Applications of Topos Theory to the Mathematical Foundations of Quantum Theory

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Abstract: This is my study notebook for a minicourse given by Prof. Walter de Siqueira Pedra (DFMA-IFUSP) at the IV Jayme Tiomno School on Physics (Pedra 2022). The school was organized at the University of São Paulo's Intitute of Physics (IFUSP) by the Dead Physicists Society (DPS), a student-driven organization at IFUSP. These notes were written as a way of practicing with the theory and keeping up with the minicourse. I claim no originality on results or presentation. Notice though that these notes are, quite naturally, my interpretation of the lectures, which means I might as well have added some original mistakes to them. This text is not endorsed by Prof. Pedra, IFUSP, or DPS.

Keywords: Mathematical Foundations of Quantum Theory, Topos Theory, Category Theory.

Version: August 15, 2022.

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Introduction

Our goal in this course is to show how the use of tools from Category Theory, and more specifically Topos Theory, can help someone understand deep issues concerning Quantum Theory. As a motivation for the mathematical constructions we'll perform, let us begin by discussing some quotes relating to both Quantum Mechanics and topoi*.

First we begin with a quote by Bohr.

> In this respect we must, on the one hand, realize that the aim of every physical experiment—to gain knowledge under reproducible and communicable conditions—leaves us no choice but to use everyday concepts, perhaps refined by the terminology of classical physics, not only in all accounts on the construction and manipulation of the measuring instruments, but also in the description of the actual experimental results. On the other hand, it is equally important to understand that just this circumstance implies that no result of an experiment concerning a phenomenon which, in principle, lies outside the range of classical physics can be interpreted as giving information about independent properties of the objects, but is inherently connected with a definite situation in the description of which the measuring instruments interacting with the objects also enter essentially. (Bohr 1957, pp. 25–26, as quoted by Faye 2017; Gomatam 2007)

* "Topoi" is the plural of "topos". Some authors prefer to use "toposes", but I find it too "Gollum-esque".

Notice how Bohr's comments lead to the notion that to describe Quantum Physics, we still need to use the language of Classical Physics. Yet, not all quantum phenomena can be interpreted on a purely classical manner. This "local-to-global" point of view will come to life in the context* of presheaves, a structure that naturally arises in Category Theory.

Let us also consider a quote by Grothendieck, who was particularly influent in Category Theory.

> Et ces "nuages probabilistes", remplaçant les rassurantes particules matérielles d'antan, me rappellent étrangement les élusifs "voisinages ouverts" qui peuplent les topos, tels des fantômes évanescents, pour entourer des "points" imaginaires, auxquels continue à se raccrocher encore envers et contre tous une imagination récalcitrante...(Grothendieck 2021)

In free translation,

> And these "probabilistic clouds", replacing the reassuring material particles of yesteryear, strangely remind me of the elusive "open neighborhoods" which populate the topoi, like evanescent ghosts, surrounding imaginary "points", to which they continue to clinge on against all odds to a recalcitrant imagination...

Grothendieck's intuition brings together the "probabilistic clouds" of quantum theory and the structures that occur in topoi, bringing forth another inspiration for our future constructions.

Let us add in a third quote.

> Sheaf theory was invented in the mid 1940s as a branch of algebraic topology to deal with the collation of local data on topological spaces. Through the success in the theory of functions of several complex variables and algebraic geometry, this theory is now indispensable in modern mathematics. However, instead of its generality dealing with local-to-global transitions, applications to other areas in science or engineering have not been well established so far except for logic and semantics in computer science with the notion of Topos. (Ghrist and Hiraoka 2011)

Our interest in topoi for modelling Quantum Mechanics will come by the means of a local-to-global transition. The basic idea will be that while, in a certain sense, Quantum Mechanics is globally different from Classical Physics, it is still described in Classical terms in a local manner. To give a more precise meaning to these terms, and to give a more precise meaning to Bohr's ideas, we'll bring in some Mathematics.

It should be mentioned that while we'll be discussing some examples of topoi, the precise definition of what a topos is is beyond the scope of these notes, given it would take further prerequisites withing Category Theory.

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*Pun intended, although it might not be clear at this stage.

1 Algebraic Notions

We'll start by defining and working with some algebraic notions that will be useful for us later.

Definition 1 [Algebra]**:**
Let $\mathcal{A}$ be a complex vector space and $\cdot\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a function. The pair $(\mathcal{A}, \cdot)$ is said to be an *algebra* if, and only if, $\cdot$ is bilinear. If $\cdot$ is commutative, the algebra is said to be *commutative*. If $\cdot$ is associative, the algebra is said to be *associative*. If there is an element $\mathbb{1} \in \mathcal{A}$ such that $\mathbb{1} \cdot A = A \cdot \mathbb{1} = A$, $\forall\, A \in \mathcal{A}$, the algebra is said to be *unital* and $\mathbb{1}$ is said to be the algebra's *identity*. ♠

Proposition 2:
An unital algebra has a unique identity. □

Once we define a mathematical structure, it is usual to also define a substructure that goes with it.

Definition 3 [Subalgebra]**:**
Let $(\mathcal{A}, \cdot)$ be an algebra and $\mathcal{B} \subseteq \mathcal{A}$ be a subspace. $\mathcal{B}$ is said to be a *subalgebra* if, and only if, it is closed under $\cdot$, *id est*, iff $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cdot B_2 \in \mathcal{B}$. If in addition to this $\mathbb{1}_{\mathcal{A}}$, the identity of $\mathcal{A}$, is an element of $\mathcal{B}$, then we say $\mathcal{B}$ is a *unital subalgebra*. ♠

It is important to point out that a subalgebra can be unital without being a unital subalgebra. This would happen because $\mathcal{B}$ would have an identity which does not coincide with that of $\mathcal{A}$.

Example?

Examples [Algebras and Subalgebras]**:**
The simplest example we can give of an algebra is the complex numbers $\mathbb{C}$ themselves, endowed with the usual complex multiplication. Notice this is a unital algebra with identity 1. It is also commutative and associative.

A more elaborate example is to consider a set Ω and pick the space of functions $f\colon \Omega \to \mathbb{C}$, denoted by $\mathcal{F}(\Omega)$. If we define

$$(f \cdot g)(\omega) \equiv f(\omega)g(\omega), \tag{1.1}$$

then $(\mathcal{F}(\Omega), \cdot)$ becomes a commutative, associative, unital algebra. An example of an unital subalgebra consists of the space of constant functions.

Another example is the space of $n \times n$ complex matrices, $\mathbb{M}_n$, equipped with the usual matrix product. This consists of an associative, unital algebra, bt it fails to be commutative for $n > 1$. The space of diagonal matrices forms a commutative unital subalgebra. Furthermore, the space of matrices of the form $\lambda\mathbb{1}_n$, for $\lambda \in \mathbb{C}$ consists of a unital subalgebra of the unital subalgebra.

An example of a non-associative algebra is obtained by equipping $\mathbb{M}_n$ with the so-called *Jordan product*, defined by

$$M_1 \circ M_2 = \frac{1}{2}(M_1M_2 + M_2M_1). \tag{1.2}$$

$(\mathbb{M}_n, \circ)$ is then a commutative unital algebra, but it fails to be associative if $n > 1$. Nevertheless, it has an associative unital subalgebra: the diagonal matrices. ♥

To proceed, we'll define a generalized notion of complex conjugation.

Definition 4 [Complex Conjugation]**:**
Let V be a complex vector space. A function $*\colon V \to V$ is said to be a *complex conjugation* if, and only if, it satisfies the following requirements:

i. $\forall\, v \in V, v^{**} = v$ ($*$ is an involution);

ii. $\forall\, v_1, v_2 \in V, \lambda \in \mathbb{C}, (v_1 + \lambda v_2)^* = v_1^* + \bar{\lambda} v_2$ ($*$ is antilinear).

If $(V, *)$ is a complex vector space with complex conjugation, a set of vectors $\Phi \subseteq V$ is said to be *self-conjugate* if, and only if, $v \in \Phi \Rightarrow v^* \in \Phi$. $v \in V$ is said to be *self-conjugate* if, and only if, $v = v^*$. ♠

Examples [Complex Conjugation]**:**
We can endow the algebras we previously considered with notions of complex conjugation.

In the complex numbers, the identification is straightforward. We define $z^* = \bar{z}$.

In the space of complex function on Ω, we define $f^*(\omega) \equiv \overline{f(\omega)}$, $\forall\, \omega \in \Omega$.

In the space of complex matrices, we define comple conjugation as Hermitian conjugation: $M^* = M^\dagger$. Notice that the diagonal matrices establish a self-conjugate set, even though their entries are not, in general, real. ♥

Having the notions of algebra and complex conjugation at hand, it is natural to require them to "work well together", leading us to the definition of a $*$-algebra.

Definition 5 [$*$-algebra]**:**
Let $\mathcal{A}$ be an algebra and $*\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a complex conjugation. The pair $(\mathcal{A}, *)$ is said to be a *$*$-algebra* if, and only if, $\forall\, A_1, A_2 \in \mathcal{A}, (A_1 A_2)^* = A_2^* A_1^*$. A *$*$-subalgebra)* of $(\mathcal{A}, *)$ is a self-conjugate subalgebra of $\mathcal{A}$. ♠

Notice that the definition of a $*$-algebra resembles the properties obtained when taking the Hermitian conjugate of matrices.

Of special interest to us will be the commutative unital $*$-subalgebras, so we'll provide them with a particular name.

Definition 6 [Context]**:**
Let $\mathcal{A}$ be a $*$-algebra. A commutative unital $*$-subalgebra of $\mathcal{A}$ is said to be a *(classical) context* of $\mathcal{A}$. ♠

Examples [$*$-algebras and Contexts]**:**
$\mathbb{C}$, $\mathcal{F}(\Omega)$, and $\mathbb{M}_n$ are $*$-algebras when equipped with the structures we defined previously.

Since $\mathcal{F}(\Omega)$ is commutative, all of its unital $*$-subalgebras are contexts.

Diagonal matrices are a context of $\mathbb{M}_n$. For $n > 1$, the algebra is not commutative, so not every unital $*$-subalgebra is a context. ♥

Some elements within a $*$-algebra present special properties, so we'll also provide them with special names.

Definition 7 [Projectors, Isometries, Resolution of the Identity]**:**
Let $\mathcal{A}$ be a $*$-algebra. We define the following terms.

i. $P \in \mathcal{A}$ is said to be an *orthogonal projector* if, and only if, $P^* = P$ (self-conjugate) and $P^2 = P$ (idempotent);

ii. $U \in \mathcal{A}$ is said to be a *partial isometry* if, and only if, U^*U and UU^* are orthogonal projectors;

iii. if $\mathcal{A}$ is unital, $U \in \mathcal{A}$ is said to be *unitary* if, and only if, $U^*U = UU^* = \mathbb{1}$;

iv. if $\{P_i\}_{i=1}^m \subseteq \mathcal{A}$ is a family of orthogonal projectors, they are said to be *mutually orthogonal projectors* if, and only if, $i \neq j \Rightarrow P_i P_j = 0$;

v. if $\mathcal{A}$ is unital and $\{P_i\}_{i=1}^m \subseteq \mathcal{A}$ are mutually orthogonal projectors, they are said to be a *resolution of the identity* if, and only if, $\sum_{i=1}^m P_i = \mathbb{1}$. ♠

Some of these names ("orthogonal", "isometry") are reminiscent of what these definitions imply when considered in the context of operators acting on Hilbert spaces.

Examples [Projectors, Isometries, Resolution of the Identity]**:**
Let us consider the unital $*$-algebra $\mathbb{M}_n$. Let $\mathbf{e} \in \mathbb{C}^n$ be non-vanishing. Then

$$P_{\mathbf{e}} = \frac{1}{|e_1|^2 + \cdots + |e_n|^2} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_n \end{bmatrix} \tag{1.3}$$

is an orthogonal projector.

Consider now a set of orthogonal vectors $\{\mathbf{e}_i\}_{i=1}^m \subseteq \mathbb{C}^n$. Then $\{P_{\mathbf{e}_i}\}_{i=1}^m$ are mutually orthogonal projectors. Furthermore, if $\{\mathbf{e}_i\}_{i=1}^n \subseteq \mathbb{C}^n$ is an orthogonal basis, then $\{P_{\mathbf{e}_i}\}_{i=1}^m$ is a resolution of the identity. ♥

Hence, the notion of a resolution of the identity translates the notion of a basis into algebraic concepts.

One of our interests with resolutions of the identity will be the fact that it can be used to generate contexts, as will some more general sets of operators as well. To see so, let us first show an auxiliary result.

Lemma 8:
Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{C}_\lambda$ be a context of $\mathcal{A}$ for each $\lambda \in \Lambda$ (which is not necessarily countable). Then

$$\mathcal{C} \equiv \inf_{\lambda \in \Lambda} \mathcal{C}_\lambda \equiv \bigcap_{\lambda \in \Lambda} \mathcal{C}_\lambda \tag{1.4}$$

is a context of $\mathcal{A}$. □

Proof:

We want to show that $\mathcal{C}$ is a commutative unital $*$-subalgebra of $\mathcal{A}$.

The intersection of subspaces is a subspace, so we know $\mathcal{C}$ is a subspace of $\mathcal{A}$. It is also a subalgebra, since

$$C_1, C_2 \in \mathcal{C} \Rightarrow C_1, C_2 \in \mathcal{C}_\lambda, \forall\, \lambda \in \Lambda \Rightarrow C_1C_2 \in \mathcal{C}_\lambda, \forall\, \lambda \in \Lambda \Rightarrow C_1C_2 \in \mathcal{C}. \tag{1.5}$$

By a similar argument, one shows $\mathcal{C}$ is a commutative $*$-subalgebra.

Since the identity of $\mathcal{A}$ is in all of the $\mathcal{C}_\lambda$, it is in $\mathcal{C}$ as well. Hence, $\mathcal{C}$ is indeed a context. ■

Lemma 8 on the preceding page allows us to define the smallest context satisfying some property. We'll then use this to define the context generated by a self-conjugate set of commuting elements.

Definition 9 [Context Generated by a Self-Conjugate Set of Commuting Elements]**:**

Let $\mathcal{A}$ be a unital $*$-algebra and $\{A_i\}_{i=1}^m$ a self-conjugate set of commuting elements (*id est*, $[A_i, A_j] \equiv A_iA_j - A_jA_i = 0$ whenever $i \neq j$). Then there is at least one context[*] containing $\{A_i\}_{i=1}^m$ and the smallest of them is said to be the *context generated by* $\{A_i\}_{i=1}^m$, denoted by $\mathcal{C}(\{A_i\}_{i=1}^m)$. ♠

Notice, in particular, that all resolutions of the identity generate a context. If $\{\mathbf{e}_i\}_{i=1}^n \subseteq \mathbb{C}^n$ is an orthogonal basis, then $\mathcal{C}\big(\{P_{\mathbf{e}_i}\}_{i=1}^m\big)$ are the diagonal matrices in this basis.

While we'll return to these constructions later, let us continue our ride through the algebraic constructions. Let us state, without proof, a particularly important theorem on decomposing matrices in terms of a resolution of the identity.

Theorem 10 [Spectral Theorem for Matrices]**:**

Let $M \in \mathbb{M}_n$ be self-adjoint, with set of eigenvalues $\sigma(M) = \{\lambda\}_{i=1}^m$. Then the following hold:

i. *there is a unitary matrix $U \in \mathbb{M}_n$ such that U^*MU is diagonal;*

ii. *there is a unique resolution of the identity $\{P_i\}_{i=1}^m \subseteq \mathbb{M}_n$ such that*

$$M = \sum_{i=1}^m \lambda_i P_i. \tag{1.6}$$

□

This decomposition allows us to define functions of matrices in a simple manner.

Definition 11 [Functional Calculus]**:**

Let $M \in \mathbb{M}_n$ be self-adjoint, with set of eigenvalues $\sigma(M) = \{\lambda\}_{i=1}^m$. Let $\{P_i\}_{i=1}^m$ be the resolution of the identity associated to M by means of the spectral theorem. Then,

[*]The space of all polynomials of $\{A_i\}_{i=1}^m$.

for any $f \in \mathcal{F}(\sigma(M))$, we define

$$f(M) \equiv \sum_{i=1}^{m} f(\lambda_i) P_i. \tag{1.7}$$

This is known as the *functional calculus* or *spectral calculus*. ♠

When working with a mathematical structure, it is always interesting to consider the properties of "structure-preserving maps". This leads us, in this context, to the notion of homomorphisms.

Definition 12 [Homomorphism]**:**
Let $\mathcal{A}$ and $\mathcal{B}$ be algebras and $\varphi \colon \mathcal{A} \to \mathcal{B}$ be a function. φ is said to be a *homomorphism* if, and only if,

i. φ is a linear transformation;

ii. $\forall\, A_1, A_2 \in \mathcal{A}$, $\varphi(A_1 A_2) = \varphi(A_1)\varphi(A_2)$.

If φ is injective, it is said to be *faithful*. If $\mathcal{A}$ and $\mathcal{B}$ are unital algebras, φ is said to be a *unital homomorphism* if, and only if, $\varphi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$. If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, φ is said to be a $*$*-homomorphism* if, and only if, $\forall\, A \in \mathcal{A}$, $\varphi(A^*) = \varphi(A)^*$. ♠

Just as we gave a specific name to characters, we'll add a specific name to some of their homomorphisms.

Definition 13 [Character]**:**
If $\mathcal{C}$ is a context, the unital $*$-homomorphisms $\varphi \colon \mathcal{C} \to \mathbb{C}$ are said to be the *characters* of $\mathcal{C}$. The set of all characters of $\mathcal{C}$ is said to be the *Gelfand spectrum of* $\mathcal{C}$, denoted $\Sigma(\mathcal{C})$. ♠

The characters provide information about the context by comparing it to the "simplest of all algebras" (here meant in a colloquial sense).

Example [Functional Calculus is a Faithful Unital $*$-homomorphism]**:**
Let $M \in \mathbb{M}_n$ be self-adjoint. The functional calculus defines a faithful unital $*$-homomorphism from $\mathcal{F}(\sigma(M))$ to $\mathbb{M}_n$.

Let us show this. Let $f, g \in \mathcal{F}(\sigma(M))$, and $\alpha \in \mathbb{C}$. Let the spectral decomposition of M be

$$M = \sum_{i=1}^{m} \lambda_i P_i. \tag{1.8}$$

The functional calculus is linear. Indeed,

$$(f + \alpha g)(M) = \sum_{i=1}^{m} (f + \alpha g)(\lambda_i) P_i, \tag{1.9a}$$

$$= \sum_{i=1}^{m} f(\lambda_i) P_i + \alpha \sum_{i=1}^{m} g(\lambda_i) P_i, \tag{1.9b}$$

$$= f(M) + \alpha g(M). \tag{1.9c}$$

To show it is a homomorphism, we must now consider its properties when dealing with products. Notice that

$$(fg)(M) = \sum_{i=1}^{m} (fg)(\lambda_i) P_i, \tag{1.10a}$$

$$= \sum_{i=1}^{m} f(\lambda_i) g(\lambda_i) P_i, \tag{1.10b}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} f(\lambda_i) g(\lambda_j) P_i \delta_{ij}, \tag{1.10c}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} f(\lambda_i) g(\lambda_j) P_i P_j, \tag{1.10d}$$

$$= \left(\sum_{i=1}^{m} f(\lambda_i) P_i \right) \left(\sum_{j=1}^{m} g(\lambda_j) P_j \right), \tag{1.10e}$$

$$= f(M) g(M). \tag{1.10f}$$

To see it is a $*$-homomorphism we notice that

$$f^*(M) = \sum_{i=1}^{m} f^*(\lambda_i) P_i, \tag{1.11a}$$

$$= \sum_{i=1}^{m} \overline{f(\lambda_i)} P_i, \tag{1.11b}$$

$$= \sum_{i=1}^{m} \overline{f(\lambda_i)} P_i^*, \tag{1.11c}$$

$$= \sum_{i=1}^{m} (f(\lambda_i) P_i)^*, \tag{1.11d}$$

$$= \left(\sum_{i=1}^{m} f(\lambda_i) P_i \right)^*, \tag{1.11e}$$

$$= f(M)^*. \tag{1.11f}$$

If $f \neq g$, then

$$f(M) - g(M) = \sum_{i=1}^{m} [f(\lambda_i) - g(\lambda_i)] P_i \neq 0, \tag{1.12}$$

for at least one of the $f(\lambda_i) - g(\lambda_i)$ won't vanish, and different values of i can't cancel each other because the P_i are linearly independent. Hence, the functional calculus is faithful.

It only remains to prove that it is unital. Let $f(\lambda) = 1$, $\forall \lambda \in \sigma(M)$. Then

$$f(M) = \sum_{i=1}^{m} f(\lambda_i) P_i, \tag{1.13a}$$

$$= \sum_{i=1}^{m} P_i, \tag{1.13b}$$

$$= \mathbb{1}_n. \tag{1.13c}$$

Hence, the functional calculus takes the identity to the identity, establishing it as a faithful unital $*$-homomorphism. ♥

Examples [Characters]**:**

Let Ω be a set* and $\omega_0 \in \Omega$. Define the function

$$\begin{aligned} \varphi_{\omega_0} \colon \mathcal{F}(\Omega) &\to \mathbb{C} \\ f &\mapsto f(\omega_0). \end{aligned} \tag{1.14}$$

φ_{ω_0} is a character on $\mathcal{F}(\Omega)$. Notice then that some characters can be understood as points on a set, so studying the characters of $\mathcal{F}(\Omega)$ can allow us to reconstruct the underlying set Ω.

As a second example, let $\psi_0 = (1, 0, \cdots, 0) \in \mathbb{C}^n$. The function

$$\begin{aligned} \varphi_{\psi_0} \colon \mathbb{M}_n &\to \mathbb{C} \\ M &\mapsto \langle \psi_0 | M | \psi_0 \rangle \equiv M_{11} \end{aligned} \tag{1.15}$$

is a character on the diagonal matrices (which is a context of $\mathbb{M}_n$). ♥

The previous example shows that some characters can be understood as points on a set. Let us pursue that fact a little bit further as an exercise.

Proposition 14:

Let Ω be a finite set. Then all of the characters of $\mathcal{F}(\Omega)$ can be understood as points on Ω. □

Proof:

Let $\varphi \in \Sigma(\mathcal{F}(\Omega))$. We want to show that there is some $\omega_0 \in \Omega$ such that $\varphi(f) = f(\omega_0)$, $\forall\, f \in \mathcal{F}(\Omega)$.

Given $\omega \in \Omega$ and define χ_ω to be the function that has $\chi_\omega(\omega) = 1$ but vanishes on all other points. Notice that

$$\mathcal{F}(\Omega) = \operatorname{span}\{\chi_\omega, \omega \in \Omega\}. \tag{1.16}$$

Furthermore, notice also that $\{\chi_\omega, \omega \in \Omega\}$ is a resolution of the identity.

Let us prove that $\varphi(\chi_\omega) \in \{0, 1\}$. This can be done by noticing that $\varphi(\chi_\omega^2) = \varphi(\chi_\omega)^2$, which implies the result. Next, we'll prove that there is a single $\omega_0 \in \Omega$ such that

*Physically, Ω could play the role of a classical phase space, so that $\mathcal{F}(\Omega)$ represents the classical observables.

$\varphi(\chi_{\omega_0}) = 1$, with $\varphi(\chi_\omega) = 0$ for the remaining cases. Notice that

$$1 = \varphi(\mathbb{1}), \tag{1.17a}$$

$$= \varphi\left(\sum_{\omega\in\Omega} \chi_\omega\right), \tag{1.17b}$$

$$= \sum_{\omega\in\Omega} \varphi(\chi_\omega). \tag{1.17c}$$

Hence, if more than one value $\omega \in \Omega$ has $\varphi(\chi_\omega) = 1$, we'll get to a contradiction. Hence, we concluded that there is some $\omega_0 \in \Omega$ such that $\varphi(\chi_\omega) = \chi_\omega(\omega_0)$.

Since the functions χ_ω span $\mathcal{F}(\Omega)$, the desired result follows by linearity. ■

Within a unital algebra, we can generalize the notion of eigenvalues to the notion of spectrum.

Definition 15 [Invertible Elements]**:**

Let $\mathcal{A}$ be a unital algebra and $A \in \mathcal{A}$. A is said to be *invertible* if, and only if, there is $A^{-1} \in \mathcal{A}$ such that $AA^{-1} = A^{-1}A = \mathbb{1}$. A^{-1} is called the inverse and, if it exists, it is unique. ♠

Definition 16 [Spectrum of an Element of a Unital Algebra]**:**

Let $\mathcal{A}$ be a unital algebra and $A \in \mathcal{A}$. We define *the spectrum of A in $\mathcal{A}$* as the set

$$\sigma_{\mathcal{A}}(A) = \{z \in \mathbb{C}; z\mathbb{1} - A \text{ is } \textit{not} \text{ invertible}\}. \tag{1.18}$$

♠

The subscript $\mathcal{A}$ in $\sigma_{\mathcal{A}}(A)$ is important, for the spectrum can change if we restrict consideration to a subalgebra. Only in some particular cases (*exempli gratia*, matrices, C^*-algebras) this doesn't happen.

Example [Spectrum]**:**

In the algebra of complex numbers, one has $\sigma_{\mathbb{C}}(z) = \{z\}$.

For any $f \in \mathcal{F}(\Omega)$, $\sigma_{\mathcal{F}(\Omega)}(f) = \operatorname{Ran} f$ (the range of f, also denoted $\operatorname{Ran} f \equiv f(\Omega)$).

For any matrix $M \in \mathbb{M}_n$ and any unital $*$-subalgebra $\mathcal{B}$, one has $\sigma_{\mathcal{B}}(M) = \sigma(M)$ (the set of eigenvalues of M). ♥

Proposition 17:

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a unital homomorphism. Then $\sigma_{\mathcal{B}}(\varphi(A)) \subseteq \sigma_{\mathcal{A}}(A)$. □

Proof:

We want to prove that if $z\mathbb{1}_{\mathcal{B}} - \varphi(A)$ is not invertible, then neither is $z\mathbb{1}_{\mathcal{A}} - A$. This is equivalent to proving that if $z\mathbb{1}_{\mathcal{A}} - A$ is invertible, then so is $z\mathbb{1}_{\mathcal{B}} - \varphi(A)$.

Suppose $z\mathbb{1}_{\mathcal{A}} - A$ is invertible with inverse C, *id est*, $(z\mathbb{1}_{\mathcal{A}} - A)C = C(z\mathbb{1}_{\mathcal{A}} - A) = \mathbb{1}_{\mathcal{A}}$. Then notice that

$$(z\mathbb{1}_{\mathcal{A}} - A)C = C(z\mathbb{1}_{\mathcal{A}} - A) = \mathbb{1}_{\mathcal{A}}, \tag{1.19a}$$

$$\varphi((z\mathbb{1}_{\mathcal{A}} - A)C) = \varphi(C(z\mathbb{1}_{\mathcal{A}} - A)) = \varphi(\mathbb{1}_{\mathcal{A}}), \tag{1.19b}$$
$$\varphi(z\mathbb{1}_{\mathcal{A}} - A)\varphi(C) = \varphi(C)\varphi(z\mathbb{1}_{\mathcal{A}} - A) = \varphi(\mathbb{1}_{\mathcal{A}}), \tag{1.19c}$$
$$(z\mathbb{1}_{\mathcal{B}} - \varphi(A))\varphi(C) = \varphi(C)(z\mathbb{1}_{\mathcal{B}} - \varphi(A)) = \mathbb{1}_{\mathcal{B}}, \tag{1.19d}$$

and hence $\varphi(C)$ is the inverse of $z\mathbb{1}_{\mathcal{B}} - \varphi(A)$. This concludes the proof. ■

Proposition 18:

Let $\mathcal{A}$ be a commutative unital $*$-algebra. Notice that $\mathcal{A}$ is a context. For every character $\varphi \in \Sigma(\mathcal{A})$ and for all $A \in \mathcal{A}$, it holds that $\varphi(A) \in \sigma_{\mathcal{A}}(A)$. □

Proof:

$\varphi(A) \in \sigma_{\mathcal{A}}(A)$ means that $\varphi(A)\mathbb{1}_{\mathcal{A}} - A$ is not invertible in $\mathcal{A}$. Let us suppose this affirmation is false, for the sake of contradiction.

Since φ is a character, it is a unital $*$-homomorphism from $\mathcal{A}$ to $\mathbb{C}$ by definition. Hence, Proposition 17 on the previous page implies $\sigma_{\mathbb{C}}(\varphi(A)) \subseteq \sigma_{\mathcal{A}}(A)$. As a consequence, if $\varphi(A) \notin \sigma_{\mathcal{A}}(A)$, then $\varphi(A) \notin \sigma_{\mathbb{C}}(\varphi(A))$. That is, if $\varphi(A) \notin \sigma_{\mathcal{A}}(A)$, then $\varphi(A)\mathbb{1}_{\mathbb{C}} - \varphi(A) = \varphi(A) - \varphi(A) = 0$ is invertible. This is a contradiction, so we conclude that $\varphi(A) \in \sigma_{\mathcal{A}}(A)$. ■

Corollary 19:

If $\mathcal{C}$ is a context of $\mathbb{M}_n$, then for all $\varphi \in \Sigma(\mathcal{C})$ and for all $M \in \mathcal{C}$ it holds that $\varphi(M)$ is an eigenvalue of M. □

Proof:

Follows from Proposition 18 by recalling that the spectrum of M is always the set of eigenvalues of M. ■

Theorem 20:

Let $M \in \mathbb{M}_n$ be self-adjoint. Then the following hold.

i. *The function*

$$\begin{aligned} \Sigma(\mathcal{C}(M)) &\to \sigma(M) \\ \varphi &\mapsto \varphi(M) \end{aligned} \tag{1.20}$$

is a bijection. Hence, we have a relation between the Gelfand spectrum and the spectrum of a matrix.

ii. *Given any $f \in \mathcal{F}(\sigma(M))$ and $\varphi \in \Sigma(\mathcal{C}(M))$, it holds that $f(M) \in \mathcal{C}(M)$ and that $\varphi(f(M)) = f(\varphi(M))$. Furthermore,*

$$\mathcal{C}(M) = \{f(M); f \in \mathcal{F}(\sigma(M))\}. \tag{1.21}$$

iii. *If $\mathcal{C}$ is a context of $\mathbb{M}_n$, then there is some $M_{\mathcal{C}} \in \mathbb{M}_n$ such that $\mathcal{C} = \mathcal{C}(M_{\mathcal{C}})$.* □

Proof:

We'll only prove the first two assertions. The third is more subtle and we should point out it does not hold for every algebra.

Let us begin by proving that $f(M) \in \mathcal{C}(M)$ for all $f \in \mathcal{F}(\sigma(M))$. Firstly, notice that

$$M - \lambda_i \mathbb{1} = \sum_j \lambda_j P_j - \sum_j \lambda_i P_i, \tag{1.22a}$$

$$= \sum_j (\lambda_j - \lambda_i) P_j, \tag{1.22b}$$

and hence $\sum_j (\lambda_j - \lambda_i) P_j \in \mathcal{C}(M)$, where $M = \sum_i \lambda_i P_i$ is the spectral decomposition of M. We can continue iterating this procedure until we isolate each one of the projectors P_i as being elements of $\mathcal{C}(M)$ (this is possible because there are finitely many projectors, so the process ends). Since all of the projectors lie on $\mathcal{C}(M)$ and $f(M)$ is a linear combination of the projectors for any $f \in \mathcal{F}(\sigma(M))$, it follows that $f(M) \in \mathcal{C}(M)$, for any $f \in \mathcal{F}(\sigma(M))$. It is straightforward to show that $\{f(M); f \in \mathcal{F}(\sigma(M))\}$ is a context itself. Hence, since $\mathcal{C}(M)$ is the smallest context containing M and it always contains the $f(M)$'s, we conclude that

$$\mathcal{C}(M) = \{f(M); f \in \mathcal{F}(\sigma(M))\}. \tag{1.23}$$

Let now $\varphi \in \Sigma(\mathcal{C}(M))$ and $f \in \mathcal{F}(\sigma(M))$. Then

$$\varphi(f(M)) = \varphi\left(\sum_i f(\lambda_i) P_i\right), \tag{1.24a}$$

$$= \sum_i f(\lambda_i) \varphi(P_i), \tag{1.24b}$$

$$= \sum_i f(\lambda_i) \varphi(P_i), \tag{1.24c}$$

$$= f\left(\sum_i \lambda_i \varphi(P_i)\right), \tag{1.24d}$$

$$= f(\varphi(M)), \tag{1.24e}$$

which establishes the second assertion.

Let us now prove the first assertion. Suppose $\varphi_1, \varphi_2 \in \Sigma(\mathcal{C}(M))$ with $\varphi_1(M) = \varphi_2(M)$. Then $\varphi_1(f(M)) = f(\varphi_1(M)) = f(\varphi_2(M)) = \varphi_2(f(M))$, which shows the two characters must coincide on all of $\mathcal{C}(M)$ if they coincide on M. Hence, $\varphi \mapsto \varphi(M)$ is injective. To show that the map is surjective, define $\varphi_i \colon \mathcal{C}(M) \to \mathbb{C}$ by imposing that $\varphi_i(P_j) = \delta_{ij}$ and extending it by linearity. One can check this defines a (family of) characters. Furthermore, $\varphi_i(M) = \lambda_i$, so these characters yield each of the eigenvalues of M when acted on with $\varphi \mapsto \varphi(M)$. Since $\sigma(M)$ is the set of eigenvalues of M, the result is proved.

at the function $\varphi \mapsto \varphi(M)$ is injective. Suppose $\varphi_1, \varphi_2 \in \Sigma(\mathcal{C}(M))$ and that $\varphi_1(M) = \varphi_2(M)$. ■

Notice then that knowing the spectrum of $M \in \mathbb{M}_n$ and knowing the characters of $\mathcal{C}(M)$ are actually the same thing. This is made particularly clear by looking at the characters $\varphi_i(M) = \lambda_i$, which are actually all of the characters of $\mathcal{C}(M)$.

This fact allows us to generalize the notion of spectrum to sets of self-adjoint commuting matrices by defining it to be the Gelfand spectrum of $\mathcal{C}(M_1, \cdots, M_n)$. Notice then that all characters will be uniquely determined by the "quantum numbers" $\varphi(M_1), \cdots, \varphi(M_n) \in \mathbb{R}$. Notice also that the term "quantum numbers" is used suggestively in here because this is precisely the sort of construction that often arises in Quantum Mechanics. The characters then represent the expectation values of operators built out of those in the set of self-adjoint commuting matrices, and we assume the quantum state to be an eigenstate of all of these operators simultaneously.

Furthermore, since the functional calculus is a faithful unitary $*$-homomorphism, we can understand $\mathcal{C}(M)$ as being a "copy" in $\mathbb{M}_n$ of $\mathcal{F}(\sigma(M))$. Even further, since any context $\mathcal{C}$ in $\mathbb{M}_n$ has the form $\mathcal{C}(M_\mathcal{C})$, we see that all contexts of $\mathbb{M}_n$ are $\mathcal{F}(\sigma(M_\mathcal{C}))$ in disguise for some $M_\mathcal{C}$.

In other words, the contexts of $\mathbb{M}_n$ are function $*$-algebras, up to a faithful $*$-homomorphism (or if, you prefer, up to a $*$-isomorphism).

2 The Bell–Kochen–Specker (BKS) Theorem

We'll start to move towards Quantum Theory by studying the Bell–Kochen–Specker theorem, one of the principal theorems leading to the conclusion that the usual formulation of Quantum Theory cannot be realist. While we'll focus mainly on the mathematical aspects, Flori 2013, Chap. 3 discusses other physical aspects.

Firstly, we'll define what is meant by a valuation function, which is out notion of a function that attributes to an observable its physical value.

Notation:

We'll denote the space of self-adjoint $n \times n$ complex matrices by

$$\mathcal{O}_n = \{M \in \mathbb{M}_n; M = M^*\}. \tag{2.1}$$

Notice this is the space of observables of an n-level quantum system. ♦

Definition 21 [Valuation Function]**:**

We say a function $V\colon \mathcal{O}_n \to \mathbb{R}$ is a *valuation function* if, and only if, for all $A \in \mathcal{O}_n$ and all functions $f\colon \mathbb{R} \to \mathbb{R}$ it holds that

$$V(f(A)) = f(V(A)). \tag{2.2}$$

♠

Physically, the condition of Eq. (2.2) requires, for example, that the value of the energy squared is equal to the square of the value of the energy. Notice that while characters also share this property (see Theorem 20 on page 11), valuation functions are not characters.

We then ask ourselves whether we can attribute to all of these observables a real value at the same time. In other words, are there valuation functions? The answer to this question will be negative for $n \geq 3$ and is given by the Bell–Kochen–Specker theorem.

Before we prove the theorem itself, we'll first deal with an auxiliary lemma.

Lemma 22:

Let V be a valuation function in $\mathcal{O}_n$. For every context $\mathcal{C}$ in $\mathbb{M}_n$, there is a unique character $\varphi^V_{\mathcal{C}} \in \Sigma(\mathcal{C})$ that coincides with V on the self-adjoint matrices of $\mathcal{C}$. □

Proof:

Let $\mathcal{C} \subseteq \mathbb{M}_n$ be a context and suppose such a $\varphi^V_{\mathcal{C}}$ exists. Then, due to linearity, we have that for all $M \in \mathcal{C}$

$$\varphi^V_{\mathcal{C}}(M) = \varphi^V_{\mathcal{C}}\left(\frac{1}{2}(M + M^*)\right) + i\varphi^V_{\mathcal{C}}\left(\frac{i}{2}(M^* - M)\right), \tag{2.3a}$$

$$= V\left(\frac{1}{2}(M + M^*)\right) + iV\left(\frac{i}{2}(M^* - M)\right), \tag{2.3b}$$

for $\frac{1}{2}(M + M^*)$ and $\frac{i}{2}(M^* - M)$ are self-adjoint. Hence, if $\varphi^V_{\mathcal{C}}$ exists, it is proven that it is unique.

We still need to prove existence. We can do so by using Eq. (2.3) as a definition of $\varphi^V_{\mathcal{C}}$ and then proceeding to show it is indeed a character.

The definition of valuation function implies that $V(0) = 0$ (pick $f(x) = 0$ and use $V(f(A)) = f(V(A)))$. A similar trick yields $V(\mathbb{1}) = 1$. Hence, given $M_1, M_2 \in \mathcal{C} \cap \mathcal{O}_n$, it is straightforward to show using Eq. (2.3) that

$$\varphi^V_{\mathcal{C}}(M_1 + iM_2) = \varphi^V_{\mathcal{C}}(M_1) + i\varphi^V_{\mathcal{C}}(M_2). \tag{2.4}$$

The definition of valuation also lets us see that if $\alpha \in \mathbb{R}$, then

$$\varphi^V_{\mathcal{C}}(\alpha M_1) = V(\alpha M_1) = \alpha V(M_1) = \alpha\varphi^V_{\mathcal{C}}(M_1). \tag{2.5}$$

Using these two results, we find that for $\alpha \in \mathbb{C}$ we have

$$\varphi^V_{\mathcal{C}}(\alpha M_1) = \varphi^V_{\mathcal{C}}(\operatorname{Re}\alpha M_1 + i \operatorname{Im}\alpha M_1), \tag{2.6a}$$

$$= \operatorname{Re}\alpha\varphi^V_{\mathcal{C}}(M_1) + i \operatorname{Im}\alpha\varphi^V_{\mathcal{C}}(M_1), \tag{2.6b}$$

$$= \alpha\varphi^V_{\mathcal{C}}(M_1). \tag{2.6c}$$

Picking $M \in \mathcal{C}$ (without the requirement of being self-adjoint) and separating it in "real and imaginary parts" one can show that $\varphi^V_{\mathcal{C}}$ preserves the complex conjugation.

We know that there is some $\tilde{M} \in \mathbb{M}_n$ such that $M_1 = f_1(\tilde{M})$ and $M_2 = f_2(\tilde{M})$. Using this fact we notice that

$$\varphi^V_{\mathcal{C}}(M_1 + M_2) = \varphi^V_{\mathcal{C}}(f_1(\tilde{M}) + f_2(\tilde{M})), \tag{2.7a}$$

$$= V(f_1(\tilde{M}) + f_2(\tilde{M})), \tag{2.7b}$$

$$= V((f_1 + f_2)(\tilde{M})), \tag{2.7c}$$

$$= (f_1 + f_2)(V(\tilde{M})), \tag{2.7d}$$

$$= f_1(V(\tilde{M})) + f_2(V(\tilde{M})), \tag{2.7e}$$

$$= V(f_1(\tilde{M})) + V(f_2(\tilde{M})), \tag{2.7f}$$

$$= V(M_1) + V(M_2), \tag{2.7g}$$
$$= \varphi^V_{\mathcal{C}}(M_1) + \varphi^V_{\mathcal{C}}(M_2). \tag{2.7h}$$

Hence, we know now that $\forall\, M_1, M_2 \in \mathcal{C} \cap \mathcal{O}_n$ and $\forall\, \alpha \in \mathbb{C}$,

$$\varphi^V_{\mathcal{C}}(M_1 + \alpha M_2) = \varphi^V_{\mathcal{C}}(M_1) + \alpha\varphi^V_{\mathcal{C}}(M_2). \tag{2.8}$$

Using this, we can obtain the general result. Similar arguments lead to the fact that $\varphi^V_{\mathcal{C}}$ is indeed an algebra homomorphism (firstly for self-adjoint matrices, then for the general case). ■

In short, Lemma 22 on the previous page ensures that, within any given context of $\mathbb{M}_n$, we can extend a valuation to a character.

Theorem 23 [Bell–Kochen–Specker]**:**
If $n \geq 3$, there are no valuation functions on $\mathcal{O}_n$. □

Proof:
Suppose V is a valuation on $\mathcal{O}_n$. Let $\{P_i\}_{i=1}^m$ be a resolution of the identity on $\mathbb{M}_n$ and define $\mathcal{C}$ to be the context generated by $\{P_i\}_{i=1}^m$. Lemma 22 on the preceding page tells us that there is some character $\varphi^V_{\mathcal{C}}$ defined on $\mathcal{C}$ that agrees with V on the self-adjoint matrices (which includes those in the resolution of the identity). Hence, we see that

$$1 = \varphi^V_{\mathcal{C}}(\mathbb{1}), \tag{2.9a}$$
$$= \varphi^V_{\mathcal{C}}\left(\sum_{i=1}^m P_i\right), \tag{2.9b}$$
$$= \sum_{i=1}^m V(P_i). \tag{2.9c}$$

However, $V(P_i) = V(P_i^2) = V(P_i)^2$, which implies $V(P_i)$ is either 0 or 1. Therefore, we learn that for any valuation fuction V and resolution of the identity $\{P_i\}_{i=1}^m$ there is a unique k, $1 \leq k \leq m$, such that $V(P_k) = 1$, with $V(P_i) = 0$ for $i \neq k$.

Our goal will now be to choose resolutions of the identity in a clever enough way that leads us to a contradiction. We'll do it in the case $n = 4$, which is particularly simple, but the spirit is the same for the other cases (up to being able to find the appropriate resolutions of the identity).

Consider the resolutions of the identity associated with the eleven bases of $\mathbb{C}^4$ listed on Table 1 on the next page.

Notice that each element on Table 1 on the following page appears an even number of times. Hence, the elements that correspond to a projector P_k with $V(P_k) = 1$ occur an even number of times. On the other hand, for each basis we know that one, and only one, of the four elements yields a projector P_k with $V(P_k) = 1$. Hence, we know that the table has an even number of entries that lead to non-vanishing $V(P_k) = 1$, but we also know that this even number happens to be eleven. Since eleven is odd, we've reached a contradiction, forcing us to conclude that there are no valuations to begin with. ■

Table 1: *The eleven bases of $\mathbb{C}^4$ to be used in the proof of the Bell–Kochen–Specker theorem.*

$\mathbf{e}_1$	1	1	1	1	−1	−1	1	1	0	0	1
	0	0	0	0	1	1	−1	1	1	0	0
	0	0	0	0	1	1	1	−1	−1	1	1
	0	0	0	0	1	1	1	1	0	−1	0
$\mathbf{e}_2$	0	0	0	0	1	1	1	1	1	1	0
	1	1	0	0	−1	1	1	1	0	−1	1
	0	0	1	0	1	−1	−1	1	0	0	0
	0	0	0	1	1	1	1	−1	−1	0	1
$\mathbf{e}_3$	0	0	0	0	1	1	0	0	1	1	1
	0	0	1	1	1	0	1	0	1	1	1
	1	1	0	1	−1	1	1	1	1	1	−1
	0	1	1	0	1	0	0	1	1	1	−1
$\mathbf{e}_4$	0	0	0	0	1	0	1	1	1	1	1
	0	0	1	1	1	1	0	−1	−1	1	−1
	0	1	0	−1	1	0	0	0	−1	−1	−1
	1	−1	−1	0	−1	−1	−1	0	1	−1	1

It is worth pointing out that before the Bell–Kochen–Specker result came around, von Neumann had already shown that there are no $*$-homomorphisms from $\mathbb{M}_n$ to $\mathbb{C}$ (which explains why our definition of character is restricted to contexts only). The BKS theorem is a considerable improvement upon this previous result.

The Bell–Kochen–Specker Theorem is a no-go theorem restricting the possibilities concerning realistic interpretations of Quantum Theory, as discussed in more detail in Flori 2013, Chap. 3. Our choices when facing a no-go theorem are to either give up and accept it, or to find a loophole to bypass it. Our goals in this course are closer to the idea of finding a loophole, and exploring more possibilities by using topoi to change the logic we're using in the problem, and allowing truth values different from $\{0, 1\}$.

3 Introduction to Category Theory

Our next steps will now be to learn a new language to better describe the structure of Quantum Mechanics: that of Category Theory.

3.1 Basic Notions

Let us begin by defining what a category is (Borceux 1994).

Definition 24 [Category]**:**
A *category* $\mathbf{C}$ is made of:

i. a collection* of *objects*, $\mathrm{Ob}_{\mathbf{C}}$;

*Notice how we are avoiding the use of the world "set": categories can be larger than sets. The details,

ii. for each pair of objects $A, B \in \mathrm{Ob}_{\mathbf{C}}$, a collection of *arrows* or *morphisms*, $\mathrm{Mor}_{\mathbf{C}}(A, B)$;

iii. for each triple of objects $A, B, C \in \mathrm{Ob}_{\mathbf{C}}$, a composition rule $\circ\colon \mathrm{Mor}_{\mathbf{C}}(A, B) \times \mathrm{Mor}_{\mathbf{C}}(B, C) \to \mathrm{Mor}_{\mathbf{C}}(A, C)$;

iv. for each object $A \in \mathrm{Ob}_{\mathbf{C}}$, a morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathbf{C}}(A, A)$, which we'll call the *identity* on A.

We also require these structures to respect the following two axioms:

i. $\forall\, A, B, C, D \in \mathrm{Ob}_{\mathbf{C}}$, $\forall\, f \in \mathrm{Mor}_{\mathbf{C}}(A, B), g \in \mathrm{Mor}_{\mathbf{C}}(B, C), h \in \mathrm{Mor}_{\mathbf{C}}(C, D)$, it holds that $h \circ (g \circ f) = (h \circ g) \circ f$ (associative law);

ii. $\forall\, A, B \in \mathrm{Ob}_{\mathbf{C}}$, $\forall\, f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, it holds that $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ (identity law).

If $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, we write $f\colon A \to B$, and A, B are called the *domain* $\mathrm{Dom}\, f$ and *codomain* $\mathrm{Cod}\, f$ of f, respectively. ♠

Definition 25 [Pre-Order Category]**:**
A category $\mathbf{C}$ is said to be a *pre-order category* if, and only if, for every pair of objects $A, B \in \mathrm{Ob}_{\mathbf{C}}$ there is at most one arrow $f\colon A \to B$. ♠

While pre-order categories might seem simple, they can be extremely important as we shall see. For example, quantum observables ($\mathcal{O}_n$) and contexts will form pre-order categories.

Given a category, we can build a new one by reversing its arrows.

Definition 26 [Opposite Category]**:**
Let $\mathbf{C}$ be a category. We define the *category opposite to* $\mathbf{C}$, denoted $\mathbf{C}^{\mathrm{op}}$, by giving it

i. the objects $\mathrm{Ob}_{\mathbf{C}^{\mathrm{op}}} \equiv \mathrm{Ob}_{\mathbf{C}}$,

ii. the morphisms $\mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(A, B) \equiv \mathrm{Mor}_{\mathbf{C}}(B, A)$,

iii. the composition law such that if $f^{\mathrm{op}} \in \mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(A, B), g^{\mathrm{op}} \in \mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(B, C)$, then $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$, where we have written h^{op} for the morphism of $\mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(C, D)$ corresponding to $h\colon D \to C$ in $\mathbf{C}$.

♠

It is useful to characterize some special types of morphisms, so we'll give them particular names.

Definition 27 [Types of Morphisms]**:**
Let $\mathbf{C}$ be a category and $f\colon A \to B$ be a morphism in $\mathbf{C}$. We say f is

i. *monic* (or a *monomorphism*) if, and only if, for any $g, h \in \mathrm{Mor}_{\mathbf{C}}(B, C)$, $f \circ g = f \circ h$ implies $g = h$, *id est*, iff f is left-simplifiable. In this case, we write $f\colon A \rightarrowtail B$;

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however, are beyond our scope.

ii. *epic* (or an *epimorphism*) if, and only if, for any $g, h \in \mathrm{Mor}_{\mathsf{C}}(C, A)$, $g \circ f = h \circ f$ implies $g = h$, *id est*, iff f is right-simplifiable. In this case, we write $f \colon A \twoheadrightarrow B$;

iii. a *bimorphism* if, and only if, f is both a monic and epic;

iv. *iso* (or an *isomorphism*) if, and only if, there is a morphism $f^{-1} \colon B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. We say that A and B are *isomorphic* or *equivalent*. $[A]$ denotes the equivalence class of objects equivalent to A;

v. an *endomorphism* if, and only if, $A = B$;

vi. an *automorphism* if, and only if, f is both an isomorphism and an endomorphism. ♠

Notation:

We'll denote $\mathrm{End}_{\mathsf{C}}(A) \equiv \mathrm{Mor}_{\mathsf{C}}(A, A)$ for the collection of endomorphisms of A and $\mathrm{Aut}_{\mathsf{C}}(A)$ for the collection of automorphisms of A. ♦

Proposition 28:

Isomorphisms define an equivalence relation in sets of objects of a category. □

Proof:

Omitted. ■

Proposition 29:

Let C *be a category and* $f \colon A \to B$ *be an isomorphism. Then* f *is both monic and epic.* □

Proof:

f is simplifiable from both sides by composing it with f^{-1}. ■

It is important to point out that not all morphisms that are monic and epic are isomorphisms. While that does hold in some categories (such as **Set**, the category of sets), it might fail in others.

Example [Bimorphisms Don't Need to be Isomorphisms]**:**

Consider the category formed by two objects A and B, each with its identity arrow, and with a single arrow from A to B, but no arrows in the other direction. Then f is a bimorphism, but not an isomorphism. ♥

Definition 30 [Initial and Terminal Objects]**:**

Let C be a category. An object $I \in \mathrm{Ob}_{\mathsf{C}}$ is said to be *initial* if, and only if, for every object $A \in \mathrm{Ob}_{\mathsf{C}}$ there is a unique arrow $i_A \colon I \to A$. Similarly, an object $T \in \mathrm{Ob}_{\mathsf{C}}$ is said to be *terminal* if, and only if, for every object $A \in \mathrm{Ob}_{\mathsf{C}}$ there is a unique arrow $t_A \colon A \to T$. ♠

Not all categories have initial and/or terminal objects.

Proposition 31:

Let C *be a category and let* I *(*T*) be an initial (terminal) object. Then all arrows with* $\operatorname{Cod} f = I$ *(*$\operatorname{Dom} f = T$*) are epic (monic).* □

Proof:

Suppose I is an initial object and let $A, B \in \mathrm{Ob}_{\mathbf{C}}$. Let $f\colon A \to I$ be a morphism and let $g, h \in \mathrm{Mor}_{\mathbf{C}}(I, B)$. We want to show that $g \circ f = h \circ f$ implies $g = h$. However, $\mathrm{Mor}_{\mathbf{C}}(I, B)$ has only one arrow so it is impossible for $g \neq h$.

A similar argument will apply to terminal objects. ■

Proposition 32:

Let $\mathbf{C}$ be a category. Any two initial (terminal) objects of $\mathbf{C}$ are equivalent. □

Proof:

Suppose I_1 and I_2 are initial objects. Then there is a single arrow $f\colon I_1 \to I_2$ and a single arrow $g\colon I_2 \to I_1$. Notice then that $f \circ g\colon I_1 \to I_1$. However, since I_1 is initial there can be only one arrow on $\mathrm{End}_{\mathbf{C}}(I_1)$: the identity arrow. Hence, $f \circ g = \mathrm{id}_{I_1}$. An analogous argument shows that $g \circ f = \mathrm{id}_{I_2}$, proving the result. ■

Due to this equivalence, it is usual to talk about initial and terminal objects in the singular (the initial object, the terminal object) even if there's more than one of them.

3.2 Diagrams

It is common in Category Theory to write diagrams instead of equations. For example, instead of writing $f\colon A \to B$ to mean that there is a morphism f from A to B, we can draw the diagram

$$A \xrightarrow{f} B. \tag{3.1}$$

Similarly, instead of writing $f\colon A \to B$, $g\colon B \to C$, and $g \circ f\colon A \to C$, we can draw

$$A \xrightarrow{f} B \xrightarrow{g} C. \tag{3.2}$$

We can state that $g \circ f = h$ by stating that the triangular diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \tag{3.3}$$

commutes. Saying that a diagram commutes mean that all "paths" between any two objects shown on the diagram are equal. Hence, if A, B, C, D are objects and $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, $g \in \mathrm{Mor}_{\mathbf{C}}(B, D)$, $h \in \mathrm{Mor}_{\mathbf{C}}(A, C)$, $k \in \mathrm{Mor}_{\mathbf{C}}(C, D)$, we can say that $g \circ f = k \circ h$ by saying that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \tag{3.4}$$

commutes.

We can just as easily cast systems of equations into diagrammatic versions. The associative law, which states that $h \circ (g \circ f) = (h \circ g) \circ f$, becomes the statement that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, "h \circ g"] & \\ & C \arrow[r, "h"'] & D \end{tikzcd} \tag{3.5}$$

commutes. The identity law, which said that $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ for any morphism $f \colon A \to B$, is the statement that

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "f"'] & A \arrow[d, "f"] \arrow[dr, "f"] & \\ & B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd} \tag{3.6}$$

commutes.

Notice that these diagrams are composed of smaller triangular and square diagrams, each one representing an equation in the system of equations.

Diagrams can be way more complicated, and possibly not even planar, but for our purposes we'll be able to keep with these simpler ones.

3.3 Examples of Categories

Example [Category of Sets]**:**

The category of sets is often written as **Set**. Its objects are the sets (all sets, so the collection of objects is definitely not a set), its morphisms are the functions from a set to another and the composition of arrows is given by the standard composition of functions.

One can show that the monomorphisms are the injective functions, that the epimorphisms are the surjective functions, and that the isomorphisms are the bijective functions (see, *exempli gratia*, Geroch 1985, Cap. 2). Hence, in **Set**, it does hold that all bimorphisms are isomorphisms. Prove

$\varnothing$, the empty set, is the only initial object. This can be seen by recalling that a function in set theory is actually a subset of the Cartesian product between domain and codomain satisfying some extra properties. If Ω is some set, a function from $\varnothing$ to Ω is then a subset of $\varnothing \times \Omega = \varnothing$, the only subset of which is $\varnothing$. Hence, there is a single function from $\varnothing$ to any other set: the empty function $\varnothing$.

The singleton sets, *id est*, those of the form $\{\cdot\}$ (sets with a single element), are the terminal objects: any function from any set to a singleton can be only the constant function.

In an analogous manner, we can also define the category **FinSet** of finite sets.

Due to **Set**, many authors consider Category Theory as an abstraction of the concept of function. Notice, for example, that we were able to define a generalized notion of injection and surjection without ever mentioning the concept of a point in an object, but rather in a quite general way. This can come in handy when we're dealing with objects that don't necessarily have points. ♥

Example [Category of Complex Vector Spaces]**:**

The category of complex vector spaces is called $\mathsf{Vect}_{\mathbb{C}}$. Its objects are complex vector spaces, its morphisms are linear transformations, and the composition rule is the standard composition of functions.

Similarly to **Set**, monomorphisms are injective linear transformations, epimorphisms are surjective linear transformations, and isomorphisms are bijective linear transformations.

Nevertheless, there is a new behaviour when it comes to initial and terminal objects: the trivial vector space $\{\mathbf{0}\}$ is now the only initial and terminal object. It is initial, because the only possible linear transformation from $\{\mathbf{0}\}$ to any vector space takes the origin to the origin. It is terminal because all linear transformations from any vector space to $\{\mathbf{0}\}$ must take all elements to the origin. Notice then how "simple" changes to the arrows and objects can lead to different categorical behaviours.

Similarly, we can also define the category of finite dimensional vector spaces, $\mathsf{FinVect}_{\mathbb{C}}$.

♥

Example [Unital $*$-Algebras]**:**

The category of unital $*$-algebras is denoted $*$-**UAlg**. Its objects are the unital $*$-algebras, its arrows the unital $*$-homomorphisms, and its composition rule the standard function composition.

This time, we have faithful unital $*$-homomorphisms for the monomorphisms, surjective unital $*$-homomorphisms for the epimorphisms, and bijective $*$-homomorphisms for the isomorphisms (bijective $*$-homomorphisms are automatically unital).

In a similar manner, we can define $*$-**UCAlg** (the category of commutative unital $*$-algebras) and $*$-**FinUCAlg** (the category of finite dimensional commutative unital $*$-algebras). ♥

Proposition 33:

The initial object in $*$-**UAlg** *is* $\mathbb{C}$*, the unital* $*$*-algebra of the complex numbers.* □

Proof:

We want to prove that there is a unique unital $*$-homomorphism from $\mathbb{C}$ into any unital $*$-algebra $\mathcal{A}$.

To prove existence, notice that in any unital $*$-algebra $\mathcal{A}$ we can consider the context generated by the identity, $\mathcal{C}(\mathbb{1})$. Define then $\varphi\colon \mathbb{C} \to \mathcal{C}(\mathbb{1})$ through $\varphi(z) = z\mathbb{1}$. It is straightforward to check this is a unital $*$-homomorphism.

However, since we want only unital $*$-homomorphisms, any other option ψ would also have to have $\psi(1) = \mathbb{1}$ and, by linearity, $\psi(z) = z\mathbb{1}$. Therefore, uniqueness is proved. ■

Example [Pre-ordered Sets]**:**

Let Ω be a set and $\preceq$ a pre-order on Ω, *id est*, a relation on Ω that is reflexive ($\forall\, p \in \Omega, p \preceq p$) and transitive ($\forall\, p, q, r \in \Omega$, $p \preceq q$ and $q \preceq r$ imply $p \preceq r$). Notice this does not need to be a partial order, since we aren't requesting antisymmetry ($\forall\, p, q \in \Omega$, $p \preceq q$ and $q \preceq p$ imply $p = q$).

We can make this pre-ordered set into a pre-order category. We simply let Ω be the collection of objects and define that, given $A, B \in \Omega$, there is a single arrow in $\mathrm{Mor}_\Omega(A, B)$

if, and only if, $A \preceq B$.

In this case, all arrows are monic and epic. If Ω has a manixum, then it is the terminal object. If it has a minimum, it is the initial object.

Due to this example, some author see categories as generalizations of pre-ordered sets. ♥

Example [Categories of Contexts]**:**

Let $\mathcal{A}$ be a unital $*$-algebra and $\mathfrak{C}$ be the family of all contexts of $\mathcal{A}$. If $C, C' \in \mathfrak{C}$, then C being a context of C' defines a pre-order on $\mathfrak{C}$, hence turning it into a pre-order category.

We'll denote by $\mathfrak{C}_n$ the category of contexts of $\mathbb{M}_n$. ♥

Proposition 34:

The initial object of $\mathfrak{C}_n$ is the context $\{\alpha\mathbb{1}_n; \alpha \in \mathbb{C}\}$. □

Proposition 35:

Since objects of $\mathfrak{C}_n$ are unital $*$-subalgebras of $\mathbb{M}_n$, they all contain $\mathbb{1}_n$, which is the identity on $\mathbb{M}_n$. Hence, linearity implies they all contain a copy of $\{\alpha\mathbb{1}_n; \alpha \in \mathbb{C}\}$, which ensures there is always at least one morphism from this object to any other one. The morphism is unique because it must be unital and the domain are multiples of the identity. □

Proposition 36:

For $n > 1$, there is no terminal object on $\mathfrak{C}_n$. For $n = 1$, the terminal context is $\mathbb{M}_1$. □

Proof:

Suppose there was. Then all contexts of $\mathbb{M}_n$ are contexts of this terminal context. Since all matrices on $\mathbb{M}_n$ generate contexts, this implies $\mathbb{M}_n$ must be contained in this terminal context. However, that would imply that $\mathbb{M}_n$ is commutative. We've reached a contradiction, so there is no such terminal context.

For $n = 1$, $\mathbb{M}_1$ is a context that includes all of the other ones, so it is the maximum of $\mathfrak{C}_1$ when thought of as a pre-ordered set. Hence, it is the terminal object. ■

Example [Category of Quantum Observables]**:**

Consider $\mathcal{O}_n$, the space of $n \times n$ self-adjoint matrices or, equivalently, the space of quantum observables. If $A, B \in \mathcal{O}_n$, we could say that "A depends completely on B" iff there is some function $f\colon \sigma(B) \to \mathbb{R}$ such that $A = f(B)$, for in this case we can determine A completely with knowledge of B. Notice that, in this case, $A \in \mathcal{C}(B)$. We may then write $A \preceq B$ and define a pre-order through this way, hence turning $\mathcal{O}_n$ into a pre-order category.

For the same reasons we had in the category of contexts, the constant observables ($\alpha\mathbb{1}_n$, $\alpha \in \mathbb{R}$) are the initial objects and there is no terminal object when $n > 1$, since that would imply on $\mathbb{M}_n$ being commutative (all observables would be on the context generated by the terminal object). For $n = 1$, all objects are terminal ($\mathcal{O}_1 = \mathbb{R}$ and there is always a function that takes a real number to any other real number by means of the functional calculus). ♥

3.4 Subobjects and Elements

Definition 37 [Monomorphism Factorization]**:**

Let C be a category, $A, B, C \in \mathrm{Ob}_{\mathsf{C}}$, and let $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Mor}_{\mathsf{C}}(C, B)$ be monic*. If there is $h \in \mathrm{Mor}_{\mathsf{C}}(A, C)$ such that $f = g \circ h$, *id est*, such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, tail, "f"'] & C \arrow[d, tail, "g"] \\ & B \end{tikzcd} \tag{3.7}$$

commutes, then we say that f *factors through* g, in which case we write $f \to g$. ♠

Notice that monomorphism factorization allows us to define arrows between morphisms of a category. In fact, it allows us to define a pre-order on the monomorphisms of C with fixed codomain B, so that these monomorphisms can be turned into a pre-order category.

Definition 38 [Equivalence of Monomorphisms]**:**

Let C be a category and $B \in \mathrm{Ob}_{\mathsf{C}}$. Let f and g be monomorphisms on C with codomain B. We'll say $f \sim g$ if, and only if, f factors through g and g factors through f. ♠

Proposition 39:

Let C be a category and $A, B, C \in \mathrm{Ob}_{\mathsf{C}}$. Let $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Mor}_{\mathsf{C}}(C, B)$. Then $f \sim g$ if, and only if, $A \sim C$. □

Proof:

Suppose $f \sim g$. Then there are morphisms h and k such that $f = g \circ h$ and $g = f \circ k$. From this expressions, we see that $f \circ \mathrm{id}_A = f = f \circ (k \circ h)$. Since f is monic, it is left-cancelling and we conclude $k \circ h = \mathrm{id}_A$. Analogously, the fact that g is a monomorphism implies $h \circ k = \mathrm{id}_C$. Hence, h is an isomorphism with $k = h^{-1}$, proving $A \sim C$.

If $A \sim C$, then we can use the isomorphism to factor f through g and vice-versa. ■

Example [Subsets are Equivalence Classes of Monomorphisms]**:**

Consider the Set category. Let $B \in \mathrm{Ob}_{\mathsf{Set}}$. For each $C \subseteq B$, let $i_C \in \mathrm{Mor}_{\mathsf{Set}}(C, B)$ be the identity mapping $x \mapsto x$.

i_C is a monomorphism for every C. Furthermore, $C \mapsto [i_C]$ defines a bijection between subsets of B and equivalence classes of monomorphisms with codomain B. It is straightforward to see that $C \mapsto [i_C]$ is surjective. To see injection, notice that if $D \subseteq B$ but $D \neq C$, then i_C and i_D have different ranges, so they can't be equivalent (at least one of them doesn't factor through the other). ♥

Due to this example, we define sub-objects in a category in the following manner.

*We could be more general, but monomorphisms will turn out to be more interesting.

Definition 40 [Subobject]**:**

Let $\mathbf{C}$ be a category. If $B \in \mathrm{Ob}_{\mathbf{C}}$, then the *subobjects* of B are the equivalence classes of monomorphisms in $\mathbf{C}$ with codomain B. We denote the collection of subobjects of B by $\mathrm{Sub}(B)$. ♠

Some categories admit more tangible definitions, but this one is completely general. Furthermore, some objects admit that we identify subobjects with objects in the category (as it happens with sets, for example), but this doesn't hold always.

Example [Points in a Set are Subobjects with Terminal Domains]**:**

Recall from Proposition 31 on page 18 that if $\mathbf{C}$ is an arbitrary category with a terminal object T, then $f \in \mathrm{Mor}_{\mathbf{C}}(T, B)$ is always monic for any object B.

Furthermore, since T is terminal, $\mathrm{End}_{\mathbf{C}}(T)$ has a single element, so the only morphism in $\mathrm{Mor}_{\mathbf{C}}(T, B)$ that f can factor through is itself. In other words, the only element of $\mathrm{Mor}_{\mathbf{C}}(T, B)$ in $[f] \in \mathrm{Sub}(B)$ is f. The domain of all other arrows in $[f]$ also needs to be terminal. To see this, consider the diagram

$$\begin{tikzcd} C & T \arrow[l, "\sim"'] \arrow[r, "\sim"] & D \\ & A, \arrow[ul, shift left] \arrow[ul, shift right] \arrow[u] & \end{tikzcd} \tag{3.8}$$

where we denoted iso arrows with a $\sim$ next to them. T is some terminal object isomorphic to C and D. A is some arbitrary object. We know there is one arrow from A to D by composing the arrows from A to T and from T to D. Uniqueness of the arrows from A to C follows from composing them with the isomorphism to T and then using the fact it is left-simplifiable.

In $\mathbf{Set}$, terminal objects are singletons. Hence, we can identify the equivalence classes $[f] \in \mathrm{Sub}(B)$ with terminal $\mathrm{Dom}\, f$ as being the points, or elements, of B. ♥

Definition 41 [Elements of an Object]**:**

Let $\mathbf{C}$ be a category and $B \in \mathrm{Ob}_{\mathbf{C}}$. The *elements* of B are the subobjects of B corresponding to arrows with terminal domain. ♠

Notice $\mathrm{Mor}_{\mathbf{C}}(T, B)$, with T being a terminal object, is identified with the collection of elements of B. Since all terminal objects are isomorphic, this is well-defined.

Example [Elements of Sets]**:**

In $\mathbf{Set}$, the elements of a set A are equivalence classes of functions $\{\cdot\} \to A$. They are identified with the points of A by interpreting them as their range. ♥

Example [Elements on the Category Opposite to Commutative Unital $*$-algebras]**:**

In $*$-$\mathbf{UCAlg}$, $\mathbb{C}$ is initial. Hence, $\mathbb{C}$ is terminal in $*$-$\mathbf{UCAlg}^{\mathrm{op}}$. Therefore, given a commutative unital $*$-algebra $\mathcal{A}$ considered as an object of $*$-$\mathbf{UCAlg}^{\mathrm{op}}$, it has as elements the unital $*$-homomorphisms with codomain[*] $\mathbb{C}$, *id est*, the elements of $\mathcal{A}$ correspond to

[*] We're stating the morphisms in $*$-$\mathbf{UCAlg}^{\mathrm{op}}$ in terms of their definition in $*$-$\mathbf{UCAlg}$.

its Gelfand spectrum,

$$\Sigma(\mathcal{A}) = \mathrm{Mor}_{*\text{-}\mathbf{UCAlg}^{\mathrm{op}}}(\mathbb{C}, \mathcal{A}). \tag{3.9}$$

♥

As well see, it turns out that categories opposite to some (sub)categories of unital $*$-algebras are equivalent (in a sense to be defined) to categories of sets. In particular, $*$-$\mathbf{FinUCAlg}^{\mathrm{op}}$ is equivalent to $\mathbf{FinSet}$.

3.5 Functors

Our next step will be to define transformations between categories.

Definition 42 [Functor]**:**
Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *covariant functor* $F\colon \mathbf{C} \to \mathbf{D}$ from $\mathbf{C}$ to $\mathbf{D}$ is a transformation such that

i. for all objects $A \in \mathrm{Ob}_{\mathbf{C}}$, the functor assigns an object $F(A) \in \mathrm{Ob}_{\mathbf{D}}$;

ii. for any two objects $A, B \in \mathrm{Ob}_{\mathbf{C}}$ and morphism $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, the functor assigns an arrow $F(f) \in \mathrm{Mor}_{\mathbf{C}}(F(A), F(B))$ such that

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \quad \text{and} \quad F(g \circ f) = F(g) \circ F(f). \tag{3.10}$$

Similarly, a *contravariant functor* $F\colon \mathbf{C} \to \mathbf{D}$ from $\mathbf{C}$ to $\mathbf{D}$ is a transformation such that

i. for all objects $A \in \mathrm{Ob}_{\mathbf{C}}$, the functor assigns an object $F(A) \in \mathrm{Ob}_{\mathbf{D}}$;

ii. for any two objects $A, B \in \mathrm{Ob}_{\mathbf{C}}$ and morphism $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, the functor assigns an arrow $F(f) \in \mathrm{Mor}_{\mathbf{C}}(F(B), F(A))$ such that

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \quad \text{and} \quad F(g \circ f) = F(f) \circ F(g). \tag{3.11}$$

♠

Notice the change in order when defining the action of the different sorts of functors on the morphisms. We could also understand contravariant functors $F\colon \mathbf{C} \to \mathbf{D}$ as covariant functors $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ or $F\colon \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$, as some author do.

Definition 43 [Presheaves]**:**
Let $\mathbf{C}$ be a category. A contravariant functor $F\colon \mathbf{C} \to \mathbf{Set}$ is said to be a *presheaf* on $\mathbf{C}$. ♠

Example [Identity Functor]**:**
The trivial covariant functor $\mathrm{Id}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$ acts trivially.

The trivial contravariant functor $\mathrm{Id}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ acts trivially by transforming $f \in \mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(A, B)$ on themselves, but seen as elements of $\mathrm{Mor}_{\mathbf{C}}(B, A)$, *id est*, the arrows are "flipped twice": once by taking the opposite category, and a second one by applying the covariant functor. ♥

Example [Forgetful Functors]**:**

Consider the functor $F\colon *\text{-}\mathbf{UAlg} \to \mathbf{Vect}_{\mathbb{C}}$ that takes each object to itself, but seen as merely a complex vector space, and each morphism to itself, but seen as merely a linear transformation. This is an example of a forgetful functor (it "forgets structure").

One can define analogous functors from $*$-**UCAlg** to $*$-**UAlg**, from $\mathbf{Vect}_{\mathbb{C}}$ to **Set**, from **FinSet** to **Set**, and so on. ♥

Example [Preimage]**:**

Consider the transformation $P\colon \mathbf{Set} \to \mathbf{Set}$ which maps each object to its powerset and each morphism $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$ to its preimage $f^{-1}\colon P(B) \to P(A)$, where $f^{-1}(C) = x \in A; f(x) \in C$. This is a contravariant functor.

Since this is a contravariant functor with codomain in **Set**, it is an example of a presheaf. ♥

Definition 44 [Locally Small Categories]**:**

Let $\mathbf{C}$ be a category. We say $\mathbf{C}$ is *locally small* if, and only if, $\forall\, A, B \in \mathrm{Ob}_{\mathbf{C}}$, $\mathrm{Mor}_{\mathbf{C}}(A, B)$ is a set. ♠

Some categories are way complex and might not be locally small, but those of our interest are.

Example [Hom Functor]**:**

Let $\mathbf{C}$ be a locally small category. For each $A \in \mathrm{Ob}_{\mathbf{C}}$, the contravariant functor $\mathrm{Hom}_{\mathbf{C}}(-, A)\colon \mathbf{C} \to \mathbf{Set}$ is the presheaf over $\mathbf{C}$ that transforms $B \in \mathrm{Ob}_{\mathbf{C}}$ in the set

$$\mathrm{Hom}_{\mathbf{C}}(B, A) \equiv \mathrm{Mor}_{\mathbf{C}}(B, A) \tag{3.12}$$

and transforms the morphism $f \in \mathrm{Mor}_{\mathbf{C}}(B, C)$ in the morfism

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(f, A)\colon \mathrm{Mor}_{\mathbf{C}}(C, A) &\to \mathrm{Mor}_{\mathbf{C}}(B, A) \\ g &\mapsto g \circ f. \end{aligned} \tag{3.13}$$

♥

Hom functors play a central role in category theory because many important presheaves are equivalent to Hom functors. While they are not central to our developments in this course, they are essential for eventual continuations, so we'll keep commenting on them.

Example [Spectral Functor]**:**

Define $S\colon *\text{-}\mathbf{FinUCAlg} \to \mathbf{FinSet}$ by mapping each commutative unital $*$-algebra $\mathcal{A}$ to its Gelfand spectrum $\Sigma(\mathcal{A})$ and by mapping each unital $*$-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ to the function Why is it finite?

$$\begin{aligned} S(\varphi)\colon \Sigma(\mathcal{B}) &\to \Sigma(\mathcal{A}) \\ \psi &\mapsto \psi \circ \varphi. \end{aligned} \tag{3.14}$$

S is a contravariant functor. With some more (topological) structure, we could do an analogous construction in infinite dimension. Show ♥

Example [Counter-Spectral Functor]**:**

Define $K\colon \textsf{FinSet} \to *\text{-}\textsf{FinUCAlg}$ by mapping each finite set Ω to its algebra of functions $\mathcal{F}(\Omega)$ and by mapping each function $f\colon \Omega \to \Omega'$ to the unital $*$-homomorphism Why is it finite dimensional?

$$\begin{aligned} K(f)\colon \mathcal{F}(\Omega') &\to \mathcal{F}(\Omega) \\ g &\mapsto g \circ f. \end{aligned} \tag{3.15}$$

K is a contravariant functor. With some more (topological) structure, we could do an analogous construction in infinite domains. Show

♥

3.6 Natural Transformations and Functor Categories

We can now start discussing transformations between functors.

Definition 45 [Natural Transformations]**:**

Let $\textsf{C}$ and $\textsf{D}$ be any two categories and let $F\colon \textsf{C} \to \textsf{D}$ and $G\colon \textsf{C} \to \textsf{D}$ be contravariant functors. A *(contravariant) natural transformation* $N\colon F \to G$ is a collection $(N_A)_{A \in \mathrm{Ob}_{\textsf{C}}}$ of morphisms $N_A \in \mathrm{Mor}_{\textsf{D}}(F(A), G(A))$ in $\textsf{D}$ such that

$$\begin{tikzcd} F(B) \arrow[r, "F(f)"] \arrow[d, "N_B"'] & F(A) \arrow[d, "N_A"] \\ G(B) \arrow[r, "G(f)"'] & G(A) \end{tikzcd} \tag{3.16}$$

commutes for all $A, B \in \mathrm{Ob}_{\textsf{C}}$ and all $f \in \mathrm{Mor}_{\textsf{C}}(A, B)$. In other words, such that $N_A \circ F(f) = G(f) \circ N_B$.

Suppose instead that $F\colon \textsf{C} \to \textsf{D}$ and $G\colon \textsf{C} \to \textsf{D}$ are covariant functors. A *(covariant) natural transformation* $N\colon F \to G$ is also a collection $(N_A)_{A \in \mathrm{Ob}_{\textsf{C}}}$ of morphisms $N_A \in \mathrm{Mor}_{\textsf{D}}(F(A), G(A))$ in $\textsf{D}$, but it is such that

$$\begin{tikzcd} F(B) \arrow[d, "N_B"'] & F(A) \arrow[l, "F(f)"'] \arrow[d, "N_A"] \\ G(B) & G(A) \arrow[l, "G(f)"] \end{tikzcd} \tag{3.17}$$

commutes for all $A, B \in \mathrm{Ob}_{\textsf{C}}$ and all $f \in \mathrm{Mor}_{\textsf{C}}(A, B)$. Notice how the arrows got reversed. In equation terms, it is such that $N_B \circ F(f) = G(f) \circ N_A$.

We denote the collection of natural transformations from F to G by $\mathrm{Nat}_{\textsf{C},\textsf{D}}(F, G)$. ♠

In diagrams, natural transformations are often written as $\Rightarrow$. Hence, the structures involved in the previous definition are

$$\begin{tikzcd} \textsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow N"] & \textsf{D}. \end{tikzcd} \tag{3.18}$$

This diagram hints at the fact that we can also consider categories that have categories as objects. That is the case of **Cat**, the category of small categories (a category is said to be small if the collection of its objects can be taken to be a set). For more details, see, *exempli gratia*, Borceux 1994; Mac Lane 1978.

Definition 46 [Natural Isomorphism]**:**
Let $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{C} \to \mathsf{D}$ be functors between the same two categories and let $N = (N_A)_{A \in \mathrm{Ob}_{\mathsf{C}}}$ be a natural transformation. If, and only if, $\forall\, A \in \mathrm{Ob}_{\mathsf{C}}$, N_A is an isomorphism, we say that N is a *natural isomorphism* and that F and G are *equivalent*. ♠

Example [Natural Transformation Between Hom Functors]**:**
Let C be a locally small category. For any two $A, B \in \mathrm{Ob}_{\mathsf{C}}$ and for any $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$ we define the natural transformation

$$\mathrm{Hom}_{\mathsf{C}}(-, f) \equiv (\mathrm{Hom}_{\mathsf{C}}(-, f)_C)_{C \in \mathrm{Ob}_{\mathsf{C}}} \colon \mathrm{Hom}_{\mathsf{C}}(-, A) \to \mathrm{Hom}_{\mathsf{C}}(-, B) \tag{3.19}$$

by

Prove

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(-, f)_C \colon \mathrm{Mor}_{\mathsf{C}}(C, A) &\to \mathrm{Mor}_{\mathsf{C}}(B, A) \\ g &\mapsto f \circ g. \end{aligned} \tag{3.20}$$

As we shall see, all natural transformations from $\mathrm{Hom}_{\mathsf{C}}(-, A)$ to $\mathrm{Hom}_{\mathsf{C}}(-, B)$ are of this form. ♥

With the notion of natural transformations at hand, we're now able to define equivalence between different categories.

Definition 47 [Equivalent Categories]**:**
Let C and D be categories. We say they are *equivalent* if, and only if, there exist covariant functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ and natural isomorphisms $N_{\mathsf{C}}\colon G \circ F \to \mathrm{Id}_{\mathsf{C}}$ and $N_{\mathsf{D}}\colon F \circ G \to \mathrm{Id}_{\mathsf{D}}$. In this case, F and G are said to be *equivalences* of C and D. ♠

This definition might seem unnatural (pun intended) at first: wouldn't it make more sense to define equivalence by imposing $F \circ G = \mathrm{Id}_{\mathsf{D}}$ and $G \circ F = \mathrm{Id}_{\mathsf{D}}$? The issue with this definition is that it becomes too restrictive and leads to few examples. It is still recovered by the less restrictive definition, but the less restrictive version also will turn out to be more useful on the long run.
Let us give an example of a theorem employing the language we've developed. It relates commutative unital $*$-algebras and the algebras of functions on sets.

Theorem 48 [Gelfand–Naimark]**:**
The spectral functor

$$S\colon \ast\text{-}\mathsf{FinUCAlg} \to \mathsf{FinSet} \tag{3.21}$$

and the counter-spectral functor

$$K\colon \mathsf{FinSet} \to \ast\text{-}\mathsf{FinUCAlg} \tag{3.22}$$

are equivalences of the categories **FinSet** *and* $\ast$-$\mathsf{FinUCAlg}^{\mathrm{op}}$. □

The reason the equivalences are between $\textbf{FinSet}$ and $*\text{-}\textbf{FinUCAlg}^{\mathrm{op}}$ (rather than $*\text{-}\textbf{FinUCAlg}$) is due to the fact that the functors are contravariant rather than covariant, so they actually lead to an equivalence on the opposite category.

The Gelfand–Naimark theorem admits a generalization for infinite dimensional algebras and infinite sets, but that would require more topological structure. Furthermore, it is usually not stated in the language of category theory. For more information on it, see, *exempli gratia*, Bratteli and Robinson 1987, Theorem 2.1.11A.

Let us nevertheless mention some implications of this result. First and foremost, it establishes the existence of $*$-isomorphisms between a commutative unital $*$-algebra $\mathcal{A}$ and the function algebra $\mathcal{F}(\Sigma(\mathcal{A}))$. Furthermore, these isomorphisms are "natural", in the sense that behave well with respect to unital $*$-homomorphisms between objects of $*\text{-}\textbf{FinUCAlg}$. It also holds that for any finite set Ω, we can identify Ω with $\Sigma(\mathcal{F}(\Omega))$ by means of a bijection, and this bijection is also "natural" in the sense of behaving well under the action of functions between finite sets.

Notice that natural transformations resemble morphisms in a category where functors would be the objects. That is, in fact, quite literal, allowing us then to define functor categories.

Definition 49 [Functor Category]**:**

Let $\mathbf{C}$ and $\mathbf{D}$ be fixed categories. We define the *covariant functor category* $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ (also denoted $\mathbf{D}^{\mathbf{C}}$) as the category whose objects are all the covariant functors $F \colon \mathbf{C} \to \mathbf{D}$, the morphisms are, for all $F, G \in \mathrm{Ob}_{\mathrm{Fun}(\mathbf{C},\mathbf{D})}$,

$$\mathrm{Mor}_{\mathrm{Fun}(\mathbf{C},\mathbf{D})}(F, G) \equiv \mathrm{Nat}_{\mathbf{C},\mathbf{D}}(F, G), \tag{3.23}$$

and the composition rule is given by

$$M \circ N = (M_A \circ N_A)_{A \in \mathrm{Ob}_{\mathbf{C}}}, \tag{3.24}$$

for all $M, N \in \mathrm{Mor}_{\mathrm{Fun}(\mathbf{C},\mathbf{D})}(F, G)$. We say that $\mathbf{C}$ is the *base category* of $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$.

In an analogous manner one defines the *contravariant functor category* $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{D})$ (or $\mathbf{D}^{\mathbf{C}^{\mathrm{op}}}$). As the notation suggests, the definition is the same, apart from the fact that both the functors and natural transformations are now contravariant. ♠

Example [Category of Presheaves]**:**

The category $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ is the category of presheaves over $\mathbf{C}$. ♥

At this stage, it is interesting for us to digress a little more about natural transformations and Hom functors to mention one of the most central results in Category Theory: the Yoneda Lemma. We refer to the literature for a proof.

Definition 50 [Yoneda Embedding]**:**

Let $\mathbf{C}$ be a locally small category. The *Yoneda embedding* is the functor $Y \colon \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ defined by mapping each $A \in \mathrm{Ob}_{\mathbf{C}}$ to $Y(A) \equiv \mathrm{Hom}_{\mathbf{C}}(-, A)$ (the Hom functor associated to A in the category of presheaves over $\mathbf{C}$) and each $f \in \mathrm{Mor}_{\mathbf{C}}(B, C)$ to $Y(f) \equiv \mathrm{Hom}_{\mathbf{C}}(-, f) \in \mathrm{Mor}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set})}(Y(B), Y(C))$ (the natural transformation between Hom functors associated to f). ♠

Lemma 51 [Yoneda]**:**

Let C *be a locally small category. For all presheaves* $F\colon \mathsf{C} \to \mathsf{Set}$ *and for all* $A \in \mathrm{Ob}_{\mathsf{C}}$*, the mapping*

$$\begin{aligned} \mathrm{Nat}_{\mathsf{C},\mathsf{Set}}(Y(A), F) &\to F(A) \\ \alpha &\mapsto \alpha_A(\mathrm{id}_A) \end{aligned} \tag{3.25}$$

is a bijection. □

Notice that $\mathrm{id}_A \in \mathrm{End}_{\mathsf{C}}(A) = [Y(A)](A)$. Also, given $\alpha \in \mathrm{Nat}_{\mathsf{C},\mathsf{Set}}(Y(A), F)$ and $B \in \mathrm{Ob}_{\mathsf{C}}$, $\alpha_B\colon \mathrm{Mor}_{\mathsf{C}}(B, A) \to F(B)$. Therefore, given any two objects $A, B \in \mathrm{Ob}_{\mathsf{C}}$, the natural transformations

$$\mathrm{Nat}_{\mathsf{C},\mathsf{Set}}(\mathrm{Hom}_{\mathsf{C}}(-, A), \mathrm{Hom}_{\mathsf{C}}(-, B)) = \mathrm{Nat}_{\mathsf{C},\mathsf{Set}}(Y(A), Y(B)) \tag{3.26}$$

are in bijection with $[Y(B)](A) = \mathrm{Hom}_{\mathsf{C}}(A, B) = \mathrm{Mor}_{\mathsf{C}}(A, B)$, *id est*, the natural transformations from $\mathrm{Hom}_{\mathsf{C}}(-, A)$ to $\mathrm{Hom}_{\mathsf{C}}(-, B)$ not only form a set, but are deep down just the morphisms from A to B.

The Yoneda lemma raises an interesting question. Let C be a locally small category and F be a presheaf over C. Choose an object $A \in \mathrm{Ob}_{\mathsf{C}}$ and a point $a \in F(A)$. By the Yoneda Lemma, a is a natural transformation from $\mathrm{Hom}_{\mathsf{C}}(-, A)$ to F. Can we choose A and a such that the natural transformation is a natural isomorphism? If so, we say F is *representable* and that a is a *universal element* of F. We'll then say that (A, a) is *universal* iff there is some other pair (B, b) such that there is a morphism $f_B \in \mathrm{Mor}_{\mathsf{C}}(B, A)$ with $[F(f_B)](a) = b$. This condition turns out to be equivalent to (A, a) being universal in a specific natural category—which depends on F—for such pairs. From the categorical point of view, this is the relevant notion of universality in the various situations one can find in Mathematics, such as the definition of tensor products in vector spaces. For more information, see, *exempli gratia*, Perrone 2021; Riehl 2017.

Let us go back to the presheaves now. We'll state a couple of results without proof.

Proposition 52:

Let F *and* G *be presheaves over some category* C*. A natural transformation* $f\colon F \to G$ *is monic if, and only if, all of the component functions* f_A*,* $A \in \mathrm{Ob}_{\mathsf{C}}$*, are monomorphisms.* □

Corollary 53:

Let F *and* G *be presheaves over some category* C*. Two monomorphisms* $f, g\colon F \to G$ *are equivalent iff* $\mathrm{Ran}\, f_A = \mathrm{Ran}\, g_A \subseteq G(A)$ *for each* $A \in \mathrm{Ob}_{\mathsf{C}}$*.* □ Can I prove this?

We then get to a further corollary.

Corollary 54 [Subobjects of Presheaves]**:**

Let G *be a presheaf over some category* C*. The subobjects of* G *can be identified with the presheaves* $F \in \mathrm{Ob}_{\mathrm{Fun}(\mathsf{C}^{\mathrm{op}},\mathsf{Set})}$ *such that,* $\forall\, A \in \mathrm{Ob}_{\mathsf{C}}$*,* $F(A) \subseteq G(A)$ *and such that there is a natural transformation* $i = (i_A)_{A \in \mathrm{Ob}_{\mathsf{C}}}$ *from* F *to* G *such that* $i_A\colon F(A) \to G(A)$ *is the inclusion function* $x \mapsto x$ *for each* $A \in \mathrm{Ob}_{\mathsf{C}}$*.* □ Prove this

Lemma 55:

Given an arbitrary category C*, the initial object of the presheaf category* $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$

is the contravariant functor $\mathbf{0_C} \colon \mathbf{C} \to \mathbf{Set}$ that maps all objects of $\mathbf{C}$ to the empty set $\varnothing$ and all morphisms of $\mathbf{C}$ to the empty function (the identity on $\varnothing$). The terminal object of the presheaf category $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ *is the contravariant functor $\mathbf{1_C} \colon \mathbf{C} \to \mathbf{Set}$ that maps all objects of $\mathbf{C}$ to the singleton set $\{\cdot\}$ and all morphisms of $\mathbf{C}$ to the identity function on* $\{\cdot\}$. □

Prove

Definition 56 [Global Section of a Presheaf]**:**

Let $\mathbf{C}$ be a category and let G be a presheaf over $\mathbf{C}$. We say a presheaf F over $\mathbf{C}$ is a *global section* of G if, and only if, for each $A \in \mathrm{Ob}_{\mathbf{C}}$, $F(A) = \{f_A\}$, for some $f_A \in G(A)$, and $i \colon F \to G$ with $i_A(f_A) = f_A$ (the inclusion function) is a natural transformation. ♠

Notice that a global section must transform the morfism $g \colon A \to B$ in $\{f_B\} \mapsto \{f_A\}$. Hence, a global section is completely determined by the collection $(f_A)_{A \in \mathrm{Ob}_{\mathbf{C}}}$ As a consequence, the naturality condition is equivalent to the equalities

$$f_A = G(h)(f_B) \tag{3.27}$$

for each $h \in \mathrm{Mor}_{\mathbf{C}}(A, B)$. Indeed, the naturality condition is simply that $i_A \circ F(f) = G(f) \circ i_B$, which due to the fact that $F(B) = \{f_B\}$ becomes

$$[i_A \circ F(f)](f_B) = [G(f) \circ i_B](f_B), \tag{3.28a}$$
$$i_A([F(f)](f_B)) = G(f)(i_B(f_B)), \tag{3.28b}$$
$$i_A(f_A) = G(f)(f_B), \tag{3.28c}$$
$$f_A = G(f)(f_B). \tag{3.28d}$$

This then implies we can identify global sections on a presheaf with the presheaf's elements.

Why?

4 Category Theory and the BKS Theorem

At last, we shall use the language of Category Theory to look at the Bell–Kochen–Specker Theorem from a different point of view, that will also shed light on a mathematical formulation of Bohr's ideas. Firstly, we'll deal with the case of algebras of observables, and later we'll move to dealing with contexts.

4.1 BKS and the Category of Observables

Let us consider the category $\mathrm{Fun}({\mathcal{O}_n}^{\mathrm{op}}, \mathbf{Set})$ of presheaves over $\mathcal{O}_n$ (considered as a preorder category in the way described on the example on page 22). We'll define a contravariant functor $S_n \colon \mathcal{O}_n \to \mathbf{Set}$ that takes $A \in \mathrm{Ob}_{\mathcal{O}_n}$ and takes it to $S_n(A) = \sigma(A)$ (its spectrum), and takes $f \in \mathrm{Mor}_{\mathcal{O}_n}(A, B)$ and takes it to the unique function $f \colon \sigma(B) \to \sigma(A)$ such that $A = f(B)$ by means of the functional calculus. This does define a functor thanks to the fact that the functional calculus commutes with taking the spectrum of a matrix $(f(\sigma(A)) = \sigma(f(A))$, which ensures that determinating the arrows uniquely by means of functional calculus will lead to the functor composition rule.

Theorem 57:

If V is a valuation on $\mathcal{O}_n$, then defining $X_V(A) = \{V(A)\}, \forall\, A \in \mathcal{O}_n$ defines a global section of S_n. Furthermore, this attribution is bijective. □

Proof:

Let V be a valuation on $\mathcal{O}_n$. For any $A \in \mathcal{O}_n$, we know that $V(A) \in \sigma(A) = S_n(A)$ as a consequence of Lemma 22 and Proposition 18 on page 11 and on page 14 and the fact that any matrix lies in a character.

Define a presheaf $X_V : \mathcal{O}_n \to \mathbf{Set}$ by attributing $X_V(A) = \{V(A)\}$ for all $A \in \mathcal{O}_n$ and

$$\begin{aligned} X_V(f) \colon X_V(B) &\to X_V(A) \\ V(B) &\mapsto V(A) \end{aligned} \tag{4.1}$$

for all $f \colon A \to B$, $A, B \in \mathcal{O}_n$. We want to show that this is a global section of S_n. To do so, we just need to show that $i \colon X_V \to S_n$ with $(i_A)_{A\in\mathcal{O}_n}$ is a natural transformation. This boils down to showing that

$$V(A) = S_n(f)(V(B)), \tag{4.2a}$$
$$V(A) = f(V(B)), \tag{4.2b}$$

for each morphism $f \colon A \to B$. However, such a morphism exists if, and only if, $A = f(B)$, so what we actually want to prove is that

$$V(f(B)) = f(V(B)) \tag{4.3}$$

for each $B \in \mathcal{O}_n$, which is true due to the definition of valuation.

Hence, we now just need to prove that $V \mapsto X_V$ is bijective. It is straightforward to prove injectivity: if $V \neq V'$ are valuations, then there is at least some $A \in \mathcal{O}_n$ for which $V(A) \neq V'(A)$, and hence $X_V(A) \neq X_{V'}(A)$. Let's see about surjectivity.

Let X be a global section of S_n. We want to find some valuation V_X such that $X = X_{V_X}$. Let's us take an attempt at building such a valuation. For each $A \in \mathcal{O}_n$, define $V_X(A) \equiv x_A$, where $X(A) = \{x_A\}$. We are now tasked with showing that this constitutes a valuation. So see so, let us consider $V(B)$, where $B = f(A)$ for some function f. Hence, we know that there is a morphism $f \in \mathrm{Mor}_{\mathcal{O}_n}(B, A)$. Since X is a global section, we know that $x_B = S_n(f)(x_B) = f(x_A)$. Hence,

$$V_X(f(A)) = V_X(B), \tag{4.4a}$$
$$= x_B, \tag{4.4b}$$
$$= f(x_A), \tag{4.4c}$$
$$= f(V_X(A)), \tag{4.4d}$$

for any $A \in \mathcal{O}_n$ and any function $f \colon \mathbb{R} \to \mathbb{R}$, proving V_X is a valuation. Since global sections are completely determined by their action on the objects and we imposed $V_X(A) = x_A$, it holds that $X = X_{V_X}$.

This concludes the proof. ■

We may then obtain a further result.

Corollary 58:
For $n \geq 3$, the presheaf of observables S_n admits no global sections, id est, *it has no elements.* □

Proof:
Follows from Theorems 23 and 57 on page 15 and on the preceding page. ■

These results were discussed (in more generality) in the first sections of Butterfield and C. J. Isham 1998, 1999, which then proceeds to a discussion of how, motivated by the topos structure, one can introduce the notion of generalized valuations, which, on the other hand, will have contextual values and involve truth values outside of the usual $\{0, 1\}$ set. For more information, see Butterfield and C. J. Isham 1998, 1999. Notice, though, how the use of contructions from Topos Theory can start to shed light on Quantum Theory. In particular, for a discussion on the main aspects of Topos Theory that occurs in these papers, see Butterfield and C. J. Isham 1998, App. A.

4.2 BKS and the Category of Contexts

Let us now consider similar a construction on contexts. This time, define the presheaf P_n on $\mathfrak{C}_n$ by assigning to each context $\mathcal{C} \in \mathfrak{C}_n$ its Gelfand algebra, $\Sigma(\mathcal{C})$, and to each arrow $f: \mathcal{C} \to \mathcal{C}'$ (unique if it exists, since $\mathfrak{C}_n$ is a pre-order category) the function

$$\begin{aligned} P_n(f)\colon \Sigma(\mathcal{C}') &\to \Sigma(\mathcal{C}) \\ \varphi &\mapsto \varphi|_{\mathcal{C}}, \end{aligned} \tag{4.5}$$

where $\varphi\big|_{\mathcal{C}}$ denotes the restriction of φ to $\mathcal{C} \subseteq \mathcal{C}'$. If $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{C}''$ are contexts and $\varphi \in \Sigma(\mathcal{C}'')$, then $\varphi\big|_{\mathcal{C}} = \big(\varphi\big|_{\mathcal{C}'}\big)\big|_{\mathcal{C}}$, so this does define a functor. Furthermore, $\varphi\big|_{\mathcal{C}} \in \Sigma(\mathcal{C})$ is ensured because the restriction doesn't destroy any algebraic properties.

Recall that $\Sigma(\mathcal{C})$ resembled a classical phase space, so what we're doing is assigning to each context a classical description. In principle, this might seem to destroy all of the quantum structure we had in the first place. Nevertheless, the functorial structure of P_n comes at our rescue: the contexts have a non-trivial structure within $\mathbb{M}_n$, given by how each of them is connected to each other. In categorical terms, the have a non-trivial structure given by the morphisms of $\mathfrak{C}_n$. Since P_n is a functor, it translates this non-trivial global structure to the classical phase spaces $\Sigma(\mathcal{C})$, hence yielding us a description that is globally quantum, but locally classical. Hence, we've obtained a mathematically precise description of Bohr's ideas.

We can now state, and prove, a theorem very similar to the one we just obtained for self-adjoint operators.

Theorem 59:
Let V be a valuation on $\mathcal{O}_n$ and $\varphi_{\mathcal{C}}^V$ be the unique character on $\mathcal{C}$ induced by V on $\mathcal{C}$ by means of Lemma 22 on page 14. Then defining $X_V(\mathcal{C}) = \varphi_{\mathcal{C}}^V$ for each $\mathcal{C} \in \mathfrak{C}_n$ defines a global section of P_n. Furthermore, this assignment is bijective. □

Proof:

Let V be a valuation on $\mathcal{O}_n$. It is straightforward to show that X_V is indeed a presheaf, so we'll begin by proving that X_V is indeed a global section. To do so, we must show that, for each morphism $f\colon \mathcal{C} \to \mathcal{C}'$,

$$\varphi_{\mathcal{C}}^{V} = P_n(f)(\varphi_{\mathcal{C}'}^{V}), \tag{4.6a}$$

$$= \varphi_{\mathcal{C}'}^{V}\Big|_{\mathcal{C}} \tag{4.6b}$$

holds. It does hold, for both $\varphi_{\mathcal{C}}^{V}$ and $\varphi_{\mathcal{C}'}^{V}\big|_{\mathcal{C}}$ are characters on $\mathcal{C}$ that extend V to $\mathcal{C}$, and Lemma 22 on page 14 ensures such extensions are unique.

Let us then proceed to prove that this assignment is a bijection. Firstly, we prove injection. Suppose $V \neq V'$ are valuations. Since they are not equal, there is at least one matrix A such that $V(A) \neq V'(A)$. Hence, $X_V(\mathcal{C}(A)) \neq X_{V'}(\mathcal{C}(A))$, where $\mathcal{C}(A)$ is the context generated by A.

As for surjection, suppose X is a global section. We want to produce a valuation V_X that will induce X by means of the procedure defined above. For each $A \in \mathcal{O}_n$, define $V_X(A) = \varphi_{\mathcal{C}(A)}(A)$, where $X(\mathcal{C}) = \{\varphi_{\mathcal{C}}\}$.

To show that V_X is indeed a valuation, let us first recall that Theorem 20 on page 11 tells us that $\mathcal{C}(A) = \{f(A); f\colon \sigma(A) \to \mathbb{R}\}$. As a consequence, $\mathcal{C}(f(A)) \subseteq \mathcal{C}(A)$, meaning there is a unique morphism $f \in \mathrm{Mor}_{\mathfrak{C}_n}(\mathcal{C}(f(A)), \mathcal{C}(A))$. Hence, since X is a global section of P_n, it holds that

$$\varphi_{\mathcal{C}(f(A))} = P_n(f)(\varphi_{\mathcal{C}(A)}), \tag{4.7a}$$

$$= \varphi_{\mathcal{C}(A)}|_{\mathcal{C}(f(A))}. \tag{4.7b}$$

With these remarks in mind, notice that given $A \in \mathcal{O}_n$ and $f\colon \mathbb{R} \to \mathbb{R}$,

$$V_X(f(A)) = \varphi_{\mathcal{C}(f(A))}(f(A)), \tag{4.8a}$$

$$= \varphi_{\mathcal{C}(A)}|_{\mathcal{C}(f(A))}(f(A)), \tag{4.8b}$$

$$= \varphi_{\mathcal{C}(A)}(f(A)), \tag{4.8c}$$

$$= f(\varphi_{\mathcal{C}(A)}(A)), \tag{4.8d}$$

$$= f(V_X(A)), \tag{4.8e}$$

hence proving that V_X is a valuation. In the fourth line, we also employed the result from Theorem 20 on page 11 that characters commute with the functional calculus.

If we now use V_X to induce a global section, we'll get back to X due to the fact that Lemma 22 on page 14 ensures uniqueness of characters extending valuations. Indeed, notice that given $A \in \mathcal{O}_n$, we have

$$\varphi_{\mathcal{C}(A)}^{V_X}(A) = V_X(A) = \varphi_{\mathcal{C}(A)}(A), \tag{4.9}$$

which ensures $\varphi_{\mathcal{C}(A)}^{V_X}$ and $\varphi_{\mathcal{C}(A)}$ agree on $\mathcal{C}(A)$, since characters commute with the functional calculus and $\mathcal{C}(A) = f(A); f\colon \sigma(A) \to \mathbb{R}$, as given by Theorem 20 on page 11. However, Theorem 20 on page 11 also ensures all contexts in $\mathfrak{C}_n$ are of the form $\mathcal{C}(A)$ for some $A \in \mathcal{O}_n$, and hence $\varphi_{\mathcal{C}}^{V_X}$ and $\varphi_{\mathcal{C}}$ holds for all contexts.

This concludes the proof. ■

Corollary 60:

For $n \geq 3$, the presheaf of contexts P_n admits no global sections, id est, *it has no elements.* □

Proof:

Follows from Theorems 23 and 59 on page 15 and on page 33. ■

Therefore, we have obtained a categorical version of the Bell–Kochen–Specker theorem in a manner that makes Bohr's ideas on Quantum Theory mathematically precise and explicit. We're now equipped with a new tool to probe Quantum Mechanics.

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