**Conditional Probability Models**

Table of Contents

[Conditional Distribution Conditioned on an Event 3](#_Toc64284880)

[Conditional Distribution Conditioned on a Random Variable 5](#_Toc64284881)

[Law of Multiplication 6](#_Toc64284882)

[Law of Total Probability 9](#_Toc64284883)

[Conditional Expectation 11](#_Toc64284884)

[Conditional Distribution Conditioned on an Event 17](#_Toc64284885)

[Conditional Distributions Conditioned on another Random Variable 20](#_Toc64284886)

[Law of Multiplication 22](#_Toc64284887)

[Law of Total Probability 23](#_Toc64284888)

[Conditional Expectation 24](#_Toc64284889)

[Two Random Variables Conditioned on an Event 26](#_Toc64284890)

We have previously covered conditional probability for events defined for a single random experiment. If and are the events,

We also learnt the multiplication law (), addition rule () and Bayes theorem ().

We shall now look into how to deal with conditional probability distributions using random variables instead. For random variables, we can have the conditional PMF of a random variable conditioned on either another random variable or on some event. We shall also look into all of the three laws.

## Conditional Distribution Conditioned on an Event

Consider an experiment where a packet is being sent repeatedly until successfully delivered. The maximum number of attempts allowed, , is . The observation is the number of attempts made.

Let be a random variable representing the number of attempts made and be the event that the experiment ends with a success.

Our goal is to find , the conditional PMF of given . This is actually just .

Since we have been given a condition, and we are dealing with a truncated geometric distribution, the distribution we are working with is called a conditional truncated distribution.

For a normal truncated geometric distribution, we had to deal with the scenario where we did not have a success at all, i.e. . However, we cannot consider that scenario in this case due to the given condition. Thus, the only outcomes we can consider are , and .

However, if we sum up these probabilities, we will find that the result is . Thus, this cannot be the PMF, since the values of a PMF need to sum up to .

We can solve this problem by normalizing the values, i.e. dividing them by their sum. Thus, , and . Thus,

Another way we could have gotten to the exact same conclusion is by dividing the probability of getting a success in attempts, i.e. the geometric distribution, by the probability of getting a success, i.e. event .

The general formula for a conditional truncated geometric distribution is thus given by

Since we have found the conditional PMF, we can easily calculate the conditional CDF.

## Conditional Distribution Conditioned on a Random Variable

Say we are given a set of values . The joint PMF is given by the following matrix:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

Say . Again, we need to normalize the values so

Thus, there is no difference between this and the formula we learnt for the conditional probability of an event.

## Law of Multiplication

We have previously seen that the law of multiplication was derived from the conditional probability of an event,

Similarly,

Example

Say a professor will answer any given question incorrectly with a probability of . As such, they decide to limit the number of questions that can be asked in a class to a maximum of . Each possibility for the number of questions asked is equally likely.

Let and . We need to find

In this scenario, the set of possible values are

Let’s say we choose to find . This is like say we are taking attempts to get the teacher to mess up, with a success being defined as the teacher messing up. That makes this conditional probability a binomial function.

For , . Thus

Since there are three possible values of , .

If we follow a similar pattern for all values of ,

Using these values, we can create the following matrix:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

Using the matrix, we can easily calculate for any combination.

## Law of Total Probability

For events, we know that the law of total probability states

There were two conditions associated with this law

1. The events must all be mutually exclusive
2. The events must be collectively exhaustive

An identical formula can be used for random variables.

This is the same as saying that is an event such that and is an event such that . Thus, this is literally the same formula.

Example

## Conditional Expectation

For a random variable ,

The expectation of a random variable is constant. If we run the experiment repeatedly for a long time, the average value we get will be .

For conditional expectation, we can have two different formulae. One is for when the random variable is conditioned on an event , and the other is for when the random variable is conditioned on another random variable .

Example

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

Notice that, even though the unconditional expectation of a random variable is a constant, the conditional expectation of a random variable conditioned on another random variable is not a constant. Rather, it is a function of , meaning if the value of changes, the expectation changes.

For simplicity, let . This means is a derived random variable.

When , , and .

When , , and .

Thus,

We just found the PMF for a function for the conditional expectation of .

Since we have a PMF, we can find an expectation for that as well.

For the given example, we were given the PMF of in a previous lecture.

From here, we can see that the expectation of is

Thus, . This is obvious, since gives us the expectation of for some value of , and gives us the expectation of for all possible values of , which is just .

This is called the law of total expectation. We had two expected values for two different values of , and we took the weighted average to find the expected value of , with the weights being the respective PMFs of the values of .

Example

Say someone will read one chapter from either a history book or a book on probability. The probability of reading either book is the same. The number of misprints in the history book is per chapter on average and in the probability book it is per chapter.

We need to find the expected number of misprints they will encounter.

Let and . Say if the probability book is read and if the history book is read.

Example

We know that the PMF for a geometric distribution is given by

The geometric distribution goes to infinity, so the expectation of can be calculated as

We know that this sums up to , but doing the actual calculations behind this is extremely cumbersome. Instead, we could try and find the value using the law of total expectations.

We already have . Let’s define a second arbitrary random variable. Say . Thus,

Thus,

, since if we get a heads in the first toss, the value of can only be .

For , consider this. We will definitely miss in the first toss, so the value is some value. After that, we fall into a loop. If we get a heads, the value is and if we get a tails, the value is some value. Thus, the expected value remains unchanged.

## Conditional Distribution Conditioned on an Event

We have been working on conditional distributions with discrete random variables so far. Now, we shall look into continuous random variables.

We have previously seen for discrete random variables

Similarly, for continuously random variables

From this, we can find the conditional CDF as well.

Example

Say we are at a bus station and a bus has just left. The time between the last bus leaving and the next one arriving is minutes, and we have arrived minutes after the last bus has left. It is given that , meaning the distribution of is .

Let , and . Let . We need to find .

Essentially, we are now trying to find the probability of the event that given that we already know that the event is true. Thus,

We need to be a little careful about this part, since if we think about it, is further down on the number line that . This means that covers a smaller area than and hence is a subset of . Thus,

Now we just need to find the two probabilities for an exponential random variable.

Similarly,

Notice that and . These two events are not the same, but their distributions functions are. It is as though the moment we arrived at the station, the event forgot that some time had already passed. Because of this property, exponential random variables are called memoryless random variables. Similar to the exponential random variable, the geometric random variable is also a memoryless random variable. It is the discrete version of this.

## Conditional Distributions Conditioned on another Random Variable

For discrete random variables, we defined

Similarly, for continuous random variables,

Obviously, we have the other version of this

Example

We need to find . Since we already know , we just need to find .

The value of ranges from to . Thus,

Given a specific value of , .

Similarly,

Again, given , .

## Law of Multiplication

We have previously seen two different versions of the law of multiplication. The first version dealt with random events.

The second dealt with discrete conditional random variables.

Similar to these, we also have a law of multiplication for continuous conditional random variables.

Example

We choose a random variable between and another random variable between . We need to find .

is the independent random variable here and is the independent variable.

## Law of Total Probability

We have previously seen the law of total probability for two case. For random events,

For discrete conditional random variables,

Similarly, for continuous conditional random variables,

Example

In continuation of the previous example, we can use the law of total probability to find .

Since the value of ranges from to , the value of must always be more than . Thus, ranges from to .

## Conditional Expectation

The conditional expectation of a continuous conditional random variable can be found for two cases, one when it is conditioned over an event, and one when it is conditioned over another random variable.

The second formula can be used along with the law of total probability, which is also called the law of total expectation, to find the expectation of .

Example

For this example, let us use one of the examples we worked with previously.

If we look back at the example the equation was taken from, it should be obvious that the range is from to .

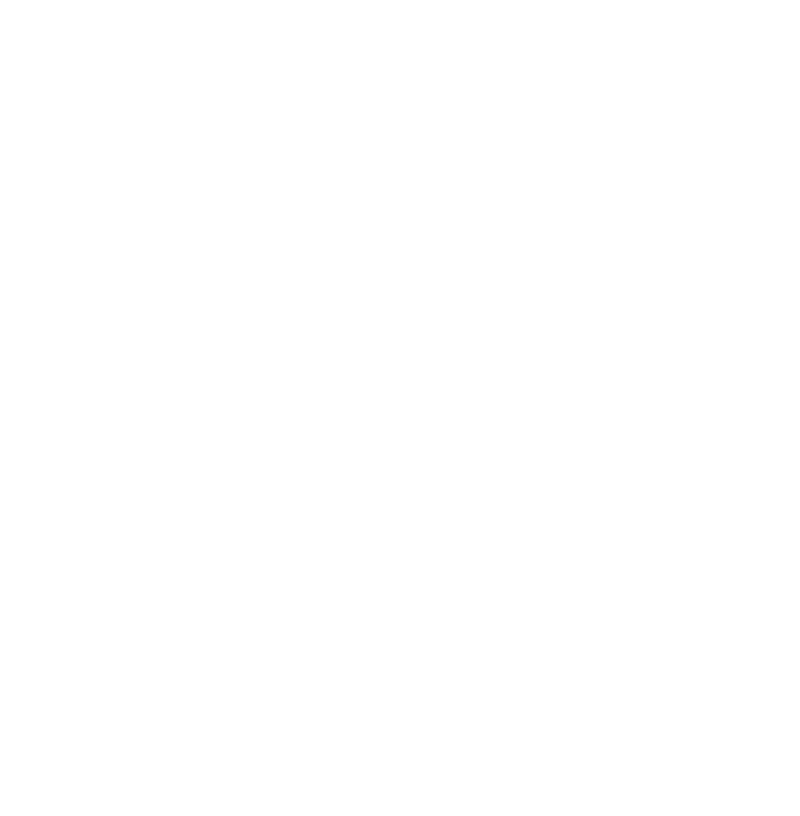
## Two Random Variables Conditioned on an Event

So far, we have only seen cases where a single random variable was conditioned on either an event or on another random variable. If instead we have two random variables, both of which are conditioned on an event, we can have two cases.

For discrete random variables,

Example

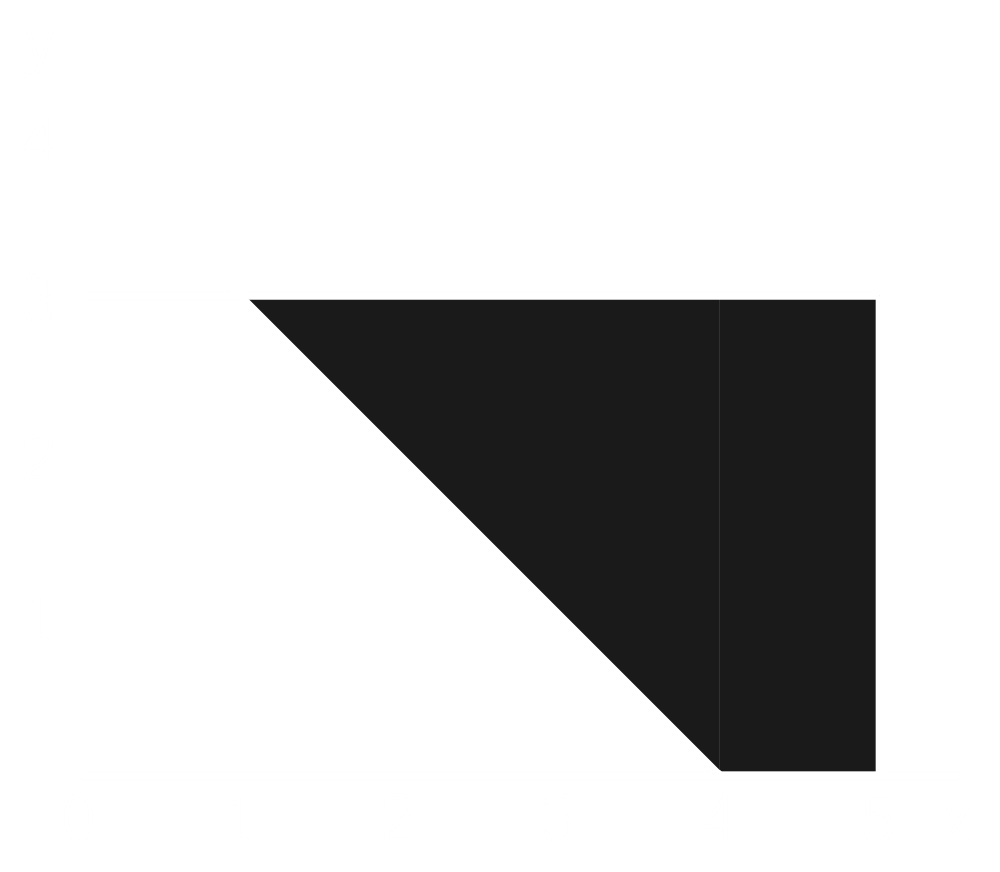
The random variables and are defined for the range with the following probabilities:



Let . Thus, there are just four points in , , , and . Thus,

Now we can calculate the conditional probability for any point. For example, for the point ,

Example



The event is defined for the shaded region. Thus,