Exercises on Penrose Diagrams

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Abstract: This document is a collection of computations of Penrose diagrams in a few spacetimes of interest. It was written as a manner of getting acquainted with the computation of Penrose diagrams in General Relativity. The only essential prerequisite should be a basic knowledge of General Relaivity.

Keywords: General Relativity, Penrose–Carter Conformal Diagrams.

Contents

1 Minkowski Spacetime

We begin by writing the Minkowski metric in spherical coordinates,

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{1.1}$$

where $\mathrm{d}\Omega^2$ is the usual metric on the unit sphere.

Our goal will be to represent the entire, infinite spacetime in a finite diagram. In order to do so, we'll need to "pull-in infinity" by means of what is called a conformal transformation, which shall keep the causal structure of spacetime unchanged while allowing us to compactify it. Roughly speaking, we'll find a new, unphysical metric $\tilde{g}_{ab}$ which relates to the spacetime metric g_{ab} through an expression of the form $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where Ω is a smooth, strictly-positive function of spacetime. Notice that a vector is null with respect to one of the metrics if, and only if, it is null with respect to the other. By exploiting this, we'll be able to rewrite the spacetime as a piece of a different, unphysical spacetime. Roughly speaking, this piece is what the call the Penrose diagram.

In order to do this procedure, we'll first find a set of null coordinates, which will make our job of compactifying easier since we'll be able to compactify these coordinates to a new set of null coordinates. By "null coordinates", we mean coordinates whose associated vector fields are null, *id est*, u is said to be null when $g_{ab}\left(\frac{\partial}{\partial u}\right)^a\left(\frac{\partial}{\partial u}\right)^b = 0$.

To obtain a pair of coordinates, we'll investigate the radial null geodesics in Minkowski spacetime, which corresponds to solving

$$0 = -\,\mathrm{d}t^2 + \mathrm{d}r^2\,, \tag{1.2}$$

where we've set $\mathrm{d}s^2 = 0$ (null) and $\mathrm{d}\Omega^2 = 0$ (radial) in the metric. We see then that null, radial geodesics in Minkowski spacetime respect

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = 1, \tag{1.3}$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm 1, \tag{1.4}$$

$$t = \pm r + \text{constant}, \tag{1.5}$$

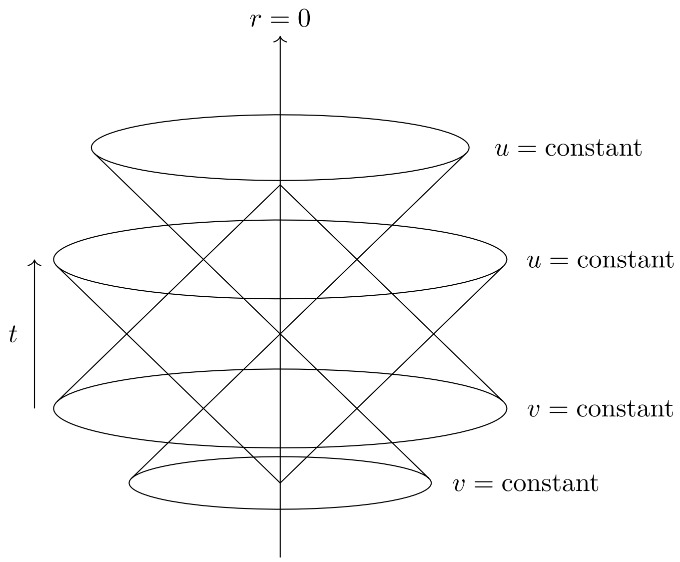

Figure 1: *Illustration, with one dimension suppressed, of the physical meaning of the null coordinates u and v. Surfaces of constant u are outgoing spherical "waves", while surfaces of constant v are their incoming analogues. The figure is based on Hawking and Ellis 1973, Fig. 12.i.*

where the sign depends on whether the geodesic is incoming or outgoing. We can then label each incoming or outgoing geodesic in terms of the constant that accompanies the solution, *id est*, we define new coordinates according to

$$v = t + r \quad \text{and} \quad u = t - r, \tag{1.6}$$

where v plays the role of "advanced time" and u plays the role of "retarded time". These are the null coordinates we were looking for. See Fig. 1 for an illustration of the physical meaning of these coordinates.

Noticing that

$$t = \frac{v+u}{2} \quad \text{and} \quad r = \frac{v-u}{2}, \tag{1.7}$$

we find that

$$\mathrm{d}t^2 = \frac{1}{4}(\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}u + \mathrm{d}u^2) \quad \text{and} \quad \mathrm{d}r^2 = \frac{1}{4}(\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}u + \mathrm{d}u^2), \tag{1.8}$$

which at last yield

$$-\,\mathrm{d}t^2 + \mathrm{d}r^2 = -\frac{1}{4}\,\mathrm{d}u\,\mathrm{d}v\,. \tag{1.9}$$

Hence, the Minkowski metric becomes

$$\mathrm{d}s^2 = -\frac{1}{4}\,\mathrm{d}v\,\mathrm{d}u + \frac{1}{4}(v-u)^2\,\mathrm{d}\Omega^2\,. \tag{1.10}$$

Notice that, since $t \in \mathbb{R}$ and $r \geq 0$, our new coordinates have the ranges

$$-\infty < u \leq v < +\infty. \tag{1.11}$$

Hence, while we are now in a more convenient coordinate system for dealing with radial null geodesics, it is still not compact. We'll compactify it by defining new coordinates U and V through

$$u = \tan U \quad \text{and} \quad v = \tan V. \tag{1.12}$$

Notice that we can still label radial geodesics with U and V, for if u is constant throughout a geodesic, so is U and vice-versa, with similar statements for v and V. However, we now have

$$-\frac{\pi}{2} < U \leq V < \frac{\pi}{2}, \tag{1.13}$$

where the ordering $U < V$ comes from the tangent function being monotonically increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Noticing that

$$\mathrm{d}u = \sec^2 U \,\mathrm{d}U \quad \text{and} \quad \mathrm{d}v = \sec^2 V \,\mathrm{d}V\,, \tag{1.14}$$

we see that the metric becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{\sec^2 U \sec^2 V}{4}\,\mathrm{d}U\,\mathrm{d}V + \frac{(\tan V - \tan U)^2}{4}\,\mathrm{d}\Omega^2\,, && \text{(1.15a)}\\
&= -\frac{\sec^2 U \sec^2 V}{4}\,\mathrm{d}U\,\mathrm{d}V + \frac{\sec^2 U \sec^2 V \sin^2(V-U)}{4}\,\mathrm{d}\Omega^2\,, && \text{(1.15b)}\\
&= \frac{\sec^2 U \sec^2 V}{4}\left[-\,\mathrm{d}U\,\mathrm{d}V + \sin^2(V-U)\,\mathrm{d}\Omega^2\right]. && \text{(1.15c)}
\end{aligned}$$

We now make a conformal transformation to keep the unphysical metric

$$\mathrm{d}\tilde{s}^2 = -\,\mathrm{d}U\,\mathrm{d}V + \sin^2(V-U)\,\mathrm{d}\Omega^2\,. \tag{1.16}$$

It is convenient to perform another change of coordinates through

$$V = T + R \quad \text{and} \quad U = T - R, \tag{1.17}$$

so that we get

$$\mathrm{d}\tilde{s}^2 = -\,\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R\,\mathrm{d}\Omega^2 \tag{1.18}$$

at last. This is a subsection of the Einstein static universe restricted to

$$-\frac{\pi}{2} < T - R \leq T + R < \frac{\pi}{2}, \tag{1.19}$$

which is compact. Notice we have $T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $R \geq 0$.

We can now represent this spacetime in a finite drawing, which is the Penrose diagram of Minkowski spacetime, shown in Fig. 2 on the next page. Notice it includes a few extra pieces that are not part of Minkowski spacetime: the past and future null infinities, $\mathcal{I}^-$ and $\mathcal{I}^+$, which are three-manifolds at $T - R = -\frac{\pi}{2}$ and $T + R = +\frac{\pi}{2}$; the spatial infinity, i^0, at $T = 0$, $R = \frac{\pi}{2}$; and the past and future timelike infinities, at $T = -\frac{\pi}{2}$, $R = 0$, and at $T = +\frac{\pi}{2}$, $R = 0$.

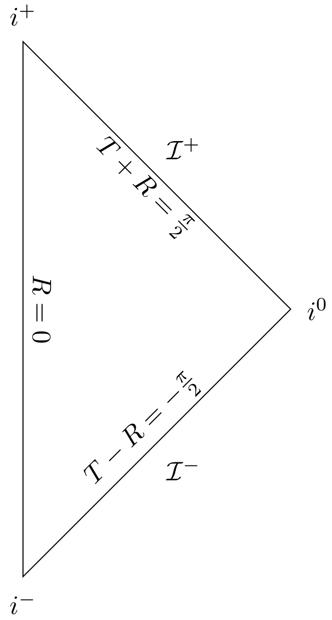

Figure 2: *Penrose diagram of Minkowski spacetime.*

2 Schwarzschild Spacetime

Next we perform an analogous analysis of Schwarzschild spacetime, which will be a bit more involved due to the presence of a coordinate singularity when we write the metric in Schwarzschild coordinates. Apart from that, the analysis is essentially identical.

We begin with the metric

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{2.1}$$

As before, our first step is to introduce null coordinates. The equation for radial null geodesics in Schwarzschild spacetime becomes

$$-\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2\,, \tag{2.2}$$

and hence

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{1}{1-\frac{2M}{r}}. \tag{2.3}$$

We can integrate this equation to

$$t = \pm\int\frac{\mathrm{d}r}{1-\frac{2M}{r}}, \tag{2.4a}$$

$$= \pm\int\frac{r\,\mathrm{d}r}{r-2M}, \tag{2.4b}$$

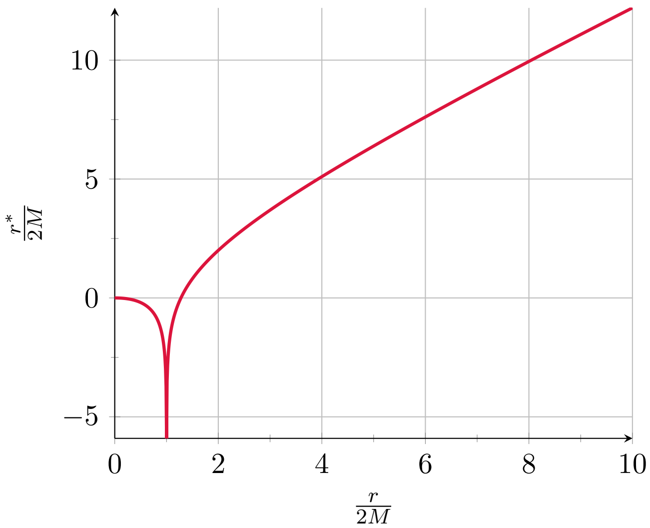

Figure 3: *Behaviour of the tortoise coordinate* $r^* = r + 2M \log\left|\frac{r}{2M} - 1\right|$ *as a function of* r*. The divergence at* $r = 2M$ *will allow us to balance the divergence coming from the bad coordinate choice in the metric.*

$$= \pm \int \frac{(\rho + 2M)\,\mathrm{d}\rho}{\rho}, \tag{2.4c}$$

$$= \pm\rho \pm 2M \log|\rho| + \text{constant}, \tag{2.4d}$$

$$= \pm r \pm 2M \log\left|\frac{r}{2M} - 1\right| + \text{constant}, \tag{2.4e}$$

where we defined $\rho = r - 2M$ and absorbed a few terms in the integration constant in the last step.

Due to this expression, we define the Regge–Wheeler "tortoise coordinate", r^*, by

$$r^* = r + 2M \log\left|\frac{r}{2M} - 1\right|, \tag{2.5}$$

which allows us to write, for the radial null geodesics,

$$t = \pm r^* + \text{constant}. \tag{2.6}$$

The tortoise coordinate as a function of the radial distance r is plotted in Fig. 3.

In analogy with our procedure for the Minkowski spacetime, we now define new coordinates

$$\tilde{v} = t + r^* \quad \text{and} \quad \tilde{u} = t - r^*, \tag{2.7}$$

where the tilde is used just because we'll make more coordinate changes this time and I'd like to keep the notation (u, v) for later. This time we have the one-forms

$$\mathrm{d}\tilde{u} = \mathrm{d}t - \frac{\mathrm{d}r}{1 - \frac{2M}{r}} \quad \text{and} \quad \mathrm{d}\tilde{v} = \mathrm{d}t + \frac{\mathrm{d}r}{1 - \frac{2M}{r}}, \tag{2.8}$$

which imply

$$-\left(1-\frac{2M}{r}\right)\mathrm{d}\tilde{u}\,\tilde{v} = -\left(1-\frac{2M}{r}\right)\left(\mathrm{d}t-\frac{\mathrm{d}r}{1-\frac{2M}{r}}\right)\left(\mathrm{d}t+\frac{\mathrm{d}r}{1-\frac{2M}{r}}\right), \tag{2.9a}$$

$$= -\left(1-\frac{2M}{r}\right)\left(\mathrm{d}t^2-\frac{\mathrm{d}r^2}{\left(1-\frac{2M}{r}\right)^2}\right), \tag{2.9b}$$

$$= -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2-\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2\,. \tag{2.9c}$$

Hence, the Schwarzschild metric can be rewritten as

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}\tilde{u}\,\tilde{v}+r^2\,\mathrm{d}\Omega^2\,, \tag{2.10}$$

where r is now seen as a function defined implicitly by

$$r+2M\log\left|\frac{r}{2M}-1\right| = \frac{\tilde{v}-\tilde{u}}{2}. \tag{2.11}$$

This last equation will allow us to get rid of the $\left(1-\frac{2M}{r}\right)$ in the metric, which is still keeping us from using the metric at the event horizon. We notice that

$$\log\left|\frac{r}{2M}-1\right| = \frac{\tilde{v}-\tilde{u}}{4M}-\frac{r}{2M}, \tag{2.12a}$$

$$\left|\frac{r}{2M}-1\right| = e^{\frac{\tilde{v}-\tilde{u}}{4M}}e^{-\frac{r}{2M}}\,, \tag{2.12b}$$

$$\left(1-\frac{2M}{r}\right) = \pm\frac{2Me^{-\frac{r}{2M}}}{r}e^{\frac{\tilde{v}-\tilde{u}}{4M}}\,, \tag{2.12c}$$

where, in the last line, the upper sign refers to the case $r > 2M$ and the lower sign refers to the case $r < 2M$. With this expression the Schwarzschild metric becomes

$$\mathrm{d}s^2 = \mp\frac{2Me^{-\frac{r}{2M}}}{r}e^{\frac{\tilde{v}-\tilde{u}}{4M}}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}+r^2\,\mathrm{d}\Omega^2\,. \tag{2.13}$$

We'll now get rid of the exponential $e^{\frac{\tilde{v}-\tilde{u}}{4M}}$, which vanishes at the horizon, by performing an auxiliary redefinition of coordinates before compactifying spacetime. Namely, we define

$$u = \mp e^{\frac{-\tilde{u}}{4M}} \quad \text{and} \quad v = e^{\frac{\tilde{v}}{4M}}\,, \tag{2.14}$$

where, as before, the upper sign refers to $r > 2M$. The one-forms become

$$\mathrm{d}u = \pm\frac{1}{4M}e^{\frac{-\tilde{u}}{4M}}\,\mathrm{d}\tilde{u} \quad \text{and} \quad \mathrm{d}v = \frac{1}{4M}e^{\frac{\tilde{v}}{4M}}\,\mathrm{d}\tilde{v}\,, \tag{2.15}$$

id est,

$$-16M^2\,\mathrm{d}u\,\mathrm{d}v = \mp e^{\frac{\tilde{v}-\tilde{u}}{4M}}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}\,, \tag{2.16}$$

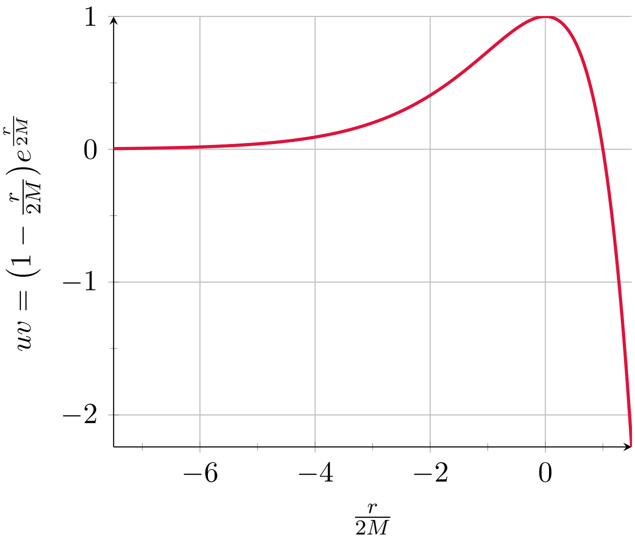

Figure 4: *Plot of the function* $uv = e^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right)$*, showing it attains a maximum value* $uv = 1$ *at* $r = 0$*. Since* $r = 0$ *is an essential singularity, the coordinates* (u, v) *are restricted to* $uv < 1$*.*

meaning the Schwarzschild metric is now given by

$$\mathrm{d}s^2 = -\frac{32M^3 e^{-\frac{r}{2M}}}{r}\,\mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\Omega^2\,, \tag{2.17}$$

which is nonsingular at the event horizon.

To properly compactify the spacetime, we should keep track of the ranges of each coordinate, so let us take a look at the ranges of u and v. Firstly we notice that

$$\begin{aligned} uv &= \mp e^{\frac{\tilde{v}-\tilde{u}}{4M}}\,, && (2.18\text{a}) \\ &= \mp e^{\frac{r^*}{2M}}\,, && (2.18\text{b}) \\ &= \mp e^{\frac{r}{2M}}\left|\frac{r}{2M} - 1\right|, && (2.18\text{c}) \\ &= e^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right), && (2.18\text{d}) \end{aligned}$$

where the last line uses the fact that the upper sign refers to $r > 2M$ (when $\left|\frac{r}{2M} - 1\right| = \left(\frac{r}{2M} - 1\right)$), while the lower sign refers to $r < 2M$ (when $\left|\frac{r}{2M} - 1\right| = \left(1 - \frac{r}{2M}\right)$).

The fact that $e^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right) \leq 1$ (see Fig. 4) imposes the condition $uv \leq 1$. However, the only point in which $uv = 1$ is $r = 0$, which is an intrinsic singularity of the Schwarzschild spacetime. This can be seen, for example, by computing the Kretschmann scalar, $K = R^{abcd}R_{abcd}$, which behaves as $M^2 r^{-6}$, and hence diverges at $r = 0$(Hawking and Ellis 1973, p. 151). Hence, we must restrict our attention to $uv < 1$.

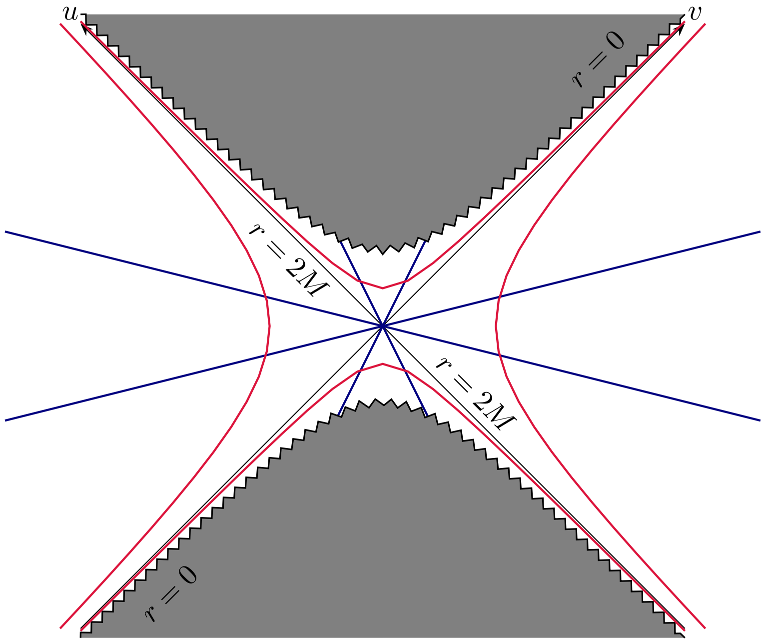

Figure 5: *Kruskal diagram of the unique analytic and locally inextendible extension of Schwarzschild spacetime. The red hyperbolae are hypersurfaces of constant r. The blue straight lines are hypersurfaces of constant t. The crossing axes correspond to $r = 2M$. The hyperbolae in zizag correspond to $r = 0$ and both them and the shaded regions they bound are not in spacetime.*

For completeness, let us also notice that

$$\frac{v}{u} = \mp e^{\frac{\tilde{v}+\tilde{u}}{4M}}, \tag{2.19a}$$

$$= \mp e^{\frac{t}{2M}}, \tag{2.19b}$$

which means

$$\frac{t}{2M} = \log\left|\frac{v}{u}\right|. \tag{2.20}$$

One should notice that the (u, v) coordinates allow us to extend the spacetime, since the same value of (t, r) allows for two different values of (u, v). For example, $r = 2M$ can be achieved via either $v = 0$ or $u = 0$. This reveals the fact that our original choice of coordinates covered only a portion of the manifold.

At this stage, albeit we have not performed a conformal transformation yet, it is instructive to draw a diagram of the current situation. Such a diagram, known as a Kruskal diagram, is show in Fig. 5. It represents the full spacetime, including now the pieces we left out in our original choice of coordinates.

To compactify our coordinates, we proceed as before by defining

$$u = \tan U \quad \text{and} \quad v = \tan V. \tag{2.21}$$

The metric can now be written as

$$\mathrm{d}s^2 = -\frac{32M^3e^{-\frac{r}{2M}}}{r}\sec^2 U \sec^2 V \,\mathrm{d}U\,\mathrm{d}V + r^2\,\mathrm{d}\Omega^2\,, \tag{2.22}$$

and the conformal transformation will come by means of defining

$$\mathrm{d}\tilde{s}^2 = -\,\mathrm{d}U\,\mathrm{d}V + \frac{r^3\cos^2 U\cos^2 V e^{\frac{r}{2M}}}{32M^3}\,\mathrm{d}\Omega^2\,, \tag{2.23}$$

which is the unphysical metric of the diagram we'll draw. We should not forget to keep track of the ranges of the coordinates. This time, we have $-\frac{\pi}{2} < U, V < +\frac{\pi}{2}$ with the additional condition that $uv = \tan U \tan V < 1$.

Suppose, temporarily, that $V > 0$. Then

$$\tan U \tan V < 1, \tag{2.24a}$$

$$\tan U < \cot V, \tag{2.24b}$$

$$U < \arctan(\cot V). \tag{2.24c}$$

Since $\cot V = \tan\left(\frac{\pi}{2} - V\right)$, it follows that

$$U < \arctan\left(\tan\left(\frac{\pi}{2} - V\right)\right). \tag{2.25a}$$

We know $\arctan(\tan(x)) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so if we ensure the argument is in this range we can compute the expression. For $0 < V < \frac{\pi}{2}$, $0 < \frac{\pi}{2} - V < \frac{\pi}{2}$, so we have the condition

$$U < \frac{\pi}{2} - V \text{ for } V > 0. \tag{2.26}$$

Similarly one gets

$$U > -\frac{\pi}{2} - V \text{ for } V < 0. \tag{2.27}$$

Taking these restrictions into account, we can draw the Penrose diagram for the maximally extended Schwarzschild spacetime. We define, for convenience, the new variables T and R by means of the expressions

$$V = T + R \quad \text{and} \quad U = T - R. \tag{2.28}$$

The diagram is given on Fig. 6 on the next page. We see we have four regions, I through IV. The union of I and III represents the original piece of spacetime we had in hands, and the same holds for the union of II and IV. Hence, we have "an Universe" I, a "parallel Universe", a black hole III, and a "white hole" IV.

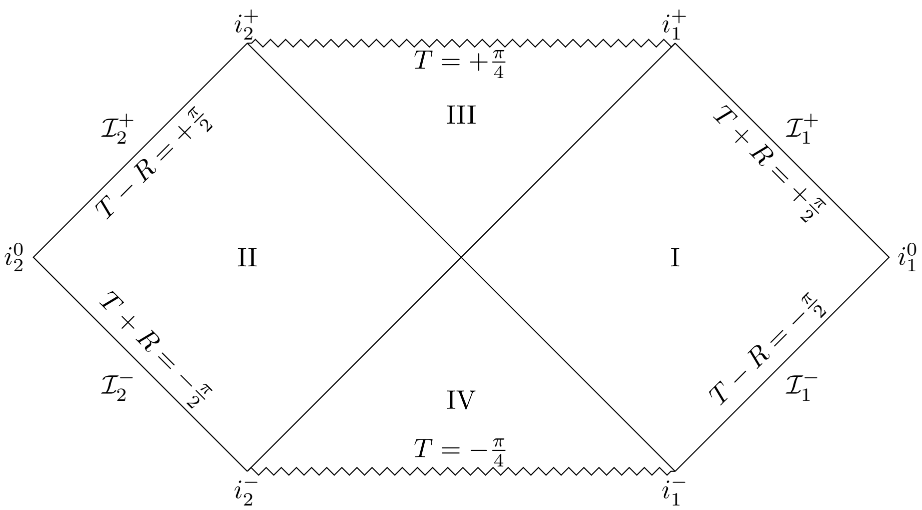

Figure 6: *Penrose diagram of the maximally extended Schwarzschild spacetime.*

3 Spatially Flat, Radiation-Filled Universe

Next we'll consider a spatially flat, radiation-filled Universe. Since it is spatially flat, the metric is given in terms of cosmic time τ by

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + a(\tau)^2\big[\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\big], \tag{3.1}$$

where $a(\tau)$ is the scale factor. It can be determined from the Friedmann equationsWald 1984, Eqs. (5.2.14) and (5.2.15),

$$\begin{cases} 3\dfrac{\dot{a}^2}{a^2} = 8\pi\rho - \dfrac{3k}{a^2}, \\ 3\dfrac{\ddot{a}}{a} = -4\pi(\rho + 3p), \end{cases} \tag{3.2}$$

where k represents the spatial curvature ($k = 0$ for spatially flat, which is our case of interest), p is the pressure of a perfect fluid filling the Universe, ρ is the fluid's energy density, and the dot denotes differentiation with respect to cosmic time τ. For radiation, one has the equation of state $p = \frac{\rho}{3}$.

Therefore, for our particular case the Friedmann equations become

$$\begin{cases} 3\dfrac{\dot{a}^2}{a^2} = 8\pi\rho, \\ 3\dfrac{\ddot{a}}{a} = -8\pi\rho, \end{cases} \tag{3.3}$$

From this pair of equations we see that

$$\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} = 0, \tag{3.4a}$$

$$\dot{a}^2\ddot{a}a = 0, \tag{3.4b}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}) = 0, \tag{3.4c}$$
$$a\dot{a} = c, \tag{3.4d}$$

where c is some constant. Notice that

$$c = a^2\frac{\dot{a}}{a}, \tag{3.5a}$$
$$= a^2 H(\tau), \tag{3.5b}$$

which, since is a constant, can be written in terms of the present-day scale factor a and Hubble parameter $H = \frac{\dot{a}}{a}$. It is conventional to pick the present-day scale factor to be $a_0 = 1$. Denoting the present-day Hubble parameter by H_0, we see we now have

$$a\dot{a} = H_0, \tag{3.6}$$

which can be integrated to

$$a(\tau) = \sqrt{2H_0(\tau - \tau_0) + 1}. \tag{3.7}$$

For any $H_0 > 0$, there is some τ at which $a(\tau) = 0$. We'll choose $\tau_0 = \frac{1}{2H_0}$, so that $a(0) = 0$. Hence, we get to

$$a(\tau) = \sqrt{2H_0\tau}. \tag{3.8}$$

Therefore, we get to the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + 2H_0\tau\left[\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right]. \tag{3.9}$$

As before, our first step is to find the radial null geodesics. We get

$$\mathrm{d}\tau^2 = 2H_0\tau\,\mathrm{d}r^2\,, \tag{3.10}$$

which gives the ordinary differential equations

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\frac{1}{\sqrt{2H_0\tau}}. \tag{3.11}$$

Integrating it leads us to

$$r = \pm\sqrt{\frac{2\tau}{H_0}} + \text{constant}. \tag{3.12}$$

As usual, we define null coordinates through

$$u = \sqrt{\frac{2\tau}{H_0}} - r \quad \text{and} \quad v = \sqrt{\frac{2\tau}{H_0}} + r, \tag{3.13}$$

leading to

$$\mathrm{d}u = \frac{\mathrm{d}\tau}{\sqrt{2H_0\tau}} - \mathrm{d}r \quad \text{and} \quad \mathrm{d}v = \frac{\mathrm{d}\tau}{\sqrt{2H_0\tau}} + \mathrm{d}r\,, \tag{3.14}$$

which in turn implies

$$-2H_0\tau\,\mathrm{d}u\,\mathrm{d}v = -\,\mathrm{d}\tau^2 + 2H_0\tau\,\mathrm{d}r^2\,. \tag{3.15}$$

Noticing that

$$\tau = \frac{H_0(v+u)^2}{8} \quad \text{and} \quad r = \frac{v-u}{2}, \tag{3.16}$$

we obtain the new expression for the metric,

$$\mathrm{d}s^2 = -\frac{H_0^2(v+u)^2}{4}\,\mathrm{d}u\,\mathrm{d}v + \frac{H_0^2(v+u)^2(v-u)^2}{16}\,\mathrm{d}\Omega^2\,. \tag{3.17}$$

Our original coordinates had the ranges $\tau > 0$, $r > 0$. For consistency, the new ones must have $u + v > 0$ and $v - u > 0$.

Next we compactify the null coordinates. As usual, we define

$$v = \tan V \quad \text{and} \quad u = \tan U, \tag{3.18}$$

with ranges in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $u > -v$, *id est*, $\tan U > \tan(-V)$, we must have $U > -V$, where we employed the fact that the tangent is crescent and odd at $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Similarly, we must have $V > U$.

Since

$$\mathrm{d}v = \sec^2 V\,\mathrm{d}V \quad \text{and} \quad \mathrm{d}u = \sec^2 U\,\mathrm{d}U\,, \tag{3.19}$$

the metric becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{H_0^2(v+u)^2}{4}\sec^2 U\sec^2 V\,\mathrm{d}U\,\mathrm{d}V + \frac{H_0^2(v+u)^2(\tan V - \tan U)^2}{16}\,\mathrm{d}\Omega^2\,, && \text{(3.20a)}\\
&= \frac{H_0^2(\tan V + \tan U)^2}{16}\sec^2 U\sec^2 V\left[-4\,\mathrm{d}U\,\mathrm{d}V + \sin^2(V-U)\,\mathrm{d}\Omega^2\right], && \text{(3.20b)}
\end{aligned}$$

where we also employed the trigonometric identity

$$(\tan V - \tan U)^2 = \sec^2 U\sec^2 V\sin^2(V-U). \tag{3.21}$$

Hence, we compactify the spacetime by defining the unphysical metric

$$\mathrm{d}\tilde{s}^2 = -4\,\mathrm{d}U\,\mathrm{d}V + \sin^2(V-U)\,\mathrm{d}\Omega^2\,. \tag{3.22}$$

At last, we define new coordinates T and R through

$$R = U - V \quad \text{and} \quad T = U + V, \tag{3.23}$$

leading to

$$\mathrm{d}\tilde{s}^2 = -\,\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R\,\mathrm{d}\Omega^2\,. \tag{3.24}$$

The ranges of these coordinates are such that

$$0 < R, T < \pi \quad \text{and} \quad R + T < \pi, \tag{3.25}$$

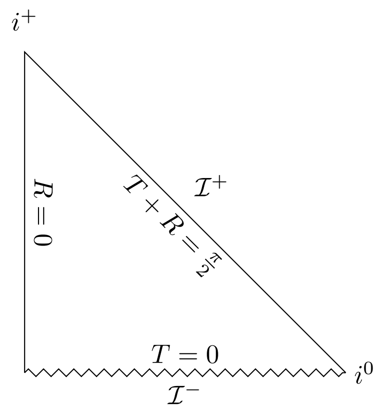

Figure 7: *Penrose diagram for a spatially flat Universe filled with radiation.*

which come from both the conditions that $U, V \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$, $U+V>0$, and $V-U>0$. With this information, we can now draw the diagram, which is given in Fig. 7.

An interesting feature of this Penrose diagram is that we can notice that there are regions in the early Universe which can't come into causal contact. This poses an interesting question then to how can the cosmic microwave background be so uniform, which is often solved by the notion of inflation: the Universe undergoes an initial phase of accelerated expansion, allowing far regions to come into causal contact, before entering a decelerating phase.

4 Spatially Closed, Radiation-Filled Universe

Once we know the causal structure of the flat radiation-filled universe, it is interesting to consider what happens for other choices of spatial curvature. If we pick a spatially closed, radiation-filled universe, the metric will be given by

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + a(\tau)^2\left[\mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\Omega^2\right], \tag{4.1}$$

where the Friedmann equations for the scale factor are now given by

$$\begin{cases} 3\dfrac{\dot{a}^2}{a^2} = 8\pi\rho - \dfrac{3}{a^2}, \\ 3\dfrac{\ddot{a}}{a} = -8\pi, \end{cases} \tag{4.2}$$

This time we get the differential equation

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = 0, \tag{4.3}$$

id est,

$$a\ddot{a} + \dot{a}^2 + 1 = 0. \tag{4.4}$$

TO integrate it, we do the same trick as for the flat universe. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}) = -1, \tag{4.5a}$$
$$a\dot{a} = -\tau + \text{constant}. \tag{4.5b}$$

To fix the constant, we notice as before that $a\dot{a} = a^2 H$, and hence

$$\text{constant} = \tau_0 + H_0, \tag{4.6}$$

where τ_0 is present-time and H_0 is the present-day value of the Hubble constant. As before, we assume $a(\tau_0) = 1$. We see then that we get

$$a\dot{a} = -\tau + \tau_0 + H_0, \tag{4.7a}$$
$$a\,\mathrm{d}a = [H_0 - (\tau - \tau_0)]\,\mathrm{d}\tau\,, \tag{4.7b}$$
$$\frac{a^2}{2} = H_0\tau + \tau_0\tau - \frac{\tau^2}{2} + c, \tag{4.7c}$$

for some constant c, which we'll fix by requiring $a(\tau_0) = 1$. We have

$$a(\tau) = \sqrt{2H_0\tau + 2\tau_0\tau - \tau^2 + 2c}. \tag{4.8}$$

Hence,

$$1 = a(\tau_0), \tag{4.9a}$$
$$= \sqrt{2H_0\tau_0 + 2\tau_0^2\tau - \tau_0^2 + 2c}, \tag{4.9b}$$
$$1 = 2H_0\tau_0 + \tau_0^2 + 2c, \tag{4.9c}$$
$$1 - 2H_0\tau_0 - \tau_0^2 = 2c. \tag{4.9d}$$

The scale factor is then given by

$$a(\tau) = \sqrt{2H_0\tau + 2\tau_0\tau - \tau^2 + 1 - 2H_0\tau_0 - \tau_0^2}, \tag{4.10a}$$
$$= \sqrt{1 + 2H_0(\tau - \tau_0) - (\tau - \tau_0)^2}. \tag{4.10b}$$

The expression can be further simplified by fixing τ_0. The scale factor has two roots, as we can see by imposing $a(\tau) = 0$. Indeed,

$$\sqrt{1 + 2H_0(\tau - \tau_0) - (\tau - \tau_0)^2} = 0, \tag{4.11a}$$
$$1 + 2H_0(\tau - \tau_0) - (\tau - \tau_0)^2 = 0, \tag{4.11b}$$

and the quadratic formula yields

$$\tau - \tau_0 = H_0 \pm \sqrt{1 + H_0^2}, \tag{4.12}$$

both of which are perfectly plausible. We then choose τ_0 such that $a(0) = 0$ and $\tau_0 > 0$, *id est*,

$$\tau_0 = \sqrt{1 + H_0^2} - H_0. \tag{4.13}$$

Notice that we also get a maximum time $\tau_{\text{max}} = 2\sqrt{1 + H_0^2}$.

The scale factor can now be written as

$$\begin{aligned}
a(\tau) &= \sqrt{1 + 2H_0(\tau - \tau_0) - (\tau - \tau_0)^2}, && \text{(4.14a)}\\
&= \sqrt{1 + (\tau - \tau_0)(2H_0 - \tau + \tau_0)}, && \text{(4.14b)}\\
&= \sqrt{1 + (\tau + H_0 - \sqrt{1 + H_0^2})(\sqrt{1 + H_0^2} + H_0 - \tau + \tau_0)}, && \text{(4.14c)}\\
&= \sqrt{1 + H_0^2 - (\tau - \sqrt{1 + H_0^2})^2}. && \text{(4.14d)}
\end{aligned}$$

The metric is then given explicitly by

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \left[1 + H_0^2 - (\tau - \sqrt{1 + H_0^2})^2\right]\big[\mathrm{d}r^2 + \sin^2 r \,\mathrm{d}\Omega^2\big]. \tag{4.15}$$

Notice the coordinate ranges are $\tau \in (0, 2\sqrt{1 + H_0^2})$ and $r \in (0, \pi)$.

Let us then compute the radial null geodesics. The differential equation is

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\frac{1}{\sqrt{1 + H_0^2 - (\tau - \sqrt{1 + H_0^2})^2}} = \pm\frac{1}{a(\tau)}, \tag{4.16}$$

and its solution will be

$$r(\tau) = \pm\int \frac{\mathrm{d}\tau}{\sqrt{1 + H_0^2 - (\tau - \sqrt{1 + H_0^2})^2}} = \pm\int \frac{\mathrm{d}\tau}{a(\tau)}. \tag{4.17}$$

To perform the integral, let us change notation for a while. We write $\Delta \equiv \sqrt{1 + H_0^2}$. With a few substitutions we see that

$$\begin{aligned}
\int \frac{\mathrm{d}\tau}{\sqrt{1 + H_0^2 - (\tau - \sqrt{1 + H_0^2})^2}} &= \int \frac{\mathrm{d}\tau}{\sqrt{\Delta^2 - (\tau - \Delta)^2}}, && \text{(4.18a)}\\
&= \int \frac{\mathrm{d}\lambda}{\sqrt{\Delta^2 - \lambda^2}}, && \text{(4.18b)}\\
&= \frac{1}{\Delta}\int \frac{\mathrm{d}\lambda}{\sqrt{1 - \frac{\lambda^2}{\Delta^2}}}, && \text{(4.18c)}\\
&= \int \frac{\mathrm{d}\mu}{\sqrt{1 - \mu^2}}, && \text{(4.18d)}\\
&= \arcsin\mu + \text{constant}, && \text{(4.18e)}
\end{aligned}$$

$$= \arcsin\left(\frac{\lambda}{\Delta}\right) + \text{constant}, \tag{4.18f}$$

$$= \arcsin\left(\frac{\tau - \Delta}{\Delta}\right) + \text{constant}, \tag{4.18g}$$

$$= \arcsin\left(\frac{\tau - \sqrt{1+H_0^2}}{\sqrt{1+H_0^2}}\right) + \text{constant}, \tag{4.18h}$$

where we performed the substitutions $\lambda = \tau - \Delta$ and $\mu = \frac{\lambda}{\Delta}$.

At this point, we can mimic our procedure for the previous cases and define

$$u = \arcsin\left(\frac{\tau - \sqrt{1+H_0^2}}{\sqrt{1+H_0^2}}\right) - r \quad \text{and} \quad v = \arcsin\left(\frac{\tau - \sqrt{1+H_0^2}}{\sqrt{1+H_0^2}}\right) + r, \tag{4.19}$$

which have

$$\mathrm{d}u = \frac{\mathrm{d}\tau}{a(\tau)} - \mathrm{d}r \quad \text{and} \quad \mathrm{d}v = \frac{\mathrm{d}\tau}{a(\tau)} + \mathrm{d}r\,, \tag{4.20}$$

where we wrote $a(\tau)$ instead of the full expression $a(\tau) = \sqrt{1 + H_0^2 - (\tau - \sqrt{1+H_0^2})^2}$ for simplicity. We then have

$$-\,a(\tau)^2\,\mathrm{d}u\,\mathrm{d}v = -\,\mathrm{d}\tau^2 + a(\tau)^2\,\mathrm{d}r^2\,, \tag{4.21}$$

and the metric becomes

$$\mathrm{d}s^2 = -a(\tau)^2\,\mathrm{d}u\,\mathrm{d}v + a(\tau)^2 \sin^2\left(\frac{v-u}{2}\right)\mathrm{d}\Omega^2\,, \tag{4.22}$$

where τ is now understood as a function of u and v.

The range of τ is automatically enconded in the fact that

$$\frac{u+v}{2} = \arcsin\left(\frac{\tau - \sqrt{1+H_0^2}}{\sqrt{1+H_0^2}}\right), \tag{4.23}$$

since it ends up being the domain of definition of the expression. We see this equation also imposes that

$$-\,\frac{\pi}{2} < \frac{u+v}{2} < +\frac{\pi}{2}, \tag{4.24}$$

since this is the range of the arcsine. The range of r also imposes that

$$0 < \frac{v-u}{2} < \pi. \tag{4.25}$$

We'll define the unphysical metric through

$$\mathrm{d}\tilde{s}^2 = -\,\mathrm{d}u\,\mathrm{d}v + \sin^2\left(\frac{v-u}{2}\right)\mathrm{d}\Omega^2\,. \tag{4.26}$$

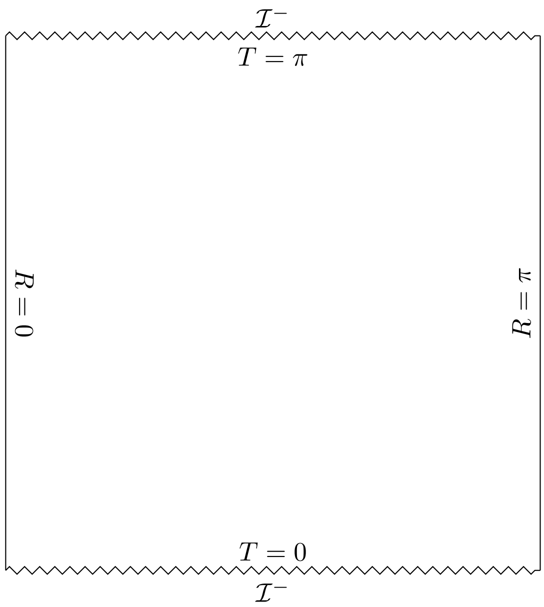

Figure 8: *Penrose diagram for a spatially closed Universe filled with radiation.*

Defining

$$T = \frac{u+v}{2} + \frac{\pi}{2} \quad \text{and} \quad R = \frac{v-u}{2} \tag{4.27}$$

we get to

$$\mathrm{d}\tilde{s}^2 = -\,\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R\,\mathrm{d}\Omega^2\,. \tag{4.28}$$

The $\frac{\pi}{2}$ term on T is there to change its range slightly. With these choices, notice we get

$$0 < T < \pi \quad \text{and} \quad 0 < R < \pi, \tag{4.29}$$

leading to the diagram Fig. 8. We have two coordinate singularities at the sides, which are the usual singularities one has in the spherical coordinate system (remember this universe is spatially a 3-sphere, so $R = 0$ and $R = \pi$ are analogous to $\theta = 0$ and $\theta = \pi$ in the 2-sphere), while at the top and bottom we have real singularities associated to the Big Bang and Big Crunch of the Universe.

This time we didn't need to compactify our coordinates, for the closed universe is already compact. Instead, we only changed coordinates in order to have the lightcones at $\frac{\pi}{4}$ angles and introduced a conformal scale for mere convenience. Notice this conformal scaling is the same one we would get if we used conformal time from the beginning, instead of cosmic time (*conferatur* Wald 1984, p. 104–105).

We can read from Fig. 8 an interesting result on the causal structure of the spatially flat, radiation-filled universe: at the moment of the Big Crunch, the particle horizons cease to exist, *id est*, at the moment of the Big Crunch a light particle emitted at the beginning of the Universe has completed a travel halfway across the Universe, and at this moment all points in space come into causal contact with each other. This result is discussed in Wald 1984, Sec. 5.3b.

5 Spatially Closed, Dust-Filled Universe

Penrose diagrams capture the causal structure of spacetime, and hence we should be able to see the differences in causal structure clearly. The spatially closed, radiation-filled universe provided a nice example, since it shows clearly the presence and disappearance of its particle horizons. As mentioned in Wald 1984, Sec. 5.3b, for a spatially closed, dust-filled universe the particle horizons disappear at the middle of the universe's lifetime, instead of at the end. It is then particularly interesting for us to take a look at how the Penrose diagram for this different universe looks like.

From the previous example, we've seen that the conformal transformation we end up performing on a closed universe is pretty much just changing from cosmic time to conformal time. Hence, this time we'll write the metric as

$$\mathrm{d}s^2 = -a(t)^2\big[-\mathrm{d}t^2 + \mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\Omega^2\big], \tag{5.1}$$

which spares us the effort of changing to null coordinates and back. We already know the range of the r coordinate is $(0, \pi)$, due to the spherical coordinate system. We just need to figure out the range of the t coordinate. This can be done by solving the Friedmann equations and looking at the roots of the scale factor.

Since we'll be working with both cosmic and conformal time, we shall make clear that we denote by $\dot{a}$ the derivative of the scale factor with respect to cosmic time τ and by a' the derivative with respect to conformal time t. We also recall that conformal time and cosmic time are related by

$$\mathrm{d}\tau = a(t)\,\mathrm{d}t\,, \tag{5.2}$$

which is the coordinate transformation one needs to perform to get to Eq. (5.1) from Eq. (4.1) on page 13.

The Friedmann equations are given for our case by

$$\begin{cases} 3\dfrac{\dot{a}^2}{a^2} = 8\pi\rho - \dfrac{3}{a^2}, \\ 3\dfrac{\ddot{a}}{a} = -4\pi\rho, \end{cases} \tag{5.3}$$

which lead us to

$$\dot{a}^2 + 2a\ddot{a} + 1 = 0, \tag{5.4a}$$

$$\dot{a}^3 + 2a\dot{a}\ddot{a} + \dot{a} = 0, \tag{5.4b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big(a\dot{a}^2 + a\big) = 0, \tag{5.4c}$$

$$a\dot{a}^2 + a = \text{constant}. \tag{5.4d}$$

From the same arguments we used in the previous cases, we see that the constant is $H_0^2 + 1$. Hence, we get to the differential equation

$$a\dot{a}^2 + a = H_0^2 + 1. \tag{5.5}$$

This equation is given in terms of cosmic time, but we are interested in figuring out what happens in conformal time. Hence, we should find the differential equation with respect to conformal time. Since

$$\begin{aligned} \dot{a} &= \frac{\mathrm{d}a}{\mathrm{d}\tau}, && (5.6a)\\ &= \frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}, && (5.6b)\\ &= \frac{a'}{a}, && (5.6c) \end{aligned}$$

and hence the Friedmann equations yield

$$a\frac{a'^2}{a^2} + a = H_0^2 + 1. \tag{5.7}$$

For simpicity, we'll write $\Delta = H_0^2 + 1$ from now on. The differential equation we want to solve is

$$a'^2 + a^2 - \Delta a = 0, \tag{5.8}$$

id est,

$$a' = \pm\sqrt{\Delta a - a^2}. \tag{5.9}$$

To solve the differential equation, we notice that

$$\mathrm{d}t = \frac{\mathrm{d}a}{\sqrt{\Delta a - a^2}}, \tag{5.10}$$

which can be integrated by completing squares in the denominator and using essentially the same tricks we used on Eq. (4.18) on page 15. At the end of the day, one gets to

$$t = \pm \arcsin\left(\frac{2a - \Delta}{\Delta}\right) + c_{\pm}, \tag{5.11}$$

where $c_{\pm}$ is a constant that could depend on the sign taken in consideration in the differential equation.

Let us isolate a. We can do this by noticing that

$$\begin{aligned} t - c_{\pm} &= \pm \arcsin\left(\frac{2a - \Delta}{\Delta}\right), && (5.12a)\\ \sin(t - c_{\pm}) &= \pm\frac{2a - \Delta}{\Delta}, && (5.12b)\\ \frac{\Delta}{2} \pm \frac{\Delta}{2}\sin(t - c_{\pm}) &= a, && (5.12c) \end{aligned}$$

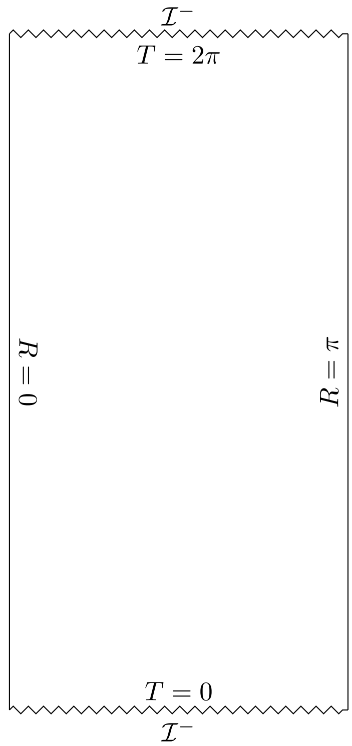

Figure 9: *Penrose diagram for a spatially closed Universe filled with dust.*

and hence,

$$a(t) = \frac{\Delta}{2}(1 \pm \sin(t - c_\pm)), \tag{5.13}$$

from which we can already see that the Universe will indeed have a Big Bang and a Big Crunch, for the sine will vanish more than once. We would like to keep the Big Bang at $t = 0$, so we pick $c_\pm = \pm\frac{\pi}{2}$. This ends up ensuring both signs yield the very same thing, for

$$+\sin\left(t - \frac{\pi}{2}\right) = \cos(t) = -\sin\left(t + \frac{\pi}{2}\right). \tag{5.14}$$

Hence, we have

$$a(t) = \frac{1 + H_0^2}{2}(1 - \cos t). \tag{5.15}$$

We have the Big Bang at $t = 0$ by construction, since we choose time to start at the Big Bang. As for the Big Crunch, it happens at the next root of $a(t)$, which is $t = 2\pi$. We are now in position to draw the Penrose diagram for this universe. It is shown in Fig. 9, which defines $T = t$ and $R = r$ to keep consistency with the notation used in the previous diagrams.

Comparing Figs. 8 and 9 on page 17 and on this page we see that the dust-filled universe has a taller diagram. This exhibits the difference between the causal structures of the two models: the particle horizons disappear at the middle of the universe's lifetime, instead of at the end. The fact that the diagram is taller reflects this, because the lightcones are always kept at a $\frac{\pi}{4}$ angle. Once one gets to the Big Crunch, all of space was already available to be seen by anyone.

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