hausdorff axiom is not essential for us to study limits, continuity or Topology in general. However, it is useful, for endowing spaces with this extra property allow us to obtain more interesting results. Generality is interesting from the point of view that having few assumptions allows us to apply our results to many different spaces, but it comes with the price of having less results.

## 3.5 What is Continuity?

Theorem 3.7 on page 23 allows us to extend the definition of what is a continuous function through the following definition:

**Definition 3.54** [Continuous Function]**:**
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. We say that $f$ is a *continuous function* if, and only if, $f^{-1}(A) \in \tau_X, \forall\, A \in \tau_Y$. ♠

Despite Theorem 3.7 on page 23, this definition might still be a bit nebulous. Thus, it could be useful for us to verify that this definition recovers what we expect from real functions.

**Example** [Topological Continuity]**:**
Consider the real line $\mathbb{R}$ with its usual metric topology (the topology of open balls

for the metric $d(x,y) = |x-y|$). Let $\Theta\colon \mathbb{R} \to \mathbb{R}$ denote Heaviside's step function:

$$\Theta(x) = \begin{cases} 1, \text{ if } x > 0, \\ \frac{1}{2}, \text{ if } x = 0, \\ 0, \text{ if } x < 0. \end{cases} \tag{3.37}$$

We know that $\Theta$ should be continuous at any point $x \neq 0$ and discontinuous at $x = 0$. Thus, if we consider its restrictions to $\mathbb{R}_+^* \equiv \{x > 0; x \in \mathbb{R}\}$ or $\mathbb{R}_-^* \equiv \{x < 0; x \in \mathbb{R}\}$, it should be a continuous function.

Let us pick an open set in the range of $\Theta$ such that its preimage is either in $\mathbb{R}_+^*$ or $\mathbb{R}_-^*$. For example, let us pick $A = \mathcal{B}_{\frac{1}{2}}(1)$, which is an open ball in $\mathbb{R}$ and thus is an open set in $\mathbb{R}$ (Lemma 3.5 on page 23). $\Theta^{-1}(A) = \mathbb{R}_+^*$. Since $\mathbb{R}_+^* = \bigcup_{n \in \mathbb{N}^*} \mathcal{B}_1(n)$, where $\mathbb{N}^*$ denotes the positive natural numbers, $\mathbb{R}_+^*$ is indeed open and we see no problem on this region*.

Let us now pick an open set that might give us a bit more trouble, for example one that has 0 in its preimage. We might choose $A = \mathcal{B}_{\frac{1}{2}}\left(\frac{1}{2}\right)$, for example. Now we have that $\Theta^{-1}(A) = \{0\}$, which is not an open set (since we are dealing with a metric topology, an easy way to see it is by proving that no open ball centered at 0 (which is the only element in the set) can be contained in $\{0\}$). Thus, $\Theta$ can't be continuous on $\mathbb{R}$.

Notice now how the continuity of a function depends on the topology we are considering: if we had chosen the discrete topology instead of the usual topology, $\Theta$ (and in fact any other function) would be a continuous function. ♥

One might also wonder whether we could write the definition as "$f(A)$ is open for every open set $A \in X$" for a function $f\colon X \to Y$. The answer is no, and we might give an example of when this fails:

**Example:**

Once more let us pick $\mathbb{R}$ with its usual topology. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2, \forall\, x \in \mathbb{R}$, which is continuous everywhere when we consider the definition of continuity commonly used in Real Analysis. We pick the open set $A = \mathcal{B}_\epsilon(0)$ for any $\epsilon > 0$. Notice that $f(A) = [0, \epsilon)$, which is not an open set ♥

We also have a name for functions $\varphi$ such that $\varphi(A)$ is open for every open set $A$:

**Definition 3.55** [Open Maps]**:**

Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. We say that $f$ is a *open* if, and only if, $f(A) \in \tau_Y, \forall\, A \in \tau_X$. ♠

**Proposition 3.56:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Let $Z \subseteq X$. If $f\colon X \to Y$ is a continuous function, its restriction $f\big|_Z : Z \to Y$ is continuous when $Z$ is equipped with the relative topology.* □

*Be careful: this argument alone doesn't imply that $\Theta$ is continuous on $\mathbb{R}_+^*$! Recall that continuity requires for $\Theta^{-1}(A)$ to be open for *any* open set $A$, not only the one we've picked.

*Proof:*
We shall denote $g \equiv f\big|_Z$.
Let $O \in \tau_Y$. We want to prove that $g^{-1}(O)$ is open in the relative topology of $Z$. Since $f$ is continuous, we know that $f^{-1}(O) \in \tau_X$.
Notice that

$$\begin{aligned} f^{-1}(O) &= \{x \in X; f(x) \in O\}, \\ g^{-1}(O) &= \{x \in Z \subseteq X; f(x) \in O\}. \end{aligned} \tag{3.38}$$

Thus, $g^{-1}(O) = f^{-1}(O) \cap Z$. Since $f^{-1}(O)$ is open relatively to $X$, the definition of the relative topology implies $g^{-1}(O)$ is open in $Z$, concluding the proof. ■

Of course, it is not unusual within Real Analysis for one to talk about a function continuous *at a given point*. Naturally, we may wonder how can we define continuity at a given point. The trick is simple: continuity at $x$ used to be defined as $\forall\, \epsilon > 0, \exists\, \delta > 0; d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$. We must simply erase the open balls and write neighborhoods instead.

**Definition 3.57** [Continuity at a Point]**:**
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. Let $x \in X$. We say $f$ is *continuous at* $x$ if, and only if, for every neighborhood $O$ of $f(x)$ there is a neighborhood $U$ of $x$ such that $f(U) \subseteq O$. ♠

**Proposition 3.58:**
*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. Let $x \in X$. If for every neighborhood $O$ of $f(x)$, $f^{-1}(O)$ is a neighborhood of $x$, then $f$ is continuous at $x$.* □

*Proof:*
Suppose that for every neighborhood $O$ of $f(x)$ it holds that $f^{-1}(O)$ is a neighborhood of $x$. Then notice that $f\big(f^{-1}(O)\big) = O \subseteq O$. Thus, for every neighborhood $O$ of $f(x)$ there is a neighborhood $U$ of $x$ such that $f(U) \subseteq O$. ■

A question that arises now is whether this new definition is compatible with the definition of continuity provided for topological spaces as a whole. This is settled in the following result:

**Proposition 3.59:**
*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. $f$ is continuous if, and only if, $f$ is continuous at $x$, $\forall\, x \in X$.* □

*Proof:*

- $\Rightarrow$: Let $x \in X$. Provided that $f$ is continuous, it holds that $f^{-1}(O) \in \tau_Y, \forall\, O \in \tau_X$. Therefore, if $O$ is a neighborhood of $f(x)$, then $f^{-1}(O)$ is an open set. Since $f(x) \in O$, it also holds that $x \in f^{-1}(O)$. Thus, $f^{-1}(O)$ is a neighborhood of $x$ and it holds that $f$ is continuous at $x$.

$\Leftarrow$: Let $O \in \tau_Y$. If $f^{-1}(O) = \varnothing$, then $f^{-1}(O)$ is trivially an open set and the proof is complete. Otherwise, let $xinf^{-1}(O)$. Notice that $O$ is a neighborhood of $f(x)$. Since $f$ is continuous at every point of $X$, it is continuous at $x$ and it holds that $f^{-1}(O)$ is a neighborhood of $x$. Therefore, $f^{-1}(O)$ is an open set and the proof is complete. ■

Of course, we could as well state the definition of continuity in terms of the closure operator or in terms of closed sets, instead of open sets.

**Theorem 3.60:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. The following statements are equivalent:*

i. *for every open set $O$ in $Y$, $f^{-1}(O)$ is a open set in $X$;*

ii. *for every closed set $F$ in $Y$, $f^{-1}(F)$ is a closed set in $X$;*

iii. $\forall\, A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$. □

*Proof:*

Let us first prove that if the first statement holds, then so does the second. Let $F$ be a closed set in $Y$. Then $F^{\mathrm{c}} \in \tau_Y$. Since the preimage of every open set under $f$ is an open set, we know that $f^{-1}(F^{\mathrm{c}}) = f^{-1}(F)^{\mathrm{c}} \in \tau_X$. Therefore, $f^{-1}(F)$ is a closed set in $X$.

Assuming the second statement, we might as well prove the first in a similar fashion. Let $O$ be a closed set in $Y$. Then $O^{\mathrm{c}}$ is a closed set. Since the preimage of every closed set under $f$ is a closed set, we know that $f^{-1}(O^{\mathrm{c}}) = f^{-1}(O)^{\mathrm{c}}$ is a closed set. Therefore, $f^{-1}(O)$ is an open set in $X$.

Assuming the second statement, we now want to prove the third. Let $A \subseteq X$. Then we have

$$\begin{aligned} f(A) &\subseteq \overline{f(A)}, \\ A \subseteq f^{-1}(f(A)) &\subseteq f^{-1}\Big(\overline{f(A)}\Big), \\ A &\subseteq f^{-1}\Big(\overline{f(A)}\Big), \\ \overline{A} &\subseteq \overline{f^{-1}\Big(\overline{f(A)}\Big)}. \end{aligned} \tag{3.39}$$

Since the preimage of a closed set under $f$ is a closed set, we know that $f^{-1}\Big(\overline{f(A)}\Big)$ is a closed set. As a consequence, $f^{-1}\Big(\overline{f(A)}\Big) = \overline{f^{-1}\Big(\overline{f(A)}\Big)}$. It follows that

$$\begin{aligned} \overline{A} &\subseteq f^{-1}\Big(\overline{f(A)}\Big), \\ f(\overline{A}) &\subseteq f\Big(f^{-1}\Big(\overline{f(A)}\Big)\Big), \\ f(\overline{A}) &\subseteq \overline{f(A)}, \end{aligned} \tag{3.40}$$

as desired.

Finally, we want to prove that the third statement implies the second. Let $F$ be a closed set in $Y$. We know that $\forall A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$, and therefore we have

$$\begin{aligned} f\left(\overline{f^{-1}(F)}\right) &\subseteq \overline{f(f^{-1}(F))} = \overline{F} = F, \\ f\left(\overline{f^{-1}(F)}\right) &\subseteq F, \\ \overline{f^{-1}(F)} \subseteq f^{-1}\left(f\left(\overline{f^{-1}(F)}\right)\right) &\subseteq f^{-1}(F), \\ \overline{f^{-1}(F)} &\subseteq f^{-1}(F). \end{aligned} \tag{3.41}$$

Since $A \subseteq \overline{A}, \forall A \subseteq X$, it follows that $\overline{f^{-1}(F)} = f^{-1}(F)$ and, therefore, $f^{-1}(F)$ is a closed set in $X$. ■

Just as is the case for real functions, the composition of continuous functions is a continuous function itself.

**Proposition 3.61:**
*Let $(X, \tau_X)$, $(Y, \tau_Y)$ and $(Z, \tau_Z)$ be topological spaces and let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Then the function $g \circ f\colon X \to Z$ is also a continuous function.* □

*Proof:*
Let $O \in \tau_Z$. Since $g$ is continuous, we know that $g^{-1}(O) \in \tau_Y$. And since $f$ is continuous, we also know that $f^{-1}\big(g^{-1}(O)\big) \in \tau_X$. Since $f^{-1}\big(g^{-1}(O)\big) = (g \circ f)^{-1}(O)$, we conclude the function $g \circ f$ is continuous. ■

If you recall, continuity was the main reason we started studying Topology in the first place. It would not be surprising if the notion of a continuous function could now lead us into deeper insights when considering the relations between different topological spaces, just like isomorphisms allow us to find "hidden" relations between linear spaces.

Indeed, continuity allows us to define what is a homeomorphism, which is the appropriate equivalence relation when we are dealing with topological spaces.

**Definition 3.62** [Homeomorphism]**:**
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f\colon X \to Y$ be a bijective function. Suppose further that both $f$ and $f^{-1}$ are continuous functions. Under these conditions, we say that $f$ is a *homeomorphism* and the topological spaces $(X, \tau_X)$ and $(Y, \tau_Y)$ are said to be *homeomorphic*. ♠

*Remark:*
Perhaps you would expect that is a function is continuous, then so should its inverse be. However, let $O \in \tau_X$. The preimage of $O$ under $f^{-1}\colon Y \to X$ is the set $\{y \in Y; f^{-1}(y) \in O\}$, which is equal to $f(O) = \{f(x); x \in O\}$. Thus, $f^{-1}$ being continuous means that if $O$ is an open set, then so is its image under $f$. We already know that this does not follow from continuity, and therefore the requirement is not superfluous.

Combining this with the fact that $f$ being continuous means that the preimage of an open set is also an open set, we see that we are demanding that $O \in \tau_X \Rightarrow f(O) \in \tau_Y$ and $O \in \tau_Y \Rightarrow f^{-1}(O) \in \tau_X$, *id est*, $O \in \tau_X \Leftrightarrow f(O) \in \tau_Y$. The homeomorphism provides us not only with a identification between points on the topological space, but also "translates" the topology in $X$ to the topology in $Y$ and vice-versa. This means that every property of $X$ that can be entirely expressed in terms of the topology $\tau_X$ also must hold for $Y$ due to the homeomorphism. Such properties are called *topological properties*. ♣

We might as well define a similar notion for the case in which a topological space is "larger" than the other.

**Definition 3.63** [Embedding]**:**
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f: X \to Y$ be a surjective function. Let $f': X \to f(X)$ be a function such that $f(x) = f'(x), \forall\, x \in X$, where $f(X)$ is considered as a topological subspace of $Y$. If $f'$ is a homeomorphism, we say that $f$ is an *embedding of $X$ in $Y$*. ♠

**Theorem 3.64:**
*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. The relation $(X, \tau_X) \simeq (Y, \tau_Y) \Leftrightarrow (X, \tau_X)$ and $(X, \tau_X)$ are homeomorphic is an equivalence relation.* □

*Proof:*
Consider first the identity mapping in the topological space $(X, \tau_X)$, given by $i_X: X \to X$ such that $i_X(x) = x$. Notice that this map is bijective and it has itself as its inverse. Furthermore, it is continuous, for if we are given an open set $O \in \tau_X$, $i_X^{-1}(O) = O \in \tau_X$. Therefore, this is a continuous map with continuous inverse, and therefore it is a homeomorphism, which means $(X, \tau_X)$ is homeomorphic to itself, *id est*, $(X, \tau_X) \simeq (X, \tau_X)$.

Let now $(X, \tau_X)$ and $(Y, \tau_Y)$ be homeomorphic topological spaces and let $f: X \to Y$ be a homeomorphism. $f$ is invertible, so let us consider the function $f^{-1}: Y \to X$. As $f$ is a homeomorphism, $f^{-1}$ is continuous. $f^{-1}$ is invertible and has $f$, which is continuous, as its inverse. Thus, $f^{-1}$ is a continuous function with continuous inverse, and therefore it is a homeomorphism between $(Y, \tau_Y)$ and $(X, \tau_X)$. Thus, $(Y, \tau_Y)$ and $(X, \tau_X)$ are homeomorphic and we see that $(X, \tau_X) \simeq (Y, \tau_Y) \Rightarrow (Y, \tau_Y) \simeq (X, \tau_X)$.

Finally, assume $(X, \tau_X)$, $(Y, \tau_Y)$ and $(Z, \tau_Z)$ are topological spaces such that $(X, \tau_X) \simeq (Y, \tau_Y)$ and $(Y, \tau_Y) \simeq (Z, \tau_Z)$. Let us denote $f: X \to Y$ for the homeomorphism between $(X, \tau_X)$ and $(Y, \tau_Y)$ and $g: Y \to Z$ for the homeomorphism between $(Y, \tau_Y)$ and $(Z, \tau_Z)$. We want to prove that $(X, \tau_X)$ and $(Z, \tau_Z)$ are homeomorphic.

Consider the map $g \circ f: X \to Z$. We know from Proposition 3.61 on the previous page that $g \circ f$ is a continuous function. Since both $f$ and $g$ are bijective, so is $g \circ f$, which has $f^{-1} \circ g^{-1}$ as its inverse. Since $f^{-1}$ and $g^{-1}$ are continuous, so is $f^{-1} \circ g^{-1}$. Thus, $g \circ f$ is a continuous function with continuous inverse, which means $(X, \tau_X)$ and $(Z, \tau_Z)$ are homeomorphic, *id est*, $(X, \tau_X) \simeq (Z, \tau_Z)$.

We see then that the following properties hold for $\simeq$ for every topological spaces $(X, \tau_X)$, $(Y, \tau_Y)$ and $(Z, \tau_Z)$:

i. $(X, \tau_X) \simeq (X, \tau_X)$;

ii. $(X, \tau_X) \simeq (Y, \tau_Y) \Rightarrow (Y, \tau_Y) \simeq (X, \tau_X)$;

iii. $(X, \tau_X) \simeq (Y, \tau_Y)$ and $(Y, \tau_Y) \simeq (Z, \tau_Z) \Rightarrow (X, \tau_X) \simeq (Z, \tau_Z)$.

Therefore, $\simeq$ is indeed an equivalence relation. ■

**Proposition 3.65:**
*Let $a, b \in \mathbb{R}, a < b$. The real line $\mathbb{R}$ equipped with the standard topology is homeomorphic to the interval $(a, b)$ with the relative topology.* □

*Proof:*
Consider the function $f\colon (a, b) \to \mathbb{R}$ given by

$$f(x) \equiv \tan\left(\frac{\pi}{b-a}x + \frac{\pi}{2}\left(\frac{a+b}{a-b}\right)\right). \tag{3.42}$$

We know from Real Analysis that such a function is a continuous bijection between $\mathbb{R}$ and $(a, b)$ with continuous inverse. Thus, the result is proven. ■

Continuous functions are also interesting because they provide us with yet another way of specifying the topology on a space.

**Theorem 3.66:**
*Let $\Lambda$ be an arbitrary collection of indices, $X$ be a set and let $(Y_\lambda, \tau_\lambda)$ be topological spaces, $\forall\, \lambda \in \Lambda$. Let $\{f_\lambda\colon X \to Y_\lambda\}_{\lambda\in\Lambda}$ be a family of maps. Then there is a unique coarsest topology $\tau_X$ in $X$ which makes every function $f_\lambda$ continuous.* □

*Proof:*
Consider the family of all topologies over $X$ that make $f_\lambda$ continuous, $\forall\, \lambda \in \Lambda$, denoted $\mathfrak{T}$. This is a set, for it is merely a restriction of the set $\mathbb{P}(\mathbb{P}(X))$. We might then just consider the intersection of all elements of $\mathfrak{T}$, denoted by $\tau_X = \bigcap_{\tau\in\mathfrak{T}} \tau$. We know the intersection of topologies is a topology, and therefore $\tau_X$ is a topology over $X$. Notice that $\tau_X$ is coarser than any element of $\mathfrak{T}$ by construction. We now only need to prove that $\tau_X$ does make $f_\lambda$ continuous, $\forall\, \lambda \in \Lambda$.

Pick $\lambda \in \Lambda$. Let $O \in \tau_\lambda$. We know that $f_\lambda^{-1}(O) \in \tau, \forall\, \tau \in \mathfrak{T}$, for every topology in $\mathfrak{T}$ makes $f_\lambda$ continuous. Thus, $O \in \bigcap_{\tau\in\mathfrak{T}} \tau = \tau_X$. Therefore, the preimage of any open set in $Y_\lambda$ under $f_\lambda$ is in $\tau_X$, which means $f_\lambda$ if continuous when we equip $X$ with $\tau_X$. Since the result holds for every $\lambda \in \Lambda$, the proof is complete. ■

**Definition 3.67** [Weak Topology]**:**
Let $\Lambda$ be an arbitrary collection of indices, $X$ be a set and let $(Y_\lambda, \tau_\lambda)$ be topological spaces, $\forall\, \lambda \in \Lambda$. Let $\{f_\lambda\colon X \to Y_\lambda\}_{\lambda\in\Lambda}$ be a family of maps. The coarsest topology $\tau_X$ in $X$ which makes every function $f_\lambda$ continuous is said to be the *weak topology* generated by $\{f_\lambda\}_{\lambda\in\Lambda}$. ♠

**Definition 3.68** [Product Topology]**:**
Let $\Lambda$ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces and let $X = \bigtimes_{\lambda\in\Lambda} X_\lambda$. The *product topology* in $X$ is the weak topology generated by the projections $\{\pi_\lambda\colon X \to X_\lambda\}_{\lambda\in\Lambda}$. ♠

*Remark:*

We provide below the definitions of the generalized Cartesian product, $\bigtimes_{\lambda\in\Lambda} X_\lambda$, and of the projections $\pi_\lambda$ in the same situation for completeness. ♣

**Definition 3.69** [Generalized Cartesian Product]**:**

Let $\Lambda$ be an arbitrary family of indices and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. We define the *Cartesian product* of $\{X_\lambda\}_{\lambda\in\Lambda}$, $\bigtimes_{\lambda\in\Lambda} X_\lambda$, as the set

$$\bigtimes_{\lambda\in\Lambda} X_\lambda = \left\{ f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda; f(\lambda) \in X_\lambda, f \text{ is a function} \right\}.$$

Let us write $X \equiv \bigtimes_{\lambda\in\Lambda} X_\lambda$ for simplicity for the rest of this definition. We define the *projections* $\pi_\lambda\colon X \to X_\lambda$ as the functions such that $\pi_\lambda(f) = f(\lambda)$.

From now on, when writing a generalized Cartesian product explicitly, we will not write the requirement that $f$ is a function, but it shall always be understood implicitly. ♠

**Lemma 3.70:**

*Let $\Lambda$ be an arbitrary family of indices and let $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_\lambda\}_{\lambda\in\Lambda}$ be two families of sets. It holds that*

$$\bigtimes_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda) = \left(\bigtimes_{\lambda\in\Lambda} X_\lambda\right) \cap \left(\bigtimes_{\lambda\in\Lambda} Y_\lambda\right).$$

□

*Proof:*

Notice that

$$\bigtimes_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda) = \left\{ f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda); f(\lambda) \in (X_\lambda \cap Y_\lambda) \right\}.$$

Furthermore,

$$\begin{aligned}
\left(\bigtimes_{\lambda\in\Lambda} X_\lambda\right) \cap \left(\bigtimes_{\lambda\in\Lambda} Y_\lambda\right) &= \left\{ f\colon \Lambda \to \left(\bigcup_{\lambda\in\Lambda} X_\lambda\right) \cap \left(\bigcup_{\lambda\in\Lambda} Y_\lambda\right); f(\lambda) \in (X_\lambda \cap Y_\lambda) \right\}, \\
&= \left\{ f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda); f(\lambda) \in (X_\lambda \cap Y_\lambda) \right\}, \\
&= \bigtimes_{\lambda\in\Lambda} (X_\lambda \cap Y_\lambda).
\end{aligned}$$

This proves the result. ■

**Theorem 3.71:**

*Let $\Lambda$ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces and let $X = \bigtimes_{\lambda\in\Lambda} X_\lambda$. The set*

$$\mathfrak{B} = \left\{ \bigtimes_{\lambda\in\Lambda} O_\lambda; O_\lambda \in \tau_\lambda, \forall\, \lambda \in \Lambda \textit{ and } \{\lambda; O_\lambda \neq X_\lambda\}_{\lambda\in\Lambda} \textit{ is finite} \right\}$$

*is a basis for the product topology on $X$.* $\square$

*Proof:*

Notice that $\mathfrak{S} \equiv \left\{\pi_\lambda^{-1}(O_\lambda); O_\lambda \in \tau_\lambda, \forall\, \lambda \in \Lambda\right\}$ is a subbasis for a topology in $X$. Indeed, since $O_\lambda \subseteq X_\lambda, \forall\, \lambda \in \Lambda$, $\pi_\lambda^{-1}(O_\lambda) \subseteq \pi_\lambda^{-1}(X_\lambda) = X$. Since $\pi_\lambda^{-1}(X_\lambda) \in \mathfrak{S}, \forall\, \lambda \in \Lambda$, we have that $X \in \mathfrak{S}$ and thus it is clear that $\mathfrak{S}$ must be a subbasis for a topology in $X$.

We shall denote $U_\lambda^\lambda \equiv O_\lambda$ and $U_\lambda^\kappa \equiv X_\lambda, \forall\, \kappa \in \Lambda, \kappa \neq \lambda$. Let $L = \{\lambda \in \Lambda; O_\lambda \neq X_\lambda\}$. We assume, as stated in the definition of $\mathfrak{B}$, that $L$ is a finite set.

The basis induced by this subbasis has elements given by

$$\begin{aligned}\bigcap_{\lambda \in L} \pi_\lambda^{-1}(O_\lambda) &= \bigcap_{\lambda \in L} \bigtimes_{\kappa \in \Lambda} U_\lambda^\kappa, \\ &= \bigtimes_{\lambda \in \Lambda} O_\lambda. \end{aligned} \tag{3.43}$$

Therefore, the basis induced by $\mathfrak{S}$ is $\mathfrak{B}$. Since $\pi_\lambda^{-1}(O_\lambda)$ must be an open set in the product topology (otherwise, the projection would not be continuous), we see the topology generated by $\mathfrak{B}$ must be the product topology. The elements of the topology generated by $\mathfrak{B}$ are given by

$$O = \bigcup_{\mu \in M} \bigcap_{\lambda \in L} \pi_\lambda^{-1}(O_\lambda^\mu), \tag{3.44}$$

where $M$ is an arbitrary set of indices, $O_\lambda^\mu \in \tau_\lambda, \forall\, \mu \in M, \forall\, \lambda \in \Lambda$. All these sets must be open in the product topology, for the finite intersection and the arbitrary union of open sets are open sets as well. The product topology also cannot be properly contained in the topology generated by $\mathfrak{B}$, for the product topology must contain $\mathfrak{S}$ and be closed under finite intersections and arbitrary unions, which means the elements of the topology generated by $\mathfrak{B}$ must be elements of the product topology. Thus, $\mathfrak{B}$ does generate the product topology. $\blacksquare$

**Lemma 3.72:**

*Let $\Lambda$ be an arbitrary family of indices and let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of sets. Suppose there is $\lambda \in \Lambda$ such that $X_\lambda = \varnothing$. Then $X = \bigtimes_{\lambda \in \Lambda} X_\lambda = \varnothing$.* $\square$

*Proof:*

$f \in X \Rightarrow f(\lambda) \in X_\lambda = \varnothing$. Since $\forall\, x, x \notin \varnothing$, $X$ must be empty. $\blacksquare$

**Theorem 3.73:**

*Let $\Lambda$ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ be a family of Hausdorff topological spaces and let $X = \bigtimes_{\lambda \in \Lambda} X_\lambda$. Then it holds that $(X, \tau)$ is Hausdorff, where $\tau$ denotes the product topology.* $\square$

*Proof:*

We want to prove that given two points $f, g \in X$ such that $f \neq g$, there are two disjoint open sets $O, U$ such that $f \in O, g \in U$.

Since $f \neq g$, there is at least one $\lambda \in \Lambda$ such that $f(\lambda) \neq g(\lambda)$. Let us write $f(\lambda) = x_\lambda, g(\lambda) = y_\lambda$. Since $X_\lambda$ is a Hausdorff space, there are two sets $O_\lambda$ and $U_\lambda$ such that $x_\lambda \in O_\lambda$ and $y_\lambda \in U_\lambda$ with $O_\lambda \cap U_\lambda = \varnothing$. Let us define $O_\kappa = U_\kappa = X_\kappa, \forall\, \kappa \in \Lambda; \kappa \neq \lambda$. Notice that $f \in \bigtimes_{\kappa\in\Lambda} O_\kappa$ and $g \in \bigtimes_{\kappa\in\Lambda} U_\kappa$. Furthermore,

$$\begin{aligned}\left(\bigtimes_{\kappa\in\Lambda} O_\kappa\right) \cap \left(\bigtimes_{\kappa\in\Lambda} U_\kappa\right) &= \bigtimes_{\kappa\in\Lambda} (O_\kappa \cap U_\kappa),\\ &= \varnothing. \end{aligned} \tag{3.45}$$

The last line follows from Lemma 3.72 on the preceding page. ■

**Proposition 3.74:**

*Let $\Lambda$ be an arbitrary set of indexes. Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces and $(X, \tau_X)$ be the product space generated by such a family. Let $(Y, \tau_Y)$ be a topological space as well. Given a function $f\colon Y \to X$, it holds that $f$ is continuous if, and only if, $\pi_\lambda \circ f$ is continuous $\forall\, \lambda \in \Lambda$.* □

*Proof:*

- $\Rightarrow$: Suppose $f$ is continuous. By definition of $(X, \tau_X)$, we know that $\pi_\lambda$ is continuous $\forall\, \lambda \in \Lambda$. Since the composition of continuous functions is continuous, $\pi_\lambda \circ f$ is continuous $\forall\, \lambda \in \Lambda$.

- $\Leftarrow$: Let us suppose that $\pi_\lambda \circ f$ is continuous $\forall\, \lambda \in \Lambda$. One then has that, given an open set $O_\lambda \in \tau_\lambda$, $f^{-1}\Big(\pi_\lambda^{-1}(O_\lambda)\Big) \in \tau_Y$. Since the product topology makes every projection a continuous function, we know $\pi_\lambda^{-1}(O_\lambda) \in \tau_X$. We now want to prove that $f^{-1}(O) \in \tau_Y, \forall\, O \in \tau_X$.

  We know from the proof to Theorem 3.71 on page 58 that an arbitrary open set in $X$ can be written as $\bigtimes_{\lambda\in\Lambda} O_\lambda = \bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)$, where $O_\lambda \in \tau_\lambda, \forall\, \lambda \in \Lambda, O_\kappa \neq X_\kappa, \forall\, \kappa \in L$ and $L \subseteq \Lambda$ is finite. Since

  $$f^{-1}\left(\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)\right) = \bigcap_{\lambda\in L} f^{-1}\Big(\pi_\lambda^{-1}(O_\lambda)\Big), \tag{3.46}$$

  $\pi_\lambda^{-1}(O_\lambda) \in \tau_X, \forall\, \lambda \in \Lambda$ and the finite intersection of open sets is an open set, we may conclude that $f^{-1}\Big(\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)\Big) \in \tau_Y$. As we mentioned earlier, any element of $\tau_X$ can be written in the form $\bigcap_{\lambda\in L} \pi_\lambda^{-1}(O_\lambda)$, and thus the proof is complete.

■

*Remark:*

Notice that if we consider the Cartesian product of a series of *equal* spaces (*id est*,

$X_\lambda = X, \forall\, \lambda \in \Lambda$, for some space $X$), then $\bigtimes_{\lambda\in\Lambda} X_\lambda$ is the set of all functions from $\Lambda$ to $X$:

$$\begin{aligned} \bigtimes_{\lambda\in\Lambda} X_\lambda &= \left\{ f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda ; f(\lambda) \in X_\lambda \right\}, \\ &= \{ f\colon \Lambda \to X ; f(\lambda) \in X \}, \\ &= \{ f\colon \Lambda \to X \}. \end{aligned} \tag{3.47}$$

♣

*Notation* $[X^\Lambda]$:

Due to the previous remark and the common notation for a finite Cartesian product: $X^n \equiv \bigtimes_{1\leq 1\leq n} X$, we shall write the set of all functions from a set $\Lambda$ to a set $X$ as $X^\Lambda \equiv \bigtimes_{\lambda\in\Lambda} X$. ♦

**Proposition 3.75:**

*Let $(X, \tau_X)$ be a topological space and $\Lambda$ be an arbitrary set. Consider the topological space $(X^\Lambda, \tau)$, where $\tau$ is the product topology. Given a sequence $(f_n)_{n\in\mathbb{N}}$ in $X^\Lambda$ and $f \in X^\Lambda$, it holds that $f_n \to f$ in the product topology if, and only if, $f_n \to f$ pointwise,* id est, $f_n(\lambda) \to f(\lambda), \forall\, \lambda \in \Lambda$. □

*Proof:*

$\Rightarrow$: Let us assume $f_n \to f$ in the product topology, *id est*, $f_n$ is eventually in every neighborhood of $f$. We want to prove that $f_n \to f$ pointwise.

Let $O_\lambda \in \tau)_X$ be such that $O = \bigtimes_{\lambda\in\Lambda} O_\lambda$ is a neighborhood of $f$ (notice that every neighborhood of $f$ can be written in such a way). There is a natural number, $n_0$, such that $f_n \in O, \forall\, n \geq n_0$. Notice thus that $f(\lambda), f_n(\lambda) \in O_\lambda, \forall\, n \geq n_0$. Thus, $O_\lambda$ is a neighborhood of $f(\lambda)$ and the sequence $(f_n(\lambda))_{n\in\mathbb{N}}$ is eventually in such a neighborhood. Notice that in $O$ is a neighborhood of $f$ if, and only if, every $O_\lambda$ is be a neighborhood of $f(\lambda)$, and thus it is proven that convergence in the product topology implies pointwise convergence.

$\Leftarrow$: Suppose $f_n(\lambda) \to f(\lambda), \forall\, \lambda \in \Lambda$. This means that, for every neighborhood $O_\lambda$ of $f(\lambda)$, there is a natural number $n_0(\lambda)$ such that $f_n(\lambda) \in O_\lambda, \forall\, n \geq n_0(\lambda)$.

For every $\lambda$, let $O_\lambda$ be a neighborhood of $f(\lambda)$ such that $O = \bigtimes_{\lambda\in\Lambda} O_\lambda$ is an open set in $X^\Lambda$. Then $O$ is a neighborhood of $f$. Let $n_0 = \max_{\lambda\in\Lambda} n_0(\lambda)$. Notice that $f_n(\lambda) \in O_\lambda, \forall\, n \geq n_0, \forall\, \lambda \in \Lambda$. Thus, $f_n \in O, \forall\, n \geq n_0$. Since every neighborhood of $f$ can be written in the previous form, this means that $f_n$ is eventually in any neighborhood of $f$, and thus $f_n \to f$ in the product topology. ■

Naturally, many of the functions we consider interesting have the real line or the complex plane as its codomain. We shall then introduce some notation.

*Notation* $[\tau_\mathbb{C}]$:

When considered as a set, without algebraic properties, $\mathbb{C} = \mathbb{R}^2$. Thus, the standard topology on $\mathbb{C}$, which we shall denote by $\tau_{\mathbb{C}}$, is the product topology obtained when we consider $(\mathbb{C}, \tau_{\mathbb{C}})$ as the product space of $(\mathbb{R}, \tau_{\mathbb{R}})$ with itself. ♦

*Notation* $[B(X)$, $C(X)$ and $BC(X))$:

Let $\mathbb{F}$ be either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. Let $(X, \tau)$ be a topological space. We denote by $B(X, \mathbb{F})$ the set of all bounded functions $f\colon X \to \mathbb{F}$. Similarly, we denote by $C(X, \mathbb{F})$ the set of all continuous functions $f\colon X \to \mathbb{F}$ when $X$ if equipped with the $\tau$ topology (which should be clear by context) and $\mathbb{F}$ is equipped with its standard topology. We might also consider the set

$$BC(X, \mathbb{F}) \equiv B(X, \mathbb{F}) \cap C(X, \mathbb{F}). \tag{3.48}$$

When considering complex-valued functions, we might drop the $\mathbb{C}$ and write simply

$$B(X) \equiv B(X, \mathbb{C}), \quad C(X) \equiv C(X, \mathbb{C}), \quad BC(X) \equiv BC(X, \mathbb{C}).$$ ♦

**Lemma 3.76:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $\mathfrak{B}$ be a basis for a topology in $Y$ generating $\tau_Y$. A function $f\colon X \to Y$ is continuous if, and only if, $f^{-1}(\mathcal{B}) \in \tau_X, \forall\, \mathcal{B} \in \mathfrak{B}$.* □

*Proof:*

- $\Rightarrow$: Suppose $f$ is continuous. Then $f^{-1}(O) \in \tau_X, \forall\, O \in \tau_Y$. Since $\mathfrak{B} \subseteq \tau_Y$, the statement holds.
- $\Leftarrow$: Let us assume $f^{-1}(\mathcal{B}) \in \tau_X, \forall\, \mathcal{B} \in \mathfrak{B}$. We know every open set $O \in \tau_Y$ can be written as $\bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda$, where $\Lambda$ is some set of indices and $\mathcal{B}_\lambda \in \mathfrak{B}, \forall\, \lambda \in \Lambda$. Thus, we have that

$$\begin{aligned} f^{-1}(O) &= f^{-1}\left(\bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda\right) \\ &= \bigcup_{\lambda \in \Lambda} f^{-1}(\mathcal{B}_\lambda). \end{aligned} \tag{3.49}$$

  We already know that $f^{-1}(\mathcal{B}_\lambda) \in \tau_X, \forall\, \lambda \in \Lambda$. Thus, since arbitrary unions of open sets are open sets, the result holds. ■

**Lemma 3.77:**

$\mathfrak{B} = \{(a, b) \times (c, d); a, b, c, d \in \mathbb{R}\}$ *is a basis for the product topology in* $\mathbb{R}^2$. □

*Proof:*

Let us first prove that $\mathfrak{B}$ is indeed a basis for a topology in $\mathbb{R}^2$.Let $(x, y) \in \mathbb{R}^2$. Since the open intervals are a basis for a topology in $\mathbb{R}$, we know that there are real numbers $a, b, c, d$ such that $a < x < y$ and $c < y < d$, and therefore $(x, y) \in (a, b) \times (c, d)$.

Let now $\mathcal{B}_1 = (a, b) \times (c, d), \mathcal{B}_2 = (e, f) \times (g, h)$. Notice that $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$. We want to prove that, $\forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$.

Let $\mathcal{B}_3 = \mathcal{B}_1 \cap \mathcal{B}_2 = \big[(a,b) \cap (e,f)\big] \times \big[(c,d) \cap (g,h)\big]$. Since the intersection of open intervals is an open interval, $\mathcal{B}_3 \in \mathfrak{B}$. Clearly it holds that $\forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Therefore, $\mathfrak{B}$ is indeed a basis for a topology in $\mathbb{R}^2$.

We know that $\mathfrak{B}' = \{O \times U; O, U \in \tau_{\mathbb{R}}\}$ is a basis for the product topology in $\mathbb{R}^2$, thanks to Theorem 3.71 on page 58. Notice that $\mathfrak{B} \subseteq \mathfrak{B}'$. Therefore, $\forall\, x \in \mathbb{R}^2$ and $\forall\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$ we know that there is $\mathcal{B}' \in \mathfrak{B}'$, given by $\mathcal{B}' = \mathcal{B}$, such that $x \in \mathcal{B}' \subseteq \mathcal{B}$. Therefore, due to Proposition 3.18 on page 30, the topology generated by $\mathfrak{B}'$ is finer than the topology generated by $\mathfrak{B}$.

However, let $x \in \mathbb{R}^2$. We shall write $x = (x_1, x_2)$. Let $\mathcal{B}' \in \mathfrak{B}'$ such that $x \in \mathcal{B}'$. Suppose $\mathcal{B}' = O \times U$, where $O, U \in \tau_{\mathbb{R}}$. Since the open intervals are a basis for the standard topology in $\mathbb{R}$, we know there are families of indices $\Lambda$ and $M$ and real numbers $a_\lambda, b_\lambda, c_\mu, d_\mu, \forall\, \lambda \in \Lambda, \forall\, \mu \in M$ such that $O = \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda)$ and $U = \bigcup_{\mu \in M} (c_\mu, d_\mu)$. Therefore, since $x \in O \times U$, we know there is some $\lambda \in \Lambda$ and some $\mu \in M$ such that $x_1 \in (a_\lambda, b_\lambda)$ and $x_2 \in (c_\mu, d_\mu)$. Thus, $x \in (a_\lambda, b_\lambda) \times (c_\mu, d_\mu) \equiv \mathcal{B} \in \mathfrak{B}$, with $\mathcal{B} \subseteq \mathcal{B}'$. Therefore, the topology generated by $\mathfrak{B}$ is finer than the topology generated by $\mathfrak{B}'$. Since the topology generated by $\mathfrak{B}'$ is also finer than the topology generated by $\mathfrak{B}$, we conclude both topologies are in fact the same, and thus $\mathfrak{B}$ is a basis for the product topology in $\mathbb{R}^2$. ■

**Lemma 3.78:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Let $f\colon X \to Y$ be a constant function,* id est, *$\exists\, y_0 \in Y; f(x) = y_0, \forall\, x \in X$. Then $f$ is continuous.* □

*Proof:*

Let $O \in \tau_Y$. If $y_0 \in O$, then $f^{-1}(O) = X \in \tau_X$. If $y_0 \notin O$, then $f^{-1}(O) = \varnothing \in \tau_X$. ■

**Proposition 3.79:**

*Let $(X, \tau)$ be a topological space. $C(X)$ and $BC(X)$ can be regarded as complex vector spaces when equipped with the usual addition and multiplication of complex-valued functions.* □

*Proof:*

We must first prove that addition and multiplication of complex-valued (bounded) continuous functions are binary operations in $C(X)$ and $BC(X)$. It is known that the addition and multiplication of bounded functions is always a bounded function, and thus we shall only bother with the proof of continuity.

Let $f, g \in C(X)$. We want to prove that $f + g \in C(X)$. In order to do so, let us consider a different function: let $h\colon X \to \mathbb{C}^2$ be the function defined by $h(x) = (f(x), g(x))$. Since $f$ and $g$ are continuous, Proposition 3.74 on page 60 guarantees $h$ is continuous as well when we equip $\mathbb{C}^2$ with the product topology.

Consider now the map $+_{\mathbb{C}}\colon \mathbb{C}^2 \to \mathbb{C}$ such that $+_{\mathbb{C}}(x, y) = x + y, \forall\, x, y \in \mathbb{C}$, *id est*, $+_{\mathbb{C}}$ is ordinary addition in the complex plane. $+_{\mathbb{C}}$ is a continuous function. In order to prove this, notice that given $x, y, z, w \in \mathbb{R}$, we have $+_{\mathbb{C}}\big((x, y), (z, w)\big) = (x + z, y + w) \in \mathbb{C}$. If we prove the coordinate functions $+(x, z) = x + z$ and $+(y, w) = y + w$, which are simply real addition, are continuous, then $+_{\mathbb{C}}$ is continuous due to Proposition 3.74 on page 60.

Due to Lemma 3.76 on page 62 and the fact that the standard topology on $\mathbb{R}$ is generated by the open intervals $(a,b) \subseteq \mathbb{R}$, we must only prove that $+^{-1}\big((a,b)\big)$ is an open set for any pair of real numbers $a,b \in \mathbb{R}$.

Notice that $+^{-1}\big((a,b)\big) = \big\{(x,y) \in \mathbb{R}^2; a < x+y < b\big\}$. We want to prove this is an open set. Due to Theorem 3.15 on page 29 Lemma 3.77 on page 62, we must simply prove that $\forall\, x,y \in \mathbb{R}; a < x+y < b, \exists\, c,d,e,f \in \mathbb{R}$ such that $(x,y) \in (c,d) \times (e,f)$ and $a < z+w < b, \forall\,(z,w) \in (c,d) \times (e,f)$.

Let $c = \frac{a-y+x}{2}, d = \frac{b-y+x}{2}, e = \frac{a-x+y}{2}, f = \frac{b-x+y}{2}$. Notice that $x > c$, for $x - c = \frac{y+x-a}{2} > 0$, for $x+y > a$. Analogous arguments guarantee that $x \in (c,d)$ and $y \in (e,f)$.

Let $z \in (c,d)$ and $w \in (e,f)$. Then

$$\begin{aligned} z+w &> \frac{a-y+x}{2} + \frac{a-x+y}{2}, \\ &= a. \end{aligned} \tag{3.50}$$

Similarly,

$$\begin{aligned} z+w &< \frac{b-y+x}{2} + \frac{b-x+y}{2}, \\ &= b. \end{aligned} \tag{3.51}$$

Thus, $+^{-1}\big((a,b)\big)$ is an open set, which implies addition, both real and complex, is a continuous function. Since $+_{\mathbb{C}}$ is continuous, the composition $+_{\mathbb{C}} \circ h \colon X \to \mathbb{C}$ is also continuous. However, notice that $(+_{\mathbb{C}} \circ h)(x) = f(x) + g(x) = (f+g)(x)$. It is thus proven that the sum of continuous complex-valued functions is a continuous function.

The proof that complex multiplication is continuous can be done in a similar manner. The same argument used to prove that the sum of continuous functions is continuous can be applied to show that the product of continuous functions is continuous as well.

We now know that usual addition is a binary operation in $C(X)$ and $BC(X)$ and that $C(X)$ and $BC(X)$ are closed under scalar multiplication (for complex scalars are simply constant functions, which are continuous due to Lemma 3.78 on the previous page). Let us now prove that they obey the conditions necessary for $C(X)$ and $BC(X)$ to be vector spaces.

A1 Complex addition is associative, and therefore so is the addition of complex-valued functions;

A2 The constant function $0(x) = 0, \forall\, x \in X$ is continuous due to Lemma 3.78 on the preceding page and is clearly bounded. Furthermore, $f + 0 = 0 + f = f$ for every continuous function $f$;

A3 For every continuous function $f$, there is a function $f^* = -f$ such that $f + f^* = f^* + f = 0$;

A4 Complex addition is commutative, and thus so is the addition of complex-valued functions;

M1 Complex multiplication is associative, and therefore it holds that, $\forall\, x \in X, \forall\, z, w \in \mathbb{C}, \forall\, f \in C(X), [zw]f(x) = z[wf(x)]$;

M2 We know that $1 \in \mathbb{C}$ is such that $1 \cdot f = f$;

D1 Since complex multiplication is distributive over complex addition, we know that $\forall\, x \in X, \forall\, z, w \in \mathbb{C}, \forall\, f \in C(X), (z+w)f(x) = zf(x) + wf(x)$;

D2 Since complex multiplication is distributive over complex addition, we know that $\forall\, x \in X, \forall\, z \in \mathbb{C}, \forall\, f, g \in C(X), z[f+g](x) = zf(x) + zg(x)$.

Since these properties hold in $C(X)$, they also hold in $BC(X)$. 0 is bounded and the operations we defined are closed in $BC(X)$, ensuring $BC(X)$ is a subspace of $C(X)$. This concludes the proof. ■

The proof to Proposition 3.79 on page 63 also showed the following results:

**Theorem 3.80:**
*Let $\mathbb{F}$ be either the real line or the complex plane and consider it equipped with the standard topology. The functions $+\colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\cdot\colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ defined as $+(x,y) = x + y$ and $\cdot(x,y) = x \cdot y$ are continuous when $\mathbb{F}^2$ is equipped with the product topology.* □

*Proof:*
See proof to Proposition 3.79 on page 63. ■

**Definition 3.81** [Uniform Norm and Metric]**:**
Let $(X, \tau)$ be a topological space. Let $f \in B(X)$. We define the *uniform norm* of $f$, denoted $\|f\|_u$, through

$$\|f\|_u = \sup_{x \in X} |f(x)|. \tag{3.52}$$

We define the *uniform metric* on $B(X)$ through $d(f,g) = \|f - g\|_u$. ♠

**Proposition 3.82:**
*Let $(X, \tau)$ be a topological space. The function $d\colon B(X) \times B(X) \to \mathbb{R}_+$ given by $d(f,g) = \|f - g\|_u$ defines a metric in $B(X)$.* □

*Proof:*
Let $f, g, h \in B(X)$. Notice that

$$\begin{aligned} d(f,g) &= \|f - g\|_u, \\ &= \sup_{x \in X} |f(x) - g(x)|, \\ &= \sup_{x \in X} |g(x) - f(x)|, \\ &= \|g - f\|_u, \\ &= d(g,f). \end{aligned} \tag{3.53}$$

It is clear that $f = g$ implies $d(f, g) = 0$, for $f(x) - f(x) = 0$ and $\sup_{x \in X} 0 = 0$. Furthermore, notice that $\forall\, x \in X, 0 \leq |f(x) - g(x)| \leq \|f - g\|_u = d(f, g)$. Thus, if $d(f, g) = 0, 0 \leq |f(x) - g(x)| \leq 0$ and we see that it holds that $d(f, g) = 0 \Leftrightarrow f = g$.

Finally, we must prove the triangle inequality. Notice that, if we write $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$, we have that $|z| = \sqrt{x^2 + y^2}$. This means that $|z - w| = d_2(z, w)$, where $d_2$ stands for the Euclidean metric, which we know satisfies the triangle inequality. Keeping this in mind, notice that:

$$\begin{aligned} d(f, g) &= \sup_{x \in X} |f(x) - g(x)|, \\ &= \sup_{x \in X} \left[d_2(f(x), g(x))\right], \\ &\leq \sup_{x \in X} \left[d_2(f(x), h(x)) + d_2(g(x), h(x))\right], \\ &\leq \sup_{x \in X} \left[d_2(f(x), h(x))\right] + \sup_{x \in X} \left[d_2(g(x), h(x))\right], \\ &= d(f, h) + d(g, h). \end{aligned} \tag{3.54}$$

Consequently, $d$ defines a metric in $B(X)$. ■

*Remark:*

Notice that the hypothesis that $f \in B(X)$ in the definition of the uniform norm is necessary if we want it to be well-defined. If we picked, *exempli gratia*, $f \in C(X)$, the supremum could be ill-defined, for the set $\{|f(x)|; x \in X\}$ would possibly be unbounded. ♣

An interesting result concerning the uniform metric requires us to define some convergence criteria on metric spaces.

**Definition 3.83** [Modes of Convergence on Metric Spaces]**:**

Let $X$ be a set and $(M, d)$ be a metric space. Consider a sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n \colon X \to M$. Let $f \colon X \to M$ be a function.

i. We say $f_n$ converges *pointwise* to $f$ whenever it holds that

$$\forall\, x \in X, \forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n(x) - f(x)) < \epsilon; \tag{3.55}$$

ii. We say $f_n$ converges *uniformly* to $f$ whenever it holds that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, d(f_n(x) - f(x)) < \epsilon. \tag{3.56}$$

Notice that uniform convergence implies pointwise convergence. ♠

**Proposition 3.84:**

*Let $(X, \tau)$ be a topological space. Convergence in $B(X)$ with respect to the uniform metric is equivalent to uniform convergence in $X$,* id est, *if $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions $f_n \colon X \to \mathbb{C}$ and $f \colon X \to \mathbb{C}$ is a function, then*

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n, f) < \epsilon \tag{3.57}$$

*if, and only if,*

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \epsilon, \tag{3.58}$$

*where $d$ denotes the uniform metric on $B(X)$.* □

*Proof:*

⇒: Assume $f_n \to f$ in the uniform metric. Let $\epsilon > 0$. Then we know that $\exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n, f) < \epsilon$, *id est*, $\forall\, n > n_0, \sup_{x\in X} |f_n(x) - f(x)| < \epsilon$. Since $\forall\, x \in X, |f_n(x) - f(x)| \leq \sup_{x\in X} |f_n(x) - f(x)|$, we see that is must hold that $\forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \epsilon$, proving $f_n \to f$ uniformly.

⇐: Suppose $f_n \to f$ uniformly. Let $\forall\, \epsilon > 0$. We know that $\exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, |f_n(x) - f(x)| < \frac{\epsilon}{2}$. Since $\forall\, x \in X, |f_n(x) - f(x)| < \frac{\epsilon}{2}$, it holds that $\sup_{x\in X} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$. Thus, we see that $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(f_n, f) < \epsilon$ and we conclude that uniform convergence implies convergence in the uniform metric. ■

We might also explore it even further and see that $B(X)$ is in fact a *complete* space, which essentially means a space with no "holes". In order to understand this statement, we must work again with some concepts regarding metric spaces.

**Definition 3.85** [Cauchy Sequences]**:**

Let $(M, d)$ be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $M$. $(x_n)_{n\in\mathbb{N}}$ is said to be a *Cauchy sequence* if, and only if, it holds that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n, m > n_0, d(x_n, x_m) < \epsilon. \tag{3.59}$$

♠

**Proposition 3.86:**

*Let $(M, d)$ be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $M$. If $(x_n)_{n\in\mathbb{N}}$ is convergent, it is a Cauchy sequence.* □

*Proof:*

Let $x \in M$ be such that $x_n \to x$. We know that, given $\epsilon > 0$,

$$\exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, d(x_n, x) < \frac{\epsilon}{2}. \tag{3.60}$$

Let $n, m > n_0$. Then we have that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x_m, x), \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned} \tag{3.61}$$

Therefore, we see that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n, m > n_0, d(x_n, x_m) < \epsilon. \tag{3.62}$$

■

**Example** [Divergent Cauchy Sequence]**:**

Not every Cauchy sequence is convergent. In fact, spaces in which Cauchy implies convergence are said to be *complete*. As a simple example, consider the metric space $(\mathbb{Q}, d_2)$: the rational numbers with the Euclidean metric $d_2(x,y) = |x-y|$.

Let us define $x_n = \frac{\lfloor 10^n\sqrt{2}\rfloor}{10^n}$, where $\forall\, x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer smaller than $x$. Notice the first few elements of this sequence are $x_0 = 1, x_1 = 1.4, x_2 = 1.41, x_3 = 1.414, x_4 = 1.4142$, *et cetera*. The sequence consists of the decimal expansion of the square root of two up to the $n$-th decimal place. Naturally, $x_n \to \sqrt{2}$, which is not a rational number, despite $x_n$ being a Cauchy sequence, a fact I leave for you to check. ♥

**Definition 3.87** [Completeness of a Metric Space]**:**

Let $(M,d)$ be a metric space. We say $(M,d)$ is *complete* if, and only if, every Cauchy sequence of elements of $M$ is convergent. ♠

**Proposition 3.88:**

*Let $(X,\tau)$ be a topological space. $B(X)$ is complete in the uniform metric.* □

*Proof:*

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of functions $f_n\colon X \to \mathbb{C}$, *id est*, $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n,m > n_0, d(f_n,f_m) < \epsilon$. Since $d(f_n,f_m) = \sup_{x\in X}|f_n(x)-f_m(x)|$ and it holds that $\forall\, x \in X, |f_n(x)-f_m(x)| \leq \sup_{x\in X}|f_n(x)-f_m(x)|$, we see that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n,m > n_0, \forall\, x \in X, |f_n(x)-f_m(x)| < \epsilon. \tag{3.63}$$

Therefore, if we consider a fixed point $x \in X$, the sequence of real numbers determined by $\left(f_n(x)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Every Cauchy sequence in the complex plane is convergent, and thus so is $\left(f_n(x)\right)_{n\in\mathbb{N}}$. Let us define $f(x) = \lim_{n\to\infty} f_n(x)$.

Let us now allow $m \to \infty$ in Eq. (3.63). It follows that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in X, |f_n(x)-f(x)| < \epsilon. \tag{3.64}$$

Therefore,

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \sup_{x\in X}|f_n(x)-f(x)| < \epsilon, \tag{3.65}$$

which means $f_n \to f$ with respect to the metric $d$. ■

**Lemma 3.89:**

*Every metric space $(M,d)$ satisfies the Hausdorff property.* □

*Proof:*

Let $x,y \in M, x \neq y$. Let $r = d(x,y)$. Consider the open sets given by $\mathcal{B}_{\frac{r}{3}}(x)$ and $\mathcal{B}_{\frac{r}{3}}(y)$. Suppose $z \in \mathcal{B}_{\frac{r}{3}}(x) \cap \mathcal{B}_{\frac{r}{3}}(y)$. Then we have that $d(x,z) < \frac{r}{3}$ and $d(y,z) < \frac{r}{3}$. It follows that

$$\begin{aligned} r &= d(x,y), \\ &\leq d(x,z) + d(y,z), \end{aligned}$$

$$
\begin{aligned}
&< \frac{r}{3} + \frac{r}{3}, \\
&= \frac{2r}{3}.
\end{aligned} \tag{3.66}
$$

We found a contradiction. Thus, the hypothesis that there is $z \in \mathcal{B}_{\frac{r}{3}}(x) \cap \mathcal{B}_{\frac{r}{3}}(y)$ and we may conclude $(M, d)$ is Hausdorff. ■

**Lemma 3.90:**

*Let $(M, d)$ be a complete metric space. Let $N \subseteq M$ and let us consider $(N, d|_N)$ as a metric subspace of $(M, d)$. For simplicity, we shall write $d|_N$ just as $d$. $(N, d)$ is complete if, and only if, $N$ is a closed set.* □

*Proof:*

$\Rightarrow$: Let us assume $N$ is a closed set. Then it holds that $N = \overline{N}$. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $N$. Since it is Cauchy and $(M, d)$ is complete, there is some $x \in M$ such that $x_n \to x$. Notice that $(x_n)_{n\in\mathbb{N}}$ is eventually in every neighborhood $O$ of $x$. However, we know from Theorem 3.43 on page 45 that $x \in \overline{N} \Leftrightarrow O \cap N \neq 0$ for every neighborhood $O$ of $x$. Since given a neighborhood $O$ of $x$ there is $n_0 \in \mathbb{N}$ such that $x_n \in O, \forall\, n > n_0$ and we already now that $x_n \in N, \forall\, n \in \mathbb{N}$, we see that $O \cap N \neq 0$ for every neighborhood $O$ of $x$. Therefore, $x \in \overline{N} = N$, and we see that every Cauchy sequence in $N$ admits a limit in $N$.

$\Leftarrow$: Suppose every Cauchy sequence of elements of $N$ is convergent. Let $x \in \overline{N}$. Then Theorem 3.43 on page 45 states $x \in \overline{N} \Leftrightarrow O \cap N \neq 0$ for every neighborhood $O$ of $x$. Let us pick, in particular, the open balls centered at $x$ and with radius $\frac{1}{n}$. $\forall\, n \in \mathbb{N}$, let $x_n \in \mathcal{B}_{\frac{1}{n}}(x) \cap N$.

This is a Cauchy sequence, for $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{1}{n} + \frac{1}{m}$. Given $\epsilon > 0$, we can pick $n_0 \in \mathbb{N}$ respecting $n_0 > \frac{2}{\epsilon}$ and it holds that $d(x_n, x_m) < \epsilon, \forall\, n, m > n_0$.

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and $(N, d)$ is complete, we have that $\lim x_n \in N$. Notice that every metric space is a Hausdorff space (Lemma 3.89 on the facing page) and every sequence in a Hausdorff space admits at most one limit (Theorem 3.53 on page 51). Thus, we may conclude that $\lim x_n = x$, and it follows that $x \in N, \forall\, x \in \overline{N}$. Since it always holds that $N \subseteq \overline{N}$, we conclude that $N = \overline{N}$ and, therefore, that $N$ is closed. ■

**Proposition 3.91:**

*Let $(X, \tau)$ be a topological space. $BC(X)$ is a closed subspace of $B(X)$ under the uniform metric. Furthermore, $BC(X)$ is complete.* □

*Proof:*

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $BC(X)$. Since $B(X)$ is complete, we know there is $f \in B(X)$ such that $f_n \to f$ in the uniform metric. If we prove that $f \in C(X)$, then we have that $f \in BC(X)$ and may conclude that $BC(X)$ is complete (and closed, due to Lemma 3.90 on the preceding page).

Let $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $d(f_n, f) = \sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{3}, \forall\, n > n_0$. Let $n > n_0$ and $x \in X$. We know $f_n$ is continuous at $x$. Thus, given a neighborhood $O$ of $f_n(x)$, we have a neighborhood $U$ of $x$ such that $U \subseteq f_n^{-1}(O)$. In particular, we might consider $O = \mathcal{B}_{\frac{\epsilon}{3}}(f_n(x))$ (where the metric considered is the Euclidean metric in $\mathbb{C}$). Thus, we see $\forall\, y \in U$ it holds that $|f_n(y) - f_n(x)| < \frac{\epsilon}{3}$. We have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x)|, \\ &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|, \\ &\leq \epsilon. \end{aligned} \tag{3.67}$$

Now, notice that $x \in U \subseteq f^{-1}(\mathcal{B}_\epsilon(f(x)))$. Thus, $f$ is continuous at $x$. ■

## 3.6 Countability and Separation Axioms

You might have noticed by now that Topology is quite a general theory. Indeed, it is too general for usual physical purposes and has too few axioms to generate interesting results. This leads us to the restrict ourselves to a smaller class of topological spaces. In the process, we do lose generality, but also are able to obtain more results and work more intensely with the spaces that do interest us.

Two categories of axioms commonly used are the countability and separation axioms.

**Definition 3.92** [First Axiom of Countability]**:**

A topological space $(X, \tau)$ is said to satisfy the *first axiom of countability*, or to be *first-countable*, whenever, $\forall\, x \in X$, there is a countable neighborhood basis for $\tau$ at $x$. ♠

**Proposition 3.93:**

*Let $(X, \tau)$ be a first-countable topological space. Then, $\forall\, x \in X$, there is a neighborhood base $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ such that $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i, \forall\, i$.* □

*Proof:*

Let $x \in X$. Since $(X, \tau)$ is first-countable, there is a countable neighborhood base $\{O_j\}_{j=1}^{\infty}$ for $\tau$ at $x$. We may then define another neighborhood basis $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ through

$$\mathcal{B}_i = \bigcap_{j=1}^{i} O_j.$$

Notice that $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ is indeed a neighborhood basis for $\tau$ at $x$: since $x \in O_j, \forall\, j$, it follows that $x \in \mathcal{B}_i, \forall\, i$. If $x \in O \in \tau, \exists\, j; x \in O_j$. Therefore, $x \in \mathcal{B}_j$. ■

**Proposition 3.94:**

*Let $(X, d)$ be a metric space. Then $(X, d)$ is first-countable.* □

*Proof:*

Let $x \in X$. The collection $\mathfrak{N} = \left\{\mathcal{B}_{\frac{1}{n}}(x); n \in \mathbb{N}\right\}$ is a countable neighborhood basis for the topology generated by $d$ at $x$. Indeed, $x \in \mathcal{B}_{\frac{1}{n}}(x), \forall\, n \in \mathbb{N}$. Furthermore, if $O$ is open and $x \in O$, then there is some $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x) \subseteq O$. The Archimedean property of the real line guarantees the existence of $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$, and we then have $x \in \mathcal{B}_{\frac{1}{n_0}}(x) \subseteq \mathcal{B}_\epsilon(x) \subseteq O$. This proves the result. ■

**Definition 3.95** [Second Axiom of Countability]**:**

A topological space $(X, \tau)$ is said to satisfy the *second axiom of countability*, or to be *second-countable*, whenever there is a countable basis for $\tau$ on $X$. ♠

**Proposition 3.96:**

*Let $(X, \tau)$ be a second-countable topological space. Let $Y \subseteq X$ and let $\tau_Y$ be the relative topology on $Y$. $(Y, \tau_Y)$ is second-countable.* □

*Proof:*

Let $\mathfrak{B}$ be a countable basis for $(X, \tau)$. Then $\mathfrak{B}_Y \equiv \{\mathcal{B} \cap Y; \mathcal{B} \in \mathfrak{B}\}$ is a basis for the relative topology in $Y$.

Indeed, since $\mathfrak{B}$ is a basis for $(X, \tau)$, it holds that $\forall\, y \in Y, \exists\, \mathcal{B} \in \mathfrak{B}; y \in \mathcal{B}$. Hence, $y \in \mathcal{B} \cap Y$.

Furthermore, $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. This implies in particular that $\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, y \in \mathcal{B}_1 \cap \mathcal{B}_2 \cap Y, \exists\, \mathcal{B}_3 \in \mathfrak{B}; y \in \mathcal{B}_3 \cap Y \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap Y$. Thus, $\mathfrak{B}_Y$ is a basis for a topology in $Y$.

We know that

$$\tau = \{O \subseteq X | \forall\, x \in O, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq O\}. \tag{3.68}$$

Also, we know

$$\begin{aligned}
\tau_Y &= \{(U \cap Y) \subseteq Y; U \in \tau\},\\
&= \{(U \cap Y) \subseteq Y | \forall\, x \in U, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq U\},\\
&= \{(U \cap Y) \subseteq Y | \forall\, y \in U \cap Y, \exists\, \mathcal{B} \in \mathfrak{B}; y \in (\mathcal{B} \cap Y) \subseteq (U \cap Y)\},\\
&= \{U \subseteq Y | \forall\, y \in U, \exists\, \mathcal{B} \in \mathfrak{B}_Y; y \in \mathcal{B} \subseteq U\}.
\end{aligned} \tag{3.69}$$

This concludes the proof. ■

We shall also define a third countability axiom which exemplifies how bad terminology can get.

**Definition 3.97** [Separable Space]**:**

A topological space $(X, \tau)$ is said to be *separable* whenever it has a countable dense subset, *id est*, whenever there is a countable set $A$ such that $\overline{A} = X$. ♠

**Proposition 3.98:**

*Let $(X, \tau)$ be a second-countable topological space. Then $(X, \tau)$ is separable.* □

*Proof:*

Since $(X, \tau)$ is second-countable, there is a countable basis for $\tau$ in $X$, which we shall write $\{\mathcal{B}_i\}_{i=1}^{+\infty}$. With the Axiom of Choice, $\forall\, i$ we choose a point $x_i \in \mathcal{B}_i$. Consider now the set $A = \{x_i\}_{i=1}^{+\infty}$. Notice that $\overline{A}^{\mathsf{c}}$ is an open set, and thus there is a family of indexes $\Lambda$ such that $\overline{A}^{\mathsf{c}} = \bigcup_{\lambda \in \Lambda} \mathcal{B}_{i_\lambda}$.

Let us suppose that $\Lambda \neq \varnothing$ and fix $\lambda \in \Lambda$. We know that $x_{i_\lambda} \in \mathcal{B}_{i_\lambda}$ and thus $x_{i_\lambda} \in \overline{A}^{\mathsf{c}}$. However, $x_{i_\lambda} \in A \subseteq \overline{A}$, which means we have reached a contradiction. We must have $\Lambda = \varnothing$, and it follows that $\overline{A}^{\mathsf{c}} = \varnothing$. Therefore, $\overline{A} = X$. Since $A$ is countable, we have shown the existence of a countable dense subset of $X$, which means $(X, \tau)$ is separable. ■

**Definition 3.99** [Lindelöf Space]**:**

Let $(X, \tau)$ be a topological space. $X$ is said to be a *Lindelöf space* if, and only if, every open cover of $X$ admits a countable subcover. ♠

**Proposition 3.100:**

*Every second-countable space is a Lindelöf space.* □

*Proof:*

Let $(X, \tau)$ be a second-countable space. Since it is second-countable, we know it possesses a countable basis, which we shall call $\mathfrak{B}$. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $X$, where $\Lambda$ is an arbitrary family of indices.

Since, $\forall\, \lambda \in \Lambda$, $A_\lambda$ is open, we know all the $A_\lambda$ are unions of elements of $\mathfrak{B}$.

Let us define

$$\mathfrak{B}' \equiv \{\mathcal{B} \in \mathfrak{B} |\, \exists\, \lambda \in \Lambda; \mathcal{B} \subseteq A_\lambda\}. \tag{3.70}$$

$\mathfrak{B}'$ covers the entire space, for the $A_\lambda$ are unions of elements of $\mathfrak{B}'$ and they cover the entire space. Since $\mathfrak{B}' \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is countable, $\mathfrak{B}'$ is countable as well.

Since $\mathfrak{B}'$ is countable, we may now enumerate its elements. Let $\mathcal{B} \colon \mathbb{N} \to \mathfrak{B}'$ be a surjective function such that $n \mapsto \mathcal{B}_n$. Then, for each $n \in \mathbb{N}$, we may pick $\lambda_n \in \Lambda; \mathcal{B}_n \subseteq A_{\lambda_n}$. The existence of $\lambda_n$ is guaranteed by the definition of $\mathfrak{B}'$. Notice now that

$$\begin{gathered} \bigcup_{n=1}^{+\infty} \mathcal{B}_n = X, \\ \therefore \mathcal{B}_n \subseteq A_{\lambda_n}, \forall\, n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{+\infty} A_{\lambda_n} = X. \end{gathered} \tag{3.71}$$

Thus, $\{A_{\lambda_n}\}_{n \in \mathbb{N}}$ is a countable subcover of $\{A_\lambda\}_{\lambda \in \Lambda}$. ■

The next result illustrates how these extra axioms can be useful when dealing with more concrete concepts such as the convergence of sequences.

**Proposition 3.101:**

*Let $(X, \tau)$ be a first-countable topological space and let $A \subseteq X$. Then $x \in \overline{A}$ if, and only if, there is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $A$ that converges to $x$.* □

*Proof:*

$\Rightarrow$: Let us assume $x \in \overline{A}$. From Theorem 3.43 on page 45, we know that every neighborhood of $x$ intersects $A$. From Proposition 3.93 on page 70, we know there is a countable neighborhood basis $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ for $\tau$ in $x$ satisfying $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$. Since $\{\mathcal{B}_i\}_{i=1}^{+\infty} \subseteq \tau$, we may conclude that $A \cap \mathcal{B}_i \neq \varnothing, \forall\, i$. We may then use the Axiom of Choice to pick, $\forall\, i$, $x_i \in A \cap \mathcal{B}_i$. Since the neighborhood basis is countable, this does define a sequence. Notice that $x_i \in \mathcal{B}_i \subseteq \mathcal{B}_j, \forall\, j < i$.

We remain to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ does converge to $x$, *id est*, we remain to prove that $(x_n)_{n\in\mathbb{N}}$ is eventually in every neighborhood of $x$. In order to do so, let $O$ be such a neighborhood. Since $x \in O \in \tau$, we know that $\exists\, i; \mathcal{B}_i \subseteq O$ (for $\{\mathcal{B}_i\}_{i=1}^{+\infty}$ is a neighborhood basis for $\tau$ in $x$). Thus, $x_i \in O$. Since $x_j \in \mathcal{B}_j \subseteq \mathcal{B}_i \subseteq O, \forall\, j > i$, we see that $(x_n)_{n\in\mathbb{N}}$ is eventually in $O$. As the argument holds for every neighborhood $O$ of $x$, we conclude that $x_n \to x$, as desired.

$\Leftarrow$: Suppose now that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $A$ with $x_n \to x$. Therefore, if $O$ is a neighborhood of $x$, $\exists\, n_0 \in \mathbb{N}; x_n \in O, \forall\, n \geq n_0$. Since $x_n \in A, \forall\, n$, this means that $A \cap O \neq \varnothing$ for every neighborhood $O$ of $x$. Thus, it follows from Theorem 3.43 on page 45 that $x \in \overline{A}$, as desired. ■

**Theorem 3.102:**

*Let $(M, d)$ be a metric space. A set $A$ is dense in the sense of metric spaces,* id est, $\forall\, \epsilon > 0, \forall\, x \in M, \exists\, p \in A; p \in \mathcal{B}_\epsilon(x)$ *if, and only if, it is dense in the sense of topological spaces,* id est, $\overline{A} = M$. □

*Proof:*

From Propositions 3.94 and 3.101 on page 70 and on the preceding page, we know $x \in \overline{A}$ if, and only if, there is some sequence of elements of $A$ converging to $x$.

Suppose $A$ is dense in the sense of metric spaces. Then, given $x \in X$, we know $\forall\, \epsilon > 0, \exists\, p \in A; p \in \mathcal{B}_\epsilon(x)$. Thus, $\forall\, n \in \mathbb{N}, \exists\, x_n \in A; x_n \in \mathcal{B}_{\frac{1}{n}}(x)$. This is a sequence with $x_n \to x$, and thus $x \in \overline{A}$.

Suppose now that $\overline{A} = X$. Then, $\forall\, x \in X$, there is some sequence of elements of $X$ with $x_n \to x$. Thus, given $x$, we know there is a sequence $x_n$ of elements of $A$ such that $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; d(x, x_n) < \epsilon$, *id est*, $\forall\, \epsilon > 0, \exists\, p \in A; p \in \mathcal{B}_\epsilon(x)$.

This concludes the proof. ■

**Proposition 3.103:**

*Let $(M, d)$ be a metric space. The following statements are equivalent:*

i. *it is a Lindelöf space;*

ii. *it is separable;*

iii. *it is second-countable.* □

*Proof:*

i. $\Rightarrow$ ii.: For each $n \in \mathbb{N}$, consider the open cover of $M$ given by $\left\{\mathcal{B}_{\frac{1}{n}}(x)\right\}_{x \in M}$. Since $(M, d)$ is a Lindelöf space, these open covers admit countable subcovers given by the collections $\left\{\mathcal{B}_{\frac{1}{n}}\left(x_m^{(n)}\right)\right\}_{m \in \mathbb{N}}$. Consider the set $A = \left\{x_m^{(n)}\right\}_{n,m \in \mathbb{N}}$, which is countable, for it is the countable union of countable sets. Let $x \in M$ and $\epsilon > 0$. The Archimedean property of the real numbers allows us to find $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Since $\left\{\mathcal{B}_{\frac{1}{n}}\left(x_m^{(n)}\right)\right\}_{m \in \mathbb{N}}$ covers $X$, there is $m \in \mathbb{N}$ such that $x \in \mathcal{B}_{\frac{1}{n}}\left(x_m^{(n)}\right) \subseteq \mathcal{B}_\epsilon\left(x_m^{(n)}\right)$. Therefore, $A$ is dense in $M$ in the sense of metric spaces. Theorem 3.102 on the preceding page guarantees $\overline{A} = M$. Since $A$ is countable, $(M, d)$ is separable.

ii. $\Rightarrow$ iii.: Since $(M, d)$ is separable, there is some countable subset $A$ such that $\overline{A} = M$. We can consider the collection $\mathfrak{B}$ defined by

$$\mathfrak{B} = \left\{\mathcal{B}_{\frac{1}{n}}(x); n \in \mathbb{N}, x \in A\right\}. \tag{3.72}$$

$\mathfrak{B}$ is a countable basis for the metric topology on $(M, d)$. It is guaranteed to cover $X$ due to the fact that $A$ is dense in the sense of metric spaces (as per Theorem 3.102 on the previous page), so $\forall\, x \in M$ and $\forall\, n \in \mathbb{N}$, there is $p \in A; x \in \mathcal{B}_{\frac{1}{n}}(p)$. The fact that

$$\forall\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}, \forall\, x \in \mathcal{B}_1 \cap \mathcal{B}_2, \exists\, \mathcal{B}_3 \in \mathfrak{B}; x \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \tag{3.73}$$

can be proven from the triangle inequality and from the Archimedean property. The topology generated by $\mathfrak{B}$ is indeed the metric topology on $(M, d)$, for the elements of $\mathfrak{B}$ are the open balls with respect to $d$.

iii. $\Rightarrow$ i.: Proposition 3.100 on page 72. ∎

**Example:**

Let $X = [0, 1]$ and let $\tau = \{O \subseteq X; O^c \text{ is countable}\}$. $\tau$ defines a topology* on $X$. Let $A = [0, 1)$. Since $A$ is not countable, $\{1\}$ is not open and, as a consequence, $A$ is not closed. Therefore, $\overline{A} = X$. I claim there are no sequences of elements of $A$ with $x_n \to 1$.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of $A$. The set $B = \{x_n; n \in \mathbb{N}\}$ is countable, and therefore $B^c$ is an open set. Since $1 \notin A$ and $B \subseteq A$, $1 \in B^c$. Therefore, $B^c$ is an open set containing 1, a neighborhood of 1, that does not contain any elements of the sequence $(x_n)_{n \in \mathbb{N}}$. Therefore, is does not hold that $(x_n)_{n \in \mathbb{N}}$ is eventually in any neighborhood of 1 and it follows that $x_n \not\to 1$. Furthermore, since no elements of $(x_n)_{n \in \mathbb{N}}$ are in $B$, 1 isn't even a cluster point of $(x_n)_{n \in \mathbb{N}}$. ♥

*Named the cocountable topology

Let us now pay attention to the separation axioms. We have already presented one of them in Definition 3.52 on page 50, and we shall define it once again so all the axioms can be together in a single definition. You may notice some resemblances between them.

**Definition 3.104** [Separation Axioms]**:**

Let $(X, \tau)$ be a topological space. If it has the property $\mathrm{T_j}$, it is said to be a $\mathrm{T_j}$ space or that the topology on $X$ is $\mathrm{T_j}$. We suppose $x, y \in X$. The axioms read:

$\mathrm{T_0}$: If $x \neq y, \exists\, O \in \tau; x \in O, y \notin O$ *or* $x \notin O, y \in O$.

$\mathrm{T_1}$: If $x \neq y, \exists\, O \in \tau; x \in O, y \notin O$.

$\mathrm{T_2}$: If $x \neq y, \exists\, O, U \in \tau; x \in O, y \in U, O \cap U = \varnothing$.

$\mathrm{T_3}$: The space is $\mathrm{T_1}$ and for every closed set $F \subseteq X$ and $\forall\, x \in F^{\mathrm{c}}, \exists\, O, U \in \tau; x \in O, F \subseteq U, O \cap U = \varnothing$.

$\mathrm{T_{3\frac{1}{2}}}$: The space is $\mathrm{T_1}$ and for every closed set $F \subseteq X$ and $\forall\, x \in F^{\mathrm{c}}$, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0, \forall\, y \in F$.

$\mathrm{T_4}$: The space is $\mathrm{T_1}$ and for every disjoint closed sets $F, G \subseteq X, \exists\, O, U \in \tau; F \subseteq O, G \subseteq U, O \cap U = \varnothing$.

A space with the $\mathrm{T_2}$ property is also called a *Hausdorff* space (and $\mathrm{T_2}$ is also called the *Hausdorff property*). A space with the $\mathrm{T_3}$ property is also called a *regular* space. A space with the $\mathrm{T_{3\frac{1}{2}}}$ property is also called a *completely regular* space or a *Tychonoff* space. A space with the $\mathrm{T_4}$ property is also called a *normal* space. Some authors might not require for a space to be $\mathrm{T_1}$ in order to be regular, completely regular or normal. ♠

*Remark:*

The definition for a Tychonoff space might seem awkward right now, and the fact that it is associated with the $\mathrm{T_{3\frac{1}{2}}}$ property is not exactly helpful. The truth is this axiom is intermediate between $\mathrm{T_3}$ and $\mathrm{T_4}$ and the mysteriousness behind this definition shall vanish once we study Urysohn's Lemma (Lemma 3.109 on page 77). ♣

**Proposition 3.105:**

*Let $(X, \tau)$ be a topological space. $(X, \tau)$ is $T_1$ if, and only if, $\{x\}$ is closed $\forall\, x \in X$.* □

*Proof:*

$\Leftarrow$: Suppose $\{x\}$ is closed $\forall\, x \in X$. Let $x, y \in X, x \neq y$. Then $\{y\}^{\mathrm{c}}$ is an open set such that $x \in \{y\}^{\mathrm{c}}$, but $y \notin \{y\}^{\mathrm{c}}$. In a similar way, $x \notin \{x\}^{\mathrm{c}}$, but $y \in \{x\}^{\mathrm{c}}$. Thus, $(X, \tau)$ is $\mathrm{T_1}$.

$\Rightarrow$: Suppose $(X, \tau)$ is $\mathrm{T_1}$. Let $x \in X$. We want to prove that $\{x\}$ is a closed set.

We know that $\forall\, y \in X, y \neq x, \exists\, O_y \in \tau; y \in O_y, x \notin O_y$ (for $(X, \tau)$ is $\mathrm{T_1}$). Since the arbitrary union of open sets is an open set, $O \equiv \bigcup_{y \in X \setminus \{x\}} O_y$ is an open set. Notice that, since $y \in O_y, x \notin O_y, \forall\, y \in X \setminus \{x\}$, we have that $O = X \setminus \{x\}$. Since $O$ is an open set, $O^c = \{x\}$ is a closed set. This concludes the proof. ■

**Proposition 3.106:**
*Let $(X, \tau)$ be a Hausdorff topological space. Let $Y \subseteq X$ and let $\tau_Y$ be the relative topology on $Y$. $(Y, \tau_Y)$ is Hausdorff.* □

*Proof:*
Let $x, y \in Y, x \neq y$. We want to prove there are open sets $O, U \in \tau_Y$ such that $x \in O, y \in U, O \cap U = \varnothing$.

Since $(X, \tau)$ is Hausdorff and $x, y \in Y \subseteq X$, we see there are open sets $O_X, U_X \in \tau$ such that $x \in O_X, y \in U_X, O_X \cap U_X = \varnothing$. The sets $O = O_X \cap Y$ and $U = U_X \cap Y$ are open in $Y$ and satisfy $x \in O, y \in U, O \cap U = \varnothing$. This concludes the proof. ■

**Proposition 3.107:**
*Let $(X, \tau)$ be a topological space. Let $\Delta = \{(x, y) \in X \times X; x = y\}$. $(X, \tau)$ is Hausdorff if, and only if, $\Delta$ is closed in the product topology.* □

*Proof:*
Suppose $(X, \tau)$ is Hausdorff. We want to prove that given any $(x, y) \in \overline{\Delta}$, it holds that $x = y$. Due to Theorem 3.43 on page 45, we know that $(x, y) \in \overline{\Delta}$ if, and only if, $O_x \times O_y$ intersects $\Delta$ for all $O_x, O_y \in \tau$ with $x \in O_x, y \in O_y$, for $\mathfrak{B} = O \times U; O, U \in \tau$ is a basis for the product topology.

Let $(x, y) \in \overline{\Delta}$. Suppose $x \neq y$. The Hausdorff property ensures the existence of $O, U \in \tau$ such that $x \in O, y \in U, O \cap U = \varnothing$. However, we see that $(O \times U) \cap \Delta \neq \varnothing$. Thus, there is $z \in X$ such that $(z, z) \in O \times U$, *id est*, $z \in O \cap U$. This contradicts the fact that $O \cap U \neq \varnothing$, proving the assumption $x \neq y$ is false. Hence, $(x, y) \in \overline{\Delta} \Rightarrow x = y$, proving $\overline{\Delta} \subseteq \Delta$ and, as a consequence, that $\Delta$ is closed.

Suppose now that $\Delta$ is closed. It follows that $\overline{\Delta} = \Delta$, and hence $(x, y) \in \overline{\Delta} \Rightarrow x = y$. Theorem 3.43 on page 45 guarantees then that, if $x \neq y$, there are $O, U \in \tau$ with $x \in O, y \in \tau$ such that $O \times U$ does not intersect $\Delta$.

Pick $x, y \in X$. We know there are $O, U \in \tau$ with the properties that $x \in O, y \in U$ and $(w, z) \in O \times U \Rightarrow w \neq z, \forall (w, z) \in O \times U$. Thus, $w \in O \Rightarrow w \notin U$ and $z \in U \Rightarrow z \notin O$. Hence, $O \cap U = \varnothing$, proving $(X, \tau)$ is Hausdorff. ■

## 3.7 Urysohn's Lemma

We might now start studying some consequences of the countability and separation axioms. Sometimes, even regular spaces admit only the constant functions as continuous functions, but we shall see that normal spaces always have a good amount of continuous functions.

**Lemma 3.108:**
*Let $(X, \tau)$ be a normal topological space. Let $A, B \subseteq X$ be a pair of disjoint closed sets.*

*Consider the set of dyadic rational numbers in the interval* $[0,1]$, *given by*

$$\Delta = \left\{ \frac{k}{2^n}; n \in \mathbb{N}^*, k \in \{i\}_{i=0}^{2^n} \right\}. \tag{3.74}$$

*There is a collection* $\{U_r\}_{r\in\Delta}$ *of open sets in* $X$ *satisfying* $A \subseteq U_r \subseteq B^c, \forall\, r \in \Delta$ *with* $r < s \Rightarrow \overline{U}_r \subseteq U_s$. ☐

*Proof:*

We shall proceed by induction.

For $n = 1$, we have $r = 0$ and $r = 1$. Since $A$ and $B$ are disjoint closed sets in a normal space, there is a pair of disjoint open sets $O$ and $U$ satisfying $A \subseteq O$ and $B \subseteq U$. Lets us define $U_0 \equiv O$ and $U_1 \equiv B^c$. Indeed, since $O \cap U = \varnothing$, it holds that $O \cap B = \varnothing$ and, as a consequence, $O \subseteq B^c$. Thus, $A \subseteq U_0 \subseteq B^c$. We still must prove that $\overline{U}_0 \subseteq U_1 = B^c$.

$\overline{U}_0 \subseteq B^c$ if, and only if, $\overline{U}_0 \cap B = \varnothing$. Let $x \in B$. We know from Theorem 3.43 on page 45 that $x \in \overline{U}_0$ if, and only if, $U_0 \cap O_x \neq \varnothing$, for every neighborhood $O_x$ of $x$. However, since $B \subseteq U$ and $U$ is an open set satisfying $U \cap O = \varnothing$, $U$ is a neighborhood of $x$ that doesn't intersect $O = U_0$. Thus, $\overline{U}_0 \cap B = \varnothing$ and it holds that $\overline{U}_0 \subseteq U_1$.

We must now prove the inductive step. Suppose there are open sets $U_r$ satisfying our desires for every rational of the form $r = \frac{k}{2^{n-1}}$. We want to prove it holds for rationals of the form $r = \frac{k}{2^n}$ as well. If $k$ is an even number, then it holds that $r = \frac{l}{2^{n-1}}$, with $l = \frac{k}{2}$, and it is done by hypothesis. Let us bother with $k$ odd then. We know that for $r_0 = \frac{k-1}{2^n}$ and $r_1 = \frac{k+1}{2^n}$ there are open sets $U_{r_0}$ and $U_{r_1}$ with $A \subseteq U_{r_0} \subseteq \overline{U}_{r_0} \subseteq U_{r_1} \subseteq B^c$.

Consider now the closed sets $\overline{U}_{r_0}$ and $U_{r_1}^c$. Since $(X, \tau)$ is normal, there are open sets $O_0$ and $O_1$ such that $\overline{U}_{r_0} \subseteq O_0$, $U_{r_1}^c \subseteq O_1$ and $O_0 \cap O_1 = \varnothing$. We already see that $A \subseteq U_{r_0} \subseteq \overline{U}_{r_0} \subseteq O_0 \subseteq U_{r_1} \subseteq B^c$ (since $U_{r_1}^c \subseteq O_1$ and $O_0 \cap O_1 = \varnothing$, it must hold that $O_0 \subseteq U_{r_1}$). Let us now show that $\overline{O}_0 \subseteq U_{r_1}$.

Let $x \in U_{r_1}^c$. If $x \in \overline{O}_0$, then it must hold that $O_x \cap O_0 \neq \varnothing$ for every neighborhood $O_x$ of $x$. However, since $x \in U_{r_1}^c \in O_1$ and $O_1$ is an open set satisfying $O_1 \cap O_0$, there is a neighborhood of $x$ that doesn't intersect $O_0$. Thus, $x \notin \overline{O}_0$. We might conclude that $\overline{O}_0 \subseteq U_{r_1}$.

Finally, we might now set $U_r \equiv O_0$, for $r = \frac{k}{2^n}$, and the proof is complete. ■

**Lemma 3.109** [Urysohn's Lemma]:

*Let* $(X, \tau)$ *be a normal topological space. Let* $A, B$ *be a pair of closed sets in* $X$. *Then there is a continuous function* $f \colon X \to [0,1]$ *satisfying* $f(A) = \{0\}$ *and* $f(B) = \{1\}$. ☐

*Proof:*

Let $\Delta$ be the set of dyadic rational numbers in the interval $[0,1]$. Due to Lemma 3.108 on the facing page, we know that there is a collection $\{U_r\}_{r\in\Delta}$ of open sets in $X$ satisfying $A \subseteq U_r \subseteq B^c, \forall\, r \in \Delta$ with $r < s \Rightarrow \overline{U}_r \subseteq U_s$. Let us define a function $f$ by

$$\begin{cases} f(x) = 0, \forall\, x \in U_0, \\ f(x) = \sup\limits_{t \notin U_r} r, \forall\, x \notin U_0. \end{cases} \tag{3.75}$$

Since $A \subseteq U_0$ and $f(U_0) = \{0\}$, it holds that $f(A) = \{0\}$. Furthermore, since $x \in B \Rightarrow x \notin U_r, \forall\, r \in [0,1]$, it holds that $f(B) = \{1\}$. Furthermore, it is simple to see that $f(X) \subseteq [0,1]$. We now must prove that $f$ is indeed a continuous function.

Let $\alpha \in [0,1]$. Notice that $f(x) < \alpha$ if, and only if, $x \in U_r$ for some $r < \alpha$, *id est*, if, and only if, $x \in \bigcup_{r<\alpha} U_r$. Thus, $f^{-1}\big((\alpha, +\infty)\big) = \bigcup_{r<\alpha} U_r$, which is an arbitrary union of open sets, and therefore is itself open.

On the other hand, if we let $\beta \in [0,1]$, $f(x) > \beta$ if, and only if, $x \in U_r^{\mathsf{c}}$ for some $r > \beta$. However, since $\overline{U}_s \subseteq U_r$ for very $s < r$, this happens if, and only if, $x \in \overline{U}_s^{\mathsf{c}}$ for some $s > \beta$, *id est*, if, and only if, $x \in \bigcup_{s>\beta} \overline{U}_s^{\mathsf{c}}$. Thus, $f^{-1}\big((-\infty, \beta)\big) = \bigcup_{s>\beta} \overline{U}_s^{\mathsf{c}}$, which is an arbitrary union of open sets, and therefore it is open as well.

Finally, notice that $f^{-1}\big((\alpha,\beta)\big) = f^{-1}\big((\alpha,+\infty)\big) \cap f^{-1}\big((-\infty,\beta)\big)$, which is a finite intersection of open sets, and thus an open set itself. Since the intervals comprise a basis for the standard topology on $\mathbb{R}$ and the preimage of the intervals are always open under $f$, it holds that $f$ is indeed continuous. ■

**Corollary 3.110:**

*Let $(X, \tau)$ be a normal topological space. Let $A, B$ be a pair of closed sets in $X$. Let $a, b \in \mathbb{R}, a < b$. Then there is a continuous function $f\colon X \to [a,b]$ satisfying $f(A) = \{a\}$ and $f(B) = \{b\}$.* □

*Proof:*

Due to Urysohn's Lemma, we know there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Since we may add and multiply continuous functions by other continuous functions without altering their continuity, let us consider the continuous function given by $g(x) = (b-a)f(x) + a$. Notice that $g(A) = \{a\}$ and $g(B) = \{b\}$. This concludes the proof. ■

*Scholium:*

We shall also refer to both Lemma 3.109 and Corollary 3.110 on the previous page and on this page as Urysohn's Lemma. ♣

**Theorem 3.111** [Tietze Extension Theorem]**:**

*Let $(X, \tau)$ be a normal topological space. Let $A \subseteq X$ be a closed set and let $f\colon A \to [a,b]$ be a continuous function. There is a continuous function $F\colon X \to [a,b]$ such that $F(x) = f(x), \forall\, x \in A$.* □

*Proof:*

Let us first bother with the case $f\colon A \to [a,b]$. The remaining cases shall be proved as consequences. We can assume $[a,b] = [0,1]$. Indeed, we would be able to extend $\frac{f-a}{b-a}$ to a function $\frac{F-a}{b-a}$ and $F$ is now the function we were looking for. Thus, we assume for simplicity that $[a,b] = [0,1]$.

I claim there is a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ on $X$ satisfying $0 \leq g_n(x) \leq \frac{2^{n-1}}{3^n}, \forall\, x \in X$, and $0 \leq f(x) - \sum_{k=1}^{n} g_k(x) \leq \left(\frac{2}{3}\right)^n, \forall\, x \in A$.

Let us consider the sets $B = f^{-1}\big(\big[0, \frac{1}{3}\big]\big)$ and $C = f^{-1}\big(\big[\frac{2}{3}, 1\big]\big)$. These sets are closed as subsets of $A$ (for $f$ is continuous and closed intervals are closed in the standard topology of the real line). Since $A$ is closed in $X$, $B$ and $C$ are closed in $X$ (Corollary 3.28 on page 36). Since $(X, \tau)$ is a normal space, Urysohn's Lemma guarantees the existence of a function $g_1 \colon X \to [0, \frac{1}{3}]$ with $g_1(B) = \{0\}$ and $g_1(C) = \left\{\frac{1}{3}\right\}$. As a consequence, $0 \leq f(x) - g_1(x) \leq \frac{2}{3}, \forall\, x \in A$.

Let us now consider the continuous function $h(x) = f(x) - g_1(x), \forall\, x \in A$. Notice that $h \colon A \to \left[0, \frac{2}{3}\right]$. We (re-)define $B = h^{-1}\left(\left[0, \frac{2^{2-1}}{3^2}\right]\right)$ and $C = h^{-1}\left(\left[\frac{2^2}{3^2}, \frac{2^{2-1}}{3^{2-1}}\right]\right)$. Once again it holds that $B$ and $C$ are closed sets in $X$ and Urysohn's Lemma guarantees the existence of a continuous function $g_2 \colon X \to \left[0, \frac{2^{2-1}}{3^2}\right]$ with $g_2(B) = \{0\}$ and $g_2(C) = \left\{\frac{2^{2-1}}{3^2}\right\}$. Now we have that $0 \leq f(x) - g_1(x) - g_2(x) \leq \left(\frac{2}{3}\right)^2$.

In general, we consider the continuous function $h(x) = f(x) - \sum_{k=1}^{n-1} g_k(x), \forall\, x \in A$. It holds that $h \colon A \to \left[[0, \left(\frac{2}{3}\right)^{n-1}]\right]$. We (re-)define $B = h^{-1}\left(\left[0, \frac{2^{n-1}}{3^n}\right]\right)$ and $C = h^{-1}\left(\left[\frac{2^n}{3^n}, \frac{2^{n-1}}{3^{n-1}}\right]\right)$. Urysohn's Lemma guarantees the existence of a continuous function $g_n \colon X \to \left[0, \frac{2^{n-1}}{3^n}\right]$ with $g_n(B) = \{0\}$ and $g_n(C) = \left\{\frac{2^{n-1}}{3^n}\right\}$. We finally have that $0 \leq f(x) - \sum_{k=1}^{n} g_k(x) \leq \left(\frac{2}{3}\right)^n$.

Let we define $F_n(x) = \sum_{k=1}^{n} g_k(x), \forall\, x \in X$. Let $d$ denote the uniform metric on $BC(X)$. Let now $\epsilon > 0$. We know from Real Analysis that there is $n_0 \in \mathbb{N}$ such that, $\forall\, n, m > 0$ (let us suppose, without any loss of generality, that $m \geq n$),

$$
\begin{aligned}
d(F_n, F_m) &\leq d(F_n, 0) + d(F_m, 0), \\
&= \sup_{x \in X} |F_n(x)| + \sup_{x \in X} |F_m(x)|, \\
&\leq 2\left(\frac{2}{3}\right)^n, \\
&< \epsilon.
\end{aligned} \tag{3.76}
$$

Therefore, $(F_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $BC(X)$, which is a complete metric space when equipped with the uniform metric (Proposition 3.91 on page 69). Therefore, there is a function $F \in BC(X)$ such that $F(x) = \sum_{k=1}^{+\infty} g_k(x), \forall\, x \in X$.

Notice that $\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; \forall\, n > n_0, \forall\, x \in A, |f(x) - F_n(x)| \leq \left(\frac{2}{3}\right)^n < \epsilon$. Therefore, $F_n \to f$ and it holds that $F(x) = f(x), \forall\, x \in A$.

We still must prove that $F \colon X \to [0, 1]$. Since $g_k(x) \geq 0, \forall\, k \in \mathbb{N}^*, \forall\, x \in X$, of course $F(x) \geq 0, \forall\, x \in X$.

In order to force the extension to be limited to $[0, 1]$, we may define a new extension which surely respects that and use it as the extension. We already know that $F$ is bounded (for $F \in BC(X)$), and thus we just need to re-scale it in a manner that preserves its continuity and without altering its value in $A$.

Consider the map $r \colon \mathbb{R}_+ \to [0, 1]$ given by

$$
\begin{cases}
r(x) = x, \text{ if } x \leq 1, \\
r(x) = 1, \text{ if } x > 1.
\end{cases} \tag{3.77}
$$

This map is continuous when $\mathbb{R}_+$ and $[0,1]$ are equipped with the relative topology (with respect to the standard topology in $\mathbb{R}$). Indeed, $r^{-1}([0,a)) = [0,a)$, which is open in $\mathbb{R}_+$. In a similar manner, $r^{-1}((a,1]) = (a,+\infty)$, which is open as well. Finally, $r^{-1}((a,b)) = (a,b)$, which is also open. These intervals are a basis in $[0,1]$ due to Lemma 3.24 on page 34, and therefore $r$ is continuous. As a consequence, the composition $F' \equiv r \circ F$ is continuous as well. Notice that $F(x) \in [0,1], \forall\, x \in A$, and thus $F'$ is also an extension of $f$.

This concludes the proof. ■

**Corollary 3.112:**

*Let $(X,\tau)$ be a normal topological space. Let $A \subseteq X$ be a closed set and let $f\colon A \to \mathbb{R}$ be a continuous function. There is a continuous function $F\colon X \to \mathbb{R}$ such that $F(x) = f(x), \forall\, x \in A$.* □

*Proof:*

Since $\mathbb{R}$ and $(-1,1)$ are homeomorphic (Proposition 3.65 on page 57), we might instead consider a function $f\colon A \subseteq (-1,1)$. The result will hold by composing the functions $f$ and $F$ with any homeomorphism between $\mathbb{R}$ and $(-1,1)$.

Due to the Tietze Extension Theorem, we know that there is a function $g\colon X \to [-1,1]$ which extends $f$ continuously. We want to find a function $h\colon X \to (-1,1)$ which extends $f$.

With $g$ given by the Tietze Extension Theorem, we define the set $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$. $g$ is continuous and $\mathbb{R}$ is $T_1$*, and therefore it holds that $D$ is closed. Since $g$ extends $f$ and $f(A) = (-1,1)$, it holds that $D \cap A = \varnothing$. Urysohn's Lemma then allows us to obtain a continuous function $\varphi\colon X \to [0,1]$ with $\varphi(D) = \{0\}$ and $\varphi(A) = \{1\}$. We might now define $h(x) \equiv \varphi(x)g(x)$. This is a product of continuous functions, and thus it continuous itself. Furthermore, if $x \in A$, $h(x) = \varphi(x)g(x) = g(x) = f(x)$, and therefore $h$ extends $f$. Notice that $h\colon X \to (-1,1)$, for $h(D) = \{0\}$ and $h(D^c) \in (-1,1)$. ■

**Corollary 3.113:**

*Let $(X,\tau)$ be a normal topological space. Let $A \subseteq X$ be a closed set and let $f\colon A \to \mathbb{C}$ be a continuous function. There is a continuous function $F\colon X \to \mathbb{C}$ such that $F(x) = f(x), \forall\, x \in A$.* □

*Proof:*

It suffices to consider real and imaginary parts separately, for a function is continuous if, and only if, the coordinate functions are continuous as well. We are now left with the same problem for real functions, which was solved in Corollary 3.112. ■

## 3.8 Nets

When dealing with Topology, the notion of sequence isn't always appropriate, due to the generality of the spaces we deal with. In order to study them, we first must define what

*This follows from the fact that $\mathbb{R}$ is a metric space and we have already proven that every metric space is Hausdorff. The proof that every Hausdorff space is $T_1$ is straightforward.

is a directed system.

**Definition 3.114** [Partial Orderings and Posets]**:**
Let $\prec$ be a relation on a set $X$. $\prec$ is said to be a *partial ordering* if, and only if, the following properties hold:

i. $\forall\, x \in X, x \prec x$ (reflexive);

ii. $\forall\, x, y, z \in X, x \prec y$ and $y \prec z \Rightarrow x \prec z$ (transitive);

iii. $\forall\, x, y \in X, x \prec y$ and $y \prec x \Rightarrow x = y$ (antisymmetric).

The pair $(X, \prec)$, where $X$ is a set and $\prec$ is a partial ordering, is said to be a *partially ordered set* or a *poset*. ♠

*Notation:*
We might write $y \succ x$ to mean $x \prec y$. Both statements are equivalent. ♦

**Definition 3.115** [Directed System]**:**
Let $I$ be a set and let $\prec$ be a partial ordering on $I$. $(I, \prec)$ is said to be a *directed system*, or *directed set*, if, and only if, $\forall\, \alpha, \beta \in I, \exists\, \gamma \in I; \alpha \prec \gamma, \beta \prec \gamma$. ♠

**Proposition 3.116:**
*Let $(X, \tau)$ be a topological space and let $x \in X$. Let $I = \{O \in \tau; x \in O\}$ and let $O \prec U$ if, and only if, $U \subseteq O$. $(I, \prec)$ is a directed system.* □

*Proof:*
Since $O \subseteq O, \forall\, O \in I$, $\prec$ is reflexive.
If $O \prec U$ and $U \prec V$, then we know that $U \subseteq O$ and $V \subseteq U$ and it follows that $V \subseteq O$, *id est*, $O \prec V$. Thus, $\prec$ is transitive.
If $O \prec U$ and $U \prec O$, we see that $U \subseteq O$ and $O \subseteq U$. Therefore, $O = U$ and we see that $\prec$ is antisymmetric.
Let now $O, U \in I$. The set $O \cap U$ is open and, since $x \in O$ and $x \in U$, it holds that $x \in O \cap U$. Therefore, $O \cap U \in I$. Notice that $O \cap U \succ O$ and $O \cap U \succ U$. Therefore, it holds that $\forall\, O, U \in I, \exists\, V = O \cap U; O \prec V, U \prec V$. ■

**Proposition 3.117:**
*Let $(I, \prec_I)$ and $(J, \prec_J)$ be directed systems. Consider the set $I \times J$ with the relation $(\alpha, \beta) \prec (\alpha', \beta') \Leftrightarrow \alpha \prec_I \alpha'$ and $\beta \prec_J \beta'$. $(I \times J, \prec)$ is a directed system.* □

*Proof:*
Let $\alpha, \alpha' \in I, \beta, \beta' \in J$.
It is simple to show that the properties of a partial order are inherited from $\prec_I$ and $\prec_J$.
We know that there are $\gamma \in I$ and $\delta \in J$ with $\gamma \succ \alpha, \gamma \succ \alpha', \delta \succ \beta, \delta \succ \beta'$. Therefore, $(\alpha, \beta) \prec (\gamma, \delta)$ and $(\alpha', \beta') \prec (\gamma, \delta)$. ■

**Definition 3.118** [Net]**:**
Let $(X, \tau)$ be a topological space and let $(I, \prec)$ be a directed system. A *net* in $(X, \tau)$ is a function $x\colon I \to X$. ♠

*Notation:*
As when dealing with sequences, instead of writing $x(\alpha)$ for the image of $\alpha \in I$ through a net $x$, it is usual to write simply $x_\alpha$. It is also customary to write $(x_\alpha)_{\alpha \in I}$ for the net, instead of $x$. ♦

*Remark:*
Notice that $(\mathbb{N}, \leq)$, where $\leq$ denotes the usual order in the natural numbers, is a directed system, and as a consequence every sequence is a net. We are simply making the theory more general. ♣

We might now make the terms frequently and eventually, which we already defined when dealing with sequences, more general.

**Definition 3.119** [Frequently and Eventually]**:**
Let $(I, \prec)$ be a directed system. Let $A \subseteq X$. Let $(x_\alpha)_{\alpha \in I}$ be a net. $(x_\alpha)_{\alpha \in I}$ is said to be *eventually* on $A$ if, and only if, $\exists\, \beta \in I; x_\alpha \in A\, \forall\, \alpha \succ \beta$. $(x_\alpha)_{\alpha \in I}$ is said to be *frequently* in $A$ if, and only if, $\forall\, \beta \in I, \exists\, \alpha \succ \beta; x_\alpha \in A$. ♠

**Definition 3.120** [Cluster Point]**:**
Let $(X, \tau)$ be a topological space, $(I, \prec)$ be a directed system and $(x_\alpha)_{\alpha \in I}$ be a net of elements of $X$. A point $x \in X$ is said to be a *cluster point* of the net $(x_\alpha)_{\alpha \in I}$ with respect to the topology $\tau$ if, and only if, $(x_\alpha)_{\alpha \in I}$ is frequently in every neighborhood of $x$. ♠

**Definition 3.121** [Limit Point]**:**
Let $(X, \tau)$ be a topological space, $(I, \prec)$ be a directed system and $(x_\alpha)_{\alpha \in I}$ be a net of elements of $X$. A point $x \in X$ is said to be an *limit point* (sometimes called simply *limit*) of the net $(x_\alpha)_{\alpha \in I}$ with respect to the topology $\tau$ if, and only if, $(x_\alpha)_{\alpha \in I}$ is eventually in every neighborhood of $x$. ♠

*Notation:*
If a net $(x_\alpha)_{\alpha \in I}$ has a point $x$ as a limit point, we write $x_\alpha \to x$ and say that $(x_\alpha)_{\alpha \in I}$ converges to $x$. ♦

Definitions are always nice, but they seem pointless when ill-motivated. As a consequence, let us prove some results concerning nets as soon as possible in order to keep our interests clear.

**Proposition 3.122:**
*Let $(X, \tau)$ be a topological space and let $A \subseteq X$. $x \in \overline{A}$ if, and only if, there is a net $(x_\alpha)_{\alpha \in I}$ of elements of $A$ with $x_\alpha \to x$.* □

*Proof:*

⇐: Suppose there is a net $(x_\alpha)_{\alpha \in I}$ of elements of $A$ with $x_\alpha \to x$. This means that, for any neighborhood $O$ of $x$, $\exists\, \beta \in I; x_\alpha \in O, \forall\, \alpha \succ \beta$. Since $x_\alpha \in A, \forall\, \alpha \in I$, this means that for any neighborhood $O$ of $x$ it holds that $O \cap A \neq \varnothing$. Thus, Theorem 3.43 on page 45 implies that $x \in \overline{A}$.

⇒: Suppose $x \in \overline{A}$. We know from Proposition 3.116 on page 81 that if we define $I = \{O \in \tau; x \in O\}$ and equip it with the partial ordering $O \prec U \Leftrightarrow U \subseteq O$, then $(I, \prec)$ is a directed system. Consider a net $(x_O)_{O \in I}$ such that $\forall\, O \in I$, $x_O \in O \cap A$.

Is there such a net? Since $x \in \overline{A}$, every neighborhood of $x$ - *id est*, every element of $I$ - intersects $A$. Therefore, none of the sets $O \cap A$ is empty. $I$ isn't empty either, for $X \in I$. Therefore, the Axiom of Choice guarantees the existence of such a net.

I claim that this net satisfies $x_O \to x$. Let $O$ be a neighborhood of $x$ (*id est*, let $O \in I$). By construction, $x_O \in O$. Furthermore, $x_U \in U \subseteq O, \forall\, U \succ O$. Therefore, $x$ is indeed a limit point of $x_O$. This concludes the proof. ■

**Theorem 3.123:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and $f \colon X \to Y$ be a function. $f$ is continuous at $x \in X$ if, and only if, the net $(f(x_\alpha))_{\alpha \in I}$ converges to $f(x)$ for every net $(x_\alpha)_{\alpha \in I}$ converging to $x$.* □

*Proof:*

⇒: Suppose $f$ is continuous at $x$. Let $(x_\alpha)_{\alpha \in I}$ be a net converging to $x$. This means that for every neighborhood $O$ of $x$, $\exists\, \beta \in I; x_\alpha \in O, \forall\, \alpha \succ \beta$.

Let $O$ be a neighborhood of $f(x)$. Since $f$ is continuous, there is some open $U$ with $x \in U \subseteq f^{-1}(O)$. Therefore, $\exists\, \beta \in I; x_\alpha \in U, \forall\, \alpha \succ \beta$. Notice that if $x_\alpha \in U \subseteq f^{-1}(O)$, then $f(x_\alpha) \in O$. Therefore, given a neighborhood $O$ of $f(x)$, $\exists\, \beta \in I; f(x_\alpha) \in O, \forall\, \alpha \succ \beta$. Thus, $f(x_\alpha) \to x$.

⇐: We shall prove the contrapositive affirmation. Suppose $f$ is not continuous at $x$. Then there is some neighborhood $O$ of $f(x)$ for which there are no neighborhoods $U$ of $x$ satisfying $x \in U \subseteq f^{-1}(O)$. Therefore, $x \notin \overset{\circ}{f^{-1}(O)}$, *id est*, $x \in \overline{f^{-1}(O^c)}$. Proposition 3.122 on the facing page guarantees that there is a net $(x_\alpha)_{\alpha \in I}$ of elements of $f^{-1}(O^c)$ converging to $x$. However, since $x_\alpha \in f^{-1}(O^c), \forall\, \alpha \in I$, it holds that $f(x_\alpha) \in O^c, \forall\, \alpha \in I$. Therefore, no element of the net $(f(x_\alpha))_{\alpha \in I}$ is ever in $O$, which is a neighborhood of $f(x)$. Therefore, we have found a net converging to $x$, but whose image does not converge to $f(x)$. ■

**Proposition 3.124:**

*Let $(X, \tau)$ be a topological space. $(X, \tau)$ is Hausdorff if, and only if, every net $(x_\alpha)_{\alpha \in I}$ in $X$ admits at most one limit point.* □

*Proof:*

$\Rightarrow$: Assume $(X, \tau)$ is a Hausdorff space. Suppose $x \in X$ is a limit point for $(x_\alpha)_{\alpha \in I}$. Let $y \in X, y \neq x$. Since $(X, \tau)$ is a Hausdorff space, there are two disjoint open sets $O, U$ satisfying $x \in O$, $y \in U$. Since $x_\alpha \to x$, we know there is $\beta \in I; x_\alpha \in O, \forall \alpha \succ \beta$. Since $O \cap U = \varnothing$, this implies that $\forall \alpha \succ \beta, x_\alpha \notin U$. Therefore, $x_\alpha$ cannot be eventually in $U$. As a consequence, $y$ can't be a limit point of $(x_\alpha)_{\alpha \in I}$.

$\Leftarrow$: We shall prove the contrapositive statement. Assume $(X, \tau)$ is not Hausdorff. Then there are two distinct points $x, y \in X$ with no disjoint neighborhoods. Let $\mathfrak{N}_x \equiv \{O \in \tau; x \in O\}$ and $\mathfrak{N}_y \equiv \{O \in \tau; y \in O\}$. We now these sets, when equipped with the reverse inclusion order, are directed systems (Proposition 3.116 on page 81). Therefore, we might use Proposition 3.117 on page 81 to consider $\mathfrak{N}_x \times \mathfrak{N}_y$ as a directed system, with the order given by $(\alpha, \beta) \prec (\alpha', \beta') \Leftrightarrow \alpha \prec_x \alpha'$ and $\beta \prec_y \beta'$.

By hypothesis, we know that given $O \in \mathfrak{N}_x, U \in \mathfrak{N}_y$, it holds that $O \cap U \neq \varnothing$. Furthermore, since $X$ is an open set, $\mathfrak{N}_x \times \mathfrak{N}_y$ is not empty. Therefore, the Axiom of Choice guarantees the existence of a net $\left(x_{(O,U)}\right)_{(O,U) \in \mathfrak{N}_x \times \mathfrak{N}_y}$ with $x_{(O,U)} \in O \cap U, \forall O \in \mathfrak{N}_x, U \in \mathfrak{N}_y$. Notice that such a sequence converges both to $x$ and to $y$.

Indeed, let $O$ be a neighborhood of $x$. Given any neighborhood $U$ of $y$, we know that $x_{(O,U)} \in O$. Furthermore, since $(O, U) \prec (O', U')$ if, and only if, $O' \subseteq O$ and $U' \subseteq U$, we have that whenever $(O, U) \prec (O', U')$, it holds that $x_{(O',U')} \in O' \cap U' \subseteq O \cap U \subseteq O$. Thus, the net is eventually in $O$ and it holds that $x$ is a limit point for it. The same argument applies to $y$, and therefore $\left(x_{(O,U)}\right)_{(O,U) \in \mathfrak{N}_x \times \mathfrak{N}_y}$ admits more than one limit. ■

We might now give a wider notion of a subsequence* by defining a subnet.

**Definition 3.125** [Subnet]**:**
Let $(X, \tau)$ be a topological space and let $(x_\alpha)_{\alpha \in I}$ be a net in $X$. We say a net $\left(y_\beta\right)_{\beta \in J}$ is a *subnet* of $(x_\alpha)_{\alpha \in I}$ whenever there is a function $\beta \mapsto \alpha_\beta$ respecting the following requirements:

i. $y_\beta = x_{\alpha_\beta}$;

ii. $\forall \alpha_0 \in I, \exists \beta_0 \in J; \alpha_\beta \succ \alpha_0, \forall \beta \succ \beta_0$. ♠

*Remark:*
Notice that we don't ask for the mapping $\beta \mapsto \alpha_\beta$ to be injective. If $\left(y_\beta\right)_{\beta \in J}$ is a subnet of $(x_\alpha)_{\alpha \in I}$, it might still hold that the cardinality of $J$ is larger than the cardinality of $I$, for example. A subnet of a sequence is not necessarily a subsequence. In fact, it is possible for a sequence to have no convergent subsequences, but still have convergent subnets. ♣

*Subsequences play an important role in the theory of Metric Spaces, but they are also not so interesting when dealing with Topology

Due to these complications, the definition of a subnet might seem useless or unnecessarily difficult, but it isn't. The following result guarantees that we are indeed working with an appropriate definition, for it generalizes a similar result found in metric spaces.

**Theorem 3.126:**
*Let $(X, \tau)$ be a topological space and $(x_\alpha)_{\alpha \in I}$ be a net in such space. A point $x \in X$ is a cluster point of X if, and only if, $(x_\alpha)_{\alpha \in I}$ admits a subnet $\left(y_\beta\right)_{\beta \in J}$ with $y_\beta \to x$.* □

*Proof:*

- $\Leftarrow$: Suppose $(x_\alpha)_{\alpha \in I}$ admits a subnet $\left(y_\beta\right)_{\beta \in J}$ with $y_\beta \to x$. We want to prove that, if $O$ is a neighborhood of $x$, $\forall\, \alpha \in I, \exists\, \beta \succ \alpha; x_\beta \in O$. Since $y_\beta \to x$, we know that $\exists\, \gamma \in J; y_\beta \in O, \forall\, \beta \succ \gamma$.

  Let $\alpha_0 \in I$. There is $\beta_0 \in J$ such that $\alpha_\beta \succ \alpha_0, \forall\, \beta \succ \beta_0$. Let $\gamma_0 \in J; y_\beta \in O, \forall\, \beta \succ \gamma_0$. We know there is $\delta \in J; \delta \succ \gamma_0$ and $\delta \succ \beta_0$. Since $\delta \succ \beta_0, \alpha_\delta \succ \alpha_0$. Since $\delta \succ \gamma_0, x_{\alpha_\delta} = y_\delta \in O$. Therefore, $\forall\, \alpha_0 \in I, \exists\, \alpha_\delta \in I; x_{\alpha_\delta} \in O$. This proves that $x$ is a cluster point of $(x_\alpha)_{\alpha \in I}$.

- $\Rightarrow$: Let us assume $x$ is a cluster point of $(x_\alpha)_{\alpha \in I}$. Then, for every neighborhood $O$ of $x$, it holds that $\forall\, \alpha \in I, \exists\, \beta \succ \alpha; x_\beta \in O$.

  Let $\mathfrak{N} = \{O \in \tau; x \in O\}$. Propositions 3.116 and 3.117 on page 81 allow us to consider $\mathfrak{N} \times I$ as a directed system. Let us choose, $\forall\, (O, \gamma) \in \mathfrak{N} \times I$, $\alpha_{(O,\gamma)} \in I$ such that $\alpha_{(O,\gamma)} \succ \gamma$ and $x_{\alpha_{(O,\gamma)}} \in O$. It is possible to make this choice due to the fact that $x$ is a cluster point of $(x_\alpha)_{\alpha \in I}$: $\forall\, \gamma \in I, \exists\, \beta \succ \gamma; x_\beta \in O$.

  Consider the net $\left(y_{(O,\gamma)}\right)_{(O,\gamma) \in \mathfrak{N} \times I}$ given by $y_{(O,\gamma)} = x_{\alpha_{(O,\gamma)}}$. Notice that $\forall\, \alpha_0 \in I$, one might pick any $O \in \mathfrak{N}$ and have $\alpha_{(O,\gamma)} \succ \alpha_0, \forall\, \gamma \succ \alpha_0$ (for $\alpha_{(O,\gamma)} \succ \gamma$ by definition). Therefore, $\left(y_{(O,\gamma)}\right)_{(O,\gamma) \in \mathfrak{N} \times I}$ is a subnet of $(x_\alpha)_{\alpha \in I}$.

  We want to prove that $y_{(O,\gamma)} \to x$. Let $O \in \mathfrak{N}$ and $\gamma \in I$. If $(U, \delta) \succ (O, \gamma)$, then it holds that $U \subseteq O$ and $\delta \succ \gamma$. Therefore, $y_{(U,\delta)} = x_{\alpha_{(U,\delta)}} \in U \subseteq O$. ■

## 3.9 Compactness

Once again, we are going to impose extra conditions on topological spaces with the intention of obtaining a richer theory.

**Definition 3.127** [Open Cover and Subcover]**:**
Let $(X, \tau)$ be a topological space. Let $\mathcal{A} \subset \tau$. We say $\mathcal{A}$ is an *open cover* of $X$ if, and only if, $\bigcup_{O \in \mathcal{A}} O = X$. A family $\mathcal{S} \subseteq \mathcal{A}$ is said to be a *subcover* of $\mathcal{A}$ if it is also an open cover. ♠

**Definition 3.128** [Compact Spaces and Subsets]**:**
Let $(X, \tau)$ be a topological space. $X$ is said to be a *compact space* if, and only if, every

open cover of $X$ admits a finite subcover. We might also say that a subset $A \subseteq X$ is a *compact subspace* if it is a compact space when equipped with the relative topology. ♠

**Proposition 3.129:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Let $f\colon X \to Y$ be a homeomorphism. $X$ is compact if, and only if, $Y$ is compact.* □

*Proof:*

Suppose $X$ is compact. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $Y$. Then $\{f^{-1}(A_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of $X$. Indeed,

$$\begin{aligned}
Y &\subseteq \bigcup_{\lambda\in\Lambda} A_\lambda, \\
f^{-1}(Y) &\subseteq f^{-1}\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right), \\
X &\subseteq \bigcup_{\lambda\in\Lambda} f^{-1}(A_\lambda),
\end{aligned} \tag{3.78}$$

where we used that $f^{-1}(Y) = X$ (for $f$ is a bijection). Notice that the sets $f^{-1}(A_\lambda)$ are indeed open, for $f$ is continuous.

Since $X$ is compact, we know there is $n \in \mathbb{N}$ and $\lambda_k \in \Lambda, k \in \{k\}_{k=1}^n$ such that $X \subseteq \bigcup_{k=1}^n f^{-1}(A_{\lambda_n})$. It follows then that

$$\begin{aligned}
X &\subseteq \bigcup_{k=1}^n f^{-1}(A_{\lambda_n}), \\
f(X) &\subseteq f\left(\bigcup_{k=1}^n f^{-1}(A_{\lambda_n})\right), \\
Y &\subseteq \bigcup_{k=1}^n f\left(f^{-1}(A_{\lambda_n})\right), \\
Y &\subseteq \bigcup_{k=1}^n A_{\lambda_n},
\end{aligned} \tag{3.79}$$

where the manipulations were possible due to the fact that $f$ is bijective. Since we found a finite subcover of $\{A_\lambda\}_{\lambda\in\Lambda}$, $Y$ is compact.

If we assumed $Y$ was compact, the same argument with $f^1$ instead of $f$ would prove $X$ is compact. This concludes the proof. ■

The definition of compactness can also be cast in a different form.

**Definition 3.130** [Finite Intersection Property]**:**

Let $(X, \tau)$ be a topological space. It is said to have the *finite intersection property* - also known as f.i.p. - if, and only if, given an arbitrary family of indices $\Lambda$ and a family of closed sets $\{F_\lambda\}_{\lambda\in\Lambda}$ such that

$$\bigcap_{k=1}^{n} F_{\lambda_k} \neq \varnothing, \tag{3.80}$$

$\forall\, n \in \mathbb{N}$ and for any choice of $\lambda_k \in \Lambda, k \in \{i\}_{i=1}^{n}$, it holds that

$$\bigcap_{\lambda\in\Lambda} F_\lambda \neq \varnothing, \tag{3.81}$$

*id est*, if the intersections of finitely many elements of the family are non-empty, then the intersection of the whole family is also non-empty. ♠

**Theorem 3.131:**
*A topological space is compact if, and only if, it has the finite intersection property.* □

*Proof:*

$\Leftarrow$: Suppose $(X, \tau)$ is a topological space endowed with the finite intersection property. Let $A_{\lambda\lambda\in\Lambda}$ be an open cover of $X$. We want to prove it admits a finite subcover.

Notice that

$$\begin{aligned}\bigcup_{\lambda\in\Lambda} A_\lambda &= X,\\ \bigcap_{\lambda\in\Lambda} A_\lambda^{\mathsf{c}} &= \varnothing.\end{aligned} \tag{3.82}$$

Since $\{A_\lambda^{\mathsf{c}}\}_{\lambda\in\Lambda}$ is a family of closed sets with empty intersection and $(X, \tau)$ has the f.i.p., there has to be some $n \in \mathbb{N}$ and $\lambda_k \in \Lambda, k \in \{i\}_{i=1}^{n}$ such that

$$\bigcap_{k=1}^{n} A_{\lambda_k}^{\mathsf{c}} = \varnothing. \tag{3.83}$$

Otherwise, the finite intersection property would imply the intersection of the whole family is non-empty, which is absurd.

Notice now that

$$\begin{aligned}\bigcap_{k=1}^{n} A_{\lambda_k}^{\mathsf{c}} &= \varnothing,\\ \bigcup_{k=1}^{n} A_{\lambda_k} &= X,\end{aligned} \tag{3.84}$$

and thus we have found a finite subcover of $\{A_\lambda\}_{\lambda\in\Lambda}$.

$\Rightarrow$: Suppose now that $(X, \tau)$ is a compact space. The contrapositive of the finite intersection property can be proven by essentially reversing the steps taken to prove that the f.i.p. implies compactness.

■

**Proposition 3.132:**

*Let $(X, \tau)$ be a compact space. If $F \subseteq X$ is closed, it is compact.* □

*Proof:*

Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets in $F$ such that $\bigcap_{\lambda \in \Lambda} A_\lambda = \varnothing$. Due to Corollary 3.28 on page 36, we know $\{A_\lambda\}_{\lambda \in \Lambda}$ is a collection of closed sets with respect to the topology on $X$ as well. Since $(X, \tau)$ is compact, Theorem 3.131 on the previous page implies it has the f.i.p. Therefore, we know there must exist $n \in \mathbb{N}, \lambda_k \in \Lambda, k \in \{i\}_{i=1}^n$ such that $\bigcap_{k=1}^n A_{\lambda_k} = \varnothing$. Thus, $F$ also respects the f.i.p., and Theorem 3.131 on the preceding page guarantees its compactness. ■

**Lemma 3.133:**

*Let $(X, \tau)$ be a Hausdorff space. If $K \subseteq X$ is compact and $x \notin K$, then there are disjoint open sets $O$ and $U$ such that $K \subseteq U, x \in O$.* □

*Proof:*

Let $y \in K$. The Hausdorff property guarantees the existence of disjoint open sets $O_y$ and $U_y$ such that $y \in U_y, x \in O_y$. Notice that the collection $\{U_y\}_{y \in K}$ defined in this manner is an open cover of $K$. Thus, since $K$ is compact, there is $n \in \mathbb{N}, y_k \in K, k \in \{i\}_{i=1}^n$ such that $\{U_{y_k}\}_{k=1}^n$ is an open cover of $K$. Since arbitrary unions of open sets are open sets, $U = \bigcup_{k=1}^n U_{y_k}$ is an open set containing $K$.

Notice now that the set $O = \bigcap_{k=1}^n O_{y_k}$ is an open set - for it is a finite intersection of open sets - and it contains $x$ - for $x \in O_y, \forall\, y \in K$.

Finally, notice now that $O \cap U = \varnothing$. Since $U_y \cap O_y = \varnothing, \forall\, y \in K$, and $O \subseteq O_{y_k}, \forall\, k \in \{i\}_{i=1}^n$, it holds that $O \cap U_{y_k} = \varnothing, \forall\, k \in \{i\}_{i=1}^n$. Since $U = \bigcup_{k=1}^n U_{y_k}$, we conclude $U \cap O = \varnothing$, and thus we have proven the claim. ■

**Proposition 3.134:**

*Let $(X, \tau)$ be a Hausdorff space. If $K \subseteq X$ is compact, it is closed.* □

*Proof:*

Given $x \in K^c$, Lemma 3.133 guarantees the existence of an open set $O_x$ such that $x \in O_x \subseteq K^c$. Notice then that $K^c = \bigcup_{x \in K^c} O_x$ and, being a union of open sets, we conclude that $K^c$ is open. Thus, $K$ is closed. ■

**Theorem 3.135:**

*Let $(X, \tau)$ be a compact and Hausdorff space. Then $(X, \tau)$ is normal.* □

*Proof:*

Let $A, B \subseteq X$ be closed sets. Since $X$ is compact, Proposition 3.132 guarantees $A$ and $B$ are compact. Pick $x \in B$. Since $X$ is Hausdorff, Lemma 3.133 guarantees there are disjoint open sets $O_x$ and $U_x$ such that $A \subseteq U_x$ and $x \in O_x$. Notice that $\{O_x\}_{x \in B}$ is an open cover of $B$, which is compact. Thus, we may conclude there are $n \in \mathbb{N}, x_k \in B, k \in \{i\}_{i=1}^n$ such that $\{O_{x_k}\}_{k=1}^n$ covers $B$.

If we take $O = \bigcup_{k=1}^n O_{x_k}$ and $U = \bigcap_{k=1}^n U_{x_k}$, we find that these are both open sets - for they are finite unions or intersections of open sets -, $A \subseteq U$ - for $A \subseteq U_x, \forall x \in B$ - and $B \subseteq O$ - $\{O_{x_k}\}_{k=1}^n$ covers $B$.

Finally, $O \cap U = \varnothing$. Since $U_x \cap O_x = \varnothing, \forall x \in B$, and $U \subseteq U_{x_k}, \forall k \in \{i\}_{i=1}^n$, it holds that $U \cap O_{x_k} = \varnothing, \forall k \in \{i\}_{i=1}^n$. Since $O = \bigcup_{k=1}^n O_{x_k}$, we conclude $U \cap O = \varnothing$.

Thus, we have proven that given any two closed sets $A, B \subseteq X$, there are $O, U \in \tau$ such that $A \subseteq U, B \subseteq O, O \cap U = \varnothing$, *id est*, $(X, \tau)$ is normal. ■

**Corollary 3.136:**

*Let $(X, \tau)$ be a compact Hausdorff space and let $A, B \subseteq X$ be closed sets. Let $a, b \in \mathbb{R}, a < b$. Then there is some continuous function $f\colon A \to B$ with $f(A) = \{a\}$ and $f(B) = \{b\}$.* □

*Proof:*

Theorem 3.135 on the facing page guarantees $(X, \tau)$ is normal. Thus, we might apply Urysohn's Lemma and the result is proven. ■

**Definition 3.137** [Weakly Sequentially Compact]**:**

Let $(X, \tau)$ be a topological space. If every sequence in $X$ has a cluster point, $(X, \tau)$ is said to be *weakly sequentially compact*. ♠

**Definition 3.138** [Sequentially Compact]**:**

Let $(X, \tau)$ be a topological space. If every sequence in $X$ has a convergent subsequence, $(X, \tau)$ is said to be *sequentially compact*. ♠

**Lemma 3.139:**

*Let $(M, d)$ be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence defined on $M$. $x$ is a cluster point of $(x_n)_{n \in \mathbb{N}}$ if, and only if, $(x_n)_{n \in \mathbb{N}}$ has a subsequence converging to $x$.* □

*Proof:*

Assuming $x$ is a cluster point of $(x_n)_{n \in \mathbb{N}}$, we know that given any neighborhood $\mathcal{B}$ of $x$, there are infinitely many $m \in \mathbb{N}$ such that $x_m \in \mathcal{B}$.

Let us consider the sets $\mathcal{B}_{\frac{1}{n}}(x)$, which are all neighborhoods of $x$. Thus, there are infinitely many elements of the sequence in each one of them. $\forall n \in \mathbb{N}$, let us define

$$m_n \equiv \min\left\{p \in \mathbb{N}; x_p \in \mathcal{B}_{\frac{1}{n}}(x)\right\}. \tag{3.85}$$

Notice that $y_{m_n}$ defines a subsequence of $x_n$. Furthermore, given $n \in \mathbb{N}$, it holds that

$$y_{m_n} \in \mathcal{B}_{\frac{1}{n}}(x) \subseteq \mathcal{B}_{\frac{1}{p}}(x), \forall p > n. \tag{3.86}$$

Thus, $y_{m_n}$ is eventually in any $\mathcal{B}_{\frac{1}{n}}(x)$. In Proposition 3.94 on page 70 we have proved this sets are a neighborhood basis for the metric topology at $x$. Thus, given a neighborhood $O$ of $x$, there is some $n \in \mathbb{N}$ such that $\mathcal{B}_{\frac{1}{n}}(x) \subseteq O$. Since $y_{m_n}$ is eventually in $\mathcal{B}_{\frac{1}{n}}(x)$, it is eventually in $O$. Since the subsequence is eventually in any neighborhood of $x$, we conclude $y_{m_n} \to x$, as desired.

Suppose now that $(x_n)_{n\in\mathbb{N}}$ has a subsequence converging to $x$. This means the subsequence is eventually in every neighborhood of $x$. Since the subsequence has infinitely many terms, this can only happen in the sequence is frequently in every neighborhood of $x$. ■

**Proposition 3.140:**
*A metric space is weakly sequentially compact if, and only if, it is sequentially compact.* □

*Proof:*
Lemma 3.139 on the preceding page. ■

**Theorem 3.141:**
*Let $(X,\tau)$ be a compact topological space. $(X,\tau)$ is weakly sequentially compact.* □

*Proof:*
Suppose $(X,\tau)$ was not weakly sequentially compact, *id est*, suppose there is some sequence $(x_n)_{n\in\mathbb{N}}$ with no cluster points, *id est*, a sequence such that

$$\forall\, x\in X, \exists\, O_x\in\tau, \exists\, n\in\mathbb{N}; x\in O_x, x_m\notin O_x, \forall\, m\geq n. \tag{3.87}$$

Thus, there are finitely many elements of the sequence in each $O_x$ (otherwise, the sequence would be frequently in some $O_x$, which is forbidden by hypothesis).

Notice that $\{O_x\}_{x\in X}$ is an open cover of $X$. Sine $X$ is compact, it admits a finite subcover, and thus $X\subseteq\bigcup_{k=1}^{n}O_{x_k}$ for some $n\in\mathbb{N}, x_k\in X, k\in\{i\}_{i=1}^{n}$. Since there are finitely many elements of $x_n$ in each $O_x$ and $\bigcup_{k=1}^{n}O_{x_k}$ is the reunion of finitely many $O_x$, there are finitely many elements of $x_n$ in $\bigcup_{k=1}^{n}O_{x_k}$. However, we know there are infinitely many elements of $x_n$ in $X$, and therefore we have arrived at a contradiction. This forces to conclude $(X,\tau)$ is indeed weakly sequentially compact. ■

**Definition 3.142** [Totally Bounded Metric Space]**:**
Let $(M,d)$ be a metric space. $(M,d)$ is said to be *totally bounded* if, and only if, $\forall\,\epsilon>0, \exists\, n\in\mathbb{N}, x_k\in M, k\in\{i\}_{i=1}^{n}$ such that $M\subseteq\bigcup_{k=1}^{n}\mathcal{B}_\epsilon(x_k)$. ♠

**Proposition 3.143:**
*Let $(M,d)$ be a totally bounded metric space. Then it is separable.* □

*Proof:*
Given $n\in\mathbb{N}$, we know there are $m_n\in\mathbb{N}, x_k^{(n)}\in X, k\in\{i\}_{i=1}^{m_n}; M\subseteq\bigcup_{k=1}^{m_n}\mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$. Notice that the set $A=\left\{x_k^{(n)}; k\in\{i\}_{i=1}^{m_n}, n\in\mathbb{N}\right\}$ is countable, for it is the countable union of finite sets. $A$ is dense in $M$.

Indeed, let $x\in M, \epsilon>0$. Due to the Archimedean property of the real numbers, we know there is some $n\in\mathbb{N}; \frac{1}{n}<\epsilon$. Since $M\subseteq\bigcup_{k=1}^{m_n}\mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$, we know there is some $k\in\{i\}_{i=1}^{m_n}$ such that $x\in\mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$. Thus, $\exists\, p\in A; x\in\mathcal{B}_{\frac{1}{n}}(p)\subseteq\mathcal{B}_\epsilon(p)$. Therefore, $A$ is dense in $M$ in the sense of metric spaces and Theorem 3.102 on page 73 implies $\overline{A}=M$. As $A$ is countable, $M$ is separable. ■

**Proposition 3.144:**

*Let $(M, d)$ be a complete and totally bounded metric space. Then $(M, d)$ is weakly sequentially compact as well.* $\square$

*Proof:*

Let $(x_l)_{l \in \mathbb{N}}$ be a sequence of points of $M$. Given $n \in \mathbb{N}$, we know there are $m_n \in \mathbb{N}, x_k^{(n)} \in X, k \in \{i\}_{i=1}^{m_n}; M \subseteq \bigcup_{k=1}^{m_n} \mathcal{B}_{\frac{1}{n}}\left(x_k^{(n)}\right)$.

Let $n = 1$. We know that $\forall\, l \in \mathbb{N}, \exists\, k \in \{k\}_{k=1}^{m_1}; x_l \in \mathcal{B}_1\left(x_k^{(1)}\right)$. Since the sequence $(x_l)_{l \in \mathbb{N}}$ has infinitely many terms and there are only finitely many $x_k^{(1)}$, there is at least one $k_1 \in \{k\}_{k=1}^{m_1}$ such that there are infinitely many terms of $(x_l)_{l \in \mathbb{N}}$ laying on $\mathcal{B}_1\left(x_{k_1}^{(1)}\right)$. These terms define a subsequence $y_p^{(1)}$.

Now, for each $n \in \mathbb{N}$, we can iterate this process. Given the sequence $(y_l^{(n)})_{l \in \mathbb{N}}$, we know that $\forall\, l \in \mathbb{N}, \exists\, k \in \{k\}_{k=1}^{m_{n+1}}; y_l^{(n)} \in \mathcal{B}_{\frac{1}{n+1}}(x_k) \cap \mathcal{B}_{\frac{1}{n}}(x_{k_n})$, since $y_l^{(n)} \in \mathcal{B}_{\frac{1}{n}}(x_{k_n}), \forall\, l \in \mathbb{N}$ an $M$ is totally bounded. There is at least one $k_n \in \{k\}_{k=1}^{m_{n+1}}$ such that infinitely many terms of $(y_l^{(n)})_{l \in \mathbb{N}}$ lay on $\mathcal{B}_{\frac{1}{n+1}}(x_{k_n})$.

The Axiom of Choice now allows us to, $\forall\, n \in \mathbb{N}$, pick $l_n \in \mathbb{N}$ in order to form a subsequence $(x_{l_n})_{n \in \mathbb{N}}$ such that $x_{l_m} \in \mathcal{B}_{\frac{1}{n}}(x_{k_n}), \forall\, m > n$. $x_{l_1}$ is an element of $(y_l^{(1)})_{l \in \mathbb{N}}$, $x_{l_2}$ is an element of $(y_l^{(2)})_{l \in \mathbb{N}}$ such that $l_2 > l_1$ (which is possible due to the fact that there are infinitely many terms on $(y_l^{(2)})_{l \in \mathbb{N}}$) and so on.

$(x_{l_n})_{n \in \mathbb{N}}$ is a Cauchy sequence. Indeed, let $\epsilon > 0$. The Archimedean property of the real numbers guarantees the existence of $m \in \mathbb{N}; \frac{2}{m} < \epsilon$. By construction, $x_{l_p}, x_{l_q} \in \mathcal{B}_{\frac{1}{m}}(x_{k_m}), \forall\, p, q > m$. Therefore, $d(x_{l_p}, x_{l_q}) < \frac{2}{m} < \epsilon, \forall\, p, q > m$.

Since $x \in M$ is a limit point of a subsequence of $(x_n)_{n \in \mathbb{N}}$, Lemma 3.139 on page 89 guarantees $x$ is a cluster point of $(x_n)_{n \in \mathbb{N}}$. $\blacksquare$

**Lemma 3.145:**

*Let $(M, d)$ be a metric space and let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. If a subsequence $(x_{n_m})_{m \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ converges to some point $x \in M$, then $x_n \to x$ as well.* $\square$

*Proof:*

Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy, we know that

$$\forall\, \epsilon > 0, \exists\, n_\epsilon \in \mathbb{N}; d(x_m, x_n) < \epsilon, \forall\, m, n > n_\epsilon. \tag{3.88}$$

Given $\epsilon > 0$, let $m_\epsilon$ be such that $n_{m_\epsilon} \geq n_\epsilon$. There is such an $m_\epsilon$ due to the fact that the subsequence $(x_{n_m})_{m \in \mathbb{N}}$ has infinitely many terms. We have that

$$\forall\, \epsilon > 0, \exists\, n_\epsilon \in \mathbb{N}; d(x_{n_m}, x_n) < \epsilon, \forall\, n > n_\epsilon, \forall\, m > m_\epsilon. \tag{3.89}$$

Since $x_{n_m} \to x$, we know that

$$\forall\, \epsilon > 0, \exists\, p_\epsilon \in \mathbb{N}; d(x_{n_m}, x) < \epsilon, \forall\, m > p_\epsilon. \tag{3.90}$$

Keeping this in mind, let $\epsilon > 0$. Then we know that

$$\begin{aligned} d(x_{n_m}, x) < \epsilon, \forall\, m > \max p_\epsilon, m_\epsilon \equiv q_\epsilon, \\ d(x_{n_m}, x_n) < \epsilon, \forall\, m > q_\epsilon, \forall\, n > n_{q_\epsilon}. \end{aligned} \tag{3.91}$$

Due to the triangle inequality, we see that $\forall\, m > q_\epsilon, \forall\, n > n_{q_\epsilon}$,

$$d(x_n, x) < d(x_{n_m}, x_n) + d(x_{n_m}, x) < 2\epsilon. \tag{3.92}$$

As a consequence, we conclude that

$$\forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}; d(x_n, x) < \epsilon, \forall\, n > n_0, \tag{3.93}$$

*id est*, $x_n \to x$, as desired. ■

**Theorem 3.146:**

*Let $(M, d)$ be a metric space. The following statements are equivalent:*

i. *it is compact;*

ii. *it is weakly sequentially compact;*

iii. *it is sequentially compact;*

iv. *it is complete and totally bounded.* □

*Proof:*

i. $\Rightarrow$ ii. Theorem 3.141 on page 90;

ii. $\Leftrightarrow$ iii. Proposition 3.140 on page 90;

iii. $\Rightarrow$ iv. Suppose $(M, d)$ is sequentially compact, *id est*, that any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $M$ has a convergent subsequence. Due to Lemma 3.145 on the previous page, this implies $(M, d)$ is complete.

Suppose $(M, d)$ is not totally bounded, *id est*,

$$\exists\, \epsilon > 0; \forall\, n \in \mathbb{N}, \forall\, x_k \in M, k \in \{k\}_{k=1}^n, \exists\, x \in M; x \notin \bigcup_{k=1}^{n} \mathcal{B}_\epsilon(x_k). \tag{3.94}$$

Let $x_1 \in M$. For each $n \in \mathbb{N}^*$, let us pick $x_{n+1} \in \left(\bigcup_{k=1}^n \mathcal{B}_\epsilon(x_k)\right)^{\mathsf{c}}$. Notice that, by construction, $\forall\, n \in \mathbb{N}, d(x_n, x_m) \geq \epsilon, \forall\, m < n$. We may rewrite this as $d(x_n, x_m) \geq \epsilon\, \forall\, n \neq m$. As a consequence, $(x_n)_{n\in\mathbb{N}}$ has no cluster points. Indeed, suppose $x_n \in \mathcal{B}_{\frac{\epsilon}{3}}(x)$ for some $n \in \mathbb{N}$. Then $d(x, x_n) \leq \frac{\epsilon}{3}$. We already know that $\epsilon \leq d(x_n, x_m)\, \forall\, m \neq n$. The triangle inequality yields, $\forall\, m \neq n$,

$$\epsilon \leq d(x_n, x_m) \leq d(x_n, x) + d(x, x_m),$$

$$\epsilon < \frac{\epsilon}{3} + d(x, x_m),$$
$$\frac{2\epsilon}{3} < d(x, x_m). \tag{3.95}$$

Therefore, $x_m \notin \mathcal{B}_{\frac{\epsilon}{3}}(x)$. Since the argument holds for every $x \in M$, there are no points in $M$ such that $(x_n)_{n\in\mathbb{N}}$ is frequently within every neighborhood. Therefore, $(x_n)_{n\in\mathbb{N}}$ has no cluster points, which implies, through Lemma 3.139 on page 89, that $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence. Therefore, $(M, d)$ is not sequentially compact. This means we have proved the contrapositive to the statement we wanted to prove, which means that, indeed, $(M, d)$ being sequentially compact implies $(M, d)$ being totally bounded.

iv. $\Rightarrow$ i. Proposition 3.143 on page 90 guarantees a totally bounded metric space is separable. Proposition 3.103 on page 73 guarantees a separable metric space is a Lindelöf space. Thus, every open cover of $(M, d)$ has a countable subcover. As a consequence, we just need to prove that every countable open cover of $(M, d)$ admits a finite subcover.

Let $\{A_n\}_{n\in\mathbb{N}}$ be an open cover of $(M, d)$. Suppose, by contradiction, $M$ is not compact, *id est*, $\forall\, n \in \mathbb{N}, M \neq \bigcup_{k=1}^{n} A_n$. We may then define $B_n \equiv X \setminus \bigcup_{k=1}^{n} A_n$. Notice that $B_{n+1} \subseteq B_n, \forall\, n \in \mathbb{N}$. Since $B_n \neq \varnothing, \forall\, n \in \mathbb{N}$, we may define a sequence by choosing $x_n \in B_n, \forall\, n \in \mathbb{N}$. Proposition 3.144 on page 91 guarantees $(x_n)_{n\in\mathbb{N}}$ has a cluster point. However, Propositions 3.94 and 3.101 on page 70 and on page 72 guarantee $x \in B_n, \forall\, n \in \mathbb{N}$, since $B_n$ is always closed (for it is the complement of a union of open sets, *id est*, the complement of an open set) and removing finitely many terms of $(x_n)_{n\in\mathbb{N}}$ doesn't change the fact that $(x_n)_{n\in\mathbb{N}}$ is frequently in any neighborhood of $x$.

Since $x \in B_n, \forall\, n \in \mathbb{N}$, we get

$$\begin{aligned} x \in \bigcap_{n=1}^{+\infty} B_n &= \bigcap_{n=1}^{+\infty} \left[\bigcup_{k=1}^{n} A_k\right]^{\mathsf{c}}, \\ &= \left[\bigcup_{k=1}^{+\infty} A_k\right]^{\mathsf{c}}, \\ &= M^{\mathsf{c}}, \\ &= \varnothing. \end{aligned} \tag{3.96}$$

This is a contradiction, and thus the hypothesis that $M$ is not compact is false. Hence, the proof is complete. ■

Finally, we may give a complete description of all compact sets in $\mathbb{R}^n$ by means of the Heine-Borel Theorem.

**Theorem 3.147** [Heine-Borel]**:**

*Consider the metric space $(\mathbb{R}^n, d)$, where $d$ is the standard Euclidean metric. A subset $K \subseteq \mathbb{R}^n$ is compact if, and only if, it is closed and bounded.* □

*Proof:*

⇒: Assume $K$ is compact. Then Lemma 3.89 and Proposition 3.134 on page 68 and on page 88 guarantee $K$ is closed. Theorem 3.146 on page 92 guarantees $K$ is totally bounded. Thus, $\exists\, n \in \mathbb{N}, x_k \in K, k \in \{k\}_{k=1}^n; K \subseteq \bigcup_{k=1}^n \mathcal{B}_1(x_k)$.

If we define $m \equiv \max_{1 \leq k \leq n} \{d(x_k, 0)\}$, then it holds that $U \subseteq \mathcal{B}_{1+m}(0)$. Indeed, suppose $x \in U$. Then $x \in \bigcup_{k=1}^n \mathcal{B}_1(x_k)$, which means $x \in \mathcal{B}_1(x_k)$ for some $k \in \{k\}_{k=1}^n$. Thus, we know that $d(x, x_k) < 1$ and $d(x_k, 0) \leq m$. The triangle inequality yields

$$\begin{aligned} d(x,0) &\leq d(x,x_k) + d(x_k,0), \\ &< 1 + m, \end{aligned} \tag{3.97}$$

as claimed. This proves $K$ is bounded.

⇐: Assume now $K$ is closed and bounded. Since $(\mathbb{R}^n, d)$ is complete and $K$ is closed, $K$ is complete as well*.

Since $K$ is bounded, we know there is some $r > 0$ such that $K \subseteq \mathcal{B}_r(0)$. As a consequence, we see that $K \subseteq (-r, r)^n$, the hypercube with side $2r$ centered at the origin.

Given $\epsilon > 0$, let $\delta < \frac{2\epsilon}{\sqrt{n}}$. Notice that a hypercube of side $\delta$ is always contained in an open ball of radius $\epsilon$. Indeed, the hypercube's diagonal is given by $D = \sqrt{n\delta^2} = \sqrt{n}\delta < 2\epsilon$, which is the diameter of an open ball of radius $\epsilon$. Thus, if we cover $K$ with finitely many cubes of side $\delta$, we may also cover it with finitely many open balls of radius $\epsilon$, and that proves $K$ is totally bounded.

The interval $(-r, r)$ can be covered by $\lceil \frac{2r}{\delta} \rceil$ intervals of size $\delta$, where $\lceil x \rceil$ denotes the smallest integer larger than $x$. Similarly, the hypercube $(-r, r)^n$ can be covered by $\lceil \frac{2r}{\delta} \rceil^n$ hypercubes of side $\delta$. Since $K \subseteq (-r, r)^n$, this means $K$ can be covered by finitely many hypercubes of side $\delta$, or, equivalently, by finitely many open balls of radius $\epsilon$. Hence, $K$ is totally bounded.

Since $K$ is complete and totally bounded, Theorem 3.146 on page 92 ensures $K$ is compact, concluding the proof.

■

**Theorem 3.148** [Bolzano-Weierstrass]**:**

*Consider the metric space $(\mathbb{R}^n, d)$, where $d$ is the Euclidean metric. If a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ is bounded,* id est*, if $\exists\, r > 0; x_n \in \mathcal{B}_r(0), \forall\, n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.* □

*This can be proven easily by employing Propositions 3.94 and 3.101 on page 70 and on page 72.

*Proof:*

Since $x_n \in \mathcal{B}_r(0), \forall\, n \in \mathbb{N}$, it holds that $x_n \in \overline{\mathcal{B}_r(0)}, \forall\, n \in \mathbb{N}$. Notice that $\overline{\mathcal{B}_r(0)} = \{x \in \mathbb{R}^n; d(x,0) \leq r\}$. Since $\overline{\mathcal{B}_r(0)}$ is closed and bounded (for $\overline{\mathcal{B}_r(0)} \subseteq \mathcal{B}_{r+1}(0)$), the Heine-Borel Theorem ensures it is compact. Since it is compact, Theorem 3.146 on page 92 ensures it is sequentially compact. Hence, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. ■

**Proposition 3.149:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Suppose $(X, \tau_X)$ is compact. Let $f\colon X \to Y$ be a continuous function. Then* $\operatorname{Ran} f = f(X)$ *is compact.* □

*Proof:*

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a open cover of $\operatorname{Ran} f$ in the relative topology. We know then there are $O_\lambda \in \tau_Y$ such that $U_\lambda = O_\lambda \cap \operatorname{Ran} f, \forall\, \lambda \in \Lambda$. We see that

$$\begin{aligned}
\operatorname{Ran} f &\subseteq \bigcup_{\lambda\in\Lambda} O_\lambda, \\
f^{-1}(\operatorname{Ran} f) &\subseteq f^{-1}\left(\bigcup_{\lambda\in\Lambda} O_\lambda\right), \\
X &\subseteq \bigcup_{\lambda\in\Lambda} f^{-1}(O_\lambda). \qquad (3.98)
\end{aligned}$$

Since $f$ is continuous, the sets $f^{-1}(O_\lambda)$ are open. Since $X$ is compact, there is a finite set $I$ such that

$$\begin{aligned}
X &\subseteq \bigcup_{i\in I} f^{-1}(O_{\lambda_i}), \\
f(X) &\subseteq f\left(\bigcup_{i\in I} f^{-1}(O_{\lambda_i})\right), \\
\operatorname{Ran} f &\subseteq \bigcup_{i\in I} O_{\lambda_i}, \\
\operatorname{Ran} f &\subseteq \bigcup_{i\in I} O_{\lambda_i} \cap \operatorname{Ran} f, \\
\operatorname{Ran} f &\subseteq \bigcup_{i\in I} U_{\lambda_i}, \qquad (3.99)
\end{aligned}$$

proving $\{U_\lambda\}_{\lambda\in\Lambda}$ admits a finite subcover, and hence that $\operatorname{Ran} f$ is compact. ■

# Four

# Differential Geometry

> Let it have been postulated [...] that if a straight line falling across two straight lines makes internal angles on the same side less than two right angles, then the two straight lines, being produced to infinity, meet on that side that is less than two right angles.
>
> EUCLID OF ALEXANDRIA. *Elements*, Book I, Postulate V (Axiom XII in certain editions).

## 4.1 Manifolds

THIS is a text on Mathematical Physics, and thus it could sound weird to talk about cartography in here. Nevertheless, I ask you to trust me that this discussion will lead us to some interesting concepts and consequences.

Our current goal is to make a good map of the Earth, which we shall consider as a perfect sphere*. This being a text on Mathematics, "good map" might sound somewhat vague, and thus we must be more specific.

Let us consider the unit 2-sphere given by

$$S^2\{x \in \mathbb{R}^3; \|x\| = 1\}, \tag{4.1}$$

where the norm $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. Our goal is to find a homeomorphism $\varphi\colon S^2 \to \mathbb{R}^2$. Such a function would allow us to map each point of the Earth's surface (here represented by $S^2$) to a single point in our map in a continuous way, so close points in the Earth are represented by close points in our map.

Unfortunately, this is an impossible task. $S^2$ is the boundary of the set $\mathcal{B}_1(0)$ (the open ball centered at the origin with unitary radius), and we know that given a set $A$, it holds that $\partial A = \overline{\partial A}$, *id est*, the boundary of $A$ is always closed. Furthermore, $S^2 \subseteq \mathcal{B}_2(0)$, meaning it is a bounded set. As a closed and bounded set, the Heine-Borel Theorem ensures it is a compact set.

*With apologies to the Flat Earth Community

On the other side, $\mathbb{R}^2$ is open, but it is not bounded. Hence, the Heine-Borel Theorem guarantees $\mathbb{R}^2$ is not a compact set.

Since compactness is preserved by homeomorphisms, it is impossible to obtain a homeomorphism between a compact set such as $S^2$ and a non-compact set such as $\mathbb{R}^2$. Thus, there is no such thing as a perfect map.

In the absence of a perfect map, we must content ourselves with simpler versions. What is our possible option?

Cartographers might not be able to produce a single map of the entire Earth at once which happens to be a homeomorphism, but it is still possible to map the Earth. We have two options: either we abandon our requirement of continuity - and then we may get something as the Mercator projection, which maps the entire Earth but is not continuous everywhere - or we may abandon the idea of mapping the entire Earth at once and stick to continuity. This second option would still allow us to make an atlas if we are careful enough. A single chart can't describe the whole Earth at once, but maybe 12 charts can.

In this text, we are interested in studying the case in which we stick to continuity and abandon the desire of mapping the whole Earth at once. The other possibility we leave for the cartographers to explore.

Our goal now is to obtain a way of finding a certain amount of "charts" $(U, \varphi)$, where $U$ is an open set in $S^2$ and $\varphi\colon U \to \operatorname{Ran}\varphi \subseteq \mathbb{R}^2$ is a homeomorphism (notice this implies $\operatorname{Ran}\varphi$ is an open set in $\mathbb{R}^2$, since $\varphi$ is an open map), such that the collection of all such charts covers $S^2$ as a whole.

**Example** [Charting $S^2$]**:**

This requirement is easy to be fulfilled Let us split $S^2$ in six parts given by

$$U_i^{\pm} \equiv \left\{(x_1, x_2, x_3) \in S^2; \operatorname{sign}(x_i) = \pm 1\right\}. \tag{4.2}$$

These sets are open, for they are of the form

$$U_1^{+} = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}; x_1 > 0\}, \tag{4.3}$$

*id est*, they are the intersection of the subspace we are considering $S^2$ with an open set in the ambient space $\mathbb{R}^3$.

These sets do cover $S^2$. Let $(x_1, x_2, x_3) \in S^2$. If $(x_1, x_2, x_3) \notin U_1^+ \cup U_1^-$, then it must hold that $x_1 = 0$. If $(x_1, x_2, x_3) \notin U_2^+ \cup U_2^-$, then it must hold that $x_2 = 0$. Since $x_1^2 + x_2^2 + x_3^2 = 1$, this implies $x_3 = \pm 1$, and hence $(x_1, x_2, x_3) \in U_3^+ \cup U_3^-$.

We may now consider the maps $\varphi_1^{\pm}\colon U_i^{\pm} \to \mathcal{B}_1(0)$ given by $\varphi_1^{\pm}((x_1, x_2, x_3)) = (x_2, x_3)$ with similar definitions for $\varphi_i^{\pm}$. We shall prove $\varphi_1^{\pm}$ is a homeomorphism. The remaining cases are similar.

Let us begin by proving $\varphi_1^+$ is invertible. $\varphi_1^+$ is onto. Indeed, given $(x_2, x_3) \in \mathcal{B}_1(0)$, it holds that

$$\left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right) \in U_1^+, \tag{4.4}$$

since

$$\sqrt{1 - x_2^2 - x_3^2}^2 + x_2^2 + x_3^2 = 1 - x_2^2 - x_3^2 + x_2^2 + x_3^2, \\ = 1 \tag{4.5}$$

and $\sqrt{1 - x_2^2 - x_3^2} > 0$. Notice that

$$\varphi_1^+\left(\left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right)\right) = (x_2, x_3). \tag{4.6}$$

Furthermore, $\varphi_1^+$ is one-to-one. Suppose $(y, x_2, x_3), (z, x_2, x_3) \in U_1^+$ are such that

$$\varphi_1^+((y, x_2, x_3)) = \varphi_1^+((z, x_2, x_3)) = (x_2, x_3). \tag{4.7}$$

Then notice that

$$z^2 + x_2^2 + x_3^2 = 1, \\ z^2 = 1 - x_2^2 - x_3^2, \\ z = +\sqrt{1 - x_2^2 - x_3^2}, \tag{4.8}$$

where the last step used the fact that $z > 0$. The same argument applies to $y$, and hence $y = z$ and we conclude $(y, x_2, x_3) = (z, x_2, x_3)$. Hence, $\varphi_1^+$ is bijective.

We now must prove $\varphi_1^+$ and its inverse are continuous. Notice that the components of $\varphi_1^+$ are simply the projections $\pi_i((x_1, x_2, x_3)) = x_i$ (which are continuous when $\mathbb{R}^3$ is equipped with the product topology, and it is) restricted to $U_1^+$, and hence they are continuous in the relative topology.

The inverse is also continuous and one can prove it by employing the fact that a function $f\colon Y \to \times_{\lambda \in \Lambda} X_\lambda$ is continuous if, and only if, all the coordinate functions $(\pi_\lambda \circ f)\colon Y \to X_\lambda$ are continuous.

Notice that $(\pi_2 \circ \varphi_1^{+-1})(x_2, x_3) = x_2$ with a similar expression for $\pi_3$ and $(\pi_2 \circ \varphi_1^{+-1})(x_2, x_3) = \sqrt{1 - x_2^2 - x_3^2}$. Thus, $(\pi_i \circ \varphi_1^{+-1})$ are simply projections restricted to a certain domain for $i = 1, 2$. $(\pi_1 \circ \varphi_1^{+-1})$ is a composition of continuous functions, and I leave for you the task of proving it. Therefore, we see that $\varphi_1^+$ is indeed a homeomorphism. ♥

This construction allowed us to chart $S^2$ in parts. By diving the Earth in six pieces, we can chart each piece continuously. Notice that the charts we chose superpose: for example, $(1, 1, 1) \in U_i^+$ for $i = 1, 2, 3$. This means we should expect some agreement between the different charts: I must be able to change from a chart to another one continuously. In cartographical terms, suppose you are following the trajectory of a ship in one of the charts of an atlas. If the trajectory reaches the edge of the page, you must be able to keep following it in another page without any contradictions. If the trajectory was depicted in a certain way in a page, it must be descripted in an analogue way in another page.

Let us put it in mathematical terms: if you have two charts $(U, \varphi)$ and $(V, \psi)$ such that $U \cap V \neq \varnothing$, you would like to be able to continuously transition from $\varphi(U \cap V)$ to $\psi(U \cap V)$. Is this possible?

Indeed it is. Let us write $\varphi(U \cap V) = A$ and $\psi(U \cap V) = B$. Notice that we have a function $(\varphi \circ \psi^{-1})\colon B \to A$ which is a homeomorphism, since it is the composition of two homeomorphisms. Thus. we can transit continuously between the pages of our atlas. We have completely described the Earth in the pages of a book in a continuous manner, despite being unable to produce a unique map describing the entire planet in a continuous way.

This might solve the problem for the cartographers, but this is a text in Mathematical Physics. The next question a mathematician would ask might be something similar to "Ok, but what if the Earth was a torus?". Can we make this more general? If so, how general?

In order to address these questions, we should step back a bit and provide precise definitions for the concepts we've developed so far.

**Definition 4.1** [Locally Euclidean Space]**:**

Let $(M, \tau)$ be a topological space. We say it is a *locally Euclidean space of dimension n* if, and only if, every point $p \in M$ has an open neighborhood $U$ which has an homeomorphism $\varphi$ onto an open subset of $\mathbb{R}^n$. The pair $(U, \varphi)$ is said to be a *chart*, $U$ is said to be a *coordinate neighborhood* and $\varphi$ is said to be a *coordinate system* on $U$. If $\varphi(p) = 0$, the chart $(U, \varphi)$ is said to be *centered* at $p$. ♠

**Theorem 4.2:**

*Let $(M, \tau)$ be a topological space. $(M, \tau)$ is a locally Euclidean space if, and only if, every point $p \in M$ has an open neighborhood $U$ which has an homeomorphism $\varphi$ onto an open ball of $\mathbb{R}^n$.* □

*Proof:*

Suppose every point $p \in M$ has an open neighborhood $U$ which has an homeomorphism $\varphi$ onto an open ball of $\mathbb{R}^n$. Since every open ball is an open set, it follows immediately that $(M, \tau)$ is locally Euclidean.

Suppose $(M, \tau)$ is locally Euclidean. Let $p \in M$. We know there is a pair $(U, \varphi)$ such that $U$ is an open neighborhood of $p$ and $\varphi\colon U \to \varphi(U)$ is a homeomorphism. Since $\varphi$ is a homeomorphism, $\varphi(U)$ is an open set. In particular, it is an open neighborhood of $\varphi(p)$. Therefore, there is some $\epsilon > 0$ such that $\varphi(p) \subseteq \mathcal{B}_\epsilon(\varphi(p)) \subseteq \varphi(U)$. Furthermore, since $\varphi$ is a homeomorphism, $\varphi^{-1}(\mathcal{B}_\epsilon(\varphi(p)))$ is an open neighborhood of $p$. Also, the restriction of $\varphi$ to $\varphi^{-1}(\mathcal{B}_\epsilon(\varphi(p)))$ is a homeomorphism, proving that $p \in M$ has an open neighborhood which has an homeomorphism onto an open ball of $\mathbb{R}^n$. ■

**Definition 4.3** [Atlas]**:**

Let $(M, \tau)$ be a locally Euclidean space of dimension $n$. An *atlas* on $(M, \tau)$ is a collection $\mathcal{A} = \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ of charts on $(M, \tau)$ such that $M = \bigcup_{\lambda \in \Lambda} U_\lambda$. ♠

Notice that, by definition, every locally Euclidean spaces admits at least one atlas.

**Definition 4.4** [Topological Manifold]**:**

A *topological manifold of dimension* $n$ is a Hausdorff, second-countable, locally Euclidean space of dimension $n$. ♠

In order to prove the well-definition of the dimension of a topological manifold, we shall employ (without proof) the Theorem of Topological Invariance of Dimension, which is Corollary 1.6.3 of [73].

**Theorem 4.5** [Topological Invariance of Dimension]**:**

*Let $n, m \in \mathbb{N}, n > m$. Let $\varnothing \neq U \subseteq \mathbb{R}^n$. There is no continuous injective mapping from $U$ to $\mathbb{R}^m$. In particular, $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.* □

The following result will also be useful:

**Lemma 4.6:**

*Let $n \in \mathbb{N}$, $p \in \mathbb{R}^n$, $\epsilon > 0$. $\mathcal{B}_\epsilon(p)$ is homeomorphic to $\mathbb{R}^n$.* □

*Proof:*

Firstly we notice that $\mathcal{B}_\epsilon(p)$ is homeomorphic to $\mathcal{B}_1(0)$. This can be proven in a simple way by employing the fact that $\mathbb{R}^n$ is a locally convex space. More details can be found at Chapter 5 (in particular, Proposition 5.39 on page 152).

We now must simply prove that $\mathcal{B}_1(0)$ and $\mathbb{R}^n$ are homeomorphic. Consider the function $\varphi\colon \mathcal{B}_1(0) \to \mathbb{R}^n$

$$\varphi(x) = \tan\left(\frac{\pi\|x\|}{2}\right)x. \tag{4.9}$$

It is a composition of continuous functions, since tan is continuous on the interval $[0, 1)$, the norm is continuous on the topology induced by itself and product by a scalar is continuous on $\mathbb{R}^n$ (Lemma 5.17 and Proposition 5.39 on page 142 and on page 152). It is also a inversible function with continuous inverse. Hence, it is a homeomorphism. ■

**Theorem 4.7:**

*The dimension of a topological manifold is well-defined.* □

*Proof:*

Let $(M, \tau)$ be a topological manifold of dimension $n$ and assume, for the sake of contradiction, that it is also a topological manifold of dimension $m \neq n$. We assume without any loss of generality that $n > m$.

Let $p \in M$. Due to Theorem 4.2 on the facing page, we know there is an open set $U$ with $p \in U$ and a homeomorphism $\varphi\colon U \to \mathcal{B}_\epsilon(x) \subseteq \mathbb{R}^n$, for some $\epsilon > 0$ and some $x \in \mathbb{R}^n$. Similarly, there is an open set $V$ with $p \in V$ and a homeomorphism $\psi\colon V \to \psi(V) \subseteq \mathcal{B}_\delta(y)$, for some $\delta > 0$ and some $y \in \mathbb{R}^m$. Due to Lemma 4.6, we know that $V$ is homeomorphic to $\mathbb{R}^m$ (let's call this homeomorphism $g$) and $U$ is homeomorphic to $\mathbb{R}^n$ (let's call this homeomorphism $f$).

We may consider the open set $U \cap V$. We know that $f\colon U \cap V \to f(U \cap V) \subseteq \mathbb{R}^n$ is a homeomorphism and so is $g\colon U \cap V \to g(U \cap V) \subseteq \mathbb{R}^m$. Hence, $(g \circ f^{-1})\colon f(U \cap V) \to \mathbb{R}^m$ is a continuous injective map. Theorem 4.5 tells us this is a contradiction.

Hence, $(M, \tau)$ can't have two different dimensions, proving the dimension of $M$ is well-defined. ■

*Notation:*

We denote the dimension of a topological manifold $(M, \tau)$ by $\dim M$. ♦

The Hausdorff and second-countability properties are demanded in order to add some structure to the manifold. It is always interesting to have uniqueness of limits (which is provided by the Hausdorff condition), especially considering we will eventually develop a generalization of Calculus on structures similar to (but more complicated than) these. Second-countability also will allow us to obtain more results (at the cost of generality), but in particular it will prove its importance later on, when we start dealing with partitions of unity.

The last requirement could be considered the soul of our interest: develop a theory of structures which locally resemble $\mathbb{R}^n$, which we are already familiar with. The notions we shall develop are closely tied to the ideas of using charts and atlases to map the Earth: you don't need to know the actual Earth if your atlas is good enough. Being able to read the information in the charts will be enough to obtain information about the real world. We may illustrate this in the following proposition.

**Definition 4.8** [Continuous Curve]**:**

Let $(M, \tau)$ be a topological manifold. A *curve* on $M$ is a function $\gamma \colon I \to M$, where $I \subseteq \mathbb{R}$. A curve is said to be *continuous* if it is continuous as a function between topological spaces, where $\mathbb{R}$ is considered to be equipped with the standard topology. Continuity of $\gamma$ at a point $\lambda \in \mathbb{R}$ is defined in a similar manner. ♠

**Proposition 4.9:**

*Let $(M, \tau)$ be a topological manifold of dimension $n$. Consider a curve $\gamma \colon \mathbb{R} \to M$. Let $\lambda \in \mathbb{R}$ and consider a chart $(U, \varphi)$ of $M$ such that $\gamma(\lambda) \in U$. $\gamma$ is continuous at $\lambda$ if, and only if, $\varphi \circ \gamma$ is continuous at $\lambda$.* □

*Proof:*

Suppose $\gamma$ is continuous at $\lambda$. Since $\varphi$ is a homeomorphism, $\varphi \circ \gamma$ is a composition of continuous functions at $x$ and hence it is continuous. On the other hand, if $\varphi \circ \gamma$ is continuous at $x$, notice that $\gamma = \varphi^{-1} \circ (\varphi \circ \gamma)$, and hence $\gamma$ is the composition of continuous functions. ■

We may illustrate this notion in the following commutative diagram*.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "\varphi\circ\gamma"'] & U \arrow[d, "\varphi"] \\ & \varphi(U) \end{tikzcd}$$

*The word "commutative" means following a path in the diagram yields the same result as following any other path with the same endpoints. These concepts turn out to be quite useful in areas such as Category Theory[43, 50]

One should notice the fact that even though we may find whether $\gamma$ is continuous by looking at $\varphi \circ \gamma$, the fact that $\gamma$ is continuous at a point $\lambda$ does not depend on the chart we choose. Suppose, for example, that $(U, \varphi)$ and $(V, \psi)$ are charts with $\gamma(\lambda) \in U \cap V$. Then $\varphi \circ \gamma$ is continuous at $\lambda$ if, and only if, $\psi \circ \gamma$ is continuous at $\lambda$. A way of noticing it is by looking at the map $\psi \circ \varphi^{-1}$ (which is a composition of continuous maps, and therefore is continuous as well):

$$\begin{aligned} \psi \circ \gamma &= \psi \circ (\varphi^{-1} \circ \varphi) \circ \gamma, \\ &= (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma). \end{aligned} \tag{4.10}$$

Since $\psi \circ \varphi^{-1}$ is continuous, continuity of $\varphi \circ \gamma$ implies continuity of $\psi \circ \gamma$.

This remark might seem pointless, since we have already proven that continuity of $\varphi \circ \gamma$ at $x$ for any chart $(U, \varphi)$ containing $\gamma(x)$ is equivalent to continuity of $\gamma$ itself at $x$, but this notion will be useful when we try to generalize these concepts, which is the reason we shall put these remarks in a more formal setting.

**Definition 4.10** [Chart Transition Maps]**:**

Let $(M, \tau)$ be a topological manifold of dimension $n$ and let $(U, \varphi)$ and $(V, \psi)$ be charts on $M$ such that $U \cap V \neq \varnothing$. The *chart transition maps*, or simply *transition maps* or *transition functions*, between $(U, \varphi)$ and $(V, \psi)$ are the maps

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V), \quad \psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V) \spadesuit$$

**Lemma 4.11:**

*Let $(M, \tau)$ be a topological manifold of dimension n and let $(U, \varphi)$ and $(V, \psi)$ be charts on M such that $U \cap V \neq \varnothing$. The chart transition maps between $(U, \varphi)$ and $(V, \psi)$ are continuous.* □

*Proof:*

Both $\varphi$ and $\psi$ are homeomorphisms, and hence both them and their inverses are continuous. Thus, the composition of any combination of $\psi, \psi^{-1}, \varphi$ and $\varphi^{-1}$ is a continuous function. ■

This provides us with a theory of spaces which resemble $\mathbb{R}^n$ in terms of continuity of functions. If we want to check what was the real trajectory of a ship on the topological manifold, we can simply see the trajectory across the charts (and use the continuous chart transition maps to "flip pages" on the atlas) and evaluate continuity by looking at this projection.

However, what if we not only wanted to know the ship's trajectory, but also its velocity? As we know from elementary Physics, this requires a theory of differentiation, which we do not possess for such general spaces, since Topology can only deal with continuity.

Nevertheless, we have already seen a possible way of *defining* differentiability of curves in this context. Proposition 4.9 on the preceding page states that we can speak of continuity without needing to pay attention to the manifold's topology, so imposing a similar result could provide a satisfactory notion of differentiability.

However, a problem arises. We must ensure our definition is chart-independent, since it should reflect a property of the curve itself, and not a property of the curve's projection through a specific chart. This can be visualized in the following diagram:

$$\begin{tikzcd}
\mathbb{R}^n \supseteq \varphi(U \cap V) \arrow[rr, "\psi\circ\varphi^{-1}"] & & \psi(U \cap V) \subseteq \mathbb{R}^n \\
& U \cap V \arrow[ul, "\varphi"'] \arrow[ur, "\psi"] & \\
& \mathbb{R} \arrow[u, "\gamma"] \arrow[uul, "\varphi\circ\gamma"] \arrow[uur, "\psi\circ\gamma"'] &
\end{tikzcd}$$

If we want to give a proper definition of differentiability at $p \in U \cap V$ to $\gamma$ by analysing, say, whether $\psi \circ \gamma$ is differentiable, then the same result should be obtained by analysing $\varphi \circ \gamma$.

Therefore, we must have that, given charts $(U, \varphi)$ and $(V, \psi)$ with $p \in U \cap V$, then $\psi \circ \gamma$ is differentiable at $p$ if, and only if, $\varphi \circ \gamma$ is differentiable at $p$.

The diagram tells us how to achieve this: $\psi \circ \gamma = (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma)$. If $\varphi \circ \gamma$ is differentiable and $\psi \circ \varphi^{-1}$ is differentiable, then $\psi \circ \gamma$ will also be differentiable. This motivates the definition of compatible charts.

**Definition 4.12** [$\mathcal{C}^k$-compatible Charts]**:**

Let $(M, \tau)$ be a locally Euclidean space. Let $(U, \varphi)$ and $(V, \psi)$ be charts on $(M, \tau)$. The charts are said to be $\mathcal{C}^k$*-compatible* if, and only if, either of the following requirements hold:

i. $U \cap V = \varnothing$;

ii. $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are of class $\mathcal{C}^k$. ♠

**Definition 4.13** [$\mathcal{C}^k$-atlas]**:**

Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be an atlas on $(M, \tau)$. $\mathcal{A}$ is said to be a $\mathcal{C}^k$*-atlas* if, and only if, the charts on $\mathcal{A}$ are pairwise $\mathcal{C}^k$-compatible.

In particular, $\mathcal{C}^\infty$-atlases are commonly referred to as *smooth atlases*. ♠

Curiously, in order to define differentiability, we are not asking for more structure. We are asking for less.

The problem we had with our original topological manifold was that some transition maps were not $\mathcal{C}^1$-compatible, and thus it was not possible to define differentiability (which we shall do soon) in a chart independent manner. We simply had too many pages on our atlas and some of them were kind of weird when we speak about differentiability. Our solution was to tear off these bad pages and keep a smaller, but more powerfull, atlas.

On the other hand, after we tear off the pages of our atlas, we'll be careful to keep all the useful pages. In other words, we will ask for our manifolds to be equipped with a maximal atlas.

**Definition 4.14** [Maximal $\mathcal{C}^k$-atlas]**:**
Let $(M, \tau)$ be a topological manifold and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on $(M, \tau)$. $\mathcal{A}$ is said to be *maximal* if, and only if, for every $\mathcal{C}^k$-atlas $\mathcal{A}'$ with $\mathcal{A} \subseteq \mathcal{A}$ it holds that $\mathcal{A} = \mathcal{A}'$.

A maximal $\mathcal{C}^k$-atlas on a topological manifold $(M, \tau)$ is also referred to as a $\mathcal{C}^k$*-structure* on $(M, \tau)$. Once again, the $k = \infty$ case is referred commonly as "smooth" instead of $\mathcal{C}^\infty$. ♠

**Definition 4.15** [$\mathcal{C}^k$-manifold]**:**
Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on $(M, \tau)$. The triple $(M, \tau, \mathcal{A})$ is said to be a $\mathcal{C}^{[}k]$*-manifold*.

In particular, $\mathcal{C}^\infty$-manifolds are commonly referred to as *smooth manifolds* or *differentiable manifolds*. ♠

*Remark:*
Notice that a $\mathcal{C}^k$-manifold is a $\mathcal{C}^l$-manifold for every $l \leq k$. ♣

Requiring a maximal atlas might seem silly, but it is well justified: it comes for free if you already have any other atlas.

**Lemma 4.16:**
*Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on $(M, \tau)$. Let $(U, \varphi)$ and $(V, \psi)$ be charts on $(M, \tau)$. If both $(U, \varphi)$ and $(V, \psi)$ are compatible with the atlas $\mathcal{A}$, then they are compatible with each other.* □

*Proof:*
If $U \cap V = \varnothing$, the proof is complete. Let us then assume $U \cap V \neq \varnothing$.

$\mathcal{A}$ covers $M$, and therefore, given $p \in U \cap V$, there is some chart $(W, \chi)$ with $p \in W$. By hypothesis, $(W, \chi)$ is compatible with both $(U, \varphi)$ and $(V, \psi)$. We may represent this in the diagram

$$\begin{tikzcd}
\varphi(U\cap V\cap W) & & \psi(U\cap V\cap W) \arrow[ll, "\varphi\circ\psi^{-1}"'] \arrow[ddl, "\chi\circ\psi^{-1}"] \\
& U\cap V\cap W \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] \arrow[d, "\chi"] & \\
& \chi(U\cap V\cap W) \arrow[uul, "\varphi\circ\chi^{-1}"] &
\end{tikzcd}$$

Since $(W, \chi)$ is compatible with both $(U, \varphi)$ and $(V, \psi)$, we know that $\chi \circ \psi^{-1}$ is $\mathcal{C}^k$ at $\psi(U \cap V \cap W)$ and $\varphi \circ \chi^{-1}$ is $\mathcal{C}^k$ at $\chi(U \cap V \cap W)$. Hence, $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$ at $\psi(U \cap V \cap W)$

and, in particular, at $\psi(p)$. Since $p \in U \cap V$ was arbitrary, we see that $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$ at $\psi(U \cap V)$. A similar argument proves that $\psi \circ \varphi^{-1}$ is $\mathcal{C}^k$ at $\varphi(U \cap V)$. Therefore, $(U, \varphi)$ and $(V, \psi)$ are $\mathcal{C}^k$-compatible. ■

**Proposition 4.17:**

*Let $(M, \tau)$ be a locally Euclidean space and let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on $(M, \tau)$. $\mathcal{A}$ is contained on a unique maximal $\mathcal{C}^k$-atlas.* □

*Proof:*

Consider the set $\bar{\mathcal{A}}$ of all charts $\mathcal{C}^k$-compatible with $\mathcal{A}$. Notice that $\mathcal{A} \subseteq \bar{\mathcal{A}}$ and, as a consequence, $\bar{\mathcal{A}}$ is an atlas, for it is a collection of charts that covers $M$. We must now prove that it is a $\mathcal{C}^k$-atlas and that it is maximal.

Let $(U, \varphi), (V, \psi) \in \bar{\mathcal{A}}$. By hypothesis, both of them are $\mathcal{C}^k$-compatible with $\mathcal{A}$ and, due to Lemma 4.16 on the preceding page, are compatible with each other. Therefore, $\bar{\mathcal{A}}$ is a $\mathcal{C}^k$-atlas.

Suppose now $\mathcal{A}'$ is a $\mathcal{C}^k$-atlas containing $\bar{\mathcal{A}}$. Notice $\mathcal{A} \subseteq \bar{\mathcal{A}} \subseteq \mathcal{A}'$. Thus, every chart $(U, \varphi)$ in $\mathcal{A}'$ is $\mathcal{C}^k$-compatible with $\mathcal{A}$. Thus, by definition of $\bar{\mathcal{A}}$, every chart $(U, \varphi)$ of $\mathcal{A}'$ is in $\bar{\mathcal{A}}$, *id est*, $\mathcal{A}' \subseteq \bar{\mathcal{A}}$. Therefore, $\bar{\mathcal{A}} = \mathcal{A}'$, proving $\bar{\mathcal{A}}$ is maximal.

Finally, we must prove $\bar{\mathcal{A}}$ is unique. Suppose $\mathcal{A}'$ is some $\mathcal{C}^k$-atlas with $\mathcal{A} \subseteq \mathcal{A}'$. Then every chart in $\mathcal{A}'$ is compatible with $\mathcal{A}$ and hence $\mathcal{A}' \subseteq \bar{\mathcal{A}}$, so either $\mathcal{A}' = \bar{\mathcal{A}}$ or $\mathcal{A}'$ is not maximal. One way or the other, the proof is complete. ■

Proposition 4.17 guarantees that, when proving some topological space is a $\mathcal{C}^k$-manifold, we do not need to bother with describing the whole maximal atlas. Instead, it suffices to find *some* atlas and the existence of a maximal atlas is guaranteed.

A result due to Hassler Whitney states that, for every $k > 0$, a maximal $\mathcal{C}^k$-atlas contains a smooth atlas[43]. As a consequence, we will be mostly interested on the theory of smooth manifolds.

The restriction $k \neq 0$ is important: there are examples of topological manifolds that do not admit a smooth structure. The first example[44] of such a manifold is a 10-dimensional manifold constructed by Michel Kervaire in 1960[39].

Let us check a few examples of manifolds.

**Example** [Euclidean Space]**:**

The first example of smooth manifold one might consider is $\mathbb{R}^n$ itself, which is a Hausdorff, second-countable space. An atlas is given by $\{(\mathbb{R}^n, \mathrm{id})\}$, where $\mathrm{id} \colon \mathbb{R}^n \to \mathbb{R}^n$ is the function that maps $x \mapsto x$. ♥

**Example** [Locally Euclidean space which is not Hausdorff]**:**

A simple example of a locally euclidean space which is not Hausdorff is the line with two origins: the real line with an extra point.

We begin by picking some set which we already know to exist. As any set will do, let ☙ denote a leaf. We write $X = \mathbb{R} \cup \{☙\}$.

We now proceed to define a topology in $X$. Let $\mathfrak{B}_{\mathbb{R}}$ be the basis of open intervals for the standard topology in $\mathbb{R}$. Let $\mathfrak{B}_{☙} \equiv \{\{☙\} \cup B \setminus \{0\}; B \in \mathfrak{B}_{\mathbb{R}}\}$. We define $\mathfrak{B} \equiv \mathfrak{B}_{\mathbb{R}} \cup \mathfrak{B}_{☙}$. $\mathfrak{B}$ is a basis for a non-Hausdorff topology in $X$. On the other hand, every point $p$ has an open neighbourhood which can be mapped with the identity (or with a quasi-identity $x \mapsto x$ for $x \neq$ ☙ and ☙ $\mapsto 0$) to $\mathbb{R}$. Thus, it is locally Euclidean. ♥

**Example** [2-sphere]:

The construction made on the beginning of this chapter can be used to prove that $S^2$ is a smooth manifold. ♥

**Definition 4.18** [Surface in $\mathbb{R}^3$]:

Let $S \subseteq \mathbb{R}^3$. $S$ is said to be a *surface* if, and only if, for every $p \in S$ there is an open neighborhood $V_p$ of $p$ in $\mathbb{R}^3$, an open subset $U_p$ of $\mathbb{R}^2$, and a smooth map $f_p \colon U_p \to \mathbb{R}^3$ such that $S \cap V_p$ is the graph of $z = f_p(x,y)$, $x = f_p(y,z)$ or $y = f_p(z,x)$. ♠

**Proposition 4.19:**

*Let $S \subseteq \mathbb{R}^3$ be a surface. For every $p \in S$, let $V_p$ be an open neighborhood of $p$ in $\mathbb{R}^3$, $U_p$ be an open subset of $\mathbb{R}^2$, and $f_p \colon U_p \to \mathbb{R}^3$ be a smooth map such that $S \cap V_p$ is the graph of $z = f_p(x,y)$, $x = f_p(y,z)$ or $y = f_p(z,x)$. $(S, \tau, \mathcal{A})$ is a smooth manifold, where $\tau$ is the relative topology of $S$ with respect to $\mathbb{R}^3$ and $\mathcal{A}$ is the maximal smooth atlas associated to the smooth atlas*

$$\mathcal{A} = \{(S \cap V_p, \varphi_p); p \in M\}, \tag{4.11}$$

*where the maps $\varphi_p$ are defined such that $(x,y,z) \mapsto (x,y)$ when $S \cap V_p$ is the graph of $z = f_p(x,y)$, with similar definitions for the other cases.* □

*Proof:*

Since $\mathbb{R}^3$ is Hausdorff and second-countable, so is $(S, \tau)$.

Pick $p \in S$. We know there is an open neighborhood $V_p$ of $p$ in $\mathbb{R}^3$, an open subset $U_p$ of $\mathbb{R}^2$, and a smooth map $f_p \colon U_p \to \mathbb{R}^3$ such that $S \cap V_p$ is the graph of $z = f_p(x,y)$, $x = f_p(y,z)$ or $y = f_p(z,x)$. Notice that $S \cap V_p$ is an open neighborhood of $p$ in $(S, \tau)$.

Let us assume, without any loss of generality, that $f_p \colon U_p \to \mathbb{R}^3$ is such that $S \cap V_p$ is the graph of $z = f_p(x,y)$. Then $S \cap V_p = \{(x,y,f_p(x,y)) \in \mathbb{R}^3; (x,y) \in U_p\}$.

Let $\varphi_p \colon S \cap V_p \to \varphi_p(S \cap V_p)$ be given by $\varphi_p((x,y,z)) = (x,y)$. One may show this function is bijective and continuous (it is a projection). The inverse is $\varphi_p^{-1}((x,y)) = (x,y,f_p(x,y))$. Since $f_p$ is a smooth map, this is just a composition of continuous functions. Hence, $\varphi_p$ is a homeomorphism and we have proven $(S, \tau)$ is a topological manifold. The homeomorphisms are onto open sets of $\mathbb{R}^2$, and thus $\dim S = 2$.

We must now prove $\mathcal{A}$ is a smooth atlas. It surely is an atlas, since it holds that $\forall\, p \in M, \exists\, (S \cap V_p, \varphi_p) \in \mathcal{A}$ with $p \in S \cap V_p$.

Let $p, q \in M$ such that $S \cap V_p \cap V_q \neq \varnothing$ (otherwise, the result is trivial). We assume, without any loss of generality, that $S \cap V_p = \{(x,y,f_p(x,y)) \in \mathbb{R}^3; (x,y) \in U_p\}$ and $S \cap V_q = \{(x, f_q(z,x), z) \in \mathbb{R}^3; (z,x) \in U_q\}$. We want to prove that $\varphi_p \circ \varphi_q^{-1}$ is smooth (the proof already applies to the other transition function by simply exchanging $p$ and $q$).

Notice that $\varphi_q^{-1}((z,x)) = (x, f_q(z,x), z)$ and $\varphi_p((x,y,z)) = (x,y)$. Hence,

$$\begin{aligned}(\varphi_p \circ \varphi_q^{-1})((z,x)) &= \varphi_p\big((x, f_q(z,x), z)\big), \\ &= (x, f_q(z,x)).\end{aligned} \tag{4.12}$$

Since $f_q$ is smooth, $\varphi_p \circ \varphi_q^{-1}$ is a composition of smooth functions between Euclidean spaces, and hence $\varphi_p \circ \varphi_q^{-1}$ is smooth. This proves $\mathcal{A}$ is smooth, and thus $(S, \tau, \mathcal{A})$ is a smooth manifold of dimension 2. This concludes the proof. ■

Now that we have the structure to develop the theory of differentiability, let us define differentiability.

**Definition 4.20** [$\mathcal{C}^k$ Maps]**:**

Let $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ be $\mathcal{C}^k$-manifolds with $\dim M = m$ and $\dim N = n$ and let $p \in M$. A map $f\colon M \to N$ is said to be *of class $\mathcal{C}^k$ at $p$* if, and only if, there are charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$ and $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$.

The map $f$ is said to be *of class $\mathcal{C}^k$* if, and only if, it is of class $\mathcal{C}^k$ at $p$ for every $p \in M$. A $\mathcal{C}^\infty$ map is often called a *smooth map* or a *differentiable map*. The map $\psi \circ f \circ \varphi^{-1}$ is said to be a *local representation of $f$*. ♠

The definition of a $\mathcal{C}^k$ map can be visualized through a diagram.

$$\begin{tikzcd}
M \supseteq U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \subseteq N \arrow[d, "\psi"] \\
\mathbb{R}^m \supseteq \varphi(U) \arrow[r, "\psi\circ f\circ\varphi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^n
\end{tikzcd}$$

**Proposition 4.21:**

*The notion of a $\mathcal{C}^k$ map between two $\mathcal{C}^k$-manifolds is well-defined,* id est, *it does not depend on the charts chosen.* □

*Proof:*

Let $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ be $\mathcal{C}^k$-manifolds with $\dim M = m$ and $\dim N = n$ and let $p \in M$. Let $f\colon M \to N$ be a map and let there be are charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$ and $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$. We want to show that if there are charts $(W, \zeta) \in \mathcal{A}_M$ and $(X, \xi) \in \mathcal{A}_N$ with $p \in W$ and $f(W) \subseteq X$ such that $\xi \circ f \circ \zeta^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\zeta^{-1}(p)$.

Notice that $p \in U \cap W$ and $f(U \cap W) \subseteq V \cap X$. Once again, it is useful for us to look at a diagram.

$$\begin{tikzcd}
\mathbb{R}^m \supseteq \zeta(U \cap W) \arrow[r, "\xi \circ f \circ \zeta^{-1}"] & \xi(V \cap X) \subseteq \mathbb{R}^n \\
M \supseteq U \cap W \arrow[r, "f"] \arrow[u, "\zeta"] \arrow[d, "\varphi"'] & V \cap X \subseteq N \arrow[u, "\xi"'] \arrow[d, "\psi"] \\
\mathbb{R}^m \supseteq \varphi(U \cap W) \arrow[r, "\psi \circ f \circ \varphi^{-1}"'] & \psi(V \cap X) \subseteq \mathbb{R}^n
\end{tikzcd}$$

The diagram then invites us to notice that

$$\xi \circ f \circ \zeta^{-1} = (\xi \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \zeta^{-1}), \tag{4.13}$$

which, due to the fact that $\mathcal{A}_M$ and $\mathcal{A}_N$ are $\mathcal{C}^k$-atlases, is a composition of $\mathcal{C}^k$-maps in the sense of Real Analysis. Hence, $\xi \circ f \circ \zeta^{-1}$ is $\mathcal{C}^k$ in $\zeta(U \cap W)$ and, in particular, in $\zeta(p)$, proving the result. ■

**Proposition 4.22:**

*Let $(M, \tau_M, \mathcal{A}_M)$ and $(N, \tau_N, \mathcal{A}_N)$ be $\mathcal{C}^k$-manifolds and let $p \in M$. Let $f\colon M \to N$ be a $\mathcal{C}^k$-map at $p \in M$. $f$ is a $\mathcal{C}^l$-map at $p \in M$ for every $0 \leq l \leq k$. In particular, $f$ is continuous at $p \in M$.* □

*Proof:*

We know there be are charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$ and $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^k$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$. We know $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^l$ (in the sense of Real Analysis) at $\varphi^{-1}(p)$, for every $0 \leq l \leq k$.

In particular, we see that $\psi \circ f \circ \varphi^{-1}$ is continuous. Since $\psi$ and $\varphi$ are homeomorphisms, it follows that

$$f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi \tag{4.14}$$

is continuous at $p$. ■

**Proposition 4.23:**

*Let $(L, \tau_L, \mathcal{A}_L)$, $(M, \tau_M, \mathcal{A}_M)$, and $(N, \tau_N, \mathcal{A}_N)$ be $\mathcal{C}^k$-manifolds and let $p \in L$. Let $f\colon L \to M$ and $g\colon M \to N$ be $\mathcal{C}^k$-maps at $p \in L$ and at $f(p) \in M$, respectively. Then the map $g \circ f\colon L \to N$ is $\mathcal{C}^k$ at $p \in L$.* □

*Proof:*

We know there are charts $(U, \varphi) \in \mathcal{A}_L$ and $(V, \psi) \in \mathcal{A}_M$ such that $p \in U, f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^k$. Furthermore, there are charts $(W, \zeta) \in \mathcal{A}_M$ and $(X, \xi) \in \mathcal{A}_N$ with $f(p) \in W, g(W) \subseteq X$ and such that $\xi \circ g \circ \zeta^{-1}$ is $\mathcal{C}^k$. This can be represented through the diagrams

$$\begin{tikzcd} L \supseteq U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \subseteq M \arrow[d, "\psi"] \\ \mathbb{R}^l \supseteq \varphi(U) \arrow[r, "\psi\circ f\circ\varphi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^m \end{tikzcd} \qquad \begin{tikzcd} M \supseteq W \arrow[r, "g"] \arrow[d, "\zeta"'] & X \subseteq N \arrow[d, "\xi"] \\ \mathbb{R}^m \supseteq \zeta(W) \arrow[r, "\xi\circ g\circ\zeta^{-1}"'] & \xi(X) \subseteq \mathbb{R}^n \end{tikzcd}$$

Fortunately, $f$ is continuous as per Proposition 4.22 on the preceding page. Thus, $f^{-1}(V \cap W)$ is an open set. Since $f(U) \subseteq V$, we see that $f^{-1}(V \cap W) \subseteq U$. Thus, we may consider the chart $(f^{-1}(V \cap W), \varphi)$ and the following diagram:

$$\begin{tikzcd} & \mathbb{R}^n \supseteq \zeta(V \cap W) \arrow[r, "\xi\circ g\circ\zeta^{-1}"] & \xi(X) \subseteq \mathbb{R}^n \\ L \supseteq f^{-1}(V \cap W) \arrow[r, "f"] \arrow[d, "\varphi"'] & V \cap W \arrow[u, "\zeta"'] \arrow[r, "g"] \arrow[d, "\psi"] & X \subseteq N \arrow[u, "\xi"'] \\ \mathbb{R}^l \supseteq \varphi(f^{-1}(V \cap W)) \arrow[r, "\psi\circ f\circ\varphi^{-1}"'] & \psi(V \cap W) \subseteq \mathbb{R}^n & \end{tikzcd}$$

We see we may write

$$\xi \circ (g \circ f) \circ \varphi^{-1} = (\xi \circ g \circ \zeta^{-1}) \circ (\zeta \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}), \tag{4.15}$$

which is just a composition of $\mathcal{C}^k$ maps in the sense of Real Analysis. Hence, we have found charts $(f^{-1}(V \cap W), \varphi) \in \mathcal{A}_L$ and $(X, \xi) \in \mathcal{A}_N$ with $p \in f^{-1}(V \cap W)$, $(g \circ f)\big(f^{-1}(V \cap W)\big) \subseteq X$ and such that $\xi \circ (g \circ f) \circ \varphi^{-1}$ is $\mathcal{C}^k$ at $p \in L$. We may thus conclude $g \circ f$ is $\mathcal{C}^k$ at $p$. ■

*Notation:*

We shall often be lazy and say "Let $M$ be a manifold" instead of "Let $(M, \tau, \mathcal{A})$ be a manifold" for simplicity. Whenever this happens, the topology and atlas of the manifold should be clear from context.

One should notice this is just depraved notation and the manifold is the triple, not simply the set. ♦

*Notation:*

Given two $\mathcal{C}^k$-manifolds $M$, $N$, we denote by $\mathcal{C}^k(M, N)$ the space of all $\mathcal{C}^k$ functions $f\colon M \to N$.

We shall often write $\mathcal{C}^k(M) \equiv \mathcal{C}^k(M, \mathbb{R})$. ♦

*Remark:*

From now on, we shall focus on the theory of differentiable (*id est*, smooth) manifolds instead of $\mathcal{C}^k$-manifolds. ♣

**Definition 4.24** [Diffeomorphisms]**:**

Let $M$ and $N$ be smooth manifolds and let $f \colon M \to N$ be a function. $f$ is said to be a *diffeomorphism* if, and only if, it is invertible and both $f$ and $f^{-1}$ are differentiable. Under this condition, $M$ and $N$ are said to be *diffeomorphic*.

$f$ is said to be a *local diffeomorphism at a point* $p \in M$ if, and only if, there are open neighbourhoods $U \subseteq M$ and $V \subseteq N$ with $p \in U$ and $f(p) \in V$ such that $f|_U : U \to V$ is a homeomorphism, where $f|_U(p) = f(p), \forall p \in U$. ♠

**Theorem 4.25:**

*Let $M$ and $N$ be smooth manifolds. The relation $M \simeq N \Leftrightarrow M$ and $N$ are diffeomorphic is an equivalence relation.* □

*Proof:*

One can check that the identity map $\mathrm{id} : M \to M$ that maps $p \mapsto p$ is a diffeomorphism, for given a chart $(U, \varphi)$, one has $\varphi \circ \mathrm{id} \circ \varphi^{-1} = \mathrm{id}_{\mathbb{R}^n}$, which is smooth. Thus, $M \simeq M$.

If $f \colon M \to N$ is a diffeomorphism (meaning $M \simeq N$), $f^{-1}$ is also a diffeomorphism and we see that $N \simeq M$.

Finally, suppose $L \simeq M$ and $M \simeq N$ with diffeomorphisms $f \colon L \to M$ and $g \colon M \to N$. Proposition 4.23 on page 109 guarantees $g \circ f \colon L \to N$ is a diffeomorphism and thus $L \simeq N$. ■

**Definition 4.26** [Support of a Function]**:**

Let $(X, \tau)$ be a topological space. Let $f \colon X \to V$, where $V$ is a vector space. The *support of $f$*, denoted $\operatorname{supp} f$, is defined through

$$\operatorname{supp} f = \overline{\{x \in X; f(x) \neq 0\}}, \tag{4.16}$$

where 0 stands for the null vector. ♠

**Lemma 4.27:**

*Let $p \in \mathbb{R}^n$ and $0 < \delta < r$. There is a function $\beta \in C^\infty(\mathbb{R}^n)$ taking values in $[0, 1]$, $\beta(\mathcal{B}_\delta(p)) = \{1\}$ and with compact support in $\mathcal{B}_r(p)$.* □

*Proof:*

Let $\epsilon > 0$ be such that $\delta < \epsilon < r$. Consider the function

$$\beta(x) = \frac{\int_{\|x\|}^{\epsilon} g(t)\,\mathrm{d}t}{\int_{\delta}^{\epsilon} g(t)\,\mathrm{d}t}, \tag{4.17}$$

where

$$g(t) = \begin{cases} e^{-(t-\delta)^{-1}} e^{(t-\epsilon)^{-1}} & \text{for } \delta < t < \epsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

It can be shown (for example by handling it as an exercise to a Calculus student) that $g(t)$ is smooth. Hence, so is $\beta(x)$.

Notice that for $\|x\| \geq \epsilon$ (*id est*, $x \notin \mathcal{B}_\epsilon(0)$), it holds that $\beta(x) = 0$. Hence,

$$\operatorname{supp} \beta \subseteq \overline{\mathcal{B}_\epsilon(0)} \subset \mathcal{B}_r(0). \tag{4.19}$$

$\operatorname{supp} \beta$ is compact by the Heine-Borel Theorem.

The generic case of a ball centered at any point follows through composition with a translation. ■

**Definition 4.28** [Diameter of a Subset of a Metric Space]**:**

Let $(M, d)$ be a metric space and $A \subseteq M$ be a bounded set, *id est*, let there be $r > 0$ and $p \in M$ such that $A \subseteq \mathcal{B}_r(p)$. We define the *diameter of* $A$, denoted $\operatorname{diam} A$, through

$$\operatorname{diam} A = \sup_{x,y \in A} d(x, y). \tag{4.20}$$

♠

**Lemma 4.29:**

*Let $K \subseteq \mathbb{R}^n$ be a compact set and $O \subseteq \mathbb{R}^n$ be an open set such that $K \subseteq O$. Then there is a function $\beta \in \mathcal{C}^\infty(\mathbb{R}^n)$ taking values in $[0, 1]$, $\beta(K) = \{1\}$ and has compact support in $O$.* □

*Proof:*

For each $p \in K$, let $U_p$ be an open ball centered at $p$ such that $U_p \subseteq O$ and $K_p$ be the closure of the open ball centered at $p$ with half the radius of $U_p$. The Heine-Borel Theorem ensures every $K_p$ is compact.

Notice that the collection $\left\{\mathring{K}_p\right\}_{p \in K}$ is an open cover of $K$. Since $K$ is compact, there is a finite subcover $\left\{\mathring{K}_{p_\lambda}\right\}_{\lambda \in \Lambda}$ of $K$. For each $\lambda \in \Lambda$, Lemma 4.27 on the preceding page ensures, $\forall\, \lambda \in \Lambda$, the existence of a function $\beta_\lambda \in \mathcal{C}^\infty(\mathbb{R}^n)$ that vanishes outside of $U_{p_\lambda}$, is constantly 1 throughout $K_{p_\lambda}$ and has compact support in $U_{p_\lambda}$.

We may now define

$$\beta(x) = 1 - \prod_{\lambda \in \Lambda} (1 - \beta_\lambda(x)). \tag{4.21}$$

This function is a composition of smooth functions (hence it is smooth), is constantly 1 throughout $K$ (for $K \subseteq \bigcup_{\lambda \in \Lambda} K_{p_\lambda}$).

Notice that $\beta(x) = 0 \Leftrightarrow \beta_\lambda(x) = 0, \forall\, \lambda \in \Lambda$. Thus, $\operatorname{supp} \beta = \bigcup_\lambda \operatorname{supp} \beta_\lambda$. Since $\operatorname{supp} \beta_\lambda \subseteq U_{p_\lambda}, \forall\, \lambda \in \Lambda$, and $\bigcup_{\lambda \in \Lambda} U_{p_\lambda} \subseteq O$, it follows that $\operatorname{supp} \beta \subseteq O$.

Since $K$ is compact, it is closed and bounded by the Heine-Borel Theorem. Let $\operatorname{diam} K = d$ and $\epsilon$ be the supremum of the radii of the open balls $U_{p_\lambda}$. Notice that $\operatorname{supp} \beta \subseteq \mathcal{B}_{d+\epsilon}(0)$, and hence $\operatorname{supp} \beta$ is bounded. Since it is already closed by definition, the Heine-Borel Theorem guarantees $\operatorname{supp} \beta$ is compact. ■

**Theorem 4.30** [Existence of Cut-Off Functions]**:**

*Let $M$ be a smooth manifold. Let $K \subseteq M$ be a compact set and $O \subseteq M$ be an open set such that $K \subseteq O$. Then there is a function $\beta \in \mathcal{C}^\infty(M)$ taking values in $[0, 1]$, $\beta(K) = \{1\}$ and has compact support in $O$.* □

*Proof:*

Suppose firstly that there is a chart $(U, \varphi)$ such that $K \subseteq U$. In this case, $\varphi(U)$ is an open set and $\varphi(K)$ is a compact set with $\varphi(K) \subseteq \varphi(U)$. Lemma 4.29 on the preceding page ensures the existence of a function $\beta^* \in \mathcal{C}^\infty(\mathbb{R}^n)$ taking values in $[0, 1]$, with $\beta^*(\varphi(K)) = \{1\}$ and with compact support in $\varphi(U)$. Therefore, $\beta = \beta^* \circ \varphi$ satisfies the requirements we have.

This notion can be illustrated in the following diagram.

$$\begin{tikzcd} M \supseteq U \arrow[r, "\varphi"] \arrow[dr, "\beta^* \circ \varphi"'] & \varphi(U) \subseteq \mathbb{R}^n \arrow[d, "\beta^*"] \\ & \mathbb{R} \end{tikzcd}$$

Suppose now $K$ is not contained in the coordinate neighborhood of any chart. Since every atlas covers the whole manifold, we know the coordinate neighborhoods of the charts that compose the manifold's atlas cover $K$. For every chart $(U_\lambda, \varphi_\lambda)$ we can attribute compact sets $K_\lambda$ with $K_\lambda \subseteq U_\lambda$.

Since $K$ is compact, we know there is a finite collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that $K \subseteq \bigcup_{i \in I} \mathring{K}_i \subseteq \bigcup_{i \in I} U_i$. We may, without any loss of generality, pick $U_i \subseteq O, \forall\, i \in I$, since $K \subseteq O$ and $(U_i \cap O, \varphi_i\big|_O)$ is a chart just as good as $(U_i, \varphi_i)$.

For each $i \in I$ we are left with the case in which $K$ lies inside the coordinate neighborhood of a chart, and this yields a collection of functions $\beta_i$ as per the beginning of the proof. One might then notice that

$$\beta(x) = 1 - \prod_{i \in I}(1 - \beta_i(x)) \tag{4.22}$$

satisfies the requires properties. ∎

## 4.2 Tangent Spaces and Fiber Bundles

With a definition of differentiability at hands, we may once more search for the velocity of a ship navigating on Earth. However, once again I'll delay the subject and present a question: what do we mean by velocity?

As usual, we shall still think of velocity as the time derivative of space, which is a function of the real parameter we call time. In a more abstract manner, we can think of velocities as the derivatives of smooth curves defined on our manifold.

This seems simple enough, but there is an issue: while on Euclidean space one doesn't need to bother with the point in which the vector is defined. Figure 4.1 illustrates this with the fact that summing velocities defined on different points will yield us something that is not the velocity of any curve on the manifold (for it is not tangent to the manifold).

In order to understand how to define vectors on manifolds, let us begin by working with surfaces in Euclidean space (Proposition 4.19 on page 107 guarantees these are manifolds if we are in 3D space) and then proceed to remove the unnecessary structure.

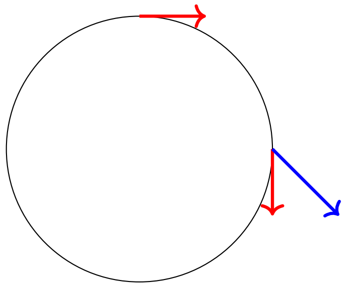


**Figure 4.1:** *Vectors in manifolds are more delicate than on Euclidean space: one can't add vectors defined on different points of the manifold*

For a given point $p \in \mathbb{R}^n$, we define the geometric tangent space to $\mathbb{R}^n$ at $p$, denoted $\mathbb{R}^n_p$, as $\mathbb{R}^n_p = \{(p, v); v \in \mathbb{R}^n\}$. A geometric tangent vector to $\mathbb{R}^n$ at $p$ is then simply an element of $\mathbb{R}^n_p$. For simplicity, we write $v_p \equiv v\big|_p \equiv (p, v) \in \mathbb{R}^n_p$.

Notice that $\mathbb{R}^n_p$ can be made into a vector space by introducing the operations $v_p + u_p = (v + u)_p$ and $\lambda \cdot v_p = (\lambda \cdot v)_p$.

Given a surface $S \subseteq \mathbb{R}^n$, the tangent vectors to $S$ at a point $p \in S$ should then be simply a subset of $\mathbb{R}^n_p$. The issue we face is we can't generalize this notion to arbitrary manifolds, since it highly depends on the ambient space. The structures we do have on manifolds are notions of functions, smoothness, coordinate charts, and so on. Thus, we should look for how tangent vectors relate to these concepts in Euclidean space.

When dealing with the theory of real-valued functions defined on $\mathbb{R}^n$, a concept that arises and is connected to the idea of a tangent vector is the notion of directional derivative. Indeed, given $v_p \in \mathbb{R}^n_p$, there is an operator $D_{v_p} \colon \mathcal{C}^\infty(M) \to \mathbb{R}$ which associated a function with its directional derivative in the direction of $v$ at the point $p$. It is such that

$$D_{v_p} f = \left[\frac{\mathrm{d}}{\mathrm{d}t} f(p + tv)\right]_{t=0}. \tag{4.23}$$

As all good derivatives, these operators respect the Leibniz rule:

$$\begin{aligned}
D_{v_p}(fg) &= \left[\frac{\mathrm{d}}{\mathrm{d}t}(f(p+tv)g(p+tv))\right]_{t=0}, \\
&= \left[g(p+tv)\frac{\mathrm{d}}{\mathrm{d}t}f(p+tv) + f(p+tv)\frac{\mathrm{d}}{\mathrm{d}t}g(p+tv)\right]_{t=0}, \\
&= g(p)\left[\frac{\mathrm{d}}{\mathrm{d}t}f(p+tv)\right]_{t=0} + f(p)\left[\frac{\mathrm{d}}{\mathrm{d}t}g(p+tv)\right]_{t=0}, \\
&= g(p)D_{v_p}f + f(p)D_{v_p}g.
\end{aligned} \tag{4.24}$$

In a similar fashion, linearity of $\frac{\mathrm{d}}{\mathrm{d}t}$ over $\mathbb{R}$ implies $D_{v_p}(f+g) = D_{v_p}f + D_{v_p}g$ and $D_{v_p}(\lambda \cdot f) = \lambda \cdot D_{v_p} f$.

Suppose now we have a basis $\{e_i\}_{i=1}^n$. We may write $v_p = v^i \, e_i\big|_p$, with summation over repeated indices implied. The chain rule implies

$$\begin{aligned} D_{v_p} f &= \left[\frac{\mathrm{d}}{\mathrm{d}t} f(p+tv)\right]_{t=0}, \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(p+tv)^i \, \frac{\partial}{\partial x^i} f(x_1, x_2, x_3)\bigg|_{(x_1,x_2,x_3)=p}, \\ &= v^i \frac{\partial}{\partial x^i} f(p). \end{aligned} \tag{4.25}$$

Motivated by these constructions, given a point $p \in \mathbb{R}^n$ we may define a derivation at $p$ as a $\mathbb{R}$-linear operator $w \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that

$$w(fg) = f(p)wg + g(p)wf. \tag{4.26}$$

This in principle seems to generalize $D_{v_p}$. We let $T_p\mathbb{R}^n$ denote the collection of all derivations at $p$. $T_p\mathbb{R}^n$ can be regarded as a linear space, for if $v$ and $w$ are derivations at $p$ and $\lambda \in \mathbb{R}$, then

$$(v+w)(f) = v(f) + w(f), \quad (\lambda \cdot v)(f) = \lambda \cdot v(f) \tag{4.27}$$

can be shown to be $\mathbb{R}$-linear. Furthermore,

$$\begin{aligned} (v+w)(fg) &= v(fg) + w(fg), \\ &= f(p)vg + g(p)vf + f(p)wg + g(p)wf, \\ &= f(p)(vg+wg) + g(p)(vf+wf), \\ &= f(p)(v+w)g + g(p)(v+w)f, \end{aligned} \tag{4.28}$$

and thus $v + w$ is a derivation. A similar proof holds for $\lambda \cdot v$.

There are some more interesting properties about derivations. For instance, notice that if $v$ is a derivation at $p$, then

$$\begin{aligned} v(1) &= v(1 \cdot 1), \\ &= 1 \cdot v(1) + 1 \cdot v(1), \\ &= 2v(1), \end{aligned} \tag{4.29}$$

which implies $v(1) = 0$. Linearity guarantees $vf = 0$ for all constant functions $f$.

Furthermore, suppose $f(p) = g(p) = 0$. Then of course

$$\begin{aligned} v(fg) &= f(p)vg + g(p)vf, \\ &= 0 + 0, \\ &= 0. \end{aligned} \tag{4.30}$$

Nevertheless, he truly remarkable result is the fact that derivations and tangent vectors are one and the same thing: the map $v_p \mapsto D_{v_p}$ is an isomorphism between $\mathbb{R}^n_p$ and $T_p\mathbb{R}^n$.

The fact that $v_p \mapsto D_{v_p}$ is linear can be seen from the decomposition of $D_{v_p}$ into a basis. Given $\lambda \in \mathbb{R}, v_p, w_p \in \mathbb{R}^n_p$ we have

$$\begin{aligned} D_{v_p+\lambda w_p} f &= D_{(v+\lambda w)_p} f, \\ &= (v^i + \lambda w^i) \frac{\partial}{\partial x^i} f(p), \\ &= v^i \frac{\partial}{\partial x^i} f(p) + \lambda w^i \frac{\partial}{\partial x^i} f(p), \\ &= D_{v_p} f + \lambda D_{w_p} f, \end{aligned} \tag{4.31}$$

$\forall f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

In order to prove it is one-to-one, let us assume $D_{v_p} = 0$, *id est*, $D_{v_p} f = 0, \forall f \in \mathcal{C}^\infty(\mathbb{R}^n)$. The decomposition of $D_{v_p}$ in a basis shows this implies $v^i_p = 0$ for all components of $v_p$, and hence $v_p$ has to be the null vector. Thus, the kernel of $v_p \mapsto D_{v_p}$ is the trivial subspace $\{0\}$ and we conclude the transformation is injective.

Finally, let $w \in T_p\mathbb{R}^n$. We want to prove there is some $v_p \in \mathbb{R}^n_p$ such that $D_{v_p} = w$.

Let $\{e_i\}_{i=1}^n$ be a basis for $\mathbb{R}^n$. Consider the functions $x^j \colon \mathbb{R}^n \to \mathbb{R}$ such that $x^j(e_i) = \delta^j_i$, where $\delta^j_i$ stands for the Kronecker delta. Consider the tangent vector $v_p = v^i \, e_i\big|_p$ where $v^i = w(x^i)$.

Given $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, Taylor's Theorem* guarantees we may write (summation is implicit over repeated indices)

$$f(x) = f(p) + \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + (x^i - p^i)(x^j - p^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(p + t(x-p)) \, \mathrm{d}t \, . \tag{4.32}$$

The last term is a product of functions that vanish at $x = p$. Thus, the last term vanishes under a derivation at $p$. Hence,

$$\begin{aligned} wf &= w(f(p)) + w\left(\frac{\partial f}{\partial x^i}(p)(x^i - p^i)\right), \\ &= 0 + \frac{\partial f}{\partial x^i}(p)\Big(w(x^i) - w(p^i)\Big), \\ &= \frac{\partial f}{\partial x^i}(p) v^i, \\ &= v^i \frac{\partial}{\partial x^i} f(p), \\ &= D_{v_p} f. \end{aligned} \tag{4.33}$$

Now we are in position to define what is a vector in an arbitrary manifold.

*See, *exempli gratia*, [48] or Appendix C of [44]

**Definition 4.31** [Tangent Space]**:**

Let $M$ be a smooth manifold and let $p \in M$. A *derivation at* $p$ is an $\mathbb{R}$-linear operator $v\colon \mathcal{C}^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(p)vg + g(p)vf. \tag{4.34}$$

The set of all derivations at a point $p \in M$ is denoted $T_pM$ and referred to as the *tangent space to M at* $p$. An element of $T_pM$ is often called a *tangent vector to M at* $p$. ♠

**Lemma 4.32:**

*Let M be a smooth manifold, $p \in M$. Let $v \in T_pM$. The following hold:*

i. *if $f \in \mathcal{C}^\infty(M)$ is a constant function, $vf = 0$;*

ii. *if $f, g \in \mathcal{C}^\infty(M)$ are such that $f(p) = g(p) = 0$, then $v(fg) = 0$.* □

*Proof:*

We begin by showing the result for $f(q) = 1, \forall\, q \in M$.

$$\begin{aligned} v(1) &= v(1 \cdot 1), \\ &= 1 \cdot v(1) + 1 \cdot v(1), \\ &= 2v(1), \end{aligned} \tag{4.35}$$

which implies $v(1) = 0$. Since $v$ is $\mathbb{R}$-linear, it follows that $vf = 0$ for all constant functions $f$.

Suppose now $f(p) = g(p) = 0$. It follows that

$$\begin{aligned} v(fg) &= f(p)vg + g(p)vf, \\ &= 0 + 0, \\ &= 0. \end{aligned} \tag{4.36}$$

This concludes the proof. ■

**Proposition 4.33:**

*Let M be a smooth manifold and $p \in M$. $T_pM$ is a real vector space when equipped with the operations $+\colon T_pM \times T_pM \to T_pM$ and $\cdot\colon \mathbb{R} \times T_pM \to T_pM$ defined through*

$$(v + w)(f) = vf + wf, \quad (\lambda \cdot v)(f) = \lambda \cdot vf \tag{4.37}$$

*for all $f \in \mathcal{C}^\infty(M)$.* □

*Proof:*

Let us begin by proving $v + w \in T_pM, \forall\, v, w \in T_pM$. Linearity of $v + w$ is easily proven and we shall focus on showing $v + w$ is a derivation. Indeed, notice that

$$\begin{aligned} (v + w)(fg) &= v(fg) + w(fg), \\ &= f(p)vg + g(p)vf + f(p)wg + g(p)wf, \\ &= f(p)(vg + wg) + g(p)(vf + wf), \\ &= f(p)(v + w)g + g(p)(v + w)f. \end{aligned} \tag{4.38}$$

A similar argument applies to $\lambda \cdot v$.

The algebraic properties that characterize $T_pM$ as a vector space comes naturally from the fact that $vf$ is a real number for any $v \in T_pM$. ■

We shall prove that $T_pM$ is not only finite dimensional, but also has the same dimension as $M$ (even though $\dim T_pM$ should be understood in an algebraic sense and $\dim M$ in a topological sense). In order to do so, we will define a specific collection of derivations which shall be similar to partial derivatives. Afterwards, we shall prove that arbitrary derivations are just linear combinations of this particular set, just as directional derivatives in Real Analysis can be written in terms of the derivatives with respect to Cartesian coordinates.

*Notation:*

Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$, $\partial_i f$ denotes the partial derivative of $f$ with respect to its $i$-th argument. For example,

$$\partial_2 f(x,y,z) = \frac{\partial f}{\partial y}(x,y,z) \tag{4.39}$$

and so on. ♦

**Definition 4.34** [Partial Derivatives]**:**

Let $M$ be a smooth manifold, $p \in M$ and $(U,\varphi)$ be a chart such that $p \in U$. Let $f \in \mathcal{C}^\infty(M)$. We define a function $\frac{\partial f}{\partial \varphi^i}\colon U \to \mathbb{R}$ through

$$\frac{\partial f}{\partial \varphi^i}(p) := \partial_i (f \circ \varphi^{-1})(\varphi(p)). \tag{4.40}$$

♠

Notice this definition could also be stated as

$$\frac{\partial f}{\partial \varphi^i}(p) := \lim_{h\to 0} \frac{(f\circ\varphi^{-1})(\varphi^1(p),\dots,\varphi^i(p)+h,\dots,\varphi^n(p)) - (f\circ\varphi^{-1})(\varphi(p))}{h}. \tag{4.41}$$

**Definition 4.35** [Partial Derivatives at a Point]**:**

Let $M$ be a smooth manifold, $p \in M$ and $(U,\varphi)$ be a chart such that $p \in U$. We define the operator $\left(\frac{\partial}{\partial\varphi^i}\right)_p\colon \mathcal{C}^\infty(M) \to \mathbb{R}$ through

$$\left(\frac{\partial}{\partial\varphi^i}\right)_p f = \frac{\partial f}{\partial \varphi^i}(p) \tag{4.42}$$

for every $f \in \mathcal{C}^\infty(M)$. ♠

**Lemma 4.36:**

*Let $M$ be a smooth manifold, $p \in M$ and $(U,\varphi)$ be a chart such that $p \in U$. Then $\left(\frac{\partial}{\partial\varphi^i}\right)_p \in T_pM$.* □

*Proof:*

Linearity of derivatives ensures linearity of $\left(\frac{\partial}{\partial \varphi^i}\right)_p$. We shall prove it is a derivation. Let $f, g \in \mathcal{C}^\infty(M)$.

$$\begin{aligned}
\left(\frac{\partial}{\partial \varphi^i}\right)_p (fg) =& \frac{\partial (f \cdot g)}{\partial \varphi^i}(p),\\
=& \partial_i [(f \cdot g) \circ \varphi^{-1}](\varphi(p)),\\
=& \partial_i [(f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1})](\varphi(p)),\\
=& [(g \circ \varphi^{-1})(\varphi(p))] \partial_i (f \circ \varphi^{-1})(\varphi(p))\\
& + [(f \circ \varphi^{-1})(\varphi(p))] \partial_i (g \circ \varphi^{-1})(\varphi(p)),\\
=& g(p) \frac{\partial f}{\partial \varphi^i}(p) + f(p) \frac{\partial g}{\partial \varphi^i}(p),\\
=& g(p) \left(\frac{\partial}{\partial \varphi^i}\right)_p f + f(p) \left(\frac{\partial}{\partial \varphi^i}\right)_p g.
\end{aligned} \tag{4.43}$$

This concludes the proof. ■

**Lemma 4.37:**

*Let $M$ be a smooth manifold and $p \in M$. Let $v \in T_pM$. The following results hold:*

i. *if $f, g \in \mathcal{C}^\infty(M)$ are equal on a neighborhood of $p$, then $v(f) = v(g)$;*

ii. *if $h \in \mathcal{C}^\infty(M)$ is constant on a neighborhood of $p$, then $v(h) = 0$.* □

*Proof:*

i. Since $v$ is linear, we want to prove that $v(f - g) = 0$ if $f = g$ on some neighborhood $U$ of $p$. Hence, we want to prove that $v(f) = 0$ whenever $f$ vanishes on some neighborhood $U$ of $p$.

   Consider a function $\beta \in \mathcal{C}^\infty(M)$ such that $\beta(p) = 1$ and with $\beta(q) = 0, \forall\, q \in U^c$. The existence of such a function is ensured by Theorem 4.30 on page 112. Notice $f \cdot \beta = 0$ (while $f$ vanishes on $U$, $\beta$ vanishes outside of it). Linearity of $v$ ensures $v(0) = 0$. We thus have

$$\begin{aligned}
0 &= v(f\beta),\\
&= f(p) v\beta + \beta(p) v f,\\
&= 0 \cdot v\beta + 1 \cdot vf,\\
&= vf.
\end{aligned} \tag{4.44}$$

ii. Linearity guarantees it suffices to prove the result for $h = 1$ on a neighborhood $U$ of $p$, for any constant function $h$ can be written as $h = \alpha \cdot 1$ for $\alpha \in \mathbb{R}$. Furthermore, the first item implies we may assume $h(p) = 1, \forall\, p \in M$. Any function constant throughout $U$ will be equal to $\alpha h$ on $U$ and the previous item will enforce the result.

We have

$$
\begin{aligned}
v(1) &= v(1 \cdot 1), \\
&= 1v(1) + 1v(1), \\
&= 2v(1).
\end{aligned} \tag{4.45}
$$

Since $v(1) = 2v(1)$, we conclude $v(1) = 0$.

■

A remark should be made at this point: even though the tangent space seems to have a local behaviour - for the derivations select the specific point under considerations and derivatives lie on the tangent space -, its elements act on functions belonging to $\mathcal{C}^\infty(M)$, which is a global property. Wouldn't it be expected that derivations may also act on elements of $\mathcal{C}^\infty(U)$ for some open neighborhood $U$ of $p$?

Consider the map $\Phi\colon T_pU \to T_pM$ defined in a way such that $\Phi(v)\colon \mathcal{C}^\infty(M) \to \mathbb{R}$ is given by $\Phi(v)(f) = v(f|_U)$, where $f|_U : U \to \mathbb{R}$ is the map given by $f|_U(p) = f(p), \forall\, p \in U$. Since $v$ is a linear derivation, it holds that $\Phi(v)$ is as well, and thus $\Phi(v) \in T_pM$. $\Phi$ is also a linear transformation. Our goal is to prove $\Phi$ is an isomorphism (and therefore $T_pU$ and $T_pM$ are identical in a linear way).

Suppose $\Phi(v) = 0$. This means $v(f|_U) = 0, \forall\, f \in \mathcal{C}^\infty(M)$. Let $g \in \mathcal{C}^\infty(U)$ and let $\beta \in \mathcal{C}^\infty(M)$ be a function such that $\operatorname{supp}\beta \subseteq U$ and $\beta(K) = \{1\}$ for a compact set $K$ with $\mathring{K} \neq \varnothing$. The existence of such a function is guaranteed by Theorem 4.30 on page 112. The existence of such $K \subseteq U$ is ensured by the fact that $M$ is locally Euclidean. $\beta g$ can be understood as a function defined on all of $M$ that coincides with $g$ on some neighborhood of $p$. Notice now that due to Lemma 4.37 on the previous page we now see that

$$
\begin{aligned}
v(g) &= v([\beta g]_U), \\
&= \Phi(v)(\beta g), \\
&= 0.
\end{aligned} \tag{4.46}
$$

Therefore, $\operatorname{Ker}\Phi$ is the trivial linear subspace, which proves $\Phi$ is one-to-one.

Let now $v \in T_pM$. We want to find $w \in T_pU$ such that $\Phi(w) = v$. Given $\beta$ as above, we may define $w(f) = v(\beta f), \forall\, f \in \mathcal{C}^\infty(U)$. This yields

$$
\begin{aligned}
\Phi(w)f &= w\left( f\Big|_U \right), \\
&= v\left( \beta\, f\Big|_U \right), \\
&= v(\beta f), \\
&= v(f),
\end{aligned} \tag{4.47}
$$

where we used the fact that $\beta f$ and $f$ coincide in a neighborhood of $p$.

Therefore, given any chart $(U, \varphi)$ with $p \in U$, it holds that $T_pU$ and $T_pM$ are isomorphic. In particular, given two charts $(U, \varphi)$ and $(V, \psi)$ with $p \in U \cap V$ it also holds that

$T_pU$ and $T_pV$ are isomorphic, since isomorphisms are an equivalence relation between linear spaces.

In particular, we may study $T_pM$ by choosing a particular chart $(U, \varphi)$ such that $x, y \in \varphi(U) \Rightarrow tx + (1-t)y \in \varphi(U), \forall\, t \in [0,1]$. The existence of such charts in ensured by the fact that $\mathbb{R}^n$ is a locally convex space. Given an arbitrary chart $(V, \psi)$, we know there is a convex open set $O \subseteq \psi(V)$. Since $\psi$ is a homeomorphism, $U = \psi^{-1}(O)$ is open and we may define $\varphi = \psi\big|_U$. For more information, see Chapter 5 (in particular, Theorem 5.38 on page 151).

**Theorem 4.38:**

*Let $M$ be a smooth manifold with* $\dim M = n$ *and let $(U, \varphi)$ be a chart with $p \in U$. It holds that*

$$\left(\left(\frac{\partial}{\partial \varphi^1}\right)_p, \dots, \left(\frac{\partial}{\partial \varphi^n}\right)_p\right) \tag{4.48}$$

*is a basis for $T_pM$. If $\varphi^i \colon M \to R$ is the i-th coordinate function of $\varphi$, we may write, for any $v \in T_pM$,*

$$v = v(\varphi^i)\left(\frac{\partial}{\partial \varphi^n}\right)_p, \tag{4.49}$$

*where once again summation is implied over repeated indices.* □

*Proof:*

The previous discussion allow us to pick, without any loss of generality, $(U, \varphi)$ to be such that $\varphi(U)$ is convex. A translation allow us to choose $\varphi$ such that the chart is centered. A derivation always vanishes once applied to a constant, so the translation is meaningless from its point of view.

Let $g \in \mathcal{C}^\infty(\varphi(U))$. The Taylor Formula with remainder in integral form allows us to write, $\forall\, x \in \varphi(U)$,

$$g(x) = g(0) + \int_0^1 \frac{\partial g}{\partial x^i}(tx)\,\mathrm{d}t\, x^i, \tag{4.50}$$

where summation is implied over repeated indices. For simplicity, we define

$$g_i(x) = \int_0^1 \frac{\partial g}{\partial x^i}(tx)\,\mathrm{d}t\,. \tag{4.51}$$

Notice that $g_i(0) = \left(\frac{\partial g}{\partial x^i}\right)_0$.

Let now $f \in \mathcal{C}^\infty(M)$. We may define $g \equiv f \circ \varphi^{-1}$, according to the diagram below.

$$\begin{tikzcd} M \supseteq U \arrow[r, "\varphi"] \arrow[dr, "f"'] & \varphi(U) \subseteq \mathbb{R}^n \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

The Taylor formula for $g$ now yields

$$\begin{aligned} g(\varphi(q)) &= g(\varphi(p)) + g_i(\varphi(q))\varphi^i(q), \\ f(q) &= f(p) + f_i(q)\varphi^i(q), \end{aligned} \tag{4.52}$$

for some functions $f_i$. We may now apply the derivations $\left(\frac{\partial}{\partial \varphi^j}\right)_p$ to see

$$\begin{aligned} \left(\frac{\partial}{\partial \varphi^j}\right)_p f &= \left(\frac{\partial}{\partial \varphi^j}\right)_p f(p) + \left(\frac{\partial}{\partial \varphi^j}\right)_p (f_i \varphi^i), \\ &= 0 + f_i(p)\left(\frac{\partial}{\partial \varphi^j}\right)_p \varphi^i + \varphi^i(p)\left(\frac{\partial}{\partial \varphi^j}\right)_p f_i, \\ &= f_i(p)\left(\frac{\partial}{\partial \varphi^j}\right)_p \varphi^i, \\ &= f_j(p), \end{aligned} \tag{4.53}$$

where we used the fact that $\left(\frac{\partial}{\partial \varphi^j}\right)_p \varphi^i = \delta^i_j$. This can be seen from the definition provided at Eq. (4.41).

Given $v \in T_pM$, we may apply it to an arbitrary function $f \in \mathcal{C}^\infty(M)$ and see that

$$\begin{aligned} vf &= vf(p) + v(f_i\varphi^i), \\ &= 0 + v(\varphi^i)f_i(p) + \varphi^i(p)v(f_i), \\ &= v(\varphi^i)f_i(p), \\ &= v(\varphi^i)\left(\frac{\partial}{\partial \varphi^j}\right)_p f. \end{aligned} \tag{4.54}$$

Hence,

$$v = v(\varphi^i)\left(\frac{\partial}{\partial \varphi^j}\right)_p, \tag{4.55}$$

for all $v \in T_pM$. In particular, we see that $v = 0$ has only null coefficients, which implies $\left(\left(\frac{\partial}{\partial \varphi^1}\right)_p, \ldots, \left(\frac{\partial}{\partial \varphi^n}\right)_p\right)$ not only generates $T_pM$, but is also linearly independent. This proves $\left(\left(\frac{\partial}{\partial \varphi^1}\right)_p, \ldots, \left(\frac{\partial}{\partial \varphi^n}\right)_p\right)$ is a basis for $T_pM$. ■

**Corollary 4.39:**

*Let $M$ be a smooth manifold and $p \in M$.* $\dim T_pM = \dim M$. □

*Proof:*

Direct consequence of Theorem 4.38 on the previous page. ■

As interesting as the theory of tangent spaces can be, it only deals with vectors one point at a time. Physics commonly needs to deal with vector *fields*, which requires a way of viewing how a vector changes from a point to another in both space and time. Thus, we would like to have a structure connecting different $T_pM$'s in a consistent way. In order to do so, we shall refer to the theory of bundles.

**Definition 4.40** [Fiber Bundle]**:**
Let $M, F, E$ be topological spaces and $\pi\colon E \to M$ be a surjective continuous map. The quadruple $(E, \pi, M, F)$ is said to be a *fiber bundle over M with model fiber F* if, and only if, $\forall\, p \in M$ there is a neighborhood $U$ of $p$ and a homeomorphism $\varphi\colon \pi^{-1}(U) \to U \times F$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

where $\pi_1\colon U \times F \to U$ maps $(u, f) \mapsto u$. $\varphi$ is said to be a *local trivialization of E under U*. $E$ is called the *total space of the bundle*, $M$ is its *base space*, $F$ its *typical fiber* and $\pi$ its *projection*. For each $p \in M$, $E_p = \pi^{-1}(p)$ is called the *fiber over p*.

If $M, F,$ and $E$ are smooth manifolds, $\pi$ is smooth and the local trivializations can be taken to be diffeomorphisms, we say $(E, \pi, M, F)$ is a *smooth fiber bundle*. If the fiber bundle admits a trivialization over the entire base (known as a *global trivialization*) $M$, it is said to be a *trivial fiber bundle*. If a smooth fiber bundle happens to admit a global trivialization which is a diffeomorphism, it is said to be *smoothly trivial*. ♠

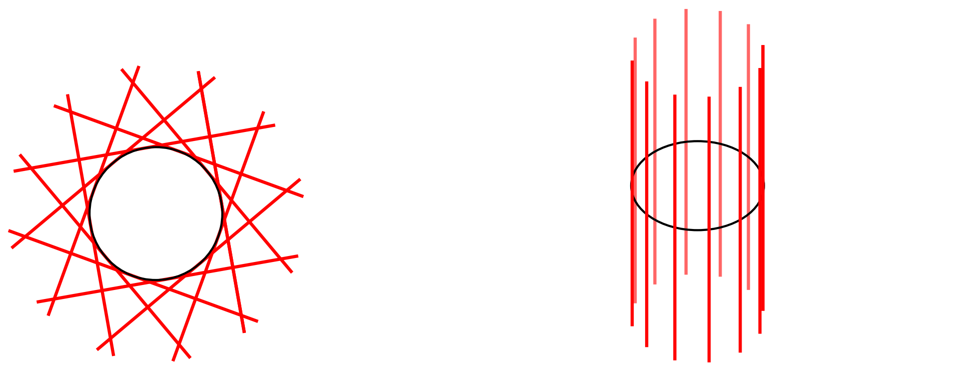
**Example** [Product Space is a Fiber Bundle]**:**
Let $M$ and $F$ be topological spaces and consider the product space $M \times F$. If we define $\pi\colon M \times F \to M$ such that $(m, f) \mapsto m$ - *id est*, if $\pi$ is the projection of $M \times F$ onto $M$ - then $(M \times F, \pi, M, F)$ is a fiber bundle.

Indeed, $\pi$ is continuous by the very definition of the product topology (the coarsest topology which maintains the projections continuous). It is surjective, for given any $m \in M$, one might pick any $f \in F$ and have $\pi((m, f)) = m$. Also, let $p \in M$ and let $U$ be a neighborhood of $p$. Notice that

$$\begin{aligned} \pi^{-1}(U) &= \{(m, f) \in M \times F; m \in U\}, \\ &= U \times F. \end{aligned} \tag{4.56}$$

Consider then the homeomorphism $\mathrm{id}\colon \pi^{-1}(U) \to U \times F$, where $\mathrm{id}(x) = x, \forall\, x \in U \times F$. The diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\mathrm{id}"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi"] \\ & U \end{tikzcd}$$



**(a)** *Depiction of the collection of tangent spaces to $S^1$ at various points*

**(b)** *Depiction of the tangent bundle as a "smooth, non-intersecting union" of tangent spaces*

**Figure 4.2:** *Depiction of how the tangent bundle arises from a "smoothed" union of tangent spaces*

commutes, for $\pi = \pi \circ \mathrm{id}$.

Notice that since id can be defined throughout all of $M \times F$, $(M \times F, \pi, M, F)$ is a trivial fiber bundle. ♥

This example might have given a hint on the idea behind fiber bundles: we want to deal with spaces that might not be product spaces, but that resemble product spaces locally.

Right now, we are particularly interested in a specific smooth fiber bundle: the tangent bundle of a manifold.

**Definition 4.41** [Tangent Bundle]**:**

Let $M$ be a smooth manifold. We define the *tangent bundle of $M$*, denoted $TM$, through

$$TM = \{(p, v); p \in M, v \in T_pM\}. \tag{4.57}$$

♠

**Example** [Tangent Bundle of $S^1$]**:**

We've seen that $S^2$ is a smooth manifold. A similar construction can be made to show that the unit circle, $S^1$, is a smooth manifold with $\dim S^1 = 1$.

For each point $p \in S^1$, we can picture the tangent space to $S^1$ at $p$, $T_pS^1$, as the tangent line to the circle at that point, given that $T_pS^1$ is a one-dimensional linear space. This is illustrated in Figure 4.2a

$TS^1$ as a set can be thought of just as depicted on Figure 4.2a. However, that isn't what one usually means when speaking of the tangent bundle. We shall now see how it can be equipped naturally with a topology and a smooth structure and be regarded as a smooth manifold. By giving it such properties, we picture $TS^1$ as in Figure 4.2b: a "smooth and non-intersecting union" of tangent spaces. No point is simultaneously on two tangent spaces and we will be able to move around the tangent bundle smoothly. ♥

**Proposition 4.42:**

*Let $M$ be a smooth manifold of dimension $n$. $TM$ can be regarded as a smooth manifold of dimension $2n$ such that the map $\pi\colon TM \to M$ defined by $\pi((p,v)) = p$ is smooth.* □

*Proof:*

Our first step will be building what will become a smooth structure for $TM$. Afterwards we shall use it to obtain a topology.

Let $(U, \varphi)$ be a smooth chart on $M$. Notice that

$$\pi^{-1}(U) = \{(p,v) \in TM; p \in U\}, \tag{4.58}$$

*id est*, $\pi^{-1}(U)$ is the collection of vectors* which are tangent to some $p \in U$.

■

*In fact, of ordered pairs $(p,v)$ with $p \in U$ and $v \in T_pM$, but we may think of it simply as the collection of vectors themselves

# Five

# Distribution Theory

Les papillons sont la seule chose importante…

Credited to LAURENT SCHWARTZ, when asked whether the butterflies are as important as Mathematics.

## 5.1 Weak Solutions of the Wave Equation

WE briefly discussed the idea of a weak solution in Section 1.1, and with our current knowledge of Topology we are able to further develop those concepts. This shall lead us to the study of Locally Convex Spaces and Distribution Theory. However, firstly let us motivate such studies through an example.

We shall study the problem of a vibrating string, under a regime of small oscillations. We suppose the string is able to move in a single dimension (which is the $u$ direction) and it vibrates along its rest position, which is the $x$ axis. Naturally, we call these transverse vibrations. The string is flexible, in the sense that it doesn't offer resistance to being bent. The deduction presented in here is the same one presented in [13] and depends mostly on Newtonian Mechanics. Another deduction, based upon Classical Field Theory, can be found at [1, 45].

Let $a$ and $b$ denote two arbitrary points on the string, $a < b$. If we denote the string linear density as $\rho(x,t)$, one should notice that, since the movement happens only in the $u$ direction, $\rho(x,t) = \rho(x)$, for the particles of the string cannot move sideways in order to "propagate mass". The total linear momentum in the piece of string between $a$ and $b$ is given by

$$p(t) = \int_a^b \rho(x) \frac{\partial u}{\partial t} \, \mathrm{d}x \,. \tag{5.1}$$

Notice that $\frac{\partial u}{\partial t}$ is the velocity of the string at the point $x$ and at time $t$.

The tension forces acting on the string are in the same direction as the string is in each point. Let us denote the tension force intensity at the point $x$ at the instant $t$ by $f(x,t)$. We understand this force as the force exerted by the rest of the string upon the piece between $a$ and $b$. As a consequence, the force at $b$ pulls the string to the right, while the force at $a$ pulls the string to the left. If the string makes an angle $\theta_x$ with the horizontal axis at the point $x$, we have that the tension forces must satisfy

$$f(a,t)\cos\theta_a = f(b,t)\cos\theta_b. \tag{5.2}$$

Indeed, $f(a,t)\cos\theta_a$ is the horizontal force in the point $a$ pulling the string to the left, while $f(b,t)\cos\theta_b$ is the horizontal force in the point $b$ pulling the string to the right. There is no movement in the horizontal direction, and as a consequence these forces should cancel each other out. We see now that the horizontal component of the tension forces ($f(x,t)\cos\theta_x$) does not depend on the position we are analyzing. We may then simply denote it by $\tau(t)$.

We might now turn our attention to the vertical forces. The vertical resultant force acting upon the piece of string between $a$ and $b$ is given by

$$\begin{aligned} f(b,t)\sin\theta_b - f(a,t)\sin\theta_a &= \tau(t)\tan(\theta_b) - \tau(t)\tan(\theta_a), \\ &= \tau(t)\left.\frac{\partial u}{\partial x}\right|_{x=a}^{x=b}, \\ &= \int_a^b \tau(t)\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x\,. \end{aligned} \tag{5.3}$$

We might as well consider extra terms in the vertical direction representing external forces. These could be a viscous drag due to the medium, gravitational effects, *et cetera*. The force in the vertical direction gains then a term of the form $\int_a^b h(x,t,u)\,\mathrm{d}x$, for some function $h$.

Finally, since $\frac{\mathrm{d}p}{\mathrm{d}t} = F$, we get the following integral equation for the movement of the string:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x)\frac{\partial u}{\partial t}\,\mathrm{d}x = \int_a^b \tau(t)\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x + \int_a^b h(x,t,u)\,\mathrm{d}x\,. \tag{5.4}$$

Let us assume that $\frac{\partial^2 u}{\partial t^2}$ is a continuous function. If so, the temporal derivative commutes with the integral. Due to the arbitrariness on $a$ and $b$, the integrands must be equal at every point and one concludes that

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \tau(t)\frac{\partial^2 u}{\partial x^2} + h(x,t,u). \tag{5.5}$$

If one defines $c(x,t) \equiv \sqrt{\frac{\tau(t)}{\rho(x)}}$ and $\varphi(x,t,u) \equiv -\frac{h(x,t,u)}{\tau(t)}$, one gets

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \varphi(x,t,u). \tag{5.6}$$

Currently, we are particularly interested in a specific case of this equation: the case for free oscillations (which implies $\varphi(x,t,u) = 0$) with small amplitude (allowing the assumption $\tau(t) = \tau(0)$) of a homogeneous ($\rho(x) = \rho(0)$) string. Under these conditions, $c$ is a constant and Eq. (5.6) becomes

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{5.7}$$

**Proposition 5.1:**

*Let $u(x,t)$ be a $C^2$ solution to the wave equation*

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{5.8}$$

*where $c$ is taken to be a real, positive constant. Then there are $C^2$ functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying*

$$u(x,t) = f(x+ct) + g(x-ct). \tag{5.9}$$

□

*Proof:*

Let us change variables in the wave equation. Let $\xi = x + ct$ and $\zeta = x - ct$. Furthermore, let us define the function $v(\xi, \zeta) = v(x+ct, x-ct) \equiv u(x,t)$. We now have

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x^2}, \\
&= \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\frac{\partial v}{\partial \xi} + \frac{\partial \zeta}{\partial x}\frac{\partial v}{\partial \zeta}\right), \\
&= \frac{\partial^2 v}{\partial \xi \partial x} + \frac{\partial^2 v}{\partial \zeta \partial x}, \\
&= \frac{\partial}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\frac{\partial v}{\partial \xi} + \frac{\partial \zeta}{\partial x}\frac{\partial v}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\partial \xi}{\partial x}\frac{\partial v}{\partial \xi} + \frac{\partial \zeta}{\partial x}\frac{\partial v}{\partial \zeta}\right), \\
&= \frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \zeta}\right), \\
&= \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \zeta}\frac{\partial^2 v}{\partial \zeta^2}, &(5.10) \\
\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= \cdots = \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \zeta} + \frac{\partial^2 v}{\partial \zeta^2}. &(5.11)
\end{aligned}$$

If we substitute these results in the wave equation, it follows that

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= 0, \\
\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \zeta} + \frac{\partial^2 v}{\partial \zeta^2} - \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \zeta} - \frac{\partial^2 v}{\partial \zeta^2} &= 0, \\
\frac{\partial^2 v}{\partial \xi \partial \zeta} &= 0. &(5.12)
\end{aligned}$$

Eq. (5.12) is far easier to be integrated. We can now see that

$$\frac{\partial v}{\partial \zeta} = h(\zeta), \tag{5.13}$$

for some function $h$. Integrating once again we have

$$v(\xi, \zeta) = f(\xi) + \int_{\zeta_0}^{\zeta} h(\zeta')\,\mathrm{d}\zeta' \,. \tag{5.14}$$

If we let $g(\zeta) \equiv \int_{\zeta_0}^{\zeta} h(\zeta')\,\mathrm{d}\zeta'$, we finally get that

$$\begin{aligned} v(\xi, \zeta) &= f(\xi) + g(\zeta), \\ u(x, t) &= f(x + ct) + g(x - ct), \end{aligned} \tag{5.15}$$

as promised. Notice that since $u$ is a $\mathcal{C}^2$ function, $f$ and $g$ must be as well. ■

We might use Proposition 5.1 on the preceding page to solve the problem of the wave equation subjected to certain initial conditions.

**Proposition 5.2** [d'Alembert's Formula]**:**
*Consider the initial valued problem determined by*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0, \forall\, x \in \mathbb{R}, \forall\, t > 0, \\ u(x, 0) = \varphi(x), \forall\, x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial t} = \psi(x), \forall\, x \in \mathbb{R}, \end{cases} \tag{5.16}$$

*where $\varphi \colon \mathbb{R} \to \mathbb{R}$ is a given $\mathcal{C}^2$ function and $\psi \colon \mathbb{R} \to \mathbb{R}$ is a given $\mathcal{C}^1$ function.*
*If there is a $\mathcal{C}^2$ solution $u(x, t)$ to the initial value problem, it is given by*

$$u(x, t) = \frac{1}{2}[\varphi(x + ct) + \varphi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s \,. \tag{5.17}$$

□

*Proof:*
We known from Proposition 5.1 on the previous page that there are real $\mathcal{C}^2$ functions $f$ and $g$ such that

$$u(x, t) = f(x + ct) + g(x - ct). \tag{5.18}$$

Let us now impose the initial conditions upon $u(x, t)$. Since $u(x, 0) = \varphi(x)$, we see that

$$f(x) + g(x) = \varphi(x). \tag{5.19}$$

The requirement that $\varphi$ is a $\mathcal{C}^2$ function is now justified: $f$ and $g$ are $\mathcal{C}^2$, and therefore their sum must be as well.

Since $\frac{\partial u}{\partial t} = \psi(x)$, we see that

$$cf'(x) - cg'(x) = \psi(x). \tag{5.20}$$

Notice that $f'$ and $g'$ are $\mathcal{C}^1$, and therefore $\psi$ must be as well. We now have

$$f(x) - g(x) = \frac{1}{c}\int_0^x \psi(s)\,\mathrm{d}s + k, \tag{5.21}$$

where $k$ is some real constant. Combining Eqs. (5.19) and (5.21) we get

$$\begin{cases} f(x) = \dfrac{1}{2}\varphi(x) + \dfrac{1}{2c}\displaystyle\int_0^x \psi(s)\,\mathrm{d}s + \dfrac{k}{2}, \\ g(x) = \dfrac{1}{2}\varphi(x) - \dfrac{1}{2c}\displaystyle\int_0^x \psi(s)\,\mathrm{d}s - \dfrac{k}{2}. \end{cases} \tag{5.22}$$

Since $u(x,t) = f(x+ct) + g(x-ct)$, we finally get d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s\,. \tag{5.23}$$

■

Let us now consider the problem of a plucked string. This represents the physical situation encountered when one is playing a harp or a guitar, for some examples. The PDE problem is given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0, \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t} = 0, \end{cases} \tag{5.24}$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a given function (notice I did not ask for it to be two times continuously differentiable this time).

*Notation:*

We define the *sign function* by

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \tag{5.25}$$

We define the *plus function* by

$$p(x) = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases} \tag{5.26}$$

We define the *Heaviside step function*, also known as *Heaviside's theta function*, by

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{5.27}$$

◆

A physically interesting problem is the obtained when we consider the plucked string with $\varphi(x) = p\left(b - \frac{b|x|}{a}\right)$, for some real numbers $a$ and $b$. This problem consists of lifting the string at a single point and then letting it vibrate freely. This initial condition does not satisfies the conditions for the d'Alembert formula to be applicable, but let us do it anyway.

The "solution" we find is

$$u(x,t) = \frac{1}{2}\left[p\left(b - \frac{b|x-ct|}{a}\right) + p\left(b - \frac{b|x+ct|}{a}\right)\right], \tag{5.28}$$

which satisfies $u(x,0) = p\left(b - \frac{b|x|}{a}\right)$.

$u$ is not differentiable everywhere, but let us try to calculate its derivatives anyway, for such a silly calculation might help us have some idea about how to make our results more general and adapt them to these situations. Notice that for $x \neq 0$ it holds that $p'(x) = \Theta(x)$. This leads us to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{-bc}{2a}\left[\Theta\left(b - \dfrac{b|x+ct|}{a}\right)\operatorname{sign}(x+ct) - \Theta\left(b - \dfrac{b|x-ct|}{a}\right)\operatorname{sign}(x-ct)\right], \\ \dfrac{\partial u}{\partial x} = \dfrac{-b}{2a}\left[\Theta\left(b - \dfrac{b|x+ct|}{a}\right)\operatorname{sign}(x+ct) + \Theta\left(b - \dfrac{b|x-ct|}{a}\right)\operatorname{sign}(x-ct)\right], \end{cases} \tag{5.29}$$

Notice that $\frac{\partial u}{\partial t} = 0, \forall\, x$. Let us write $\delta(x) \equiv \Theta'(x)$. If we "differentiate" the expressions again, one would get

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{b^2c^2}{2a^2}\left[\delta\left(b - \dfrac{b|x+ct|}{a}\right) + \delta\left(b - \dfrac{b|x-ct|}{a}\right)\right], \\ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{b^2}{2a^2}\left[\delta\left(b - \dfrac{b|x+ct|}{a}\right) + \delta\left(b - \dfrac{b|x-ct|}{a}\right)\right], \end{cases} \tag{5.30}$$

We see now that these weird derivatives led us to the conclusion that the wave equation is being satisfied in some sense by the solution we proposed. However, in this process we employed many results without respecting the necessary conditions, and we repeatedly differentiated functions that were not differentiable. Can we still make sense of these ideas?

In fact, we can. However, notice that the solutions we are proposing now are not classic solutions anymore, but weak solutions instead. In particular, the solution for a $k$-th order PDE doesn't even need to be a $C^k$ function.

This might seem like an issue, but it really isn't. When we deduced the wave equation, it was not a differential equation that we found, but the integral equation given on Eq. (5.4) on page 128.

In order to obtain the wave equation in its differential form, we assumed that $\frac{\partial^2 u}{\partial t^2}$ is continuous in order to commute the derivative and the integral sign. Only after some more manipulation we obtained Eq. (5.6) on page 128, which is more familiar.

Let us consider the space $L^2(\Omega)$ of square integrable real-valued functions - in the sense of Lebesgue - defined on some non-empty, open and connected subset $\Omega$ of $\mathbb{R}^n$. As usual, it is considered to be equipped with the norm

$$\|u\|_{L^2(\Omega)} \equiv \sqrt{\int_\Omega |f(x)|^2 \,\mathrm{d}x}. \tag{5.31}$$

Once again, the integral must be understood in the sense of Lebesgue.

We shall further consider the space $L^2_{\mathrm{loc}}(\Omega)$ of the functions which are in $L^2(K)$ for every compact set $K \subseteq \Omega$. Notice that $L^2(\Omega) \subseteq L^2_{\mathrm{loc}}(\Omega)$.

Let $\mathscr{D}(\Omega)$ denote the collection of smooth functions with compact support in $\Omega$. It is not trivial that there are nontrivial functions in this space. Nevertheless, notice that

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & \text{if } \|x\| < 1, \\ 0,, & \text{if } \|x\| \geq 1, \end{cases} \tag{5.32}$$

is such that $\varphi \in \mathscr{D}(\mathcal{B}_1(0))$. Given an open set $\Omega \subseteq \mathbb{R}^n$ and any point $p \in \Omega$, we can find $\epsilon > 0$ such that $\mathcal{B}_\epsilon(p) \subseteq \Omega$ and compose $\varphi$ with a translation and a multiplication by a scalar to obtain a smooth function with compact support in $\Omega$.

Given a function $f \in L^2_{\mathrm{loc}}(\Omega)$, we may define a notion of weak derivative. Let $\varphi \in \mathscr{D}(\Omega)$. If $f$ were differentiable, integration by parts would yield

$$\int_\Omega \frac{\partial f}{\partial x_i} \varphi \,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial x_i}(f\varphi) \,\mathrm{d}x - \int_\Omega f \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x\,. \tag{5.33}$$

Since $\varphi$ has compact support on $\Omega$, the first integral on the RHS vanishes, and we are left with

$$\int_\Omega \frac{\partial f}{\partial x_i} \varphi \,\mathrm{d}x = -\int_\Omega f \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x\,. \tag{5.34}$$

This motivates us to define $g \in L^2_{\mathrm{loc}}(\Omega)$ to be the $D_j$ generalized derivative of $f \in L^2_{\mathrm{loc}}(\Omega)$ if, and only if,

$$\int_\Omega g\varphi \,\mathrm{d}x = -\int_\Omega f D_j \varphi \,\mathrm{d}x\,, \tag{5.35}$$

for every $\varphi \in \mathscr{D}(\Omega)$, where $D_j \equiv \frac{\partial}{\partial x_j}$.

We denote by $W^{1,2}_{\text{loc}}(\Omega)$ the subspace of $L^2_{\text{loc}}(\Omega)$ of the functions $u$ with generalized derivatives $D_j u$ lying on $L^2_{\text{loc}}(\Omega)$. The norm considered on $W^{1,2}_{\text{loc}}(\Omega)$ is not the same considered on $L^2_{\text{loc}}(\Omega)$. Instead, we define

$$\|u\|_{W^{1,2}(\Omega)} = \sqrt{\int_\Omega |u|^2 \, \mathrm{d}x + \sum_{i=1}^{n} \int_\Omega |D_i u|^2 \, \mathrm{d}x}. \tag{5.36}$$

This is an example of a Sobolev space. Under this new picture, we may define a generalized, or weak, solution of the wave equation $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = 0$ to be any function $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} u \cdot \left( \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} \right) \mathrm{d}x \, \mathrm{d}t = 0 \tag{5.37}$$

for every $\varphi \in \mathscr{D}(\mathbb{R}^2)$.

Notice that we may understand these generalized solutions as linear functionals acting on elements of $\mathscr{D}(\Omega)$. In particular, we may write

$$\langle u, \varphi \rangle = \int_\Omega u\varphi \, \mathrm{d}x \, . \tag{5.38}$$

Linearity of the integral implies $\langle u, \varphi + \lambda \cdot \psi \rangle = \langle u, \varphi \rangle + \lambda \langle u, \psi \rangle, \forall \lambda \in \mathbb{R}, \forall \varphi, \psi \in \mathscr{D}(\Omega)$.

However, this particular formulation doesn't need anymore all of the structure we were dealing with earlier. We may now simply consider the topological dual $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ and define a notion of derivative where $\frac{\partial u}{\partial x_j}$ is the derivative of $u \in \mathscr{D}'(\Omega)$ with respect to $x$ whenever

$$\left\langle \frac{\partial u}{\partial x_j}, \varphi \right\rangle = -\left\langle u, \frac{\partial \varphi}{\partial x_j} \right\rangle \tag{5.39}$$

for every $\varphi \in \mathscr{D}(\Omega)$.

One should notice that any distribution - which is the name we give to elements of $\mathscr{D}'(\Omega)$ - $u$ admits derivatives $\frac{\partial u}{\partial x_j}, 1 \leq j \leq n$ and every such derivative is a distribution as well. Indeed, given a continuous linear functional $u$, we may always define $\frac{\partial u}{\partial x_j}$ by defining its action on any element of $\mathscr{D}(\Omega)$ through Eq. (5.39). Repeating the same procedure for the derivatives allow us to obtain the second derivatives, and at the end of the day one might simply write

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle \tag{5.40}$$

for any multiindex $\alpha \in \mathbb{N}^n$. This is possible due to the hypothesis that the test functions - which is the name we give to elements of $\mathscr{D}(\Omega)$ - $\varphi$ are always smooth, and it is consistent with the usual notion of derivative from real analysis due to the hypothesis that they are compact supported in $\Omega$. Thanks to this hypothesis, if $u$ is a $\mathcal{C}^{|\alpha|}(\Omega)$ function, then Eq. (5.40) follows from the divergence theorem.

Eq. (5.38) yields a way of obtaining examples of distributions. For example, a possible distribution in $\mathscr{D}'(\mathbb{R})$ would be the one defined by*

$$\left\langle e^{x^2}, \varphi \right\rangle = \int_{-\infty}^{+\infty} e^{x^2} \varphi(x) \, dx \,, \tag{5.41}$$

where $\varphi \in \mathscr{D}(\mathbb{R})$. A similar construction will work for a vast amount of functions: we only need the integral to converge and to be able to apply Lebesgue's Dominated Convergence Theorem in order to ensure continuity. Thus, as an example, any continuous function $f\colon \mathbb{R} \to \mathbb{R}$ will induce a distribution.

In order to understand this construction and notion of generalized solution, it seems natural for us to investigate the structure of the space $\mathscr{D}(\Omega)$.

We won't focus so much on $\mathscr{D}(\Omega)$, though. While it is an interesting space of test functions, it has poor properties when we want to deal with Fourier transforms, as we shall see on Section 5.7 on page 212. We'll rather deal with a similar space denoted $\mathscr{S}(\mathbb{R}^n)$ and known as Schwartz space, consisting of the so-called rapidly decreasing functions.

**Definition 5.3** [Functions of Rapid Decrease and Schwartz Space]**:**

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. $f$ is said to be a *function of rapid decrease* if, and only if, it holds that

$$\sup_{x \in \mathbb{R}^n} |p(x) D^{\alpha} f(x)| < +\infty \tag{5.42}$$

for every polynomial $p(x) = p(x_1, \ldots, x_n)$ and every multiindex $\alpha \in \mathbb{N}^n$.

The collection of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ of rapid decrease is named *Schwartz space* and usually denoted by $\mathscr{S}(\mathbb{R}^n)$ - or simply $\mathscr{S}$. ♠

**Proposition 5.4:**

*Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. $f$ is of rapid decrease if, and only if,*

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^{\alpha} D^{\beta} f(x) \right| < +\infty, \forall\, \alpha, \beta \in \mathbb{N}^n, \tag{5.43}$$

*where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $x^{\alpha} \equiv \prod_{i=1}^{n} x_i^{\alpha_i}$.* □

*Proof:*

Assume $f \in \mathscr{S}$ and let $\alpha \in \mathbb{N}^n$. Then $\sup_{x \in \mathbb{R}^n} \left| p(x) D^{\beta} f(x) \right| < +\infty$ for every polynomial $p(x) = p(x_1, \ldots, x_n)$ and every multiindex $\beta \in \mathbb{N}^n$. Pick the polynomial $p(x) = x^{\alpha}$. It follows that $\|f\|_{\alpha,\beta} < +\infty$.

Let us assume now that $\|f\|_{\alpha,\beta} < +\infty, \forall\, \alpha, \beta \in \mathbb{N}^n$. Notice that a polynomial in $n$ variables $p(x)$ can always be written in the form $p(x) = \sum_{i=1}^{m} a_i x^{\alpha_i}$ for some $m \in \mathbb{N}$,

*In order to state that this construction is a distribution, we should prove continuity. For now, this is just a motivational example, so we won't bother with rigour just yet.

appropriate multiindices $\alpha_i \in \mathbb{N}^n$ and coefficients $a_i \in \mathbb{R}$. Notice that given $\beta \in \mathbb{N}^n$, it holds $\forall\, x \in \mathbb{R}^n$ that

$$\begin{aligned} \left|p(x)D^\beta f(x)\right| &= \left|\sum_{i=1}^{m} a_i x^{\alpha_i} D^\beta f(x)\right|, \\ &\leq \sum_{i=1}^{m} |a_i| \left|x^{\alpha_i} D^\beta f(x)\right|. \end{aligned} \tag{5.44}$$

If we take the supremum on each side, it follows that

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right| &\leq \sup_{x\in\mathbb{R}^n} \left\{\sum_{i=1}^{m} |a_i| \left|x^{\alpha_i} D^\beta f(x)\right|\right\}, \\ &\leq \sum_{i=1}^{m} |a_i| \sup_{x\in\mathbb{R}^n} \left|x^{\alpha_i} D^\beta f(x)\right|. \end{aligned} \tag{5.45}$$

Hence, $\sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right|$ is less than or equal to a finite sum of finite terms. Therefore, $\sup_{x\in\mathbb{R}^n} \left|p(x)D^\beta f(x)\right| < +\infty$, and we conclude $f \in \mathscr{S}$. ■

**Theorem 5.5:**

*Consider the space $\mathscr{S}$ of functions of rapid decrease. Let $f, g \in \mathscr{S}$, $\lambda \in \mathbb{R}$. The following statements hold*

i. $\|\lambda \cdot f\|_{\alpha,\beta} = |\lambda| \cdot \|f\|_{\alpha,\beta}, \forall\, \alpha, \beta \in \mathbb{N}^n$;

ii. $\|f + g\|_{\alpha,\beta} \leq \|f\|_{\alpha,\beta} + \|g\|_{\alpha,\beta}, \forall\, \alpha, \beta \in \mathbb{N}^n$;

iii. *$\mathscr{S}$ is a real vector space;*

iv. $\|f\|_{\alpha,\beta} = 0, \forall\, \alpha, \beta \in \mathbb{N}^n \Rightarrow f = 0$. □

*Proof:*

Let $\lambda \in \mathbb{R}$. Notice that, for any $f \in \mathscr{S}$,

$$\begin{aligned} \|\lambda \cdot f\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta(\lambda \cdot f)(x)\right|, \\ &= \sup_{x\in\mathbb{R}^n} \left|\lambda \cdot x^\alpha D^\beta f(x)\right|, \\ &= \sup_{x\in\mathbb{R}^n} |\lambda| \left|x^\alpha D^\beta f(x)\right|, \\ &= |\lambda| \cdot \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta f(x)\right|, \\ &= |\lambda| \cdot \|f\|_{\alpha,\beta}. \end{aligned} \tag{5.46}$$

Let now $g \in \mathscr{S}$. With $f$ defined as before, we see that

$$\|f + g\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta (f+g)(x)\right|,$$

$$= \sup_{x\in\mathbb{R}^n}\left|x^\alpha\left(D^\beta f(x)+D^\beta g(x)\right)\right|. \tag{5.47}$$

Notice that, $\forall\, x \in \mathbb{R}^n$, it holds that

$$\left|x^\alpha D^\beta f(x)+x^\alpha D^\beta g(x)\right| \leq \left|x^\alpha D^\beta f(x)\right| + \left|x^\alpha D^\beta g(x)\right|. \tag{5.48}$$

Thus, we may take the supremum on both sides and see that

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta f(x)+x^\alpha D^\beta g(x)\right| &\leq \sup_{x\in\mathbb{R}^n}\left[\left|x^\alpha D^\beta f(x)\right| + \left|x^\alpha D^\beta g(x)\right|\right],\\
\|f+g\|_{\alpha,\beta} &\leq \sup_{x\in\mathbb{R}^n}\left[\left|x^\alpha D^\beta f(x)\right| + \left|x^\alpha D^\beta g(x)\right|\right],\\
&\leq \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta f(x)\right| + \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta g(x)\right|,\\
&= \|f\|_{\alpha,\beta}+\|g\|_{\alpha,\beta}.
\end{aligned} \tag{5.49}$$

This not only proves that $\|\lambda\cdot f\|_{\alpha,\beta} = |\lambda|\cdot\|f\|_{\alpha,\beta}$ and $\|f+g\|_{\alpha,\beta} \leq \|f\|_{\alpha,\beta}+\|g\|_{\alpha,\beta}$, but also that $\mathscr{S}$ is closed under pointwise addition and multiplication by scalar, since finiteness of $\|f\|_{\alpha,\beta}$ and $\|g\|_{\alpha,\beta}$ implies finiteness of $\|f+\lambda\cdot g\|_{\alpha,\beta}$. Since $0 \in \mathscr{S}$ - for constant functions are smooth and $\|0\|_{\alpha,\beta} = 0$ - and the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}$ is a real vector space, we see $\mathscr{S}$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$.

Suppose $f \in \mathscr{S}$ is such that $\|f\|_{\alpha,\beta} = 0, \forall\, \alpha,\beta \in \mathbb{N}^n$. Then, in particular, it holds that $\|f\|_{0,0} = 0$, where the multiindex 0 should be understood as the $n$-uple $(0,\dots,0)$. This means that $\sup_{x\in\mathbb{R}^n}|f(x)| = 0$. Since $\sup_{x\in\mathbb{R}^n}|f(x)| \geq |f(y)| \geq 0, \forall\, y \in \mathbb{R}^n$, we conclude $f(y) = 0, \forall\, y \in \mathbb{R}^n$. ■

**Definition 5.6** [Tempered Distribution]**:**

Consider the space $\mathscr{S}$ of functions of rapid decrease and let $\varphi\colon \mathscr{S} \to \mathbb{R}$ be a linear functional. $\varphi$ is said to be a *tempered distribution* if, and only if, it holds for every function $f \in \mathscr{S}$ that

$$\lim_{n\to+\infty}\langle\varphi, f_n\rangle = \langle\varphi, f\rangle \tag{5.50}$$

for every sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$ with the property that

$$\lim_{n\to\infty}\|f_n - f\|_{\alpha,\beta} = 0. \tag{5.51}$$

♠

As we shall see, this definition is equivalent to stating that tempered distributions are the continuous linear functionals acting on $\mathscr{S}$ under a topology induced by the functions $\|\cdot\|_{\alpha,\beta}$ defined on Eq. (5.43) on page 135. Since $\mathscr{D}(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$, every element of $\mathscr{S}'(\mathbb{R}^n)$ can have its domain restricted from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{D}(\mathbb{R}^n)$ and it will be continuous in the relative topology. Hence, we see $\mathscr{S}'(\mathbb{R}^n) \subseteq \mathscr{D}'(\mathbb{R}^n)$, meaning every tempered distribution is a distribution. Nevertheless, notice that the distribution defined on

Eq. (5.41) is not a tempered distribution, despite being a distribution. The function $e^{-x^2} \in \mathscr{S}(\mathbb{R})$ is not in the domain of that distribution, for the integral given by $\left\langle e^{x^2}, e^{-x^2} \right\rangle$ does not converge. This reflects the fact that $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$, where $\subset$ denotes proper inclusion.

The upshot of this discussion is that if we want to understand generalized solutions to differential equations — such as the wave equation — we should first understand distributions. However, playing around with distributions, and in particular with tempered distributions, led us to the study of Schwartz's space. In particular, it led us to trying to comprehend how a topology can be generated by functions such as the ones given on Eq. (5.43) on page 135. This shall lead us to the study of locally convex spaces.

## 5.2 Locally Convex Spaces

Let $\Lambda$ be a family of indexes. Suppose we have a vector space $V$ equipped with a family of norms $(\|\cdot\|_\lambda)_{\lambda \in \Lambda}$. If we want to turn such a space into a topological space, it would be natural to ask that a net $(x_\alpha)_{\alpha \in I}$ satisfies $x_\alpha \to x$ if, and only if, $\|x_\alpha - x\|_\lambda \to 0, \forall\, \lambda \in \Lambda$. In general, it is useful to weaken the condition required for a norm that $\|x\| = 0 \Leftrightarrow x = 0$. This should be done carefully though, due to the following results.

**Definition 5.7** [Pseudometric Space]**:**

Let $M$ be a set and $d \colon M \times M \to \mathbb{R}_+$ be a function satisfying the following axioms, $\forall\, x, y, z \in M$,

i. $d(x, x) = 0$;

ii. $d(x, y) = d(y, x)$;

iii. $d(x, y) \leq d(x, z) + d(y, z)$.

$(M, d)$ is said to be a *pseudometric space* and $d$ is said to be a *pseudometric* on $M$. ♠

**Lemma 5.8:**

*Let $(X, \tau)$ be a topological space. Let $\mathfrak{B}$ be a basis for the topology $\tau$ on $X$. The following statements are equivalent:*

i. *$(X, \tau)$ is Hausdorff;*

ii. $\forall\, x, y \in X, x \neq y, \exists\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}; x \in \mathcal{B}_1, y \in \mathcal{B}_2, \mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$. □

*Proof:*

$\Leftarrow$: Since every basis element is an open set, the Hausdorff condition holds.

$\Rightarrow$: Given $x, y \in X, x \neq y$, we know there are open sets $O$ and $U$ such that $x \in O, y \in U, O \cap U = \varnothing$. Since every open set is can be written as an union of elements of $\mathfrak{B}$, there is $\mathcal{B}_1 \in \mathfrak{B}; x \in \mathcal{B}_1 \subseteq O$ and $\mathcal{B}_2 \in \mathfrak{B}; y \in \mathcal{B}_2 \subseteq U$. As a consequence, notice that $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$, proving the result. ■

**Theorem 5.9:**

*Let $(M, d)$ be a pseudometric space. Let, $\forall\, \epsilon > 0, \forall\, x \in M$,*

$$\mathcal{B}_\epsilon(x) = (y \in M; d(x, y) < \epsilon). \tag{5.52}$$

*The set*

$$\mathfrak{B} = \{\mathcal{B}_\epsilon(x); \epsilon > 0, x \in M\} \tag{5.53}$$

*is a basis for a topology on M. Furthermore, the topology generated by $\mathfrak{B}$ is Hausdorff if, and only if, $(M, d)$ is a metric space.* □

*Proof:*

Let $x \in M$. Since $d(x, x) = 0$, we see that $\forall\, \epsilon > 0$, $x \in \mathcal{B}_\epsilon(x)$. Therefore, $\forall\, x \in M, \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$.

Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$. There are $\epsilon_1, \epsilon_2 > 0$ and $x_1, x_2 \in M$ such that $\mathcal{B}_1 = \mathcal{B}_{\epsilon_1}(x_1)$ and $\mathcal{B}_2 = \mathcal{B}_{\epsilon_2}(x_2)$. Let $x \in \mathcal{B}_i$. Notice that $\mathcal{B}_{\delta_i}(x) \subseteq \mathcal{B}_{\epsilon_i}(x_i)$, for $\delta_i = \epsilon_i - d(x_i, x)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\mathcal{B}_\delta(x) \subseteq \mathcal{B}_{\epsilon_1}(x_1) \cap \mathcal{B}_{\epsilon_2}(x_2)$. Thus, $\mathfrak{B}$ is indeed a basis for a topology in $M$.

We already know that if $(M, d)$ is a metric space, then the topology generated by $\tau$ is Hausdorff (Lemma 3.89 on page 68). Let us now assume the topology is Hausdorff. Lemma 5.8 on the facing page guarantees that $\forall\, x, y \in M, x \neq y, \exists\, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}; x \in \mathcal{B}_1, y \in \mathcal{B}_2, \mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$. This proof already showed that if, *exempli gratia*, $x \in \mathcal{B}_1$, then there is $\delta > 0$ such that $\mathcal{B}_\delta(x) \subseteq \mathcal{B}_1$. Since $y \in \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$, we know that $y \neq \mathcal{B}_\delta(x)$. Therefore, $d(x, y) \geq \delta > 0$, *id est*, $x \neq y \Rightarrow d(x, y) \neq 0$. Since $x = y \Rightarrow d(x, y) = 0$ by hypothesis, we may conclude $d(x, y) = 0 \Leftrightarrow x = y$, and therefore the Hausdorff axiom implies $(M, d)$ is a metric space. ■

**Definition 5.10** [Seminorm]**:**

Let $(V, \mathbb{F}, +, \cdot)$ be a vector space, with $\mathbb{F}$ being either the real line or the complex plane. A function $\|\cdot\|: V \to \mathbb{R}_+$ is said to be *seminorm* on $(V, \mathbb{F}, +, \cdot)$ whenever the following conditions hold, $\forall\, x, y \in V$:

i. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality);

ii. $\|\alpha \cdot x\| = |\alpha|\|x\|, \forall\, \alpha \in \mathbb{F}$. ♠

**Proposition 5.11:**

*Let $V$ be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\|\cdot\|$ be a seminorm on $V$. If we define $d: V \to \mathbb{R}_+$ by $d(x, y) := \|x - y\|$, $(V, d)$ is a pseudometric space. $(V, d)$ is a metric space if, and only if, $\|\cdot\|$ is a norm,* id est, *if it holds that $\|x\| = 0 \Rightarrow x = 0$.* □

*Proof:*

As mentioned in Section 3.1, it is sufficient to prove the triangle inequality and that $d(x, x) = 0$, for the remaining axioms are merely consequences of these[4].

Let $x, y, z \in V$.

$$\begin{aligned} d(x,y) &= \|x-y\|, \\ &= \|x-z+z-y\|, \\ &\leq \|x-z\| + \|z-y\|, \\ &= \|x-z\| + |-1|\|y-z\|, \\ &= d(x,z) + d(y,z). \end{aligned} \tag{5.54}$$

This proves the triangle inequality.

We also have

$$\begin{aligned} d(x,x) &= \|x-x\|, \\ &= \|0\|, \\ &= |0|\|0\|, \\ &= 0. \end{aligned} \tag{5.55}$$

Thus, $d$ is a pseudometric.

Suppose now that $d$ is a metric, *id est*, $d(x,y) = 0 \Rightarrow x = y$. If $\|x\| = 0$, we have that $d(x,0) = 0$ and thus $x = 0$. Conversely, if $\|\cdot\|$ is a norm, $\|x\| = 0 \Rightarrow x = 0$ means that $\|x-y\| = d(x,y) = 0$, then $x - y = 0$, *id est*, $x = y$, and we conclude $d$ is a metric. ■

*Remark:*

Given a vector space $V$ equipped with a seminorm $\|\cdot\|$, we know $d(x,y) = \|x-y\|$ is a pseudometric on $V$ and that this pseudometric induces a topology on $V$ through its open balls. We say such topology is induced by the seminorm. ♣

**Corollary 5.12:**

*Let $V$ be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\|\cdot\|$ be a seminorm on $V$. Let $\tau$ be the topology induced by $\|\cdot\|$. $(V, \tau)$ is Hausdorff if, and only if, $\|\cdot\|$ is a norm.* □

*Proof:*

Follows directly from Theorem 5.9 and Proposition 5.11 on the previous page. ■

Since the Hausdorff axiom guarantees uniqueness of limits (Proposition 3.124 on page 83), we would like to preserve it. Therefore, we will allow for $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ to be a family of seminorms, instead of a family of norms, but we shall make a further requirement in order to guarantee the uniqueness of limits.

**Definition 5.13** [Separate Points]**:**

Let $V$ be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\Lambda$ be a family of indices. For every $\lambda \in \Lambda$, let $\|\cdot\|_\lambda$ be a seminorm on $V$. The family $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ of seminorms is said to *separate points* if, and only if, $\|x\|_\lambda = 0, \forall\, \lambda \in \Lambda \Rightarrow x = 0$. ♠

With these definitions, we might now define what is a locally convex space.

**Definition 5.14** [Locally Convex Space]**:**

Let $X$ be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\Lambda$ be a family of indices. Let $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ be a family of seminorms on $X$ that separates points. $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ (which we will usually denote simply as $X$, whenever $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ is well understood) is said to be a *locally convex space*. The *natural topology* on such a space (which is always supposed to be equipped in it, unless we state otherwise) is the coarsest topology that maintains all of the seminorms, the addition of vectors and the multiplication by scalars continuous. ♠

*Remark:*

When stating that addition and scalar multiplication are continuous, what we mean is that the functions $+ \colon V \times V \to V$ (such that $+(x, y) = x + y$) and $\cdot \colon \mathbb{F} \times V$ (such that $\cdot(\alpha, x) = \alpha \cdot x$) are continuous when $V \times V$ and $\mathbb{F} \times V$ are equipped with the product topology. ♣

This definition allowed us to weaken the requirement that $\|x\|_\lambda = 0 \Rightarrow x = 0$, but without losing the Hausdorff property.

**Theorem 5.15:**

*Every locally convex space is Hausdorff.* □

*Proof:*

Let $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ be a locally convex space. For each $\lambda \in \Lambda, y \in X$ let $\rho_{\lambda,y}(x) \equiv \|x - y\|_\lambda, \forall x \in X$. Every such function $\rho_{y,\lambda}$ is a continuous function. Indeed, $\rho_{y,\lambda}$ is the composition of the functions $\|\cdot\|_\lambda$ and $x \mapsto x - y$, which are both continuous in the natural topology by hypothesis.

Let $x, y \in X, x \neq y$. We know that there is $\lambda \in \Lambda$ such that $\|x - y\|_\lambda \neq 0$ (otherwise, since the family of seminorms separates points, it would hold that $x = y$). Let $\epsilon = 3\|x - y\|_\lambda$. Consider the sets $O = \rho_{y,\lambda}^{-1}([0, \epsilon))$ and $U = \rho_{x,\lambda}^{-1}([0, \epsilon))$. Since $\rho_{z,\lambda}(z) = \|z - z\|_\lambda = 0, \forall z \in X$, it holds that $x \in O$ and $y \in U$. We want to prove that $O \cap U = \varnothing$. Suppose $z \in O \cap U$.

Since $z \in O$, it holds that $\|x - z\|_\lambda < \epsilon$. Since $z \in U$, it also holds that $\|y - z\|_\lambda < \epsilon$. The triangle inequality yields us that

$$\begin{aligned} 3\epsilon &= \|x - y\|_\lambda, \\ &\leq \|x - z\|_\lambda + \|y - z\|_\lambda, \\ &< 2\epsilon, \end{aligned} \tag{5.56}$$

which is a contradiction. Therefore, we conclude $\nexists z \in O \cap U$, *id est*, $O \cap U = \varnothing$. This proves that $X$ is indeed a Hausdorff space. ■

It is worth mentioning that locally convex spaces are a particular case of a wider class of spaces: the linear topological spaces.

**Definition 5.16** [Linear Topological Space]**:**

Let $X$ be a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is taken to be either the real line or the complex plane. Let $\tau$ be a topology on $X$. If the addition of vectors and scalar multiplication are continuous, $(X, \tau)$ is said to be a *linear topological space* and $\tau$ is said to be a *vector topology*. ♠

**Lemma 5.17:**

*Let $X$ be a linear topological space over $\mathbb{F}$. Let $a \in X, \alpha \in \mathbb{F}, \alpha \neq 0$. The functions $x \mapsto x + a$ and $x \mapsto \alpha \cdot x$ are homeomorphisms between $X$ and itself.* □

*Proof:*

Consider firstly the map $x \mapsto (x, a)$. Since the identity is always continuous (for every open set $O$, $\mathrm{id}^{-1}(O) = O$, which is clearly open) and so are continuous functions, Proposition 3.74 on page 60 guarantees that such a map is continuous. The addition of vector is always continuous in a linear topological space. Therefore, the map $x \mapsto x + a$ is a composition of continuous functions, which means it is continuous itself. Since the same argument holds for $x \mapsto x - a$ and these maps are the inverse of each other, we see $x \mapsto x + a$ is indeed a homeomorphism.

A similar argument holds for the multiplication by a non-zero scalar. It fails if the scalar is zero because there would be no inverse function, and thus the application would be continuous, but not a homeomorphism. ■

*Notation:*

Let $X$ be a vector space over $\mathbb{F}$. For a set $A \subseteq X$, we write

$$A + a \equiv \{x + a \in X; x \in A\}, \tag{5.57}$$
$$\alpha \cdot A \equiv \{\alpha \cdot x \in X; x \in A\}. \tag{5.58}$$

Equivalently, if $\mathcal{F} \subseteq \mathbb{P}(X)$,

$$\mathcal{F} + a \equiv \{A + a \in \mathbb{P}(X); A \in \mathcal{F}\}, \tag{5.59}$$
$$\alpha \cdot \mathcal{F} \equiv \{\alpha \cdot A \in \mathbb{P}(X); A \in \mathcal{F}\} \tag{5.60}$$

can also be written. ♦

**Proposition 5.18:**

*Let $X$ be a linear topological space over $\mathbb{F}$. Let $O \subseteq X$ be an open set and let $F \subseteq X$ be a closed set. Given $a \in X, \alpha \in \mathbb{F}, \alpha \neq 0$, $O + a$ and $\alpha \cdot O$ are open sets and $F + a$ and $\alpha \cdot F$ are closed sets.* □

*Proof:*

Since $x \mapsto x + a$ and $x \mapsto \alpha \cdot x$ are homeomorphisms between $X$ and itself, the image of an open (or closed) set under any of such functions is also open (closed). Since $A + a$ is the image of $A$ under $x \mapsto x + a$ and $\alpha \cdot A$ is the image of $A$ under $x \mapsto \alpha \cdot x$, the result follows. ■

**Corollary 5.19:**

*Let $X$ be a linear topological space over $\mathbb{F}$. Given a point $x \in X$, let $\mathfrak{N}_x \equiv \{O \in \tau; x \in O\}$. $\forall\, a \in X$, it holds that $\mathfrak{N}_a = a + \mathfrak{N}_0$.* □

*Proof:*

Follows from Proposition 5.18 on the facing page. ■

This result motivates the following definition.

**Definition 5.20** [Local Basis or System of Nuclei]:

Let $(X, \tau)$ be a linear topological space over $\mathbb{F}$. A neighborhood basis for $\tau$ at 0 is said to be a *local basis for* $\tau$ or a *system of nuclei*. ♠

**Proposition 5.21:**

*Let $(X, \tau)$ be a linear topological space over $\mathbb{F}$ and let $\mathfrak{N}$ be a system of nuclei for $\tau$. A set $O \subseteq X$ is an open set if, and only if, $\forall\, x \in O, \exists\, N \in \mathfrak{N}; (x + N) \subseteq O$.* □

*Proof:*

⇒: Let $O$ be an open set and let $x \in O$. Consider the set $\mathfrak{N}_x = x + \mathfrak{N}$, which is a neighborhood basis for $\tau$ at $x$, and therefore $\exists\, N' \in \mathfrak{N}_x; x \in N' \subseteq O$. Since $N' \in x + \mathfrak{N}, \exists\, N \in \mathfrak{N}; x \in (x + N \subseteq O)$, as claimed.

⇐: Suppose $\forall\, x \in O, \exists\, N_x \in \mathfrak{N}; (x + N_x) \subseteq O$. We know $\mathfrak{N} \subseteq \tau$ and that translations of open sets are open sets (Proposition 5.18 on the preceding page). Notice that $O = \bigcup_{x \in O} N_x$, which, as any arbitrary union of open sets, is an open set. This proves the result. ■

*Remark:*

Notice that the "deeper meaning" of Proposition 5.21 is that a system of nuclei describes the topology of a linear topological system completely. Given a topology, one might obtain a system of nuclei, but the opposite holds as well. ♣

The "locality" of the topology in linear topological spaces is reflected in the fact that a linear transformation between two linear topological spaces is continuous at some point $x$ if, and only if, it is continuous at the origin.

**Theorem 5.22:**

*Let $(X, \tau_X)$ and $(Y, \tau_Y))$ be linear topological spaces. Let $T\colon X \to Y$ be a linear transformation. Given $x \in X$, it holds that $T$ is continuous at $x$ if, and only if, it is continuous at 0.* □

*Proof:*

Let $\mathfrak{N}_X = \{O \in \tau_X; 0 \in O\}$ and $\mathfrak{N}_Y = \{O \in \tau_Y; 0 \in O\}$.

⇒: Assume $T$ is continuous at $x$. Then for every neighborhood $O'$ of $T(x)$ there is a neighborhood $U'$ of $x$ such that $T(U) \subseteq O$. Corollary 5.19 on the preceding page guarantees the existence of $O \in \mathfrak{N}_Y$ and $U \in \mathfrak{N}_X$ such that $U' = x + U$ and $O' = T(x) + O$. Notice that $U$ is a neighborhood of 0 and $O$ is neighborhood of $T(0) = 0$. Furthermore, let $u \in U$. I claim $T(u) \in O$. Indeed, $T(u) = T(x + u) - T(x)$ and we already know that $T(x + u) \in O' = T(x) + O$. As a consequence,

$T(x+u) - T(x) \in O$, proving the claim. Notice this implies $T(U) \subseteq O$, which proves the result.

$\Leftarrow$: Assume $T$ is continuous at 0. Then for every neighborhood $O'$ of $T(0)$ there is a neighborhood $U'$ of 0 such that $T(U) \subseteq O$. Corollary 5.19 on page 142 guarantees that $U = x + U'$ and $O = T(x) + O'$ are neighborhoods of $x$ and $T(x)$, respectively. Let $u \in U'$. I claim $T(u+x) \in O$. Indeed, $T(x+u) = T(x) + T(u)$ and we already know that $T(u) \in O'$. As a consequence, $T(u) + T(x) \in O$, proving the claim. Notice this implies $T(U) \subseteq O$, which proves the result. ∎

**Corollary 5.23:**
*Let $(X, \tau_X)$ and $(Y, \tau_Y))$ be linear topological spaces. Let $T\colon X \to Y$ be a linear transformation. T is continuous if, and only if, it is continuous at* 0. □

*Proof:*
Follows from Proposition 3.59 and Theorem 5.22 on page 53 and on the preceding page. ∎

Since the topology can be described by analysing the system of nuclei, it is useful to derive some of its properties.

**Definition 5.24** [Balanced or Circled Set]**:**
Let $V$ be a vector space over $\mathbb{F}$, which is taken to be either the real line or the complex plane. Let $A \subseteq V$. $A$ is said to be a *balanced*, or *circled*, set if, and only if, it holds that $\lambda A \subseteq A, \forall\, \lambda \in \mathbb{F}; |\lambda| \leq 1$. ♠

**Theorem 5.25:**
*Let $(X, \tau)$ be a linear topological space over $\mathbb{F}$. Let $\mathfrak{N}$ be a system of nuclei for $\tau$. The following properties hold:*

i. $\forall\, O, U \in \mathfrak{N}, \exists\, V \in \mathfrak{N}; V \subseteq O \cap U;$

ii. $\forall\, O \in \mathfrak{N}, \exists\, U \in \mathfrak{N}; U + U \subseteq O;$

iii. $\forall\, O \in \mathfrak{N}, \exists\, V \in \mathfrak{N}; \lambda V \subseteq O, \forall\, \lambda \in \mathbb{F}, |\lambda| \leq 1;$

iv. $\forall\, x \in X, \forall\, O \in \mathfrak{N}, \exists\, \lambda \in \mathbb{F}; x \in \lambda O;$

v. *$\forall\, O \in \mathfrak{N}$, there is $U \in \mathfrak{N}$ and a balanced set $V$ such that $U \subseteq V \subseteq O$.*

vi. *$(X, \tau)$ is Hausdorff if, and only if, $\bigcap_{O \in \mathfrak{N}} O = \{0\}$.*

*On the other hand, let $X$ be a vector space and let $\mathfrak{N} \subseteq \mathbb{P}(X)$ be a non-empty set satisfying the first four properties listed above. Let $\tau = \{O \in \mathbb{P}(X) \,|\, \forall\, x \in O, \exists\, U \in \mathfrak{N}; x + U \subseteq O\}$. Then it holds that $\tau$ is a vector topology on $X$ and $\mathfrak{N}$ is a system of nuclei for $\tau$.* □

*Proof:*
Let us begin by assuming $(X, \tau)$ is a linear topological space and $\mathfrak{N}$ is a system of nuclei. We shall first prove the first five properties.

i. Since $\mathfrak{N}$ is a neighborhood basis for $\tau$ at $0$, it holds that $\mathfrak{N} \subseteq \tau$ and we know that $\forall O \in \tau; 0 \in \tau, \exists N \in \mathfrak{N}; N \subseteq O$. If $O, U \in \mathfrak{N}$, then $O \cap U \in \tau, 0 \in O \cap U$ and as a consequence we know there is $V \in \mathfrak{N}; V \subseteq O \cap U$.

ii. Recall that a function is continuous if, and only if, it is continuous at every point of the domain (Proposition 3.59 on page 53). Let $x, y \in X$. Let $O \in \tau; x + y \in O$. Since addition is continuous, we know there is an open set $U \in \mathbb{P}(X \times X)$ (the product topology is being considered) such that $(x, y) \in U, +(U) \subseteq O$. We know there are sets $V_x, V_y \in \tau$ such that $V_x \times V_y \subseteq O$, due to Theorem 3.71 on page 58. Notice that $V_x + V_y = +(V_x \times V_y) \subseteq O$. Let $x = y = 0$. Since $\mathfrak{N}$ is a neighborhood basis at $0$ and both $V_x$ and $V_y$ are neighborhoods of $0$, there is $U \in \mathfrak{N}$ such that $U \subseteq V_x \cap V_y$. As a consequence, $U + U \subseteq V_x + V_y \subseteq O$, proving the result.

iii. Let $\lambda \in \mathbb{F}$, $x \in X$. Let $O \in \tau; \lambda \cdot x \in O$. Since scalar multiplication is continuous, there is some open set $U \in \mathbb{P}(\mathbb{F} \times X)$ such that $(\lambda, x) \in U, \cdot(U) \subseteq O$. Due to Theorem 3.71 on page 58, we know there is $V_x \in \tau$ and $\lambda_0 \in \mathbb{F}, \epsilon > 0$ such that $\lambda \in \mathcal{B}_\epsilon(\lambda_0), x \in V_x$ and $\mathcal{B}_\epsilon(\lambda_0) \times V_x \subseteq U$. Notice that $\cdot(\mathcal{B}_\epsilon(\lambda_0) \times V_x) \subseteq O$. Furthermore, notice that

$$\cdot(\mathcal{B}_\epsilon(\lambda_0) \times V_x) = \bigcup_{|\lambda - \lambda_0| < \epsilon} \lambda V_x. \tag{5.61}$$

If we let $x = 0$ and $\lambda_0$, we get that $V_x$ is a neighborhood of $0$. Furthermore, we see that the set

$$V = \bigcup_{|\lambda| < \epsilon} \lambda V_x \tag{5.62}$$

is a neighborhood of $0$ and it satisfies $\alpha V \subseteq V, \forall |\alpha| \leq 1$. Since $X$ is also a neighborhood of $0$, we see from the first item that there is $W \in \mathfrak{N}; W \subseteq \bigcup_{|\lambda| < \epsilon} \lambda V_x$. As $\alpha V \subseteq V, \forall |\alpha| \leq 1$ and $W \subseteq V \subseteq O$, the result holds. This also proves the fifth item.

iv. Let $x \in X, O \in \mathfrak{N}$. Notice that, for a given $\lambda \in \mathbb{F}, \lambda \neq 0$, it holds that $x \in \lambda O$ if, and only if, $\frac{1}{\lambda} x \in O$.

Let us consider the sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{F}$ given by $x_n = \frac{1}{n}$. In the standard topology, we know $x_n \to 0$. Since $\lambda \mapsto (\lambda, x)$ is continuous for each fixed $x$, we see (Theorem 3.123 on page 83) that $\left(\frac{1}{n}, x\right) \to (0, x)$. Since multiplication by scalars is continuous, Theorem 3.123 on page 83 guarantees that $\frac{1}{n} x$ converges to $0$, *id est*, for every neighborhood $O$ of $0$ there is $n \in \mathbb{N}$ such that $\frac{1}{m} x \in O, \forall m > n$. Thus, given $O \in \mathfrak{N}, \exists n \in \mathbb{N}; x \in nO$.

v. See the proof for the third item.

Let us now assume a vector space $X$ is equipped with a non-empty $\mathfrak{N} \subseteq \mathbb{P}(X)$ satisfying the first four properties enunciated. We want to prove that

$$\tau = \{O \in \mathbb{P}(X) | \forall x \in O, \exists U \in \mathfrak{N}; x + U \subseteq O\} \tag{5.63}$$

is a vector topology on $X$ and $\mathfrak{N}$ is a system of nuclei for $\tau$.

$X \in \tau$ is trivially true, and $\varnothing \in \tau$ is vacuously true.

Let $O, U \in \tau$. Let $x \in O \cap U$. We know there are $O', U' \in \mathfrak{N}$ such that $x \in (x + O') \cap (x + U') = x + (O' \cap U')$. We also know there is $V \in \mathfrak{N}; V \subseteq O' \cap U'$, and thus $x + V \subseteq O \cap U$, which proves $O \cap U \in \tau$.

Let $\Lambda$ be a family of indices and let $O_\lambda \in \tau, \forall\, \lambda \in \Lambda$. Notice that $\forall\, x \in \bigcup_{\lambda\in\Lambda} O_\lambda, \exists\, \lambda \in \Lambda; x \in O_\lambda$, and thus there is $U \in \mathfrak{N}; x + U \subseteq O_\lambda \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda$. Thus, $\bigcup_{\lambda\in\Lambda} O_\lambda \in \tau$. Therefore, $\tau$ is a topology.

Let $y \in X$ be fixed. For each $x \in X$, let $O' \in \tau; x + y \in O'$. We know there is $O \in \mathfrak{N}; x + y + O \subseteq O'$. We also know there is $U \in \mathfrak{N}; U + U \subseteq O$. Notice that $+\big((x + U) \times (y + U)\big) \subseteq O'$. This means that, for every $y$, the operation of translating a vector $x$ by $y$ is continuous at each $x$. As a consequence, addition is a continuous function.

In order to prove the continuity of the scalar multiplication, notice that $\forall\, O \in \mathfrak{N}; \exists\, V \in \mathfrak{N}; \sum_{i=1}^{n} V \subseteq O$. We already know it holds for $n = 2$. Furthermore, if it holds for $n - 1$, we might simply apply the theorem again to each of the factors (if, for example, $U + U \subseteq O, \exists\, V \in \mathfrak{N}; V + V \subseteq U$ and we have $V + V + V \subseteq O$).

Let $\lambda_0 \in \mathbb{F}$, $x_0 \in X$. let $O' \in \tau; \lambda_0 \cdot x_0 \in O'$. Let $n \in \mathbb{N}; |\lambda| < n$. We know there is $O \in \mathfrak{N}; \lambda_0 \cdot x_0 + O \subseteq O'$. Let $V \in \mathfrak{N}$ be such that $\sum_{i=1}^{n+2} V \subseteq O$. It holds as a consequence that $nV + V + V \subseteq O$. Since $V \in \mathfrak{N}, \exists\, U \in \mathfrak{N}; \epsilon U \subseteq V, \forall\, |\epsilon| < 1$.

Let $\mu \in \mathbb{F}; x_0 \in \mu O$. Let $\lambda \in \mathcal{B}_{\frac{1}{m}}(\lambda_0), x \in x_0 + U$. Notice that

$$\lambda x = \lambda_0 x_0 + (\lambda - \lambda_0)x_0 + (\lambda - \lambda_0)(x - x_0) + \lambda_0(x - x_0). \tag{5.64}$$

Notice also that $(\lambda - \lambda_0) \leq \frac{1}{m}$, and thus, since $\epsilon U \subseteq V, \forall\, |\epsilon| < 1$, it holds that $(\lambda - \lambda_0)x_0 \in V$, $(\lambda - \lambda_0)(x - x_0) \in \frac{1}{m} U \subseteq V$ and $\lambda_0(x - x_0) \in nU \subseteq nV$. Therefore, we have that

$$\begin{aligned} \lambda x &\in \lambda_0 x_0 + V + V + nV, \\ &\subseteq O. \end{aligned} \tag{5.65}$$

We might now conclude that the multiplication by a scalar is continuous at $(\lambda_0, x_0)$. Since the result holds for every point of $\mathbb{F} \times X$, we conclude $\tau$ is indeed a vector topology.

Since $\forall\, O \in \mathfrak{N}, \exists\, V \in \mathfrak{N}; \lambda V \subseteq O, \forall\, |\lambda| < 1$, we may take $\lambda = 0$ and we see that $0 \in O, \forall\, O \in \mathfrak{N}$. If $O \in \tau; 0 \in O$, it holds trivially that $\exists\, N \in \mathfrak{N}; N \subseteq O$. Thus, $\mathfrak{N}$ is a neighborhood basis for $\tau$ at 0 and, being such, is a system of nuclei for $\tau$.

Finally, we must prove the sixth item. Let $x \in \bigcap_{O\in\mathfrak{N}} O$. Suppose $x \neq 0$. For any $O, U \in \mathfrak{N}$, we see that $0 \in (-x + O)$ and $0 \in (0 + U)$, and it follows one can't separate $-x$ from 0. Therefore, $(X, \tau)$ is not Hausdorff.

Suppose now $(X, \tau)$ is not Hausdorff. Then there are $y, z \in X, y \neq z$ such that $y + O \cap z + U \neq \varnothing, \forall\, O, U \in \mathfrak{N}$. As a consequence, $y - z + O \cap U \neq \varnothing, \forall\, O, U \in \mathfrak{N}$. It follows that $y - z \in O, \forall\, O \in \mathfrak{N}$, proving the result. ■

**Proposition 5.26:**

*Let $X$ be a vector space and let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be systems of nuclei for a topology on $X$ (*id est, *assume both $\mathfrak{N}_1$ and $\mathfrak{N}_2$ satisfy the first four properties of Theorem 5.25 on page 144). Suppose it holds that $\forall\, N \in \mathfrak{N}_1, \exists\, O \in \mathfrak{N}_2; O \subseteq N$ and, conversely, $\forall\, N \in \mathfrak{N}_2, \exists\, O \in \mathfrak{N}_1; O \subseteq N$. Then it holds that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ generate the same topology.* □

*Proof:*

Let $\tau_1 \equiv \{O \in \mathbb{P}(X) \,|\, \forall\, x \in O, \exists\, N \in \mathfrak{N}_1; x + N \subseteq O\}$, with a similar definition for $\tau_2$.

Let $O \in \tau_1$ and $x \in O$. We know there is $N \in \mathfrak{N}_1; x + N \subseteq O$. However, we know there is $U \in \mathfrak{N}_2; U \subseteq N$. As a consequence, $x + U \subseteq O$. Therefore, $O \in \tau_2$, *id est*, $\tau_1 \subseteq \tau_2$. Since the argument holds both ways, we conclude $\tau_1 = \tau_2$. ■

**Definition 5.27** [Equivalent System of Nuclei]**:**

Let $X$ be a vector space and let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be systems of nuclei for a topology on $X$ (*id est*, assume both $\mathfrak{N}_1$ and $\mathfrak{N}_2$ satisfy the first four properties of Theorem 5.25 on page 144). Suppose it holds that $\forall\, N \in \mathfrak{N}_1, \exists\, O \in \mathfrak{N}_2; O \subseteq N$ and, conversely, $\forall\, N \in \mathfrak{N}_2, \exists\, O \in \mathfrak{N}_1; O \subseteq N$. Then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are said to be *equivalent* system of nuclei. ♠

**Proposition 5.28:**

*Let $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ be a locally convex space. Let*

$$N_{\lambda_1,\ldots,\lambda_n;\epsilon} \equiv \Big\{x \in X; \|x\|_{\lambda_i} < \epsilon, 1 \leq i \leq n\Big\}. \tag{5.66}$$

*The set $\mathfrak{N} = \{N_{\lambda_1,\ldots,\lambda_n;\epsilon}; \lambda_i \in \Lambda, 1 \leq i \leq n, \epsilon > 0\}$ is a system of nuclei for the natural topology on $X$.* □

*Proof:*

Firstly, let us prove that $\mathfrak{N}$ is a system of nuclei for some topology on $X$. This shall be done by applying Theorem 5.25 on page 144.

i. If $N_{\lambda_1,\ldots,\lambda_n;\epsilon}, N_{\mu_1,\ldots,\mu_m;\delta} \in \mathfrak{N}$, it holds that $M \equiv N_{\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_m;\min\{\epsilon,\delta\}} \in \mathfrak{N}$ and $M \subseteq N_{\lambda_1,\ldots,\lambda_n;\epsilon} \cap N_{\mu_1,\ldots,\mu_m;\delta}$.

ii. If $N_{\lambda_1,\ldots,\lambda_n;\epsilon} \in \mathfrak{N}$, then $N_{\lambda_1,\ldots,\lambda_n;\frac{\epsilon}{3}} \in \mathfrak{N}$ with $N_{\lambda_1,\ldots,\lambda_n;\frac{\epsilon}{3}} + N_{\lambda_1,\ldots,\lambda_n;\frac{\epsilon}{3}} \subseteq N_{\lambda_1,\ldots,\lambda_n;\epsilon}$.

iii. $\forall\, N \in \mathfrak{N}, \lambda N \subseteq N, \forall\, |\lambda| \leq 1$.

iv. Let $x \in X, N_{\lambda_1,\ldots,\lambda_n;\epsilon} \in \mathfrak{N}$. If $\alpha = \|x\|_{\lambda_1}$, we know that $\frac{\epsilon}{2\alpha}x \in N_{\lambda_1,\ldots,\lambda_n;\epsilon}$. Thus, $x \in \frac{2\alpha}{\epsilon} N_{\lambda_1,\ldots,\lambda_n;\epsilon}$.

We see now that $\tau = \{O \in \mathbb{P}(X) \,|\, \forall\, x \in O, \exists\, U \in \mathfrak{N}; x + U \subseteq O\}$ is a vector topology having $\mathfrak{N}$ as a system of nuclei. Notice that the system of nuclei forces every seminorm $\|\cdot\|_\lambda$ to be continuous. Furthermore, if we wish for the seminorms to be continuous, all the open sets we picked through $\mathfrak{N}$ must indeed be open. Thus, the topology induced by $\mathfrak{N}$ is the weakest topology keeping the seminorms, the addition and the scalar multiplication continuous, which means $\mathfrak{N}$ is indeed a system of nuclei for the natural topology. ■

*Notation:*

In a locally convex space, the notation $N_{\lambda_1,\ldots,\lambda_n;\epsilon}$ shall always be understood as

$$N_{\lambda_1,\ldots,\lambda_n;\epsilon} \equiv \left\{x \in X; \|x\|_{\lambda_i} < \epsilon, 1 \leq i \leq n\right\}. \tag{5.67}$$

◆

This allows us to prove the following result about locally convex spaces.

**Proposition 5.29:**

*Let $(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda})$ be a locally convex space. Let $(x_\alpha)_{\alpha\in I}$ be a net. Given a point $x \in X$, it holds that $x_\alpha \to x$ if, and only if, $\|x_\alpha - x\|_\lambda \to 0, \forall\, \lambda \in \Lambda$.* □

*Proof:*

⇒: Suppose $x_\alpha \to x$. Theorem 3.123 on page 83 guarantees that, since translations are homeomorphisms on any linear topological space (Lemma 5.17 on page 142), $x_\alpha - x \to 0$. Since $\|\cdot\|_\lambda$ is continuous, $\forall\, \lambda \in \Lambda$, it holds, also by Theorem 3.123 on page 83, that $\|x_\alpha - x\|_\lambda \to 0, \forall\, \lambda \in \Lambda$.

⇐: Let $N_{\lambda_1,\ldots,\lambda_n;\epsilon} \equiv \left\{x \in X; \|x\|_{\lambda_i} < \epsilon, \forall\, i \in \{i\}_{i=1}^n\right\}$ and

$$\mathfrak{N} = \left\{N_{\lambda_1,\ldots,\lambda_n;\epsilon} \in \tau; \lambda_i \in \Lambda, \forall\, i \in \{i\}_{i=1}^n, \epsilon > 0\right\}. \tag{5.68}$$

Proposition 5.28 on the preceding page guarantees $\mathfrak{N}$ is a system of nuclei for the natural topology on $X$. We also know that Theorem 3.123 on page 83 guarantees that, since translations are homeomorphisms on any linear topological space (Lemma 5.17 on page 142), $x_\alpha - x \to 0 \Leftrightarrow x_\alpha \to x$.

Since $\|x_\alpha - x\|_\lambda \to 0$, we know that, given $\epsilon > 0, \exists\, \alpha \in I; \left\|x_\beta - x\right\|_\lambda \in [0, \epsilon), \forall\, \beta \succ \alpha$. As a consequence, we see that given $N \in \mathfrak{N}, \exists\, \alpha \in I; x_\beta - x \in N, \forall\, \beta \succ \alpha$. Since every neighborhood of 0 can be written in terms of such sets $N \in \mathfrak{N}$, it follows that $x_\alpha - x \to 0$, proving the result.

■

This result means that our previous wish has been attended and our chosen topology is, in this sense, consistent with the family of seminorms.

Nevertheless, our interest in systems of nuclei wasn't due only to this result. Since systems of nuclei specify completely the topology in linear topological spaces, it is not surprising that this holds as well to locally convex spaces. In fact, the structure of a system of nuclei of a locally convex space is what justifies the name locally convex space.

In order to study these properties, some more results and definitions are necessary.

**Lemma 5.30:**

*Every linear topological space admits a system of nuclei composed only by balanced sets.* □

*Proof:*

Let $(X, \tau)$ be a linear topological space and let $\mathfrak{N}_0$ be an arbitrary system of nuclei for $\tau$. We define

$$\mathfrak{N} \equiv \left\{ \bigcup_{|\lambda| \le 1} \lambda O ; O \in \mathfrak{N}_0 \right\}. \tag{5.69}$$

It is clear that $0 \in N, \forall N \in \mathfrak{N}$ (for $0 \in O, \forall O \in \mathfrak{N}_0$).

If $O \in \tau, 0 \in O$, we know there is $U \in \mathfrak{N}_0; U \subseteq O$. Theorem 5.25 on page 144 guarantees that $\exists V \in \mathfrak{N}_0; \lambda V \subseteq U, \forall |\lambda| \le 1$. Thus, $N = \bigcup_{|\lambda| \le 1} \in \mathfrak{N}$ is such that $N \subseteq U \subseteq O$. Therefore, $\mathfrak{N}$ is a neighborhood basis for $\tau$ at 0 and, as such, is a system of nuclei for $\tau$. Notice every element of $\mathfrak{N}$ is balanced. ■

**Definition 5.31** [Convex, Absolutely Convex and Absorbing Sets]**:**

Let $X$ be a vector space and let $A \subseteq X$.

We say $A$ is a *convex* set if, and only if, it holds that $x, y \in A \Rightarrow tx + (1-t)y \in A, \forall t \in [0, 1]$.

We say $A$ is an *absolutely convex* set if, and only if, $A$ is both convex and balanced.

We say $A$ is an *absorbing*, or *absorbent*, set if, and only if, $\forall x \in X$, there is some $\lambda \in \mathbb{F}; x \in \lambda A$. ♠

**Lemma 5.32:**

*Let $(X, \tau)$ be a linear topological space. Suppose $\mathfrak{N}_0$ is a system of nuclei for $\tau$ comprised of convex sets. $X$ admits a system of nuclei comprised of absolutely convex absorbent sets.* □

*Proof:*

Lemma 5.30 on the facing page shows we can get a system of absolutely convex nuclei. Notice that its proof provides a construction that keeps the elements of the system of nuclei convex. Theorem 5.25 on page 144 guarantees that all the elements of the system of nuclei are absorbent. ■

**Lemma 5.33:**

*Let $X$ be a vector space over $\mathbb{F}$, with $\mathbb{F}$ being either the real line or the complex plane, and let $\Lambda$ be a family of indices. Let $\{\|\cdot\|_i\}_{i=1}^n$ be a family of seminorms on $X$ and $\epsilon > 0$. The set*

$$N = \{x \in X; \|x\|_i < \epsilon, 1 \le i \le n\} \tag{5.70}$$

*is convex.* □

*Proof:*

Let $x, y \in N$ and $i \in \{i\}_{i=1}^n$. Then $\|x\|_i, \|y\|_i < \epsilon$. Notice that, $\forall t \in [0, 1]$,

$$\begin{aligned} \|tx + (1-t)y\|_i &\le t\|x\|_i + (1-t)\|y\|_i, \\ &< t\epsilon + (1-t)\epsilon, \\ &= \epsilon. \end{aligned} \tag{5.71}$$

Therefore, $tx + (1-t)y \in N, \forall t \in [0, 1]$, and therefore $N$ is convex. ■

*Scholium:*

In fact, the sets $\{x \in X; \|x\|_i < \epsilon, 1 \leq i \leq n\}$ are absolutely convex and absorbent, but since the existence of a convex system of nuclei implies the existence of an absolutely convex, absorbent system of nuclei, we do not need to prove it. ♣

**Definition 5.34** [Gauge or Minkowski Functional]**:**

Let $X$ be a vector space. Let $A \subseteq X$ be an absorbent, balanced set. The *gauge*, or *Minkowski functional*, of $A$ is the function $\rho_A \colon X \to \mathbb{R}_+$ determined by

$$\rho_A(x) = \inf\{\lambda \in \mathbb{R}_+; x \in \lambda A\}. \tag{5.72}$$

♠

**Proposition 5.35:**

*Let $X$ be a vector space. Let $A \subseteq X$ be an absorbent, balanced set. The Minkowski functional of $A$ is well-defined.* □

*Proof:*

We want to prove the set $\Lambda \equiv \{\lambda \in \mathbb{R}_+; x \in \lambda A\}$ is non-empty and bounded below. Notice that $0 \leq \lambda, \forall\, \lambda \in \Lambda$, and therefore $\Lambda$ is bounded below.

Since $A$ is absorbent, $\forall\, x \in X, \exists\, \lambda \in \mathbb{F}; x \in \lambda A$, *id est*, $\frac{1}{\lambda}x \in A$. Since $A$ is balanced, $\mu A \subseteq A, \forall\, |\mu| = 1$. Thus, $\frac{\mu}{\lambda}x \in A, \forall\, |\mu| = 1$. If we pick $\mu = \frac{\lambda}{|\lambda|}$, we see that $x \in |\lambda| A$. Since $|\lambda| \in \mathbb{R}_+$, we have that $|\lambda| \in \Lambda$, which is, therefore, not empty. ■

**Lemma 5.36:**

*Let $X$ be a vector space over $\mathbb{F}$, which is either the real line or the complex plane, and let $A \subseteq X$ be a convex set. It holds that $\lambda A + \mu A = (\lambda + \mu)A, \forall\, \lambda, \mu \in \mathbb{F}$.* □

*Proof:*

Since

$$\lambda A + \mu A \equiv \{\lambda x + \mu y; x, y \in A\}, \tag{5.73}$$
$$(\lambda + \mu)A \equiv \{\lambda x + \mu x; x \in A\}, \tag{5.74}$$

we know $(\lambda + \mu)A \subseteq \lambda A + \mu A$.

Notice that, since $A$ is convex, so is $(\lambda + \mu)A$. Therefore, if $x, y \in A$, it follows that $(\lambda + \mu)tx + (\lambda + \mu)(1 - t)y \in (\lambda + \mu)A, \forall\, t \in [0, 1]$. Let $t = \frac{\lambda}{\lambda + \mu}$. We conclude that $\lambda x + \mu y \in (\lambda + \mu)A$. Therefore, $\lambda A + \mu A \subseteq (\lambda + \mu)A$. This proves the result. ■

**Lemma 5.37:**

*Let $X$ be a vector space over $\mathbb{F}$, which stands for either the real line or the complex plane. Let $A \subseteq X$ be an absorbent set and let $\rho_A$ denote its Minkowski functional. The following statements hold:*

i. $\rho_A(tx) = t\rho_A(x), \forall\, t > 0$;

ii. *if $A$ is balanced*, $\rho_A(\lambda x) = |\lambda|\rho_A(x), \forall\, \lambda \in \mathbb{F}, \forall\, x \in X$;

iii. *if $A$ is convex, $\rho_A(x+y) \leq \rho_A(x) + \rho_A(y), \forall\, x, y \in X$;*

iv. *if $A$ is balanced or convex, $\rho_A^{-1}([0,1)) \subseteq A \subseteq \rho_A^{-1}([0,1])$.* □

*Proof:*

Let $t \in \mathbb{R}_+$. Notice that, if $x \in \lambda A \Leftrightarrow tx \in t\lambda A$. Therefore, it holds that $\rho_A(tx) = t\rho_A(x)$.

Let $\mu \in \mathbb{R}_+$ and suppose $x \in \mu A$. We know that $\lambda x \in \lambda\mu A$. However, since $A$ is balanced, it holds that $\eta A \subseteq A, \forall\, |\eta| \leq 1$. If we pick $\eta = \frac{|\lambda|}{\lambda}$, we can see that it holds that $\lambda x \in |\lambda|\mu A$, and we know that $|\lambda|\mu \in \mathbb{R}_+$. It follows that $\rho_A(\lambda x) = |\lambda|\rho_A(x), \forall\, \lambda \in \mathbb{C}$.

Suppose $A$ is convex. Lemma 5.36 on the facing page guarantees that $\lambda A + \mu A = (\lambda + \mu)A, \forall\, \lambda, \mu \in \mathbb{F}$. Suppose $x \in \lambda A$ and $y \in \mu A$. Then $x + y \in \lambda A + \mu A = (\lambda + \mu)A$. By taking the greatest lower bounds, we may conclude that $\rho_A(x+y) \leq \rho_A(x) + \rho_A(y)$.

Suppose that $x \in \rho_A^{-1}([0,1))$. Then $\exists\, \lambda < 1; x \in \lambda A$. If $A$ is balanced, $\lambda A \subseteq A$ and we conclude $x \in A$.

If $A$ is convex, then $\frac{1}{\lambda}x \in A$. Since $A$ is absorbent, $0 \in A$ (given $x \in X; x \neq 0$, there is no $\mu \in \mathbb{F}$ such that $0 = \frac{1}{\mu}$, so it is necessary for 0 to be an element of $A$ itself). As $A$ is convex, then $t\frac{1}{\lambda}x + (1-t)0 = \frac{t}{\lambda}x \in A, \forall\, t \in [0,1]$. Take $t = \lambda$ and we get $x \in A$.

Suppose $x \in A$. Then $\rho_A(x) \leq 1$. As a consequence, $x \in \rho_A^{-1}([0,1])$. ■

**Theorem 5.38:**

*Let $(X, \tau)$ be a linear topological space satisfying the Hausdorff property. $X$ is a locally convex space if, and only if, it admits a system of nuclei of convex sets.* □

*Proof:*

Assume $X$ is a locally convex space. Proposition 5.28 and Lemma 5.33 on page 147 and on page 149 guarantee that $X$ admits a nuclei of convex sets.

Assume $X$ admits a nuclei of convex sets. Lemma 5.32 on page 149 guarantees the existence of a system of nuclei $\mathfrak{N}$ of absolutely convex, absorbent sets. Lemma 5.37 on the facing page allows us to define a family $\{\rho_A\}_{A \in \mathfrak{N}}$ of seminorms by using the Minkowski functionals of the elements of $\mathfrak{N}$.

Since the topology on $X$ is Hausdorff, we know from Theorem 5.25 on page 144 that $\bigcap_{A \in \mathfrak{N}} A = \{0\}$. Notice that if $\rho_A(x) = 0, \forall\, A \in \mathfrak{N}$, this means $\forall\, \lambda > 0, x \in \lambda A, \forall\, A \in \mathfrak{N}$. As a consequence, $x \in A, \forall\, A \in \mathfrak{N}$. Thus, $x \in \bigcap_{A \in \mathfrak{N}} A = \{0\}$, *id est*, $x = 0$. We conclude $\{\rho_A\}_{A \in \mathfrak{N}}$ separates points.

Let

$$N_{A_1,\ldots,A_n;\epsilon} \equiv \{x \in X; \rho_{A_i}(x) < \epsilon, 1 \leq i \leq n\} \tag{5.75}$$

and $\mathfrak{N}' = \{N_{A_1,\ldots,A_n;\epsilon}; A_i \in \mathfrak{N}, 1 \leq i \leq n, \epsilon > 0\}$. Proposition 5.28 on page 147 states $\mathfrak{N}'$ is a system of nuclei for the natural topology on $(X, \{\rho_A\}_{A \in \mathfrak{N}})$.

$\mathfrak{N}$ and $\mathfrak{N}'$ are equivalent. Indeed, let $A \in \mathfrak{N}$. Then $\rho_A^{-1}([0,1)) = N_{A;1} \subseteq A$ per Lemma 5.37 on the preceding page.

Conversely, let $N_{A_1,\ldots,A_n;\epsilon} \in \mathfrak{N}$. Induction on the first item of Theorem 5.25 on page 144 guarantees the existence of $U \in \mathfrak{N}; U \subseteq \bigcap_{i=1}^n A_i$. The same item guarantees the existence of $V \in \mathfrak{N}; V \subseteq \frac{\epsilon}{2} U \cap U$. As $\rho_{A_i}^{-1}([0,1)) \subseteq U \subseteq \rho_{A_i}^{-1}([0,1]), 1 \leq i \leq n$, and $\rho_{A_i}(\epsilon x) = |\epsilon| \rho_{A_i}(x)$ we now have that $V \subseteq \rho_{A_i}^{-1}([[0, \frac{\epsilon}{2}]]), 1 \leq i \leq n$. As a consequence, $V \subseteq N_{A_1,\dots,A_n;\epsilon}$.

Therefore, $\mathfrak{N}$ and $\mathfrak{N}'$ generate the same topology. Since $\mathfrak{N}'$ generates the natural topology on $(X, \{\rho_A\}_{A \in \mathfrak{N}})$, we see that $X$ is indeed a locally convex space. ■

*Scholium:*

This result justifies the name "locally convex space": a space equipped with a local basis of convex sets. ♣

**Proposition 5.39:**

*Let $n \in \mathbb{N}$, $p \in \mathbb{R}^n$ and $\epsilon > 0$. $\mathcal{B}_\epsilon(p)$ is homeomorphic to $\mathcal{B}_\epsilon(0)$.* □

*Proof:*

The topology on $\mathbb{R}^n$ is induced by the Euclidean norm $\|\cdot\|$. Since it is a norm, the family $\{\|\cdot\|\}$ is a family of seminorms that separates points, and hence $\mathbb{R}^n$ is a locally convex space, which implies it is a linear topological space. Since addition is a homeomorphism in a linear topological space, $\mathcal{B}_\epsilon(p)$ is homeomorphic to $\mathcal{B}_\epsilon(0)$. Since multiplication by a non-vanishing scalar is also a homeomorphism, $\mathcal{B}_\epsilon(0)$ is homeomorphic to $\mathcal{B}_1(0)$. As a consequence, $\mathcal{B}_\epsilon(p)$ is homeomorphic to $\mathcal{B}_1(0)$. ■

Now that the name has been explained, let us turn our attention back to the properties of locally convex spaces.

We have seen a locally convex space can be defined without ever mentioning the seminorms that gave origin to it, and thus it is not surprising to say that the natural topology is more interesting than the seminorms themselves. Thus, just as we can abandon a given system of nuclei in favor of another equivalent one, we might do the same thing with the seminorms.

**Definition 5.40** [Equivalence of Families of Seminorms]**:**

Let $X$ be a vector space and let both $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ and $\{\|\cdot\|_\mu\}_{\mu \in M}$ be families of seminorms defined on $X$. $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ and $\{\|\cdot\|_\mu\}_{\mu \in M}$ are said to be *equivalent* if, and only if, they generate the same natural topology on $X$. ♠

**Proposition 5.41:**

*Let $X$ be a vector space and let both $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ and $\{\|\cdot\|_\mu\}_{\mu \in M}$ be families of seminorms defined on $X$. The following statements are equivalent:*

i. $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ *and* $\{\|\cdot\|_\mu\}_{\mu \in M}$ *are equivalent;*

ii. $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ *are continuous in the natural topology generated by* $\{\|\cdot\|_\mu\}_{\mu \in M}$ *(denoted $\tau_M$) and* $\{\|\cdot\|_\mu\}_{\mu \in M}$ *are continuous in the natural topology generated by* $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ *(denoted $\tau_\Lambda$);*

iii. $\forall \lambda \in \Lambda, \exists \mu_k \in M, 1 \leq k \leq n, \exists \alpha > 0; \forall x \in X$

$$\|x\|_\lambda \leq \alpha \sum_{k=1}^{n} \|x\|_{\mu_k} \tag{5.76}$$

*and* $\forall\, \mu \in M, \exists\, \lambda_k \in \Lambda, 1 \leq k \leq m, \exists\, \beta > 0; \forall\, x \in X$

$$\|x\|_\mu \leq \beta \sum_{k=1}^{n} \|x\|_{\lambda_k}. \tag{5.77}$$

□

*Proof:*

For simplicity, we shall denote $\|\cdot\|_\lambda \equiv \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ and $\|\cdot\|_\mu \equiv \{\|\cdot\|_\mu\}_{\mu\in M}$. If $\|\cdot\|_\lambda$ and $\|\cdot\|_\mu$ are equivalent, then $\tau_\Lambda = \tau_M$. Since $\|\cdot\|_\mu$ are continuous under $\tau_M$, we see that they are continuous under $\tau_\Lambda$, with a similar argument holding for $\|\cdot\|_\lambda$. Thus, the first item implies the second.

Suppose now that each family of norms is continuous in the natural topology generated by the other. Let $\rho_\lambda \in \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$. We know that $\rho_\lambda^{-1}\big([0,\epsilon)\big) \in \tau_M$ and $0 \in \rho_\lambda^{-1}\big([0,\epsilon)\big)$. If we write

$$N_{\mu_1,\ldots,\mu_n;\delta} \equiv \Big\{x \in X; \|x\|_{\mu_i} < \delta, 1 \leq i \leq n\Big\}, \tag{5.78}$$

we know (Proposition 5.28 on page 147) that the collection

$$\mathfrak{N} = \big\{N_{\mu_1,\ldots,\mu_n;\delta}; \mu_i \in M, 1 \leq i \leq n, \delta > 0\big\} \tag{5.79}$$

is a system of nuclei for $\tau_M$. Therefore, $\exists\, N_{\mu_1,\ldots,\mu_n;\delta} \in \mathfrak{N}; N_{\mu_1,\ldots,\mu_n;\delta} \subseteq \rho_\lambda^{-1}\big([0,\epsilon)\big)$.

Let us define the function

$$d(x) \equiv \sum_{k=1}^{n} \|x\|_{\mu_k} + \frac{\delta}{4\epsilon}\rho_\lambda(x) \tag{5.80}$$

and let $D \equiv \{x \in X; d(x) = 0\}$. Notice that $x \in D \Rightarrow \rho_\lambda(x) = 0$, for $d$ is a sum of non-negative functions.

Let $x \in X \setminus D$. Pick $y = \frac{\delta x}{2d(x)}$. Notice that $\|y\|_{\mu_k} < \frac{\delta}{2} < \delta, 1 \leq k \leq n$, and therefore it holds that $y \in N_{\mu_1,\ldots,\mu_n;\delta} \subseteq \rho_\lambda^{-1}\big([0,\epsilon)\big)$. We see then that $\rho_\lambda(y) < \epsilon$. It follows that

$$\begin{aligned}
\rho_\lambda(y) &< \epsilon,\\
\frac{\delta}{2d(x)}\rho_\lambda(x) &< \epsilon,\\
\rho_\lambda(x) &< \frac{2\epsilon}{\delta} d(x),\\
\rho_\lambda(x) &< \frac{2\epsilon}{\delta} \sum_{k=1}^{n} \|x\|_{\mu_k} + \frac{1}{2}\rho_\lambda(x),\\
\rho_\lambda(x) &< \frac{4\epsilon}{\delta} \sum_{k=1}^{n} \|x\|_{\mu_k}.
\end{aligned} \tag{5.81}$$

Since the argument works equally well when we exchange $\mu \leftrightarrow \lambda$, this proves the second item implies the third.

Finally, assume the third item holds. Let $\lambda \in \Lambda$ and let us write $\rho_\lambda(x) \equiv \|x\|_\lambda$. We know that $\exists\, \mu_k \in M, 1 \leq k \leq n, \exists\, \alpha > 0; \forall\, x \in X$

$$\rho_\lambda(x) \leq \alpha \sum_{k=1}^{n} \|x\|_{\mu_k}. \tag{5.82}$$

Let $\delta > 0$ and $D = \left\{x \in X; \sum_{k=1}^{n} \|x\|_{\mu_k} < \delta\right\}$. Notice that $N_{\mu_1,\dots,\mu_n;\delta} \subseteq D$. Furthermore, notice that, $\forall\, x \in D$, $\rho_\lambda(x) < \frac{\delta}{\alpha}$. Therefore,

$$N_{\mu_1,\dots,\mu_n;\delta} \subseteq D \subseteq \rho_\lambda^{-1}\left(\left[0, \frac{\delta}{\alpha}\right)\right). \tag{5.83}$$

Therefore, given $N_{\lambda_1,\dots,\lambda_n;\epsilon} \in \mathfrak{N}_\Lambda$, we might pick each one of the $\lambda_i$, apply this construction and use the first item of Theorem 5.25 on page 144 to find an element of $\mathfrak{N}_M$ contained in $N_{\lambda_1,\dots,\lambda_n;\epsilon}$. Therefore, the system of nuclei for $\tau_\Lambda$ and $\tau_M$ are equivalent and Proposition 5.26 on page 146 guarantees that the first item is implied by the third, concluding the proof. ■

Expressions of the form $\alpha \sum_{k=1}^{n} \|x\|_{\mu_k}$ appear often in the theory of locally convex spaces, motivating the following definition.

**Definition 5.42** [Directed Family of Seminorms]**:**
Let $X$ be a vector space and let $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ be a family of seminorms defined on $X$. $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ is said to be *directed* if, and only if, $\forall\, \lambda, \mu \in \Lambda, \exists\, \eta \in \Lambda, \alpha > 0; \|x\|_\lambda + \|x\|_\mu \leq \alpha\|x\|_\eta, \forall\, x \in X$. ♠

*Remark:*
Induction allows one to prove that a family of seminorms is directed if, and only if, given $\lambda_i \in \Lambda, 1 \leq i \leq n, \exists\, \mu \in \Lambda, \alpha > 0; \sum_{i=1}^{n} \|x\|_{\lambda_i} \leq \alpha\|x\|_\mu, \forall\, x \in X$. ♣

One of the reasons we are interested in the study of locally convex spaces is the fact they provide us a richer theory than the one provided by linear topological spaces, without the restrictions demanded by Banach spaces*. In fact, locally convex spaces can be viewed as generalizations of normed spaces, and as such it is interesting for us to study the generalizations of some results of the theory of normed spaces. In particular, it holds in the theory of normed linear spaces that a linear operator $T\colon X \to Y$ is continuous if, and only if, it is bounded. A similar result holds in the theory of locally convex spaces.

**Definition 5.43** [Bounded Linear Operator]**:**
Let $X$ and $Y$ be normed spaces. Let $T\colon X \to Y$ be a linear operator. $T$ is said to be *bounded* if, and only if, there is some $\alpha \in \mathbb{R}$ such that $\|T(x)\| \leq \alpha\|x\|, \forall\, x \in X$. ♠

*Notation:*
When dealing with different normed spaces at once, we shall denote both norms by $\|\cdot\|$ whenever there is no risk of confusion. ♦

*A Banach space is a complete normed space.

This definition might seem awkward, since the image of a bounded operator needs not to be bounded. In our case, the term is justified by the fact that the image of a bounded set by a bounded linear operator is a bounded set as well.

**Definition 5.44** [Bounded Set]**:**
Let $X$ be a normed space. $B \subseteq X$ is sad to be a *bounded* set if, and only if, there is $\alpha \in \mathbb{R}; \|x\| \leq \alpha, \forall\, x \in B$. ♠

**Lemma 5.45:**
*Let $X$ and $Y$ be normed spaces. Let $T\colon X \to Y$ be a linear operator. $T$ is bounded if, and only if, $T(B)$ is bounded for every bounded set $B \subseteq X$.* □

*Proof:*
Suppose $T$ is bounded. Then there is $\alpha \in \mathbb{R}; \|T(X)\| \leq \alpha\|x\|, \forall\, x \in X$. Therefore, if $y \in T(B)$, it holds that $\|y\| \leq \alpha\|x\|$ for some $x \in B$. Since $B$ is bounded, $\exists\, \beta \in \mathbb{R}; \|x\| \leq \beta, \forall\, x \in B$. As a consequence, we see that $\|y\| \leq \alpha\beta, \forall\, y \in T(B)$. Thus, $T(B)$ is bounded.
Suppose that $T(B)$ is bounded for every bounded set $B$. In particular, consider $B = \{x \in X; \|x\| = 1\}$, which is clearly bounded. Since $T(B)$ is bounded, there is $\alpha \in \mathbb{R}; \|T(x)\| \leq \alpha, \forall\, x \in B$. Let $y \in X, y \neq 0$. We know that $\left\|T\left(\frac{y}{\|y\|}\right)\right\| \leq \alpha$. Since $T$ is linear and the norm is homogeneous, it follows that $\|T(y)\| \leq \alpha\|y\|, \forall\, y \neq 0$. The case $y = 0$ is trivially satisfied, for $T(0) = 0$ (by linearity). ■

**Definition 5.46** [Norm of an Operator]**:**
Let $X$ and $Y$ be normed spaces. Let $T\colon X \to Y$ be a bounded linear operator. We define the *norm* of $T$ as

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}. \tag{5.84}$$

♠

Notice this definition is such that $\|T\|$ is the smallest real number with the property that $\|T(x)\| \leq \|T\|\|x\|, \forall\, x \in X$.

*Notation:*
Given two vector spaces $X$ and $Y$, we denote the space of all the linear operators from $X$ to $Y$ by $L(X, Y)$. ♦

*Notation:*
Given two normed spaces $X$ and $Y$, we denote the space of all the bounded linear operators from $X$ to $Y$ by $B(X, Y)$. ♦

**Proposition 5.47:**
*Let $X$ and $Y$ be vector spaces over a field $\mathbb{F}$. When equipped with usual addition and scalar multiplication of scalar operator, $L(X, Y)$ is a vector space over $\mathbb{F}$ and, if $X$ and $Y$ are normed*

*spaces, it has $B(X,Y)$ as a subspace. Furthermore, when equipped with the norm*

$$\|T\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|T(x)\|}{\|x\|}, \tag{5.85}$$

*$\forall\, T \in B(X,Y)$, $B(X,Y)$ is a normed space.* $\square$

*Proof:*

Let $A, B, C \in L(X,Y), \lambda, \mu \in \mathbb{F}, x \in X$.

A1. $[A+(B+C)](x) = A(x)+(B+C)(x) = A(x)+B(x)+C(x) = (A+B)(x)+C(x) = [(A+B)+C](x)$;

A2. $(A+B)(x) = A(x)+B(x) = (B+A)(x)$;

A3. $(A+0)(x) = A(x)+0 = A(x) = (0+A)(x)$;

A4. $(A-A)(x) = A(x)-A(x) = 0 = (-A+A)(x)$;

M1. $(1\cdot A)(x) = 1\cdot A(x) = A(x)$;

M2. $[(\lambda\mu)\cdot A](x) = (\lambda\mu)A(x) = \lambda(\mu\cdot A)(x)$;

D1. $[(\lambda+\mu)A](x) = (\lambda+\mu)A(x) = \lambda A(x)+\mu A(x) = [\lambda\cdot A)+(\mu\cdot A](x)$;

D2. $[\lambda\cdot(A+B)](x) = \lambda(A+B)(x) = \lambda A(x)+\lambda B(x) = [\lambda\cdot A+\lambda\cdot B](x)$.

This proves $L(X,Y)$ is a vector space over $\mathbb{F}$.

From now on, let $A, B, C \in B(X,Y)$. Since 0 is bounded, $0 \in B(X,Y)$. The sum of bounded operators is a bounded operator. Indeed,

$$\begin{aligned}\|A(x)+B(x)\| &\leq \|A(x)\|+\|B(x)\|,\\ &\leq \|A\|\|x\|+\|B\|\|x\|,\\ &\leq (\|A\|+\|B\|)\|x\|.\end{aligned} \tag{5.86}$$

The product of a bounded operator by a scalar is a bounded operator as well.

$$\begin{aligned}\|\lambda A(x)\| &= |\lambda|\|A(x)\|,\\ &\leq |\lambda|\|A\|\|x\|.\end{aligned} \tag{5.87}$$

Thus, since $B(X,Y) \subseteq L(X,Y)$ is non-empty and is closed under the sum of vectors and multiplication by scalar, it is a subspace of $L(X,Y)$.

Since $\|T(x)\|, \|x\| \geq 0, \forall\, x \in X$, it holds that $\|T\| \geq 0$. Notice now that

$$\|\lambda\cdot T\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|\lambda\cdot T(x)\|}{\|x\|},$$

$$= |\lambda| \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|},$$
$$= |\lambda| \|T\|. \tag{5.88}$$

Furthermore,

$$\begin{aligned}
\|A + B\| &= \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|A(x) + B(x)\|}{\|x\|}, \\
&\leq \sup_{\substack{x \in X \\ x \neq 0}} \left( \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \right), \\
&\leq \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|A(x)\|}{\|x\|} + \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|B(x)\|}{\|x\|}, \\
&\leq \|A\| + \|B\|.
\end{aligned} \tag{5.89}$$

Finally, if $\|T\| = 0$, it holds that $\|T(x)\| = 0, \forall\, x \in X$. Since $\|T(x)\| = 0 \Leftrightarrow T(x) = 0$, this means $T = 0$. Notice that $\|0\| = 0$ is satisfied automatically due to the homogeneity of the norm. We may then conclude that $B(X, Y)$ is indeed a normed space. ■

**Proposition 5.48:**

*Let $X$ and $Y$ be normed spaces. Let $T\colon X \to Y$ be a linear map. $T$ is continuous if, and only if, it is bounded.* □

*Proof:*

We already know $T = 0$ is both bounded and continuous, and therefore we may assume $\|T\| \neq 0$ without loss of generality.

Let us assume $T$ is bounded. Notice that, since we are dealing with a metric space, we might prove that $T$ is continuous in the sense of metric spaces (which, in a metric space, is equivalent to topological continuity).

Let $\epsilon > 0$ and consider $x \in \mathcal{B}_\delta(0)$, where $\delta = \frac{\epsilon}{\|T\|}$. Notice now that

$$\begin{aligned}
\|T(x)\| &\leq \|T\| \|x\|, \\
&< \|T\| \frac{\epsilon}{\|T\|}, \\
&= \epsilon.
\end{aligned} \tag{5.90}$$

Therefore, $T$ is continuous at 0. Corollary 5.23 on page 144 guarantees $T$ is continuous.

Suppose $T$ is continuous. Then $T$ is continuous at 0 and we know that $\forall\, x \in X, \forall\, \epsilon > 0, \exists\, \delta > 0; \|x\| < \delta \Rightarrow \|T(x)\| < \epsilon$.

Let $y \in X, y \neq 0$. Pick $x = \frac{\delta y}{2\|y\|}$. Notice that $\|x\| < \delta$, and therefore it holds that $\|T(x)\| < \epsilon$. However, since $T$ is linear, we have that

$$\begin{aligned} \|T(x)\| &= \left\| \frac{\delta}{2\|y\|} T(y) \right\|, \\ &= \frac{\delta}{2\|y\|} \|T(y)\|. \end{aligned} \tag{5.91}$$

Thus, we see that, $\forall\, y \in X, y \neq 0$,

$$\|T(y)\| < \frac{2\epsilon}{\delta} \|y\|. \tag{5.92}$$

Since $T(0) = 0$, we conclude that

$$\|T(y)\| \leq \frac{2\epsilon}{\delta} \|y\| \tag{5.93}$$

for all $y \in X$, which proves $T$ is bounded. ■

**Theorem 5.49:**
*Let $(X, \{\|\cdot\|_\lambda\}_{\lambda \in \Lambda})$ and $(Y, \{\|\cdot\|_\mu\}_{\mu \in M})$ be locally convex spaces. Let $T\colon X \to Y$ be a linear transformation. It holds that $T$ is continuous if, and only if, $\forall\, \mu \in M, \exists\, \lambda_i \in \Lambda, 1 \leq i \leq n, \alpha > 0$ such that, $\forall\, x \in X$,*

$$\|T(x)\|_\mu \leq \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i}. \tag{5.94}$$

*Furthermore, if $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ is directed, $T$ is continuous if, and only if, $\forall\, \mu \in M, \exists\, \lambda \in \Lambda, \alpha > 0; \|T(x)\|_\mu \leq \alpha \|x\|_\lambda, \forall\, x \in X$.* □

*Proof:*
We shall write $\mathfrak{N}_\Lambda \equiv \{O \in \tau_\Lambda; 0 \in O\}$ and $\mathfrak{N}_M \equiv \{O \in \tau_M; 0 \in O\}$.

Suppose $T$ is continuous. Then it is continuous at 0 and we know that for any $O \in \tau_M$, there is $U \in \tau_\Lambda; T(U) \subseteq O$.

Let $\mu \in M$ and $\epsilon > 0$. Consider the set $N_{\mu;\epsilon} = \left\{y \in Y; \|y\|_\mu < \epsilon\right\}$. We know there is $U \in \mathfrak{N}_\Lambda$ such that $T(U) \subseteq N_{\mu;\epsilon}$ and, from Theorem 5.25 and Proposition 5.28 on page 144 and on page 147, that there is $N_{\lambda_1,\ldots,\lambda_n;\delta}$ such that $N_{\lambda_1,\ldots,\lambda_n;\delta} \subseteq U$.

Consider the function $d$ given by

$$d(x) = \sum_{k=1}^{n} \|x\|_{\lambda_k} + \frac{\delta}{4\epsilon} \|T(x)\|_\mu. \tag{5.95}$$

Notice that $d$ is the sum of non-negative functions, and therefore $d(x) = 0 \Rightarrow \|T(x)\|_\mu = 0$. Let us write $D \equiv \{x \in X; d(x) = 0\}$.

Pick $x \in X \setminus D$ and let $y = \frac{\delta x}{2d(x)}$. We then have $\|y\|_{\lambda_k} \leq \frac{\delta}{2} < \delta, 1 \leq k \leq n$. Therefore, $y \in N_{\lambda_1,\ldots,\lambda_n;\delta} \subseteq U$. Since $T(U) \subseteq N_{\mu;\epsilon}$, we conclude $\|T(y)\|_\mu < \epsilon$. It follows from linearity and homogeneity that

$$\begin{aligned}
\|T(y)\|_\mu &< \epsilon, \\
\frac{\delta}{2d(x)}\|T(x)\|_\mu &< \epsilon, \\
\|T(x)\|_\mu &< \frac{2\epsilon}{\delta}\sum_{k=1}^{n}\|x\|_{\lambda_k} + \frac{1}{2}\|T(x)\|_\mu, \\
\|T(x)\|_\mu &< \frac{4\epsilon}{\delta}\sum_{k=1}^{n}\|x\|_{\lambda_k}.
\end{aligned} \tag{5.96}$$

Since $\|T(x)\|_\mu = 0, \forall\, x \in D$, we may conclude that

$$\|T(x)\|_\mu \leq \frac{4\epsilon}{\delta}\sum_{k=1}^{n}\|x\|_{\lambda_k}, \tag{5.97}$$

for every $x \in X$.

Suppose now that $\forall\, \mu \in M, \exists\, \lambda_i \in \Lambda, 1 \leq i \leq n, \alpha > 0$ such that, $\forall\, x \in X$,

$$\|T(x)\|_\mu \leq \alpha \sum_{i=1}^{n}\|x\|_{\lambda_i}. \tag{5.98}$$

We want to prove that $T$ is continuous. Since this is equivalent to $T$ being continuous at the origin (Corollary 5.23 on page 144), we want to prove that for every $O \in \mathfrak{N}_M, \exists\, U \in \mathfrak{N}_\Lambda; T(U) \subseteq O$. Notice we don't need to prove it for every element of $\mathfrak{N}_M$: proving it in a system of nuclei, such as the one provided by Proposition 5.28 on page 147, is enough, for it implies the result holds in general.

Suppose $T(x) \in N_{\mu_1,\ldots,\mu_m;\epsilon}$. We want to find some open set $N$ with $T(N) \subseteq N_{\mu_1,\ldots,\mu_m;\epsilon}$.

Let $k \in \{i\}_{i=1}^m, \delta > 0$. We shall write $\mu \equiv \mu_k$ for simplicity. Let us then define $D = \left\{x \in X; \sum_{i=1}^n \|x\|_{\lambda_i} < \delta\right\}$. Notice that $D \subseteq N_{\lambda_1,\ldots,\lambda_n;\delta}$. Notice that, if $x \in N_{\lambda_1,\ldots,\lambda_n;\delta}$, then it holds that

$$\begin{aligned}
\|T(x)\|_\mu &\leq \alpha \sum_{i=1}^{n}\|x\|_{\lambda_i}, \\
\|T(x)\|_\mu &< \alpha\delta.
\end{aligned} \tag{5.99}$$

We might now pick $\delta = \frac{\epsilon}{\alpha}$ and see that, for each $1 \leq k \leq m$, we found a set in which $\|T(x)\|_{\mu_\epsilon} < \epsilon$. The first item of Theorem 5.25 on page 144 allows us to find some set $N$ which lies in the intersection of all these sets we've just built. Notice this $N$ has the property that $\forall\, x \in N, \|T(x)\|_{\mu_\epsilon} < \epsilon, 1 \leq k \leq m$. Thus, $T(N) \subseteq N_{\mu_1,\ldots,\mu_m;\epsilon}$. This proves $T$ is continuous at 0, and thus continuous as a whole.

If $\forall\, \mu \in M, \exists\, \lambda \in \Lambda, \alpha > 0; \|T(x)\|_\mu \leq \alpha\|x\|_\lambda, \forall\, x \in X$, the same proof holds, regardless of the family of seminorms being directed.

Suppose now the $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ is directed and $T$ is continuous. Since $T$ is continuous, we know that $\forall\, \mu \in M, \exists\, \lambda_i \in \Lambda, 1 \leq i \leq n, \alpha > 0$ such that, $\forall\, x \in X$,

$$\|T(x)\|_\mu \leq \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i}. \tag{5.100}$$

Since $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ is directed, we know there are $\lambda_0 \in \Lambda, \beta > 0$ such that $\sum_{i=1}^{n} \|x\|_{\lambda_i} \leq \beta\|x\|_{\lambda_0}, \forall\, x \in X$. As a consequence, we see that, $\forall\, x \in X$,

$$\|T(x)\|_\mu \leq \alpha\beta\|x\|_{\lambda_0}, \tag{5.101}$$

which concludes the proof. ■

## 5.3 The Hahn-Banach Theorem

As stated earlier, the theory of locally convex spaces is an intermediate step between the theory of linear topological spaces and the theory of Banach spaces. Just like we proved earlier that normal spaces have plenty of continuous functions, we shall now prove that locally convex spaces have plenty of linear functionals (also known as linear forms) and, as a consequence, they provide us with a rich theory of duality.

**Definition 5.50** [Linear Functional or Linear Form]**:**
Let $X$ be a vector space over a field $F$, supposed to be either the real line or the complex plane. A *linear functional*, or *linear form*, $f$ is a linear operator with domain on $X$ and codomain on $\mathbb{F}$, *id est*, a linear operator of the form $f\colon X \to \mathbb{F}$. ♠

**Definition 5.51** [Algebraic Dual]**:**
Let $X$ be a vector space over $\mathbb{F}$. We define the *algebraic dual* of $X$ as the space $X^* \equiv L(X,\mathbb{F})$ of all linear functionals defined on $X$. ♠

*Remark:*
Proposition 5.47 on page 155 implies $X^*$ is a vector space. If we defined $X^* = B(X,\mathbb{F})$ (the norm on $\mathbb{F}$ is the absolute value), as is done in the theory of normed spaces, $X^*$ would be a normed space. ♣

**Definition 5.52** [Topological Dual]**:**
Let $X$ be a linear topological space over $\mathbb{F}$. We define the *topological dual* of $X$ as the space $X'$ of all continuous linear functionals defined on $X$. ♠

*Remark:*
Since, due to Proposition 3.79 on page 63, the addition of real and complex-valued continuous functions is continuous, and so is the multiplication by a scalar, we see that the topological dual of a linear topological space is a vector space as well. In fact, notice that $X' = C(X)$.
Whenever we refer to the dual of a linear topological space, the topological dual should be understood. ♣

In order to prove the result we are seeking, we will need to use some concepts and results from set theory.

**Definition 5.53** [Total Orderings and Chains]**:**
Let $(X, \prec)$ be a poset. If, given any two elements $x, y \in X$, it holds that either $x \prec y$ or $y \prec x$, we say that $\prec$ is a *total ordering* in $X$ and that $(X, \prec)$ is a *totally ordered set* or a *chain*. ♠

*Remark:*
If $(X, \prec)$ is a poset, a chain in $X$ is some subset $A \subseteq X$ equipped with the same ordering $\prec$ such that $(A, \prec)$ is a totally ordered set. ♣

**Definition 5.54** [Lower and Upper Bounds]**:**
Let $(X, \prec)$ be a poset and let $A \subseteq X$. An element $a \in X$ is said to be a *lower bound* of $A$ if it holds that $a \prec x, \forall\, x \in A$. Similarly, $a$ is said to be a *upper bound* of $A$ if it holds that $a \succ x, \forall\, x \in A$. ♠

**Definition 5.55** [Minimal and Maximal Elements]**:**
Let $(X, \prec)$ be a poset. An element $a \in X$ is said to be *minimal* if $\forall\, x \in X, x \prec a \Rightarrow x = a$. Similarly, $a$ is said to be *maximal* if $\forall\, x \in X, x \succ a \Rightarrow x = a$. ♠

**Example** [Lower and Upper Bounds, Minimal and Maximal Elements]**:**
Consider the set $X = \mathbb{P}(\mathbb{N}) \setminus \{\varnothing\}$ ordered by inclusion, *id est*, $A \prec B \Leftrightarrow A \subseteq B$. $(\mathbb{P}(\mathbb{N}), \prec)$ is not totally ordered, for $\{0\} \not\prec \{1\}$ and $\{1\} \not\prec \{0\}$.

The set $\{\{0,1\}, \{1,2\}\}$ has $\{1\}$ as a lower bound and every set $A$ satisfying $0, 1, 2 \in A$ as an upper bound. All singletons, *id est*, sets of the form $\{n\}, n \in \mathbb{N}$, are minimal elements of $X$ and $\mathbb{N}$ is its only maximal element.

Consider now the set $X = \left\{q \in \mathbb{Q}; q^2 < 2\right\}$ with the usual ordering in $\mathbb{R}$, denoted by $\leq$. $(X, \leq)$ is a chain. Notice, however, that $X$ does not possess lower or upper bounds, nor maximal or minimal elements. This changes if we consider $X$ as a subset of $\mathbb{Q}$, for $2 > x, \forall\, x \in X$, and therefore 2 is an example of an upper bound for $X$ in $\mathbb{Q}$. ♥

**Lemma 5.56** [Zorn's Lemma]**:**
*Let $(X, \prec)$ be a poset such that every chain in $X$ has an upper bound. Then $X$ contains a maximal element.* □

*Proof:*
Firstly, notice that $\varnothing \subseteq X$ is a chain in $X$, and as a consequence it has an upper bound. Thus, there is $a \in X; a \succ x, \forall\, x \in \varnothing$. As a consequence, we see that $X \neq \varnothing$.

Let us consider the function $s\colon X \to \mathbb{P}(X)$ such that $s(x) = \{y \in X; y \prec x\}$. Let $S \equiv \operatorname{Ran} s = \{s(x) \in \mathbb{P}(X); x \in X\}$. We might now equip $S$ with the inclusion order.

Notice now that $x \prec y \Leftrightarrow s(x) \subseteq s(y)$. Indeed, if $x \prec y$, then transitivity implies that $z \prec y$ (and therefore $z \in s(y)$) for every $z \prec x$ (*id est*, for every $z \in s(x)$). Conversely, if $s(x) \subseteq s(y)$, then reflexivity implies $x \in s(x) \subseteq s(y)$, *id est*, $x \in s(y)$, and therefore $x \prec y$. We have thus found an ordering-preserving bijective relation between $X$ and $S$. If we prove Zorn's Lemma for the special case of $(S, \subseteq)$, we will have proven it for the general case.

Let $\mathcal{X}$ be the collection of all chains in $X$. If $C \in \mathcal{X}$, then $C$ has an upper bound in $X$, say $a$, and as a consequence it follows that $C \subseteq s(a)$, *id est*, $\forall\, C \in \mathcal{X}, \exists\, s \in S; C \subseteq s$.

We might now regard $\mathcal{X}$ as a poset by ordering it by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{X}$, then $\bigcup_{A \in \mathcal{C}} A \in \mathcal{X}$. Indeed, $A \in \mathcal{C} \Rightarrow A \in \mathcal{X} \Rightarrow A \subseteq X$. Given $x, y \in \bigcup_{A \in \mathcal{C}} A$, there are $A, B \in \mathcal{C}$ such that $x \in A, y \in B$. Since $\mathcal{C}$ is a chain, we either have $A \subseteq B$ or $B \subseteq A$. Suppose, without any loss of generality, that $A \subseteq B$. Then $x, y \in B$. Since $B \in \mathcal{C} \subseteq \mathcal{X}$, $B$ is a chain and we have that either $x \prec y$ or $y \prec x$.

As every element of $\mathcal{X}$ is dominated by an element of $S$, consider $\mathcal{X}$ instead of $S$ will not bring any new maximal elements. If $M \in \mathcal{X}$ is maximal, then there is some $m \in X$ such that $M \subseteq s(m)$. Since $s(m)$ is a chain in $X$, $s(m) \in \mathcal{X}$ and we see that $M = s(m)$. Thus, $M$ is a maximal element in $S$ as well.

The interest in dealing with $\mathcal{X}$ instead of $S$ is the fact that we do not need to consider only that each chain has some upper bound in $S$, but we may instead work with the stronger assumption that $\bigcup_{A \in \mathcal{C}} A \in \mathcal{X}$, for every chain $\mathcal{C}$ in $\mathcal{X}$. Notice that $\bigcup_{A \in \mathcal{C}} A$ is an upper bound for $\mathcal{C}$. Furthermore, we also have that any subset of $\mathcal{C}$ is also an element of $\mathcal{X}$, which is going to be useful in order to enlarge non-maximal sets of $\mathcal{X}$ an element at a time.

Thus, from now on we are going to deal with another set. Given a non-empty set $X$, we are going to consider a collection $\mathcal{X} \subseteq \mathbb{P}(X)$ such that $\bigcup_{A \in \mathcal{C}} A \in \mathcal{X}$, for every chain $\mathcal{C}$ in $\mathcal{X}$, and every subset of an element of $\mathcal{X}$ is also a subset of $\mathcal{X}$ (as a consequence, it holds that $\varnothing \in \mathcal{X}$). We want to prove that $\mathcal{X}$ admits a maximal element.

Let $f \colon \mathbb{P}(X) \setminus \{\varnothing\} \to X$ be a function such that $f(A) \in A, \forall\, A \in \mathbb{P}(X) \setminus \{\varnothing\}$. The existence of such a function is guaranteed by the Axiom of Choice. Given $A \in \mathcal{X}$, let $\hat{A} \equiv \{x \in X; \{x\} \cup A \in \mathcal{X}\}$. Let us define a function $g \colon \mathcal{X} \to \mathcal{X}$ through

$$g(A) = \begin{cases} A \cup \{f(\hat{A} \setminus A)\}, \text{ if } \hat{A} \setminus A \neq \varnothing; \\ A, \text{ if } \hat{A} \setminus A = \varnothing. \end{cases} \tag{5.102}$$

Notice that $A \subseteq \hat{A}$ and $\hat{A} = A$ if, and only if, $A$ is maximal. Indeed, if $\hat{A} = A$ and $A \subseteq B$ for some $B \in \mathcal{X}$, then $A \cup \{x\} \in \mathcal{X}, \forall\, x \in B$. Since every such $x$ is already an element of $\hat{A} = A$, we see that $B = A$, and thus $A$ is maximal. Therefore, we are interested in finding some set $A \in \mathcal{X}$ such that $A = g(A)$. Notice also that, given any $A \in \mathcal{X}$, $g(A)$ has at most one more element than $A$.

Given $\mathcal{T} \subseteq \mathcal{X}$, we say $\mathcal{T}$ is a *tower* if the following conditions hold:

i. $\varnothing \in \mathcal{T}$;

ii. $A \in \mathcal{T} \Rightarrow g(A) \in \mathcal{T}$;

iii. if $\mathcal{C}$ is a chain in $\mathcal{T}$, then $\bigcup_{A \in \mathcal{C}} A \in \mathcal{T}$.

Notice that $\mathcal{X}$ itself is a tower. Furthermore, let $\Lambda$ be some arbitrary set and let $\mathcal{T}_\lambda$ be a tower, $\forall\, \lambda \in \Lambda$. Then $\mathcal{T} = \bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$ is a tower as well. Indeed, $\varnothing \in \mathcal{T}$, for

$\varnothing \in \mathcal{T}_\lambda, \forall\, \lambda \in \Lambda$. If $A \in \mathcal{T}$, then $A \in \mathcal{T}_\lambda, \forall\, \lambda \in \Lambda$. As a consequence, $g(A) \in \mathcal{T}_\lambda, \forall\, \lambda \in \Lambda$, and thus $g(A) \in \mathcal{T}$. Finally, if $\mathcal{C}$ is a chain in $\mathcal{T}$, it is also a chain in $\mathcal{T}_\lambda, \forall\, \lambda \in \Lambda$. Therefore, $\bigcup_{A \in \mathcal{C}} A \in \mathcal{T}_\lambda, \forall\, \lambda \in \Lambda$. We conclude $\bigcup_{A \in \mathcal{C}} A \in \mathcal{T}$. We might then consider $\mathcal{T}_0$: the intersection of all towers.

Given $A \in \mathcal{T}_0$, we say $A$ is *comparable* if it holds that either $A \subseteq B$ or $B \subseteq A, \forall\, B \in \mathcal{T}_0$. Naturally, $\mathcal{T}_0$ is a chain if, and only if, every element of $\mathcal{T}_0$ is comparable. Since $\varnothing \in \mathcal{T}_0$ and $\varnothing \subseteq A, \forall\, A \in \mathcal{X}$, we know there are comparable sets in $\mathcal{T}_0$.

Let $C \in \mathcal{T}_0$ be comparable. If $A \in \mathcal{T}_0$ is a proper subset of $C$, *id est*, $A \subset C$, then $g(A) \subseteq C$. Indeed, since $C$ is comparable, we either have $g(A) \subseteq C$ or $C \subseteq g(A)$. If $g(A) = C$, then $g(A) \subseteq C$. Let us then suppose by contradiction that $C \subset g(A)$. Then $A \subset C \subset g(A)$, which is a contradiction, for $g(A)$ has at most one more element than $A$. Therefore, it is true that $g(A) \subseteq C$ for every $A \in \mathcal{T}_0$ with $A \subset C$.

Let us now define $\mathcal{U} \equiv \{A \in \mathcal{T}_0; A \subseteq C \text{ or } g(C) \subseteq A\}$. Notice that $A \in \mathcal{U} \Rightarrow A \subseteq g(C)$ or $g(C) \subseteq A$, for $A \subseteq C \Rightarrow A \subseteq C \subseteq g(C)$.

$\mathcal{U}$ is a tower. Indeed, since $\varnothing \in \mathcal{T}_0$ and $\varnothing \subseteq C, \varnothing \in \mathcal{U}$. If $A \in \mathcal{U}$, then either $A \subset C$ (and thus $g(A) \subseteq C$) or $A = C$ (and thus $g(A) = g(C)$, which means $g(C) \subseteq g(A)$) or $g(C) \subseteq A \subseteq g(A)$. Thus, $A \in \mathcal{U} \Rightarrow g(A) \in \mathcal{U}$. Finally, suppose $\mathcal{C}$ is a chain in $\mathcal{U}$. If all elements of $\mathcal{C}$ are contained in $C$, then the union of all elements of $\mathcal{C}$ is also contained in $C$ and thus $\bigcup_{A \in \mathcal{C}} A \in \mathcal{U}$. On the other hand, if there is $A \in \mathcal{C}$ such that $g(C) \subseteq A$, then it holds that $g(C) \subseteq \bigcup_{A \in \mathcal{C}} A$. Either way, the union of all elements of $\mathcal{C}$ is an element of $\mathcal{U}$, which concludes the proof that $\mathcal{U}$ is a tower. Since $\mathcal{U} \subseteq \mathcal{T}_0$ and $\mathcal{T}_0$ is, by definition, the smallest tower, we see that $\mathcal{T}_0 = \mathcal{U}$.

As a consequence, we see that if $C$ is comparable, then so is $g(C)$. Indeed, if $A \subseteq C$, then $A \subseteq g(C)$ (for $C \subseteq g(C)$). If $A \nsubseteq C$, then the equality $\mathcal{U} = \mathcal{T}_0$ guarantees that $g(C) \subseteq A$. Therefore, given any $A \in \mathcal{T}_0$, we see that it holds that either $A \subseteq g(C)$ or $g(C) \subseteq A$, *id est*, $g(C)$ is comparable.

$\varnothing$ is comparable and $g$ maps comparable sets onto comparable sets. Furthermore, the union of a chain of comparable sets is comparable. Indeed, suppose $\mathcal{C}$ is such a chain and let $A \in \mathcal{T}_0$. If $B \subseteq A, \forall\, B \in \mathcal{C}$, then $\bigcup_{B \in \mathcal{C}} B \subseteq A$. If there is any $B \in \mathcal{C}$ such that $A \subseteq B$, then $A \subseteq \bigcup_{B \in \mathcal{C}} B$. We see then that the collection of all comparable sets is a tower, which we denote $\mathcal{T}_C$. Since every comparable set is an element of $\mathcal{T}_0$, it holds that $\mathcal{T}_C \subseteq \mathcal{T}_0$. However, $\mathcal{T}_0$ is the smallest tower, and thus we see that $\mathcal{T}_C = \mathcal{T}_0$, *id est*, every element of $\mathcal{T}_0$ is comparable. This means that $\mathcal{T}_0$ is a chain.

As $\mathcal{T}_0$ is a chain in $\mathcal{T}_0$, the union of all elements of $\mathcal{T}_0$, denoted $U$, is in $\mathcal{T}_0$. Since $A \subseteq U, \forall\, A \in \mathcal{T}_0$, we see that $g(U) \subseteq U$. Since it always holds that $A \subseteq g(A), \forall\, A \in \mathcal{X}$, we conclude that $g(U) = U$, *id est*, $U$ is maximal.

This concludes the proof. ∎

Finally, let us make one last definition before stating, and proving, the Hahn-Banach Theorem.

**Definition 5.57** [Sublinear Functional]**:**

Let $X$ be a real vector space over. A *sublinear functional* $p$ is a function $p\colon X \to \mathbb{R}$ such that

i. $p(x+y) \leq p(x) + p(y), \forall\, x, y \in X$;

ii. $p(\alpha \cdot x) = \alpha \cdot p(x), \forall\, x \in X, \forall\, \alpha > 0$. ♠

**Theorem 5.58** [Hahn-Banach Theorem]**:**
*Assume $X$ is a real vector space, $Y \subseteq X$ is a linear subspace of $X$ and $p\colon X \to \mathbb{R}$ is a sublinear functional. Given a linear functional $f\colon Y \to \mathbb{R}$ such that $f(x) \leq p(x), \forall\, x \in Y$, there is a linear functional $\bar{f}\colon X \to \mathbb{R}$ such that $\bar{f}(x) \leq p(x), \forall\, x \in X$, and satisfying $f(x) = \bar{f}(x), \forall\, x \in Y$.* □

*Proof*:
We say a linear functional $g\colon Z \to \mathbb{R}$ linearly extends $f$ if $g$ is defined on some subspace $Z$ of $X$ such that $Y \subseteq Z$, $g(x) \leq p(x), \forall\, x \in Z$ and $g(x) = f(x), \forall\, x \in Y$.

Consider the set

$$\mathcal{F} = \{g\colon Z \to \mathbb{R}; g \text{ linearly extends } f\}. \tag{5.103}$$

$f \in \mathcal{F}$, and thus $\mathcal{F} \neq \varnothing$. We might order $\mathcal{F}$ if we define a partial ordering $\prec$ by stating that $g \prec h$ if, and only if, $\operatorname{Dom} g \subseteq \operatorname{Dom} h$ with $h(x) = g(x), \forall\, x \in \operatorname{Dom} g$, where $\operatorname{Dom} g$ denotes the domain of $g$. The partial ordering properties of inclusion guarantee that $\prec$ is a partial ordering as well.

Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Let us define $G = \bigcup_{g \in \mathcal{C}} \operatorname{Dom} g$. Since $\mathcal{C}$ is a chain, we know that given any $g_1, g_2 \in \mathcal{C}$ it holds that either $\operatorname{Dom} g_1 \subseteq \operatorname{Dom} g_2$ or $\operatorname{Dom} g_2 \subseteq \operatorname{Dom} g_1$. As a consequence, it follows that $G$ is a subspace of $X$. Since $\operatorname{Dom} g \subseteq X, \forall\, g \in \mathcal{C}$, it holds that $G \subseteq X$. Furthermore, given any two $x, y \in G$, we can find some $g \in \mathcal{C}$ such that $x, y \in \operatorname{Dom} g$ and it follows that $x + y, \alpha \cdot x \in \operatorname{Dom} g \subseteq G, \forall\, \alpha \in \mathbb{R}$. Thus, $G$ is indeed a subspace of $X$.

Now, $\forall\, g \in \mathcal{C}$, define a linear functional $\hat{g}\colon G \to \mathbb{R}$ through $\hat{g}(x) = g(x), \forall\, x \in \operatorname{Dom} g$. $\hat{g}$ is well-defined, for if $x \in \operatorname{Dom} g_1 \cap \operatorname{Dom} g_2$, the fact that $\mathcal{C}$ is a chain means that we either have $g_1 \prec g_2$ or $g_2 \prec g_1$. Either way, we have that $g_1(x) = g_2(x), \forall\, x \in \operatorname{Dom} g_1 \cap \operatorname{Dom} g_2$. Notice that, in particular, it holds that $\hat{g}(x) = f(x), \forall\, x \in Y$, *id est*, $\hat{g} \in \mathcal{F}$. Since, given any $g \in \mathcal{C}$, it holds that $\operatorname{Dom} g \subseteq \operatorname{Dom} \hat{g}$ and $\hat{g}(x) = g(x), \forall\, x \in \operatorname{Dom} g$, we see that $g \prec \hat{g}$. The condition that $\hat{g}(x) \leq p(x), \forall\, x \in G$ comes for free, as, for each $x \in G$, there is some $g \in \mathcal{C}$ with $g(x) \leq p(x)$ such that $\hat{g}(x) = g(x)$. Thus, $\hat{g}$ is an upper bound for $\mathcal{C}$.

We just showed that any chain $\mathcal{C}$ in $\mathcal{F}$ admits an upper bound. Therefore, Zorn's Lemma implies that $\mathcal{F}$ admits a maximal element. We shall denote such an element by $\bar{f}$.

Notice that $\bar{f}$ is such that $\bar{f}(x) \leq p(x), \forall\, x \in \operatorname{Dom} \bar{f}$, and satisfies $f(x) = \bar{f}(x), \forall\, x \in Y$. We remain to prove that, in fact, $\operatorname{Dom} \bar{f} = X$.

Suppose, by contradiction, that $\operatorname{Dom} \bar{f} \neq X$. Then there is some $k \in X \setminus \operatorname{Dom} \bar{f}$. Let us consider the vector space spanned by $k$ and $\operatorname{Dom} \bar{f}$, which we shall denote by $Z$. Given any $z \in Z$, there are $x \in \operatorname{Dom} \bar{f}$ and $\alpha \in \mathbb{R}$ such that $z = x + \alpha k$. In fact, there is a single $x$ and a single $\alpha$ with these properties. Indeed, suppose $z = \tilde{x} + \tilde{\alpha} k$. Then we have that $x - \tilde{x} + (\alpha - \tilde{\alpha})k = 0$. Since $k \notin \operatorname{Dom} \bar{f}$, it follows that $x = \tilde{x}$ and $\alpha = \tilde{\alpha}$, meaning the representation $z = x + \alpha k$ is indeed unique.

Let us now define a linear functional $g\colon Z \to \mathbb{R}$ such that $g(x+\alpha k) = \bar{f}(x) + \alpha c, \forall\, x \in \operatorname{Dom}\bar{f}, \forall\, \alpha \in \mathbb{R}$, where $c \in \mathbb{R}$ is given. Notice $g(x) = \bar{f}(x), \forall\, x \in \operatorname{Dom}\bar{f}$, and therefore $g(x) = f(x), \forall\, x \in Y$. If we prove that $g(x) \le p(x), \forall\, x \in \operatorname{Dom} g$, then it holds that $g \in \mathcal{F}$ and $f \prec g$ with $g \neq f$, which is a contradiction, meaning that, indeed, $\operatorname{Dom}\bar{f} = X$ and concluding the proof.

Let us then prove that $g(x) \le p(x), \forall\, x \in \operatorname{Dom} g$ for some appropriate choice of $c$.

Let $x, y \in \operatorname{Dom}\bar{f}$. Then we have

$$\begin{aligned}
\bar{f}(x) - \bar{f}(y) &= \bar{f}(x-y),\\
&\le p(x-y),\\
&\le p(x+k-k-y),\\
&\le p(x+k) + p(-k-y),\\
-\bar{f}(y) - p(-k-y) &\le p(x+k) - \bar{f}(x),\\
m \equiv \sup_{y\in\operatorname{Dom}\bar{f}} \left\{-\bar{f}(y) - p(-k-y)\right\} &\le \inf_{x\in\operatorname{Dom}\bar{f}} \left\{p(x+k) - \bar{f}(x)\right\} \equiv M. \qquad (5.104)
\end{aligned}$$

If we pick $c$ such that $m \le c \le M$, then it holds for every $x \in \operatorname{Dom}\bar{f}$ that

$$-\bar{f}(x) - p(-k-x) \le c \le p(x+k) - \bar{f}(x). \qquad (5.105)$$

We want to prove that $g(x+\alpha k) = \bar{f}(x) + \alpha c \le p(x+\alpha k), \forall\, x \in \operatorname{Dom}\bar{f}, \forall\, \alpha \in \mathbb{R}$. For $\alpha = 0$, we already know that $g(x) = \bar{f}(x) \le p(x)$. Let us then suppose, firstly, that $\alpha < 0$. If we write $x = \frac{1}{\alpha}y$, we get

$$\begin{aligned}
-\bar{f}(x) - p(-k-x) &\le c,\\
-\bar{f}\left(\frac{1}{\alpha}y\right) - p\left(-k - \frac{1}{\alpha}y\right) &\le c,\\
\alpha\bar{f}\left(\frac{1}{\alpha}y\right) + \alpha p\left(-k - \frac{1}{\alpha}y\right) &\le -\alpha c,\\
\bar{f}(y) - p(\alpha k + y) &\le -\alpha c,\\
\bar{f}(y) + \alpha c &\le p(\alpha k + y),\\
g(\alpha k + y) &\le p(\alpha k + y). \qquad (5.106)
\end{aligned}$$

On the other hand, if $\alpha > 0$, then writing $x = \frac{1}{\alpha}y$ yields

$$\begin{aligned}
c &\le p(x+k) - \bar{f}(x),\\
\alpha c &\le \alpha p\left(\frac{1}{\alpha}y + k\right) - \alpha\bar{f}\left(\frac{1}{\alpha}y\right),\\
\alpha c &\le p(y+\alpha k) - \bar{f}(y),\\
\bar{f}(y) + \alpha c &\le p(y+\alpha k),\\
g(y+\alpha k) &\le p(y+\alpha k). \qquad (5.107)
\end{aligned}$$

Thus, it does hold that $g(x) \le p(x), \forall\, x \in \operatorname{Dom} g$, proving that $g \in \mathcal{F}$ with $\bar{f} \prec g$ and $g \neq \bar{f}$, despite $\bar{f}$ being a maximal element of $\mathcal{F}$. This inconsistency proves that $\operatorname{Dom}\bar{f} = X$, concluding the proof. ■

**Theorem 5.59** [Generalized Hahn-Banach Theorem]**:**

*Let $X$ be a vector space over some field $\mathbb{F}$, where $\mathbb{F}$ is either the real line or the complex plane, $Y \subseteq X$ be a subspace and $p\colon X \to \mathbb{R}$ be a function such that*

i. $p(x+y) \leq p(x) + p(y), \forall\, x, y \in X$;

ii. $p(\alpha \cdot x) = |\alpha| \cdot p(x), \forall\, x \in X, \forall\, \alpha \in \mathbb{F}$.

*Given a linear functional $f\colon Y \to \mathbb{R}$ such that $|f(x)| \leq p(x), \forall\, x \in Y$, there is a linear functional $\bar{f}\colon X \to \mathbb{R}$ such that $\left|\bar{f}(x)\right| \leq p(x), \forall\, x \in X$, and satisfying $f(x) = \bar{f}(x), \forall\, x \in Y$.* $\square$

*Proof:*

Let us first assume $\mathbb{F} = \mathbb{R}$. We shall prove the complex case afterwards.

We know that $f(x) \leq |f(x)| \leq p(x), \forall\, x \in Y$. Thus, the Hahn-Banach Theorem implies the existence of a linear functional $\bar{f}\colon X \to \mathbb{R}$ extending $f$ such that $\bar{f}(x) \leq p(x), \forall\, x \in X$. Since $p(\alpha \cdot x) = |\alpha| \cdot p(x), \forall\, x \in X, \forall\, \alpha \in \mathbb{F}$, we see that

$$\begin{aligned} -\bar{f}(x) &= \bar{f}(-x), \\ &\leq p(-x), \\ &\leq |-1| \cdot p(x), \\ &\leq p(x), \end{aligned} \tag{5.108}$$

which, when combined with $\bar{f}(x) \leq p(x)$, allows us to conclude that $\left|\bar{f}(x)\right| \leq p(x), \forall\, x \in X$.

From now on, we assume $\mathbb{F} = \mathbb{C}$.

Since $X$ is a complex vector space, so is $Y$. Thus, $f$ is complex-valued and we might write, for every $x \in Y$,

$$f(x) = u(x) + iv(x). \tag{5.109}$$

Let us consider for a moment $X$ and $Y$ as real vector spaces, *id est*, let us restrict the multiplication by a scalar to only real scalars. We shall denote such real spaces by $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$. Under this conditions, both $u$ and $v$ are linear functionals defined on $Y_{\mathbb{R}}$. Since $u(x) \leq |f(x)| \leq p(x), \forall\, x \in Y_{\mathbb{R}}$, we might apply the Hahn-Banach Theorem. Therefore, there is some linear functional $\bar{u}\colon X_{\mathbb{R}} \to \mathbb{R}$ extending $u$ and respecting the condition that $\bar{u}(x) \leq p(x), \forall\, x \in X_{\mathbb{R}}$.

Back to $Y$, we can see that, $\forall\, x \in Y$,

$$\begin{aligned} iu(x) - v(x) &= i[u(x) + iv(x)], \\ &= if(x), \\ &= f(ix), \\ &= u(ix) + iv(ix). \end{aligned} \tag{5.110}$$

Taking the real part of both sides of this expression, we find that $v(x) = -u(ix), \forall\, x \in Y$. This motivates us to consider the function $\bar{f}\colon X \to \mathbb{C}$ given by

$$\bar{f}(x) = \bar{u}(x) - i\bar{u}(ix). \tag{5.111}$$

We know $\bar{u}(x) = u(x), \forall\, x \in Y$ and $v(x) = -u(ix), \forall\, x \in Y$. Therefore, it holds that $\bar{f}(x) = f(x), \forall\, x \in Y$. Furthermore, we know $\bar{f}$ is a real linear functional. We still must prove $\bar{f}$ to be a complex linear functional (*id est*, $\alpha f(x) = f(\alpha x), \forall\, \alpha \in \mathbb{C}$) and that $|\bar{f}(x)| \leq p(x), \forall\, x \in X$.

Notice that, $\forall\, a, b \in \mathbb{R}, \forall\, x \in X$,

$$\begin{aligned}
\bar{f}\big((a+ib)x\big) &= \bar{u}\big((a+ib)x\big) - i\bar{u}\big((ia-b)x\big), \\
&= \bar{u}(ax) + \bar{u}(ibx) - i\bar{u}(iax) + i\bar{u}(bx), \\
&= a\bar{u}(x) + b\bar{u}(ix) - ia\bar{u}(ix) + ib\bar{u}(x), \\
&= a[\bar{u}(x) - i\bar{u}(ix)] + ib[\bar{u}(x) - i\bar{u}(ix)], \\
&= (a+ib)[\bar{u}(x) - i\bar{u}(ix)], \\
&= (a+ib)\bar{f}(x).
\end{aligned} \tag{5.112}$$

Thus, $\bar{f}$ is a complex linear functional on $X$.

Finally, we still must prove $|\bar{f}(x)| \leq p(x)$. Firstly, notice that $p(x) \geq 0, \forall\, x \in X$. Indeed, we have from the first axiom imposed on $p$ that

$$p(0) \leq p(x) + p(-x), \forall\, x \in X. \tag{5.113}$$

However, $p(0) = |0|p(0) = 0$. Furthermore, $p(-x) = |-1|p(x) = p(x)$. Thus, we get that

$$2p(x) \geq 0 \Rightarrow p(x) \geq 0, \forall\, x \in X. \tag{5.114}$$

Therefore, if $x$ is such that $\bar{f}(x) = 0$, then $|\bar{f}(x)| \leq p(x)$ is automatically satisfied. Let us then turn our attention to the situation in which $\bar{f}(x) \neq 0$. Then we might write, for such a fixed $x$, $\bar{f}(x) = |\bar{f}(x)|e^{i\theta}$. As a consequence, it follows that $|\bar{f}(x)| = \bar{f}(x)e^{-i\theta} = \bar{f}(e^{-i\theta}x)$. Since $|\bar{f}(x)|$ is real, such an expression is equal to its real part (for the imaginary part vanishes) and it follows that

$$\begin{aligned}
|\bar{f}(x)| &= \bar{f}\left(e^{-i\theta}x\right), \\
&= \bar{u}\left(e^{-i\theta}x\right), \\
&\leq p\left(e^{-i\theta}x\right), \\
&= \left|e^{-i\theta}\right| p(x), \\
&= p(x),
\end{aligned} \tag{5.115}$$

which concludes the proof. ■

*Remark:*

We shall refer to both the Hahn-Banach Theorem and the Generalized Hahn-Banach Theorem by simply "Hahn-Banach Theorem". The specific theorem we are using should be clear from context. ♣

We might now consider the implications of the Hahn-Banach Theorem to the theory of locally convex spaces.

**Corollary 5.60:**
*Let $(X, \{\|\cdot\|_\lambda\})_{\lambda\in\Lambda}$ be a locally convex space over some field $\mathbb{F}$, which is taken to be either the real line or the complex plane. Let $Y \subseteq X$ be a linear subspace. Let $f\colon Y \to \mathbb{F}$ be a continuous linear functional. Then there is some continuous linear functional $\bar{f}\colon X \to \mathbb{F}$ such that $\bar{f}(x) = f(x), \forall\, x \in Y$.* □

*Proof:*
The relative topology induced in $Y$ can be obtained through the restriction of the seminorms to $Y$ (for this is equivalent to intersecting the elements of the system of nuclei, $N_{\lambda_1,\ldots,\lambda_n;\epsilon}$, with $Y$).

Notice that $\mathbb{F}$ is a locally convex space with a single seminorm: the absolute value, $|\cdot|$. Thus, Theorem 5.49 on page 158 implies there are $\lambda_i \in \Lambda, \alpha > 0$ such that

$$|f(x)| \leq \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i}. \tag{5.116}$$

Notice that, if we define $p(x) := \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i}, \forall\, x \in X$, it follows that

$$\begin{aligned}
p(x+y) &= \alpha \sum_{i=1}^{n} \|x+y\|_{\lambda_i}, \\
&\leq \alpha \sum_{i=1}^{n} \left(\|x\|_{\lambda_i} + \|y\|_{\lambda_i}\right), \\
&= \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i} + \alpha \sum_{i=1}^{n} \|y\|_{\lambda_i}, \\
&= p(x) + p(y).
\end{aligned} \tag{5.117}$$

Furthermore, for $\beta \in \mathbb{F}$,

$$\begin{aligned}
p(\beta x) &= \alpha \sum_{i=1}^{n} \|\beta x\|_{\lambda_i}, \\
&= \alpha \sum_{i=1}^{n} |\beta| \|x\|_{\lambda_i}, \\
&= |\beta| \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i}, \\
&= |\beta| p(x).
\end{aligned} \tag{5.118}$$

Therefore, we might now apply the Hahn-Banach Theorem to guarantee the existence of an extension $\bar{f}$ of $f$ which is not only a linear functional defined on $X$, but also respects $|\bar{f}(x)| \leq p(x)$, *id est*,

$$|\bar{f}(x)| \leq \alpha \sum_{i=1}^{n} \|x\|_{\lambda_i}. \tag{5.119}$$

Thus, Theorem 5.49 on page 158 implies $\bar{f}$ is continuous, proving the theorem. ■

We might as well study a geometric implication of the theorem: the separation of disjoint sets by hyperplanes.

**Definition 5.61** [Hyperplane]**:**
Let $X$ be some vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is taken to be either the real line or the complex plane. A *hyperplane* is a set of the form $\{x \in X; f(x) = a\}$, for some real-valued linear functional $f$ and some number $a \in \mathbb{R}$. ♠

*Remark:*
Notice hyperplanes are defined in terms of real-valued linear functionals even when we are dealing with complex vector spaces. ♣

**Definition 5.62** [Sets Separated by a Hyperplane]**:**
Let $(X, \{\|\cdot\|_\lambda\})_{\lambda\in\Lambda}$ be a locally convex space over some field $\mathbb{F}$, which is taken to be either the real line or the complex plane. Let $A, B \subseteq X$. $A$ and $B$ are said to be *separated by a hyperplane* if there is some continuous linear functional $f\colon X \to \mathbb{R}$ and some $a \in \mathbb{R}$ such that $f(x) \leq a, \forall\, x \in A$ and $f(x) \geq a, \forall\, x \in B$. Furthermore, if $f(x) < a, \forall\, x \in A$ and $f(x) > a, \forall\, x \in B$, $A$ and $B$ are said to be *strictly separated*. ♠

**Lemma 5.63:**
*$I \subseteq \mathbb{R}$ is convex if, and only if, it is an interval.* □

*Proof:*
We shall prove the case in which $I$ is bounded below, but not bounded above. The remaining cases are similar.

Since $I$ is not bounded above, it is not empty. Pick $x \in I$. $u \in I, \forall\, u > x$. Indeed, since $I$ is not bounded above, there is $y \in I; y > u$. Since $I$ is convex, $tx + (1-t)y \in I, \forall\, t \in [0,1]$. In particular, $t = \frac{u-y}{x-y}$ yields that $u \in I$.

Since $I$ is bounded below, it admits an infimum. We shall write $a = \inf_{x\in I} x$. $u \in I, \forall\, u > a$. Indeed, notice that $\forall\, \epsilon > 0, \exists\, x \in I \cap \mathcal{B}_\epsilon(a)$, for $a$ is the infimum of $I$. Pick $\epsilon = \frac{u-a}{2}$. This yields us $x \in I \cap \mathcal{B}_\epsilon(a)$ which satisfies $a < x < u$. Due to the previous argument, we see that $u \in I$, for $u > x$ and $x \in I$.

Now we see that, if $a \in I$, then $I = [a, +\infty)$. On the other hand, if $a \notin I$, then $I = (a, +\infty)$.

Assume now $I = (a, +\infty)$ (or $I = [a, +\infty)$). By definition, $x \in I, \forall\, x > a$. Pick $x, y \in I$. Let us assume, without any loss of generality, that $y > x$. Then, $\forall\, t \in [0,1]$, we have that $x \leq tx + (1-t) \leq y$. Since $a < x$, order transitivity implies $a < tx + (1-t)$ and therefore $tx + (1-t) \in I$. ■

**Lemma 5.64:**
*Let $X$ and be a linear topological space over some field $\mathbb{F}$, which is taken to be either the real line or the complex plane, $Y$ be some vector space and let $f\colon X \to Y$ be a linear function. If $f$ vanishes in an open set $A \subseteq X$, then $f(x) = 0, \forall\, x \in X$.* □

*Proof:*

Let $\mathfrak{N}$ denote a system of nuclei for $X$. As per Theorem 5.25 on page 144, the elements of $\mathfrak{N}$ are absorbent.

Since $A$ is open, $\forall\, x \in A$ there is some $N \in \mathfrak{N}$ such that $N + x \subseteq A$. Notice that $f(N + x) = \{0\}$ and, in particular, $f(x) = 0$. Let $z \in N$. Notice that

$$\begin{aligned} f(z + x) &= 0, \\ f(z) + f(x) &= 0, \\ f(z) &= 0. \end{aligned} \tag{5.120}$$

However, $N$ is absorbent. Thus, $\forall\, x \in X, \exists\, \lambda \in \mathbb{F}; \frac{1}{\lambda}x \in N$. As a consequence, we see that

$$\begin{aligned} f\left(\frac{1}{\lambda}x\right) &= 0, \\ \frac{1}{\lambda}f(x) &= 0, \\ f(x) &= 0, \end{aligned} \tag{5.121}$$

which proves the claim. ■

**Lemma 5.65:**

*Let $X$ be a topological vector space and let $f\colon X \to \mathbb{R}$ be a nonzero linear functional. If $A$ is open and convex, then $f(A)$ is an open interval.* □

*Proof:*

Firstly let us prove that $f(A)$ is an interval.

Let $x, y \in A$. We want to prove that $tf(x) + (1-t)f(y) \in A$. Linearity of $f$ immediately implies $f\big(tx + (1-t)y\big)$ and convexity of $A$ implies $tx + (1-t)y \in A$, meaning $f\big(tx + (1-t)y\big) \in f(A)$. Thus, $f(A)$ is convex. Since $f(A) \subseteq \mathbb{R}$, Lemma 5.63 on the previous page implies that $f(A)$ is an interval.

In order to prove that $f(A)$ is open, we want to pick some arbitrary $y \in f(A)$ and find $\epsilon > 0$ such that $\mathcal{B}_\epsilon(y) \subseteq f(A)$.

Let $\mathfrak{N}$ denote a system of nuclei for $X$. Since $A$ is open and $f$ is nonzero, Lemma 5.64 on the preceding page implies there is some $x_0 \in A$ with $f(x_0) \neq 0$. Given $y \in f(A)$, pick $x \in A$ with $f(x) = y$. We know there is some $O \in \mathfrak{N}$ such that $x + O \subseteq A$. Theorem 5.25 on page 144 guarantees both the existence of a balanced set $N \in \mathfrak{N}$ with $N \subseteq O$ and that $N$ is absorbent (for $N \in \mathfrak{N}$), and thus there is some $\lambda \in \mathbb{F}$ such that $\frac{1}{\lambda}x_0 \in N$. Since $N$ is balanced, we get $\frac{\pm 1}{|\lambda|}x_0 \in N$, and thus $x \pm \frac{1}{|\lambda|}x_0 \in A$. Let us write $\epsilon = \frac{f(x_0)}{|\lambda|}$. We see that $x \pm \frac{\epsilon}{f(x_0)}x_0 \in A$. As a consequence, we see that

$$f\left(x \pm \frac{\epsilon}{f(x_0)}x_0\right) = f(x) \pm \epsilon \in f(A). \tag{5.122}$$

Due to $N$ being absorbent and balanced, the same holds for every $\mu \in \mathbb{F}; |\mu| > \lambda$, *id est*, $\forall\, \delta \in (0, \epsilon)$. Thus, $\mathcal{B}_\epsilon(f(x)) \subseteq f(A)$, which proves $f(A)$ is open. ■

**Theorem 5.66** [Separating Hyperplane Theorem]**:**

*Let $(X, \{\|\cdot\|_\lambda\})_{\lambda\in\Lambda}$ be a locally convex space over some field $\mathbb{F}$, which is taken to be either the real line or the complex plane. Let $A, B \subseteq X$ be disjoint convex sets. Then it holds that*

i. *if $A$ is open, then $A$ and $B$ are separated by a hyperplane;*

ii. *if both $A$ and $B$ are open, then $A$ and $B$ are strictly separated by a hyperplane.* $\square$

*Proof:*

We begin by supposing $A$ is open. Consider the set $A - B \equiv \{y - z; y \in A, z \in B\}$. Pick $-x \in A - B$ and let $C \equiv (A - B) + x$. $C$ is an open set.

Indeed, since $A$ is open, it holds that $A - z$ is also open, for each $z \in X$ (and, in particular, for each $z \in B$). Notice that

$$A - B = \bigcup_{z\in B}(A - z). \tag{5.123}$$

Since the arbitrary union of open sets is open, $A - B$ is open. As a consequence, $(A - B) + x = C$ is open as well.

Notice that $0 \in C$. Indeed, since $-x \in A - B$ by construction, $-x + x = 0 \in C$. Furthermore, $x \notin C$, for the fact that $A$ and $B$ are disjoint implies $0 \notin A - B$.

As $C$ is open, there is some element $N$ of the system of nuclei (which we take to be made of absorbent and absolutely convex sets) such that $0 \in N \subseteq C$. As $N$ is absorbent and $\lambda N \subseteq \lambda C, \forall \lambda \in \mathbb{F}$, we see $C$ is absorbent as well.

Furthermore, $C$ is convex. Let $t \in [0, 1]$, $u, u' \in C$. We want to prove that $tu + (1 - t)u' \in C$.

Since $u \in C$, there are $y \in A, z \in B$ such that $u = y - z + x$. Similarly, there are $y' \in A, z' \in B$ with $u' = y' - z' + x$. Thus, notice that

$$\begin{aligned} tu + (1-t)u' &= t(y - z + x) + (1 - t)\cdot(y' - z' + x), \\ &= ty - tz + (1 - t)y' - (1 - t)z' + x, \\ &= ty + (1 - t)y' - \left[tz + (1 - t)z'\right] + x, \\ &= y'' - z'' + x \in C. \end{aligned} \tag{5.124}$$

In the last line, we defined $y'' = ty + (1 - t)y'$, which is an element of $A$ due to the hypothesis that $A$ is convex. The same idea goes for $z''$.

As $C$ is absorbent, we might consider its Minkowski functional, $\rho_C$. Since $C$ is convex, it holds (Lemma 5.37 on page 150) that $\rho_C(y + z) \leq \rho_C(y) + \rho_C(z), \forall y, z \in X$. Also, for any $t > 0$, it holds that $\rho_C(tz) = t\rho_C(z)$ (Lemma 5.37 on page 150 again). Thus, $\rho_C$ is a sublinear functional.

Consider now the linear space $Z = \{\lambda x; \lambda \in \mathbb{R}\}$ and define the linear functional $f : Z \to \mathbb{R}$ such that $f(\lambda x) = \lambda$. As $x \notin C$, it holds that $\rho_C(x) \geq 1$, for $\rho_C^{-1}\big([0, 1)\big) \subseteq C$ (Lemma 5.37 on page 150). Since $f(x) = 1$, we see that $f(x) \leq \rho_C(x)$. As a consequence, notice that, for $\lambda > 0$,

$$\begin{aligned} f(\lambda x) &= \lambda f(x), \\ &\leq \lambda \rho_C(x), \\ &= \rho_C(\lambda x). \end{aligned} \tag{5.125}$$

For $\lambda \leq 0$, $f(\lambda x) = \lambda \leq 0$. However, $\rho_C(y) \geq 0, \forall\, y \in X$ by construction. Thus, we see that $f(y) \leq \rho_C(y), \forall\, y \in Z$. Therefore, the Hahn-Banach Theorem guarantees the existence of a linear functional $\bar{f} \colon X \to \mathbb{R}$ extending $f$ and satisfying the condition that $f(y) \leq \rho_C(y), \forall\, y \in X$.

A generic neighborhood of $\bar{f}(0)$ can be written in terms of $\mathcal{B}_\epsilon(0) \subseteq \mathbb{R}$, for $\epsilon \in \mathbb{R}$ (for the sets of the form $\mathcal{B}_\epsilon(0)$ are a local basis at 0 for the standard topology in $\mathbb{R}$). Thus, if we find an open set $O \subseteq X$ such that $\bar{f}(O) \subseteq \mathcal{B}_\epsilon(0)$, we will have proven $\bar{f}$ is continuous at 0 and, as a consequence (Corollary 5.23 on page 144) that $\bar{f}$ is continuous.

$C$ is open and absorbent, and thus so is $\frac{\epsilon}{2}C$. Since $\bar{f}(y) \leq \rho_C(y), \forall\, y \in X$, and it holds that $\rho_C(y) \leq \frac{\epsilon}{2} < \epsilon, \forall\, y \in \frac{\epsilon}{2}C$, we see that $\frac{\epsilon}{2}C$ is an open set with $\bar{f}\big(\frac{\epsilon}{2}C\big) \subseteq \mathcal{B}_\epsilon(0)$, and thus $\bar{f}$ is continuous.

Given $y \in A, z \in B$, we have $u = y - z + x \in C$ and we know that $\bar{f}(u) \leq \rho_C(u) \leq 1$. Thus, since $\bar{f}$ is a linear functional with $\bar{f}(x) = 1$,

$$\begin{aligned} \bar{f}(y - z + x) &\leq 1, \\ \bar{f}(y) &\leq \bar{f}(z) + 1 - \bar{f}(x), \\ \bar{f}(y) &\leq \bar{f}(z), \\ \sup_{y \in A} \bar{f}(y) &\leq \inf_{z \in B} \bar{f}(z). \end{aligned} \tag{5.126}$$

We might now pick $c \in \mathbb{R}$ such that $\sup_{y \in A} \bar{f}(y) \leq c \leq \inf_{z \in B} \bar{f}(z)$ and conclude that $\bar{f}(x) \leq c, \forall\, x \in A$ while $\bar{f}(x) \geq c, \forall\, x \in B$, which proves the first item.

If both $A$ and $B$ are open, Lemma 5.65 on page 170 guarantees both $\bar{f}(A)$ and $\bar{f}(B)$ are open intervals. Since $\sup_{y \in A} \bar{f}(y) \leq \inf_{z \in B} \bar{f}(z)$, $\bar{f}(A)$ and $\bar{f}(B)$ intersect at no more than one point. Since they are open, it must follow that they are disjoint, and we conclude $A$ and $B$ are strictly separated by a hyperplane. ■

## 5.4 Fréchet Spaces

We are now moving our interest to a smaller class of locally convex spaces: those who are also a complete metric space. As one might know, complete metric spaces have some interesting results, such as fixed point theorems, which are going to be useful when developing Distribution Theory.

In order to know whether some locally convex space is also a complete metric space, we must study the conditions for a locally convex space to be metrizable, *id est*, the conditions for the existence of a metric which generates the natural topology of the locally convex space.

A simple case is that of a space whose topology is generated by a norm. Since every norm induces a metric, normed spaces are automatically metric spaces. However, these are not all the possible cases, as a metric $d\colon M \times M \to \mathbb{R}_+$ doesn't necessarily respect the relation $d(\lambda x, 0) = \lambda d(x, 0)$.

**Theorem 5.67:**

*Let $X$ be a locally convex space. The following statements are equivalent:*

i. *$X$ is metrizable;*

ii. *$X$ is first-countable;*

iii. *the topology on $X$ is generated by a countable family of seminorms.* □

*Proof:*

We shall denote the topology on $X$ by $\tau$.

Assume $X$ is metrizable. Since every metric space is first-countable (Proposition 3.94 on page 70), we see $X$ is first-countable as well.

Let $\mathfrak{N}$ be a system of absolutely convex, absorbent nuclei for $X$, the existence of which is guaranteed by Lemma 5.32 and Theorem 5.38 on page 149 and on page 151. Assume now that $X$ is first-countable. Then it admits a countable system of nuclei, which we shall call $\mathfrak{N}'$. Since every element of $\mathfrak{N}'$ is an open set, it holds that $\forall N' \in \mathfrak{N}', \forall x \in N', \exists N \in \mathfrak{N}; (x + N) \subseteq N'$. Since $\mathfrak{N}'$ is a system of nuclei, we know $0 \in N', \forall N' \in \mathfrak{N}'$. Thus, $\forall N' \in \mathfrak{N}', \exists N \in \mathfrak{N}; N \subseteq N'$. Thus, the Axiom of Choice guarantees the existence of a function $f\colon \mathfrak{N}' \to \mathfrak{N}$ with $f(N') \subseteq N', \forall N' \in \mathfrak{N}'$. The collection $\mathfrak{N}'' \equiv \{f(N'); N' \in \mathfrak{N}\}$ is a countable system of absolutely convex, absorbent nuclei.

$\mathfrak{N}''$ is certainly countable, for we defined it as the image of a function with a countable domain. The elements of $\mathfrak{N}''$ surely are absolutely convex and absorbent, for $\mathfrak{N}'' \subseteq \mathfrak{N}'$, and the elements of $\mathfrak{N}'$ are absolutely convex and absorbent by construction. Furthermore, since $\mathfrak{N}'$ is a system of nuclei, we know $0 \in N, \forall N \in \mathfrak{N}''$. Furthermore, if $O \in \tau; 0 \in O$, then there is some $N' \in \mathfrak{N}'$ with $N' \subseteq O$ (for $\mathfrak{N}'$ is a system of nuclei). However, $f(N') \subseteq N' \subseteq O$. Thus, $\forall O \in \tau; 0 \in O, \exists N \in \mathfrak{N}''; N \subseteq O$. This proves $\mathfrak{N}''$ is a system of nuclei.

In possession of this countable system of nuclei, we can take its Minkowski functionals as seminorms and proceed as in the proof of Theorem 5.38 on page 151. This guarantees that the topology on $X$ is generated by a countable family of seminorms.

Finally, suppose the topology on $X$ is generated by a countable family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Consider the function $d\colon X \times X \to \mathbb{R}_+$ given by

$$d(x, y) \equiv \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}. \tag{5.127}$$

Since $\frac{a}{1+a} < 1, \forall a > 0$, we have that

$$d(x, y) \equiv \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n} < \sum_{n=1}^{+\infty} \frac{1}{2^n} = 1 < +\infty, \tag{5.128}$$

and thus $d(x, y)$ converges. Furthermore, since every term of the sum is non-negative, $d(x, y)$ is non-negative as well. Let us now prove it is indeed a metric.

$d(x, x) = 0$ comes from the fact that $\|0\|_n = 0, \forall n \in \mathbb{N}$, which comes from the homogeneity of the seminorms. Since $\frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n} \geq 0, \forall n \in \mathbb{N}, \forall x, y \in X$, $d(x, y) = 0 \Leftrightarrow \|x - y\|_n = 0, \forall n \in \mathbb{N}$. However, since $X$ is a locally convex space, the family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ separates points, and thus $\|x - y\|_n = 0, \forall n \in \mathbb{N} \Leftrightarrow x = y$. We might then conclude that $d(x, y) = 0 \Leftrightarrow x = y$.

Notice then that

$$d(x, y) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n} = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|y - x\|_n}{1 + \|y - x\|_n} = d(y, x). \tag{5.129}$$

Finally, we must prove the triangle inequality. Let us consider the function $l \colon \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$l(x) = \frac{x}{1 + x}, \tag{5.130}$$

which is crescent and subadditive (as proven in Lemma 5.68 on page 176). We may write

$$d(x, y) = \sum_{n=1}^{+\infty} \frac{1}{2^n} l(\|x - y\|_n). \tag{5.131}$$

We know that $\forall x, y, z \in X, \forall n \in \mathbb{N}, \|x - y\|_n \leq \|x - z\|_n + \|x - z\|_n$, since $\|\cdot\|_n$ is a seminorm. Therefore, using the facts that $l$ is crescent and subadditive, we have, $\forall x, y, z \in X$,

$$\begin{aligned}
d(x, y) &= \sum_{n=1}^{+\infty} \frac{1}{2^n} l(\|x - y\|_n), \\
&\leq \sum_{n=1}^{+\infty} \frac{1}{2^n} l(\|x - z\|_n + \|y - z\|_n), \\
&\leq \sum_{n=1}^{+\infty} \frac{1}{2^n} [l(\|x - z\|_n) + l(\|y - z\|_n)], \\
&= \sum_{n=1}^{+\infty} \frac{1}{2^n} l(\|x - z\|_n) + \sum_{n=1}^{+\infty} \frac{1}{2^n} l(\|y - z\|_n), \\
&= d(x, z) + d(y, z).
\end{aligned} \tag{5.132}$$

This proves $d$ is a metric. We still must prove it generates the same topology as the seminorms.

Consider a system of nuclei $\mathfrak{N}$ for the locally convex space. We know that the collection $\mathfrak{B} = \{N + x; N \in \mathfrak{N}, x \in X\}$ is a basis for the natural topology on $X$.

Let us define $\mathfrak{B}' = \{\mathcal{B}_\epsilon(x); \epsilon > 0, x \in X\}$, where the open balls are taken with respect to the metric $d$. $\mathfrak{B}'$ is a basis for the topology generated by the metric $d$.

Pick $x \in X, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}$. We know $\mathcal{B}$ has the form $N_{\lambda_1,\ldots,\lambda_n;\epsilon} + x$ (Proposition 5.28 on page 147). Thus, if $y \in \mathcal{B}$, we know that $y - x \in N_{\lambda_1,\ldots,\lambda_n;\epsilon}$. Notice that we have, for this fixed $x$ and any $y \in \mathcal{B}$,

$$\begin{aligned} d(x,y) &= \sum_{m=1}^{+\infty} \frac{1}{2^m} l(\|x-y\|_m), \\ &= \sum_{i=1}^{n} \frac{1}{2^{\lambda_i}} l(\|x-y\|_{\lambda_i}), \\ &\leq \sum_{i=1}^{+n} l(\epsilon), \\ &= n\frac{\epsilon}{1+\epsilon}. \end{aligned} \tag{5.133}$$

Thus, the set $\mathcal{B}_\delta(x) \in \mathfrak{B}$, where $\delta = n\frac{\epsilon}{1+\epsilon}$, satisfies $\mathcal{B}_\delta(x) \subseteq \mathcal{B}$. Of course we have $x \in \mathcal{B}_\delta(x)$.

Consider now some $x \in X$ and some $\epsilon > 0$. We want to obtain a set $\mathcal{B} \in \mathfrak{B}$ such that $\mathcal{B} \subseteq \mathcal{B}_\epsilon(x)$, for this will allow us to employ Proposition 3.18 on page 30 to conclude the topologies coincide. Thus, we want to find a set $N_{\lambda_1,\ldots,\lambda_n;\delta}$ such that $N_{\lambda_1,\ldots,\lambda_n;\delta} + x \subseteq \mathcal{B}_\epsilon(x)$. Let us use the enumeration of the seminorms chosen when defining $d$ and look for $N_{1,\ldots,n;\delta}$. The reason for this is that the contribution of each seminorm to $d$ falls with $\frac{1}{2^n}$, so seminorms with very large $n$ are less meaningfull.

Since $l(x) \leq 1, \forall\, x \geq 0$, we know that, given some $m \in \mathbb{N}$,

$$\sum_{n=m}^{+\infty} \frac{1}{2^n} l(\|x-y\|_n) \leq \sum_{n=m}^{+\infty} \frac{1}{2^n}. \tag{5.134}$$

Since the sequence $s_n = \sum_{k=1}^{n} \frac{1}{2^k}$ is crescent and convergent (with $s_n \to 1$), we know that $\forall\, \delta > 0, \exists\, m \in \mathbb{N}; (1 - s_m) < \delta$. Notice that

$$\begin{aligned} 1 - s_m &= \sum_{k=1}^{+\infty} \frac{1}{2^k} - \sum_{k=1}^{m} \frac{1}{2^k}, \\ &= \sum_{k=m+1}^{+\infty} \frac{1}{2^k}. \end{aligned} \tag{5.135}$$

Thus, $\forall\, \delta > 0, \exists\, m \in \mathbb{N}; \sum_{k=m+1}^{+\infty} \frac{1}{2^k} < \delta$.

Let us now pick that previously chosen $\epsilon$. We now there is some $m \in \mathbb{N}$ such that $\sum_{k=m+1}^{+\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. Therefore, we have that, for that fixed $x \in X$ and $\forall\, y \in X$,

$$d(x,y) = \sum_{n=1}^{m} \frac{1}{2^n} l(\|x-y\|_n) + \frac{\epsilon}{2}. \tag{5.136}$$

Let now we pick $y \in (x + N_{1,\ldots,m;\delta})$, where $\delta = \frac{\epsilon}{2-\epsilon}$, so that $\epsilon = 2l(\delta)$. Then we have

$$\begin{aligned} d(x,y) &= \sum_{n=1}^{m} \frac{1}{2^n} l(\|x-y\|_n) + \frac{\epsilon}{2}, \\ &\leq \sum_{n=1}^{m} \frac{1}{2^n} l(\delta) + \frac{\epsilon}{2}, \\ &= \frac{\epsilon}{2} \sum_{n=1}^{m} \frac{1}{2^n} + \frac{\epsilon}{2}, \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}, \\ &= \epsilon. \end{aligned} \tag{5.137}$$

Therefore, $(x + N_{1,\ldots,m;\delta}) \subseteq \mathcal{B}_\epsilon(x)$. It is clear that $x \in (x + N_{1,\ldots,m;\delta})$

We have proven that $\forall\, x \in X, \forall\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B}, \exists\, \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}' \subseteq \mathcal{B}$. Reversely, we have also proven that $\forall\, x \in X, \forall\, \mathcal{B}' \in \mathfrak{B}'; x \in \mathcal{B}', \exists\, \mathcal{B} \in \mathfrak{B}; x \in \mathcal{B} \subseteq \mathcal{B}'$. Therefore, Proposition 3.18 on page 30 guarantees the topologies are indeed the same, and concludes the proof. ■

**Lemma 5.68:**

*The function* $l\colon \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$l(x) = \frac{x}{1+x} \tag{5.138}$$

*is crescent and subadditive,* id est,

$$\forall\, x, y \in \mathbb{R}_+, x \leq y \Rightarrow l(x) \leq l(y), \tag{5.139}$$
$$\forall\, x, y \in \mathbb{R}_+, l(x+y) \leq l(x) + l(y). \tag{5.140}$$

□

*Proof:*

Suppose $x \leq y$, for $x, y \in \mathbb{R}_+$. We have

$$\begin{aligned} x &\leq y, \\ x + xy &\leq y + xy, \\ x(1+y) &\leq y(1+x), \\ \frac{x}{1+x} &\leq \frac{y}{1+y}, \\ l(x) &\leq l(y), \end{aligned} \tag{5.141}$$

notice we were able to divide both sides by $(1+x)(1+y)$ due to the hypothesis that $x, y \geq 0$. This proves $l$ is crescent.

Notice now that

$$l(x+y) = \frac{x+y}{1+x+y},$$

$$= \frac{x}{1+x+y} + \frac{y}{1+x+y}. \tag{5.142}$$

Notice though that, since $y \geq 0$, it holds that

$$\frac{x}{1+x+y} \leq \frac{x}{1+x} = l(x), \tag{5.143}$$

and a similar argument yields

$$\frac{y}{1+x+y} \leq \frac{y}{1+y} = l(y). \tag{5.144}$$

Therefore,

$$l(x+y) \leq l(x) + l(y), \tag{5.145}$$

*id est*, $l$ is subadditive. ■

**Definition 5.69** [Fréchet Metric]**:**
Let $X$ be a locally convex space whose topology is generated by a countable family of seminorms. The metric

$$d(x,y) \equiv \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n} \tag{5.146}$$

is said to be the *Fréchet metric defined by the family of seminorms*. ♠

*Remark:*
Let $X$ be a metrizable locally convex space. Unless stated otherwise, we shall always consider it equipped with the Fréchet metric and refer to it as its metric. ♣

An interesting remark is that even in the absence of a metric, we might define completeness for a metric space, for a locally convex space always admits a notion of distance provided by the family of seminorms. We are thus able to define a Cauchy net (for general topological spaces require us to work with nets instead of sequences) and introduce a notion of completeness.

**Definition 5.70** [Cauchy Nets and Completeness of Locally Convex Space]**:**
Let $\left(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}\right)$ be a locally convex space. Let $\{x_\alpha\}_{\alpha\in I}$ be a net of elements of $X$. $\{x_\alpha\}_{\alpha\in I}$ is said to be a *Cauchy net* if, and only if,

$$\forall \epsilon > 0, \forall \lambda \in \Lambda, \exists \beta \in I; \|x_\gamma - x_\delta\|_\lambda < \epsilon, \forall \gamma, \delta \succ \beta. \tag{5.147}$$

If every Cauchy net converges, the locally convex space $\left(X, \{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}\right)$ is said to be *complete*. ♠

We shall also consider a wider notion of Cauchy nets, which does not depend on the linear structure.

**Definition 5.71** [Cauchy Nets in Pseudometric Spaces]**:**

Let $(M, d)$ be a pseudometric space. Let $\{x_\alpha\}_{\alpha \in I}$ be a net of elements of $M$. $\{x_\alpha\}_{\alpha \in I}$ is said to be a *Cauchy net with respect to the pseudometric* $d$, and only if,

$$\forall \epsilon > 0, \exists \beta \in I; d(x_\gamma, x_\delta) < \epsilon, \forall \gamma, \delta \succ \beta. \tag{5.148}$$

♠

**Proposition 5.72:**

*Let $(M, d)$ be a complete metric space. Every Cauchy net defined on $M$ converges on $M$.* □

*Proof:*

Let $(x_\alpha)_{\alpha \in I}$ be a Cauchy net defined on $M$. Then, $\forall n \in \mathbb{N}^*$, we know there is some $\alpha_0^{(n)} \in I$ such that $d(x_\beta, x_\gamma) < \frac{1}{n}, \forall \beta, \gamma \succ \alpha_0^{(n)}$.

Since $(I, \prec)$ is a directed system, we know there is some $\alpha_1 \in I; \alpha_1 \succ \alpha_0^1$. Furthermore, $\forall n > 1$, we know there is some $\alpha_n \in I; \alpha_n \succ \alpha_0^{(n)}, \alpha_n \succ \alpha_{n-1}$. We might now consider the sequence determined by $y_n \equiv x_{\alpha_n}$. $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

Pick $N \in \mathbb{N}$ and let $n, m \in \mathbb{N}, n, m > N$. Notice that, due to the inductive way in which we constructed the sequence of indices, it holds that $\alpha_n, \alpha_m \succ \alpha_N \succ \alpha_0^{(N)}$. However, by definition of $\alpha_0^{(N)}$, we know that

$$d(x_\gamma, x_\delta) < \frac{1}{N}, \forall \gamma, \delta \succ \alpha_0^{(N)}. \tag{5.149}$$

Therefore, we see that

$$d(x_{\alpha_n}, x_{\alpha_m}) < \frac{1}{N}, \forall n, m > N. \tag{5.150}$$

We have then proven that

$$\forall N \in \mathbb{N}, d(y_n, y_m) < \frac{1}{N}, \forall n, m > N. \tag{5.151}$$

Pick now $\epsilon > 0$. Due to the Archimedean property of the real line, we know there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Due to Eq. (5.151), we see then that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}; d(y_n, y_m) < \epsilon, \forall n, m > N, \tag{5.152}$$

*id est*, $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since it is defined in a complete metric space, it certainly has a limit. Let us call it $y$.

We must now prove that $x_\alpha \to y$. Notice that if $y$ is a limit point of $(x_\alpha)_{\alpha \in I}$, it is the only one, due to Proposition 3.124 and Theorem 5.9 on page 83 and on page 139.

We want to prove that $x_\alpha$ is eventually in every neighborhood of $y$. If we prove $x_\alpha$ is eventually in every element of a neighborhood basis of $y$, it follows that $x_\alpha$ is eventually in every open set containing $y$ and, as a consequence, in every neighborhood of $y$. Since we are in a metric space, a neighborhood basis for the metric topology at $y$ is provided by the open balls centered at $y$. Therefore, we want to prove that

$$\forall \epsilon > 0, \exists \beta \in I; x_\alpha \in \mathcal{B}_\epsilon(y), \forall \alpha \succ \beta. \tag{5.153}$$

In other words, we want to prove that

$$\forall \epsilon > 0, \exists \beta \in I; d(x_\alpha, y) < \epsilon, \forall \alpha \succ \beta. \tag{5.154}$$

Pick $\epsilon > 0$. We know there is some $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Since $y_n \to y$, we know there is some $N' \in \mathbb{N}; d(y_n, y) < \frac{1}{2N}, \forall n > N'$. Let us pick $n_0 = \max\{2N, N'\}$. $\forall \alpha, \beta \succ \alpha_0^{(n_0)}$, we know that $d(x_\alpha, x_\beta) < \frac{1}{n_0} \leq \frac{1}{2N}$. In particular, notice that $\alpha_{n_0} \succ \alpha_0^{(n_0)}$. For all $\alpha \succ \alpha_0^{(n_0)}$, we have

$$\begin{aligned}
d(x_\alpha, y) &\leq d(x_\alpha, x_{\alpha_{n_0}}) + d(y, x_{\alpha_{n_0}}), \\
&= d(x_\alpha, x_{\alpha_{n_0}}) + d(y, y_{n_0}), \\
&< \frac{1}{2N} + \frac{1}{2N}, \\
&= \frac{1}{N}, \\
&< \epsilon.
\end{aligned} \tag{5.155}$$

In this manner, we have shown that $\forall \epsilon > 0, \exists \beta \in I$ (which was given by $\beta = \alpha_0^{(n_0)}$) such that $d(x_\alpha, y) < \epsilon, \forall \alpha \succ \beta$. This proves that $x_\alpha \to y$, as desired, and concludes the proof. ■

**Proposition 5.73:**

*Let $X$ be a metrizable locally convex space and let $(x_\alpha)_{\alpha \in I}$ be a net defined on $X$. The net $(x_\alpha)_{\alpha \in I}$ is a Cauchy net in the Fréchet metric if, and only if, it is a Cauchy net in all of the seminorms $\|\cdot\|_n$.* □

*Proof:*

$\Leftarrow$: Suppose the net $(x_\alpha)_{\alpha \in I}$ is Cauchy in every seminorm. As shown in the proof for Theorem 5.67 on page 172, $\forall \delta > 0, \exists m \in \mathbb{N}; \sum_{k=m+1}^{+\infty} \frac{1}{2^k} < \delta$.

Let $\epsilon > 0$. Let us define $\delta = \frac{\epsilon}{2-\epsilon}$, in order to obtain $l(\delta) = \frac{\epsilon}{2}$, where $l$ is the function defined in Lemma 5.68 on page 176. We know there is $m \in \mathbb{N}$ such that $\sum_{k=m+1}^{+\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. Thus, $\forall \zeta, \xi \in I$,

$$\begin{aligned}
d(x_\zeta, x_\xi) &= \sum_{n=1}^{\infty} \frac{1}{2^n} l(\|x_\zeta - x_\xi\|_n), \\
&< \sum_{n=1}^{m} \frac{1}{2^n} l(\|x_\zeta - x_\xi\|_n) + \frac{\epsilon}{2}.
\end{aligned} \tag{5.156}$$

Since $(x_\alpha)_{\alpha\in I}$ is Cauchy in every seminorm, we know that

$$\forall\,\delta > 0, \forall\, n \in \mathbb{N}, \exists\, \beta_n \in I; \left\|x_\zeta - x_\xi\right\|_n < \delta, \forall\, \zeta, \xi \succ \beta_n. \tag{5.157}$$

Since $(I, \prec)$ is a directed system, we know that there is some $\gamma \in I$ with $\gamma \succ \beta_1, \gamma \succ \beta_2$. Induction allows us to prove that there is some $\gamma \succ \beta_n, \forall\, n \in \{i\}_{i=1}^m$. Thus, we know that

$$\forall\,\delta > 0, \exists\,\gamma \in I; \left\|x_\zeta - x_\xi\right\|_n < \delta, \forall\,\zeta, \xi \succ \gamma, \forall\, n \in \{i\}_{i=1}^m. \tag{5.158}$$

This means that $\forall\,\epsilon > 0, \exists\,\gamma \in I$ such that, $\forall\,\zeta, \xi \succ \gamma$,

$$\begin{aligned} d(x_\zeta, x_\xi) &< \sum_{n=1}^m \frac{1}{2^n} l(\left\|x_\zeta - x_\xi\right\|_n) + \frac{\epsilon}{2}, \\ &< \sum_{n=1}^m \frac{1}{2^n} l(\delta) + \frac{\epsilon}{2}, \\ &= \frac{\epsilon}{2} \sum_{n=1}^m \frac{1}{2^n} + \frac{\epsilon}{2}, \\ &< \epsilon. \end{aligned} \tag{5.159}$$

Thus, we have concluded that

$$\forall\,\epsilon > 0, \exists\,\gamma \in I; d(x_\zeta, x_\xi) < \epsilon, \forall\,\zeta, \xi \succ \gamma, \tag{5.160}$$

*id est*, $(x_\alpha)_{\alpha\in I}$ is Cauchy in the Fréchet metric.

$\Rightarrow$: We shall prove the contrapositive implication. Assume

$$\exists\, m \in \mathbb{N}, \exists\,\epsilon > 0; \forall\,\beta \in I, \exists\,\zeta, \xi \succ \beta; \left\|x_\zeta - x_\xi\right\|_m \geq \epsilon. \tag{5.161}$$

We want to show that

$$\exists\,\delta > 0; \forall\,\beta \in I, \exists\,\zeta, \xi \succ \beta; d(x_\zeta, x_\xi) \geq \delta. \tag{5.162}$$

Our assumption says that there are some $m \in \mathbb{N}$ and some $\epsilon > 0$ which allows us to, given some $\beta \in I$, always find elements $x_\zeta, x_\xi$ in the net with the properties that $\zeta, \xi \succ \beta$ and $\left\|x_\zeta - x_\xi\right\|_m \geq \epsilon$, *id est*, we can't make the points get arbitrarily close just by picking a "good" index $\beta \in I$.

Given some $\beta \in I$, let $\zeta, \xi \succ \beta$ be such that $\left\|x_\zeta - x_\xi\right\|_m \geq \epsilon$. Since $l$ is crescent, we know that $l(\left\|x_\zeta - x_\xi\right\|_m) \geq l(\epsilon)$. Notice then that we have

$$d(x_\zeta, x_\xi) = \frac{1}{2^m} l(\left\|x_\zeta - x_\xi\right\|_m) + \sum_{n\neq m} \frac{1}{2^n} l(\left\|x_\zeta - x_\xi\right\|_n),$$

$$\begin{aligned} &\leq \frac{1}{2^m} l(\epsilon) + \sum_{n \neq m} \frac{1}{2^n} l(\|x_\zeta - x_\xi\|_n), \\ &\leq \frac{1}{2^m} l(\epsilon), \\ &\equiv \delta. \end{aligned} \tag{5.163}$$

Therefore, $\exists\, \delta > 0$, given by $\delta = \frac{1}{2^m} l(\epsilon)$ with the property that $\forall\, \beta \in I, \exists\, \zeta, \xi \succ \beta$ such that $d(x_\zeta, x_\xi) \geq \delta$, *id est*,

$$\exists\, \delta > 0; \forall\, \beta \in I, \exists\, \zeta, \xi \succ \beta; d(x_\zeta, x_\xi) \geq \delta, \tag{5.164}$$

concluding the proof.

■

**Proposition 5.74:**
*A metrizable locally convex space is complete as a locally convex space if, and only if, it is complete as a metric space.* □

*Proof:*
Let $X$ be a metrizable locally convex space. If it is complete as a metric space, Proposition 5.72 on page 178 guarantees every Cauchy net with respect to the Fréchet metric is convergent. If a net is Cauchy in all of the seminorms (and thus is a Cauchy net in the locally convex space), Proposition 5.73 on page 179 guarantees it is also Cauchy in the Fréchet metric. Since all nets Cauchy in the Fréchet metric converge, we see that all nets that are Cauchy in the locally convex space converge, and thus $X$ is complete as a locally convex space.

Suppose now that $X$ is complete as a locally convex space. Then every Cauchy net in the locally convex space converges. Since these are the nets that are Cauchy in every seminorm, and these are always Cauchy in the Fréchet metric, we see that every net that is Cauchy in the Fréchet metric converges. Since every sequence is a net, we may conclude that every Cauchy sequence, in the sense of the Fréchet metric, defined on $X$ converges, and thus $X$ is complete as a metric space. ■

**Definition 5.75** [Fréchet Space]**:**
A complete metrizable locally convex space is said to be a *Fréchet space*. ♠

The requirement of completeness for a Fréchet space allows us to apply results from the theory of complete metric spaces.

**Theorem 5.76** [Baire Category Theorem]**:**
*Let $(M, d)$ be a complete metric space. A countable intersection of dense open sets is dense.* □

*Proof:*

$\forall\, n \in \mathbb{N}$, let $A_n \subseteq M$ be a dense open set. We know that

$$\forall\, n \in \mathbb{N}, \forall\, \epsilon > 0, \forall\, x \in M, \exists\, x_n \in A_n; d(x, x_n) < \epsilon. \tag{5.165}$$

Let us fix $x \in X$.

We know $\exists\, x_0 \in A_0; d(x, x_0) < \frac{\epsilon}{3}$, for $A_0$ is dense. Since $A_0$ is open, $x_0$ is an interior point and we can pick $2\delta_0 < \frac{\epsilon}{3}$ such that $\mathcal{B}_{2\delta_0}(x_0) \subseteq A_0$.

Keeping this process in mind, we can build a sequence inductively. Since $A_n$ is dense, $\exists\, x_n \in A_n; d(x_{n-1}, x_n) < \delta_{n-1}$. Notice that $x_n \in \mathcal{B}_{\delta_{n-1}}(x_{n-1})$ by construction. Since $A_n$ and $\mathcal{B}_{\delta_{n-1}}(x_{n-1})$ are open, their intersection is as well and we may pick $2\delta_n < \frac{\epsilon}{3^n}; \mathcal{B}_{2\delta_n}(x_n) \subseteq \left[A_n \cap \mathcal{B}_{\delta_{n-1}}(x_{n-1})\right]$.

Notice that this construction implies

$$\begin{aligned} d(x_{n+l}, x_{n+k}) &\leq \sum_{m=l+1}^{k} d(x_{n+m-1}, x_{n+m}), \\ &< \sum_{m=l+1}^{k} \frac{\epsilon}{2 \cdot 3^{n+m-1}}, \\ &= \frac{\epsilon}{2 \cdot 3^n} \sum_{m=l+1}^{k} \frac{1}{3^{m-1}}, \\ &< \frac{\epsilon}{2 \cdot 3^n}. \end{aligned} \tag{5.166}$$

Pick $\delta > 0$. Since $3^n$ is crescent and unbounded, there is always $n_0 \in \mathbb{N}$ such that $\frac{\epsilon}{2 \cdot 3^{n_0}} < \delta$. Thus, we get that, $\forall\, l, k \in \mathbb{N}$,

$$d(x_{n_0+l}, x_{n_0+k}) < \delta, \tag{5.167}$$

which means $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(M, d)$ is complete, every Cauchy sequence has a limit. Let us denote the limit of $(x_n)_{n\in\mathbb{N}}$ by $x_\infty$.

Notice that, $\forall\, n \in \mathbb{N}$,

$$\mathcal{B}_{2\delta_{n+1}}(x_{n+1}) \subseteq [A_{n+1} \cap \mathcal{B}_{\delta_n}(x_n)] \subseteq \overline{\mathcal{B}_{\delta_n}(x_n)} \subseteq \mathcal{B}_{2\delta_n}(x_n) \subseteq \left[A_n \cap \mathcal{B}_{\delta_{n-1}}(x_{n-1})\right], \tag{5.168}$$

which means, inductively, that if we fix $n \in \mathbb{N}$, then $x_m \in \overline{\mathcal{B}_{\delta_n}(x_n)}, \forall\, m \geq n$. Propositions 3.94 and 3.101 on page 70 and on page 72 imply that

$$x_\infty \in \overline{\mathcal{B}_{\delta_n}(x_n)} \subseteq \mathcal{B}_{2\delta_n}(x_n) \subseteq \left[A_n \cap \mathcal{B}_{\delta_{n-1}}(x_{n-1})\right]. \tag{5.169}$$

Since the argument holds for every $n \in \mathbb{N}$, we conclude that $x_\infty \in \bigcap_{n=0}^{+\infty} A_n$ and $x_\infty \in \mathcal{B}_{\delta_n}(x_n), \forall\, n \in \mathbb{N}$. In particular, $x_\infty \in \mathcal{B}_{\delta_0}(x_0)$. Since $\delta_0 < \frac{\epsilon}{3}$ and $d(x, x_0) < \frac{\epsilon}{3}$, the triangle inequality yields

$$d(x, x_\infty) \leq d(x, x_0) + d(x_\infty, x_0),$$

$$
\begin{aligned}
&< \frac{\epsilon}{3} + \frac{\epsilon}{3}, \\
&< \epsilon.
\end{aligned} \tag{5.170}
$$

Thus, we have shown that

$$
\forall\, \epsilon > 0, \forall\, x \in M, \exists\, x_\infty \in \bigcap_{n=0}^{+\infty} A_n; d(x, x_\infty) < \epsilon, \tag{5.171}
$$

proving that $\bigcap_{n=0}^{+\infty} A_n$ is indeed dense. ■

The name "Baire Category Theorem" is clearer when stated in terms of categories of sets.

**Definition 5.77** [Nowhere Dense Sets, First and Second Category Sets]**:**

Let $(M, d)$ be a metric space and let $A \subseteq M$. $A$ is said to be *nowhere dense* if, and only if, it holds that $\operatorname{int}(\overline{A}) = \varnothing$.

A countable union of nowhere dense sets is said to be a *meager* or of *first category*. Sets which are not of first category are said to be of *second category*. ♠

**Theorem 5.78** [Baire Category Theorem - Strong Category Form]**:**

*Let $(M, d)$ be a complete metric space. If $A \subseteq M$ is of first category, then $\mathring{A} = \varnothing$.* □

*Proof:*

$\forall\, n \in \mathbb{N}$, let $A_n$ be a nowhere dense set. We want to prove that $A = \bigcup_{n=0}^{\infty} A_n$ has null interior.

By definition, $\operatorname{int}(\overline{A_n}) = \varnothing, \forall\, n \in \mathbb{N}$. Since $\overline{X}^{\mathsf{c}} = \operatorname{int}(X^{\mathsf{c}})$ for any set $X$, we see that $\overline{(\overline{A_n})^{\mathsf{c}}} = M$, and thus the sets $B_n \equiv (\overline{A_n})^{\mathsf{c}}$ are dense. Since they are the complements of closed sets, they are open. As a consequence, Theorem 5.76 on page 181 implies $B = \bigcap_{n=0}^{+\infty} B_n$ is dense.

We now have that

$$
\begin{aligned}
\overline{B} &= M, \\
(\operatorname{int}(B^{\mathsf{c}}))^{\mathsf{c}} &= M, \\
\operatorname{int}(B^{\mathsf{c}}) &= \varnothing, \\
\operatorname{int}\left(\bigcup_{n=0}^{+\infty} {B_n}^{\mathsf{c}}\right) &= \varnothing, \\
\operatorname{int}\left(\bigcup_{n=0}^{+\infty} \overline{A_n}\right) &= \varnothing.
\end{aligned} \tag{5.172}
$$

However,

$$
\begin{aligned}
A_n &\subseteq \overline{A_n}, \forall n \in \mathbb{N}, \\
\bigcup_{n=0}^{+\infty} A_n &\subseteq \bigcup_{n=0}^{+\infty} \overline{A_n}, \\
\operatorname{int}\left(\bigcup_{n=0}^{+\infty} A_n\right) &\subseteq \operatorname{int}\left(\bigcup_{n=0}^{+\infty} \overline{A_n}\right), \\
\mathring{A} &\subseteq \varnothing, \\
\mathring{A} &= \varnothing,
\end{aligned}
\tag{5.173}
$$

concluding the proof. ■

**Theorem 5.79** [Baire Category Theorem - Weak Category Form]:
*Every complete metric space is of second category.* □

*Proof:*
Let $(M, d)$ be a complete metric space. Notice $M \subseteq M$. Suppose $M$ is of first category. Then Theorem 5.78 on the preceding page implies $\mathring{M} = \varnothing$. Since $M$ is open, $M = \mathring{M} = \varnothing$. In the definition of a metric space (Definition 3.1 on page 19) we demanded that every metric space must be non-empty. Thus, we arrived at a contradiction and conclude that $M$ can't be of first category. As a consequence, $M$ must be of second category. ■

The fact that Fréchet spaces are complete metric spaces allows us to use the Baire Category Theorem to conclude interesting result.

**Theorem 5.80:**
*Let $X$ and $Y$ be Fréchet spaces and let $f \colon X \to Y$ be a continuous linear surjection. Then $f$ is open.* □

*Proof:*
We want to prove that given any open set $O \subseteq X$, it holds that $f(O) \subseteq Y$ is open as well. In order to do so, it is enough to show that if $N$ is a neighborhood of $x \in X$, then $f(N)$ is a neighborhood of $f(x)$. After all, open sets can always be written as arbitrary unions of neighborhoods.

Since $f$ is linear and we are dealing with Fréchet spaces, it is enough to prove the result for a neighborhood $O$ of the origin of $X$. Indeed, if $U$ is a neighborhood for $x \in X$, we can write it as $U = x + O$, and have $f(U) = f(x) + f(O)$, which will be a neighborhood of $f(x)$ if and only if $f(O)$ is a neighborhood of the origin of $Y$.

Notice then that, given a neighborhood $O$ of the origin of $X$, we just need to prove that $\operatorname{int}\{f(O)\} \neq \varnothing$. It suffices to do this for elements of a system of nuclei $\mathfrak{N}$. From Proposition 5.28 on page 147, we see then that it is enough to prove that $f(N_{\lambda_1,\dots,\lambda_n;\epsilon})$ is a neighborhood of the origin, where

$$
N_{\lambda_1,\dots,\lambda_n;\epsilon} \equiv \left\{x \in X; \|x\|_{\lambda_i} < \epsilon, 1 \leq i \leq n\right\}, \lambda_i \in \Lambda, \epsilon > 0, \tag{5.174}
$$

and $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ is the family of seminorms of the Fréchet space. Notice that Theorem 5.67 on page 172 ensures $\Lambda$ is countable.

Notice, however, that

$$N_{\lambda_1,\dots,\lambda_n;\epsilon} = \epsilon \cdot N_{\lambda_1,\dots,\lambda_n;1}, \tag{5.175}$$

which means it is sufficient for us to prove that $f(N_{\lambda_1,\dots,\lambda_n;1})$ is a neighborhood of the origin for every possible finite collection $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$.

Given $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$ and $x \in X$, we know, by the Archimedean property of the real numbers, that there is $m \in \mathbb{N}$ such that

$$\|x\|_{\lambda_i} \leq \max_{1\leq i\leq n} \|x\|_{\lambda_i} < m. \tag{5.176}$$

As a consequence, we see that

$$X = \bigcup_{m=1}^{+\infty} N_{\lambda_1,\dots,\lambda_n;m}, \tag{5.177}$$

for every finite collection $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$. Since $f$ is surjective, we know that $f(X) = Y$, and hence

$$\begin{aligned} Y &= f\left(\bigcup_{m=1}^{+\infty} N_{\lambda_1,\dots,\lambda_n;m}\right), \\ &= \bigcup_{m=1}^{+\infty} f(N_{\lambda_1,\dots,\lambda_n;m}), \end{aligned} \tag{5.178}$$

for any finite collection $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$.

We know, from the Baire Category Theorem in Weak Category Form, that $Y$ is of second category. Since Eq. (5.178) writes $Y$ as a countable union of sets, we may conclude that at least one of such sets is not a nowhere dense set. Let us suppose it is $f\left(N_{\lambda_1,\dots,\lambda_n;p}\right)$. Hence, we know that

$$\operatorname{int}\left\{\overline{f\left(N_{\lambda_1,\dots,\lambda_n;p}\right)}\right\} \neq \varnothing. \tag{5.179}$$

If we denote the family of seminorms on $Y$ as $\{\|\cdot\|_\kappa\}_{\kappa\in K}$ and its system of nuclei as $\mathfrak{M}$, we know the elements of $\mathfrak{M}$ will be the sets

$$M_{\kappa_1,\dots,\kappa_m;\epsilon} \equiv \left\{y \in Y; \|y\|_{\kappa_i} < \epsilon, 1 \leq i \leq m\right\}, \kappa_i \in K, \epsilon > 0. \tag{5.180}$$

We know now that there is $y_0 \in Y$, $\epsilon > 0$ and $\{\kappa_i\}_{i=1}^m \subseteq K$ such that

$$y_0 + M_{\kappa_1,\dots,\kappa_m;\epsilon} \subseteq \overline{f\left(N_{\lambda_1,\dots,\lambda_n;p}\right)}. \tag{5.181}$$

Pick $y \in M_{\kappa_1,\dots,\kappa_m;\epsilon}$. Since $\overline{f\left(N_{\lambda_1,\dots,\lambda_n;p}\right)}$ is closed, we can find a sequence $(x_l)_{l\in\mathbb{N}}$, $x_l \in N_{\lambda_1,\dots,\lambda_n;p}$, such that $f(x_l) \to y_0 + y$. Since $0 \in M_{\kappa_1,\dots,\kappa_m;\epsilon}$, there is also a sequence

$(z_l)_{l\in\mathbb{N}}, z_l \in N_{\lambda_1,\ldots,\lambda_n;p}$, such that $f(z_l) \to y_0$. As a consequence, we see that linearity of $f$ implies $f(x_l - z_l) \to y$. Notice that if $x_l, z_l \in N_{\lambda_1,\ldots,\lambda_n;p}$, then $x_l - z_l \in N_{\lambda_1,\ldots,\lambda_n;2p}$. Hence, we may conclude that

$$M_{\kappa_1,\ldots,\kappa_m;\epsilon} \subseteq \overline{f\left(N_{\lambda_1,\ldots,\lambda_n;2p}\right)}. \tag{5.182}$$

One should notice that, since we are able to rescale these sets according to

$$f\left(N_{\lambda_1,\ldots,\lambda_n;\alpha\beta}\right) = \alpha f\left(N_{\lambda_1,\ldots,\lambda_n;\beta}\right), \forall\, \alpha, \beta > 0, \tag{5.183}$$

we can conclude that there is some $\delta > 0$ with

$$M_{\kappa_1,\ldots,\kappa_m;\delta} \subseteq \overline{f\left(N_{\lambda_1,\ldots,\lambda_n;\frac{1}{2}}\right)}, \tag{5.184}$$

and, rescaling once again,

$$M_{\kappa_1,\ldots,\kappa_m;\frac{\delta}{2^r}} \subseteq \overline{f\left(N_{\lambda_1,\ldots,\lambda_n;\frac{1}{2^{r+1}}}\right)}, \tag{5.185}$$

for any $r \in \mathbb{N}$.

Let now $y \in M_{\kappa_1,\ldots,\kappa_m;\delta}$. Eq. (5.184) ensures there is $x_0 \in N_{\lambda_1,\ldots,\lambda_n;\frac{1}{2}}$ such that

$$\|y - f(x_0)\|_{\kappa_i} < \frac{\delta}{2}, \forall\, i \in \{i\}_{i=1}^m. \tag{5.186}$$

Similarly, we can inductively choose $x_r \in N_{\lambda_1,\ldots,\lambda_n;\frac{1}{2^{r+1}}}$ such that

$$\left\| y - f\left(\sum_{i=0}^{r} x_i\right)\right\|_{\kappa_i} < \frac{\delta}{2^{r+1}}, \forall\, i \in \{i\}_{i=1}^m. \tag{5.187}$$

This is ensured by Eq. (5.185) and the fact that

$$y - f\left(\sum_{i=0}^{r-1} x_i\right) \in N_{\lambda_1,\ldots,\lambda_n;\frac{1}{2^r}}, \tag{5.188}$$

which holds by construction.

One should now notice that this process allowed us to define a Cauchy sequence

$$\left(\sum_{i=0}^{r} x_i\right)_{r\in\mathbb{N}} \tag{5.189}$$

in $N_{\lambda_1,\ldots,\lambda_n;1}$. As any Cauchy sequence in a complete metric space, we see that there is $x \in N_{\lambda_1,\ldots,\lambda_n;1}$ with $x = \sum_{i=0}^{+\infty} x_i$. Eq. (5.188) then ensures $f(x) = y$, showing that

$$M_{\kappa_1,\ldots,\kappa_m;\delta} \subseteq f(N_{\lambda_1,\ldots,\lambda_n;1}), \tag{5.190}$$

and hence concluding the proof. ∎

## 5.5 Tempered Distributions

**Theorem 5.81:**

*The space $\mathscr{S}$ of functions of rapid decrease is a Fréchet space when equipped with the weak topology generated by the seminorms $\|\cdot\|_{\alpha,\beta}$.* $\square$

*Proof:*

The fact that $\mathscr{S}$ is a locally convex space follows from Theorem 5.5 on page 136. Since $\mathbb{N}^n \times \mathbb{N}^n$ is a countable set, Theorem 5.67 on page 172 ensures $\mathscr{S}$ is metrizable. Hence, we only nee to prove that $\mathscr{S}$ is complete.

Propositions 5.73 and 5.74 on page 179 and on page 181 ensure it is sufficient to prove that given a sequence $(f_m)_{m\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$ such that, $\forall\, \alpha, \beta \in \mathbb{N}^n$,

$$\forall\, \epsilon > 0, \exists\, p \in \mathbb{N}; \|f_m - f_l\|_{\alpha,\beta} < \epsilon, \forall\, m, l > p, \tag{5.191}$$

then there is some $f \in \mathscr{S}$ with $f_m \to f$. From Proposition 5.29 on page 148, we know $f_m \to f$ if, and only if,

$$\|f_m - f\|_{\alpha,\beta} \to 0, \forall\, \alpha, \beta \in \mathbb{N}^n, \tag{5.192}$$

*id est*, if and only if

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta f_m(x) - x^\alpha D^\beta f(x) \right| \to 0, \forall\, \alpha, \beta \in \mathbb{N}^n. \tag{5.193}$$

Since $f_m \in \mathscr{S}, \forall\, m \in \mathbb{N}$ and every element of $\mathscr{S}$ is smooth, we know the functions $x^\alpha D^\beta f_m(x)$ are continuous. Furthermore, by the very definition of $\mathscr{S}$, they are bounded. Hence, we see that $x^\alpha D^\beta f_m(x) \in BC(\mathbb{R}^n), \forall\, \alpha, \beta \in \mathbb{N}^n, m \in \mathbb{N}$. Since we know $BC(\mathbb{R}^n)$ is complete under the uniform metric due to Proposition 3.91 on page 69, we know that, $\forall\, \alpha, \beta \in \mathbb{N}^n, \exists\, g_{\alpha,\beta} \in BC(\mathbb{R}^n)$ such that $x^\alpha D^\beta f_m \to g_{\alpha,\beta}$ in the uniform metric.

Suppose now, hypothetically, that it happened to hold that $g_{\alpha,\beta}(x) = x^\alpha D^\beta g_{0,0}(x)$, $\forall\, \alpha, \beta \in \mathbb{N}^n$. If that were true, we would know, by picking $\alpha = (0, \ldots, 0)$, that $g_{0,0}$ is smooth. In addition, we would know $g_{0,0}$ is of rapidly decrease, since every $g_{\alpha,\beta}$ is bounded by construction. Finally, we would also be able to rewrite the condition $x^\alpha D^\beta f_m \to g_{\alpha,\beta}$ as

$$\begin{aligned} x^\alpha D^\beta f_m &\to g_{\alpha,\beta}, \\ x^\alpha D^\beta f_m &\to x^\alpha D^\beta g_{0,0}, \\ \sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta f_m(x) - D^\beta g_{0,0}(x) \right| &\to 0, \\ \|f_m - g_{0,0}\|_{\alpha,\beta} &\to 0, \end{aligned} \tag{5.194}$$

and hence we would conclude $g_{0,0} \in \mathscr{S}$ is the limit, in the natural topology of $\mathscr{S}$, of the Cauchy sequence $(f_m)_{m\in\mathbb{N}}$.

We know that

$$\lim_{m\to\infty} f_m(x) = g_{0,0}(x) \tag{5.195}$$

uniformly and also that $\left(D^\beta f_m\right)_{m\in\mathbb{N}}$ converges uniformly. Hence, we know that

$$\begin{aligned}\lim_{m\to\infty} D^\beta f_m(x) &= D^\beta g_{0,0}(x),\\ \lim_{m\to\infty} x^\alpha D^\beta f_m(x) &= x^\alpha D^\beta g_{0,0}(x),\\ g_{\alpha,\beta}(x) &= x^\alpha D^\beta g_{0,0}(x),\end{aligned} \tag{5.196}$$

which is the result we desired, and hence concludes the proof. ■

**Theorem 5.82:**
*A linear functional $\varphi\colon \mathscr{S} \to \mathbb{R}$ is a tempered distribution if, and only if, it is continuous in the natural topology of $\mathscr{S}$.* □

*Proof:*
Let us begin by assuming $\varphi$ is a tempered distribution. Given a function $f \in \mathscr{S}$, we know

$$\lim_{n\to+\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle \tag{5.197}$$

for every sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$ with the property that

$$\lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0. \tag{5.198}$$

Proposition 5.29 on page 148 allows us to conclude that such sequences are precisely the sequences with $f_n \to f$ in the natural topology of $\mathscr{S}$. Therefore, we know $\lim_{n\to+\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle$ for every sequence $(f_n)_{n\in\mathbb{N}}$ converging to $f$.

We know $\mathscr{S}$ is metrizable, and thus there is a metric $d\colon \mathscr{S} \times \mathscr{S} \to \mathbb{R}_+$ that generates the natural topology on $\mathscr{S}$. The statement on Theorem 3.123 on page 83 can be weakend in a metric space: a function $\psi\colon \mathscr{S} \to \mathbb{R}$ is continuous at a point $g \in \mathscr{S}$ if, and only if, the *sequence* $(\langle \psi, g_n \rangle)_{n\in\mathbb{N}}$ converges to $\langle \psi, g \rangle$ for every *sequence* $(g_n)_{n\in\mathbb{N}}$ converging to $g$.

Since $\lim_{n\to+\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle$ for every sequence $(f_n)_{n\in\mathbb{N}}$ converging to $f$, we conclude $\varphi$ is continuous at $f$ in the natural topology of $\mathscr{S}$. Since the argument holds for every $f \in \mathscr{S}$, $\varphi$ is continuous.

One should notice that all results used in this proof were equivalences, and thus the same argument holds in the opposite direction. Therefore, if $\varphi\colon \mathscr{S} \to \mathbb{R}$ is a continuous linear functional, it is a tempered distribution. ■

*Notation:*
As one might have realized, we shall stick to the notion that tempered distributions are functionals acting on $\mathscr{S}$ and write $\langle \varphi, f \rangle$ for a tempered distribution $\varphi$ evaluated at a rapidly decreasing functions $f$. ♦

**Corollary 5.83:**
*The topological dual $\mathscr{S}'$ of the Schwartz space $\mathscr{S}$ is precisely the space of tempered distributions.* □

*Proof:*
Follows from Theorem 5.82 on the preceding page. ■

For technical reasons, it would be interesting for us to deal with a directed family of seminorms instead of the usual $\|\cdot\|_{\alpha,\beta}$ seminorms of Schwartz's space. For example, dealing with a directed family simplifies the usage of Theorem 5.49 on page 158.

**Lemma 5.84:**
*Let $k,l \in \mathbb{N}$ and define the functions $\|\cdot\|_{k,l} \colon \mathscr{S} \to \mathbb{R}_+$ through*

$$\|f\|_{k,l} = \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|f\|_{\alpha,\beta}. \tag{5.199}$$

*Then it holds that $\left\{\|\cdot\|_{k,l}\right\}_{k,l\in\mathbb{N}}$ is a directed family of seminorms that separates points and is equivalent to $\left\{\|\cdot\|_{\alpha,\beta}\right\}_{\alpha,\beta\in\mathbb{N}^n}$.* □

*Proof:*
Let us begin by proving $\|f\|_{k,l}$ are seminorms. Suppose $f,g \in \mathscr{S}$, $\lambda \in \mathbb{R}$. Since the functions $\|\cdot\|_{\alpha,\beta}$ are seminorms, we have that

$$\begin{aligned}
\|f+g\|_{k,l} &= \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|f+g\|_{\alpha,\beta}, \\
&\leq \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|f\|_{\alpha,\beta} + \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|g\|_{\alpha,\beta}, \\
&= \|f\|_{k,l} + \|g\|_{k,l}.
\end{aligned} \tag{5.200}$$

Furthermore,

$$\begin{aligned}
\|\lambda \cdot f\|_{k,l} &= \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|\lambda \cdot f\|_{\alpha,\beta}, \\
&\leq \lambda \cdot \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|f\|_{\alpha,\beta}, \\
&= \lambda \cdot \|f\|_{k,l}.
\end{aligned} \tag{5.201}$$

Let us now show that $\left\{\|\cdot\|_{k,l}\right\}_{k,l\in\mathbb{N}}$ separates points. Suppose $\|f\|_{k,l} = 0, \forall\, k,l \in \mathbb{N}$. Then we know that

$$\sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \|f\|_{\alpha,\beta} = 0, \forall\, k,l \in \mathbb{N}. \tag{5.202}$$

Since $\|f\|_{\alpha,\beta} \geq 0, \forall\, \alpha, \beta \in \mathbb{N}^n$, this implies that

$$\|f\|_{\alpha,\beta} = 0, \forall\, \alpha, \beta \in \mathbb{N}^n. \tag{5.203}$$

Since the seminorms $\|\cdot\|_{\alpha,\beta}$ separate points, we conclude that $\|f\|_{k,l} = 0, \forall\, k, l \in \mathbb{N}$ implies $f = 0$, and hence $\left\{\|\cdot\|_{k,l}\right\}_{k,l\in\mathbb{N}}$ separates points.

Next we prove the two families of seminorms are equivalent. By construction, we see that, $\forall\, k, l \in \mathbb{N}, \forall\, f \in \mathscr{S}$,

$$\|f\|_{k,l} \leq \sum_{\substack{|\alpha|\leq k\\ |\beta|\leq l}} \|f\|_{\alpha,\beta}. \tag{5.204}$$

Similarly, $\forall\, \alpha, \beta \in \mathbb{N}^n, \forall\, f \in \mathscr{S}$,

$$\begin{aligned}\|f\|_{\alpha,\beta} &\leq \sum_{\substack{|\gamma|\leq|\alpha|\\ |\delta|\leq|\beta|}} \|f\|_{\gamma,\delta},\\ &= \|f\|_{|\alpha|,|\beta|}.\end{aligned} \tag{5.205}$$

Hence, we conclude from Proposition 5.41 on page 152 that the two families of seminorms are equivalent.

Finally, we still must prove $\left\{\|\cdot\|_{k,l}\right\}_{k,l\in\mathbb{N}}$ is directed.

Let $k_1, l_1, k_2, l_2 \in \mathbb{N}$. Let us denote $k \equiv \max k_1, k_2$ and $l \equiv \max l_1, l_2$. Notice that, $\forall\, f \in \mathscr{S}$,

$$\begin{aligned}\|f\|_{k_1,l_1} + \|f\|_{k_2,l_2} &= \sum_{\substack{|\alpha|\leq k_1\\ |\beta|\leq l_1}} \|f\|_{\alpha,\beta} + \sum_{\substack{|\gamma|\leq k_2\\ |\delta|\leq l_2}} \|f\|_{\gamma,\delta},\\ &\leq \sum_{\substack{|\alpha|\leq k\\ |\beta|\leq l}} \|f\|_{\alpha,\beta} + \sum_{\substack{|\gamma|\leq k\\ |\delta|\leq l}} \|f\|_{\gamma,\delta},\\ &= 2\|f\|_{k,l},\end{aligned} \tag{5.206}$$

which is what we wanted to prove. ■

**Theorem 5.85:**

*Let $\varphi\colon \mathscr{S} \to \mathbb{R}$ be a linear functional. $\varphi$ is a tempered distribution if, and only if, there are $k, l \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ such that $|\langle\varphi, f\rangle| \leq \lambda\|f\|_{k,l}, \forall\, f \in \mathscr{S}$.* □

*Proof:*

Follows from Theorems 5.49 and 5.82 and Lemma 5.84 on page 158, on page 187 and on page 188. ■

**Lemma 5.86:**

*The surface area of a unit sphere in n dimensions, denoted $\Omega_n$, is given by*

$$\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \tag{5.207}$$

*where $\Gamma$ is the gamma function.* □

*Proof:*

One knows that

$$\pi^{\frac{1}{2}} = \int_{-\infty}^{+\infty} e^{-x^2} \, \mathrm{d}x \,. \tag{5.208}$$

Hence,

$$\begin{aligned} \pi^{\frac{n}{2}} &= \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n x_i^2} \, \mathrm{d}^n x \,, \\ &= \int_{\mathbb{R}^n} e^{-|x|} \, \mathrm{d}^n x \,, \end{aligned} \tag{5.209}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

In spherical coordinates, we have

$$\begin{aligned} \pi^{\frac{n}{2}} &= \int e^{-r^2} r^{n-1} \, \mathrm{d}r \, \mathrm{d}\Omega_n \,, \\ &= \frac{1}{2} \int e^{-u} u^{\frac{n}{2}-1} \, \mathrm{d}u \, \mathrm{d}\Omega_n \,, \\ &= \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \int \mathrm{d}\Omega_n \,. \end{aligned} \tag{5.210}$$

Finally,

$$\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \tag{5.211}$$

as claimed. ■

**Lemma 5.87:**

*Let $p, q \in \mathbb{C}$, $2\,\mathrm{Re}(q) > \mathrm{Re}(p) > 0$. Then*

$$\int_0^{\infty} \frac{r^{p-1}}{(1+r^2)^q} \, \mathrm{d}r = \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(q - \frac{p}{2}\right)}{2\Gamma(q)}, \tag{5.212}$$

*where $\Gamma$ is the gamma function.* □

*Proof:*

It is known* that for $m, n \in \mathbb{C}$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$, the beta function is given by

$$B(m,n) = \int_0^{+\infty} \frac{u^{m-1}}{(1+u)^{p+q}} \, \mathrm{d}u \tag{5.213}$$

and it holds that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{5.214}$$

*See, *exempli gratia*, [6].

By picking $q = n + m$, it immediately follows that, if $\mathrm{Re}(q) > \mathrm{Re}(m) > 0$, then

$$\begin{aligned}\frac{\Gamma(m)\Gamma(q-m)}{\Gamma(q)} &= \int_0^{+\infty} \frac{u^{m-1}}{(1+u)^q}\,\mathrm{d}u\,,\\ &= 2\int_0^{+\infty} \frac{r^{2m-1}}{(1+r^2)^q}\,\mathrm{d}r\,.\end{aligned} \tag{5.215}$$

Picking $p = 2m$ yields

$$\int_0^{\infty} \frac{r^{p-1}}{(1+r^2)^q}\,\mathrm{d}r = \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(q-\frac{p}{2}\right)}{2\Gamma(q)} \tag{5.216}$$

whenever $p, q \in \mathbb{C}$, $2\,\mathrm{Re}(q) > \mathrm{Re}(p) > 0$. ■

**Proposition 5.88:**
$\mathscr{S} \subseteq L^1$. □

*Proof*:
Since every element of $\mathscr{S}$ is continuous, we know $f$ is Borel-measurable, and the question that remains is whether the integral of $f$ converges.

From Hölder's inequality we know that, $\forall\, m \in \mathbb{N}^*$,

$$\begin{aligned}\int |f(x)|\,\mathrm{d}^n x &= \int \frac{(1+|x|^2)^m |f(x)|}{(1+|x|^2)^m}\,\mathrm{d}^n x\,,\\ &\leq \int \frac{1}{(1+|x|^2)^m}\,\mathrm{d}^n x \cdot \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|(1+|x|^2)^m f(x)\right|,\end{aligned} \tag{5.217}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$ and the integrals should be understood as Lebesgue integrals over all $\mathbb{R}^n$. $\operatorname{ess\,sup}$ denotes the essential supremum, which is the usual norm on $L^\infty$.

The integral on the right-hand side can be calculated explicitly by employing Lemmas 5.86 and 5.87 on page 190 and on the previous page. Changing the integral to spherical coordinates yields

$$\begin{aligned}\int \frac{1}{(1+|x|^2)^m}\,\mathrm{d}^n x &= \int \frac{r^{n-1}}{(1+r^2)^m}\,\mathrm{d}r\,\mathrm{d}\Omega_n\,,\\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(m-\frac{n}{2}\right)}{2\Gamma(m)},\\ &= \frac{\pi^{\frac{n}{2}}}{(m-1)!}\Gamma\left(m-\frac{n}{2}\right),\end{aligned} \tag{5.218}$$

and hence the integral converges whenever $m > \frac{n}{2}$. Notice the Riemann integral coincides with the Lebesgue integral, since the integrand is strictly positive.

Eq. (5.217) then tells us that the question of whether $f$ is integrable or not comes down to knowing whether $\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\left|(1+|x|^2)^m f(x)\right|$ is finite. We know that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left|(1+|x|^2)^m f(x)\right| \leq \sup_{x\in\mathbb{R}^n}\left|(1+|x|^2)^m f(x)\right|. \tag{5.219}$$

Notice now that $(1+|x|^2)^m$ is an $n$-variable polynomial, for $|x| = \sum_{i=1}^n x_i^2$. Hence, by the very definition of a rapidly decreasing function, it holds that

$$\sup_{x\in\mathbb{R}^n}\left|(1+|x|^2)^m f(x)\right| < +\infty, \tag{5.220}$$

which then ensures, by the previous arguments, that $f$ is integrable. ■

*Remark:*

All integrals we write down should be understood as Lebesgue integrals, unless explicitly stated otherwise. Furthermore, whenever we mention an $L^p$ space, we assume the measure space to be $\mathbb{R}^n$ equipped with the Borel $\sigma$-algebra* and the Lebesgue measure. ♣

We are now in position to investigate the elements of $\mathscr{S}'$.

**Theorem 5.89:**

*Let $\varphi \in \mathscr{S}$. One can interpret $\varphi$ as a tempered distribution by defining the functional*

$$\langle \varphi, f\rangle = \int_{\mathbb{R}^n} \varphi(x) f(x)\,\mathrm{d}x\,. \tag{5.221}$$

□

*Proof:*

If the integral converges, linearity of $\varphi$ — which is the symbol we shall also use to refer to the functional induced by the element $\varphi \in \mathscr{S}$ — follows directly from the linearity of the integral. Let us bother then with integrability and continuity.

Luckily, Hölder's inequality solves both issues at once. We know that

$$\begin{aligned}
|\langle \varphi, f\rangle| &\leq \int |\varphi(x) f(x)|\,\mathrm{d}x\,,\\
&\leq \|\varphi\|_{L^1} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|,\\
&\leq \|\varphi\|_{L^1} \sup_{x\in\mathbb{R}^n} |f(x)|,\\
&= \|\varphi\|_{L^1} \|f\|_{0,0}.
\end{aligned} \tag{5.222}$$

We know $\|f\|_{0,0}$ is finite due to the hypothesis that $f$ is rapidly decreasing. We also know that $\|\varphi\|_{L^1}$ is finite due to Proposition 5.88 on the preceding page. Hence, the integral is well-defined. Furthermore, Theorem 5.85 on page 190 ensures continuity of the linear functional defined in this way. ■

*[68, pp. 205–210] convinced me to drop the Lebesgue $\sigma$-algebra.

*Notation:*

Given $\varphi \in \mathscr{S}$, we shall use the same symbol $\varphi$ to denote both the rapidly decreasing function and the tempered distribution it induces. This is another reason for us to denote the value of a tempered distribution $\varphi$ on a rapidly decreasing function $f$ by $\langle \varphi, f \rangle$ — it avoids confusion with $\varphi(x)$ for $x \in \mathbb{R}^n$. ♦

**Lemma 5.90:**

*Let $p, q, r \in [1, +\infty]$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $fg \in L^r(\mathbb{R}^n)$ and it holds that*

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}. \tag{5.223}$$

□

*Proof:*

Notice that

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{1}{r}, \\ \frac{r}{p} + \frac{r}{q} &= 1. \end{aligned} \tag{5.224}$$

This manipulation is allowed, since $r \neq +\infty$. If one had $r = +\infty$, it would follow that $p = -q$, and hence it would be impossible to have $p, q \geq 1$. For now, we shall also assume $p, q < +\infty$.

Notice also that one has

$$0 < \frac{r}{p}, \frac{r}{q} \leq 1, \tag{5.225}$$

and hence

$$\frac{p}{r}, \frac{q}{r} \geq 1. \tag{5.226}$$

Therefore, Hölder's inequality tells us that

$$\begin{aligned} \int |f(x)g(x)|^r \, dx &= \int |f(x)|^r |g(x)|^r \, dx, \\ &\leq \||f|^r\|_{L^{\frac{p}{r}}} \||g|^r\|_{L^{\frac{q}{r}}}, \\ &= \left( \int |f(x)|^p \, dx \right)^{\frac{r}{p}} \left( \int |g(x)|^q \, dx \right)^{\frac{r}{q}}, \\ \left( \int |f(x)g(x)|^r \, dx \right)^{\frac{1}{r}} &\leq \left( \int |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q \, dx \right)^{\frac{1}{q}}, \\ \|fg\|_{L^r} &\leq \|f\|_{L^p} \|g\|_{L^q}, \end{aligned} \tag{5.227}$$

as desired.

Let us now deal with the case in which $p$ and $q$ need not to be finite. Without any loss of generality, we suppose $q = +\infty$. This implies $p = r$.

We have

$$\begin{aligned} |fg|^r &\leq |f|^r |g|^r, \\ &\leq |f|^r (\operatorname{ess\,sup} |g|)^r \quad \text{a.e.w.}, \\ \int |f(x)g(x)|^r \,\mathrm{d}x &\leq \int |f(x)|^r \,\mathrm{d}x \,(\operatorname{ess\,sup} |g|)^r, \\ \left( \int |f(x)g(x)|^r \,\mathrm{d}x \right)^{\frac{1}{r}} &\leq \left( \int |f(x)|^r \,\mathrm{d}x \right)^{\frac{1}{r}} \operatorname{ess\,sup} |g|, \\ \|fg\|_{L^r} &\leq \|f\|_{L^p} \|g\|_{L^\infty}, \end{aligned} \tag{5.228}$$

where the last line employs $p = r$. This proves the result for the remaining case, and hence concludes the proof. ■

**Lemma 5.91** [Interpolation Theorem]**:**

*Let $1 \leq p \leq q \leq +\infty$. Suppose $f \in L^p \cap L^q$. Then it holds that $f \in L^r$ for $\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$, $t \in [0, 1]$. Furthermore, it holds that*

$$\|f\|_{L^r} \leq \|f\|_{L^p}^t \|f\|_{L^q}^{1-t}. \tag{5.229}$$

□

*Proof:*

The case $p = q$ is trivial, and so are the cases in which $r = p$ or $r = q$. Hence, we may assume $p < r < q$, *id est*, $p \neq q$ and $t \in (0, 1)$.

Let us begin by writing $|f| = |f|^t |f|^{1-t}$. Notice that $|f|^t \in L^{\frac{p}{t}}$ and $|f|^{1-t} \in L^{\frac{q}{1-t}}$. Since $\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$, Lemma 5.90 on page 193 yields

$$\begin{aligned} \|f\|_{L^{p(r)}} &\leq \left\| |f|^t \right\|_{L^{\frac{p}{t}}} \left\| |f|^{1-t} \right\|_{L^{\frac{q}{1-t}}}, \\ &= \left( \int |f(x)|^p \,\mathrm{d}x \right)^{\frac{t}{p}} \left( \int |f(x)|^q \,\mathrm{d}x \right)^{\frac{1-t}{q}}, \\ &= \|f\|_{L^p}^t \|f\|_{L^q}^{1-t}, \end{aligned} \tag{5.230}$$

which is the desired result. ■

**Theorem 5.92:**

*Given $1 \leq p \leq +\infty$, There is a continuous injection of $\mathscr{S}$ into $L^p(\mathbb{R}^n)$.* □

*Proof:*

Proposition 5.88 on page 192 ensures there is an injection from $\mathscr{S}$ into $L^1(\mathbb{R}^n)$. The very definition of rapidly decreasing function ensures also that, given $f \in \mathscr{S}$,

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| \leq \sup_{x \in \mathbb{R}^n} |f(x)| = \|f\|_{0,0} < \infty. \tag{5.231}$$

Since $f \in L^1 \cap L^\infty$, Lemma 5.91 ensures $f \in L^p, \forall\, p \in [1, +\infty]$. Notice that while we are writing $f \in L^p$, what we really mean is that $f$ is a representative of one of the equivalence classes composing $L^p$, *id est*, we don't strictly mean $\mathscr{S} \subseteq L^p$, but rather that there is an one-to-one function $i \colon \mathscr{S} \to L^p$.

We now want to prove that the function $i$ is continuous. Let us begin by noticing that it is linear. Since $L^p$ spaces are Banach spaces, they are locally convex spaces, and hence we may apply Theorem 5.49 on page 158. Therefore, we want to prove that given $p \in [1, +\infty]$, there are $k, l \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+$ such that

$$\|f\|_{L^p} \leq \lambda \|f\|_{k,l}. \tag{5.232}$$

Given $p \in [1, +\infty]$ and $f \in L^p$, Lemma 5.90 on page 193 tells us that

$$\|f\|_{L^p} \leq \|f\|_{L^1} \|1\|_{L^\infty}. \tag{5.233}$$

Since $\|1\|_{L^\infty} = \operatorname{ess\,sup} |1| = 1$, we see that $\|f\|_{L^p} \leq \|f\|_{L^1}$, and hence we only need to prove that there are $k, l \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ such that

$$\|f\|_{L^1} \leq \lambda \|f\|_{k,l}. \tag{5.234}$$

Luckily, on the proof to Proposition 5.88 on page 192 we have shown that, $\forall\, m \in \mathbb{N}$,

$$\|f\|_{L^1} \leq \frac{\pi^{\frac{n}{2}}}{\Gamma(m)} \Gamma\left(m - \frac{n}{2}\right) \sup_{x \in \mathbb{R}^n} \left|(1 + |x|)^m f(x)\right|. \tag{5.235}$$

Notice that

$$\sup_{x \in \mathbb{R}^n} \left|\left(1 + |x|^2\right)^m f(x)\right| \leq \|f\|_{2m,0}. \tag{5.236}$$

Hence, $\forall\, m \in \mathbb{N}$,

$$\|f\|_{L^1} \leq \frac{\pi^{\frac{n}{2}}}{\Gamma(m)} \Gamma\left(m - \frac{n}{2}\right) \|f\|_{2m,0}. \tag{5.237}$$

We might then pick $m = 1 + \frac{n}{2}$ to conclude that

$$\|f\|_{L^1} \leq \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})} \|f\|_{2+n,0}, \tag{5.238}$$

which proves the result. ■

**Theorem 5.93:**

*Let $p \in [1, +\infty]$. There is an injection of $L^p(\mathbb{R}^n)$ into $\mathscr{S}'$.* □

*Proof:*

Let $f \in \mathscr{S}$ and $\varphi \in L^p$. Notice that

$$f \mapsto \langle \varphi, f \rangle \equiv \int \varphi(x) f(x)\, \mathrm{d}x \tag{5.239}$$

is a linear functional, due to linearity of the integral. Convergence of the integral is ensured by Theorem 5.92 on the previous page and Hölder's inequality, and so is continuity. Indeed, defining $q$ such that $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
|\langle \varphi, f\rangle| &= \left|\int \varphi(x) f(x)\,\mathrm{d}x\right|, \\
&\leq \int |\varphi(x) f(x)|\,\mathrm{d}x\,, \\
&\leq \|\varphi\|_{L^p}\|f\|_{L^q}, \\
&\leq \|\varphi\|_{L^p}\|f\|_{L^1}, \\
&\leq \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}\|\varphi\|_{L^p}\|f\|_{2+n,0},
\end{aligned} \tag{5.240}$$

where we employed the fact proven in Theorem 5.92 on page 195 that, $\forall\, q \in [1,+\infty]$,

$$\|f\|_{L^q} \leq \|f\|_{L^1} \leq \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}\|f\|_{2+n,0}. \tag{5.241}$$

Theorem 5.85 on page 190 ensures $\langle \varphi, \cdot\rangle : \mathscr{S} \to \mathbb{R}$ is continuous, and hence it is a tempered distribution, *id est*, $L^p \subseteq \mathscr{S}'$. Notice that we are abusing notation and if we wanted to be as rigorous as possible we should actually write that there is an injection from $L^p$ into $\mathscr{S}'$. ■

**Definition 5.94** [Polynomially Bounded Function]**:**
Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function. $\varphi$ is said to be polynomially bounded if, and only if, there are constants $c \in \mathbb{R}$ and $m \in \mathbb{N}$ such that such that $|\varphi(x)| \leq c\left(1+|x|^2\right)^m, \forall\, x \in \mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean norm. ♠

**Proposition 5.95:**
*Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Then it is polynomially bounded.* □

*Proof:*
We want to prove the existence of $c \in \mathbb{R}$ and $m \in \mathbb{N}$ such that

$$|p(x)| \leq c\left(1+|x|^2\right)^m, \forall\, x \in \mathbb{R}^n, \tag{5.242}$$

which is equivalent to

$$\sup_{x\in\mathbb{R}^n}\left|\frac{p(x)}{\left(1+|x|^2\right)^m}\right| \leq c, \tag{5.243}$$

and hence we just want to prove that the left-hand side of the previous equation is finite.

Since polynomials are smooth and $\left(1+|x|^2\right)^m$ won't vanish at any point regardless of $m$, we know that

$$\sup_{x\in K}\left|\frac{p(x)}{\left(1+|x|^2\right)^m}\right| < +\infty \tag{5.244}$$

for any compact $K \subseteq \mathbb{R}^n$, which is a consequence of Proposition 3.149 on page 95 and Theorem 3.147 on page 94. Hence, we remain to study whether $p(x)\cdot(1+|x|)^{-m}$ diverges at infinity.

By infinity, we mean outside of the compact set $K = \{x \in \mathbb{R}^n; \max_i x_i \leq 1\}$. Let us assume the polynomial $p$ has degree $l$, so we may write

$$p(x) = \sum_{|\alpha|\leq l} c_\alpha x^\alpha, \tag{5.245}$$

for some coefficients $c_\alpha$. We then have, while outside of $K$,

$$\begin{aligned} |p(x)| &= \left|\sum_{|\alpha|\leq l} c_\alpha x^\alpha\right|, \\ &\leq \sum_{|\alpha|\leq l} |c_\alpha x^\alpha|, \\ &\leq \sum_{|\alpha|\leq l} |c_\alpha| x^{2\alpha}, \end{aligned} \tag{5.246}$$

where we employed the fact that, outside of $K$, $|x_i|^r < |x_i|^s$ whenever $r < s$.

On the other hand, we notice that

$$\left(1+|x|^2\right)^m = \sum_{k=0}^m \binom{m}{k} |x|^{2k}. \tag{5.247}$$

Recall that $|x|^{2k} = \left(\sum_{i=1}^n x_i^2\right)^k$, which means the expression can be further simplified with the aid of the multinomial theorem, given in Proposition 1.1 on page 3. We get to

$$\begin{aligned} \left(1+|x|^2\right)^m &= \sum_{k=0}^m \sum_{|\alpha|=k} \binom{m}{k}\binom{|\alpha|}{\alpha} x^{2\alpha}, \\ &= \sum_{|\alpha|\leq m} \frac{m!}{\alpha!(m-|\alpha|)!} x^{2\alpha}. \end{aligned} \tag{5.248}$$

Comparing this expression with our estimate for $|p(x)|$, we are invited to pick $m = l$ and choose $\kappa \in \mathbb{R}$ such that

$$|c_\alpha| \leq \kappa \frac{l!}{\alpha!(l-|\alpha|)!}, \forall\, \alpha \in \mathbb{N}; |\alpha| \leq l. \tag{5.249}$$

We then have

$$|p(x)| \leq \sum_{|\alpha|\leq l} |c_\alpha| x^{2\alpha} \leq \kappa \sum_{|\alpha|\leq l} \frac{l!}{\alpha!(l-|\alpha|)!} x^{2\alpha} = \kappa \left(1+|x|^2\right)^l. \tag{5.250}$$

Therefore,

$$\left| \frac{p(x)}{\left(1+|x|^2\right)^l} \right| \leq \kappa, \forall\, x \notin K, \tag{5.251}$$

$$\sup x \in \mathbb{R}^n \left| \frac{p(x)}{\left(1+|x|^2\right)^l} \right| < +\infty, \tag{5.252}$$

where the supremum can be taken throughout $\mathbb{R}^n$ because we've already cleared up any issues within compact subsets. Hence, the proof is complete. ■

**Lemma 5.96:**

*Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function with all of its derivarives polynomially bounded. Let $g \in \mathscr{S}$. Then $fg \in \mathscr{S}$. Furthermore, once $f$ is held fixed, the map $g \mapsto fg$ is continuous.* □

*Proof:*

We want to show that

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta (f(x)g(x)) \right| < +\infty, \forall\, \alpha, \beta \in \mathbb{N}^n. \tag{5.253}$$

The Leibniz rule, Proposition 1.2 on page 4, tells us that

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta (f(x)g(x)) \right| &= \sup_{x\in\mathbb{R}^n} \left| x^\alpha \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} D^\gamma f(x) D^{\beta-\gamma} g(x) \right|, \\
&\leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\gamma f(x) D^{\beta-\gamma} g(x) \right|, \\
&\leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_{x\in\mathbb{R}^n} \left| x^\alpha p_\gamma(x) D^{\beta-\gamma} g(x) \right|,
\end{aligned} \tag{5.254}$$

where $p_\gamma(x)$ is the polnomial bounding $D^\gamma f$. Notice that $x^\alpha p_\gamma(x)$ is a polynomial as well, which means there are constants $l, k \in \mathbb{N}$, $\kappa \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta (f(x)g(x)) \right| \leq \sum_{\substack{|\gamma|\leq k \\ \kappa|\delta|\leq l}} \sup_{x\in\mathbb{R}^n} \left| x^\gamma D^\delta (g(x)) \right|, \tag{5.255}$$

so Theorem 5.49 on page 158 ensures continuity. Naturally, the fact that $g$ is rapidly decreasing ensures that

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta (f(x)g(x)) \right| < +\infty, \tag{5.256}$$

which means $fg$ is rapidly decreasing. ■

**Lemma 5.97:**

*Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial and $f \in \mathscr{S}$. Then $p \cdot f \in \mathscr{S}$.* □

*Proof:*

All polynomials are smooth, and all of their derivatives are polynomials. Furthermore, Proposition 5.95 on page 197 ensures all polynomials are polynomially bounded. All things considered, the result follows from Lemma 5.96. ■

**Theorem 5.98:**

*Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a measurable, polynomially bounded function. Then $\varphi$ can be regarded as a tempered distribution through the usual assignment*

$$\langle \varphi, f \rangle = \int \varphi(x) f(x) \,\mathrm{d}x\,, \tag{5.257}$$

*for every $f \in \mathscr{S}$.* □

*Proof:*

If the integral converges, linearity will immediately follow. Since estimates on the integral's convergence allow us not only to prove well-definition, but also continuity, this shall be our focus. Notice convergence is all we need to bother when proving well-definedness of the integral, for we already known the integrand to be measurable.

Since $\varphi$ is polinomially bounded, we know there are $c \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $|\varphi(x)| \leq c\left(1 + |x|^2\right)^m, \forall x \in \mathbb{R}^n$. We then notice that

$$\begin{aligned} |\langle \varphi, f \rangle| &= \left| \int \varphi(x) f(x) \,\mathrm{d}x \right|, \\ &\leq \int |\varphi(x) f(x)| \,\mathrm{d}x\,, \\ &\leq c \int \left| \left(1 + |x|^2\right)^m f(x) \right| \mathrm{d}x\,. \end{aligned} \tag{5.258}$$

Lemma 5.97 on the previous page ensures $\left(1 + |x|^2\right)^m f(x)$ is rapidly decreasing, and hence it is integrable due to Proposition 5.88 on page 192, which implies $|\langle \varphi, f \rangle| < +\infty$, and we conclude the integral is well-defined.

Finally, we show continuity. The assumption of polynomial boundedness allows us to conclude that

$$\begin{aligned} |\langle \varphi, f \rangle| &= \left| \int \varphi(x) f(x) \,\mathrm{d}x \right|, \\ &\leq \int |\varphi(x) f(x)| \,\mathrm{d}x\,, \\ &\leq c \int \left(1 + |x|^2\right)^m |f(x)| \,\mathrm{d}x\,, \\ &\leq c \int \frac{1}{\left(1 + |x|^2\right)^m} \left(1 + |x|^2\right)^2 m |f(x)| \,\mathrm{d}x\,. \end{aligned} \tag{5.259}$$

Hölder's inequality leads to

$$|\langle \varphi, f\rangle| \leq c \int \frac{1}{\left(1+|x|^2\right)^l} \, \mathrm{d}x \cdot \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^{m+l} f(x) \right|. \tag{5.260}$$

This might remind you of the proof to Proposition 5.88 on page 192. There we've shown that the integral converges for $l > \frac{n}{2}$, which allows us to conclude that

$$|\langle \varphi, f\rangle| \leq \kappa \cdot \sup_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^{m+l} f(x) \right|, \tag{5.261}$$

where $\kappa$ is a constant and we used the fact that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^{m+l} f(x) \right| \leq \sup_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^{m+l} f(x) \right| < +\infty, \tag{5.262}$$

where finiteness is ensured by the fact that $f$ is rapidly decreasing.

If we apply the binomial theorem on $\left(1+|x|^2\right)^{m+l}$, we can bound the supremum from above in the form

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^{m+l} f(x) \right| &\leq \sum_\alpha c_\alpha \sup_{x\in\mathbb{R}^n} |x^\alpha f(x)|, \\ &= \sum_\alpha c_\alpha \|f\|_{\alpha,0}, \end{aligned} \tag{5.263}$$

where the multiindices $\alpha$ are determined by the binomial expansion and $c_\alpha$ are constants. Theorem 5.49 on page 158 now ensures the functional defined by $\varphi$ is continuous, and hence a tempered distribution. ■

**Corollary 5.99:**

*$\Theta$, the Heaviside step function, can be interpreted as a tempered distribution.* □

*Proof:*

$\Theta$ is measurable and polynomially bounded, for $|\Theta(x)| < 2, \forall\, x \in \mathbb{R}$. Hence, Theorem 5.98 on page 199 ensures $\Theta$ describes a tempered distribution. ■

Of course, we don't need to stop at the Heaviside step function. $\Theta(x-a)$ as a function of $x$, which is just the composition of $\Theta$ with a translation, can be interpreted as a tempered distribution in the exact same way.

Whenever we have a function $\varphi$ defining a tempered distribution, we interpret it as

$$\langle \varphi, f\rangle = \int \varphi(x) f(x) \, \mathrm{d}x\,, \tag{5.264}$$

which means that two functions that differ on a set of measure zero describe the same tempered distribution. In particular, the distribution described by $\Theta$ is unaffected by the particular value the function $\Theta$ assumes at the point of discontinuity, for example.

## 5.6 Operating With Tempered Distributions

We may define translations of tempered distributions in general, without any need to stick only to the Heaviside distribution.

*Notation* [Translation]*:*
Let $f : \mathbb{R}^n \to \mathbb{R}^n$. For any $a \in \mathbb{R}^n$ we define

$$T_a f(x) = f(x - a). \tag{5.265}$$

◆

If we were considering a tempered distribution induced by a function $f$, we would have, $\forall\, g \in \mathscr{S}$,

$$\begin{aligned} \langle T_a f, g \rangle &= \int f(x-a) g(x) \,\mathrm{d}x\,, \\ &= \int f(y) g(y+a) \,\mathrm{d}y\,, \\ &= \langle f, T_{-a} g \rangle\,. \end{aligned} \tag{5.266}$$

By building up on this expression, we can get a definition for the general case.

**Theorem 5.100** [Translation of a Tempered Distribution]**:**
*Let $\varphi \in \mathscr{S}'$. Given $a \in \mathbb{R}^n$, $T_a\varphi$ defined according to*

$$\langle T_a \varphi, f \rangle = \langle \varphi, T_{-a} f \rangle\,, \forall\, f \in \mathscr{S} \tag{5.267}$$

*is a tempered distribution.* □

*Proof:*
It holds that $f \in \mathscr{S} \Rightarrow T_a f \in \mathscr{S}, \forall\, a \in \mathbb{R}^n$, which means the expression is well-defined. Linearity comes from linearity of $\varphi$ and of $T_a$. If we prove $T_a \colon \mathscr{S} \to \mathscr{S}$ is continuous, we're done.

To do so, we shall, as usual, refer to Theorem 5.49 on page 158. Notice that, for any particular fixed $a \in \mathbb{R}^n$,

$$\begin{aligned} \|T_a f\|_{\alpha,\beta} &= \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x-a) \right|, \\ &= \sup_{x \in \mathbb{R}^n} \left| (x+a)^\alpha D^\beta f(x) \right|, \\ &= \sup_{x \in \mathbb{R}^n} \left| \prod_{i=1}^n [(x_i + a_i)^{\alpha_i}] D^\beta f(x) \right|, \\ &= \sup_{x \in \mathbb{R}^n} \left| \prod_{i=1}^n \left[ \sum_{l_i=1}^{\alpha_i} \binom{\alpha_i}{l_i} x^{l_i} a^{\alpha_i - l_i} \right] D^\beta f(x) \right|. \end{aligned} \tag{5.268}$$

Notice we may now write

$$\prod_{i=1}^{n}\left[\sum_{l_i=1}^{\alpha_i}\binom{\alpha_i}{l_i}x^{l_i}a^{\alpha_i-l_i}\right]=\sum_{\gamma\leq\alpha}\kappa_\gamma x^\gamma \tag{5.269}$$

for appropriate constants $\kappa_\gamma$, where $\gamma \leq \alpha \Leftrightarrow \gamma_i \leq \alpha_i, 1 \leq i \leq n$. We then get to

$$\begin{aligned}\|T_a f\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n}\left|\sum_{\gamma\leq\alpha}\kappa_\gamma x^\gamma D^\beta f(x)\right|,\\ &\leq \sum_{\gamma\leq\alpha}\kappa_\gamma \sup_{x\in\mathbb{R}^n}\left|x^\gamma D^\beta f(x)\right|,\\ &\leq \sum_{\gamma\leq\alpha}\kappa_\gamma\|f\|_{\gamma,\beta},\end{aligned} \tag{5.270}$$

and we conclude $T_a\colon \mathscr{S} \to \mathscr{S}$ is indeed continuous, meaning $T_a\varphi$ is indeed a distribution. ■

Another natural operation to consider is a linear change of coordinates. For example, in Special Relativity, one might want to consider what happens if we consider a Lorentz transformation of the coordinates we are working with.

*Notation:*
Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. We define $\tilde{A}\colon \mathscr{S} \to \mathscr{S}$ to be the operator such that $\tilde{A}f(x) = f(A^{-1}x)$. ♦

Our goal is to extend $\tilde{A}$ to $\mathscr{S}'$. We begin by looking at the properties of the tempered distributions given by elements of $\mathscr{S}$. For any $f, g \in \mathscr{S}$,

$$\begin{aligned}\langle \tilde{A}f, g\rangle &= \int f(A^{-1}x)g(x)\,\mathrm{d}x\,,\\ &= \int f(y)g(Ay)|\det A|\,\mathrm{d}y\,,\\ &= |\det A|\left\langle f, \tilde{A}^{-1}g\right\rangle\end{aligned} \tag{5.271}$$

(notice that $\tilde{A}^{-1} = \widetilde{A^{-1}}$). This suggests a generalization for the arbitrary case.

**Theorem 5.101** [Linear Changes of Coordinates on Tempered Distributions]**:**
*Let $\varphi \in \mathscr{S}'$ and $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then $\tilde{A}\varphi$ defined by*

$$\langle \tilde{A}\varphi, f\rangle = |\det A|\left\langle \varphi, \tilde{A}^{-1}f\right\rangle, \forall f \in \mathscr{S}, \tag{5.272}$$

*is a tempered distribution.* □

*Proof:*
Linearity comes from the usual properties of $\varphi$ and the definition of $\tilde{A}^{-1}$. Our focus is to prove continuity (and well-definedness).

We would like to compute

$$\left\|\tilde{A}^{-1}f\right\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta[f(Ax)]\right|, \tag{5.273}$$

where $D^\beta[f(Ax)]$ means the derivative needs to be performed with a chain rule: one first evaluates $f$ at $Ax$ and only then proceeds with differentiation.

To compute the explicit form of this derivative would be extremely painful. Luckily though, we don't need all of the details, for we just need an estimate to $\|\tilde{A}^{-1}f\|_{\alpha,\beta}$ in terms of some not necessarily known constants. This suffices for Theorem 5.49 on page 158 to help us. Let us define $y = Ax$ and notice then that, for a linear transformation, the partial derivatives behave according to

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= \sum_k \frac{\partial f}{\partial y_k}\frac{\partial y_k}{\partial x_i}, \\ &= \sum_k a_{ki}\frac{\partial f}{\partial y_k},\end{aligned} \tag{5.274}$$

where $a_{ki}$ are simply constants. Proceeding with the process one can get

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_{k,l} a_{ki}a_{lj}\frac{\partial^2 f}{\partial y_k \partial y_l}, \tag{5.275}$$

and so on. Hence, in general, we get to

$$D^\beta_x f = \sum_{|\gamma|=|\beta|} \kappa_\gamma D^\gamma_y f, \tag{5.276}$$

for some constants $\kappa_\gamma$.

Therefore, we find that

$$\begin{aligned}\left\|\tilde{A}^{-1}f\right\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta[f(Ax)]\right|, \\ &= \sup_{x\in\mathbb{R}^n}\left|x^\alpha \sum_\gamma \kappa_\gamma D^\gamma f(Ax)\right|, \\ &\leq \sum_\gamma \kappa_\gamma \sup_{x\in\mathbb{R}^n}|x^\alpha D^\gamma f(Ax)|, \\ &\leq \sum_\gamma \kappa_\gamma \sup_{x\in\mathbb{R}^n}\left|\left(A^{-1}x\right)^\alpha D^\gamma f(x)\right|.\end{aligned} \tag{5.277}$$

Now we must deal with $\left(A^{-1}x\right)^\alpha$. Denoting the matrix elements of $A^{-1}$ as $b_{ij}$ we have

$$\left(A^{-1}x\right)^\alpha = \prod_{i=1}^n \left(\sum_{j=i}^n b_{ij}x_j\right)^{\alpha_i}. \tag{5.278}$$

The largest power $x_1$ can achieve in this expression is no more than $|\alpha|$, for if we pick the largest power of $x_1$ from each $i$ factor and multiply all of them we'll get a term proportional to $x_1^{|\alpha|}$. Naturally, the same argument holds for $x_i$ for any other value of $i$. Hence, there are constants $\sigma_\delta$ such that

$$\left(A^{-1}x\right)^{\alpha} = \sum_{|\delta| \leq |\alpha|} \sigma_\delta x^\delta. \tag{5.279}$$

This expression leads us to

$$\begin{aligned}
\left\|\tilde{A}^{-1}f\right\|_{\alpha,\beta} &\leq \sum_{\gamma} \kappa_\gamma \sup_{x\in\mathbb{R}^n} \left|\sum_{\delta} \sigma_\delta x^\delta D^\gamma f(x)\right|, \\
&\leq \sum_{\gamma,\delta} c_{\gamma,\delta} \sup_{x\in\mathbb{R}^n} \left|x^\delta D^\gamma f(x)\right|, \\
&= \sum_{\gamma,\delta} c_{\gamma,\delta} \sup_{x\in\mathbb{R}^n} \left|x^\delta D^\gamma f(x)\right|, \\
&= \sum_{\gamma,\delta} c_{\gamma,\delta} \|f\|_{\delta,\gamma},
\end{aligned} \tag{5.280}$$

and now Theorem 5.49 on page 158 ensures continuity. Furthermore, since $\|f\|_{\alpha,\beta} < +\infty, \forall\, \alpha, \beta \in \mathbb{N}^n$, this also ensures $\left\|\tilde{A}^{-1}f\right\|_{\alpha,\beta} < +\infty, \forall\, \alpha, \beta \in \mathbb{N}^n$, meaning $\tilde{A}^{-1}f \in \mathscr{S}$, which then implies $\tilde{A}\varphi$ is well-defined. ■

We might then start wondering what sorts of algebraic operations we can do with tempered distributions. Our goal would be to use them to study differential equations, and therefore we'd better be able to multiply them by functions, add them, differentiate them, *et cetera*. We begin with the algebraic operations.

**Theorem 5.102** [Algebraic Operations with Tempered Distributions]**:**
*Let $\varphi, \psi \in \mathscr{S}'$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth and have all its derivatives polynomially bounded. Then $\varphi + \psi$ and* $f \cdot \varphi$, defined according to*

$$\langle \varphi + \psi, g\rangle = \langle \varphi, g\rangle + \langle \psi, g\rangle\,, \tag{5.281}$$

*and*

$$\langle f \cdot \varphi, g\rangle = \langle \varphi, fg\rangle\,, \tag{5.282}$$

*for all $g \in \mathscr{S}$, are tempered distributions.* □

*Proof:*
The linearity properties are straighforward to prove for both operations. Continuity for the addition of tempered distributions is ensured by the continuity of addition on the real line.

*As usual, we'll often drop the dot indicating the product.

Let us then focus on the product of a tempered distribution by a function. $\langle f\varphi, g\rangle = \langle \varphi, fg\rangle$ is well-defined, for Lemma 5.96 on page 199 ensures $fg \in \mathscr{S}$.

The operation is also continuous. Notice is is just the composition of $\langle \varphi, \cdot\rangle$, which is known to be continuous, with a multiplication by $f$, which is also known to be continuous (Lemma 5.96 on page 199). If the pointwise product of elements of $\mathscr{S}$ is continuous, then $f\varphi$ is a tempered distribution. The product is indeed continuous, as one can show with the aid of the Leibniz rule, in a way similar to what was done in the proof to Lemma 5.96 on page 199. Therefore, $f\varphi$ does define a tempered distribution. ■

This solves the issue with algebraic operations. We are able to add tempered distributions to each other and multiply them by a wide range of functions. However, we should notice that the requirement of $f$ having polynomially bounded derivatives is essential for $f\varphi$ to be a tempered distribution, where $\varphi \in \mathscr{S}'$.

**Example** [Smooth Functions Which Are Not Tempered Distributions]**:**

All constants are measurable and polynomially bounded. Hence, 1, for example, defines a tempered distribution. Let us try to multiply it by another smooth function to see whether we can get a new tempered distribution.

Our pathological example of choice will be $e^{x^2}$. It won't define a tempered distribution, for

$$\left\langle e^{x^2}, e^{-x^2}\right\rangle = \int e^{x^2} e^{-x^2} \,\mathrm{d}x = \int \mathrm{d}x\,, \tag{5.283}$$

which is not convergent. Since $e^{-x^2} \in \mathscr{S}$, we see $e^{x^2}$ can't be defined as a tempered distribution throughout the space of rapidly decreasing functions.

One can fix this issue by allowing less test functions, and considering $\mathscr{D}$, the space of smooth functions with compact support, instead of $\mathscr{S}$. This solves the issue of multiplication by an arbitrary smooth function — the product of a smooth function with a smooth function of compact support will always be smooth and compactly supported — but leads to issues with the Fourier transform, as we shall see on Section 5.7 on page 212. ♥

The next step is for us to learn how to differentiate distributions. After all, we would like to use them to study partial differential equations.

When dealing with weak solutions for the wave equation, integration by parts suggested how we could define derivatives of distributions (Eq. (5.40) on page 134). Since rapidly decreasing functions vanish at infinity, if $\varphi \in \mathscr{S}'$ can be understood as a smooth function, we would have, $\forall\, f \in \mathscr{S}$,

$$\langle \varphi, D^\alpha f\rangle = \int \varphi(x) D^\alpha f(x) \,\mathrm{d}x = (-1)^{|\alpha|} \int D^\alpha \varphi(x) f(x) \,\mathrm{d}x \equiv (-1)^{|\alpha|} \langle D^\alpha \varphi, f\rangle\,. \tag{5.284}$$

This suggests that we can define derivatives of tempered distributions, regardless of whether they can be interpreted as smooth functions, according to $\langle D^\alpha \varphi, f\rangle \equiv (-1)^{|\alpha|} \langle \varphi, D^\alpha f\rangle$.

To show this prescription actually yields a tempered distribution we will first prove an auxiliary result, which partially justifies our choice of dealing with $\mathscr{S}$ as the source of our test functions.

**Proposition 5.103:**
*Let $\alpha \in \mathbb{N}^n$. The map $D^\alpha : \mathscr{S} \to \mathscr{S}$ is continuous.* □

*Proof:*
The proof is an application of Theorem 5.49 on page 158. It suffices to notice that

$$\|D^\gamma f\|_{\alpha,\beta} = \|f\|_{\alpha,\beta+\gamma}, \tag{5.285}$$

which also implies the map is well-defined, for the hypothesis that $f$ is rapidly decreasing implies $D^\gamma f$ is as well. ■

**Theorem 5.104** [Derivatives of Tempered Distributions]**:**
*Let $\varphi \in \mathscr{S}'$. Define $D^\alpha \varphi$ through*

$$\langle D^\alpha \varphi, f \rangle = (-1)^{|\alpha|} \langle \varphi, D^\alpha f \rangle, \forall f \in \mathscr{S}. \tag{5.286}$$

*$D^\alpha \varphi$ is a tempered distribution. Furthermore, $D^\alpha : \mathscr{S}' \to \mathscr{S}'$ is continuous in the weak topology generated by $\mathscr{S}$.* □

*Proof:*
$D^\alpha \varphi$ is well-defined due to Proposition 5.103 on the preceding page. It is continuous because it is the composition of $\varphi$, derivatives on $\mathscr{S}$ (continuous due to Proposition 5.103 on the facing page), and a multiplication by a scalar. Linearity comes from linearity of $\varphi$ and of the derivative.

We remain to prove continuity. Let $(\varphi_\beta)_{\beta \in I} \in \mathscr{S}'^I$ be a net with $\varphi_\beta \to \varphi \in \mathscr{S}'$ weakly, *id est*, $\langle \varphi_\beta, f \rangle \to \langle \varphi, f \rangle, \forall f \in \mathscr{S}$. Notice then that

$$\langle D^\alpha \varphi_\beta, f \rangle = (-1)^{|\alpha|} \langle \varphi_\beta, D^\alpha f \rangle \to (-1)^{|\alpha|} \langle \varphi, D^\alpha f \rangle = \langle D^\alpha \varphi, f \rangle \tag{5.287}$$

and Theorem 3.123 on page 83 ensures $D^\alpha$ is convergent. ■

This is particularly interesting. Since $\mathscr{S}$ is composed of smooth functions, all tempered distributions can be differentiated infinitely many times, regardless of whether they were not differentiable when treated as ordinary functions. As Barry Simon puts,

> This volume can be thought of as the infinitesimal calculus of the twentieth century. From that point of view, the key chapters are Chapter 4, which covers measure theory – the consummate integral calculus – and the first part of Chapter 6 on distribution theory – the ultimate differential calculus. [68, p. xvii]

Let us take a moment to appreciate this and differentiate the non differentiable.

Let us consider the Heaviside step funcion, $\Theta$. As a function, it has vanishing derivative for $|x| > 0$ and has no derivative at $x = 0$. What about as a distribution?

Per definition, we have, for all $f \in \mathscr{S}$,

$$\begin{aligned}
\langle \Theta', f \rangle &= - \langle \Theta, f' \rangle \,, \\
&= - \int_{-\infty}^{+\infty} \Theta(x) f'(x) \, \mathrm{d}x \,, \\
&= - \int_{0}^{+\infty} f'(x) \, \mathrm{d}x \,, \\
&= f(0) - \lim_{x \to +\infty} f(x), \\
&= f(0).
\end{aligned} \tag{5.288}$$

Its derivative simply picks the value of the function at the origin! If we did the same procedure with $\Theta(x-a)$, we would find that its derivative simply picks the value of the function at $a$, $f(a)$.

We may generalize this concept to $n$ dimensions as well.

**Definition 5.105** [Dirac Delta Distribution]**:**

We define the Dirac delta distribution, $\delta$, through

$$\langle \delta, f \rangle = f(0), \forall f \in \mathscr{S}. \tag{5.289}$$

♠

We already know $\delta$ is a tempered distribution in $\mathbb{R}$, but we should prove it in $\mathbb{R}^n$.

**Proposition 5.106:**

*$\delta$ is a tempered distribution.* □

*Proof:*

Linearity comes from the fact that $(f + \lambda \cdot g)(0) = f(0) + \lambda \cdot g(0), \forall f, g \in \mathscr{S}$, by the very definition of addition of real-valued functions.

Furthermore, notice that, $\forall f \in \mathscr{S}$,

$$\begin{aligned}
|f(0)| &\leq \sup_{x \in \mathbb{R}^n} |f(x)|, \\
|\langle \delta, f \rangle| &\leq \|f\|_{0,0},
\end{aligned} \tag{5.290}$$

and hence Theorem 5.85 on page 190 ensures $\delta$ is a tempered distribution. ■

One can naturally define $\langle \delta_a, f \rangle = f(a)$ and show it is a tempered distribution in a similar manner.

We can write the action of the Dirac distribution in terms of an integral if we agree to not restrain ourselves to the Lebesgue measure. If we define the Dirac measure through

$$\delta(A) = \begin{cases} 1, & \text{if } 0 \in A, \\ 0, & \text{if } 0 \notin A, \end{cases} \tag{5.291}$$

for any $A$ in the Borel $\sigma$-algebra, we can then write

$$\langle \delta, f \rangle = \int f \, \mathrm{d}\delta \,. \tag{5.292}$$

One might get tempted to push this notion and try to get a function $\delta(x)$ so we can write

$$\langle \delta, f \rangle = \int \delta(x) f(x) \, \mathrm{d}x \, , \tag{5.293}$$

with integration taking place over the Lebesgue measure. In spite of the temptation, this is impossible.

Let us recall the Lebesgue–Radon–Nikodym Theorem, which we state here without proof. Recall that two signed measures $\mu$ and $\nu$ defined on the same measurable space $(X, \mathcal{A})$ are said to be mutually singular (denoted $\mu \perp \nu$) if, and only if, $\exists\, E, F \in \mathcal{A}; E \cap F = \varnothing, E \cup F = X$, $E$ is null for $\mu$, $F$ is null for $\nu$. Furthermore, if $\mu$ is a positive measure, $\nu$ is said to be absolutely continuous with respect to $\mu$ (written $\nu \ll \mu$) if, and only if, $\forall\, E \in \mathcal{A}, \mu(E) = 0 \Rightarrow \nu(E) = 0$.

Given this, one has[22, p. 90]

**Theorem 5.107** [Lebesgue–Radon–Nikodym]**:**

*Let $(X, \mathcal{A})$ be a measurable space, $\nu$ be a $\sigma$-finite signed measure on $(X, \mathcal{A})$ and $\mu$ be a $\sigma$-finite positive measure on $(X, \mathcal{A})$. Then there are unique $\sigma$-finite signed measures $\tau$ and $\rho$ on $(X, \mathcal{A})$ such that $\tau \perp \mu$, $\rho \ll \mu$ and $\nu = \tau + \rho$. Furthermore, there is a measurable function $f \colon X \to \mathbb{R}$ such that either $\int f_+ \, \mathrm{d}\mu$ or $\int f_- \, \mathrm{d}\mu$ is finite and $\rho(E) = \int_E f \, \mathrm{d}\mu, \forall\, E \in \mathcal{A}$. Finally, $f$ is unique almost everywhere with respect to $\mu$.* □

The process of splitting $\nu$ as $\nu = \tau + \rho$ with $\tau \perp$ and $\rho \ll \mu$ is known as the Lebesgue decomposition of $\nu$ with respect to $\mu$. In the case $\nu \ll \mu$, $\nu = \rho$ and the function $f$ is often denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ and referred to as the Radon–Nikodym derivative of $\nu$ with respect to $\mu$. One should notice that it allows us to have a notion of how to "change variables" in a Lebesgue integral[22, p. 91]: if $f \in L^1(\nu)$, it holds that $f \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \in L^1(\nu)$ and

$$\int f \, \mathrm{d}\nu = \int f \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu \, . \tag{5.294}$$

Hence, we would like to have $\delta\,(x) = \frac{\mathrm{d}\delta}{\mathrm{d}\lambda}$, where $\lambda$ is the Lebesgue measure, so that

$$\langle \delta, f \rangle = \int f \, \mathrm{d}\delta = \int \delta\,(x)\, f(x) \, \mathrm{d}x \, . \tag{5.295}$$

However, one should notice that $\delta \perp \lambda$, since $\delta(\mathbb{R} \setminus \{0\}) = 0$, $\lambda(\{0\}) = 0$. Hence, the Lebesgue decomposition of $\delta$ with respect to $\mu$ is simply $\delta = \delta$, and since the Lebesgue–Radon–Nikodym Theorem ensures uniqueness implies there is not even a component of $\delta$ that admits a Radon–Nikodym derivative with respect to the Lebesgue measure.

Could there be another measurable function $g$ with

$$\int f \, \mathrm{d}\nu = \int f g \, \mathrm{d}\mu \tag{5.296}$$

for every $f \in L^1(\nu)$? If we pick $f = \chi_E$, the indicator function for the set $E \in \mathcal{A}$, we see that the previous equation reduces to $\nu(E) = \int_E g \, \mathrm{d}\mu$, regardless of any further properties of the signed measures $\nu$ and $\mu$. The expression also implies $\nu \ll \mu$, which certainly does not hold for $\delta$ and $\lambda$.

Hence, we cannot represent the Dirac distribution with a function. Tempered distributions do generalize ordinary functions in this sense.

We didn't really have to deal with Radon–Nikodym derivatives and so much measure theory to conclude this, we could have simply notice that $\delta(x)$ would need to vanish everywhere but at the origin, where it would take an infinite value. One could say we used a sledgehammer to crack a nut, but the important thing is that, at the end of the process, the nut is open. And come on, wasn't it cool?

Regardless, of these observations, we will be stubborn and write $\delta(x)$ anyway. We'll call this the generalized function associated to $\delta$, but I should warn you this nomenclature isn't universal, and many texts will use "generalized function" to refer to the (tempered) distribution itself.

*Notation* [Generalized Functions]*:*

Given a (tempered) distribution $\varphi$, we'll often write $\varphi(x)$ to refer to the would-be-function that relates to $\varphi$ through

$$\langle \varphi, f \rangle = \int \varphi(x) f(x) \, \mathrm{d}x \, , \tag{5.297}$$

where the integral should be understood in a formal, but not rigorous, manner. ♦

Rigorously speaking, this notation is nuts, but from a formal point of view it can simplify calculations a lot. If you are bothered by it due to lack of rigorousness, don't worry: it gets way worse.

Since the Dirac delta is a tempered distribution, we can go on differentiating it. We have

$$\begin{aligned} \langle D^\alpha \delta, f \rangle &= (-1)^{|\alpha|} \langle \delta, D^\alpha f \rangle \, , \\ &= (-1)^{|\alpha|} D^\alpha f(0). \end{aligned} \tag{5.298}$$

The derivatives of the Dirac delta are examples of tempered distributions that can't even be described in terms of measures. Nevertheless, we will write

$$\langle D^\alpha \delta, f \rangle = \int D^\alpha \delta(x) f(x) \, \mathrm{d}x \tag{5.299}$$

and even $D^\alpha \delta(x) = (-1)^{|\alpha|} \delta(x) D^\alpha$ anyway.

Notice the existence of such distributions exhibit the fact we won't be able to define the product of two arbitrary tempered distributions, even though we were able to define the product with a reasonable amount of functions. Suppose $f$ is measurable and polynomially bounded, so it is a tempered distribution due to Theorem 5.98 on page 199. We would like to define $f \cdot \varphi$ such that

$$\langle f \cdot \varphi, g \rangle = \langle \varphi, f g \rangle \, . \tag{5.300}$$

However, $\Theta$ satisfies this requirements, and such a definition would lead to

$$\begin{aligned}\left\langle \Theta\delta', g\right\rangle &= \left\langle \delta', \Theta g\right\rangle, \\ &= -\Theta(0)g'(0) - \Theta'(0)g(0),\end{aligned} \tag{5.301}$$

which is ill-defined: when viewed as a tempered distribution, $\Theta$ doesn't even has a well-defined value at the origin, let alone a derivative.

By playing with the Dirac delta and its derivatives, we see a myriad of tempered distributions can be obtained through differentiation. In fact, the Heaviside distribution can be obtained through distributional differentiation of the plus function $p$, given by

$$p(x) = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{5.302}$$

where we notice that we only need to define the function almost everywhere to get the distribution. Differentiating it yields

$$\begin{aligned}\left\langle p', f\right\rangle &= -\left\langle p, f'\right\rangle, \\ &= -\int_0^{+\infty} x f'(x), \\ &= \int_0^{+\infty} f(x)\,\mathrm{d}x - \lim_{x\to+\infty} x f(x) + 0, \\ &= \int_0^{+\infty} f(x)\,\mathrm{d}x, \\ &= \langle \Theta, f\rangle .\end{aligned} \tag{5.303}$$

Hence, the Heaviside distribution, the one-dimensional Dirac delta, and all of their derivatives are obtained by repeatedly differentiating a continuous, polynomially bounded function.

In fact, this is a particular general fact. One has

**Theorem 5.108** [Regularity Theorem for Tempered Distributions]**:**
*Let $\varphi \in \mathscr{S}'$. There is a multiindex $\alpha \in \mathbb{N}$ and a polynomially bounded continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi = D^\alpha f$, where both $f$ and the differentiation operator are meant in the sense of distributions.* □

*Proof:*
See, *exempli gratia*, [56, 68]. ■

We can work out a particular example, which is easier to state if we first define the notion of support of a tempered distribution. In general, it is meaningless to speak of the values a tempered distribution takes on particular points — $\Theta$ is ill-defined at the origin, for example, and when we begin differentiating it the issues just get worse. Nevertheless, we can get a feeling of its behavior by looking at the distribution's support.

**Definition 5.109** [Support of a Tempered Distribution]**:**
Let $\varphi \in \mathscr{S}'$. Let $\Omega \subseteq \mathbb{R}^n$. One says $\varphi$ *vanishes* in $\Omega$ if, and only if, $\langle \varphi, f\rangle = 0, \forall\, f \in$

$\mathscr{S}$; $\operatorname{supp} f \subseteq \Omega$. The *support* of $\varphi$, $\operatorname{supp}\varphi$, is defined as the complement of the largest open set where $\varphi$ vanishes. ♠

**Proposition 5.110:**

*If $\varphi \in \mathscr{S}'$ is a continuous function, then the support of $\varphi$ as a distribution coincides with the support of $\varphi$ as a function.* □

*Proof:*

We'll denote the support as a distribution through $\operatorname{supp}_d \varphi$ and the support as a function through $\operatorname{supp}_f \varphi$.

Let $x \in \operatorname{supp}_f \varphi$. Since $\varphi$ is continuous, $\operatorname{supp}_f \varphi$ is a neighborhood of $x$. Hence, there are $\epsilon > \delta > 0$ such that $\mathcal{B}_\epsilon(x) \subseteq \operatorname{supp}_f$. Lemma 4.29 on page 112 lets us find a smooth function $g\colon \mathbb{R}^n \to [0,1]$ which has $g(\mathcal{B}_\delta(x)) = \{1\}$ and vanishes outside of $\mathcal{B}_\epsilon(x)$, hence being of rapid decrease. Since $\mathcal{B}_\epsilon(x) \subseteq \operatorname{supp}_f \varphi$, $\varphi$ can't change sign throughout $\mathcal{B}_\epsilon(x)$ and we conclude $\langle \varphi, g\rangle \neq 0$. Therefore, for every $x \in \operatorname{supp}_f \varphi$ we can find $g \in \mathscr{S}$ such that $\operatorname{supp} g \subseteq \mathcal{B}_\epsilon(x)$ with $\langle \varphi, g\rangle \neq 0$, which then implies $\operatorname{supp}_f \varphi \subseteq \operatorname{supp}_d \varphi$.

We now want to prove that $\operatorname{supp}_d \varphi \subseteq \operatorname{supp}_f \varphi = \overline{\{x; \varphi(x) \neq 0\}}$. Let us instead show that $\operatorname{int}(\{x; \varphi(x) \neq 0\}) = \left(\operatorname{supp}_f \varphi\right)^{\mathsf{c}} \subseteq (\operatorname{supp}_d \varphi)^{\mathsf{c}}$.

Suppose $x \in \left(\operatorname{supp}_f \varphi\right)^{\mathsf{c}}$. Get any function $g \in \mathscr{S}$ with $\operatorname{supp} g \subseteq \left(\operatorname{supp}_f \varphi\right)^{\mathsf{c}}$. Then $\langle \varphi, g\rangle = 0$, since

$$\begin{aligned}
\langle \varphi, g\rangle &= \int \varphi(x) g(x)\,\mathrm{d}x\,, \\
&= \int_{\operatorname{supp} g} \varphi(x) g(x)\,\mathrm{d}x\,, \\
&= \int_{\operatorname{supp} g} 0 g(x)\,\mathrm{d}x\,, \\
&= 0. 
\end{aligned} \tag{5.304}$$

Hence, $\varphi$ vanishes on the open set $\left(\operatorname{supp}_f \varphi\right)^{\mathsf{c}}$. Since $(\operatorname{supp}_d \varphi)^{\mathsf{c}}$ is the largest open set where $\varphi$ vanishes, we must have $\left(\operatorname{supp}_f \varphi\right)^{\mathsf{c}} \subseteq (\operatorname{supp}_d \varphi)^{\mathsf{c}}$, which concludes the proof. ■

**Proposition 5.111:**

*Let $\varphi \in \mathscr{S}'$ and $\alpha \in \mathbb{N}^n$.* $\operatorname{supp} D^\alpha \varphi \subseteq \operatorname{supp} \varphi$. □

*Proof:*

Suppose $\varphi$ vanishes on $\Omega$. Then necessarily $D^\alpha \varphi$ vanishes on $\Omega$, for if $\operatorname{supp} f \subseteq \Omega$, then $\operatorname{supp} D^\alpha f \subseteq \Omega$, since $\operatorname{supp} D^\alpha f \subseteq \operatorname{supp} f$ and we know $\langle D^\alpha \varphi, f\rangle = (-1)^{|\alpha|} \langle \varphi, D^\alpha f\rangle$. ■

**Proposition 5.112:**

*Let $\varphi \in \mathscr{S}'$ have* $\operatorname{supp} \varphi = \{0\}$*. Then $\varphi$ can be written as*

$$\varphi = \sum_{|\alpha| \leq m} \kappa_\alpha D^\alpha \delta \tag{5.305}$$

*for appropriate $m \in \mathbb{N}$ and coefficients $\kappa_\alpha$.* □

*Proof:*

Once more we refer to [56, 68]. ■

## 5.7 Fourier Analysis of Functions and Distributions

We mentioned earlier, back on page 206, when dealing with multiplication of a tempered distribution by a function, that one could multiply distributions by more functions if we considered a different space of test functions. Namely, $\mathscr{D}(\Omega)$, the space of smooth functions with compact support contained on $\Omega$.

That wasn't the first time we've encountered such space. It was also the space we first considered on Section 5.1 on page 127 when dealing with weak solutions of the wave equation.

**Definition 5.113** [Test Functions]**:**

We define the space of *test functions* on $\Omega \subseteq \mathbb{R}^n$, $\mathscr{D}(\Omega)$, as the space of smooth functions with compact support on $\Omega$. ♠

I've often used the term "test function" more loosely to also refer to rapidly decreasing functions and I don't plan on stopping, but hopefully the distinction will be made clear by context.

Given $\mathscr{D}$, we would like to build distributions just as we've build the tempered distributions with $\mathscr{S}$. The new space of distributions — which are called just "distributions" or "ordinary distributions" if you want to be specific — would then be $\mathscr{D}'$. Naturally, there is some general notion we would expect of an ordinary distribution.

**Definition 5.114** [Ordinary Distribution]**:**

Let $\varphi \colon \mathscr{D}(\Omega) \to \mathbb{R}$ be a linear map. $\varphi$ is said to be a *distribution* if, and only if, $\forall\, r > 0, \exists\, m \in \mathbb{N}, \kappa > 0$ such that

$$|\langle \varphi, f \rangle| \leq \kappa \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha f(x)| \tag{5.306}$$

for every $f \in \mathscr{D}$ with $\operatorname{supp} f \subseteq \{x \in \Omega; |x| \leq r\}$. ♠

**Definition 5.115** [Locally Integrable Functions]**:**

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and suppose $p \in [1, +\infty]$. We define

$$L^p_{\text{loc}}(\Omega) := \left\{ f \in \Omega^{\mathbb{R}}; f|_K \in L^p(K), \text{for every compact } K \subseteq \Omega \right\}. \tag{5.307}$$

If $f \in L^p_{\text{loc}}(\Omega)$, we say it is locally $p$-integrable on $\Omega$. If $\Omega = \mathbb{R}^n$, we'll simply say it is locally $p$-integrable. If $p = 1$, we'll say just locally integrable, instead of locally 1-integrable. Finally, we will often write $L^p_{\text{loc}} \equiv L^p_{\text{loc}}(\mathbb{R}^n)$. ♠

**Proposition 5.116:**

*Let $\varphi \in L^p_{loc}(\Omega)$. Then $\varphi \in \mathscr{D}'(\Omega)$ if we consider*

$$\langle \varphi, f \rangle = \int \varphi(x) f(x) \, \mathrm{d}x \tag{5.308}$$

*for ever $f \in \mathscr{D}(\Omega)$.* □

*Proof:*

Let $f \in \mathscr{D}$. Since $f$ is continuous and compactly supported, it is bounded. We find that

$$\begin{aligned} |\langle \varphi, f \rangle| &= \left| \int_{\operatorname{supp} f} \varphi(x) f(x) \, \mathrm{d}x \right|, \\ &\leq \int_{\operatorname{supp} f} |\varphi(x) f(x)| \, \mathrm{d}x \,, \\ &\leq \sup_{x \in \Omega} |f| \cdot \int_{\operatorname{supp} f} |\varphi(x)| \, \mathrm{d}x \,, \\ &< +\infty, \end{aligned} \tag{5.309}$$

and hence the expression is well-defined.

Linearity comes from the properties of the integral. Continuity can be shown just as we've shown well-definedness. ■

To find the specific topology on $\mathscr{D}$ that leaves these as the elements on $\mathscr{D}'$ we refer to [68, pp. 705–712]. Nevertheless, this structure is enough for us to prove every tempered distribution is a distribution.

**Theorem 5.117:**

*Every tempered distribution can be interpreted as a distribution.* □

*Proof:*

Since test functions are smooth and have compact support, they are rapidly decreasing. Therefore, tempered distributions already are linear functionals defined on $\mathscr{D}$. It remains for us to find out whether they are continuous.

Let $\varphi \in \mathscr{S}'$. Theorem 5.85 and Lemma 5.84 on page 188 and on page 190 ensure that there are $k, l \in \mathbb{N}, \kappa \in \mathbb{R}$ such that

$$|\langle \varphi, f \rangle| \leq \lambda \sum_{\substack{|\alpha| \leq \lambda \\ |\beta| \leq l}} \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| \tag{5.310}$$

for every $f \in \mathscr{S}$. In particular, for every $f \in \mathscr{D}$.

Let us now choose $r > 0$ and restrict our attention to functions $f \in \mathscr{D}$ such that $\operatorname{supp} f \subseteq \{x \in \mathbb{R}^n; |x| \leq r\}$. Under these conditions, we know that

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| \leq r^{|\alpha|} \sup_{x \in \mathbb{R}^n} \left| D^\beta f(x) \right|, \tag{5.311}$$

$$|\langle \varphi, f \rangle| \leq \lambda \sum_{\substack{|\alpha|\leq k \\ |\beta|\leq l}} \sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| \leq \lambda \left( \sum_{|\alpha|\leq k} r^{|\alpha|} \right) \sum_{|\beta|\leq l} \sup_{x\in\mathbb{R}^n} \left| D^\beta f(x) \right|,$$

$$\leq \kappa \sum_{|\beta|\leq l} \sup_{x\in\mathbb{R}^n} \left| D^\beta f(x) \right|, \tag{5.312}$$

which concludes the proof. ■

Hence, $\mathscr{D}$ allows for even more distributions than $\mathscr{S}$. Moreover, we can multiply them by any smooth function, for products of smooth functions with compact support with smooth functions are still smooth functions with compact support. Hence, while $e^{x^2}$ does not define a tempered distribution due to issues with convergence, it does define an ordinary distribution. So why are we focusing so much on tempered distributions instead of ordinary distributions?

Once we start dealing with differential equations, a particularly useful tool to have at hand is the Fourier transform, which allows us to turn differential equations into algebraic equations, solve them, and then transform them back. Given a function $f \in \mathscr{D}$, its Fourier transform is given by

$$\hat{f}(\xi) = \int f(x) e^{-i\langle x,\xi\rangle} \, \mathrm{d}\xi \,, \tag{5.313}$$

which in general does not have compact support, and hence is not on $\mathscr{D}$. We would like the Fourier transform to be a continuous map $\mathcal{F} \colon \mathscr{D} \to \mathscr{D}$, but this is not going to happen: the range is simply wrong.

Let us now give $\mathscr{S}$ a try.

**Definition 5.118** [Fourier Transform of Rapidly Decreasing Functions]**:**
Let $f \in \mathscr{S}(\mathbb{R}^n)$. We define its Fourier transform, denoted $\hat{f}$ or $\mathcal{F}f$, through

$$\hat{f}(\xi) \equiv [\mathcal{F}f](\xi) \equiv \int f(x) e^{-i\langle x,\xi\rangle} \, \mathrm{d}x \,, \tag{5.314}$$

where $\langle x, \xi \rangle \equiv \sum_{i=1}^{n} x_i \xi_i$. ♠

*Remark:*

While dealing with Fourier transforms, we shall consider a function taking values in $\mathbb{C}$ to be rapidly decreasing whenever both its real and imaginary parts are.

Notice this is equivalent to Eq. (5.42) on page 135 or Eq. (5.43) on page 135 if we interpret $|\cdot|$ as being the absolute value of complex numbers. Indeed, one has

$$\|\mathrm{Re}[f]\|_{\alpha,\beta}, \|\mathrm{Im}[f]\|_{\alpha,\beta} \leq \|f\|_{\alpha,\beta} \leq \|\mathrm{Re}[f]\|_{\alpha,\beta} + \|\mathrm{Im}[f]\|_{\alpha,\beta} \tag{5.315}$$

due to the usual properties of the absolute value. ♣

The Fourier transform is well defined, since $e^{-i\langle x,\xi\rangle}$ is smooth and polynomially bounded as a real function of $x$ for each fixed $\xi \in \mathbb{R}$, and hence the integrand is indeed integrable.

**Proposition 5.119:**
*The Fourier transform is linear.* □

*Proof:*
Follows from linearity of the integral. ■

Before we prove a few properties of the Fourier transform on $\mathscr{S}$ it will be particularly useful if we prove a lemma concerning differentiation under the integral sign.

**Lemma 5.120:**
*Let $\Omega \subseteq \mathbb{R}^m$ be an open set and consider a function $f\colon \mathbb{R}^n \times \Omega \to \mathbb{R}$ such that*

i. *$\forall\, y \in \Omega$, $x \mapsto f(x,y)$ is integrable;*

ii. *$\forall\, (x,y) \in \mathbb{R}^n \times \Omega$, the partial derivatives $D^\alpha_y f(x,y)$ exist for every $\alpha \in \mathbb{N}^n$;*

iii. *$\forall\, \alpha \in \mathbb{N}^n$, $\exists\, g \in L^1$, $g \geq 0$, such that $g(x) \geq \left| D^\alpha_y f(x,y) \right|$, $\forall\, (x,y) \in \mathbb{R}^n \times \Omega$.*

*It then holds that $y \mapsto \int f(x,y)\,\mathrm{d}x$ is smooth on $\Omega$. Furthermore, $\forall\, y \in \Omega$, $x \mapsto D^\alpha_y f(x,y)$ is $\mu$-integrable and the equation*

$$D^\alpha_y \int f(x,y)\,\mathrm{d}x = \int D^\alpha_y f(x,y)\,\mathrm{d}x \tag{5.316}$$

*holds.* □

*Proof:*
We only need to prove the result for the special case in which $|\alpha| = 1$, for the remaining cases can be obtained from this one through induction. In particular, we may take $\Omega \subseteq \mathbb{R}$, for only one component of $y$ varies while the remaining ones can be held constant. Furthermore, we can take $\Omega$ to be an interval, since all open sets in the real line are arbitrary unions of open intervals.

Let us define $F\colon \Omega \to \mathbb{R}$ through

$$F(y) \equiv \int f(x,y)\,\mathrm{d}x\,. \tag{5.317}$$

Let $(y_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ with $y_n \to y_\infty \in \Omega$. Notice that, as long as $y_n \neq y_\infty$,

$$\frac{F(y_n) - F(y_\infty)}{y_n - y_\infty} = \int \frac{f(x,y_n) - f(x,y_\infty)}{y_n - y_\infty}\,\mathrm{d}x\,. \tag{5.318}$$

We know $y \mapsto f(x,y)$ is differentiable on $\Omega$ for every fixed $x \in \mathbb{R}^n$, since it is a single variable function and its derivative exists. We may employ the Mean Value Theorem to find out there is some $z \in \Omega$ such that

$$\frac{f(x,y_n) - f(x,y_\infty)}{y_n - y_\infty} = \frac{\partial f}{\partial y}(x,z), \tag{5.319}$$

and hence $\exists\, g \in L^1$, $g \geq 0$, such that

$$\left| \frac{f(x, y_n) - f(x, y_\infty)}{y_n - y_\infty} \right| \leq g(x), \tag{5.320}$$

for every $(x, y) \in \mathbb{R}^n \times \Omega$.

This means we may now take the limit on both sides of Eq. (5.318) and employ the Dominated Convergence Theorem to deal with the right hand side. It implies that $x \mapsto \frac{\partial f}{\partial y}(x, y)$ is integrable and

$$\frac{\mathrm{d}}{\mathrm{d}y} \int f(x, y)\,\mathrm{d}x = \int \frac{\partial f}{\partial y}(x, y)\,\mathrm{d}x\,, \tag{5.321}$$

which also implies $y \mapsto \int f(x, y)\,\mathrm{d}x$ is differentiable.

Iterating the argument leads to higher derivatives and the general multivariate result. ■

**Definition 5.121** $[p(D)]$**:**

If $p(x)$ is the polynomial given by $p(x) = \sum_{|\alpha| \leq k} \kappa_\alpha x^\alpha$, we may write $p(D) = \sum_{|\alpha| \leq k} \kappa_\alpha D^\alpha$. ♠

**Proposition 5.122:**

*Let $f\colon \mathbb{R} \to \mathbb{C}$ be rapidly decreasing. Then $\hat{f}$ is smooth and it holds that*

$$p(D_\xi)\mathcal{F}f(\xi) = \mathcal{F}[p(-ix)f]\,(\xi) \tag{5.322}$$

*for any polynomial $p$.* □

*Proof:*

We'll prove the statement in the particular case where $p(x) = x^\alpha$ for some multiindex $\alpha$, since the general case will then follow by linearity.

We want to compute

$$D_\xi^\alpha \hat{f}(\xi) = D_\xi^\alpha \int f(x) e^{-i\langle x, \xi \rangle}\,\mathrm{d}x\,, \tag{5.323}$$

which requires commuting the derivative and the integral. Lemma 5.120 on page 215 allows us to exchange the derivative with the integral sign. Notice that Theorem 5.98 on page 199 ensures the partial derivatives of the integrand are integrable for each fixed $\xi$. Since we mus allow $\xi$ to vary, we can consider the derivative in a small neighborhood of $\xi$ and consider the supremum of the partial derivatives of the integrand with respect to $\xi$.

We'll get to

$$\begin{aligned} D_\xi^\alpha \mathcal{F}f(\xi) &= \int f(x) D_\xi^\alpha e^{-i\langle x, \xi \rangle}\,\mathrm{d}x\,, \\ &= \int (-ix)^\alpha f(x) e^{-i\langle x, \xi \rangle}\,\mathrm{d}x\,, \\ &= \mathcal{F}[(-ix)^\alpha f(x)](\xi), \end{aligned} \tag{5.324}$$

which is the result we desired. ■

**Proposition 5.123:**

*Let $f\colon \mathbb{R} \to \mathbb{C}$ be rapidly decreasing. Then it holds that*

$$\mathcal{F}[p(D_x)f]\,(\xi) = p(i\xi)\mathcal{F}f(\xi) \tag{5.325}$$

*for any polynomial $p$.* □

*Proof:*

Once again, we can prove for the particular case of $p(x) = x^\alpha$ and the general case follows by linearity.

If we integrate by parts we see that

$$\begin{aligned}\mathcal{F}[D_x^\alpha f]\,(\xi) &= \int D_x^\alpha f(x)e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,,\\ &= (-1)^{|\alpha|}\int f(x)D_x^\alpha e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,(i\xi)^\alpha\int f(x)e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,,\\ &= (i\xi)^\alpha\mathcal{F}f(\xi),\end{aligned} \tag{5.326}$$

as desired. ■

**Theorem 5.124:**

*The Fourier transform of rapidly decreasing functions is a continuous map $\mathcal{F}\colon \mathscr{S} \to \mathscr{S}$.* □

*Proof:*

Let us firstly prove that $f \in \mathscr{S} \Rightarrow \hat{f} \in \mathscr{S}$.

We want to prove that, $\forall\,\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{\xi\in\mathbb{R}^n}\left|(i\xi)^\alpha D_\xi^\beta \hat{f}(\xi)\right| < +\infty, \tag{5.327}$$

where we inserted the factors of $i$ for convenience.

Due to Propositions 5.122 and 5.123 on page 216 and on the preceding page we have

$$(i\xi)^\alpha D_\xi^\beta\hat{f}(\xi) = \int D_x^\alpha[(-ix)^\beta f(x)]e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,, \tag{5.328}$$

$$= \int D_x^\alpha[(-ix)^\beta]f(x)e^{-i\langle x,\xi\rangle} + (-ix)^\beta D_x^\alpha f(x)e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,, \tag{5.329}$$

which is the sum of Fourier transforms of rapidly decreasing functions. Hence, if we prove $\sup_{\xi\in\mathbb{R}^n}\left|\hat{f}(\xi)\right| < +\infty$, $\forall\, f \in \mathscr{S}$, the general result holds.

Notice that, $\forall\,\xi \in \mathbb{R}^n$,

$$\begin{aligned}\left|\hat{f}(\xi)\right| &= \left|\int f(x)e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\right|,\\ &\leq \int\left|f(x)e^{-i\langle x,\xi\rangle}\right|\mathrm{d}x\,,\\ &= \int|f(x)|\,\mathrm{d}x\,,\end{aligned}$$

$$= \|f\|_{L^1}, \tag{5.330}$$

which we know is finite due to the hypothesis that $f \in \mathscr{S} \subseteq L^1$. Since this holds for every $\xi \in \mathbb{R}^n$, we find that, $\forall\, f \in \mathscr{S}$,

$$\sup_{\xi \in \mathbb{R}^n} \left|\hat{f}(\xi)\right| \leq \|f\|_{L^1} < +\infty. \tag{5.331}$$

We now proceed to show continuity. From Eq. (5.328) we get that

$$\begin{aligned}
\left\|\hat{f}\right\|_{\alpha,\beta} &= \sup_{\xi \in \mathbb{R}^n} \left|(i\xi)^\alpha D_\xi^\beta \hat{f}(\xi)\right|, \\
&= \sup_{\xi \in \mathbb{R}^n} \left|\int D_x^\alpha[(-ix)^\beta f(x)] e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\right|, \\
&\leq \sup_{\xi \in \mathbb{R}^n} \int \left|D_x^\alpha[(-ix)^\beta f(x)]\right| \mathrm{d}x\,, \\
&= \int \left|D_x^\alpha[(-ix)^\beta f(x)]\right| \mathrm{d}x\,, \\
&= \int \frac{(1+|x|^2)^m}{(1+|x|^2)^m} \left|D_x^\alpha[(-ix)^\beta f(x)]\right| \mathrm{d}x\,.
\end{aligned} \tag{5.332}$$

From the proof to Proposition 5.88 on page 192 we know $\int (1+|x|^2)^{-m}\,\mathrm{d}x$ converges for $m > \frac{n}{2}$. Picking such an $m$, we see there is a constant $\kappa$ such that

$$\left\|\hat{f}\right\|_{\alpha,\beta} \leq \kappa \sup_{x \in \mathbb{R}^n} \left|(1+|x|^2)^m D_x^\alpha[(-ix)^\beta f(x)]\right|, \tag{5.333}$$

and the Leibniz rule ensures there are some constants $l \in \mathbb{N}$, $\kappa_\gamma \in \mathbb{R}$ such that

$$\left\|\hat{f}\right\|_{\alpha,\beta} \leq \sum_{|\gamma|,|\delta| \leq l} \kappa_\gamma \|f\|_{\gamma,\delta}, \tag{5.334}$$

and therefore Theorem 5.49 on page 158 ensures $\mathcal{F}$ is continuous, hence concluding the proof. ■

Proposition 5.123 on page 217 allows us to pick a linear differential equation such as

$$p(D_x)f(x) = g(x), \tag{5.335}$$

Fourier transform it to

$$p(i\xi)\hat{f}(\xi) = \hat{g}(\xi) \tag{5.336}$$

and get the solution

$$\hat{f}(\xi) = \frac{\hat{g}(\xi)}{p(i\xi)}, \tag{5.337}$$

which is great, but not exactly what we were looking for. We would like to get $f$, not $\hat{f}$. Can we recover $f$ from $\hat{f}$? In other words, is $\mathcal{F}$ invertible?

**Definition 5.125** [Fourier Inverse of Rapidly Decreasing Functions]**:**
Let $f \in \mathscr{S}(\mathbb{R}^n)$. We define its Fourier inverse, denoted $\check{f}$ or $\mathcal{F}^{-1}f$, through

$$\check{f}(x) \equiv \left[\mathcal{F}^{-1}f\right](\xi) \equiv \frac{1}{(2\pi)^n} e^{+i\langle x,\xi\rangle}\,\mathrm{d}\xi\,, \tag{5.338}$$

where $\langle x, \xi\rangle = \sum_{i=1}^n x_i\xi_i$. ♠

**Theorem 5.126:**
*The Fourier inverse of rapidly decreasing functions is a continuous linear map $\mathcal{F}^{-1}\colon \mathscr{S} \to \mathscr{S}$. Furthermore, if $f\colon \mathbb{R} \to \mathbb{C}$ is rapidly decreasing, then $\check{f}$ is smooth and one has*

$$p(D_x)\mathcal{F}^{-1}f(x) = \mathcal{F}^{-1}[p(i\xi)f](x) \tag{5.339}$$

*and*

$$\mathcal{F}^{-1}\left[p(D_\xi)f\right](x) = p(-ix)\mathcal{F}^{-1}f(x) \tag{5.340}$$

*for any polynomial $p$.* □

*Proof:*
Follows from the fact that, for every $f \in \mathscr{S}$,

$$\check{f}(x) = \frac{1}{(2\pi)^n}\hat{f}(-x) \tag{5.341}$$

for all $x \in \mathbb{R}^n$. ■

Naturally, one usually does not define an inverse, but rather shows it is an inverse. Hence, let us prove $\mathcal{F}^{-1}$ is indeed the inverse of $\mathcal{F}$.

To show $\mathcal{F}$ is injective is to show it has a left inverse, *id est*, to prove $\mathcal{F}$ is injective we want to show that $\mathcal{F}^{-1}\mathcal{F}f = f$. We could try to do that by attempting to write

$$\begin{aligned}\mathcal{F}^{-1}\mathcal{F}f(x) &= \frac{1}{(2\pi)^n}\int \hat{f}(\xi)e^{+i\langle y,\xi\rangle}\,\mathrm{d}\xi\,,\\ &= \frac{1}{(2\pi)^n}\int f(y)e^{-i\langle y,\xi\rangle}\,\mathrm{d}y\, e^{+i\langle x,\xi\rangle}\,\mathrm{d}\xi\,,\\ &= \frac{1}{(2\pi)^n}\int f(y)e^{+i\langle x-y,\xi\rangle}\,\mathrm{d}y\,\mathrm{d}\xi\,,\end{aligned} \tag{5.342}$$

and then we would proceed to try to change the order of integration and compute $\int e^{+i\langle x-y,\xi\rangle}\,\mathrm{d}\xi$, which unfortunately diverges. Therefore, we should try a different approach: instead of tackling the integral directly, we shall make it behave a little better by introducing an extra factor of $e^{-\frac{|\xi|^2}{m^2}}$ with $m \in \mathbb{N}$, and eventually take the limit $m \to \infty$.

It will be useful for us to first deal with an auxiliary lemma, concerning the properties of the Gaussian $e^{-\frac{|x|^2}{m^2}}$ under Fourier transforms.

**Lemma 5.127:**
$e^{-\frac{|x|^2}{m^2}} \in \mathscr{S}$. □

*Proof:*
$e^{-\frac{|x|^2}{m^2}}$ is smooth, since it is the composition of an exponential and a polynomial. Its derivatives are simply products of itself with polynomials, and hence to prove it is rapidly decreasing it suffices to show that $\left\| e^{-\frac{|x|^2}{m^2}} \right\|_{\alpha,0} < +\infty, \forall\, \alpha \in \mathbb{N}^n$. Since all polynomials are polynomially bounded due to Proposition 5.95 on page 197, we only need to worry with showing

$$\sup_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^l e^{-\frac{|x|^2}{m^2}} \right| < +\infty \tag{5.343}$$

for all $l \in \mathbb{N}$. This has reduced to a one-dimensional problem. Namely, the one of showing that

$$\sup_{r\geq 0} \left| (1+r)^l e^{-\frac{r}{m^2}} \right| < +\infty, \tag{5.344}$$

which we know to hold, since exponentials decay faster than any polynomial. Another argument would be that the supremum is finite in every compact (for its argument is continuous) and to use L'Hôpital's rule at infinity. ■

**Lemma 5.128:**

$$\mathcal{F}\left[e^{-\frac{|x|^2}{m^2}}\right](\xi) = \left(m^2\pi\right)^{\frac{n}{2}} e^{-\frac{m^2|\xi|^2}{4}}.$$ □

*Proof:*
We know the Fourier transform is well defined, since $e^{-\frac{|x|^2}{m^2}}$ is rapidly decreasing. We want to compute

$$\mathcal{F}\left[e^{-\frac{|x|^2}{m^2}}\right](\xi) = \int e^{-i\langle x,\xi\rangle - \frac{|x^2|}{m^2}}\, \mathrm{d}^n x\,. \tag{5.345}$$

Notice that the $n$-dimensional case is just the product of $n$ one-dimensional cases. Hence, we'll focus on computing

$$\mathcal{F}\left[e^{-\frac{x^2}{m^2}}\right](\xi) = \int e^{-ix\xi - \frac{x^2}{m^2}}\, \mathrm{d}x\,. \tag{5.346}$$

Notice that

$$D_x e^{-\frac{x^2}{m^2}} = -\frac{2x}{m^2} e^{-\frac{x^2}{m^2}}. \tag{5.347}$$

Hence, using Propositions 5.122 and 5.123 on page 216 and on page 217 leads to

$$\begin{aligned}
D_\xi \mathcal{F}\left[e^{-\frac{x^2}{m^2}}\right](\xi) &= \mathcal{F}\left[-ixe^{-\frac{x^2}{m^2}}\right](\xi), \\
&= \frac{im^2}{2}\mathcal{F}\left[D_x e^{-\frac{x^2}{m^2}}\right](\xi), \\
&= -\frac{m^2\xi}{2}\mathcal{F}\left[e^{-\frac{x^2}{m^2}}\right](\xi),
\end{aligned} \tag{5.348}$$

which is a differential equation for the Fourier transform. It can be solved with the aid of an integrating factor to yield

$$\mathcal{F}\left[e^{-\frac{x^2}{m^2}}\right](\xi) = \kappa e^{-\frac{m^2\xi^2}{4}}, \tag{5.349}$$

for some constant $\kappa$. To fix $\kappa$, we may notice that

$$\begin{aligned}
\kappa &= \mathcal{F}\left[e^{-\frac{x^2}{m^2}}\right](0), \\
&= \int e^{-\frac{x^2}{m^2}}\,\mathrm{d}x\,, \\
&= \sqrt{m^2\pi},
\end{aligned} \tag{5.350}$$

and the claim follows. ■

We will also need another auxiliary lemma to deal with the integrals involving $e^{-\frac{x^2}{m^2}}$ that will occur on our derivation.

**Lemma 5.129:**

*Let $g \in \mathscr{S}$ be a spherically symmetric function* — id est, *a function $g\colon \mathbb{R}^n \to \mathbb{R}$ that depends on $x \in \mathbb{R}^n$ only through $|x|$ — such that $\int g(x)\,\mathrm{d}x = 1$. We denote $g_m(x) \equiv mg(mx), \forall\, m \in \mathbb{N}$. Let $f \in \mathscr{S}$. Then the sequence $(f * g_m)_{m\in\mathbb{N}}$, defined through*

$$(f * g_m)(x) = \int f(x-y)g_m(y)\,\mathrm{d}y \tag{5.351}$$

*converges uniformly to $f(x)$.* □

*Proof:*

The product of rapidly decreasing functions is rapidly decreasing, as one can check with the aid of the Leibniz rule. In particular, the expression $(f * g_m)(x)$ is well-defined.

Let us begin by noticing that $\int g_m(y)\,\mathrm{d}y = 1$, since

$$\begin{aligned}
\int g_m(y)\,\mathrm{d}y &= \int g(my)\,\mathrm{d}(my)\,, \\
&= 1.
\end{aligned} \tag{5.352}$$

With this in mind, we see that

$$(f * g_m)(x) - f(x) = \int [f(x-y) - f(x)] g_m(y)\, \mathrm{d}y\,, \tag{5.353}$$

and hence our goal is to prove that $\sup_{x\in\mathbb{R}^n} \left|\int [f(x-y) - f(x)] g_m(y)\, \mathrm{d}y\right|$ gets arbitrarily small as $m \to \infty$.

We begin with the proof for dimension $n > 1$. Notice that

$$\begin{aligned}
|(f * g_m)(x) - f(x)| &\leq \int |f(x-y) - f(x)||g_m(y)|\, \mathrm{d}y\,,\\
&\leq 2\|f\|_{0,0} \int |g_m(y)|\, \mathrm{d}y\,,\\
&\leq 2\|f\|_{0,0} \int |g(mr)| r^{n-1}\, \mathrm{d}r\, \mathrm{d}\Omega_n\,,\\
&\leq \frac{2\|f\|_{0,0}}{m^{n-1}} \int |g(mr)|(mr)^{n-1}\, \mathrm{d}r\, \mathrm{d}\Omega_n\,,\\
&\leq \frac{2\|f\|_{0,0}\|g\|_{L^1}}{m^{n-1}}, \qquad (5.354)\\
\sup_{x\in\mathbb{R}^n} |(f * g_m)(x) - f(x)| &\leq \frac{2\|f\|_{0,0}\|g\|_{L^1}}{m^{n-1}}, \qquad (5.355)\\
\lim_{m\to+\infty} \sup_{x\in\mathbb{R}^n} |(f * g_m)(x) - f(x)| &= 0, \qquad (5.356)
\end{aligned}$$

as we desired.

This proof fails for $n = 1$, since the estimate won't vanish when we take $m \to \infty$. Let us consider then this particular case. Notice that our condition of $g$ being spherically symmetric now reduces to $g$ being even.

The quantity of interest is now

$$|(f * g_m)(x) - f(x)| \leq \int |f(x-y) - f(x)||g_m(y)|\, \mathrm{d}y\,. \tag{5.357}$$

Since $f$ is differentiable and has a bounded derivative, it is Lipschitz continuous and, as a consequence, it is uniformly continuous. Therefore, $\forall\, \epsilon > 0, \exists\, \delta > 0; \forall\, x \in \mathbb{R}, |y| < \delta \Rightarrow |f(x-y) - f(x)| < \epsilon$. With this in mind, we write

$$\begin{aligned}
|(f * g_m)(x) - f(x)| &\leq 2\epsilon \int_0^{\delta} |g_m(y)|\, \mathrm{d}y + 4\|f\|_{0,0} \int_{\delta}^{+\infty} |g_m(y)|\, \mathrm{d}y\,,\\
&\leq 2\epsilon \int_0^{m\delta} |g(y)|\, \mathrm{d}y + 4\|f\|_{0,0} \int_{m\delta}^{+\infty} |g(y)|\, \mathrm{d}y\,,\\
&\leq \epsilon\|g\|_{L^1} + 4\|f\|_{0,0} \int_{m\delta}^{+\infty} |g(y)|\, \mathrm{d}y\,, \qquad (5.358)\\
\sup_{x\in\mathbb{R}} |(f * g_m)(x) - f(x)| &\leq \epsilon\|g\|_{L^1} + 4\|f\|_{0,0} \int_{m\delta}^{+\infty} |g(y)|\, \mathrm{d}y\,, \qquad (5.359)\\
\lim_{m\to+\infty} \sup_{x\in\mathbb{R}} |(f * g_m)(x) - f(x)| &\leq \epsilon\|g\|_{L^1}. \qquad (5.360)
\end{aligned}$$

Since $\epsilon$ is arbitrary, $\lim_{m\to+\infty}\sup_{x\in\mathbb{R}}|(f*g_m)(x)-f(x)|=0$, which concludes the proof. ∎

**Theorem 5.130:**

*The Fourier transform of rapidly decreasing functions, $\mathcal{F}\colon \mathscr{S}\to\mathscr{S}$, is bijective and has $\mathcal{F}^{-1}$ as its inverse.* □

*Proof:*

Let us consider the integral

$$f_m(x)\equiv\frac{1}{(2\pi)^n}\int f(y)e^{+i\langle x-y,\xi\rangle}e^{-\frac{|\xi|^2}{m^2}}\,\mathrm{d}y\,\mathrm{d}\xi=\mathcal{F}^{-1}\left[e^{-\frac{|\xi|^2}{m^2}}\mathcal{F}f\right]. \tag{5.361}$$

$e^{-\frac{|\xi|^2}{m^2}}\mathcal{F}f\in\mathscr{S}$, and therefore $f_m\in\mathscr{S}$ due to Theorem 5.126 on page 219. Since the integrand is measurable in the product $\sigma$-algebra (it is continuous), we get from Tonelli's Theorem[22, p. 67] that

$$\begin{aligned}\int\left|f(y)e^{+i\langle x-y,\xi\rangle}e^{-\frac{|\xi|^2}{m^2}}\right|\mathrm{d}(y\otimes\xi)&=\int\left|f(y)e^{-\frac{|\xi|^2}{m^2}}\right|\mathrm{d}(y\otimes\xi)\,,\\&=\int\left|f(y)e^{-\frac{|\xi|^2}{m^2}}\right|\mathrm{d}y\,\mathrm{d}\xi\,,\\&=\int\left|f(y)e^{-\frac{|\xi|^2}{m^2}}\right|\mathrm{d}\xi\,\mathrm{d}y\,,\end{aligned} \tag{5.362}$$

and the last two expressions are known for us to be finite, since $f(y)\in\mathscr{S}$, $e^{-\frac{|\xi|^2}{m^2}}\in\mathscr{S}$.

Given this, Fubini's Theorem[22, p. 67] allows us to exchange the order of integration and get to

$$f_m(x)=\frac{1}{(2\pi)^n}\int f(y)e^{+i\langle x-y,\xi\rangle}e^{-\frac{|\xi|^2}{m^2}}\,\mathrm{d}\xi\,\mathrm{d}y\,, \tag{5.363}$$

meaning we should compute the integral

$$\int e^{i\sum_{j=1}^n z_j\xi_j-\frac{1}{m^2}\sum_{j=1}^n\xi_j^2}\,\mathrm{d}^n\xi\,, \tag{5.364}$$

where we've written $\mathrm{d}^n\xi$ instead of $\mathrm{d}\xi$ to make explicit the dimensionality of the space in which we are integrating and $z=x-y$ for simplicity. By completing squares we get to

$$\begin{aligned}\int e^{i\sum_{j=1}^n z_j\xi_j-\frac{1}{m^2}\sum_{j=1}^n\xi_j^2}\,\mathrm{d}^n\xi&=e^{-\frac{m^2|z|^2}{4}}\int e^{-\sum_{j=1}^n\left(\frac{\xi_j}{m}-\frac{imz_j}{2}\right)^2}\,\mathrm{d}^n\xi\,,\\&=\left(m^2\pi\right)^{\frac{n}{2}}e^{-\frac{m^2|z|^2}{4}}\,,\end{aligned} \tag{5.365}$$

which implies

$$f_m(x)=\left(\frac{m^2}{4\pi}\right)^{\frac{n}{2}}\int f(y)e^{-\frac{m^2|x-y|^2}{4}}\,\mathrm{d}y\,. \tag{5.366}$$

A change of variables $y \mapsto x - y$ puts us in condition to apply Lemma 5.129 on page 222 and conclude that

$$\lim_{m\to+\infty} f_m(x) = f(x) \tag{5.367}$$

uniformly, and hence pointwise.

On the other hand, notice that the Dominated Convergence Theorem tells us that

$$\begin{aligned}
\lim_{m\to+\infty} f_m(x) &= \frac{1}{(2\pi)^n} \lim_{m\to+\infty} \int e^{+i\langle x,\xi\rangle} e^{-\frac{|\xi|^2}{m^2}} \hat{f}(y)\,\mathrm{d}\xi\,,\\
&= \frac{1}{(2\pi)^n} \int \lim_{m\to+\infty} e^{+i\langle x,\xi\rangle} e^{-\frac{|\xi|^2}{m^2}} \hat{f}(y)\,\mathrm{d}\xi\,,\\
&= \frac{1}{(2\pi)^n} \int e^{+i\langle x,\xi\rangle} \hat{f}(y)\,\mathrm{d}\xi\,,\\
&= \mathcal{F}^{-1}\mathcal{F}f(x)
\end{aligned} \tag{5.368}$$

pointwise. Therefore, we conclude that

$$\mathcal{F}^{-1}\mathcal{F}f(x) = f(x) \tag{5.369}$$

for every $f \in \mathscr{S}$.

This allows us to conclude that $\mathcal{F}$ is injective, $\mathcal{F}^{-1}$ being its left-inverse. We remain to prove it is surjective. To do so, let $F \in \mathscr{S}$ and define $f = \mathcal{F}^{-1}F$. One then has

$$\begin{aligned}
\mathcal{F}f(\xi) &= \int f(x) e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,,\\
&= \frac{1}{(2\pi)^n} \int F(\eta) e^{i\langle x,\eta\rangle}\,\mathrm{d}\eta\, e^{-i\langle x,\xi\rangle}\,\mathrm{d}x\,,\\
&= \frac{1}{(2\pi)^n} \int \mathcal{F}F(-x) e^{i\langle -x,\xi\rangle}\,\mathrm{d}x\,,\\
&= \mathcal{F}^{-1}\mathcal{F}F(\xi),\\
&= F(\xi),
\end{aligned} \tag{5.370}$$

where we employed our knowledge of the fact that $\mathcal{F}^{-1}$ is the left-inverse of $\mathcal{F}$. Hence, we see that $\mathcal{F}$ is also surjective, hence bijective. Since $\mathcal{F}^{-1}$ is its left-inverse, it is its inverse, as claimed. ■

The occurrence of expressions such as $\int f(x-y)g(y)\,\mathrm{d}y$ in Lemma 5.129 and Theorem 5.130 on page 222 and on page 223 is frequent within Fourier Analysis and, therefore, deserves a definition.

**Definition 5.131** [Convolution of Rapidly Decreasing Functions]:

Let $f, g \in \mathscr{S}$. We'll define their *convolution product* as the function $f * g \colon \mathbb{R}^n \to \mathbb{R}$ given by

$$(f * g)(x) = \int f(x-y)g(y)\,\mathrm{d}y\,, \tag{5.371}$$

for every $x \in \mathbb{R}^n$. ♠

**Proposition 5.132:**

*Given $f, g, h \in \mathscr{S}$,*

i. *the map $g \mapsto f * g$ maps $\mathscr{S}$ to itself continuously;*

ii. *$f * g = g * f$;*

iii. *$f * (g * h) = (f * g) * h$;*

iv. *$D^\alpha (f * g) = D^\alpha f * g = f * D^\alpha g$.* □

*Proof:*

$f * g$ is well-defined due to the fact that $f, g \in \mathscr{S} \subseteq L^p$, $p \in [1, +\infty]$ (Theorem 5.92 on page 195) and hence Hölder's inequality ensures convergence of the integral.

We still must prove $f * g$ is rapidly decreasing. Lemma 5.120 on page 215 allows us to conclude that $f * g$ is smooth. We still must check whether $\|f * g\|_{\alpha,\beta} < +\infty$.

Notice that, given $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned}\left|x^\alpha D^\beta (f * g)(x)\right| &\leq \int \left|x^\alpha D^\beta f(x-y)\right| |g(y)| \, \mathrm{d}y \,, \\ &\leq \int \|f\|_{\alpha,\beta} |g(y)| \, \mathrm{d}y \,, \\ &= \|f\|_{\alpha,\beta} \|g\|_{L^1} \,, \qquad (5.372) \\ \|f * g\|_{\alpha,\beta} &\leq \|f\|_{\alpha,\beta} \|g\|_{L^1} \,, \qquad (5.373)\end{aligned}$$

and hence $f * g \in \mathscr{S}$. Continuity is ensured by Theorem 5.49 on page 158.

$f * g = g * f$ through a coordinate change:

$$\begin{aligned}(f * g)(x) &= \int f(x-y) g(y) \, \mathrm{d}y \,, \\ &= \int f(z) g(x-z) \, \mathrm{d}z \,, \\ &= (g * f)(x). \qquad (5.374)\end{aligned}$$

As for associativity we have

$$\begin{aligned}[f * (g * h)](x) &= \int f(x-y)(g * h)(y) \, \mathrm{d}y \,, \\ &= \int f(x-y) g(y-z) h(z) \, \mathrm{d}z \, \mathrm{d}y \,, \\ &= \int f(x-y) g(y-z) h(z) \, \mathrm{d}y \, \mathrm{d}z \,, \\ &= \int f(x-w-z) g(w) h(z) \, \mathrm{d}w \, \mathrm{d}z \,, \\ &= \int (f * g)(x-z) h(z) \, \mathrm{d}z \,, \\ &= [(f * g) * h](x). \qquad (5.375)\end{aligned}$$

We now must only deal with the result concerning differentiation. Lemma 5.120 on page 215 yields

$$
\begin{aligned}
D_x^\alpha (f * g)(x) &= D_x^\alpha \int f(x-y) g(y)\, \mathrm{d}y\,, \\
&= \int D_x^\alpha f(x-y) g(y)\, \mathrm{d}y\,, \\
&= (D_x^\alpha f * g)(x)
\end{aligned}
\tag{5.376}
$$

and a similar argument can be applied to obtain $D^\alpha(f * g) = f * D^\alpha g$ by employing the fact that $f * g = g * f$. ■

The convolution product arises in Fourier analysis due to the fact that it reflects, on Fourier space, the effect of taking a product. Let us state this more precisely.

**Theorem 5.133:**
*Let $f, g \in \mathscr{S}$. Then*

$$
\mathcal{F}[fg] = (2\pi)^{-n} \hat{f} * \hat{g}, \qquad \mathcal{F}[f * g] = \hat{f}\hat{g}, \tag{5.377}
$$

$$
\mathcal{F}^{-1}[fg] = \check{f} * \check{g}, \qquad \mathcal{F}^{-1}[f * g] = (2\pi)^n \check{f}\check{g}, \tag{5.378}
$$

id est, *the Fourier transform and the Fourier take products to convolutions and vice-versa.* □

*Proof:*
We have

$$
\begin{aligned}
\mathcal{F}[f * g](\xi) &= \int (f * g)(x) e^{-i\langle x, \xi\rangle}\, \mathrm{d}x\,, \\
&= \int f(x-y) g(y) e^{-i\langle x, \xi\rangle}\, \mathrm{d}y\, \mathrm{d}x\,, \\
&= \int f(x-y) g(y) e^{-i\langle x, \xi\rangle}\, \mathrm{d}x\, \mathrm{d}y\,, \\
&= \int f(z) g(y) e^{-i\langle z+y, \xi\rangle}\, \mathrm{d}z\, \mathrm{d}y\,, \\
&= \hat{f}(\xi) \int g(y) e^{-i\langle y, \xi\rangle}\, \mathrm{d}y\,, \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}
\tag{5.379}
$$

Since $\hat{f}(\xi) = (2\pi)^n \check{f}(-\xi), \forall f \in \mathscr{S}$, it follows that

$$
\mathcal{F}^{-1}[f * g](\xi) = (2\pi)^n \check{f}(\xi)\check{g}(\xi). \tag{5.380}
$$

Furthermore, applying $\mathcal{F}$ to both sides of Eq. (5.380) leads to

$$
(f * g)(x) = (2\pi)^n \mathcal{F}[\check{f}\check{g}](x). \tag{5.381}
$$

Since $\mathcal{F}$ is surjective, we can take $\check{f}$ and $\check{g}$ to be generic functions of $\mathscr{S}$ and get to

$$
\mathcal{F}[fg](x) = (2\pi)^{-n} (\hat{f} * \hat{g})(x). \tag{5.382}
$$

Using $\hat{f}(x) = (2\pi)^n \check{f}(x)$ once more leads to

$$\mathcal{F}^{-1}[fg](x) = (\check{f} * \check{g})(x), \tag{5.383}$$

concluding the proof. ■

**Corollary 5.134:**
*Fix $f \in \mathscr{S}$. The map $g \mapsto f * g$ is continuous.* □

*Proof:*
Lemma 5.96 on page 199 ensures that, it $f$ is any smooth function with all of its derivatives being polynomially bounded, the map $g \mapsto f \cdot g$ is continuous. In particular, notice we can have $f \in \mathscr{S}$: elements of $\mathscr{S}$ are always smooth and have derivatives bounded by constants.

We also know the Fourier transform is a homeomorphism, due to Theorems 5.124, 5.126 and 5.130 on page 217, on page 219 and on page 223. Theorem 5.133 on the preceding page then teaches us how to write any convolution as a composition of Fourier transforms, inverses and usual products. Hence, convolution is a composition of continuous mappings, being itself continuous as a consequence. ■

With the essential theory of Fourier analysis built on $\mathscr{S}$, we can extend it to $\mathscr{S}'$ in a straightforward way by exploiting the behavior of distributions induced by functions. Let $f, g \in \mathscr{S}$. Notice that

$$\begin{aligned}
\langle \mathcal{F} f, g \rangle &= \int \hat{f}(x) g(x) \,\mathrm{d}x\,, \\
&= \int f(\xi) e^{-i\langle x, \xi \rangle} g(x) \,\mathrm{d}\xi \,\mathrm{d}x\,, \\
&= \int f(\xi) e^{-i\langle x, \xi \rangle} g(x) \,\mathrm{d}x \,\mathrm{d}\xi\,, \\
&= \int f(\xi) \hat{g}(\xi) \,\mathrm{d}\xi\,, \\
&= \langle f, \mathcal{F} g \rangle\,,
\end{aligned} \tag{5.384}$$

where Tonelli's and Fubini's theorems can be used to exchange the order of integration.

Given this, we define

**Definition 5.135** [Fourier Transform and Fourier Inverse of Tempered Distributions]:
Let $\varphi \in \mathscr{S}'$. We define the *Fourier transform* of $\varphi$, denoted $\hat{\varphi}$ or $\mathcal{F}\varphi$, and the *Fourier inverse* of $\varphi$, $\check{\varphi}$ or $\mathcal{F}^{-1}\varphi$, through

$$\langle \mathcal{F}\varphi, f \rangle = \langle \varphi, \mathcal{F} f \rangle\,, \tag{5.385}$$

and

$$\left\langle \mathcal{F}^{-1}\varphi, f \right\rangle = \left\langle \varphi, \mathcal{F}^{-1} f \right\rangle \tag{5.386}$$

for every $f \in \mathscr{S}$. ♠

This is well-defined thanks to Theorems 5.124 and 5.126 on page 217 and on page 219. The Fourier inverse is the inverse of the Fourier transform in the space of tempered distributions, since

$$\begin{aligned}\left\langle \mathcal{F}^{-1}\mathcal{F}\varphi, f\right\rangle &= \left\langle \mathcal{F}\varphi, \mathcal{F}^{-1}f\right\rangle, \\ &= \left\langle \varphi, \mathcal{F}\mathcal{F}^{-1}f\right\rangle, \\ &= \langle \varphi, f\rangle . \end{aligned} \tag{5.387}$$

**Proposition 5.136:**

*If $\mathscr{S}'$ is equipped with the weak topology induced by $\mathscr{S}$, the Fourier transform of tempered distributions is a linear homeomorphism $\mathcal{F}\colon \mathscr{S}' \to \mathscr{S}'$ corresponding to the unique continuous extension of $\mathcal{F}\colon \mathscr{S} \to \mathscr{S}'$.* □

*Proof:*

The fact that $\mathcal{F}$ is a linear bijection $\mathcal{F}\colon \mathscr{S}' \to \mathscr{S}'$ comes from the properties of distributions and of the Fourier transform on $\mathscr{S}$, particularly Theorem 5.124 on page 217.

We'll focus on proving continuity of $\mathcal{F}$. $\mathcal{F}^{-1}$ is analogous. Suppose one has a net $(\varphi_\alpha)_{\alpha\in I} \in \mathscr{S}'^I$ with $\varphi_\alpha \to \varphi \in \mathscr{S}'$ weakly. This is equivalent to stating that $\langle \varphi_\alpha, f\rangle \to \langle \varphi, f\rangle, \forall f \in \mathscr{S}$. Notice then that

$$\langle \mathcal{F}\varphi_\alpha, f\rangle = \langle \varphi_\alpha, \mathcal{F}f\rangle \to \langle \varphi, \mathcal{F}f\rangle = \langle \mathcal{F}\varphi, f\rangle , \tag{5.388}$$

meaning $\mathcal{F}$ takes convergent nets to convergent nets and, as such, is continuous (follows from Theorem 3.123 on page 83). ■

**Proposition 5.137:**

$$\mathcal{F}\delta = 1.$$ □

*Proof:*

$$\begin{aligned}\langle \mathcal{F}\delta, f\rangle &= \langle \delta, \mathcal{F}f\rangle , \\ &= \mathcal{F}f(0), \\ &= \int f(x)e^0 \,\mathrm{d}x , \\ &= \int 1 f(x)\,\mathrm{d}x , \\ &= \langle 1, f\rangle , \end{aligned} \tag{5.389}$$

$\forall f \in \mathscr{S}$. ■

**Corollary 5.138:**

$$\delta = \mathcal{F}^{-1}1.$$ □

If we pick the expression $\delta = \mathcal{F}^{-1}1$ and write in terms of generalized functions, we get to

$$\delta\,(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x,\xi\rangle}\,\mathrm{d}\xi\,. \tag{5.390}$$

Neither side of the equation makes sense on its own, but in the middle of computations they allow us to move around with ease.

Given convolutions on $\mathscr{S}$, we can define $*\colon \mathscr{S} \times \mathscr{S}' \to \mathscr{S}'$ in a similar way. We begin by investigating how they behave for distributions induced by functions. Let $f, g, h \in \mathscr{S}$. Then

$$\begin{aligned}
\langle f * g, h\rangle &= \int (f*g)(x)h(x)\,\mathrm{d}x\,,\\
&= \int f(x-y)g(y)h(x)\,\mathrm{d}y\,\mathrm{d}x\,,\\
&= \int f(x-y)g(y)h(x)\,\mathrm{d}x\,\mathrm{d}y\,, &(5.391)\\
&= \int f^{\neg}(y-x)g(y)h(x)\,\mathrm{d}x\,\mathrm{d}y\,,\\
&= \int g(y)(f^{\neg}*h)(y)\,\mathrm{d}y\,,\\
&= \langle g, f^{\neg}*h\rangle\,, &(5.392)
\end{aligned}$$

where we introduced the notation

$$f^{\neg}(x) = f(-x). \tag{5.393}$$

With this in mind, the definition for the general case is a mere repetition.

**Definition 5.139** [Convolution of a Rapidly Decreasing Function with a Tempered Distribution]**:**

Let $\varphi \in \mathscr{S}'$, $f \in \mathscr{S}$. We define the *convolution product* of $f$ and $\varphi$, $f * \varphi$, through

$$\langle f*\varphi, g\rangle = \langle \varphi, f^{\neg}*g\rangle\,, \tag{5.394}$$

for all $g \in \mathscr{S}$. ♠

**Proposition 5.140:**

*The convolution product of a rapidly decreasing function with a tempered distribution is a tempered distribution. Furthermore, if $f \in \mathscr{S}$ and $\varphi \in \mathscr{S}'$, then the formula*

$$D^\alpha(f*\varphi) = D^\alpha f * \varphi = f * D^\alpha\varphi \tag{5.395}$$

*holds.* □

*Proof:*

Proposition 5.132 on page 225 ensures convolution with a fixed function is a continuous mapping from $\mathscr{S}$ into itself. Linearity comes from the fact that convolution distributes over pointwise addition, as one can check. ■

**Proposition 5.141:**
*Let $f, g \in \mathscr{S}$, $\varphi \in \mathscr{S}'$. Then $f * (g * \varphi) = (f * g) * \varphi$.* □

*Proof:*
For any $h \in \mathscr{S}$, Proposition 5.132 on page 225 allows us to see that

$$\begin{aligned}\langle f * (g * \varphi), h\rangle &= \langle g * \varphi, f^{\neg} * h\rangle\,,\\ &= \langle \varphi, g^{\neg} * (f^{\neg} * h)\rangle\,,\\ &= \langle \varphi, (g^{\neg} * f^{\neg}) * h\rangle\,.\end{aligned} \tag{5.396}$$

However,

$$\begin{aligned}(g^{\neg} * f^{\neg})(x) &= \int g^{\neg}(x-y) f^{\neg}(y)\,\mathrm{d}y\,,\\ &= \int g(y-x) f(-y)\,\mathrm{d}y\,,\\ &= \int g(-x-y) f(y)\,\mathrm{d}y\,,\\ &= (g * f)^{\neg}(x).\end{aligned} \tag{5.397}$$

Hence,

$$\begin{aligned}\langle f * (g * \varphi), h\rangle &= \langle \varphi, (g * f)^{\neg} * h\rangle\,,\\ &= \langle \varphi, (f * g)^{\neg} * h\rangle\,,\\ &= \langle (f * g) * \varphi, h\rangle\,,\end{aligned} \tag{5.398}$$

concluding the proof. ■

**Proposition 5.142:**
*Let $f \in \mathscr{S}$, $\varphi \in \mathscr{S}'$. Then $D^\alpha(f * \varphi) = D^\alpha f * \varphi = f * D^\alpha \varphi$.* □

*Proof:*
Let $g \in \mathscr{S}$. Notice how

$$\begin{aligned}\langle D^\alpha(f * \varphi), g\rangle &= (-1)^{|\alpha|}\langle f * \varphi, D^\alpha g\rangle\,,\\ &= (-1)^{|\alpha|}\langle \varphi, f^{\neg} * D^\alpha g\rangle\,.\end{aligned} \tag{5.399}$$

Proposition 5.132 on page 225 then allow us to write

$$\begin{aligned}\langle D^\alpha(f * \varphi), g\rangle &= (-1)^{|\alpha|}\langle \varphi, D^\alpha(f^{\neg} * g)\rangle\,,\\ &= \langle D^\alpha \varphi, f^{\neg} * g\rangle\,,\\ &= \langle f * D^\alpha \varphi, g\rangle\,,\end{aligned} \tag{5.400}$$

which proves $D^\alpha(f * \varphi) = f * D^\alpha \varphi$.

Moreover,

$$\begin{aligned}
\langle D^\alpha (f * \varphi), g\rangle &= (-1)^{|\alpha|} \langle \varphi, D^\alpha (f^\neg * g)\rangle\,,\\
&= (-1)^{|\alpha|} \langle \varphi, D^\alpha (f^\neg * g)\rangle\,,\\
&= \langle \varphi, (D^\alpha f)^\neg * g\rangle\,,\\
&= \langle D^\alpha f * \varphi, g\rangle\,,
\end{aligned} \tag{5.401}$$

which concludes the proof. ■

**Lemma 5.143:**

*Let $\varphi \in \mathscr{S}'$, $f \in \mathscr{S}$, $a \in \mathbb{R}^n$. It holds that*

$$(T_a f) * \varphi = T_a (f * \varphi).$$ □

*Proof:*

Let $g \in \mathscr{S}$. For simplicity, we shall aid the proof with generalized function notation. Notice how

$$\begin{aligned}
\langle (T_a f) * \varphi, g\rangle &= \int [(T_a f) * \varphi] (x) g(x) \,\mathrm{d}x\,,\\
&= \int (T_a f)(x-y)\varphi(y) g(x) \,\mathrm{d}y \,\mathrm{d}x\,,\\
&= \int f(x-a-y)\varphi(y) g(x) \,\mathrm{d}y \,\mathrm{d}x\,,\\
&= \int f(x-a-y)\varphi(y) g(x) \,\mathrm{d}x \,\mathrm{d}y\,,\\
&= \int f(x-y)\varphi(y) g(x+a) \,\mathrm{d}x \,\mathrm{d}y\,,\\
&= \int f^\neg(y-x)\varphi(y) g(x+a) \,\mathrm{d}x \,\mathrm{d}y\,,\\
&= \int \varphi(y) [f^\neg * T_{-a} g] (y) \,\mathrm{d}y\,,\\
&= \langle \varphi, f^\neg * T_{-a} g\rangle\,,\\
&= \langle f * \varphi, T_{-a} g\rangle\,,\\
&= \langle T_a (f * \varphi), g\rangle\,,
\end{aligned} \tag{5.402}$$

which is the result we desired. ■

**Proposition 5.144:**

*The convolution product of a rapidly decreasing function with a tempered distribution is a smooth function with all its derivatives being polynomially bounded.* □

*Proof:*

Let $f \in \mathscr{S}$ and $s \colon \mathbb{R}^n \to \mathbb{R}$ be a polynomially bounded continuous function. Theorem 5.98 on page 199 ensures it induces a tempered distribution. Notice now that, $\forall\, g \in \mathscr{S}$,

$$\langle f * s, g\rangle = \langle s, f^\neg * g\rangle\,,$$

$$
\begin{aligned}
&= \int s(x) f(y-x) g(y)\, \mathrm{d}y\, \mathrm{d}x\,, \\
&= \int s(x) f(y-x) g(y)\, \mathrm{d}x\, \mathrm{d}y\,, \\
&= \int \left\langle s_x, T_{-y} f_x^{\neg} \right\rangle g(y)\, \mathrm{d}y\,, \\
&= \left\langle \left\langle s_x, T_{-y} f_x^{\neg} \right\rangle, g_y \right\rangle .
\end{aligned} \tag{5.403}
$$

Let $\varphi \in \mathscr{S}'$. Theorem 5.108 on page 211 ensures the existence of a polynomially bounded continuous function $s\colon \mathbb{R}^n \to \mathbb{R}$ such that $\varphi = D^\alpha s$ for an appropriate multiindex $\alpha$. One then has, by using Proposition 5.142 and Eq. (5.403) on the previous page and on this page,

$$
\begin{aligned}
\langle f * \varphi, g \rangle &= \langle f * D^\alpha s, g \rangle\,, \\
&= \langle D^\alpha f * s, g \rangle\,, \\
&= \langle D^\alpha f * s, g \rangle\,, \\
&= \left\langle \left\langle s_x, T_{-y} (D_x^\alpha f)_x^{\neg} \right\rangle, g_y \right\rangle, \\
&= (-1)^{|\alpha|} \left\langle \left\langle s_x, D_x^\alpha \left( T_{-y} f_x^{\neg} \right) \right\rangle, g_y \right\rangle, \\
&= \left\langle \left\langle \varphi_x, T_{-y} f_x^{\neg} \right\rangle, g_y \right\rangle .
\end{aligned} \tag{5.404}
$$

Now we just need to prove that $\left\langle \varphi_x, T_{-y} f_x^{\neg} \right\rangle$ is a smooth function of $y$ with all its derivatives being polynomially bounded. Theorem 5.108 on page 211 ensures the existence of a continuous bounded function $h \in BC(\mathbb{R}^n)$, some number $m \in \mathbb{N}$ and a multiindex $\alpha$ such that

$$
\left\langle \varphi_x, T_{-y} f_x^{\neg} \right\rangle = \int h(x) \left(1 + \|x\|^2\right)^m (D^\alpha f)(y-x)\, \mathrm{d}x\,. \tag{5.405}
$$

$D^\alpha f \in \mathscr{S}$ and Lemma 5.120 on page 215 imply smoothness of $\left\langle \varphi_x, T_{-y} f_x^{\neg} \right\rangle$ as a function of $y$.

Furthermore, notice that

$$
\begin{aligned}
\left| \left\langle \varphi_x, T_{-y} f_x^{\neg} \right\rangle \right| &\leq \|h\|_\infty \int \left(1 + \|x\|^2\right)^m (D^\alpha f)(y-x)\, \mathrm{d}x\,, \\
&= \|h\|_\infty \int \left(1 + \|y-z\|^2\right)^m (D^\alpha f)(z)\, \mathrm{d}z\,,
\end{aligned} \tag{5.406}
$$

from which one can show $\left| \left\langle \varphi_x, T_{-y} f_x^{\neg} \right\rangle \right|$ is indeed polynomially bounded. ■

We were not able to define usual products of two arbitrary tempered distributions, so it is not surprising that we won't be able to define the convolution product of two arbitrary tempered distributions: the Fourier transform would lead to a product of arbitrary tempered distributions. Nevertheless, it is possible to define it when one of the distributions is not only tempered, but also of compact support.

**Definition 5.145** [Distributions of Compact Support]**:**

Let $\varphi \in \mathscr{D}'(\Omega)$. $\varphi$ is said to be a *distribution of compact support* if, and only if, there is a compact set $K \subseteq \Omega$ such that $\operatorname{supp} \varphi \subseteq K$. ♠

**Lemma 5.146:**
*The Dirac delta distribution is of compact support.* □

*Proof:*
$\operatorname{supp} \delta = \{0\}$, which is compact. ■

**Definition 5.147** [Space of Smooth Functions]**:**
Given $\Omega \in \mathbb{R}^n$, we denote by $\mathscr{E}(\Omega)$ the locally convex space $\mathcal{C}^\infty(\Omega)$ of smooth functions endowed with the topology generated by the seminorms

$$\|f\|_{\alpha,K} = \sup_{x \in K} |D^\alpha f(x)| \tag{5.407}$$

where $\alpha \in \mathbb{N}^n$ and $K \subseteq \Omega$ is compact. ♠

**Proposition 5.148:**
*There is a linear continuous injection of $\mathscr{S}$ into $\mathscr{E}$.* □

*Proof:*
When considered as sets, there is a natural linear mapping $i \colon \mathscr{S} \to \mathscr{E}$, since every rapidly decreasing function is smooth. We only need to prove this map is continuous.
This can be done with Theorem 5.49 on page 158. Notice that

$$\|f\|_{\alpha,K} = \sup_{x \in K} |D^\alpha f(x)| \leq \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| = \|f\|_{0,\alpha} \tag{5.408}$$

for any compact set $K$ and any multiindex $\alpha$. ■

**Corollary 5.149:**
*There is a continuous injection of $\mathscr{E}'$ into $\mathscr{S}'$.* □

*Proof:*
Suppose $\varphi \in \mathscr{E}'$. Proposition 5.148 ensures there is a linear continuous mapping $i \colon \mathscr{S} \to \mathscr{E}$. Notice then that $\varphi \circ i \in \mathscr{S}'$. This concludes the proof. ■

**Lemma 5.150:**
*$\varphi \in \mathscr{E}'(\Omega)$ if, and only if, there is a compact $K \subseteq \Omega$ and constants $\kappa > 0$, $m \in \mathbb{N}$ such that*

$$|\langle \varphi, f \rangle| \leq \kappa \sum_{|\alpha| \leq m} \|f\|_{\alpha,K} \tag{5.409}$$

*for every $f \in \mathscr{E}$.* □

*Proof:*
Follows from Theorem 5.49 on page 158 by noticing that if $K_1 \subseteq K_2$ are compacts, then $\|f\|_{\alpha,K_1} \leq \|f\|_{\alpha,K_2}$. Furthermore, if $K_3$ is also a compact, then $\|f\|_{\alpha,K_1} + \|f\|_{\alpha,K_3} \leq \|f\|_{\alpha,K}$, where $K = K_1 \cup K_3$, which is also a compact. ■

**Theorem 5.151:**
*$\mathscr{E}'$ coincides with the space of distributions of compact support.* □

*Proof:*

Suppose $\varphi \in \mathscr{D}'$ has compact support. Lemma 4.29 on page 112 allows us to conclude there is a smooth function $g\colon \mathbb{R}^n \to [0,1]$ with compact support such that $g(\operatorname{supp}\varphi) = \{1\}$. Hence, notice that given $f \in \mathscr{E}$ we may define

$$\langle \varphi, f\rangle = \langle \varphi, gf\rangle - \langle \varphi, (1-g)f\rangle = \langle \varphi, gf\rangle\,, \tag{5.410}$$

for $gf \in \mathscr{D}$.

We also see that there is a compact $K$ and constants $m \in \mathbb{N}$ and $\kappa > 0$ such that

$$|\langle \varphi, f\rangle| \leq \kappa \sum_{|\alpha|\leq m} \sup_{x\in K} |D^\alpha f(x)|, \tag{5.411}$$

where $K = \operatorname{supp} g$. Lemma 5.150 ensures $\varphi \in \mathscr{E}'$.

Let now $\varphi \in \mathscr{E}'$. Corollary 5.149 on the facing page ensures $\varphi$ is a tempered distribution and Theorem 5.117 on page 214 ensures it is a distribution. We just need to prove it is of compact support.

Lemma 5.150 on the facing page ensures there is a compact $K$ and constants $\kappa > 0$ and $m \in \mathbb{N}$ such that

$$|\langle \varphi, f\rangle| \leq \kappa \sum_{|\alpha|\leq m} \|f\|_{\alpha,K}, \tag{5.412}$$

for every $f \in \mathscr{E}$. In particular, for every compactly supported $f$. Suppose now that $\operatorname{supp} f \subseteq K^c$. Then the previous equation implies $|\langle \varphi, f\rangle| \leq 0$, which in turn implies $\operatorname{supp}\varphi \subseteq K$. Hence, $\varphi$ is indeed a compactly supported distribution. ■

**Definition 5.152** [Convolution of a Smooth Function with a Compactly Supported Distribution]**:**

Let $\varphi \in \mathscr{E}'$, $f \in \mathscr{E}$. We define the *convolution product* of $f$ and $\varphi$, $f * \varphi$, through

$$\langle f * \varphi, g\rangle = \langle \varphi, f^{\neg} * g\rangle\,, \tag{5.413}$$

for all $g \in \mathscr{E}$. ♠

**Proposition 5.153:**

*If $f \in \mathscr{E}$, $\varphi \in \mathscr{E}'$, then $f * \varphi \in \mathscr{E}'$.* □

*Proof:*

Propositions 5.132 and 5.140 on page 225 and on page 230 can be adapted from to $\mathscr{E}$ with few alterations. ■

**Theorem 5.154:**

*Let $U\colon \mathscr{S} \to \mathscr{E}$ be a continuous linear map. If $U$ commutes with all translations,* id est, *$UT_a f = T_a U f, \forall\, a \in \mathbb{R}^n, \forall\, f \in \mathscr{S}$, then there is a unique $\varphi \in \mathscr{S}'$ such that $Uf = f * \varphi, \forall\, f \in \mathscr{S}$.* □

*Proof:*

Proposition 5.144 and Lemma 5.143 on page 231 and on page 232 ensure that convolution with an particular tempered distribution is indeed a continuous linear map from $\mathscr{S}$ to $\mathscr{E}$ that commutes with translation.

Given $U$, suppose there is a tempered distribution with $Uf = f * \varphi$. From Eq. (5.404) on page 233 we see that one must have

$$Uf(0) = \langle \varphi, f^{\neg} \rangle \,, \tag{5.414}$$

and as a consequence

$$Uf^{\neg}(0) = \langle \varphi, f \rangle \,. \tag{5.415}$$

Let us then define a tempered distribution $\varphi \in \mathscr{S}'$ through

$$\langle \varphi, f \rangle = Uf^{\neg}(0). \tag{5.416}$$

Linearity comes from linearity of $U$ and of the $\neg$ operation. Continuity can be shown by using the fact that

$$\begin{aligned} \|f^{\neg}\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n} \left| x^{\alpha} D^{\beta} f^{\neg}(x) \right|, \\ &= \sup_{x\in\mathbb{R}^n} \left| (-1)^{|\alpha|+|\beta|} x^{\alpha} D^{\beta} f(x) \right|, \\ &= \|f\|_{\alpha,\beta}, \end{aligned} \tag{5.417}$$

and hence $f \mapsto Uf^{\neg}$ is continuous. The map from $\mathscr{E}$ to $\mathbb{R}$ given by $f \mapsto f(0)$ is continuous as well, since

$$|f(0)| \leq \sup_{x\in K} |f(x)| = \|f\|_{0,K} \tag{5.418}$$

for an compact $K$ with $0 \in K$. Furthermore, by construction, we have $(\varphi * f)(0) = \langle \varphi, f^{\neg} \rangle = Uf(0)$.

Now that we know $\varphi$ is a tempered distribution, let us show its convolution does yield $U$. We have

$$\begin{aligned} (Uf)(x) &= (T_{-x}Uf)(0), \\ &= (UT_{-x}f)(0), \\ &= (\varphi * T_{-x}f)(0), \\ &= (T_{-x}\varphi * f)(0), \\ &= (\varphi * f)(x), \end{aligned} \tag{5.419}$$

for every $x \in \mathbb{R}^n$. Uniqueness holds due to the fact we defined $\varphi$ on every $f \in \mathscr{S}$. ■

**Theorem 5.155:**

*Let $\varphi \in \mathscr{S}'$, $\psi \in \mathscr{E}'$. There is a unique $\chi \in \mathscr{S}'$ such that*

$$f * \chi = (f * \varphi) * \psi \tag{5.420}$$

*holds for every $f \in \mathscr{S}$.* □

*Proof:*

For every $f \in \mathscr{S}$, the expression $(f * \varphi) * \psi$ is well defined due to Propositions 5.144 and 5.153 on page 232 and on the previous page.

The map $f \mapsto (f * \varphi) * \psi$ is continuous, linear and commutes with all translations, since it is the composition of two maps with such properties. Theorem 5.154 on the preceding page then ensures existence and uniqueness of $\chi$. ■

**Definition 5.156** [Convolution of a Tempered Distribution and a Compactly Supported Distribution]**:**

Let $\varphi \in \mathscr{S}'$ and $\psi \in \mathscr{E}'$. We define $\varphi * \psi$ to be the unique tempered distribution such that

$$f * (\varphi * \psi) = (f * \varphi) * \psi \tag{5.421}$$

holds for every $f \in \mathscr{S}$. ♠

**Proposition 5.157:**

*Let $\varphi \in \mathscr{S}'$. $\varphi * \delta = \varphi$.* □

*Proof:*

Let us first prove the case in which $\varphi \in \mathscr{E}$. We have, $\forall f \in \mathscr{E}$,

$$\begin{aligned}\langle \varphi * \delta, f \rangle &= \langle \delta, \varphi^{\neg} * f \rangle , \\ &= (\varphi^{\neg} * f)(0), \\ &= \int \varphi^{\neg}(x) f(-x)\, \mathrm{d}x , \\ &= \int \varphi(x) f(x)\, \mathrm{d}x , \\ &= \langle \varphi, f \rangle . \end{aligned} \tag{5.422}$$

As for the general case,

$$\begin{aligned} f * (\varphi * \delta) &= (f * \varphi) * \delta, \\ &= f * \varphi, \end{aligned} \tag{5.423}$$

$$\varphi * \delta = \varphi, \tag{5.424}$$

for $f * \varphi \in \mathscr{E}$. ■

## 5.8 Division of Distributions

Consider the function $\log |x|$. Despite being undefined at the origin, it defines a distribution, since it is locally integrable. Indeed,

$$\int_0^a \log |x|\, \mathrm{d}x = a \log a - a. \tag{5.425}$$

**Lemma 5.158:**

$\log |x|$ *defines a tempered distribution.* □

*Proof:*
Since $\log|x|$ is locally integrable, its integral against a rapidly decreasing function converges on a compact. On a compact's complement, it also converges, since $\log|x|$ grows slower than polynomials and rapidly decreasing functions can be integrated against polynomials and still converge. ■

As a consequence, $\log|x|$ has a derivative in the sense of distributions. While we expect it to be related to $\frac{1}{x}$, the usual prescription of expecting

$$\left\langle \frac{1}{x}, f(x) \right\rangle = \int \frac{f(x)}{x} \, \mathrm{d}x \tag{5.426}$$

fails, since $\frac{1}{x}$ is not locally integrable near the origin.
Let us try to compute the derivative in the distributional sense. We have

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}x} \log|x|, f \right\rangle &= -\left\langle \log|x|, f' \right\rangle, \\ &= -\int \log|x| f'(x) \, \mathrm{d}x, \\ &= -\lim_{\epsilon\to 0^+} \left( \int_{-\infty}^{-\epsilon} \log|x| f'(x) \, \mathrm{d}x + \int_{+\epsilon}^{+\infty} \log|x| f'(x) \, \mathrm{d}x \right), \\ &= \lim_{\epsilon\to 0^+} \left( \int_{-\infty}^{-\epsilon} \frac{f(x)}{x} \, \mathrm{d}x + \int_{+\epsilon}^{+\infty} \frac{f(x)}{x} \, \mathrm{d}x + \log(\epsilon) f(\epsilon) - \log(\epsilon) f(-\epsilon) \right).\end{aligned} \tag{5.427}$$

Since

$$\log(\epsilon) f(\epsilon) = \log(\epsilon) f(0) + \frac{f(\epsilon) - f(0)}{\epsilon} \epsilon \log(\epsilon) \overset{\epsilon\to 0}{=} \log(\epsilon) f(0), \tag{5.428}$$

we get to

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}x} \log|x|, f \right\rangle &= \lim_{\epsilon\to 0^+} \left( \int_{-\infty}^{-\epsilon} \frac{f(x)}{x} \, \mathrm{d}x + \int_{+\epsilon}^{+\infty} \frac{f(x)}{x} \, \mathrm{d}x + \log(\epsilon) f(0) - \log(\epsilon) f(0) \right), \\ &= \lim_{\epsilon\to 0^+} \left( \int_{-\infty}^{-\epsilon} \frac{f(x)}{x} \, \mathrm{d}x + \int_{+\epsilon}^{+\infty} \frac{f(x)}{x} \, \mathrm{d}x \right).\end{aligned} \tag{5.429}$$

To deal with this expression, we shall define the so-called Cauchy's principal value.

**Definition 5.159** [Cauchy's Principal Value]**:**
Given $\Omega \subseteq \mathbb{R}^n$, let $f\colon \Omega \to \mathbb{R}$ be a function and $z \in \overline{\Omega}$. Suppose $f$ is integrable on $\Omega \setminus \mathcal{B}_\epsilon(z)$, $\forall\, \epsilon > 0$. We define *Cauchy's principal value* of the integral $\int_\Omega f(x) \, \mathrm{d}x$ as

$$\text{P.V.} \int_\Omega f(x) \, \mathrm{d}x \equiv \lim_{\epsilon\to 0^+} \int_{\Omega\setminus\mathcal{B}_\epsilon(z)} f(x) \, \mathrm{d}x \tag{5.430}$$

whenever the limit exists. ♠

Therefore, we see we got to

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} \log|x|, f \right\rangle = \text{P.V.} \int \frac{f(x)}{x} \,\mathrm{d}x \,. \tag{5.431}$$

While $\frac{1}{x}$ is not a distribution, it can be used to define one with the aid of the principal value.

**Proposition 5.160:**
*The principal value distribution,*

$$\left\langle \text{P.V.}\left(\frac{1}{x}\right), f \right\rangle \equiv \text{P.V.} \int \frac{f(x)}{x} \,\mathrm{d}x \tag{5.432}$$

*is a tempered distribution in one-dimension.* □

*Proof:*
Linearity comes from the integral. We just need to prove it is well-defined and continuous.

We begin with well-definedness. Notice that

$$\text{P.V.} \int \frac{f(x)}{x} \,\mathrm{d}x = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{+\infty} \frac{f(x) - f(-x)}{x} \,\mathrm{d}x \,, \tag{5.433}$$

and we know $f'(0) = \lim_{\epsilon \to 0^+} \frac{f(\epsilon) - f(-\epsilon)}{\epsilon}$. Hence,

$$\text{P.V.} \int \frac{f(x)}{x} \,\mathrm{d}x = \int_0^{+\infty} \frac{f(x) - f(-x)}{x} \,\mathrm{d}x \,, \tag{5.434}$$

which we now know to be finite, since convergence at infinity was already ensured due to $f$ decreasing rapidly and convergence at the origin is now ensured due to the derivative appearing.

We now move to continuity by noticing that

$$\left| \left\langle \text{P.V.}\left(\frac{1}{x}\right), f \right\rangle \right| \leq \int_0^1 \left| \frac{f(x) - f(-x)}{x} \right| \mathrm{d}x + \left| \int_1^{+\infty} \frac{f(x) - f(-x)}{x} \,\mathrm{d}x \right|, \tag{5.435}$$

but we also have

$$\begin{aligned} \int_0^1 \left| \frac{f(x) - f(-x)}{x} \right| \mathrm{d}x &\leq \int_0^1 \frac{1}{x} \int_{-x}^{x} \left| f'(t) \right| \mathrm{d}t \,\mathrm{d}x \,, \\ &\leq \int_0^1 \frac{1}{x} \int_{-x}^{x} \|f\|_{0,1} \,\mathrm{d}t \,\mathrm{d}x \,, \\ &= 2\|f\|_{0,1} \end{aligned} \tag{5.436}$$

and

$$\begin{aligned}\left|\int_1^{+\infty}\frac{f(x)-f(-x)}{x}\,\mathrm{d}x\right| &\le \int_1^{+\infty}\frac{|f(x)-f(-x)|}{x}\,\mathrm{d}x\,,\\ &\le \int_1^{+\infty}\frac{|xf(x)|}{x^2}\,\mathrm{d}x+\int_1^{+\infty}\frac{|-xf(-x)|}{x^2}\,\mathrm{d}x\,,\\ &\le 2\int_1^{+\infty}\frac{\|f\|_{1,0}}{x^2}\,\mathrm{d}x\,,\\ &= 2\|f\|_{1,0}. \end{aligned}\tag{5.437}$$

Therefore,

$$\left|\left\langle \mathrm{P.V.}\left(\frac{1}{x}\right), f\right\rangle\right| \le 2\|f\|_{0,1}+2\|f\|_{1,0},\tag{5.438}$$

which ensures continuity through Theorem 5.49 on page 158. ■

The principal value is also related to the problem of finding a distribution $\varphi$ such that

$$f\varphi=\psi,\tag{5.439}$$

where $\psi$ is a given distribution and $f$ is a smooth function.

We would expect that, if $f(x)=x$ and $\psi=1$, then $\varphi$ should be somehow related to $\frac{1}{x}$. It turns out P.V.$\left(\frac{1}{x}\right)$ does solve the problem. Indeed, let $f\in\mathscr{S}$. One has

$$\begin{aligned}\left\langle x\mathrm{P.V.}\left(\frac{1}{x}\right), f\right\rangle &= \left\langle \mathrm{P.V.}\left(\frac{1}{x}\right), xf\right\rangle\,,\\ &= \mathrm{P.V.}\int\frac{xf(x)}{x}\,\mathrm{d}x\,,\\ &= \mathrm{P.V.}\int f(x)\,\mathrm{d}x\,,\\ &= \int f(x)\,\mathrm{d}x\,,\\ &= \langle 1, f\rangle\,, \end{aligned}\tag{5.440}$$

as claimed.

However, this is not the only solution. Consider the equation

$$x\chi=0,\tag{5.441}$$

where $\chi$ is some unknown distribution. This equation can be solved, for example, by $\chi=c\delta$, for any $c\in\mathbb{R}$. Indeed,

$$\begin{aligned}\langle xc\delta, f\rangle &= c\,\langle\delta, xf\rangle\,,\\ &= 0. \end{aligned}\tag{5.442}$$

Hence, $\varphi=\mathrm{P.V.}\left(\frac{1}{x}\right)+c\delta$ solves $x\varphi=\psi$ for any constant $c$.

This is an example of how the existence of zeroes on the function $f(x)$ leads to non-uniqueness of solutions to the equation $f\varphi=\psi$.

**Theorem 5.161:**

*Let $\psi \in \mathscr{D}'$ be given and let $f \in \mathscr{E}$ be not identically zero. If the equation $f\varphi = \psi$ has a solution in $\mathscr{D}'$, it is unique if, and only if, $f$ has no real zeroes. Furthermore, if $f$ has finitely many zeros, namely at $\{a_i\}_{i=1}^{m}$, and $\varphi_1$ and $\varphi_2$ are solutions, then they respect*

$$\varphi_1 - \varphi_2 = \sum_{i=1}^{m} \sum_{|\alpha_i| \leq k_i} \kappa_{\alpha_i} D^{\alpha_i}(T_{a_i}\delta), \tag{5.443}$$

*for constants $\kappa_{\alpha_i}$ and $k_i$.* □

*Proof:*

Let $\varphi_1$ and $\varphi_2$ solve said equation. Then

$$f\varphi_1 - f\varphi_2 = \psi - \psi, \tag{5.444}$$

$$f(\varphi_1 - \varphi_2) = 0. \tag{5.445}$$

Suppose $f$ vanishes nowhere. Then, $\forall\, g \in \mathscr{D}$,

$$0 = \langle f(\varphi_1 - \varphi_2), g\rangle = \langle \varphi_1 - \varphi_2, fg\rangle\,, \tag{5.446}$$

*id est*, $\varphi_1 - \varphi_2$ vanishes on $f\mathscr{D}$. However, $f\mathscr{D} = \mathscr{D}$: given any $g \in \mathscr{D}$, $\frac{g}{f} \in \mathscr{D}$ is well-defined, since $f$ never vanishes and $g \in \mathscr{D}, f \in \mathscr{E} \Rightarrow \frac{g}{f} \in \mathscr{D}$. Therefore, we find $\langle \varphi_1 - \varphi_2, g\rangle = 0, \forall\, g \in \mathscr{D}$, which means $\varphi_1 = \varphi_2$.

Suppose now $f(a) = 0$ for some $a \in \mathbb{R}^n$. We assume this is the only zero of $f$, for the general case works similarly. Without loss of generality we assume $a = 0$, and the general case can be worked out by applying translations. We still have

$$\langle \varphi_1 - \varphi_2, fg\rangle = 0, \tag{5.447}$$

$\forall\, g \in \mathscr{D}$, but we no longer have $f\mathscr{D} = \mathscr{D}$, since $\frac{g}{f}$ is not necessarily smooth.

Suppose, however, that $g \in \mathscr{D}$ has $\operatorname{supp} g \subseteq \mathbb{R}^n \setminus \{0\}$. Then $\frac{g}{f} \in \mathscr{D}$, which implies

$$\langle \varphi_1 - \varphi_2, g\rangle = 0, \tag{5.448}$$

$\forall\, g \in \mathscr{D}; \operatorname{supp} g \subseteq \mathbb{R}^n \setminus \{0\}$. Hence, $\operatorname{supp}(\varphi_1 - \varphi_2) \subseteq \{0\}$.

Since $\operatorname{supp}(\varphi_1 - \varphi_2) \subseteq \{0\}$, $\varphi_1 - \varphi_2$ has compact support and, as a consequence, is a tempered distribution. Proposition 5.112 on page 212 then implies

$$\varphi_1 - \varphi_2 = \sum_{|\alpha| \leq m} \kappa_\alpha D^\alpha \delta. \tag{5.449}$$

This concludes the proof. ■

**Corollary 5.162:**

*Theorem 5.161 on the preceding page still holds if one replaces $\mathscr{D}'$ with $\mathscr{E}'$. It also holds if one replaces $\mathscr{D}'$ with $\mathscr{S}'$ if one restricts the function $f$ to polynomials.* □

*Proof:*
This is more a corollary to the proof than to the statement of Theorem 5.161 on the facing page. The proof is completely analogous to Theorem 5.161 on the preceding page. ■

If existence is assumed, we already have Theorem 5.161 and **??** on the facing page and on page ?? to ensure uniqueness. Nevertheless, we would like to know whether existence of solution is ensured. To prove it, we first quote Theorem 1 of [32].

**Theorem 5.163:**
*Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial that does not vanish identically. Then the map*

$$\mathscr{S} \ni f \mapsto pf \in \mathscr{S} \tag{5.450}$$

*has a continuous inverse.* □

*Proof:*
See [32]. ■

**Theorem 5.164:**
*Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial that does not vanish identically and let $\psi \in \mathscr{S}'$. There exists $\varphi \in \mathscr{S}'$ such that $p\varphi = \psi$.* □

*Proof:*
Consider the mapping $pf \mapsto \langle \psi, f \rangle$, defined on $p\mathscr{S}$. One can show it is linear. Furthermore, it is continuous, for it is the composition of the maps $pf \mapsto f$ and $f \mapsto \langle \psi, f \rangle$. The latter is continuous due to $\psi$ being a tempered distribution, while the former is continuous due to Theorem 5.163 on the preceding page.

We see we've found a continuous linear functional $\tilde{\varphi}\colon p\mathscr{S} \to \mathbb{R}$ with $\langle \tilde{\varphi}, pf \rangle = \langle \psi, f \rangle$. Corollary 5.60 on page 167, which is just the Hahn–Banach Theorem in the particular context of locally convex spaces, then implies that there is a continuous linear functional $\varphi\colon \mathscr{S} \to \mathbb{R}$ extending $\tilde{\varphi}$. Hence, there is a tempered distribution $\varphi$ such that, $\forall f \in \mathscr{S}$,

$$\langle \varphi, pf \rangle = \langle \psi, f \rangle, \tag{5.451}$$

$$\langle p\varphi, f \rangle = \langle \psi, f \rangle, \tag{5.452}$$

$$p\varphi = \psi, \tag{5.453}$$

concluding the proof. ■

Now that we already know there always is a solution, we might work out a few examples in one-dimension. For example, let us consider

$$x^m \varphi = f, \tag{5.454}$$

for some $m \in \mathbb{N}$ and a polynomially bounded smooth function $f$. Inspired by our solution for the case $m = 1$, one could guess

$$\varphi = f \cdot \mathrm{P.V.}\left(\frac{1}{x^m}\right), \tag{5.455}$$

but this fails: P.V.$\left(\frac{1}{x^2}\right)$, for example, does not exist.

When we were dealing with odd functions, the principal value was able to regularize* and renormalize† the integrals we were dealing with, but the integrals of even functions do not have cancelling divergences.

Let us revisit the problem of differentiating the logarithm. However, this time let us consider the functions

$$(\log x)_{\pm} = \begin{cases} \log|x|, & \text{if } \pm x > 0, \\ 0, & \text{if } \pm x < 0 \end{cases} \tag{5.456}$$

and try to differentiate them.

**Lemma 5.165:**

$(\log x)_{\pm}$ *define tempered distributions.* □

*Proof:*

The argument is identical to the proof of Lemma 5.158 on page 237. ■

$\log|x|$ led us to an odd derivative and symmetry allowed the principal value procedure to cancel the divergences, but that won't work here. This time we have, $\forall\, g \in \mathscr{S}$,

$$\begin{aligned} \left\langle \frac{\mathrm{d}}{\mathrm{d}x}(\log x)_+, g \right\rangle &= -\left\langle (\log x)_+, g' \right\rangle, \\ &= -\int_0^{+\infty} \log(x) g'(x)\, \mathrm{d}x, \\ &= -\lim_{\epsilon\to 0^+} \int_\epsilon^{+\infty} \log(x) g'(x)\, \mathrm{d}x, \\ &= \lim_{\epsilon\to 0^+} \left( g(0)\log(\epsilon) + \int_\epsilon^{+\infty} \frac{g(x)}{x}\, \mathrm{d}x \right), \end{aligned} \tag{5.457}$$

and we are not able to make sense of the expression unless we first split the integral in a divergent and a convergent part. If we are not careful with our procedures, we might end up splitting the integral in a non-unique way, which would lead to ill-definedness. We thus begin by stating a preliminary lemma.

**Lemma 5.166:**

*Let* $\kappa_i \in \mathbb{C}$, $1 \le i \le n$. *Let* $\alpha_i, \beta_i \in \mathbb{C}$, $1 \le i \le n$, *be such that, for any given* $i$, $\mathrm{Re}[\alpha_i], \mathrm{Re}[\beta_i] \le 0$, *but* $\mathrm{Re}[\alpha_i]$ *and* $\mathrm{Re}[\beta_i]$ *are not mutually zero. Then, if*

$$\lim_{\epsilon\to 0^+} \sum_{i=1}^{n} \kappa_i x^{-\alpha_i} (\log x)^{\beta_i} = k \in \mathbb{C}, \tag{5.458}$$

*it must hold that* $k = 0$ *and* $\kappa_i = 0$, $1 \le i \le n$. □

*In field theory, and often in distribution theory as well, it is common to refer to the process of giving meaning to divergent integrals as "regularization".

†In field theory, having "opposite divergences" to cancel out and leave a finite contribution is often called "renormalization".

*Proof:*
See [4, pp. 1902–1906]. ■

*Notation:*
We shall denote

$$\mathcal{I}_H \equiv \operatorname{span}\left(x^{-\alpha}(\log x)^{\beta}; \operatorname{Re}[\alpha], \operatorname{Re}[\beta] \leq 0, (\operatorname{Re}[\alpha], \operatorname{Re}[\beta]) \neq (0,0)\right), \tag{5.459}$$

where $\operatorname{span}(X)$ denotes the vector space of finite linear combinations of the set $X$. The linear combinations are meant in the sense of a complex vector space. ♦

**Proposition 5.167:**
*Let $f\colon (0,a) \to \mathbb{C}$. Assume $f$ can be decomposed in the form*

$$f(x) = h(x) + s(x) \tag{5.460}$$

*with $|\lim_{x\to 0^+} h(x)| < +\infty$ and $s \in \mathcal{I}_H$. Then this decomposition is unique.* □

*Proof:*
Assume $f$ can be decomposed as

$$f(x) = h(x) + s(x), \tag{5.461}$$

and as

$$f(x) = g(x) + r(x), \tag{5.462}$$

both satisfying the conditions we imposed. Then

$$s(x) - r(x) = g(x) - h(x), \tag{5.463}$$

which must have a finite limit as $x \to 0^+$, since $h$ and $g$ have. Hence, Lemma 5.166 on the previous page ensures $\lim_{x\to 0^+}[s(x) - r(x)]$ vanishes and, as a consequence, uniqueness of representation. ■

**Definition 5.168** [Hadamard's Finite Part]**:**
Given $\Omega \subseteq \mathbb{R}^n$, let $f\colon \Omega \to \mathbb{R}$ be a function and $z \in \overline{\Omega}$. Suppose, that, for every $\epsilon > 0$, we can write

$$\int_{\Omega\setminus\mathcal{B}_\epsilon(z)} f(x)\,\mathrm{d}x = I(\epsilon) + F(\epsilon), \tag{5.464}$$

where $I \in \mathcal{I}_H$ and $|\lim_{\epsilon\to 0^+} F(\epsilon)| < +\infty$. We define *Hadamard's finite part*, also known as *Hadamard's partie finie*, of the integral $\int_\Omega f(x)\,\mathrm{d}x$ as

$$\text{P.F.} \int_\Omega f(x)\,\mathrm{d}x \equiv \lim_{\epsilon\to 0^+} F(\epsilon), \tag{5.465}$$

which is well defined due to Proposition 5.167 on the preceding page. ♠

Returning to Eq. (5.457) on the previous page, we must consider the integral

$$\int_{\epsilon}^{+\infty} \frac{g(x)}{x}\,\mathrm{d}x \tag{5.466}$$

as $\epsilon \to 0^+$.

Let us notice that, given some small $\delta > \epsilon$, we can write

$$\begin{aligned}\int_{\epsilon}^{+\infty} \frac{g(x)}{x}\,\mathrm{d}x &= \int_{\epsilon}^{\delta} \frac{g(x)}{x}\,\mathrm{d}x + \int_{\delta}^{+\infty} \frac{g(x)}{x}\,\mathrm{d}x\,,\\ &= \int_{\epsilon}^{\delta} \frac{g(0)}{x} + g'(0) + \mathcal{O}(x)\,\mathrm{d}x + \int_{\delta}^{+\infty} \frac{g(x)}{x}\,\mathrm{d}x\,,\\ &= -g(0)\log(\epsilon) + \text{finite terms},\end{aligned} \tag{5.467}$$

where "finite terms" are terms that remain bounded in the limit $\epsilon \to 0^+$. We see they will actually correspond to the integral's partie finie. Eq. (5.457) on the preceding page then tells us that

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}x}(\log x)_+, g\right\rangle &= \text{P.F.} \int_{0}^{+\infty} \frac{g(x)}{x}\,\mathrm{d}x\,,\\ &= \text{P.F.} \int \frac{g(x)\Theta(x)}{x}\,\mathrm{d}x\,,\end{aligned} \tag{5.468}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log x)_+ = \text{P.F.}\left(\frac{\Theta(x)}{x}\right), \tag{5.469}$$

where we define

$$\text{P.F.}\left(\frac{\Theta(\pm x)}{x}\right) = \text{P.F.} \int \frac{g(x)\Theta(\pm x)}{x}\,\mathrm{d}x\,. \tag{5.470}$$

A similar procedure leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log x)_- = \text{P.F.}\left(\frac{\Theta(-x)}{x}\right). \tag{5.471}$$

Noticing that $\log|x| = (\log x)_- + (\log x)_+$ we see that

$$\text{P.V.}\left(\frac{1}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \text{P.F.}\left(\frac{\Theta(x)}{x}\right) + \text{P.F.}\left(\frac{\Theta(-x)}{x}\right) = \text{P.F.}\left(\frac{1}{x}\right). \tag{5.472}$$

**Proposition 5.169:**

$\text{P.F.}\left(\frac{\Theta(x)}{x}\right), \text{P.F.}\left(\frac{\Theta(-x)}{x}\right), \text{P.F.}\left(\frac{1}{x}\right) \in \mathscr{S}'(\mathbb{R})$. □

*Proof:*

They are derivatives of tempered distributions (Lemmas 5.158 and 5.165 on page 237 and on page 243). ■

**Proposition 5.170:**

*Given* $n \in \mathbb{N}^*$, $\text{P.F.}\left(\frac{\Theta(x)}{x^n}\right), \text{P.F.}\left(\frac{\Theta(-x)}{x^n}\right), \text{P.F.}\left(\frac{1}{x^n}\right) \in \mathscr{S}'(\mathbb{R})$. □

*Proof:*

The case $m = 1$ is solved by Proposition 5.169.

Let us try to differentiate P.F.$\left(\frac{\Theta(x)}{x}\right)$. One has

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\text{P.F.}\left(\frac{\Theta(x)}{x}\right), f\right\rangle &= -\left\langle \text{P.F.}\left(\frac{\Theta(x)}{x}\right), f'\right\rangle, \\
&= -\lim_{\epsilon\to 0+}\left[f'(0)\log(\epsilon) + \int_{\epsilon}^{+\infty}\frac{f'(x)}{x}\,\mathrm{d}x\right], \\
&= -\lim_{\epsilon\to 0+}\left[f'(0)\log(\epsilon) - \frac{f(\epsilon)}{\epsilon} + \int_{\epsilon}^{+\infty}\frac{f(x)}{x^2}\,\mathrm{d}x\right], \\
&= -\lim_{\epsilon\to 0+}\left[f'(0)\log(\epsilon) - \frac{f(0)}{\epsilon} - f'(0) + \int_{\epsilon}^{+\infty}\frac{f(x)}{x^2}\,\mathrm{d}x\right]. \quad (5.473)
\end{aligned}$$

Given some small $\delta > \epsilon$, we can write

$$\begin{aligned}
\int_{\epsilon}^{+\infty}\frac{f(x)}{x^2}\,\mathrm{d}x &= \int_{\epsilon}^{\delta}\frac{f(x)}{x^2}\,\mathrm{d}x + \int_{\delta}^{+\infty}\frac{f(x)}{x^2}\,\mathrm{d}x\,, \\
&= \int_{\epsilon}^{\delta}\frac{f(0)}{x^2} + \frac{f'(0)}{x} + \frac{f''(0)}{2} + \mathcal{O}(x)\,\mathrm{d}x + \int_{\delta}^{+\infty}\frac{g(x)}{x}\,\mathrm{d}x\,, \\
&= \frac{f(0)}{\epsilon} - f'(0)\log(\epsilon) + \text{P.F.}\int\frac{f(x)\Theta(x)}{x^2}\,\mathrm{d}x\,. \quad (5.474)
\end{aligned}$$

Therefore,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\text{P.F.}\left(\frac{\Theta(x)}{x}\right), f\right\rangle = -\text{P.F.}\int\frac{f(x)\Theta(x)}{x^2}\,\mathrm{d}x + f'(0), \quad (5.475)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\text{P.F.}\left(\frac{\Theta(x)}{x}\right) = -\text{P.F.}\left(\frac{\Theta(x)}{x^2}\right) - \delta', \quad (5.476)$$

and a similar process leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\text{P.F.}\left(\frac{\Theta(-x)}{x}\right) = -\text{P.F.}\left(\frac{\Theta(-x)}{x^2}\right) + \delta'. \quad (5.477)$$

If we now add both expressions,

$$\frac{\mathrm{d}}{\mathrm{d}x}\text{P.F.}\left(\frac{1}{x}\right) = -\text{P.F.}\left(\frac{1}{x^2}\right). \quad (5.478)$$

Hence, P.F.$\left(\frac{\Theta(x)}{x^2}\right)$, P.F.$\left(\frac{\Theta(-x)}{x^2}\right)$, and P.F.$\left(\frac{1}{x^2}\right)$ are derivatives of tempered distributions. A similar process applies for the general case. ■

Let us now turn again our attention to Eq. (5.454) on page 242.

**Proposition 5.171:**

*Every distribution $\varphi \in \mathscr{D}'(\mathbb{R})$ that solves the equation $x^n \varphi = f$, where $f$ is a polynomially bounded smooth function, is of the form*

$$\varphi = f \cdot \text{P.F.}\left(\frac{1}{x^n}\right) + \sum_{k=0}^{n-1} \kappa_k D^k \delta, \tag{5.479}$$

*for any constants $\kappa_k \in \mathbb{C}$. The same holds replacing $\mathscr{D}'(\mathbb{R})$ by $\mathscr{S}'(\mathbb{R})$.* □

*Proof:*

We begin by showing $f \cdot \text{P.F.}\left(\frac{1}{x^n}\right)$ is a particular solution. Notice that, for any $g \in \mathscr{D}$,

$$\begin{aligned} \left\langle x^n \cdot f \cdot \text{P.F.}\left(\frac{1}{x^n}\right), g\right\rangle &= \left\langle \text{P.F.}\left(\frac{1}{x^n}\right), x^n \cdot f \cdot g\right\rangle, \\ &= \text{P.F.} \int \frac{x^n f(x) g(x)}{x^n} \, \mathrm{d}x, \\ &= \int f(x) g(x) \, \mathrm{d}x, \\ &= \langle f, g\rangle. \end{aligned} \tag{5.480}$$

Notice the same argument holds if one replaces $\mathscr{D}$ by $\mathscr{S}$.

Theorem 5.161 and Corollary 5.162 on page 240 and on page 241 now deal with the rest. Any other solution to the equation must have the form

$$\varphi_1 = f \cdot \text{P.F.}\left(\frac{1}{x^n}\right) + \sum_{k=0}^{l} \kappa_k D^k \delta \tag{5.481}$$

for some $l \in \mathbb{N}$. However, since

$$x^n D^l \delta = 0 \tag{5.482}$$

is only solved for $l < n$, the general solution has to be

$$f \cdot \text{P.F.}\left(\frac{1}{x^n}\right) + \sum_{k=0}^{n-1} \kappa_k D^k \delta, \tag{5.483}$$

as previously claimed. ■

# Six

# Hyperbolic Equations

> In order to solve this differential equation you look at it until a solution occurs to you.
>
> Attributed to GEORGE PÓLYA.

## 6.1 Fundamental Solutions

SUPPOSE you have a partial differential equation with constant coefficients. It can be written as

$$\sum_{|\alpha|\leq k} \kappa_\alpha D^\alpha u(x) = f(x). \tag{6.1}$$

We recall that if $p(x)$ is the polynomial given by $p(x) = \sum_{|\alpha|\leq k} \kappa_\alpha x^\alpha$, we may write $p(D) = \sum_{|\alpha|\leq k} \kappa_\alpha D^\alpha$ so that the equation becomes*

$$p(D)u(x) = f(x). \tag{6.2}$$

We already know distribution theory, so there is no reason to restrain ourselves to usual functions. Let us consider the distributional equation

$$p(D)u = f, \tag{6.3}$$

where we now allow $f \in \mathscr{S}'$.

The case $f = \delta$ is of particular interest and defines what we call a fundamental solution.

**Definition 6.1** [Fundamental Solution]**:**

Let $p$ be some polynomial of order $k$ in $n$ variables. We say $F \in \mathscr{D}'$ is a *fundamental solution* for the operator $p(D)$ at a point $x \in \mathbb{R}^n$ if, and only if,

$$p(D)F = T_x\delta, \tag{6.4}$$

*Some of the literature on partial differential equations would rather write $p(\partial) = \sum_{|\alpha|\leq k}(-i)^{|\alpha|}\kappa_\alpha D^\alpha$, so the Fourier transform of $p(\partial)$ is $p(x)$. We won't adopt this convention.

where $\delta$ is the Dirac delta distribution and the derivatives are meant in the sense of distributions.

We'll often refer to a fundamental solution at the origin as simply a fundamental solution. ♠

**Proposition 6.2:**

*Let $F$ be a fundamental solution for the differential operator $p(D)$, where $p$ is some polynomial. Given $x \in \mathbb{R}^n$, $T_x F$ is a fundamental solution at $x$.* □

*Proof:*

We know we can write $p(D)$ in the form

$$p(D) = \sum_{|\alpha| \leq k} \kappa_\alpha D^\alpha. \tag{6.5}$$

With this in mind, we notice that, $\forall\, f \in \mathscr{S}$,

$$\begin{aligned} \langle p(D) T_x F, f \rangle &= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \langle F, T_{-x} D^\alpha f \rangle, \\ &= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \langle F, D^\alpha T_{-x} f \rangle, \\ &= \langle p(D) F, T_{-x} f \rangle, \\ &= \langle \delta, T_{-x} f \rangle, \\ &= \langle T_x \delta, f \rangle, \end{aligned} \tag{6.6}$$

which is the result we desired. ■

**Proposition 6.3:**

*Let $F$ be a fundamental solution for the differential operator $p(D)$, where $p$ is some polynomial. Given $f \in \mathscr{D}$, the equation*

$$p(D)(f * F) = f \tag{6.7}$$

*holds.* □

*Proof:*

One has

$$\begin{aligned} p(D)F &= \delta, \\ f * (p(D)F) &= f * \delta, \\ p(D)(f * F) &= f * \delta, \\ &= f, \end{aligned} \tag{6.8}$$

concluding the proof. ■

Naturally, if the fundamental solution turns out to be a tempered distribution or a distribution of compact support we can consider even more general sources $f$.

One may then wonder whether this is indeed a useful concept. Do we know when one can find a fundamental solution? Furthermore, do we have any information on uniqueness?

**Definition 6.4** [Principal Part of a Polynomial]:
Let $p(x) = \sum_{|\alpha|\leq k} \kappa_\alpha x^\alpha$ be a polynomial. We call

$$p_k(x) = \sum_{|\alpha|=k} \kappa_\alpha x^\alpha \tag{6.9}$$

its *principal part*. ♠

**Definition 6.5** [Symbol of a Differential Operator with Constant Coefficients]:
Let $p(D) = \sum_{|\alpha|\leq k} \kappa_\alpha D^\alpha$ be a differential operator. We define its *symbol* to be the polynomial $p(i\xi) = \sum_{|\alpha|\leq k} \kappa_\alpha (i\xi)^\alpha$. Furthermore, the *principal symbol* of a differential operator is defined to be the principal part of the operator's symbol. ♠

**Theorem 6.6:**
*Let $p$ be a polynomial that does not vanish identically and let $\psi \in \mathscr{S}'$. There is a tempered distribution $\varphi$ which solves the equation $p(D)\varphi = \psi$. The solution is unique if, and only if, the symbol of the operator $p(D)$ has no zeroes on $\mathbb{R}^n$.* □

*Proof:*
We want to find $\varphi \in \mathscr{S}'$ such that

$$p(D)\varphi = \psi. \tag{6.10}$$

Since both sides involve tempered distributions, we can apply the Fourier transform to both sides and find

$$p(i\xi)\hat{\varphi} = \hat{\psi}. \tag{6.11}$$

Theorem 5.164 on page 242 now ensures the existence of a tempered distribution $\hat{\varphi}$ with such properties. Hence, we find that

$$\varphi = \mathcal{F}^{-1}\left[\frac{\hat{\psi}}{p(i\xi)}\right] \tag{6.12}$$

solves the distributional differential equation, with the division understood in the sense of distributions.

If the symbol of $p(D)$ has no zeroes, Theorem 5.161 and Corollary 5.162 on page 240 and on page 241 ensure the uniqueness of solution. ■

**Corollary 6.7:**
*Let $p$ be a polynomial that does not vanish identically. There is a tempered distribution which is a fundamental solution for the differential operator $p(D)$. If the symbol of $p(D)$ has no real zeroes, then the fundamental solution is unique.* □

*Proof:*
Follows from Theorem 6.6, since $\delta$ is a tempered distribution. ■

**Corollary 6.8** [Malgrange–Ehrenpreis Theorem]:
*Let $p$ be a polynomial. There is an ordinary distribution which is a fundamental solution for the differential operator $p(D)$.* □

*Proof:*

Follows from Corollary 6.7 on the previous page, since every tempered distribution is a distribution. For alternative and more direct proofs, see [21, pp. 62–63], [33, pp. 189–191], and/or [68, pp. 603–606]. ■

One should notice fundamental solutions are not always unique. For instance, let $p(D)F = \delta$ and $p(D)H = 0$. Then $p(D)(F+H) = \delta$, and hence $F+H$ is also a fundamental solution. In the case of tempered distributions, this is just a reflection of the fact that the division of distributions is not always uniquely defined. Furthermore, if uniqueness holds for tempered distributions, it doesn't necessarily holds for ordinary distributions.

**Example:**

Let us consider the differential operator $p(D) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)$, which has $-i\xi - 1$ as its symbol. The symbol vanishes at

$$\xi = i \tag{6.13}$$

and there only. Hence, there is a single tempered fundamental solution. Let us find it.

Employing generalized function notation, the Fourier transform tells us that

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right) F &= \delta, \qquad (6.14)\\
(i\xi - 1)\hat{F} &= 1,\\
F(x) &= \frac{1}{2\pi}\int \frac{e^{ix\xi}}{i\xi - 1}\,\mathrm{d}\xi\,, \qquad (6.15)\\
&= -e^{x}\Theta(-x). \qquad (6.16)
\end{aligned}$$

To check the Fourier inverse is right, we can just apply the Fourier transform to the result:

$$\begin{aligned}
\mathcal{F}[-e^{x}\Theta(-x)]\,(\xi) &= -\int_{-\infty}^{0} e^{x(1-i\xi)}\,\mathrm{d}x\,,\\
&= \left.\frac{e^{x(1-i\xi)}}{i\xi - 1}\right|_{-\infty}^{0},\\
&= \frac{1}{i\xi - 1}, \qquad (6.17)
\end{aligned}$$

as expected.

We see then that one fundamental solution is the tempered distribution $-e^{x}\Theta(-x)$. This is the only tempered solution, but there are more solutions if we consider ordinary distributions. Indeed, the homogeneous equation has

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right) f(x) = 0, \tag{6.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}f(x)\right) = 0, \tag{6.19}$$
$$e^{-x}f(x) = c, \tag{6.20}$$
$$f(x) = ce^{x}, \tag{6.21}$$

for any constant $c$. Since $e^x$ is locally integrable, we have infinitely many fundamental solutions when considering ordinary distributions. Namely,

$$F = (c - \Theta(-x))e^x \tag{6.22}$$

is a fundamental solution for every $c \in \mathbb{C}$. ♥

Notice that we are dealing with uniqueness of solution without mentioning boundary conditions. In general, we do want to impose initial or boundary conditions when dealing with our solutions. If we want to be able to fix such conditions, we usually must be able to add solutions of the homogeneous equation to manipulate a few properties according to our desire. However, how can one define initial and boundary conditions when dealing with distributions?

We have more than one option. The first one involves classifying the kinds of solutions we have.

**Definition 6.9** [Classical, Weak and Distributional Solutions of a PDE]**:**
Let $p$ be a $k$-th degree polynomial. A solution of the differential equation $p(D)\varphi = \psi$, with $\psi \in \mathscr{D}'$, can be

i. a *classical solution*, if $\psi$ is actually a function and $\varphi$ is a $k$-times differentiable function;

ii. a *weak solution*, if $\varphi$ is a function, but it fails to be $k$-times differentiable;

iii. a *distributional solution*, if $\varphi$ can't be interpreted as a function. ♠

One could then, for example, define boundary conditions on $F$ by imposing boundary conditions on $f * F$, as long as it configures at least a weak solution sufficiently differentiable for the boundary conditions to be computed.

We'll now choose to specialize to a case of interest for example on Local Quantum Field Theory: hyperbolic operators. For these cases, the natural problem to deal with is the Cauchy problem: we specify data on a particular manifold and require it to be satisfied when solving the PDE.

For data to be satisfied, we roughly mean that on a suitable surface $S = \{x; \rho(x)\}$ we assign a function $u_0$ and require that a solution $u$ of $p(D)u = f$ has $u(x) - u_0(x) = \mathcal{O}(\rho^k)$ as $\rho(x) \to 0$, where $k$ is the degree of the polynomial $p$. We may then write that $u = u_0 + \rho^k v$. Differentiating and setting $\rho = 0$, we find that $f = p(D)u_0 + k!p_k(\rho')v$. If the surface is non-characteristic, which means $p_k(\rho') \neq 0$, the equation can be solved for $v$ and we can iterate the process, obtaining $u$ order by order. This provides the intuition behind the Cauchy problem.

In particular, notice that by setting $u = \varphi + \rho^k v$ we can just bother with solving the problem for $v$, and hence it suffices to consider the homogeneous case. We can then split the problem into an advanced and one retarded part: we solve for homogeneous data imposing $u$ and $f$ both vanish for $\rho \leq 0$ (retarded) and for homogeneous data imposing $u$ and $f$ both vanish for $\rho \geq 0$ (advanced).

This makes things easier: we can now impose boundary conditions by requiring for the supports of the fundamental solutions to vanish on this or that side of the surface $S$.

## 6.2 Hyperbolic Operators

We are particularly interested in the case of hyperbolic operators due to their physical motivations, which we will leave to Chapter 7 on page 259. For now, we develop the essential theory.

**Definition 6.10** [Hyperbolic Polynomial]**:**

Let $p(x) = \sum_{|\alpha|\leq k} \kappa_\alpha x^\alpha$ be a polynomial. $p$ is said to be *hyperbolic* with respect to $\xi \in \mathbb{R}^n$, $\xi \neq 0$, if, and only if, there is some $\tau_0 \in \mathbb{R}$ such that

$$p(\zeta + i\tau\xi) \neq 0, \forall\, \zeta \in \mathbb{R}^n, \forall\, \tau \leq \tau_0 \tag{6.23}$$

and

$$p_k(\xi) \equiv \sum_{|\alpha|\leq m} \kappa_\alpha \xi^\alpha \neq 0, \tag{6.24}$$

hold. ♠

**Definition 6.11** [Hyperbolic Equation]**:**

A linear differential operator with constant coefficients $p(D)$ is said to be *hyperbolic* if, and only if, there is at least one $\xi \in \mathbb{R}^n$ where the polynomial $p$ is hyperbolic. ♠

The template for a hyperbolic equation is the wave equation. Let us show it is indeed hyperbolic.

**Proposition 6.12:**

*The operator* $-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *is hyperbolic.* □

*Proof:*

We just need to show that the polynomial $p(x) = -x_0^2 + x_1^2$ is hyperbolic in at least one point. Let us consider $\xi = (1, 0)$.

The polynomial equals its principal part. We see that $p(\xi) = -\xi_0^2 + \xi_1^2 = -1 \neq 0$. Furthermore, given any $\eta \in \mathbb{R}^n$,

$$\begin{aligned} p(\zeta + i\tau\xi) &= -(\zeta_0 + i\tau)^2 + \zeta_1^2, \\ &= -\zeta_0^2 - 2i\zeta_0\tau + \tau^2 + \zeta_1^2, \\ &= [\tau - i(\zeta_0 + \zeta_1)][\tau - i(\zeta_0 - \zeta_1)], \end{aligned} \tag{6.25}$$

which, since $t \in \mathbb{R}$ and $\zeta \in \mathbb{R}^2$, is non-vanishing whenever $t \neq 0$. ■

A class of operators in which we are particularly interested is that of generalized d'Alembert operators, or normally hyperbolic operators.

**Definition 6.13** [Normally Hyperbolic Operator]**:**

Let $p\colon \mathbb{R}^{d+1} \to \mathbb{R}$ be a second degree polynomial. We say the differential operator $p(D)$ is a *normally hyperbolic operator*, or a *generalized d'Alembert operator*, if the principal part of $p$ can be written as

$$p_2(x) = \eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu, \tag{6.26}$$

where $\eta_{\mu\nu} = \text{diag}(-, +, \cdots, +)$ is the Minkowski metric tensor in $(d+1)$-dimensional spacetime. ♠

In the previous definition we've employed standard index notation from relativistic theories.

**Proposition 6.14:**

*Every normally hyperbolic operator is hyperbolic.* □

*Proof:*

We begin by noticing that, $\forall\, x \in \mathbb{R}^{d+1}$,

$$p(x) = \eta_{\mu\nu} x^\mu x^\nu + a_\mu x^\mu + b, \tag{6.27}$$

for some coefficients $a_\mu$ and $b$. Hence, $\forall\, \xi \in \mathbb{R}^{d+1}$,

$$\begin{aligned} p(\zeta + i\tau\xi) &= \eta_{\mu\nu}(\zeta^\mu + i\tau\xi^\mu)(\zeta^\nu + i\tau\xi^\nu) + a_\mu(\zeta^\mu + i\tau\xi^\mu) + b, \\ &= \zeta_\mu\zeta^\mu + 2i\tau\xi_\mu\zeta^\mu - \tau^2\xi_\mu\xi^\mu + a_\mu\zeta^\mu + ia_\mu\xi^\mu + b, \\ &= i(2\xi_\mu + a_\mu)\zeta^\mu + \zeta_\mu\zeta^\mu - \tau^2\xi_\mu\xi^\mu + a_\mu\zeta^\mu + b. \end{aligned} \tag{6.28}$$

We see then that the polynomial will be hyperbolic for any $\xi$ such that $2\xi_\mu \neq -a_\mu$ and $\xi_\mu\xi^\mu$. This can be accomplished by fixing all components, but one, of $\xi$ to be zero and choosing the remaining one such that $2\xi_\mu \neq -a_\mu$. ■

Our particular interest will now be to look for fundamental solutions of normally hyperbolic operators that are supported on the causal future, $J_+(0)$, or past, $J_(0)$, of the origin.

**Definition 6.15** [Causal Future and Past]**:**

Given $x \in \mathbb{R}^{d+1}$, we define its *causal future*, $J_+(x)$, and its *causal past*, $J_-(x)$, as the collections of points $y \in \mathbb{R}^{d+1}$ which can be reached from $x$ through future-oriented causal curves* or past-oriented causal curves, respectively.

Let $A \subseteq \mathbb{R}^{d+1}$. We define the *causal future* of $A$, $J_+(A)$, and the *causal past* of $A$, $J_-(A)$, according to the equation

$$J_\pm(A) \equiv \bigcup_{x\in A} J_\pm(x). \tag{6.29}$$

We'll often denote the causal future (past) of the origin as simply $J_\pm$. ♠

*We recall that a curve is causal if it is either timelike or lightlike

**Definition 6.16** [Advanced and Retarded Fundamental Solutions]**:**

Let $p(D)$ be a linear differential operator. We say a fundamental solution $F_+(x)$ for $p(D)$ at $x$ is *retarded* if, and only if,

$$\operatorname{supp} F_+(x) \subseteq J_+(x) \tag{6.30}$$

holds. Similarly, a fundamental solution $F_-(x)$ for $p(D)$ at $x$ is *advanced* if, and only if,

$$\operatorname{supp} F_-(x) \subseteq J_-(x) \tag{6.31}$$

holds. ♠

**Definition 6.17** [Advanced and Retarded Green's Operators]**:**

Let $p(D)$ be a linear differential operator. We say a linear map $G_+ \colon \mathscr{D} \to \mathscr{E}$ such that

i. $p(D) \circ G_+ = \mathrm{id}_{\mathscr{D}}$;

ii. $G_+ \circ p(D)\big|_{\mathscr{D}} = \mathrm{id}_{\mathscr{D}}$;

iii. $\operatorname{supp}(G_+ f) \subseteq J_+(\operatorname{supp} f), \forall f \in \mathscr{D}$

is a *retarded Green's operator* for $p$.

We define an *advanced Green's operator* for $p$ similarly, by simply changing the $+$ signs for $-$ signs. ♠

*Remark:*

On Local Quantum Field Theory, one often defines advanced and retarded Green's operators and fundamental solution according to the opposite convention[2, 3, 25]. Our naming convention is aimed at preserving the usual names one finds when dealing with Classical Physics. ♣

Finally, we state two results related to Green's operators and fundamental solutions of normally hyperbolic operators[25, pp. 81–82].

**Theorem 6.18:**

*Let $p(D) = \sum_{|\alpha| \leq k} \kappa_\alpha D^\alpha$ be a normally hyperbolic operator. The advanced (retarded) Green's operators $G_\pm$ have a one-to-one correspondence with retarded (advanced) fundamental solutions of $p^*(D) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \kappa_\alpha D^\alpha$,* id est, *if there is, $\forall x \in \mathbb{R}^{d+1}$, a fundamental solution $F_\pm(x)$ of $p^*(D)$ at $x$ such that*

i. $\operatorname{supp} F_\pm(x) \subseteq J_\pm(x)$;

ii. $x \mapsto \langle F_\pm(x), f \rangle$ *is smooth* $\forall f \in \mathscr{D}$;

iii. $p(D)(x \mapsto \langle F_\pm(x), f \rangle) = f, \forall f \in \mathscr{D}$,

*then*

$$(G_\mp f)(x) \equiv \langle F_\pm(x), f \rangle, \forall x \in \mathbb{R}^{d+1}, \forall f \in \mathscr{D} \tag{6.32}$$

*defines an advanced (retarded) Green's operator. Similarly, every advanced (retarded) Green's operator defines through this formula a fundamental solution of $p^*(D)$ satisfying the enumerated properties.* □

**Corollary 6.19:**
*A normally hyperbolic operators admits a unique advanced and a unique retarded Greens operator.* □

We'll often refer to advanced (retarded) fundamental solutions as advanced (retarded) Green's functions.

# Seven

# Physical Applications

> Mathematicians are only dealing with the structure of reasoning, and they do not really care what they are talking about. [...] But the physicist has meaning to all his phrases.
>
> RICHARD FEYNMAN. *The Character of Physical Law*, Chapter 2: The Relation of Mathematics to Physics.

## 7.1 Small Oscillations on a String

WE have already encountered the wave equation on Section 5.1 on page 127, when we were discussing small oscillations of a one-dimensional string. It reads, according to Eq. (5.6) on page 128,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \varphi(x, t, u). \tag{7.1}$$

We will assume the string undergoes small oscillations is homogeneous, so that $c$ can be taken to be constant. Under these conditions, notice that the wave equation is normally hyperbolic, and hence Theorem 6.18 and Corollary 6.19 on page 256 and on page 257 ensure existence and uniqueness of advanced and retarded Green's operators.

Let us then search for the retarded Green's function for the wave equation, *id est*, we want to find $G\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(t, x) = \delta\left(t\right)\delta\left(x\right), \tag{7.2}$$

where $G(t, x)$ is a distribution written in generalized function notation. It might or might not turn out to be a regular function.

We'll have the Fourier transform $\hat{G}$ of $G$ defined as

$$\hat{G}(\omega; k) \equiv \int G(t, x) e^{-i(\omega t + kx)} \, \mathrm{d}t \, \mathrm{d}x\,, \tag{7.3}$$

so that

$$G(t,x) = \frac{1}{(2\pi)^2}\int \hat{G}(\omega;k)e^{i(\omega t+kx)}\,\mathrm{d}\omega\,\mathrm{d}k\,. \tag{7.4}$$

We also recall that one can formally write, in generalized function notation,

$$\delta\,(t) = \frac{1}{2\pi}\int e^{i\omega t}\,\mathrm{d}\omega\,. \tag{7.5}$$

Fourier transforming Eq. (7.2) on the previous page leads to

$$\left(-k^2+\frac{\omega^2}{c^2}\right)\hat{G}(\omega,k) = 1,$$

$$\hat{G}(\omega,k) = \frac{c^2}{\omega^2-c^2k^2}, \tag{7.6}$$

and an inverse Fourier transform can then be used to yield

$$G(t,x) = \frac{c^2}{(2\pi)^2}\int \frac{e^{i(\omega t+kx)}}{\omega^2-c^2k^2}\,\mathrm{d}k\,\mathrm{d}\omega\,. \tag{7.7}$$

This leads us to consider the integral

$$\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2-c^2k^2}\,\mathrm{d}\omega\,, \tag{7.8}$$

which is ill-defined. This is not surprising: we are dividing a distribution by a polynomial which vanishes on the real line, which means the process is ill-defined. It simply means the fundamental solution is not unique. We know, however, that we are interested in the particular case of a retarded Green's function, meaning we want it to be supported on the future lightcone.

The integrand has poles at $\omega = \pm kc$. We could shift these poles by an infinitesimal imaginary amount $i\epsilon$ — the $i\epsilon$ prescription[67] —, we could shift the integration path by an infinitesimal amount, we could circle the poles in a carefully chosen way, and so on. The important point is we must avoid the poles somehow. To obtain a retarded Green's function, we shall choose $G(t,x) = 0, \forall\, t < 0$, which will lead to the conclusion the solution is supported on the future lightcone. Therefore, we choose the integration path shown in Fig. 7.1 on the facing page, which is slightly shifted from the real axis by an infinitesimal amount.

This choice essentially corresponds to computing

$$\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{(\omega-ck+i\epsilon)(\omega+ck+i\epsilon)}\,\mathrm{d}\omega \tag{7.9}$$

instead of the integral on Eq. (7.8). The Residue Theorem then ensures that, for $t > 0$, we can interpret

$$\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2-c^2k^2}\,\mathrm{d}\omega = 2\pi i\lim_{\epsilon\to 0^+}\left[\frac{e^{i(ck+i\epsilon)t}}{2(ck+i\epsilon)}-\frac{e^{-i(ck+i\epsilon)t}}{2(ck+i\epsilon)}\right],$$

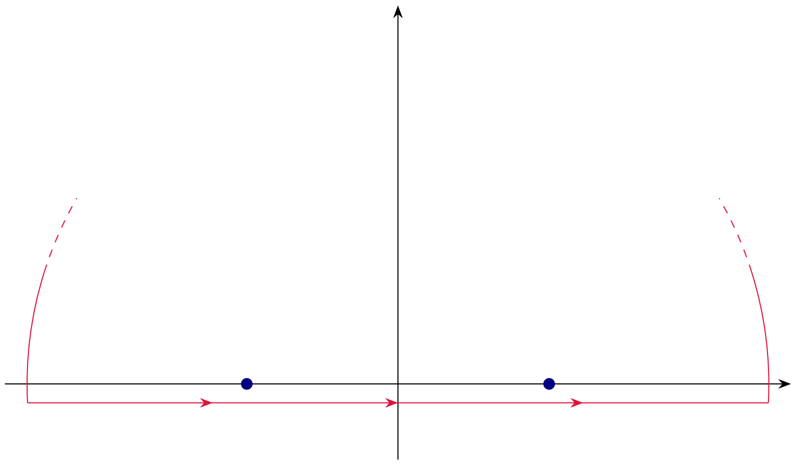


**Figure 7.1:** *Contour for the integral on Eq. (7.8) on the facing page. The straight line piece of the contour is parallel to the real axis, but has imaginary part $-i\epsilon$, for some infinitesimal $\epsilon > 0$. At the end of the computation, one might take $\epsilon \to 0$.*

$$
\begin{aligned}
&= 2\pi i \frac{e^{ickt} - e^{-ickt}}{2ck}, \\
&= -2\pi \frac{\sin ckt}{ck}, \qquad (7.10)
\end{aligned}
$$

while the integral vanishes for $t < 0$.

Replacing Eq. (7.10) on Eq. (7.7) on the preceding page yields

$$G(t,x) = \frac{-c}{2\pi} \int \frac{\sin ckt}{k} e^{ikx}\, \mathrm{d}k\,, \qquad (7.11)$$

for $t > 0$, and $G(t,x) = 0, \forall\, t < 0$.

This integral can be solved through Fourier analysis. Consider Heaviside's step function, $\Theta$. Notice that

$$
\begin{aligned}
\int \Theta(a - |z|) e^{-iz\xi}\, \mathrm{d}z &= \int_{-a}^{a} e^{-iz\xi}\, \mathrm{d}z\,, \\
&= 2\frac{\sin a\xi}{\xi}, \qquad (7.12)
\end{aligned}
$$

and therefore the inverse Fourier transform will yield

$$\frac{1}{\pi} \int \frac{\sin a\xi}{\xi} e^{+iz\xi}\, \mathrm{d}\xi = \Theta(a - |z|). \qquad (7.13)$$

Therefore,

$$G(t,x) = \frac{-c\Theta(ct - |x|)}{2}. \qquad (7.14)$$

We don't even need to bother inserting another step function to account for the fact that $G(t,x)$ vanishes for $t < 0$: $ct - |x| > 0$ already implies $t > 0$, since $c > 0$.

Eq. (7.14) provides the Green's function for the one-dimensional wave equation.

## 7.2 Oscillations of a Drum

A somewhat harder problem we might be interested in solving is that of the vibrations of some two-dimensional membrane, such as the surface of a drum.

Given some piece $S$ of the membrane, we shall assume the rest of the membrane applies on it some uniform tension. By uniform we mean its horizontal component, $\tau$, is independent of both space and time. We assume that, everywhere at the boundary $\partial S$ of $S$, the tension is directed along the normal of $\partial S$ and tangent to $S$. Essentially, different portions of the membrane are able to push and pull each other, but can't exert, *exempli gratia*, shears.

In addition to the hypothesis of uniform tension, we'll also assume the membrane has uniform density $\rho$.

The overall vertical component of the tension on $S$ will be given by*

$$\oint_{\partial S} \tau \boldsymbol{\nabla} u \cdot \mathbf{n} \, \mathrm{d}l \,, \tag{7.15}$$

where $\mathbf{n}$ is the vector normal to $\partial S$ and $\boldsymbol{\nabla} u$ is meant with respect to $x$ and $y$ only.

Green's Theorem implies that Eq. (7.15) can be written as

$$\oint_{\partial S} \tau \boldsymbol{\nabla} u \cdot \mathbf{n} \, \mathrm{d}l = \int_S \boldsymbol{\nabla} \cdot (\tau \boldsymbol{\nabla} u) \, \mathrm{d}S \,, \tag{7.16}$$

while Newton's Second Law leads to

$$\int_S \tau \nabla^2 u \, \mathrm{d}S = \int \rho \frac{\partial^2 u}{\partial t^2} \, \mathrm{d}S \,, \tag{7.17}$$

where we already used the facts that $\tau$ is constant and $\nabla^2 u = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

Since the equality holds for every piece of membrane $S$, it follows that

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \tag{7.18}$$

which is the two-dimensional homogeneous wave equation. As in the one-dimensional case, we have defined $c = \frac{\tau}{\rho}$.

Naturally, we could have added extra forces acting on the membrane — such as a viscous drag, gravity, *et cetera*— and the equation would read

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \varphi(x, t, u), \tag{7.19}$$

where the term $\varphi$ depends on the particular external force acting on the membrane. This equation is normally hyperbolic by simply defining $x^0 \equiv ct$.

*If the $\mathbf{n} \cdot \boldsymbol{\nabla} u$ term seems weird, you might want to check again on the one-dimensional case, particularly Eq. (5.3) on page 128.

Let us then look for the Green's function for the two-dimensional wave equation. We want to solve
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(t, \mathbf{x}) = \delta\left(t\right)\delta^{(2)}\left(\mathbf{x}\right). \tag{7.20}$$
Fourier transforming the equation in all variables leads us to

$$\left(-\|\mathbf{k}\|^2 + \frac{\omega^2}{c^2}\right)\hat{G}(\omega, \mathbf{k}) = 1, \tag{7.21}$$

and therefore

$$\hat{G}(\omega, \mathbf{k}) = \frac{c^2}{\omega^2 - c^2\|\mathbf{k}\|^2}. \tag{7.22}$$

We are an inverse Fourier transform away from the Green's function. $G(t, \mathbf{x})$ will be given by
$$G(t, \mathbf{x}) = \frac{c^2}{(2\pi)^3}\int \frac{e^{i(\omega t + \mathbf{k}\cdot\mathbf{x})}}{\omega^2 - c^2\|\mathbf{k}\|^2}\,\mathrm{d}\omega\,\mathrm{d}^2 k. \tag{7.23}$$

Once again we face an ill-defined integral. The physical situation we are interested in is still very similar to the vibrating string, and hence our previous prescription of choosing $G(t, \mathbf{x}) = 0, \forall\, t < 0$ will do just as well. Hence, we shift the integration contour according to Fig. 7.1 on page 261 and conclude from the Residue Theorem that

$$\int \frac{e^{i\omega t}}{\omega^2 - c^2\|\mathbf{k}\|^2}\,\mathrm{d}\omega = -\frac{2\pi}{c}\frac{\sin(c\|\mathbf{k}\|t)}{\|\mathbf{k}\|}, \tag{7.24}$$

when $t > 0$, vanishing for $t < 0$, and therefore

$$G(t, \mathbf{x}) = \frac{-c\Theta(t)}{(2\pi)^2}\int \frac{\sin(c\|\mathbf{k}\|t)e^{i\mathbf{k}\cdot\mathbf{x}}}{\|\mathbf{k}\|}\,\mathrm{d}^2 k. \tag{7.25}$$

Let us compute the integral in polar coordinates in the $\mathbf{k}$-plane. We define the polar angle, $\theta$, such that it is zero when $\mathbf{k}$ is parallel to $\mathbf{x}$. Denoting $s \equiv \|\mathbf{x}\|$ and $k = \|\mathbf{k}\|$ for simplicity, we get to

$$\begin{aligned} G(t, \mathbf{x}) &= \frac{-c\Theta(t)}{(2\pi)^2}\int \frac{\sin(ckt)e^{iks\cos\theta}}{k} k\,\mathrm{d}k\,\mathrm{d}\theta\,, \\ &= \frac{-c\Theta(t)}{(2\pi)^2}\int_0^{2\pi}\int_0^{+\infty} \sin(ckt)e^{iks\cos\theta}\,\mathrm{d}k\,\mathrm{d}\theta\,. \end{aligned} \tag{7.26}$$

To deal with this integral, let us recall from the theory of Bessel functions that[4, 6]

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x)t^n, \tag{7.27}$$

where $J_n(x)$ denotes the $n$-th order Bessel function of the first kind.

Picking $t = ie^{i\theta}$ we find that

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(x) e^{in\theta}, \tag{7.28}$$

and we notice the right-hand side of the equation is nothing but a Fourier series in $\theta$, which implies

$$J_n(x) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{ix\cos\theta - in\theta}\,\mathrm{d}\theta\,, \tag{7.29}$$

and lets us conclude from Eq. (7.26) on the preceding page that

$$G(t, \mathbf{x}) = \frac{-c\Theta(t)}{2\pi} \int_0^{+\infty} \sin(ckt) J_0(ks)\,\mathrm{d}k\,. \tag{7.30}$$

This still doesn't look particularly easy to integrate. Nevertheless, it can be shown[77, p. 170] that

$$J_0(x) = \frac{2}{\pi} \int_1^{+\infty} \frac{\sin(x\xi)}{\sqrt{\xi^2 - 1}}\,\mathrm{d}\xi\,, \tag{7.31}$$

hence allowing us to write

$$G(t, \mathbf{x}) = \frac{-c\Theta(t)}{\pi^2} \int_1^{+\infty} \frac{1}{\sqrt{\xi^2 - 1}} \int_0^{+\infty} \sin(ckt)\sin(ks\xi)\,\mathrm{d}k\,\mathrm{d}\xi\,. \tag{7.32}$$

To solve the integral over $k$, we shall use some properties of the Fourier sine transform. Given some function $f$ on a suitable space, we'll write

$$f^s(\xi) \equiv \int_0^{+\infty} f(x)\sin(x\xi)\,\mathrm{d}x\,. \tag{7.33}$$

Notice that $f^s(\xi)$ is an odd function.

Let $g$ be such that $g(x) = f(x), \forall\, x > 0$, but $g(x) = -f(-x), \forall\, x < 0$ and $g(0) = 0$, so $g$ is odd by construction. Notice that the Fourier transform $\hat{g}$ of $g$ has

$$\begin{aligned} \hat{g}(\xi) &= \int_{-\infty}^{+\infty} g(x) e^{-ix\xi}\,\mathrm{d}x\,, \\ &= -2i \int_0^{+\infty} g(x)\sin(x\xi)\,\mathrm{d}x\,, \\ &= -2i f^s(\xi). \end{aligned} \tag{7.34}$$

We also know that

$$\begin{aligned} g(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}(\xi) e^{i\xi x}\,\mathrm{d}\xi\,, \\ &= \frac{-i}{\pi} \int_{-\infty}^{+\infty} f^s(\xi) e^{i\xi x}\,\mathrm{d}\xi\,, \end{aligned}$$

$$= \frac{-2i}{\pi} \int_0^{+\infty} f^s(\xi) \sin(\xi x)\, \mathrm{d}\xi\,. \tag{7.35}$$

Replacing Eq. (7.33) on the preceding page on Eq. (7.35) then yields

$$\begin{aligned} g(x) &= \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} f(y) \sin(y\xi) \sin(x\xi)\, \mathrm{d}y\, \mathrm{d}\xi\,, \\ &= \int_0^{+\infty} f(y) \left[ \frac{2}{\pi} \int_0^{+\infty} \sin(y\xi) \sin(x\xi)\, \mathrm{d}\xi \right] \mathrm{d}y\,. \end{aligned} \tag{7.36}$$

We see then that

$$\frac{2}{\pi} \int_0^{+\infty} \sin(y\xi) \sin(x\xi)\, \mathrm{d}\xi \tag{7.37}$$

can be thought of as a generalized function similar to the translated Dirac delta $\delta(x-y)$, but different nevertheless: notice it doesn't take some function $f$ to $f(x)$ if $x \leq 0$. It is a different distribution which, nevertheless, we can treat here as being equal to the translated Dirac delta: Eq. (7.32) on the preceding page tells us that, luckily, our application only involves the function on the positive real line — since the integral over $\xi$ has $\xi > 1$ and $s > 0$ — meaning we can treat it as if it were a Dirac delta. Employing Eq. (7.36) on Eq. (7.32) on the preceding page leads to

$$G(t, \mathbf{x}) = \frac{-c\Theta(t)}{2\pi} \int_1^{+\infty} \frac{\delta(s\xi - ct)}{\sqrt{\xi^2 - 1}}\, \mathrm{d}\xi \tag{7.38}$$

and defining $\eta = s\xi - ct$ yields

$$\begin{aligned} G(t, \mathbf{x}) &= \frac{-c\Theta(t)}{2\pi s} \int_{s-ct}^{+\infty} \frac{\delta(\eta)}{\sqrt{\left(\frac{\eta+ct}{s}\right)^2 - 1}}\, \mathrm{d}\eta\,, \\ &= \frac{-c\Theta(t)}{2\pi} \frac{\Theta(ct-s)}{\sqrt{c^2t^2 - s^2}}. \end{aligned} \tag{7.39}$$

Since $s > 0$ and $c > 0$, $ct - s > 0$ implies $t > 0$, and the Green's function can be written simply as

$$G(t, \mathbf{x}) = \frac{-\Theta(ct-s)}{2\pi\sqrt{t^2 - \frac{s^2}{c^2}}}. \tag{7.40}$$

## 7.3 Electromagnetic Waves

In Heaviside–Lorentz units, Maxwell's equations read[36]

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho, \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \boldsymbol{\nabla} \times \mathbf{B} &= \frac{1}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{7.41}$$

where $\mathbf{E}$ stands for the electric field, $\mathbf{B}$ for the magnetic field, $\rho$ for the charge density, $\mathbf{J}$ for the current density and $c$ for the speed of light.

Applying the curl operator to the Faraday law and using the remaining equations to simplify the expression yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B}), \tag{7.42}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right), \tag{7.43}$$

$$\nabla\rho - \nabla^2\mathbf{E} = -\frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{7.44}$$

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla\rho + \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}. \tag{7.45}$$

Similarly, we can use the Ampère–Maxwell law to obtain

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c}\nabla \times \mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}), \tag{7.46}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} = \frac{1}{c}\nabla \times \mathbf{J} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}, \tag{7.47}$$

$$-\nabla^2\mathbf{B} = \frac{1}{c}\nabla \times \mathbf{J} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}, \tag{7.48}$$

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{1}{c}\nabla \times \mathbf{J}. \tag{7.49}$$

Eqs. (7.45) and (7.49) both share a similar structure: each component of the electric and magnetic fields satisfies an equation of the form

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \varphi(t, \mathbf{x}, u), \tag{7.50}$$

which is the three-dimensional wave equation. Once again a normally hyperbolic operator if one defines $x^0 \equiv ct$.

To find the Green's function for the three-dimensional wave equation, we must solve

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(t, \mathbf{x}) = \delta(t)\,\delta^{(3)}(\mathbf{x}). \tag{7.51}$$

Comparing Eqs. (7.20) and (7.51) on page 263 and on this page, we see at least part of the problem is no different from what we did in the two-dimensional case. *Mutatis mutandis*, we can borrow Eq. (7.25) on page 263 and get

$$G(t, \mathbf{x}) = \frac{-c\Theta(t)}{(2\pi)^3}\int \frac{\sin(c\|\mathbf{k}\|t)e^{i\mathbf{k}\cdot\mathbf{x}}}{\|\mathbf{k}\|}\,\mathrm{d}^3k. \tag{7.52}$$

Since we faced the two-dimensional integral in polar coordinates, this time we'll go spherical. Naturally, we pick the coordinate system such that the polar axis points in the same direction as $\mathbf{x}$. Denoting $r = \|\mathbf{x}\|$ and $k = \|\mathbf{k}\|$ we have

$$
\begin{aligned}
G(t,\mathbf{x}) &= \frac{-c\Theta(t)}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} k\sin(ckt)e^{ikr\cos\theta}\sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\varphi\,,\\
&= \frac{-c\Theta(t)}{(2\pi)^2}\int_{-1}^{1}\int_0^{+\infty} k\sin(ckt)e^{ikr\cos\theta}\,\mathrm{d}k\,\mathrm{d}(\cos\theta)\,,\\
&= \frac{-2c\Theta(t)}{(2\pi)^2 r}\int_0^{+\infty}\sin(ckt)\sin(kr)\,\mathrm{d}k\,.
\end{aligned}
\tag{7.53}
$$

Pulling on our result from Eq. (7.36) on page 265 and noticing that $r$, $t$ and $c$ are all positive, we see we can write

$$
G(t,\mathbf{x}) = \frac{-\Theta(t)\delta\left(t-\frac{r}{c}\right)}{4\pi r}, \tag{7.54}
$$

where we employed $a\delta\left(x\right) = \delta\left(\frac{x}{a}\right)$ for a positive constant $a$. Since the Dirac delta will enforce $ct = r \geq 0$, we can drop the Heaviside step function and simply write

$$
G(t,\mathbf{x}) = \frac{-\delta\left(t-\frac{r}{c}\right)}{4\pi r}. \tag{7.55}
$$

## 7.4 General Solution for Maxwell's Equations in Three Spatial Dimensions

Eq. (7.55) is a Green's function for the three-dimensional wave equation with the boundary conditions that the function must vanish at spatial infinity and be causal. We can now employ it to obtain a solution to Eqs. (7.45) and (7.49) on the facing page. In order to do it, let us first treat the problem abstractly and then specialize to the particular case of Electrodynamics.

Suppose we want to find a solution of

$$
\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \varphi(t,\mathbf{x}), \tag{7.56}
$$

where we choose the source term to be independent of $u$ for simplicity. A solution to the equation with the properties that it is causal and vanishes at spatial infinity will be provided by

$$
\begin{aligned}
u(t,x) &= \int G(t-t',\mathbf{x}-\mathbf{x}')\varphi(t',\mathbf{x}')\,\mathrm{d}t\,\mathrm{d}^3x\,,\\
&= \frac{-1}{4\pi}\int\frac{\delta\left(t-t'-\frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right)}{\|\mathbf{x}-\mathbf{x}'\|}\varphi(t',\mathbf{x}')\,\mathrm{d}t\,\mathrm{d}^3x\,,\\
&= \frac{-1}{4\pi}\int\frac{\varphi\left(t-\frac{\|\mathbf{x}-\mathbf{x}'\|}{c},\mathbf{x}'\right)}{\|\mathbf{x}-\mathbf{x}'\|}\,\mathrm{d}^3x'\,,\\
&= \frac{-1}{4\pi}\int\frac{\lfloor\varphi\rfloor}{\|\mathbf{x}-\mathbf{x}'\|}\,\mathrm{d}^3x'\,,
\end{aligned}
\tag{7.57}
$$

where we introduce the notation

$$\lfloor\varphi\rfloor \equiv \varphi\left(t - \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}, \mathbf{x}'\right), \tag{7.58}$$

*id est*, $\lfloor\varphi\rfloor$ represents $\varphi$ evaluated at the so-called retarded time. For future reference, it is useful to notice from that Eq. (7.57) on the previous page

$$\lfloor\varphi\rfloor \equiv \int \delta\left(t - t' - \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right) \varphi(t', \mathbf{x}')\, \mathrm{d}t', \tag{7.59}$$

which will be useful to derive some properties of retarded quantities.

Employing these results on Eqs. (7.45) and (7.49) on page 266 we find that

$$\mathbf{E}(t, \mathbf{x}) = \frac{-1}{4\pi} \int \frac{\lfloor\boldsymbol{\nabla}'\rho\rfloor + \frac{1}{c^2}\lfloor\dot{\mathbf{J}}\rfloor}{\|\mathbf{x}-\mathbf{x}'\|}\, \mathrm{d}^3x' \tag{7.60}$$

and

$$\mathbf{B}(t, \mathbf{x}) = \frac{1}{4\pi} \int \frac{\lfloor\boldsymbol{\nabla}' \times \mathbf{J}\rfloor}{c\|\mathbf{x}-\mathbf{x}'\|}\, \mathrm{d}^3x', \tag{7.61}$$

where, as one could expect, we denote $\dot{\varphi} \equiv \frac{\partial\varphi}{\partial t}$. The primes in the gradient and curl mean the spatial derivatives should be taken with respect to $\mathbf{x}'$ rather than $\mathbf{x}$.

Eqs. (7.60) and (7.61) do not provide as much insight as we'd like. It would be convenient if we could express them in a manner that reduces to the Coulomb and Biot–Savart laws in the static case, *id est*, we would like to express the electromagnetic fields in terms of integrals of the charge and current densities and their time derivatives.

In order to do so, we must rewrite the retarded spatial derivatives. We'll make use of the following equalities, derived on Section 7.A on page 289. Furthermore, we'll denote $\mathbf{R} \equiv \mathbf{x} - \mathbf{x}'$, $R \equiv \|\mathbf{R}\|$, $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R}$ to avoid writing $\mathbf{x} - \mathbf{x}'$ all the time.

As per Section 7.A on page 289, we have

$$\lfloor\boldsymbol{\nabla}'\rho\rfloor = \boldsymbol{\nabla}'\lfloor\rho\rfloor - \frac{\hat{\mathbf{R}}}{c}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor, \tag{7.193}$$

and

$$\lfloor\boldsymbol{\nabla}' \times \mathbf{J}\rfloor = \boldsymbol{\nabla}' \times \lfloor\mathbf{J}\rfloor - \frac{\hat{\mathbf{R}}}{c} \times \left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor, \tag{7.194}$$

which allow us to write Eqs. (7.60) and (7.61) as

$$\mathbf{E}(t, \mathbf{x}) = \frac{-1}{4\pi} \int \frac{\boldsymbol{\nabla}'\lfloor\rho\rfloor}{R} - \frac{\lfloor\dot{\rho}\rfloor\,\hat{\mathbf{R}}}{cR} + \frac{\lfloor\dot{\mathbf{J}}\rfloor}{c^2R}\, \mathrm{d}^3x' \tag{7.62}$$

and

$$\mathbf{B}(t, \mathbf{x}) = \frac{1}{4\pi c} \int \frac{\boldsymbol{\nabla}' \times \lfloor\mathbf{J}\rfloor}{R} - \frac{\hat{\mathbf{R}} \times \lfloor\dot{\mathbf{J}}\rfloor}{cR}\, \mathrm{d}^3x'. \tag{7.63}$$

The last two terms of Eq. (7.62) on the preceding page are already in a convenient form: if we pick the static limit, $\dot{\rho} = 0$ and $\dot{\mathbf{J}} = \mathbf{0}$, they'll vanish. We must only worry with the gradient term. Notice that

$$
\begin{aligned}
\int \frac{\boldsymbol{\nabla}' \lfloor \rho \rfloor}{R} \, \mathrm{d}^3 x' &= \int \boldsymbol{\nabla}' \left( \frac{\lfloor \rho \rfloor}{R} \right) \mathrm{d}^3 x' - \int \lfloor \rho \rfloor \, \boldsymbol{\nabla}' \left( \frac{1}{R} \right) \mathrm{d}^3 x' \,, \\
&= \oint \frac{\lfloor \rho \rfloor}{R} \, \mathrm{d}S - \int \lfloor \rho \rfloor \, \frac{\hat{\mathbf{R}}}{R^2} \, \mathrm{d}^3 x' \,, \\
&= - \int \lfloor \rho \rfloor \, \frac{\hat{\mathbf{R}}}{R^2} \, \mathrm{d}^3 x' \,,
\end{aligned}
\tag{7.64}
$$

where the surface integral vanishes if we assume the charge distribution vanishes sufficiently fast at spatial infinity.

Eq. (7.64) lets us rewrite Eq. (7.62) on the preceding page as

$$
\mathbf{E}(t, \mathbf{x}) = \frac{1}{4\pi} \int \frac{\lfloor \rho \rfloor \, \hat{\mathbf{R}}}{R^2} + \frac{\lfloor \dot{\rho} \rfloor \, \hat{\mathbf{R}}}{cR} - \frac{\lfloor \dot{\mathbf{J}} \rfloor}{c^2 R} \, \mathrm{d}^3 x' \,,
\tag{7.65}
$$

which immediately reduces to the Coulomb law for $\dot{\rho} = 0$ and $\dot{\mathbf{J}} = \mathbf{0}$.

As for the magnetic field, the second term on Eq. (7.63) on the facing page already looks like what we would like. We've got o deal with the curl, though. Proceeding as in the electric case we get

$$
\begin{aligned}
\int \frac{\boldsymbol{\nabla}' \times \lfloor \mathbf{J} \rfloor}{R} \, \mathrm{d}^3 x' &= \int \boldsymbol{\nabla}' \times \left( \frac{\lfloor \mathbf{J} \rfloor}{R} \right) \mathrm{d}^3 x' + \int \lfloor \mathbf{J} \rfloor \times \boldsymbol{\nabla}' \left( \frac{1}{R} \right) \mathrm{d}^3 x' \,, \\
&= - \oint \frac{\lfloor \mathbf{J} \rfloor}{R} \times \mathrm{d}\mathbf{S} + \int \lfloor \mathbf{J} \rfloor \times \frac{\hat{\mathbf{R}}}{R^2} \, \mathrm{d}^3 x' \,, \\
&= \int \lfloor \mathbf{J} \rfloor \times \frac{\hat{\mathbf{R}}}{R^2} \, \mathrm{d}^3 x' \,,
\end{aligned}
\tag{7.66}
$$

where once again the surface integral will vanish under the assumption that the current densities vanish sufficiently fast at spatial infinity.

Eq. (7.66) allows us to cast Eq. (7.63) on the facing page into the form

$$
\mathbf{B}(t, \mathbf{x}) = \frac{1}{4\pi c} \int \frac{\lfloor \mathbf{J} \rfloor \times \hat{\mathbf{R}}}{R^2} + \frac{\lfloor \dot{\mathbf{J}} \rfloor \times \hat{\mathbf{R}}}{cR} \, \mathrm{d}^3 x' \,,
\tag{7.67}
$$

which reduces to the Biot–Savart law if $\dot{\mathbf{J}} = \mathbf{0}$.

Eqs. (7.65) and (7.67) are known as Jefimenko's equations[37] and can be used to understand to which extent the Coulomb and Biot–Savart laws fail when we allow charges and currents to depend on time[28].

For example, suppose the current $\mathbf{J}$ is time independent. From Eq. (7.67), we see the Biot–Savart law holds, Eq. (7.65) also implies the Coulomb law holds. While the current term vanishes immediately, the contributions coming from the charges are less trivial. Let us study them.

Conservation of charge demands that the charge and current densities respect the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}. \tag{7.68}$$

However, if $\mathbf{J}$ does not depend on time, then neither does $\frac{\partial \rho}{\partial t}$. If $\frac{\partial \rho}{\partial t}$ is constant on time, then we have the differential equation

$$\frac{\partial \rho}{\partial t}(t, \mathbf{x}) = \dot{\rho}(0, \mathbf{x}), \tag{7.69}$$

which can be immediately integrated to yield

$$\rho(t, \mathbf{x}) = \rho(0, \mathbf{x}) + \dot{\rho}(0, \mathbf{x})t, \tag{7.70}$$

and hence the charge density is at most linear on time.

Hence, under the hypothesis that $\dot{\mathbf{J}} = \mathbf{0}$, Eq. (7.65) on the preceding page reduces to

$$\begin{aligned}
\mathbf{E}(t, \mathbf{x}) &= \frac{1}{4\pi} \int \frac{(\rho(0, \mathbf{x}) + t_{\mathbf{x}'}\dot{\rho}(0, \mathbf{x}'))\hat{\mathbf{R}}}{R^2} + \frac{\dot{\rho}(0, \mathbf{x}')\hat{\mathbf{R}}}{cR} \, \mathrm{d}^3 x', \\
&= \frac{1}{4\pi} \int \frac{\rho(0, \mathbf{x})\hat{\mathbf{R}}}{R^2} + \frac{(R + ct_{\mathbf{x}'})\dot{\rho}(0, \mathbf{x}')\hat{\mathbf{R}}}{cR^2} \, \mathrm{d}^3 x', \\
&= \frac{1}{4\pi} \int \frac{\rho(0, \mathbf{x})\hat{\mathbf{R}}}{R^2} + \frac{t\dot{\rho}(0, \mathbf{x}')\hat{\mathbf{R}}}{R^2} \, \mathrm{d}^3 x', \\
&= \frac{1}{4\pi} \int \frac{\rho(t, \mathbf{x})\hat{\mathbf{R}}}{R^2} \, \mathrm{d}^3 x',
\end{aligned} \tag{7.71}$$

which is Coulomb's law.

This also allows us to realize that accelerated charges emit radiation. Eqs. (7.65) and (7.67) on the previous page show that time-dependent sources lead to terms that decay as $\frac{1}{R}$ instead of the usual $\frac{1}{R^2}$ we get in the static case, but this occurs in a way that cancels out for charges moving at a constant velocity.

Eqs. (7.65) and (7.67) on the preceding page can be seen as the solutions to Maxwell's equations in three spatial dimensions. Notice this is not due to the fact that they were derived from the wave equations: satisfying the wave equations is a necessary condition to solve Maxwell's equations, but not a sufficient condition. For example, in principle we could have waves without $\boldsymbol{\nabla} \cdot \mathbf{B} = \mathbf{0}$. In fact, Eq. (7.65) on the previous page is the solution of a wave equation, but should have $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$. Nevertheless, it turns out Jefimenko's equations are indeed solutions of Maxwell's equations, as one can check through explicit computation.

## 7.5 General Solution for Maxwell's Equations in Less Spatial Dimensions

How would Jefimenko's equations look like if we lived in less than three spatial dimensions? We may answer this question by resorting to Eqs. (7.14) and (7.40) on page 261 and on page 265. Well, this in fact solves the math, but we also need to have some physics. How would Maxwell's equations look like in less than three spatial dimensions?

We can get some inspiration by looking at the field strenght tensor of covariant Electrodynamics. It reads, in three spatial dimensions,

$$F_{\mu_\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{7.72}$$

Hence, in two dimensions we would get

$$F^{(2)}_{\mu_\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B \\ E_y & -B & 0 \end{pmatrix}, \tag{7.73}$$

while in one dimension it simplifies even further to

$$F^{(1)}_{\mu_\nu} = \begin{pmatrix} 0 & -E_x \\ E_x & 0 \end{pmatrix}. \tag{7.74}$$

In two dimensions, the magnetic field is a pseudoscalar, while it doesn't even exist in one dimension. In both cases, we would expect Maxwell's equations to hold just as they do in three dimensions, but with the absent fields set to zero and the present fields being independent of the absent spatial coordinates. We get

$$\begin{cases} \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = \rho, \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = -\dfrac{1}{c}\dfrac{\partial B}{\partial t}, \\ \dfrac{\partial B}{\partial y} = \dfrac{1}{c}J_x + \dfrac{1}{c}\dfrac{\partial E_x}{\partial t}, \\ \dfrac{\partial B}{\partial x} = -\dfrac{1}{c}J_y - \dfrac{1}{c}\dfrac{\partial E_y}{\partial t}, \end{cases} \tag{7.75}$$

for the two-dimensional case and

$$\begin{cases} \dfrac{\partial E}{\partial x} = \rho, \\ \dfrac{1}{c}\dfrac{\partial E}{\partial t} = -\dfrac{1}{c}J, \end{cases} \tag{7.76}$$

in a single dimension.

With these expressions, one can show that in two-dimensions we have the wave equations

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \boldsymbol{\nabla}\rho + \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}, \tag{7.77}$$

$$\nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = -\frac{1}{c}\boldsymbol{\nabla} \times \mathbf{J}, \tag{7.78}$$

where the curl should be understood as the $\hat{\mathbf{z}}$-component only: $\boldsymbol{\nabla} \times \mathbf{J} \equiv \frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y}$.

One could say the one-dimensional case has

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial \rho}{\partial x} + \frac{1}{c^2}\frac{\partial J}{\partial t}, \tag{7.79}$$

but one should notice that the occurrence of the speed of light $c$ is now entirely arbitrary: Eq. (7.76) on the preceding page allows us to pick whichever speed of propagation we want. The presence of a magnetic field is what constrains the speed of propagation of signals to a particular value, and there is no such thing as a magnetic field in a single spatial dimension.

Furthermore, there is no wave propagation in one dimension. Eq. (7.76) on the previous page tell us the electric field changes according only to the presence of charges and currents. In vacuum, the electric field is constant, and therefore no wave can propagate.

Nevertheless, there is nothing to worry about: the equations turned out to be so simple we don't need the wave equations to solve them. Up to an additive constant — which is just the solution of the homogeneous equations that we've ignored in the three-dimensional case* — we have

$$E(t,x) = \frac{1}{2}\int_{-\infty}^{x} \rho(t,x')\,\mathrm{d}x' - \frac{1}{2}\int_{x}^{+\infty} \rho(t,x')\,\mathrm{d}x' - \frac{1}{2}\int_{-\infty}^{t} J(t')\,\mathrm{d}t' + \frac{1}{2}\int_{t}^{+\infty} J(t')\,\mathrm{d}t', \tag{7.80}$$

where we choose the constants of the integral limits such that all charges and currents are considered. Notice, however, that

$$E(t,x) = \int_0^x \rho(t,x')\,\mathrm{d}x' - \int_0^t J(t')\,\mathrm{d}t', \tag{7.81}$$

is also a perfectly valid solution, up to, of course, the integration constant corresponding to the solution of the homogeneous equation.

Let us give the two-dimensional case a try. Eq. (7.20) on page 263 tells us the solution to

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \varphi(t,\mathbf{x}) \tag{7.82}$$

is ($\mathbf{r} = \mathbf{x} - \mathbf{x}'$)

$$\begin{aligned} u(t,\mathbf{x}) &= \int G(t-t',\mathbf{x}-\mathbf{x}')\varphi(t',\mathbf{x}')\,\mathrm{d}t'\,\mathrm{d}^2x', \\ &= \frac{-1}{2\pi}\int \frac{\Theta(t-t'-\frac{r}{c})}{\sqrt{(t-t')^2-\frac{r^2}{c^2}}}\varphi(t',\mathbf{x}')\,\mathrm{d}t'\,\mathrm{d}^2x', \qquad (7.83) \\ &= \frac{-1}{2\pi}\iint\int_{-\infty}^{t-\frac{r}{c}} \frac{\varphi(t',\mathbf{x}')}{\sqrt{(t-t')^2-\frac{r^2}{c^2}}}\,\mathrm{d}t'\,\mathrm{d}^2x'. \qquad (7.84) \end{aligned}$$

*Those would be there to fit the initial and boundary conditions of the Universe. We are in general more interested in the forced oscillations due to the presence of charges and currents.

Therefore, Eq. (7.78) on page 271 is solved by

$$\mathbf{E}(t,\mathbf{x}) = \frac{-1}{2\pi} \iint \int_{-\infty}^{t-\frac{r}{c}} \frac{\boldsymbol{\nabla}'\rho(t',\mathbf{x}')}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} + \frac{1}{c^2\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \frac{\partial \mathbf{J}}{\partial t'} \,\mathrm{d}t'\,\mathrm{d}^2x' \tag{7.85}$$

and

$$B(t,\mathbf{x}) = \frac{1}{2\pi c} \iint \int_{-\infty}^{t-\frac{r}{c}} \frac{\boldsymbol{\nabla}' \times \mathbf{J}(t',\mathbf{x}')}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \,\mathrm{d}t'\,\mathrm{d}^2x' \,. \tag{7.86}$$

To simplify these expressions, we'll want to take a step back to the expression with the Heaviside step function, so we can employ the chain rule. For the charge gradient we get

$$\begin{aligned}
\int \frac{\Theta(t-t'-\frac{r}{c})\boldsymbol{\nabla}'\rho(t',\mathbf{x}')}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \,\mathrm{d}t'\,\mathrm{d}^2x' &= \int \boldsymbol{\nabla}' \left( \frac{\Theta(t-t'-\frac{r}{c})\rho(t',\mathbf{x}')}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \right) \mathrm{d}t'\,\mathrm{d}^2x' \\
&\quad - \int \rho(t',\mathbf{x}')\boldsymbol{\nabla}' \left( \frac{\Theta(t-t'-\frac{r}{c})}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \right) \mathrm{d}t'\,\mathrm{d}^2x' \,, \\
&= \int \oint \frac{\Theta(t-t'-\frac{r}{c})\rho(t',\mathbf{x}')}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \,\mathrm{d}\mathbf{l}\,\mathrm{d}t' \\
&\quad - \int \rho(t',\mathbf{x}')\boldsymbol{\nabla}' \left( \frac{\Theta(t-t'-\frac{r}{c})}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \right) \mathrm{d}t'\,\mathrm{d}^2x' \,, \\
&= - \int \rho(t',\mathbf{x}')\boldsymbol{\nabla}' \left( \frac{\Theta(t-t'-\frac{r}{c})}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \right) \mathrm{d}t'\,\mathrm{d}^2x' \,.
\end{aligned} \tag{7.87}$$

Naturally, the gradient in the remaining term should be understood in the distributional sense. We get

$$\begin{aligned}
\boldsymbol{\nabla}' \left( \frac{\Theta(t-t'-\frac{r}{c})}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \right) &= \frac{\boldsymbol{\nabla}'\Theta(t-t'-\frac{r}{c})}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} + \Theta(t-t'-\frac{r}{c})\boldsymbol{\nabla}' \left( \frac{1}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \right), \\
&= \frac{\delta\left(t-t'-\frac{r}{c}\right)}{c\sqrt{(t-t')^2 - \frac{r^2}{c^2}}}\hat{\mathbf{r}} - \frac{\Theta(t-t'-\frac{r}{c})}{2c\left[(t-t')^2 - \frac{r^2}{c^2}\right]^{\frac{3}{2}}}\hat{\mathbf{r}}.
\end{aligned} \tag{7.88}$$

Therefore,

$$\iint \int_{-\infty}^{t-\frac{r}{c}} \frac{\boldsymbol{\nabla}'\rho(t',\mathbf{x}')}{\sqrt{(t-t')^2 - \frac{r^2}{c^2}}} \,\mathrm{d}t'\,\mathrm{d}^2x' = \int \frac{\lfloor\rho\rfloor}{c\sqrt{(t-t')^2 - \frac{r^2}{c^2}}}\hat{\mathbf{r}} - \frac{\Theta(t-t'-\frac{r}{c})}{2c\left[(t-t')^2 - \frac{r^2}{c^2}\right]^{\frac{3}{2}}}\hat{\mathbf{r}} \tag{7.89}$$

## 7.6 Waves in $d$ Spatial Dimensions

Let us begin with the one-dimensional case. Eq. (7.14) on page 261 tells us a solution to

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \varphi(t,x) \tag{7.90}$$

is given by ($r \equiv x - x'$)

$$\begin{aligned} u(t,x) &= \int G(t-t', x-x')\varphi(t',x')\,\mathrm{d}t'\,\mathrm{d}x'\,, \\ &= \frac{-c}{2}\int \Theta\Big(t - t' - \frac{r}{c}\Big)\varphi(t',x')\,\mathrm{d}t'\,\mathrm{d}x'\,, \\ &= \frac{-c}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{t-\frac{r}{c}} \varphi(t',x')\,\mathrm{d}t'\,\mathrm{d}x'\,. \end{aligned} \tag{7.91}$$

From our discussions of waves in two and three spatial dimensions, we see that the $d$-dimensional case will have the Green's function determined by the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(t,\mathbf{x}) = \delta\,(t)\,\delta^{(d)}\,(\mathbf{x})\,, \tag{7.92}$$

which is solved by

$$G(t,\mathbf{x}) = \frac{-c\Theta(t)}{(2\pi)^d}\int \frac{\sin(c\|\mathbf{k}\|t)e^{i\mathbf{k}\cdot\mathbf{x}}}{\|\mathbf{k}\|}\,\mathrm{d}^d k\,. \tag{7.93}$$

## 7.7 Telegraph Signals

Suppose we want to study the transmission of signals on a telegraph line. While at first sight we could expect it to behave according to a wave equation, we must not forget that real transmission lines involve energetic losses, and hence the equation must be somewhat more complicated.

We can model the transmission line as having uniform resistivity $\rho$, inductance per unit length $\ell$, capacitance per unit length $\kappa$ and leakage per unit length $\sigma$. An infinitesimal piece of wire of length $\Delta x$ can then be see as the circuit on Fig. 7.2 on the next page. Our goal is to obtain the differential equations respected by the electric potential, $V(t,x)$, and by the electric current, $I(t,x)$.

At any particular instant $t$, the potential difference between two points $x_1$ and $x_2$ shall be given by

$$V(t,x_1) - V(t,x_2) = \rho\int_{x_1}^{x_2} I(t,x)\,\mathrm{d}x + \ell\int_{x_1}^{x_2}\frac{\partial I}{\partial t}(t,x)\,\mathrm{d}x\,, \tag{7.94}$$

but we also know that

$$V(t,x_1) - V(t,x_2) = -\int_{x_1}^{x_2}\frac{\partial V}{\partial x}(t,x)\,\mathrm{d}x\,. \tag{7.95}$$

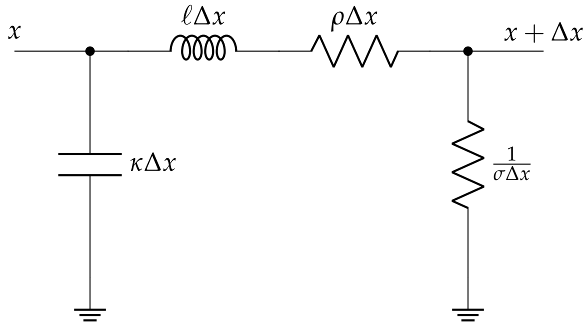



**Figure 7.2:** *Piece with infinitesimal length* $\Delta x$ *of a transmission line.*

Eqs. (7.94) and (7.95) on the facing page then yield

$$\int_{x_1}^{x_2} \frac{\partial V}{\partial x} + \rho I + \ell \frac{\partial I}{\partial t}\, \mathrm{d}x = 0. \tag{7.96}$$

Since this holds for arbitrary endpoints $x_1$ and $x_2$, it follows that

$$\frac{\partial V}{\partial x} + \rho I + \ell \frac{\partial I}{\partial t} = 0. \tag{7.97}$$

Let us also consider the variation of the current $I$ between the endpoints $x_1$ and $x_2$. We know

$$I(t, x_1) - I(t, x_2) = -\int_{x_1}^{x_2} \frac{\partial I}{\partial x}(t, x)\, \mathrm{d}x\,, \tag{7.98}$$

but we also know that the difference between the current entering the piece of wire and the current leaving it must correspond to the charge stored in that section of wire through capacitance effects and to the charge that leaked due to imperfect insulation of the wire. Therefore,

$$I(t, x_1) - I(t, x_2) = \sigma \int_{x_1}^{x_2} V(t, x)\, \mathrm{d}x + \kappa \int_{x_1}^{x_2} \frac{\partial V}{\partial t}\, \mathrm{d}x\,. \tag{7.99}$$

Eqs. (7.98) and (7.99) can then be combined to yield

$$\int_{x_1}^{x_2} \frac{\partial I}{\partial x} + \sigma V + \kappa \frac{\partial V}{\partial t}\, \mathrm{d}x = 0 \tag{7.100}$$

and once again arbitrariness of $x_1$ and $x_2$ allows us to get to a differential equation. Namely,

$$\frac{\partial I}{\partial x} + \sigma V + \kappa \frac{\partial V}{\partial t} = 0. \tag{7.101}$$

If we differentiate Eq. (7.97) with respect to $x$ and Eq. (7.101) with respect to $t$, we get the equations

$$\frac{\partial^2 V}{\partial x^2} + \rho \frac{\partial I}{\partial x} + \ell \frac{\partial^2 I}{\partial t \partial x} = 0 \tag{7.102}$$

and

$$\frac{\partial^2 I}{\partial x \partial t} + \sigma \frac{\partial V}{\partial t} + \kappa \frac{\partial^2 V}{\partial t^2} = 0. \tag{7.103}$$

If we substitute Eqs. (7.101) and (7.103) on the preceding page and on the current page on Eq. (7.102) on the preceding page to eliminate $I$, we get to

$$\frac{\partial^2 V}{\partial x^2} = \ell\kappa \frac{\partial^2 V}{\partial t^2} + (\rho\kappa + \ell\sigma)\frac{\partial V}{\partial t} + \rho\sigma V, \tag{7.104}$$

and a similar procedure shows that $I$ satisfies the very same equation.

Eq. (7.104) is known as the telegraph equation. It is convenient for us to define the constants

$$c = \frac{1}{\sqrt{\kappa\ell}}, \qquad \beta = \rho\sigma, \qquad \gamma = \frac{\ell\sigma + \kappa\rho}{\sqrt{\kappa\ell}}, \tag{7.105}$$

which are strictly positive in realistic physical situations, and deal with the slightly more abstract equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\gamma}{c}\frac{\partial u}{\partial t} - \beta u = 0. \tag{7.106}$$

Once again we are dealing with a normally hyperbolic operator, as one can see with the usual replacement $x^0 \equiv ct$.

Let us work out the Green function for the telegraph equation. We'd like to solve

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\gamma}{c}\frac{\partial}{\partial t} - \beta\right) G(t,x) = \delta(t)\,\delta(x)\,. \tag{7.107}$$

Taking the Fourier transform leads to

$$\begin{aligned} \left(-k^2 + \frac{\omega^2}{c^2} - \frac{i\omega\gamma}{c} - \beta\right)\hat{G}(\omega,k) &= 1, \\ \hat{G}(\omega,k) &= \frac{1}{\frac{\omega^2}{c^2} - \frac{i\omega\gamma}{c} - k^2 - \beta}, \end{aligned} \tag{7.108}$$

which means the Green's function will be given by

$$G(t,x) = \frac{1}{(2\pi)^2}\int \frac{e^{i(\omega t + kx)}}{\frac{\omega^2}{c^2} - \frac{i\omega\gamma}{c} - k^2 - \beta}\,\mathrm{d}\omega\,\mathrm{d}k\,. \tag{7.109}$$

The integrand has poles at

$$\frac{\omega_\pm}{c} = \pm\sqrt{k^2 + \beta - \frac{\gamma^2}{4}} + \frac{i\gamma}{2}, \tag{7.110}$$

which have $\mathrm{Im}[\omega_\pm] > 0$ whenever $\beta > 0, \gamma > 0$, which we assume to be true on physical grounds.

Since the poles are not on the real line, we only need to choose how to close the integration contour when computing the integral with the aid of the Reside Theorem. For $t > 0$, we close it on the upper half-plane and the poles contribute to the integral. For $t < 0$, we close it on the lower half-plane, where the integrand is analytic, and hence the integral vanishes. Therefore,

$$\begin{aligned} G(t,x) &= \frac{ic^2}{2\pi}\Theta(t)\int \frac{e^{i\omega_+ t} - e^{i\omega_- t}}{\omega_+ - \omega_-} e^{ikx}\,\mathrm{d}k\,, \\ &= \frac{ic}{4\pi} e^{-\frac{\gamma ct}{2}}\Theta(t)\int \frac{e^{ict\sqrt{k^2+\beta-\frac{\gamma^2}{4}}} - e^{-ict\sqrt{k^2+\beta-\frac{\gamma^2}{4}}}}{\sqrt{k^2+\beta-\frac{\gamma^2}{4}}} e^{ikx}\,\mathrm{d}k\,. \end{aligned} \tag{7.111}$$

We shall compute the integral of each exponential term separately. The integrals are

$$\int \frac{e^{ikx \pm ict\sqrt{k^2+\beta-\frac{\gamma^2}{4}}}}{\sqrt{k^2+\beta-\frac{\gamma^2}{4}}}\,\mathrm{d}k \tag{7.112}$$

and have branch points at

$$k_\pm = \pm\sqrt{\frac{\gamma^2}{4} - \beta}, \tag{7.113}$$

which leads us to the conclusion that the solution of the integral will depend on whether these points are pure real or pure imaginary.

We begin by dealing with the case $\Delta^2 \equiv \beta - \frac{\gamma^2}{4} > 0$. By writing $k = \Delta \sinh u$ we find that

$$\int_{-\infty}^{+\infty} \frac{e^{ikx \pm ict\sqrt{k^2+\Delta^2}}}{\sqrt{k^2+\Delta^2}}\,\mathrm{d}k = \int_{-\infty}^{+\infty} e^{i\Delta(x\sinh u \pm ct\cosh u)}\,\mathrm{d}u\,. \tag{7.114}$$

The Heaviside step function on Eq. (7.111) allows us to assume $ct > 0$. Depending on the sign of $x$ and of $ct - |x|$ we must use different substitutions. Namely,

$$\begin{cases} x = \sqrt{c^2t^2 - x^2}\sinh\theta, \\ ct = \sqrt{c^2t^2 - x^2}\cosh\theta, \end{cases} \qquad \text{if } x > 0 \text{ and } ct > |x|, \tag{7.115}$$

$$\begin{cases} x = -\sqrt{c^2t^2 - x^2}\sinh\theta, \\ ct = \sqrt{c^2t^2 - x^2}\cosh\theta, \end{cases} \qquad \text{if } x < 0 \text{ and } ct > |x|, \tag{7.116}$$

$$\begin{cases} x = \sqrt{x^2 - c^2t^2}\cosh\theta, \\ ct = \sqrt{x^2 - c^2t^2}\sinh\theta, \end{cases} \qquad \text{if } x > 0 \text{ and } ct < |x|, \tag{7.117}$$

and

$$\begin{cases} x = -\sqrt{x^2 - c^2t^2}\cosh\theta, \\ ct = \sqrt{x^2 - c^2t^2}\sinh\theta, \end{cases} \qquad \text{if } x < 0 \text{ and } ct < |x|. \tag{7.118}$$

With this substitutions we get to

$$\int_{-\infty}^{+\infty} \frac{e^{ikx\pm ict\sqrt{k^2+\Delta^2}}}{\sqrt{k^2+\Delta^2}}\,\mathrm{d}k = \begin{cases} \int_{-\infty}^{+\infty} e^{\pm i\Delta\sqrt{c^2t^2-x^2}\cosh(u\pm\theta)}\,\mathrm{d}u\,, & \text{if } x>0 \text{ and } ct>|x|, \\ \int_{-\infty}^{+\infty} e^{\pm i\Delta\sqrt{c^2t^2-x^2}\cosh(u-\theta)}\,\mathrm{d}u\,, & \text{if } x<0 \text{ and } ct>|x|, \\ \int_{-\infty}^{+\infty} e^{+i\Delta\sqrt{x^2-c^2t^2}\sinh(u\pm\theta)}\,\mathrm{d}u\,, & \text{if } x>0 \text{ and } ct<|x|, \\ \int_{-\infty}^{+\infty} e^{-i\Delta\sqrt{x^2-c^2t^2}\sinh(u\mp\theta)}\,\mathrm{d}u\,, & \text{if } x<0 \text{ and } ct<|x|. \end{cases} \tag{7.119}$$

These integrals can be computed with the aid of the theory of Bessel functions. For $z>0$ one knows that[77, pp. 180, 182]

$$H_0^{(1)}(z) = J_0(z) + iY_0(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} e^{iz\cosh\xi}\,\mathrm{d}\xi\,, \tag{7.120}$$

$$H_0^{(2)}(z) = J_0(z) - iY_0(z) = \frac{-1}{\pi i}\int_{-\infty}^{+\infty} e^{-iz\cosh\xi}\,\mathrm{d}\xi\,, \tag{7.121}$$

and

$$K_0(z) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{\pm iz\sinh\xi}\,\mathrm{d}\xi\,, \tag{7.122}$$

the $\pm$ sign on the last equation comes from the fact that if the equation holds with the negative sign (as stated in [77, p. 182]), one can just change the variable of integration according to $\xi \to -\xi$ and get the result for the positive sign as well.

Therefore,

$$\int_{-\infty}^{+\infty} \frac{e^{ikx\pm ict\sqrt{k^2+\Delta^2}}}{\sqrt{k^2+\Delta^2}}\,\mathrm{d}k = \begin{cases} \pm i\pi J_0(\Delta\sqrt{c^2t^2-x^2}) - Y_0(\Delta\sqrt{c^2t^2-x^2}), & \text{if } ct>|x|, \\ 2K_0(\Delta\sqrt{x^2-c^2t^2}), & \text{if } ct<|x|, \end{cases} \tag{7.123}$$

and we already see the result does not depend on the particular sign of $x$.

Eqs. (7.111) and (7.123) on the previous page and on this page now lead to

$$G(t,x) = -\frac{c}{2}e^{-\frac{\gamma ct}{2}}\Theta(ct-|x|)J_0\left(\sqrt{\beta-\frac{\gamma^2}{4}}\sqrt{c^2t^2-x^2}\right), \quad \text{if } \beta-\frac{\gamma^2}{4}>0. \tag{7.124}$$

Now let us deal with the case $\Delta^2 \equiv \frac{\gamma^2}{4} - \beta > 0$ (notice we changed our definition of $\Delta^2$).

We shall compute the integral of each exponential term separately. Let us begin by looking at

$$\int \frac{e^{ikx-ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\,\mathrm{d}k\,, \tag{7.125}$$

which has real branch points at

$$k_\pm = \pm\Delta. \tag{7.126}$$

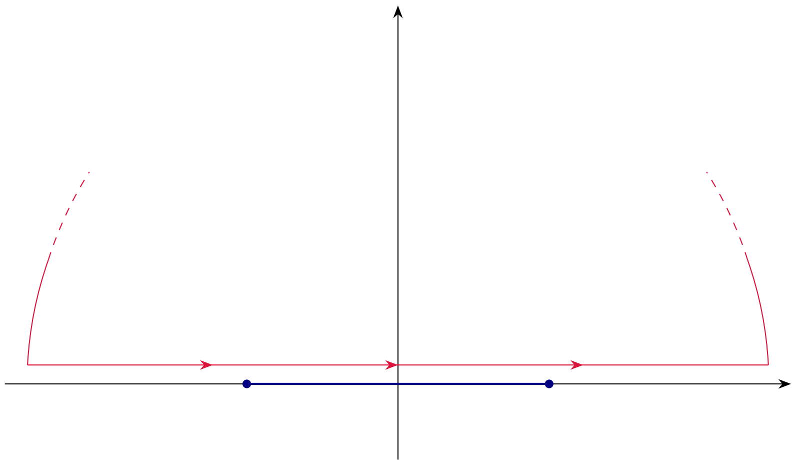


**Figure 7.3:** *Contour for the integral on Eq. (7.125) on the facing page with* $\Delta^2 \equiv \frac{\gamma^2}{4} - \beta > 0$. *The straight line piece of the contour is parallel to the real axis, but has imaginary part* $+i\epsilon$, *for some infinitesimal* $\epsilon > 0$.

We'll pick a branch cut connecting both branch points and shift the integration contour infinitesimaly towards the upper half-plane.

We'll want to close the contour on the upper or lower half-plane with the help of a large semicircle, and eventually take the limit as its radius goes to infinity. Hence, we want to take a look at the integrand's behavior as $|k| \to \infty$. For $|k| \gg |\Delta|$, the integrand can be regarded as

$$\frac{e^{ik(x-ct)}}{k}. \tag{7.127}$$

Hence, $x > ct$ leads us to close the contour on the upper half-plane, where the integrand is analytic, and hence the integral vanishes. $x < ct$ requires us to close the contour on the lower half-plane.

To deal with the integral once the contour has been closed on the lower half-plane, let us deform it without crossing any singularities so that it extends along the negative imaginary axis, as shown on Fig. 7.4 on the next page.

We have then written, for $\Delta^2 \equiv \frac{\gamma^2}{4} - \beta > 0$,

$$\int_{-\infty}^{+\infty} \frac{e^{ikx - ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\, \mathrm{d}k = \Theta(ct - x) \int_{\infty \cdot e^{\frac{3\pi i}{2}}}^{\infty \cdot e^{-\frac{\pi i}{2}}} \frac{e^{ikx - ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\, \mathrm{d}k\,. \tag{7.128}$$

If we now define $\theta$ and $\varphi$ such that

$$x = \sqrt{x^2 - c^2t^2}\cosh\theta \tag{7.129}$$

and

$$k = \Delta\cosh\varphi, \tag{7.130}$$

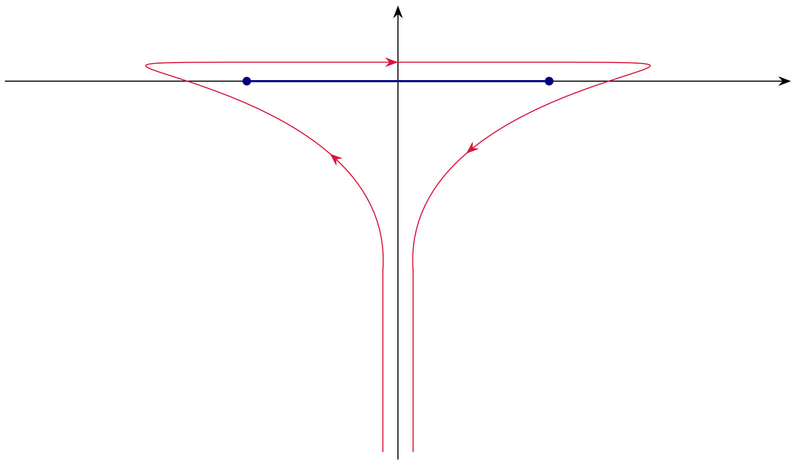


**Figure 7.4:** *Contour for the integral on Eq. (7.125) on page 278 with $\frac{\gamma^2}{4} > \beta$ and $|x| < ct$.*

then we may write, after manipulating the hyperbolic functions,

$$\int_{-\infty}^{+\infty} \frac{e^{ikx-ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\,\mathrm{d}k = \Theta(ct-x)\int_{\infty+\frac{3\pi i}{2}}^{\infty-\frac{\pi i}{2}} e^{i\Delta\sqrt{x^2-c^2t^2}\cosh(\varphi-\theta)}\,\mathrm{d}\varphi\,. \tag{7.131}$$

$\theta$ might not be real, since the definition we chose requires $i\sqrt{c^2t^2-x^2}\cosh\theta$ to be real and we know this expression will correspond only to the case with $x < ct$. Nevertheless, $\theta$ is certainly finite and cosh is periodic with respect to the imaginary part of its argument. Hence, we may write

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{e^{ikx-ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\,\mathrm{d}k &= \Theta(ct-x)\int_{\infty+\frac{3\pi i}{2}-\theta}^{\infty-\frac{\pi i}{2}-\theta} e^{-\Delta\sqrt{c^2t^2-x^2}\cosh\xi}\,\mathrm{d}\xi\,,\\ &= -\Theta(ct-x)\int_{\infty-\pi i}^{\infty+\pi i} e^{-\Delta\sqrt{c^2t^2-x^2}\cosh\xi}\,\mathrm{d}\xi\,.\end{aligned} \tag{7.132}$$

The theory of Bessel functions tells us that[77, p. 181]

$$I_0(z) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} e^{z\cosh\xi}\,\mathrm{d}\xi\,, \tag{7.133}$$

where $I_0$ is the zeroth order modified Bessel function of the first kind. We now see that

$$\int_{-\infty}^{+\infty} \frac{e^{ikx-ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\,\mathrm{d}k = -2\pi i\Theta(ct-x)I_0(\Delta\sqrt{c^2t^2-x^2}). \tag{7.134}$$

We still must solve the integral

$$\int_{-\infty}^{+\infty} \frac{e^{ikx+ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\,\mathrm{d}k\,. \tag{7.135}$$

Which is the same integral as before, except for a flipped sign in front of the only occurrence of time in the integral. This time, the integral will vanish for $x > -ct$, but not for $x < -ct$. The remaining steps shall carry out as before and we'd eventually get to

$$\int_{-\infty}^{+\infty} \frac{e^{ikx+ict\sqrt{k^2-\Delta^2}}}{\sqrt{k^2-\Delta^2}}\,\mathrm{d}k = -2\pi i\Theta(-ct-x)I_0(\Delta\sqrt{c^2t^2-x^2}). \tag{7.136}$$

Eqs. (7.134) and (7.136) on this page and on the preceding page can then be combined to yield

$$\int \frac{e^{ict\sqrt{k^2+\beta-\frac{\gamma^2}{4}}} - e^{-ict\sqrt{k^2+\beta-\frac{\gamma^2}{4}}}}{\sqrt{k^2+\beta-\frac{\gamma^2}{4}}} e^{ikx}\,\mathrm{d}k = 2\pi i\Theta(ct-|x|)I_0\left(\sqrt{\frac{\gamma^2}{4}-\beta}\sqrt{c^2t^2-x^2}\right), \tag{7.137}$$

which can be substituted on Eq. (7.111) on page 277 to yield

$$G(t,x) = -\frac{c}{2}e^{-\frac{\gamma ct}{2}}\Theta(ct-|x|)I_0\left(\sqrt{\frac{\gamma^2}{4}-\beta}\sqrt{c^2t^2-x^2}\right),, \quad \text{if } \frac{\gamma^2}{4}-\beta > 0. \tag{7.138}$$

which recovers Eq. (7.14) on page 261 if we take $\beta \to 0$ followed by $\gamma \to 0$.

## 7.8 A Relativistic Theory of Spinless Particles

While the Schrödinger Equation is capable of describing a myriad of physical phenomena to a good precision, it is certainly not an appropriate description of reality on a fundamental level, since it is not adequate under the light of Special Relativity.

This can be seen from the equation without the need of doing any calculations. Schrödinger's equation reads

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x},t)\psi, \tag{7.139}$$

which is of second-order on spatial variables, but of first-order on time. Relativistic equations should treat time and space on an equal stance, and hence we see this will break down for particles moving at speeds comparable to the speed of light.

Eq. (7.139) on the preceding page corresponds to the non-relativistic dispersion relation

$$E\psi = \frac{\|\mathbf{p}\|}{2m}\psi + V\psi, \tag{7.140}$$

where the operators $E$, and $\mathbf{p}$ are formally* defined according to

$$E = i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} - i\hbar\boldsymbol{\nabla} \tag{7.141}$$

*Since this derivation is, in fact, just a physical motivation for the Klein–Gordon equation, we won't bother with being rigorous.

and we denote $\|\mathbf{p}\| \equiv \mathbf{p} \cdot \mathbf{p} \equiv -\hbar^2 \nabla^2$.

Let us bother with finding a relativistic equation for a free quantum particle and leave the presence of a potential to another work. Special Relativity tells us that*

$$p^\mu p_\mu = -m^2 c^2, \tag{7.142}$$

$$\|\mathbf{p}\|^2 - \frac{E^2}{c^2} = -m^2 c^2, \tag{7.143}$$

$$E^2 = m^2 c^4 + \|\mathbf{p}\|^2 c^2, \tag{7.144}$$

where $p_\mu = \left(\frac{E}{c}, \mathbf{p}\right)$ is the particle's four-momentum, $m$ is its mass and $c$ is the speed of light.

If we replace Eq. (7.141) on the previous page on Eq. (7.144), we get to

$$-\hbar^2 \frac{\partial^2 \varphi}{\partial t^2} = m^2 c^4 \varphi - \hbar^2 c^2 \nabla^2 \varphi, \tag{7.145}$$

where $\varphi$ is the wavefunction. If we define $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$, we may then write Eq. (7.145) as

$$\left(\partial_\mu \partial^\mu - \frac{m^2 c^2}{\hbar^2}\right) \varphi(t, \mathbf{x}) = 0, \tag{7.146}$$

which is known as the Klein–Gordon equation. It is particularly appropriate for the description of spinless — or scalar — particles.

For simplicity, from now on we shall work on units with $c = \hbar = 1$.

One should notice, as usual, that the Klein–Gordon equation is a normally hyperbolic equation. In fact, it is one of the normally hyperbolic equations of most interest when dealing with Quantum Field Theory, since it describes free scalar fields, which are the simplest† possible theory one could come up with.

We want now to find the Green's function associated with the Klein–Gordon operator, *id est*, the function $G : \mathbb{M}^4 \to \mathbb{C}$ such that

$$\left(\partial_\mu \partial^\mu - m^2\right) G(x) = \delta^{(4)}(x), \tag{7.147}$$

where $\mathbb{M}^4$ denotes Minkowski spacetime.

Eq. (7.147) can be readily generalized for an arbitrary number of dimensions. Instead of working on a four-dimensional spacetime, let us for now consider the problem in $d$-dimensional Minkowski spacetime. Eq. (7.147) on page 282 becomes

$$\left(\partial_\mu \partial^\mu - m^2\right) G(x) = \delta^{(d)}(x), \tag{7.148}$$

and the partial derivatives $\partial_\mu$ now have $d-1$ spatial components.

Taking a Fourier transform leads to

$$-\left(k^2 + m^2\right) \hat{G}(k) = 1,$$

*We shall employ the convention in which the Minkowski metric tensor, $\eta_{\mu\nu}$, has signature $(-+++)$.

†In some ambiguous sense.

$$\hat{G}(k) = \frac{-1}{k^2 + m^2}, \tag{7.149}$$

where $k = (\omega, \mathbf{k})$ is a "$d$-vector". While dealing with Special Relativity, we shall convention that

$$\hat{G}(k) \equiv \int G(x) e^{-ik \cdot x} \, \mathrm{d}^d x \tag{7.150}$$

and

$$G(x) = \frac{1}{(2\pi)^d} \int \hat{G}(k) e^{ik \cdot x} \, \mathrm{d}^d k \,, \tag{7.151}$$

where $k \cdot x = k^\mu x_\mu$.

Therefore, we see that

$$G(x) = \frac{-1}{(2\pi)^d} \int \frac{e^{ik \cdot x}}{k^2 + m^2} \, \mathrm{d}^d k \,. \tag{7.152}$$

It is now particularly convenient for us to split the $d$-vectors into spatial and time components, leading us to

$$G(t, \mathbf{x}) = \frac{1}{(2\pi)^d} \int \frac{e^{-i\omega t} e^{i\mathbf{k} \cdot \mathbf{x}}}{\omega^2 - \|\mathbf{k}\|^2 - m^2} \, \mathrm{d}\omega \, \mathrm{d}^{d-1} k \,. \tag{7.153}$$

We'll now specialize the expression for two cases: $d = 1 + 1$ and $d = 3 + 1$, the latter being the one we actually want to compute. If we have a single spatial dimension, then

$$G^{(1)}(t, x) = \frac{1}{(2\pi)^2} \int \frac{e^{-i\omega t} e^{ikx}}{\omega^2 - k^2 - m^2} \, \mathrm{d}\omega \, \mathrm{d}k \,, \tag{7.154}$$

where $x$ now denotes only the spatial coordinate. Notation should be clear, since we are explicitly writing time as well. As the odd part of the integrand vanishes due to the symmetric integration interval, we can write

$$G^{(1)}(t, x) = \frac{1}{2\pi^2} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} \cos(kx)}{\omega^2 - k^2 - m^2} \, \mathrm{d}\omega \, \mathrm{d}k \,. \tag{7.155}$$

In three spatial dimensions we get

$$G^{(3)}(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int \frac{e^{-i\omega t} e^{i\mathbf{k} \cdot \mathbf{x}}}{\omega^2 - \|\mathbf{k}\|^2 - m^2} \, \mathrm{d}\omega \, \mathrm{d}^3 k \,. \tag{7.156}$$

To see why this is interesting, let us write the integral over $\mathbf{k}$ on $G^{(3)}(t, \mathbf{x})$ in spherical coordinates. We get to

$$G^{(3)}(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int_\Omega \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} e^{ikr\cos\theta}}{\omega^2 - k^2 - m^2} k^2 \sin\theta \, \mathrm{d}\omega \, \mathrm{d}k \, \mathrm{d}\theta \, \mathrm{d}\varphi \,, \tag{7.157}$$

where we chose the polar axis pointing in the same direction as $\mathbf{x}$ and denoted $r \equiv \|\mathbf{x}\|$, $k \equiv \|\mathbf{k}\|$. There is no risk of mistaking it for the four-vector, since we are explicitly writing $\omega$.

Performing the angular integrals yields

$$G^{(3)}(t,\mathbf{x}) = \frac{1}{4\pi^3 r} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} \cdot k \sin(kr)}{\omega^2 - k^2 - m^2} \, \mathrm{d}\omega \, \mathrm{d}k \, . \tag{7.158}$$

Notice how we are now able to write

$$\begin{aligned} G^{(3)}(t,\mathbf{x}) &= \frac{-1}{4\pi^3 r} \frac{\partial}{\partial r} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} \cdot \cos(kr)}{\omega^2 - k^2 - m^2} \, \mathrm{d}\omega \, \mathrm{d}k \, , \\ &= \frac{-1}{2\pi r} \frac{\partial}{\partial r} G^{(1)}(t,r), \end{aligned} \tag{7.159}$$

which uses Eq. (7.155) on the preceding page. Now we can use the Green's function from the problem with a single spatial dimension to obtain the solution to the problem in three-dimensions.

Furthermore, we have already solved the problem in one spatial dimension. Take a look again at the telegraph equation. In particular, consider Eq. (7.107) on page 276 with $\gamma = 0$, $\beta = m^2$ and $c = 1$ — the last two being due to our choices of units. This is just the Klein–Gordon equation in one spatial dimension, and the Green's function will be given by Eq. (7.124) on page 278, which, under the notation of the Klein–Gordon equation and with our choices of constants, becomes

$$G^{(1)}(t,r) = -\frac{1}{2} \Theta(t-r) J_0\left(m\sqrt{t^2 - r^2}\right), \tag{7.160}$$

where we take $r$ to be positive (it is the distance to the origin, regardless of sign).

Eq. (7.159) on the preceding page now tells us that

$$\begin{aligned} G^{(3)}(t,\mathbf{x}) &= \frac{-1}{2\pi r} \frac{\partial}{\partial r} G^{(1)}(t,r), \\ &= \frac{1}{4\pi r} \left[ \frac{\partial}{\partial r} \Theta(t-r) \cdot J_0\left(m\sqrt{t^2 - r^2}\right) + \Theta(t-r) \frac{\partial}{\partial r} J_0\left(m\sqrt{t^2 - r^2}\right) \right], \\ &= \frac{1}{4\pi r} \left[ -\delta\,(t-r) \cdot J_0\left(m\sqrt{t^2 - r^2}\right) + \frac{\Theta(t-r) \cdot mr J_1\left(m\sqrt{t^2 - r^2}\right)}{\sqrt{t^2 - r^2}} \right], \\ &= \frac{-\delta\,(t-r)}{4\pi r} + \frac{\Theta(t-r) \cdot m J_1\left(m\sqrt{t^2 - r^2}\right)}{4\pi\sqrt{t^2 - r^2}}. \end{aligned} \tag{7.161}$$

This is the retarded Green's function for the Klein–Gordon equation in three spatial dimensions.

## 7.9 A Relativistic Theory of Spin $\frac{1}{2}$ Particles

In spite of the fact that the Klein–Gordon equation allows for a relativistic description of quantum particles, some of its features hint that there is something more. This is not a text on Relativistic Quantum Mechanics or Quantum Field Theory*, and hence we shall not focus on many details, but let us mention, for example, that the fact the Klein–Gordon equation is of second-order on time immediately implies we should supply it with not only $\varphi(0, \mathbf{x})$, but also $\dot{\varphi}(0, \mathbf{x})$, for example. This contradicts one of the axioms of Quantum Mechanics, which claims the wavefunction should completely describe the state of the system at any given time. Therefore, it would be interesting for us to search for a partial differential equation, or for a system of partial differential equations, that is of first order in each variable.

Let us consider the ansatz†

$$\left(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}\right)\psi = 0, \tag{7.162}$$

where $\psi$ is the wavefunction and $\gamma^\mu$ are some coefficients that we still must determine. Eq. (7.162) is known as the Dirac equation‡, and it is suitable for the description of spin-$\frac{1}{2}$ fermions, such as electrons, neutrinos and quarks.

The coefficients $\gamma^\mu$ certainly can't be "just numbers", for this would imply $\gamma^\mu$ is a four-vector and there would be a preferred direction in spacetime. It turns out[27, 78, 80] these coefficients must be $4 \times 4$ matrices.

If we apply the operator $\left(\gamma^\mu \partial_\mu - m\right)$ on Eq. (7.162), we get

$$\left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - \frac{m^2c^2}{\hbar^2}\right)\psi = 0, \tag{7.163}$$

which can be written as

$$\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu - \frac{m^2c^2}{\hbar^2}\right)\psi = 0, \tag{7.164}$$

with $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$, due to the fact that the derivatives $\partial_\mu$ and $\partial_\nu$ commute (as all distributional derivatives do).

If we compare Eqs. (7.146) and (7.164) on page 282 and on the current page, we notice that if the $\gamma$ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}, \tag{7.165}$$

then Eq. (7.164) on the preceding page implies each component of $\psi$ respects the Klein–Gordon equation, and hence the Dirac equation described the relativistic motion of a

*For those, one can see, *exempli gratia*, [27, 55, 78, 80].

†It is possible to start from a more general ansatz, but we will take a shorter route by cheating and skipping a few developments. For a more careful approach, see, *exempli gratia*, [27, 78].

‡Beware we are working with $\eta_{\mu\nu} = (-+++)$, which is the reason the equation reads $\left(\gamma^\mu\partial_\mu + m\right)\psi = 0$ instead of $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$ ($c = \hbar = 1$).

quantum particle with mass $m$. As with the Klein–Gordon equation, we will now focus on working with units $c = \hbar = 1$.

Let us find the Green's function for the Dirac operator, *id est*, the function $S\colon \mathbb{M}^4 \to \mathbb{C}^4$ such that

$$\left(\gamma^\mu \partial_\mu + m\right) S(x) = \delta^{(4)}(x), \tag{7.166}$$

where a $4 \times 4$ identity matrix $\mathbb{1}$ should be understood whenever necessary for the equation to make sense. Notice also that $S(x)$ is a four-component object, and the same goes for $\delta^{(4)}(x)$.

Instead of proceeding as usual with the Fourier transforms and integrals galore, let us recall that the Dirac equation is, in some sense, a "square root" of the Klein–Gordon equation. In fact, it turns out that given a Green's function for the Klein–Gordon equation, $G$, we can obtain a Green's function $S$ for the Dirac equation.

Let $G$ be a Green's function for the Klein–Gordon equation. Define $S$ according to

$$S(x) \equiv \left(\gamma^\mu \partial_\mu - m\right) G(x). \tag{7.167}$$

Then it follows that

$$\begin{aligned}
\left(\gamma^\mu \partial_\mu + m\right) S(x) &= \left(\gamma^\mu \partial_\mu + m\right)(\gamma^\nu \partial_\nu - m) G(x), \\
&= \left(\partial^\mu \partial_\mu - m^2\right) G(x), \\
&= \delta^{(4)}(x). \qquad (7.168)
\end{aligned}$$

Hence, to obtain the Green's function for the Dirac operator we just need to differentiate the Green's function for the Klein–Gordon operator. This must be done with care, though: the term $\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}$ on Eq. (7.161) on page 284 means blindly differentiating formally, as we have done on most of these calculations, won't work, since it would lead us to a term such as $\frac{\delta(t-r)}{\sqrt{t^2-r^2}}$, which is ill-defined even in the sense of distributions.

Since Eq. (7.161) on page 284 treats time and space differently, there is not reason for us to stick with relativistic notation. We write

$$S(t, x) = \left(\gamma^0 \frac{\partial}{\partial t} + \vec{\gamma} \cdot \hat{\mathbf{r}} \frac{\partial}{\partial r} - m\right) G(t, x), \tag{7.169}$$

where $\hat{\mathbf{r}}$ stands for the unit radial vector.

Let us split $G$ in two parts, $G_\delta$ and $G_\Theta$, defined by

$$G_\delta(t, x) = -\frac{\delta(t - r)}{4\pi r}, \tag{7.170}$$

and

$$G_\Theta(t, x) = \frac{\Theta(t - r) \cdot m J_1\left(m\sqrt{t^2 - r^2}\right)}{4\pi\sqrt{t^2 - r^2}}. \tag{7.171}$$

The derivatives of $G_\delta$ can be computed "blindly" by just relying on generalized function notation. One has

$$\frac{\partial G_\delta}{\partial t} = -\frac{\delta'(t-r)}{4\pi r}, \tag{7.172}$$

and

$$\frac{\partial G_\delta}{\partial r} = \frac{\delta'(t-r)}{4\pi r} + \frac{\delta(t-r)}{4\pi r^2}. \tag{7.173}$$

$\frac{\partial G_\Theta}{\partial t}$ requires a little more care. We shall compute the derivative by the definition of distributional derivative. Given an arbitrary test function $f$ on a suitable space we have

$$\begin{aligned}\left\langle \frac{\partial G_\Theta}{\partial t}, f \right\rangle &= -\left\langle G_\Theta, f' \right\rangle, \\ &= -\frac{m}{4\pi}\int_r^{+\infty} \frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} f'(t)\,\mathrm{d}t, \\ &= \frac{m}{4\pi}\int_r^{+\infty} \frac{\partial}{\partial t}\left(\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\right) f(t)\,\mathrm{d}t + \frac{m}{4\pi}\lim_{t\to r^+}\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} f(t).\end{aligned} \tag{7.174}$$

The function $J_1$ can be expressed as the series

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n}}{4^n n!(n+1)!}, \tag{7.175}$$

meaning its behaviour close to the origin is given by

$$J_1(x) = \frac{x}{2} + \mathcal{O}(x^3). \tag{7.176}$$

Therefore,

$$\lim_{t\to r^+}\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} f(t) = \frac{m}{2} f(r). \tag{7.177}$$

To compute the derivative in the integrand of Eq. (7.174) on page 287, let us recall the recurrence relation[77, p. 45]

$$J_1'(z) = -J_2(z) + \frac{1}{z}J_1(z). \tag{7.178}$$

With it in mind, we notice that

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\right) &= J_1(m\sqrt{t^2-r^2})\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{t^2-r^2}}\right)+\frac{1}{\sqrt{t^2-r^2}}\frac{\partial}{\partial t}\Big(J_1(m\sqrt{t^2-r^2})\Big),\\
&= -\frac{tJ_1(m\sqrt{t^2-r^2})}{[t^2-r^2]^{\frac{3}{2}}}\\
&\quad+\frac{m\frac{\partial}{\partial t}\left(\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}}\left(-J_2(m\sqrt{t^2-r^2})+\frac{J_1(m\sqrt{t^2-r^2})}{m\sqrt{t^2-r^2}}\right),\\
&= -\frac{mtJ_2(m\sqrt{t^2-r^2})}{t^2-r^2}. \qquad (7.179)
\end{aligned}$$

Hence,

$$\left\langle\frac{\partial G_\Theta}{\partial t},f\right\rangle = -\frac{m^2}{4\pi}\int\frac{t\Theta(t-r)J_2(m\sqrt{t^2-r^2})}{t^2-r^2}f(t)\,\mathrm{d}t+\frac{m^2}{8\pi}f(r), \qquad (7.180)$$

implying that

$$\frac{\partial G_\Theta}{\partial t} = -\frac{m^2}{4\pi}\frac{t\Theta(t-r)J_2(m\sqrt{t^2-r^2})}{t^2-r^2}+\frac{m^2}{8\pi}\delta\left(t-r\right). \qquad (7.181)$$

Similarly,

$$\begin{aligned}
\left\langle\frac{\partial G_\Theta}{\partial r},f\right\rangle &= -\frac{m}{4\pi}\int_{-\infty}^{t}\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}f'(r)\,\mathrm{d}r\,,\\
&= \frac{m}{4\pi}\int_{-\infty}^{t}\frac{\partial}{\partial r}\left(\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\right)f(r)\,\mathrm{d}r-\frac{m}{4\pi}\lim_{r\to t^-}\frac{J_1(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}f(r),\\
&= \frac{m}{4\pi}\int\frac{r\Theta(t-r)J_2(m\sqrt{t^2-r^2})}{t^2-r^2}f(r)\,\mathrm{d}r-\frac{m^2}{8\pi}f(t), \qquad (7.182)
\end{aligned}$$

so we conclude that

$$\frac{\partial G_\Theta}{\partial r} = \frac{m^2}{4\pi}\frac{r\Theta(t-r)J_2(m\sqrt{t^2-r^2})}{t^2-r^2}-\frac{m^2}{8\pi}\delta\left(t-r\right). \qquad (7.183)$$

Therefore, we conclude that the retarded Green's function for the Dirac operator is

$$S(t,x) = \gamma^0\frac{\partial G}{\partial t}(t,x)+\vec{\gamma}\cdot\hat{\mathbf{r}}\frac{\partial G}{\partial r}(t,x)-mG(t,x), \qquad (7.184)$$

where

$$\frac{\partial G}{\partial t}(t,x) = -\frac{\delta'\left(t-r\right)}{4\pi r}+\frac{m^2}{8\pi}\delta\left(t-r\right)-\frac{m^2}{4\pi}\frac{t\Theta(t-r)J_2(m\sqrt{t^2-r^2})}{t^2-r^2}, \qquad (7.185)$$

$$\frac{\partial G}{\partial t}(t,x) = \frac{\delta'(t-r)}{4\pi r} + \frac{1}{4\pi}\left(\frac{1}{r^2} - \frac{m^2}{2}\right)\delta(t-r) + \frac{m^2}{4\pi}\frac{r\Theta(t-r)J_2(m\sqrt{t^2-r^2})}{t^2-r^2}, \quad (7.186)$$

and

$$G(t,x) = \frac{-\delta(t-r)}{4\pi r} + \frac{\Theta(t-r)\cdot mJ_1\left(m\sqrt{t^2-r^2}\right)}{4\pi\sqrt{t^2-r^2}}. \quad (7.187)$$

One should notice $S$ is a matrix. To compute each entry one would need to choose a particular representation of the $\gamma$ matrices.

## 7.A Retarded Quantities

In order to write Eqs. (7.60) and (7.61) on page 268,

$$\mathbf{E}(t,\mathbf{x}) = \frac{-1}{4\pi}\int \frac{\lfloor \boldsymbol{\nabla}'\rho\rfloor + \frac{1}{c^2}\lfloor \dot{\mathbf{J}}\rfloor}{\|\mathbf{x}-\mathbf{x}'\|}\, \mathrm{d}^3x' \quad (7.60)$$

and

$$\mathbf{B}(t,\mathbf{x}) = \frac{1}{4\pi}\int \frac{\lfloor \boldsymbol{\nabla}'\times\mathbf{J}\rfloor}{c\|\mathbf{x}-\mathbf{x}'\|}\, \mathrm{d}^3x', \quad (7.61)$$

in a way that resembles the Coulomb and Biot–Savart laws, it is necessary to rewrite some quantities evaluated on retarded time. We'll derive the necessary identities in this appendix through the use of Eq. (7.59) on page 268,

$$\lfloor \varphi\rfloor \equiv \int \delta\left(t - t' - \frac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right)\varphi(t',\mathbf{x}')\,\mathrm{d}t'. \quad (7.59)$$

Since writing $\mathbf{x}-\mathbf{x}'$ can make the following expressions quite cumbersome, we'll write $\mathbf{R}\equiv\mathbf{x}-\mathbf{x}'$, $R\equiv\|\mathbf{R}\|$, $\hat{\mathbf{R}}\equiv\frac{\mathbf{R}}{R}$.

Let us employ Eq. (7.59) on page 268 to open up the spatial derivatives on Eqs. (7.60) and (7.61) on page 268. For simplicity, we shall denote retarded time as $t_{\mathbf{x}'}\equiv t-\frac{R}{c}$. We begin with the charge gradient, which has

$$\begin{aligned}\lfloor \boldsymbol{\nabla}'\rho\rfloor &= \int \delta\left(t_{\mathbf{x}'}-t'\right)\boldsymbol{\nabla}'\rho(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \int \boldsymbol{\nabla}'\left(\delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\right) - \boldsymbol{\nabla}'\left(\delta\left(t_{\mathbf{x}'}-t'\right)\right)\rho(t',\mathbf{x}')\,\mathrm{d}t'. \quad (7.188)\end{aligned}$$

Since $\mathbf{x}$ and $\mathbf{x}'$ occur on equal footings within $\delta\left(t_{\mathbf{x}'}-t'\right)$, except for a minus sign. Hence, we can drop the prime in its derivative if we agree to flip a sign, getting to

$$\begin{aligned}\lfloor \boldsymbol{\nabla}'\rho\rfloor &= \boldsymbol{\nabla}'\left(\int \delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\,\mathrm{d}t'\right) + \int \boldsymbol{\nabla}\left(\delta\left(t_{\mathbf{x}'}-t'\right)\right)\rho(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \boldsymbol{\nabla}'\lfloor\rho\rfloor + \int \boldsymbol{\nabla}\left(\delta\left(t_{\mathbf{x}'}-t'\right)\right)\rho(t',\mathbf{x}')\,\mathrm{d}t'. \quad (7.189)\end{aligned}$$

We still must deal with the Dirac delta derivative, which, of course, should be understood in the sense of distributions. Using the chain rule we see that

$$\begin{aligned}\boldsymbol{\nabla}\delta\left(t_{\mathbf{x}'}-t'\right) &= \delta'\left(t_{\mathbf{x}'}-t'\right)\boldsymbol{\nabla}\left(t-t'-\frac{R}{c}\right),\\ &= -\frac{\hat{\mathbf{R}}}{c}\frac{\partial}{\partial t}\delta\left(t_{\mathbf{x}'}-t'\right).\end{aligned} \tag{7.190}$$

Therefore,

$$\begin{aligned}\lfloor\boldsymbol{\nabla}'\rho\rfloor &= \boldsymbol{\nabla}'\lfloor\rho\rfloor-\frac{\hat{\mathbf{R}}}{c}\frac{\partial}{\partial t}\int\delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \boldsymbol{\nabla}'\lfloor\rho\rfloor-\frac{\hat{\mathbf{R}}}{c}\frac{\partial\lfloor\rho\rfloor}{\partial t}.\end{aligned} \tag{7.191}$$

Notice the time derivative is done in the $t$ variable, while the integral is over $t'$.

However, one should notice that

$$\begin{aligned}\frac{\partial\lfloor\rho\rfloor}{\partial t} &= \int\frac{\partial}{\partial t}\delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\,\mathrm{d}t',\\ &= -\int\frac{\partial}{\partial t'}\delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \int\delta\left(t_{\mathbf{x}'}-t'\right)\frac{\partial}{\partial t'}\rho(t',\mathbf{x}')\,\mathrm{d}t'-\int\frac{\partial}{\partial t'}\left(\delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\right)\mathrm{d}t',\\ &= \left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor-\delta\left(t_{\mathbf{x}'}-t'\right)\rho(t',\mathbf{x}')\Big|_{-\infty}^{+\infty},\\ &= \left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor,\end{aligned} \tag{7.192}$$

and therefore we can write

$$\lfloor\boldsymbol{\nabla}'\rho\rfloor = \boldsymbol{\nabla}'\lfloor\rho\rfloor-\frac{\hat{\mathbf{R}}}{c}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor. \tag{7.193}$$

Next we do the current curl. Proceeding as we have with the charge gradient, we see that

$$\begin{aligned}\lfloor\boldsymbol{\nabla}'\times\mathbf{J}\rfloor &= \int\delta\left(t_{\mathbf{x}'}-t'\right)\boldsymbol{\nabla}'\times\mathbf{J}(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \boldsymbol{\nabla}'\times\int\delta\left(t_{\mathbf{x}'}-t'\right)\mathbf{J}(t',\mathbf{x}')\,\mathrm{d}t'+\int\mathbf{J}(t',\mathbf{x}')\times\boldsymbol{\nabla}'\delta\left(t_{\mathbf{x}'}-t'\right)\mathrm{d}t',\\ &= \boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor+\int\boldsymbol{\nabla}\delta\left(t_{\mathbf{x}'}-t'\right)\times\mathbf{J}(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor-\frac{\hat{\mathbf{R}}}{c}\times\frac{\partial}{\partial t}\int\delta\left(t_{\mathbf{x}'}-t'\right)\mathbf{J}(t',\mathbf{x}')\,\mathrm{d}t',\\ &= \boldsymbol{\nabla}'\times\lfloor\mathbf{J}\rfloor-\frac{\hat{\mathbf{R}}}{c}\times\frac{\partial\lfloor\mathbf{J}\rfloor}{\partial t},\end{aligned}$$

$$= \nabla' \times \lfloor \mathbf{J} \rfloor - \frac{\hat{\mathbf{R}}}{c} \times \left\lfloor \frac{\partial \mathbf{J}}{\partial t} \right\rfloor . \tag{7.194}$$

# Bibliography


1. Alves, N. *A Brief Introduction to Classical Fied Theory* http://fma.if.usp.br/~nickolas/pdf/A_Brief_Introduction_to_Classical_Field_Theory.pdf.
2. Bär, C. & Ginoux, N. *Classical and Quantum Fields on Lorentzian Manifolds* in *Global Differential Geometry* (eds Bär, C., Lohkamp, J. & Schwarz, M.) (Springer, Berlin, 2012), 359–400. DOI: 10.1007/978-3-642-22842-1_12.
3. Bär, C., Ginoux, N. & Pfäffle, F. *Wave Equations on Lorentzian Manifolds and Quantization* (EMS Publishing House, Zürich, 2007).
4. Barata, J. C. A. *Notas de Física-Matemática* http://denebola.if.usp.br/~jbarata/Notas_de_aula/notas_de_aula.html.
5. Bogoliubov, N. N. & Shirkov, D. V. *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980).
6. Braga, C. L. R. *Notas de Física-Matemática: Equações Diferenciais, Funções de Green e Distribuições* (eds Wrezinski, W. F., Perez, J. F., Marchetti, D. H. U. & Barata, J. C. A.) (Ed. Livraria da Física, São Paulo, 2006).
7. Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (Springer, New York, 2011).
8. Cavenaghi, L. F. *O Básico de Geometria Riemanniana para Cálculo Variacional* https://www.ime.usp.br/~piccione/Downloads/O_b_sico_de_Geometria_Riemanniana_para_m_todos_variacionais.pdf.
9. Coelho, F. U. & Lourenço, M. L. *Um Curso de Álgebra Linear* (Editora da Universidade de São Paulo, São Paulo, 2013).
10. Conway, J. *A Course in Functional Analysis* (Springer Science+Business Media, New York, 2010).
11. Couto, R. T. Green's functions for the wave, Helmholtz and Poisson equations in a two-dimensional boundless domain. *Revista Brasileira de Ensino de Física* **35,** 01–08 (2013). DOI: 10.1590/S1806-11172013000100004.
12. Couto, R. T. Sobre a Dedução da Equação da Onda e da Solução Segundo a Fórmula de Kirchhoff. *TEMA (São Carlos)* **11,** 49–58 (2010).

13. De Figueiredo, D. G. *Análise de Fourier e Equações Diferenciais Parciais* (Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2007).
14. De Oliveira, O. R. B. *Medida e Integração* (lecture notes). https://www.ime.usp.br/~oliveira/Disciplinas.html.
15. Dieudonné, J. *Treatise on Analysis Vol. II* (Academic Press, New York, 1976).
16. Engelking, R. *General Topology* (Heldermann Verlag, Berlin, 1989).
17. Evans, L. C. *Partial Differential Equations* (American Mathematical Society, Providence, 2010).
18. Feldman, J. *Derivation of the Telegraph Equation* https://www.math.ubc.ca/~feldman/apps/telegrph.pdf.
19. Fernandez, P. J. *Medida e Integração* (Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2015).
20. Folland, G. B. *Fourier Analysis and its Applications* (Brooks/Cole, Pacific Grove, 1992).
21. Folland, G. B. *Introduction to Partial Differential Equations* (Princeton University Press, Princeton, 1995).
22. Folland, G. B. *Real Analysis: Modern Techniques and Their Applications* (Wiley, New York, 1999).
23. Friedman, A. *Generalized Functions and Partial Differential Equations* (Dover Publications, Inc., Mineala, New York, 1963).
24. Gårding, L. Linear hyperbolic partial differential equations with constant coefficients. *Acta Mathematica* **85,** 01–62 (1951). DOI: 10.1007/BF02395740.
25. Ginoux, N. in *Quantum Field Theory on Curved Spacetimes: Concepts and Mathematical Foundations* (eds Bär, C. & Fredenhagen, K.) 59–84 (Springer, Berlin, 2009). DOI: 10.1007/978-3-642-02780-2_3.
26. Godinho, L. & Natário, J. *An Introduction to Riemannian Geometry With Applications to Mechanics and Relativity* (Springer, Cham, 2014).
27. Greiner, W. *Relativistic Quantum Mechanics: Wave Equations* (Springer, Berlin, 2000).
28. Griffiths, D. J. & Heald, M. A. Time-dependent generalizations of the Coulomb and Biot–Savart laws. *American Journal of Physics* **59,** 111–117 (1991). DOI: 10.1119/1.16589.
29. Hadamard, J. *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).
30. Halmos, P. R. *Naive Set Theory* (Springer, New York, 1974).
31. Hoffman, K. & Kunze, R. *Linear Algebra* (Prentice-Hall, Inc., Englewood Cliffs, 1971).
32. Hörmander, L. On the division of distributions by polynomials. *Arkiv för Matematik* **3,** 555–568 (1958). DOI: 10.1007/BF02589517.
33. Hörmander, L. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis* (Springer, Berlin, 2003).

34. Hörmander, L. *The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients* (Springer, Berlin, 2005).

35. Isnard, C. *Introdução à Medida e Integração* (Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2013).

36. Jackson, J. D. *Classical Electrodynamics* (Wiley, New York, 1999).

37. Jefimenko, O. D. *Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields* (Electret Scientific Company, Star City, 1989).

38. Kelley, J. & Namioka, I. *Linear Topological Spaces* (Springer-Verlag, New York, 1963).

39. Kervaire, M. A. A Manifold Which Does Not Admit Any Differentiable Structure. *Commentarii Mathematici Helvetici* **34,** 257–270 (1960). DOI: 10.1007/BF02565940.

40. Koshlyakov, N. S., Smirnov, M. M. & Gliner, E. B. *Differential Equations of Mathematical Physics* (North-Holland, Amsterdam, 1964).

41. Köthe, G. *Topological Vector Spaces I* (Springer-Verlag, Berlin, 1983).

42. Kreyszig, E. *Introductory Functional Analysis With Applications* (Wiley, New York, 1978).

43. Lee, J. *Manifolds and Differential Geometry* (American Mathematical Society, Providence, Rhode Island, 2009).

44. Lee, J. *Introduction to Smooth Manifolds* (Springer, New York, 2012).

45. Lemos, N. *Analytical Mechanics* (Cambridge University Press, Cambridge, United Kingdom, 2018).

46. Levandosky, J. *Course Page for "Partial Differential Equations of Applied Mathematics"* Last visited on June 08, 2019. https://web.stanford.edu/class/math220a/.

47. Lima, E. L. *Curso de Análise Vol. 1* (Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2017).

48. Lima, E. L. *Curso de Análise Vol. 2* (Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2018).

49. Lima, E. L. *Espaços Métricos* (Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2017).

50. Mac Lane, S. *Categories for the Working Mathematician* (Springer, New York, 1978).

51. Mendelson, B. *Introduction to Topology* (Dover Publications, New York, 1990).

52. Morse, P. M. & Feshbach, H. *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

53. Munkres, J. *Topology* (Prentice Hall, Inc, Upper Saddle River, NJ, 2000).

54. Nakahara, M. *Geometry, Topology and Physics* (Institute of Physics Publishing, Bristol, 2003).

55. Peskin, M. E. & Schroeder, D. V. *An Introduction to Quantum Field Theory* (Westview Press, Boulder, 1995).

56. Reed, M. & Simon, B. *Methods of Modern Mathematical Physics Vol. 1: Functional Analysis, Revised and Enlarged Edition* (Academic Press, San Diego, 1980).
57. Reed, M. & Simon, B. *Methods of Modern Mathematical Physics Vol. 2: Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975).
58. Robertson, A. P. & Robertson, W. *Topological Vector Spaces* (Cambridge University Press, Cambridge, 1973).
59. Rother, T. *Greens Functions in Classical Physics* (Springer, Cham, 2017).
60. Rudin, W. *Functional Analysis* (McGraw-Hill, New York, 1991).
61. Rudin, W. *Principles of Mathematical Analysis* (McGraw-Hill, New York, 1976).
62. Schaefer, H. H. & Wolff, M. P. *Topological Vector Spaces* (Springer Science+Business Media, New York, 1999).
63. Schindler, R. *Set Theory: Exploring Independence and Truth* (Springer, Cham, 2014).
64. Schuller, F. P. *The Mathematics and Physics of Gravity and Light* (lecture course). https://www.youtube.com/playlist?list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_.
65. Schwartz, L. *Théorie des Distributions* (Hermann, Paris, 1966).
66. Seibold, B. *Course Page for "Numerical Methods for Partial Differential Equations"* Last visited on June 27, 2019. https://ocw.mit.edu/courses/mathematics/18-336-numerical-methods-for-partial-differential-equations-spring-2009/.
67. Shankar, R. *Principles of Quantum Mechanics* (Springer, New York, 1994).
68. Simon, B. *A Comprehensive Course in Analysis, Part 1: Real Analysis* (American Mathematical Society, Providence, 2015).
69. Sobolev, S. L. *Partial Differential Equations of Mathematical Physics* (Dover, Mineola, 1989).
70. Strauss, W. *Partial Differential Equations: An Introduction* (Wiley, New York, 2008).
71. Suppes, P. *Axiomatic Set Theory* (Dover Publications, New York, 1972).
72. Tao, T. *Analysis I* (Hindustan Book Agency, New Delhi, 2006).
73. Tao, T. *Hilbert's Fifth Problem and Related Topics* (preliminary version). https://terrytao.files.wordpress.com/2014/11/gsm-153.pdf (American Mathematical Society, Providence, Rhode Island, 2014).
74. Tu, L. *An Introduction to Manifolds* (Springer, New York, 2011).
75. Wald, R. M. *General Relativity* (University of Chicago Press, Chicago, 1984).
76. Wald, R. M. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).
77. Watson, G. N. *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944).
78. Weinberg, S. *The Quantum Theory of Fields Vol. I: Foundations* (Cambridge University Press, Cambridge, 1995).

79. Yosida, K. *Functional Analysis* (Springer, Berlin, 1995).
80. Zee, A. *Quantum Field Theory in a Nutshell* (Princeton University Press, Princeton, 2010).

# List of Definitions