**Chapter 10**

Table of Contents

[10.1 Graphs and Graph Models 3](#_Toc66760545)

[Graph Terminologies 4](#_Toc66760546)

[Simple Graph 4](#_Toc66760547)

[Multigraph 4](#_Toc66760548)

[Pseudographs 4](#_Toc66760549)

[Simple Directed Graph 4](#_Toc66760550)

[Directed Multigraph 4](#_Toc66760551)

[Mixed Graph 4](#_Toc66760552)

[10.2 Graph Terminology and Special Types of Graphs 5](#_Toc66760553)

[Special Graphs 7](#_Toc66760554)

[Complete Graphs 7](#_Toc66760555)

[Cycles 7](#_Toc66760556)

[Wheels 7](#_Toc66760557)

[-Dimensional Hypercubes (-cubes) 7](#_Toc66760558)

[Bipartite Graphs 8](#_Toc66760559)

[10.3 Representing Graphs and Graph Isomorphism 9](#_Toc66760560)

[Adjacency Lists 9](#_Toc66760561)

[Adjacency Matrices 10](#_Toc66760562)

[Incidence Matrices 10](#_Toc66760563)

[Isomorphism in Graphs 11](#_Toc66760564)

[10.4 Connectivity 14](#_Toc66760565)

[Directed Graph 14](#_Toc66760566)

[10.5 Euler – Hamilton Paths and Circuits 16](#_Toc66760567)

[10.8 Graph Coloring 18](#_Toc66760568)

## 10.1 Graphs and Graph Models

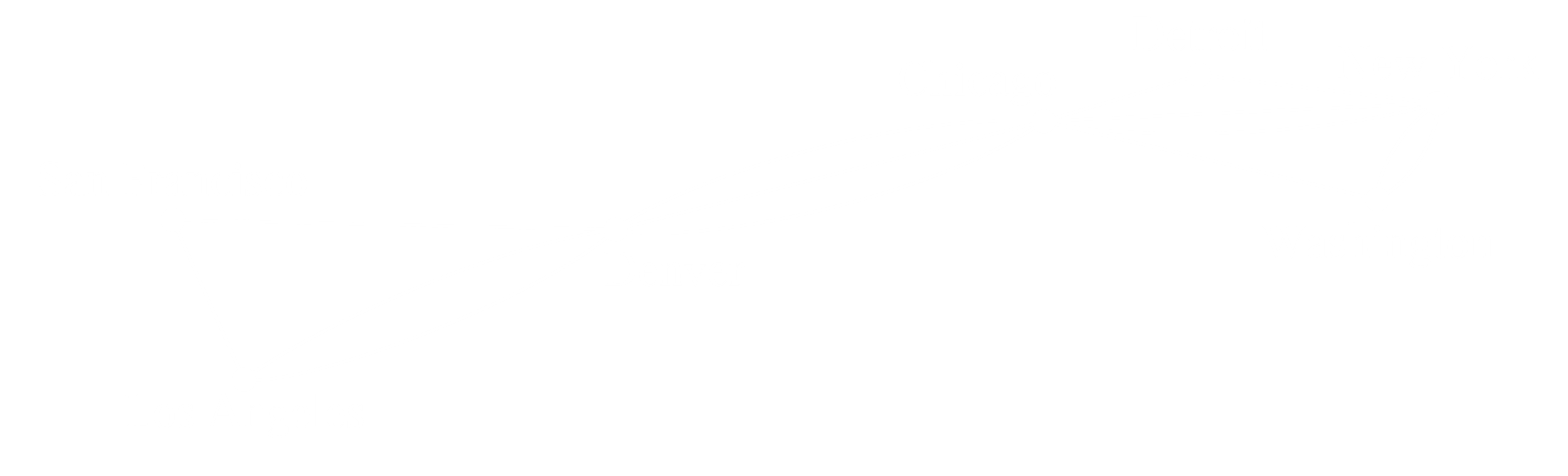
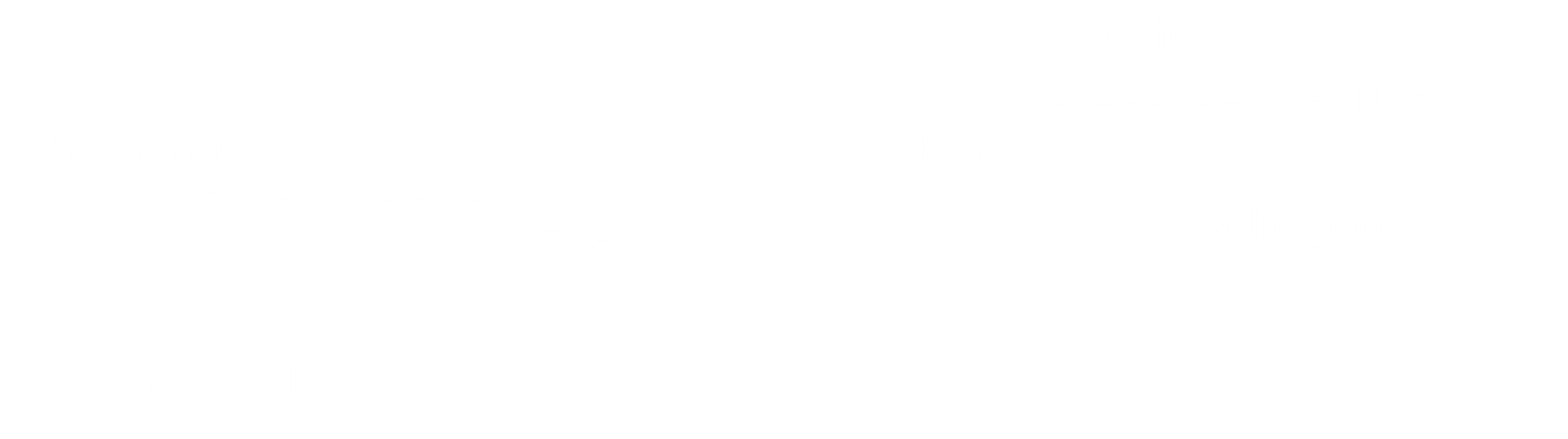
Definition 1: A graph consists of , a non-empty set of vertices (or nodes) and , a set of edges. Each edge has one or two vertices associated with it, called its endpoints. An edge connects its endpoints.

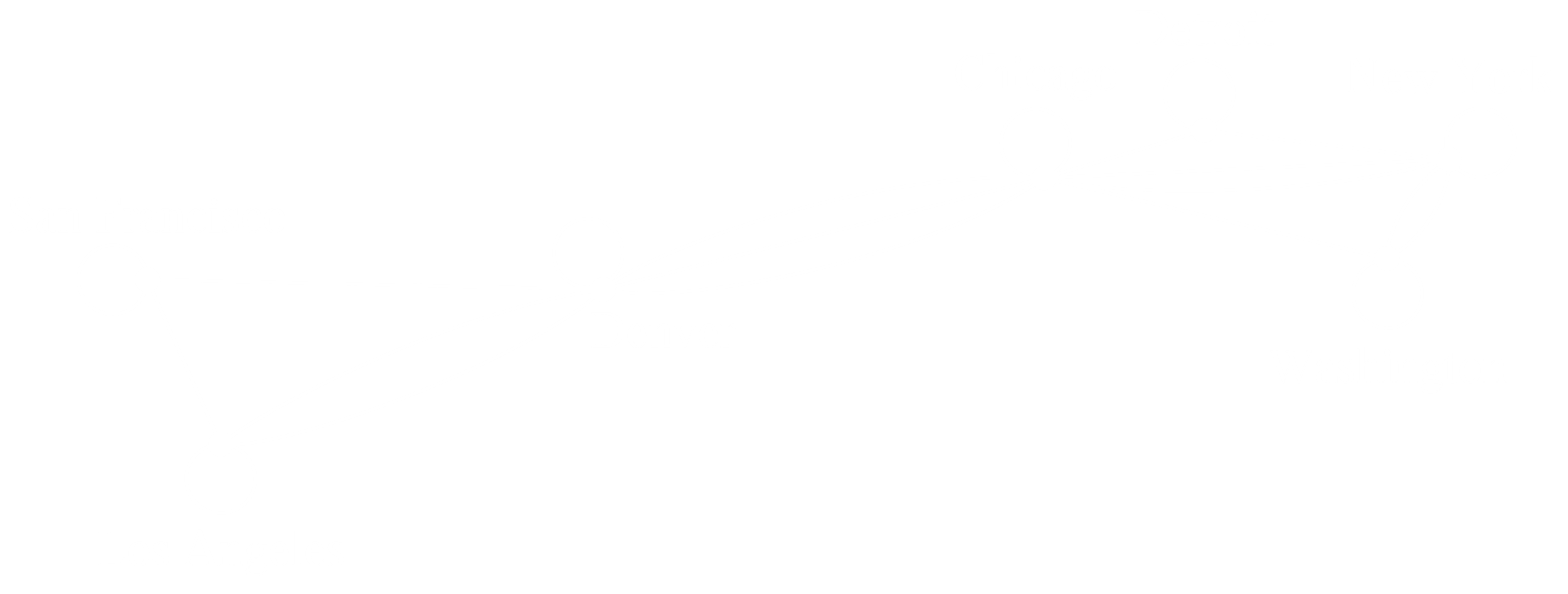
Two vertices may also have more than one edge between them, meaning they are connected by more than one edge. Such a graph is called a multigraph (as opposed to a simple graph). A single vertex may also be connected to itself by an edge. This edge is called a loop.

Definition 2: A directed graph consists of a non-empty set of vertices and a set of directed edges . Each directed edge is associated with an ordered pair of vertices. For example, a directed edge associated with the ordered pair is said to start at and end at . In the graph, a directed edge is depicted by an arrow in the required direction.

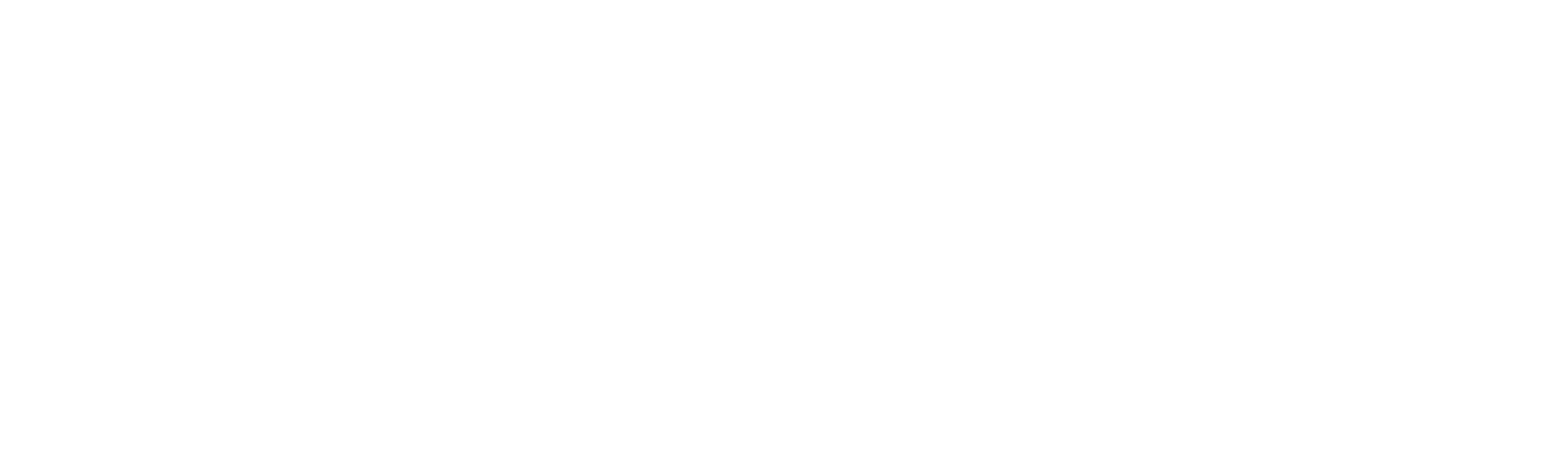
A directed graph may be a simple directed graph, with no multiple directed edges or loops, or a multiple directed graph, with multiple directed edges and/or loops. When there are directed edges associated with an order pair of vertices, is said to be the multiplicity of that pair.

### Graph Terminologies

Simple Graph: Any two vertices are connected by a single edge, and there are no vertices with two edges between them. There are no loops present. The edges are undirected.

Multigraph: Vertices may be connected by multiple edges, but have no loops. The edges are undirected.

Pseudographs: Vertices may be connected by multiple edges, including by loops. The edges are undirected.

Simple Directed Graph: There are only single edges between the vertices, with no multiple edges or loops. The edges are directed.

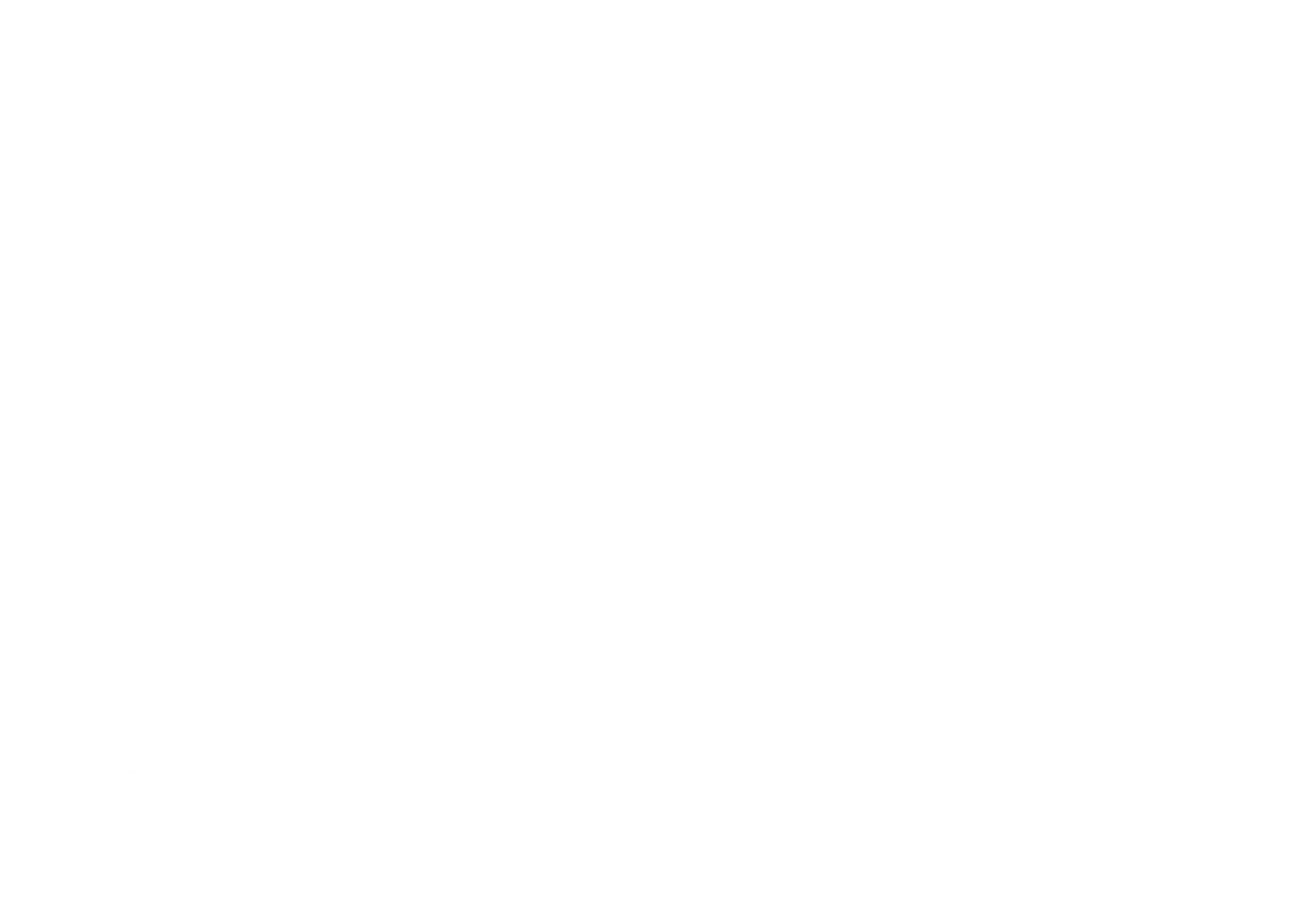
Directed Multigraph: There are multiple edges between vertices, including loops. The edges are directed.

Mixed Graph: This contains a mixture of directed and undirected edges.

## 10.2 Graph Terminology and Special Types of Graphs

Definition 1: Two vertices and in an undirected graph are said to be adjacent or neighbors if they are connected by an edge. Such an edge is called incident with the vertices and and is said to connect them.

Definition 2: The set of all neighbors of a vertex is denoted by , and is called the neighborhood of .



Here, .

Definition 3: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop contributes twice to the degree of that vertex. The degree of vertex is denoted by .

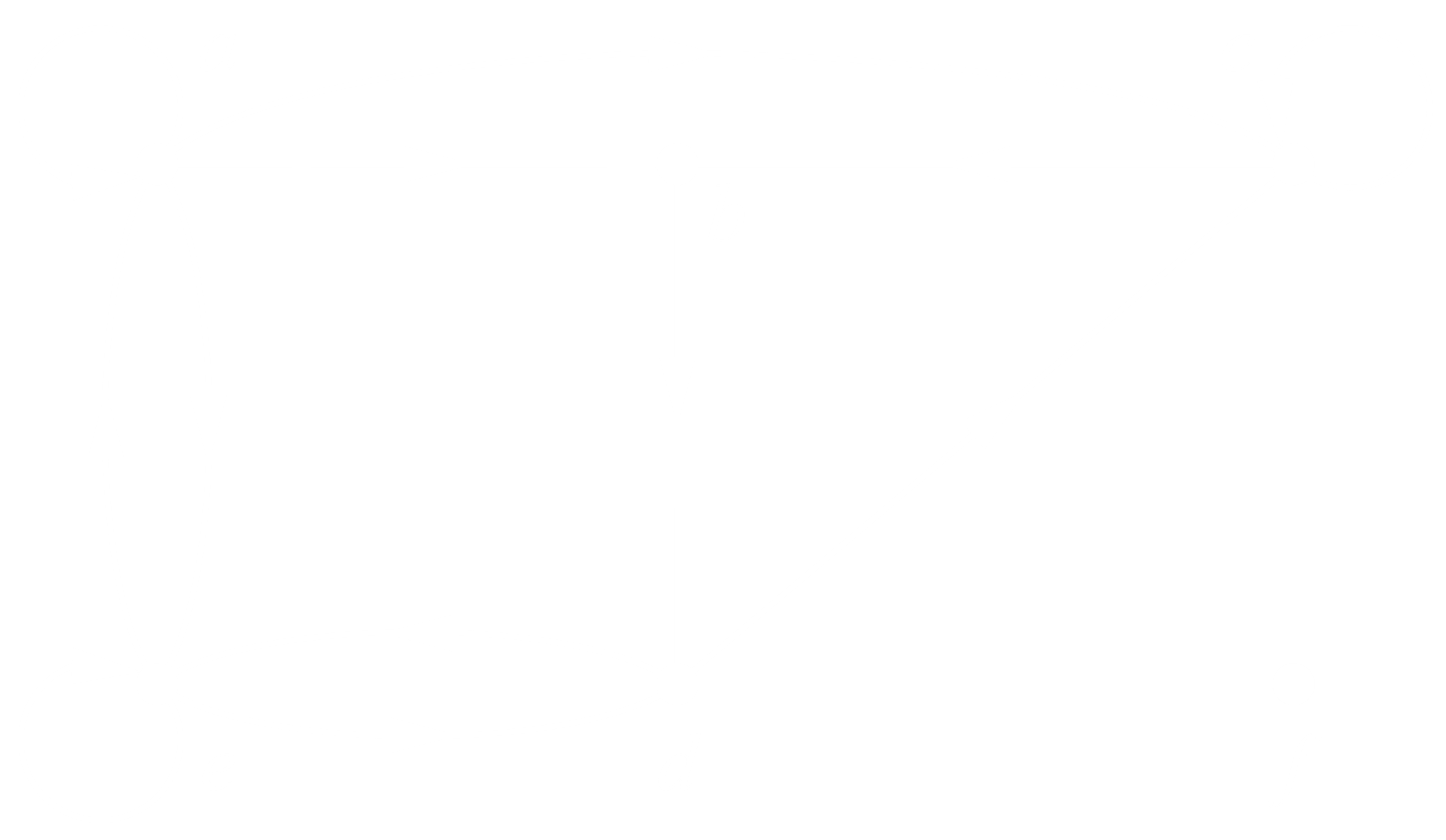
In the diagram above, , and . A vertex may also have a degree of , meaning it has no neighbors. Such a vertex is called an isolated vertex. A vertex with a degree of is called a pendant. Vertex is a pendant.

Theorem 1: For an undirected graph of edges and vertices, . Thus, for a graph with vertices, each of degree , .

Theorem 2: An undirected graph has an even number of vertices of odd degree.

Definition 4: For a directed graph, if is an edge, is said to be the initial vertex and adjacent to and is said to be the terminal vertex and adjacent from . For a loop, the initial and terminal vertices are the same.

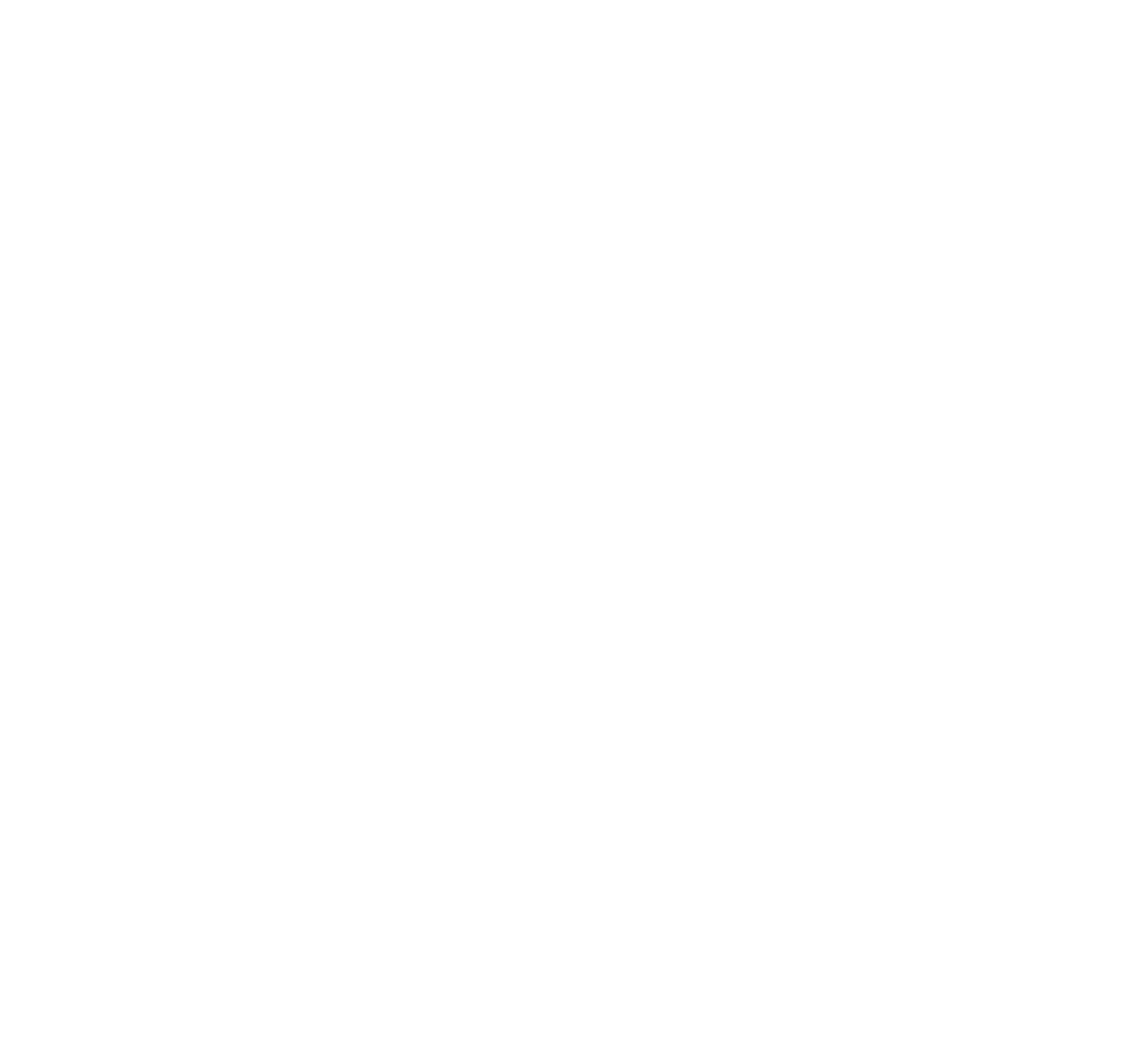
Definition 5: For a directed graph, the in-degree of a vertex is the the number of edges with as the terminal vertex, and is denoted by , and the out-degree of a vertex is the number of edges with as the initial vertex and is denoted by . Note that a loop contributes to both the in-degree and the out-degree.



Theorem 3: For a graph with directed edges, .

### Special Graphs

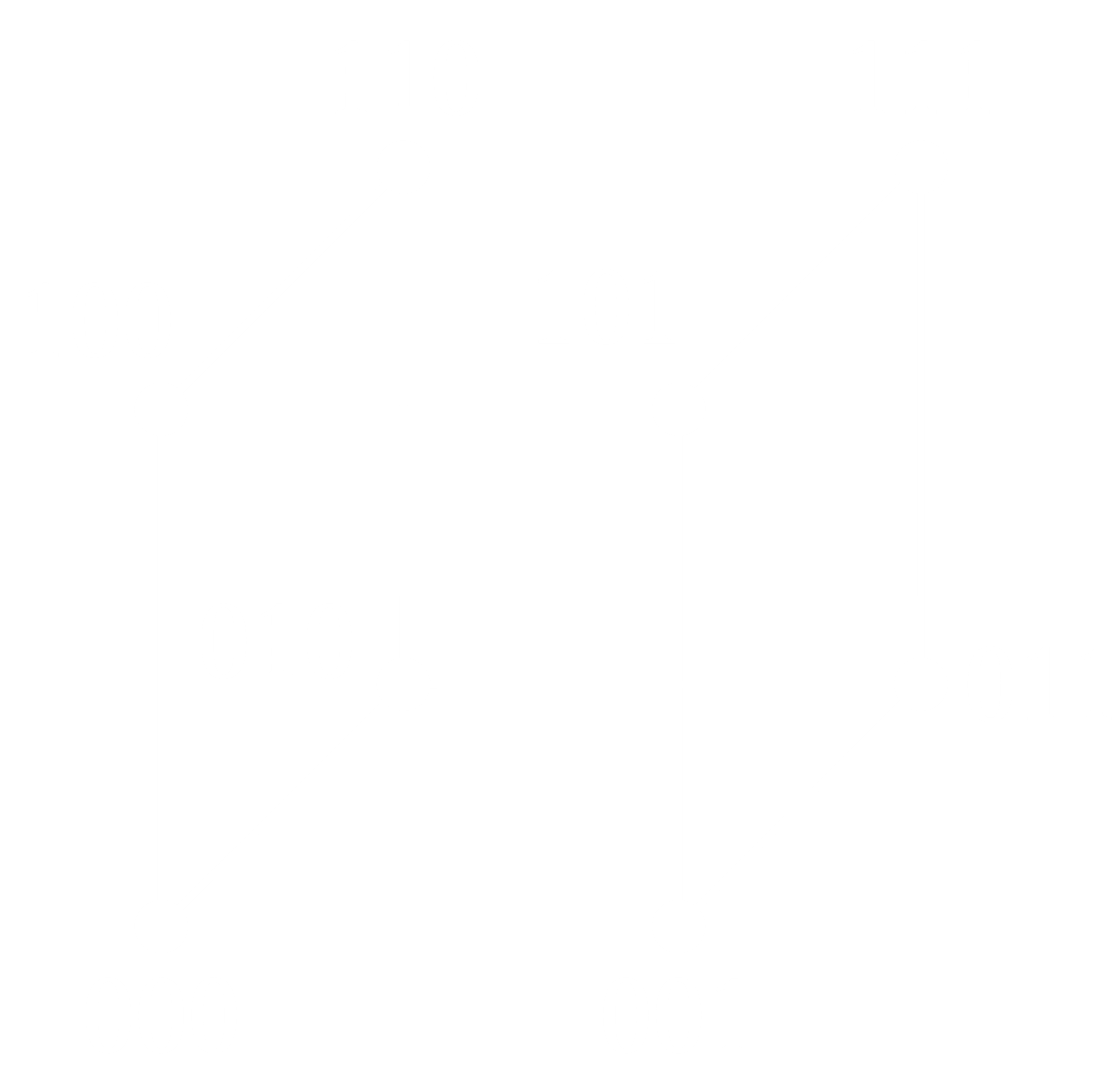
Complete Graphs: Exactly edge between each pair of distinct vertices. This is a complete graph .



Cycles: This is a cycle .



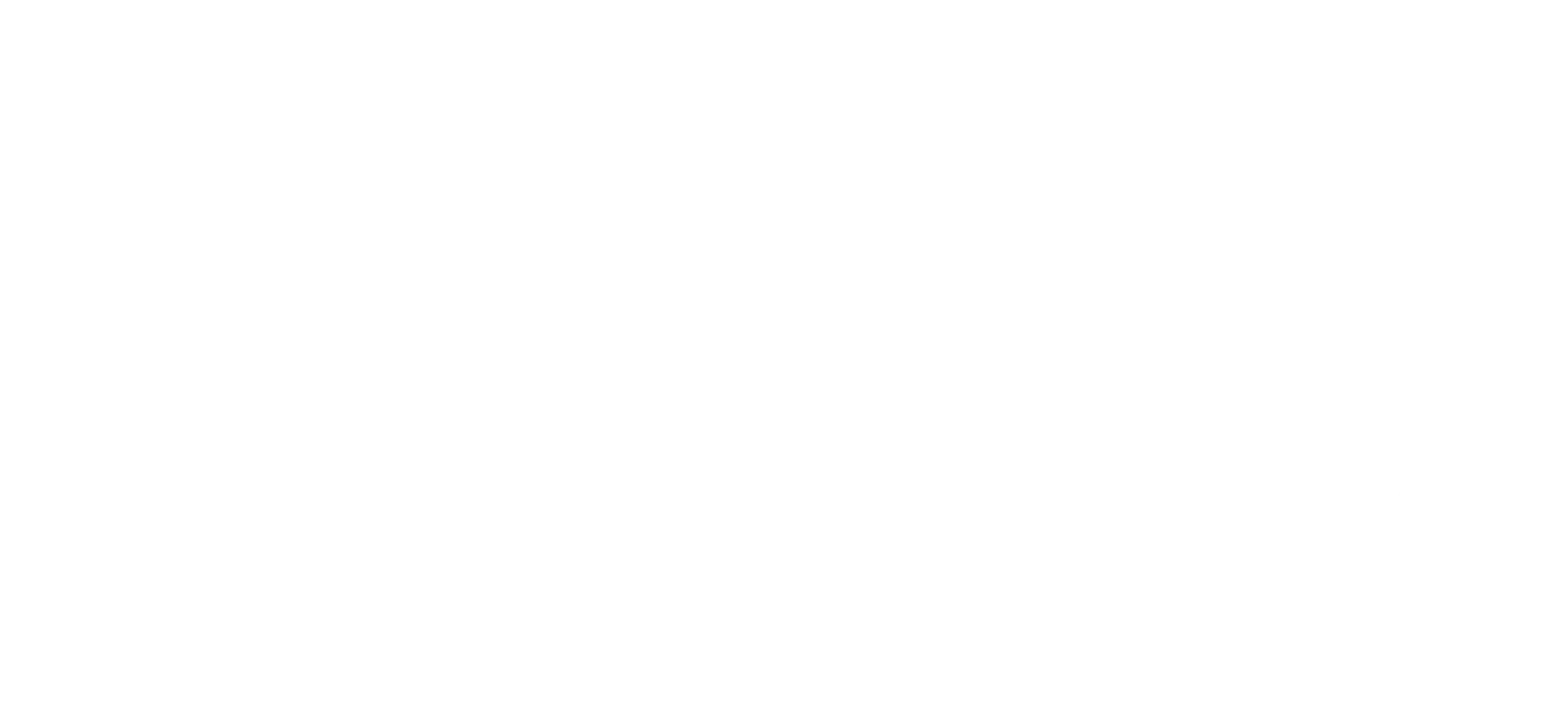
Wheels: A wheel is a cycle with an extra vertex to which each of the other vertices are connected. The extra vertex is not counted. Thus, this is a wheel .



-Dimensional Hypercubes (-cubes): A graph with vertices representing the bit strings of length . Two vertices are adjacent only if the bit strings they represent differ by exactly one bit position. This is an -cube, . Graphs for cannot be drawn.

### Bipartite Graphs

Definition 6: A simple graph is called bipartite if its vertex can be divided into two disjoint subsets and such that each edge connects a vertex from one subset to a vertex in another, and no edge connects two vertices from the same subset.The pair is said to be a bipartition of the vertex set .



Another way to check this is to ensure that, if one of two colors is assigned to each vertex of a graph, no two adjacent vertices are the same color.

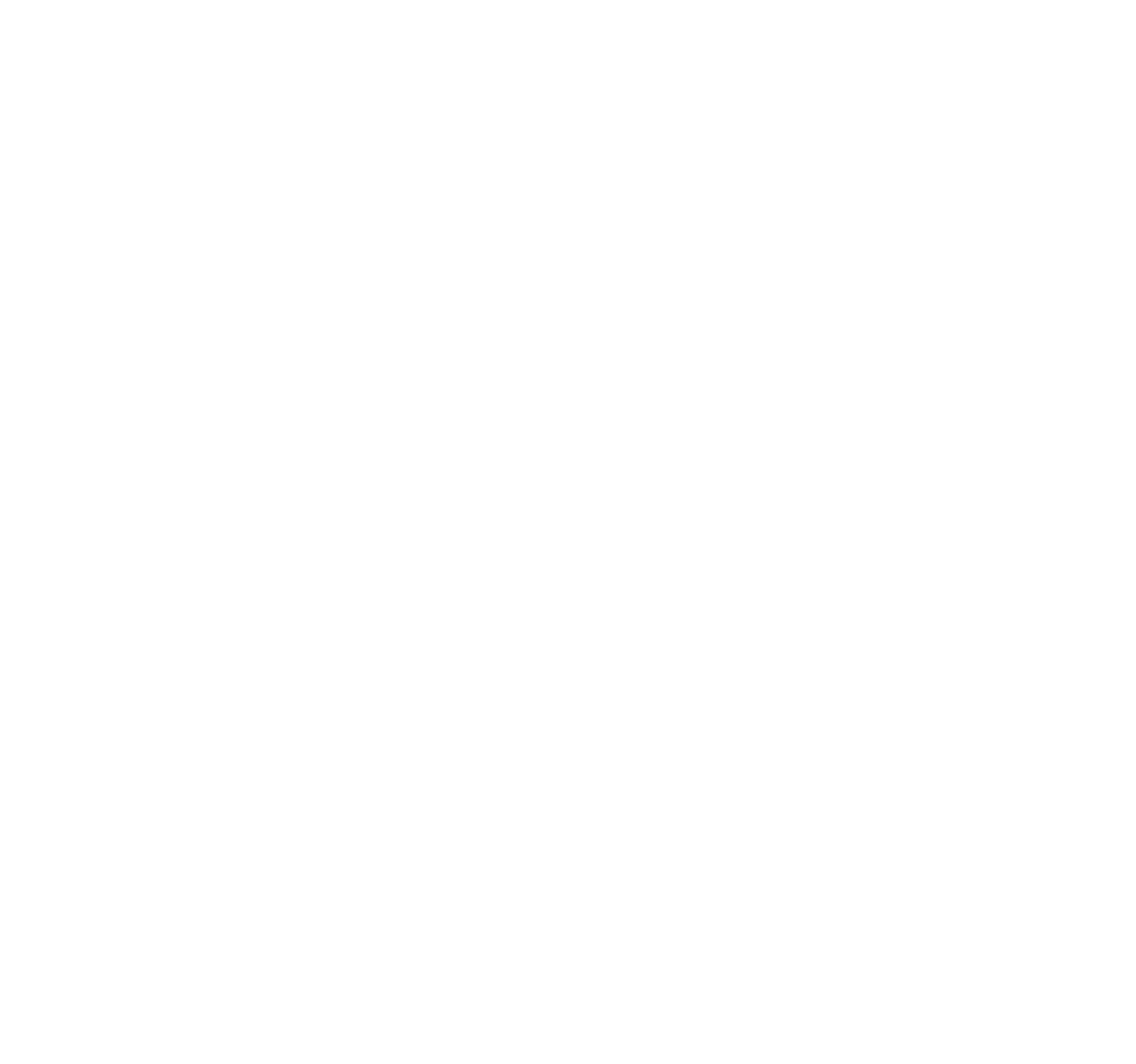
A complete bipartite graph is a graph with its vertex set partitioned into two sets and vertices such that each vertex of one set is connected to every vertex in the other, with no connections between vertices in the same set.

## 10.3 Representing Graphs and Graph Isomorphism

### Adjacency Lists

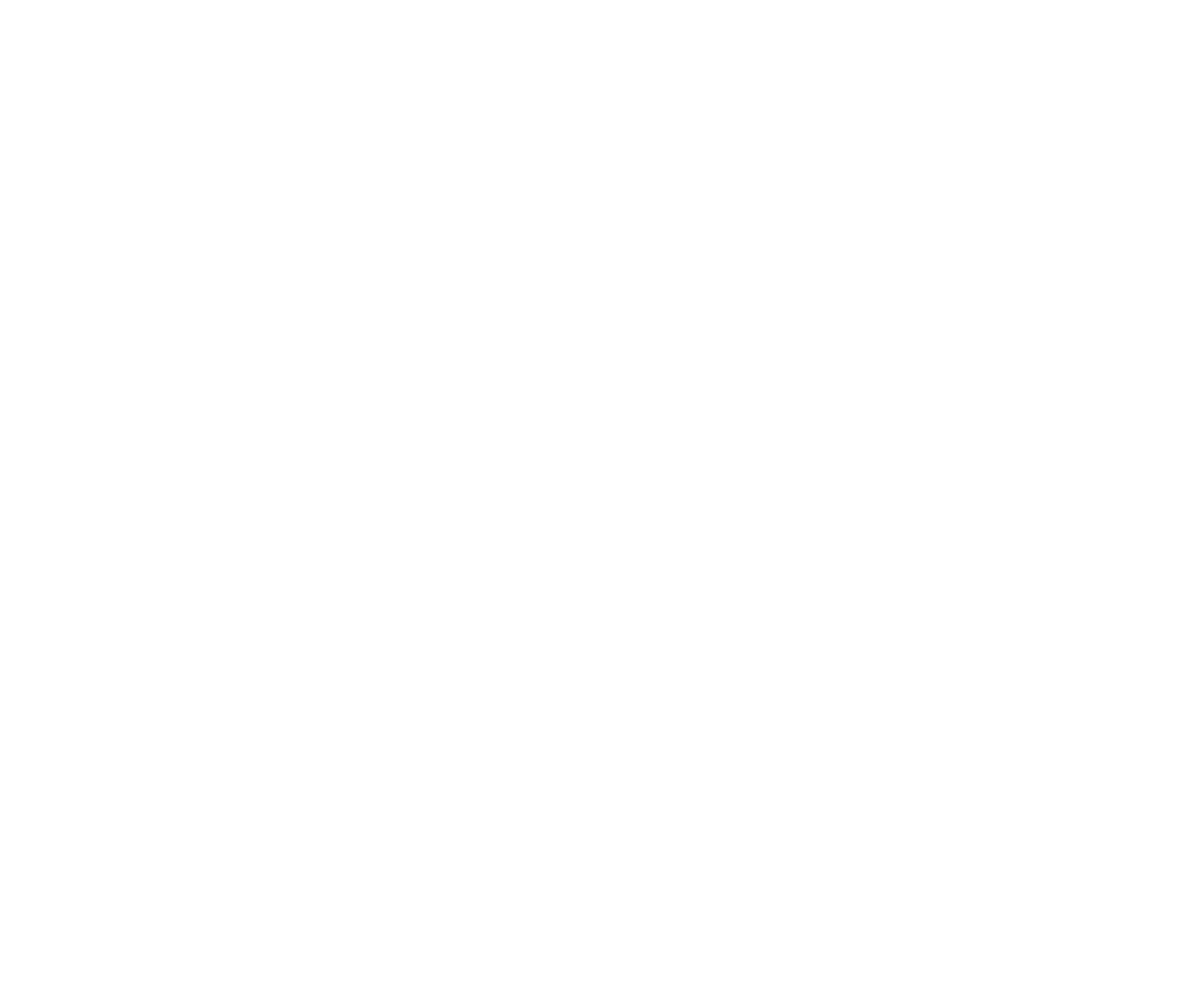
For simple graphs, these list the vertices adjacent to each vertex.

|  |  |
| --- | --- |
| Vertex | Adjacent Vertices |
|  | , , |
|  |  |
|  | , , |
|  | , |
|  | , , |



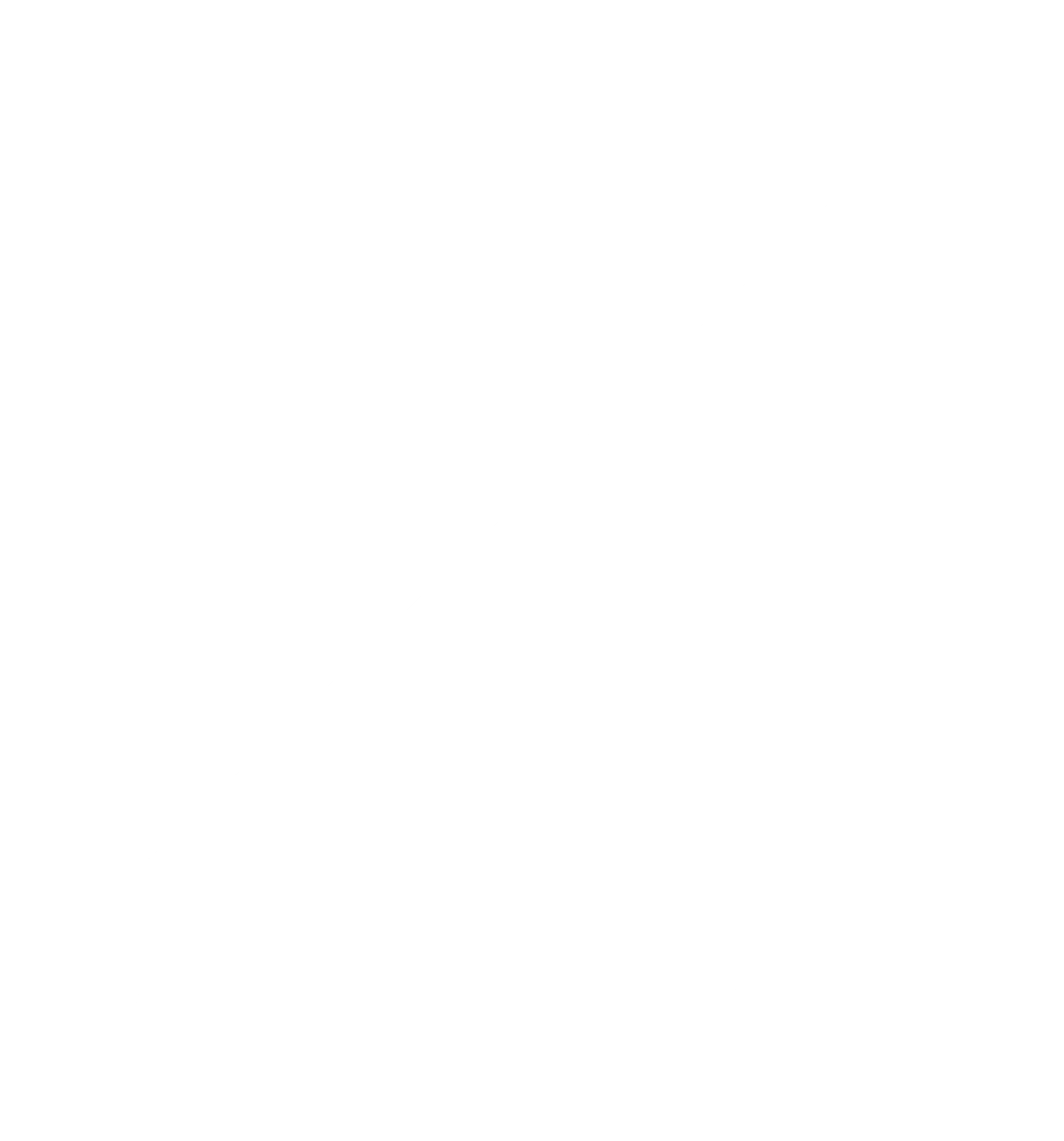
For a directed graph, the terminal vertices for each vertex are listed.

|  |  |
| --- | --- |
| Initial Vertex | Terminal Vertices |
|  | , , , |
|  | , |
|  | , , |
|  | - |
|  | , , |



### Adjacency Matrices

An adjacency matric for a graph with vertices is an by matrix, with each cell representing the number of the column member has with the row member.

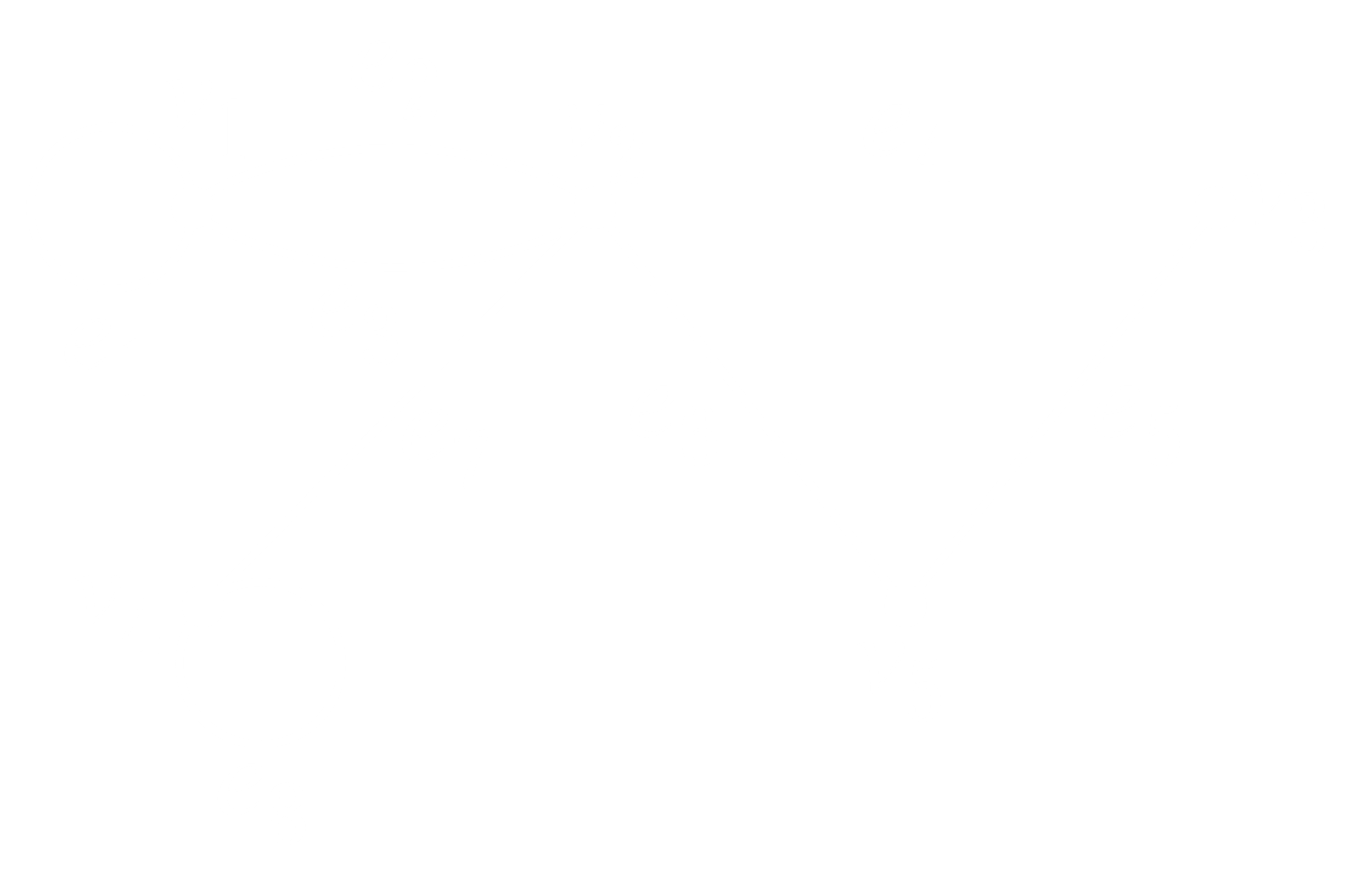
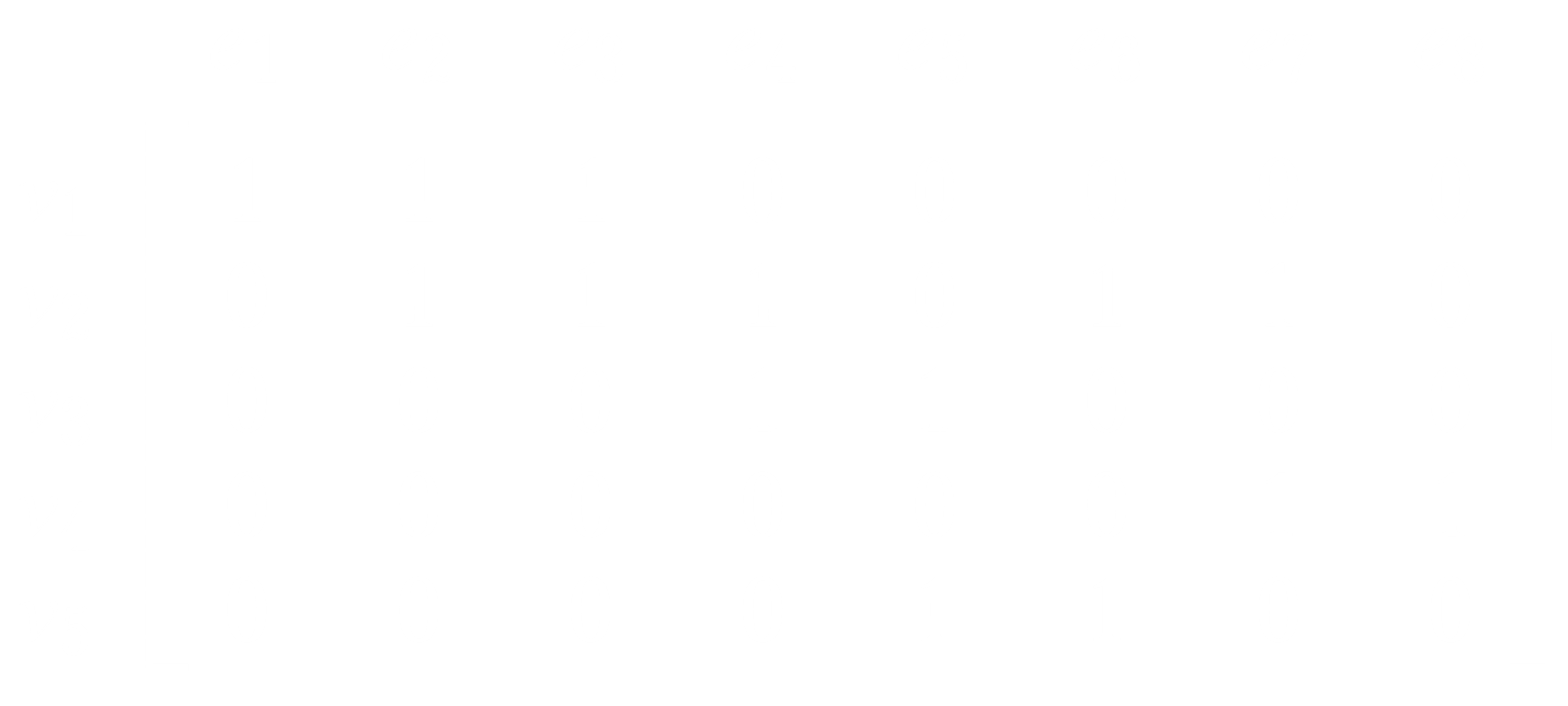


Column 2, row 1 represent connections between and , and thus has a value of . Column , row reprsents connects of with itself, a self-loop, and thus has a value of .

For a directed graph, the row numbers are considers to be initial vertices, and the column numbers to be terminal vertices. Thus, a cell would represent the number of edges that begin at the row member and end at the column member.

### Incidence Matrices

Here, the columns of the matrix represent the vertices, and the rows represent edges. A cell has a value of if the vertex is associated with the corresponding edge.



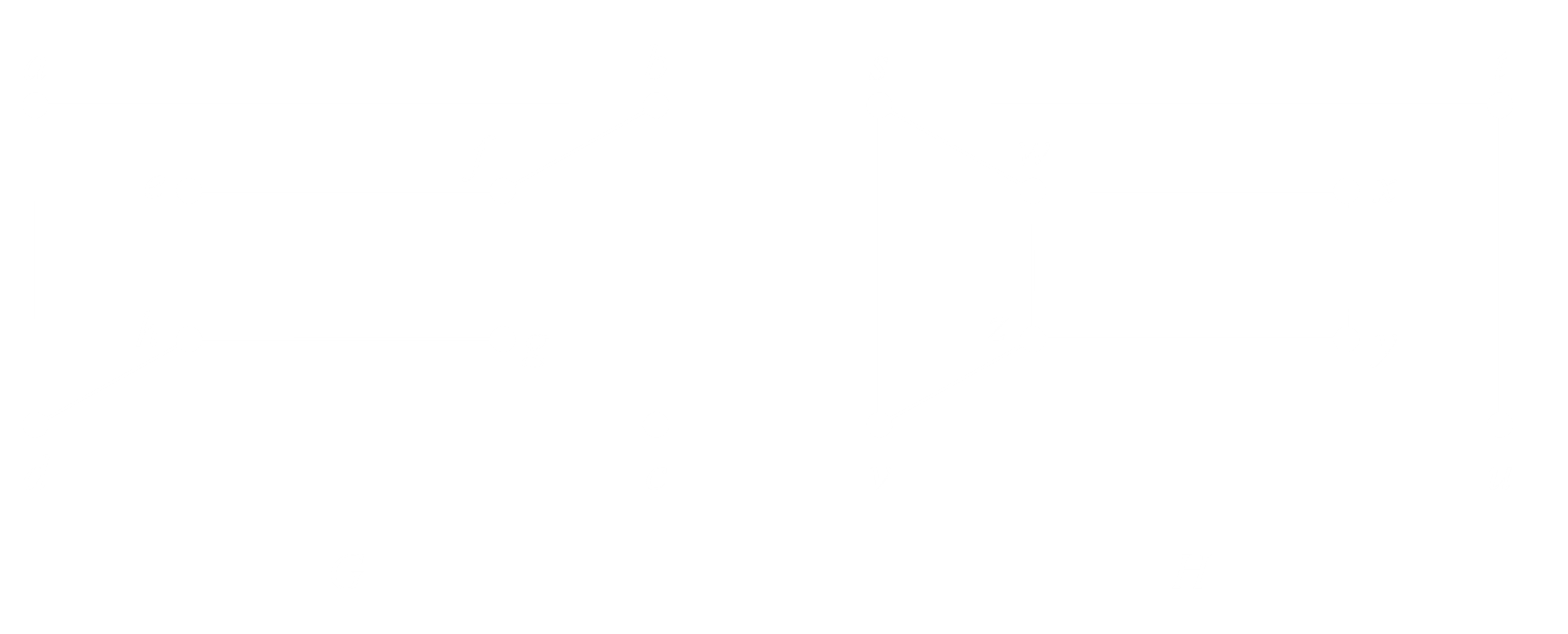
### Isomorphism in Graphs

Two graphs are isomorphic if they have the same number of elements despite having different structures.

It is possible to show that two graphs are not isomorphic, if we can find a property that only one of the two graphs has, but which is preserved by isomorphism. A property preserved by isomorphism is called a graph invariant

Identifying two isomorphic graphs:

* Same number of vertices
* Same number of edges
* Same number vertices with same degree
* Possible to derive an isomorphic function



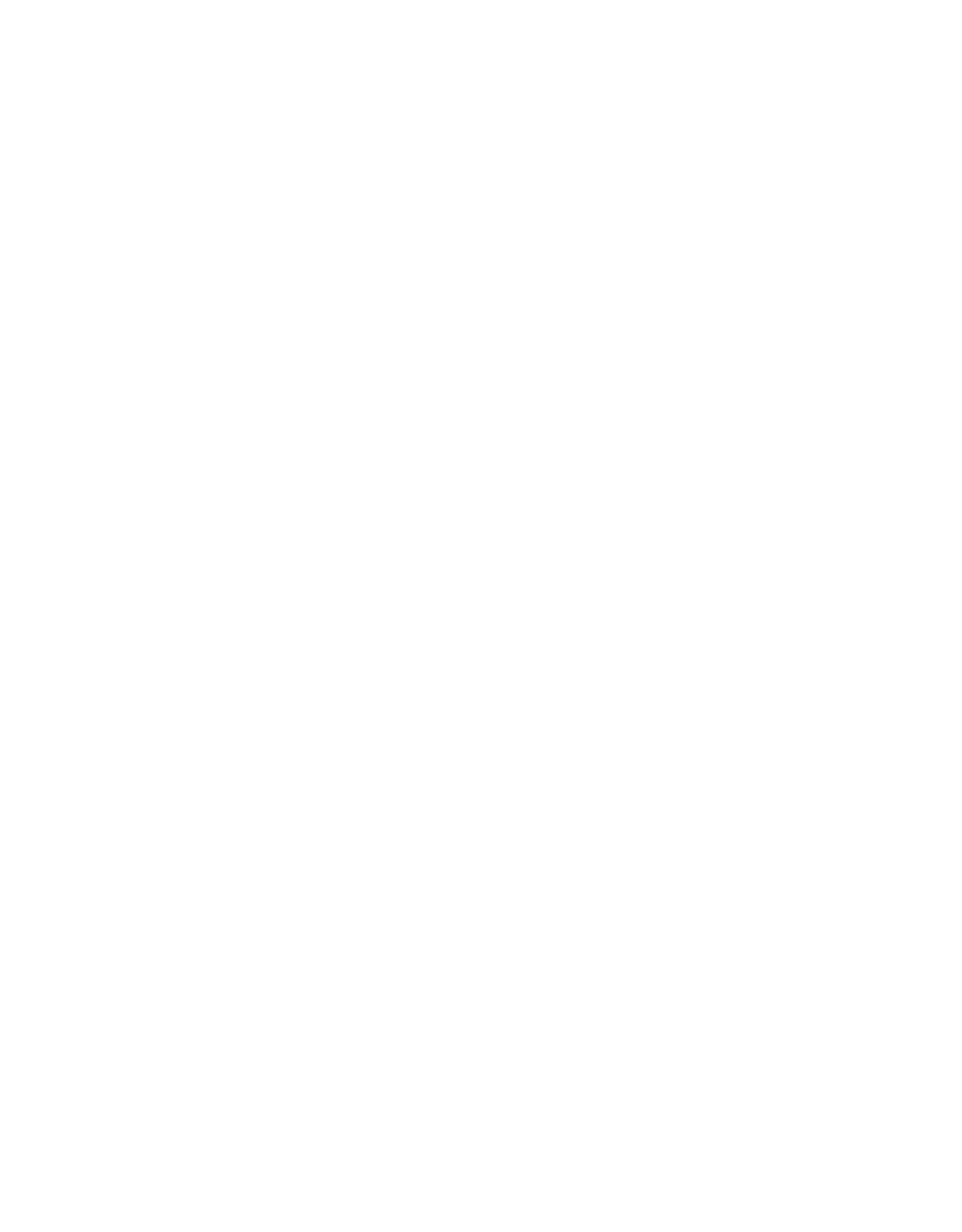
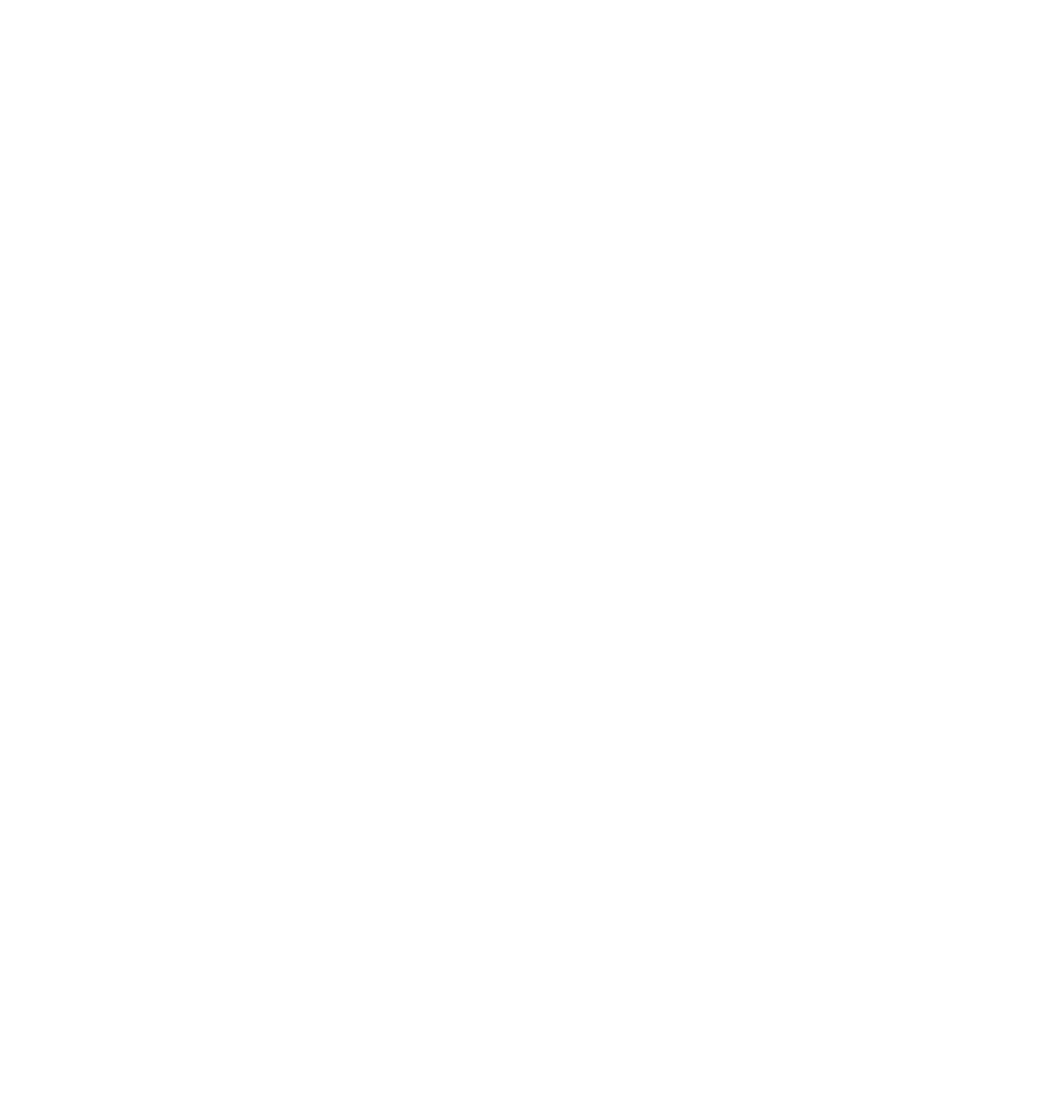
For ,

For ,

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

However, it is not possible to derive an isomorphic function. What this means is that, the rows and columns of the adjacency matrix of cannot be rearranged in such a way as to make it the same as the adjacency matrix of . Thus, these are not isomorphic graphs.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

Switching and columns and rows,

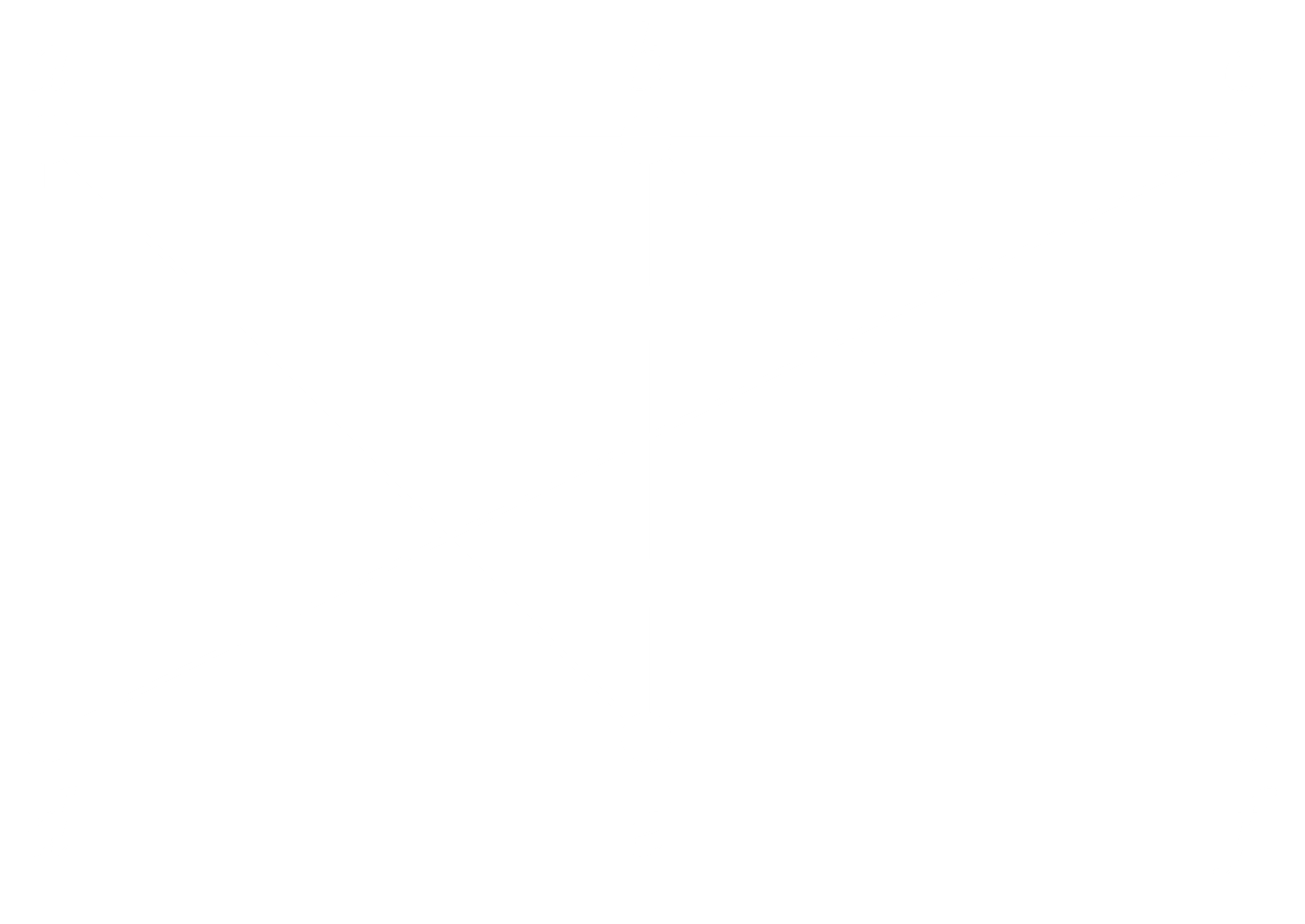
|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

Thus, the graphs are isomorphic.

## 10.4 Connectivity

A path is a sequence of edges that begin at a vertex and travel from vertex to vertex along the edges.

A circuit is a closed path, thus beginning and ending at the same vertex.

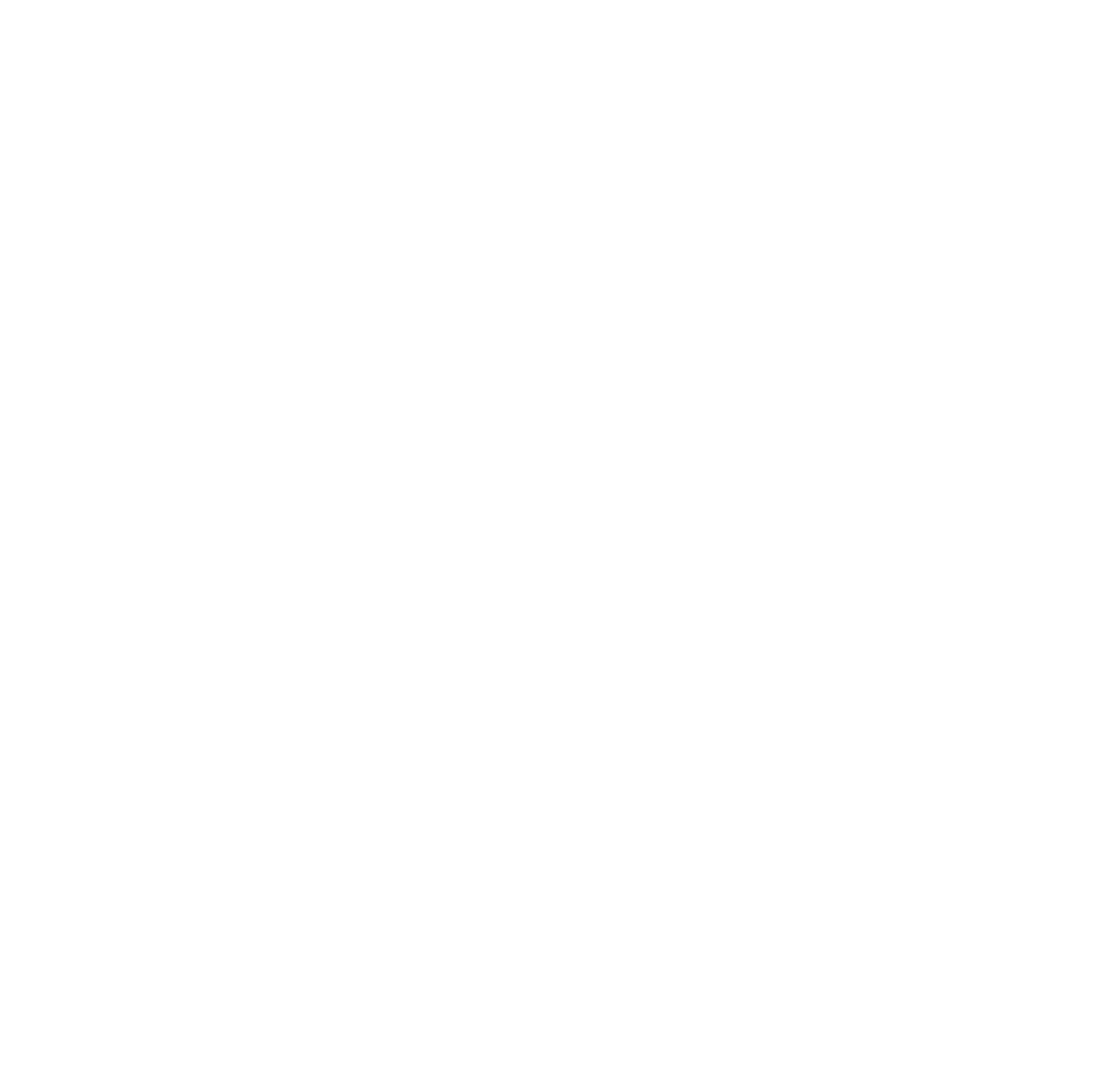


– path

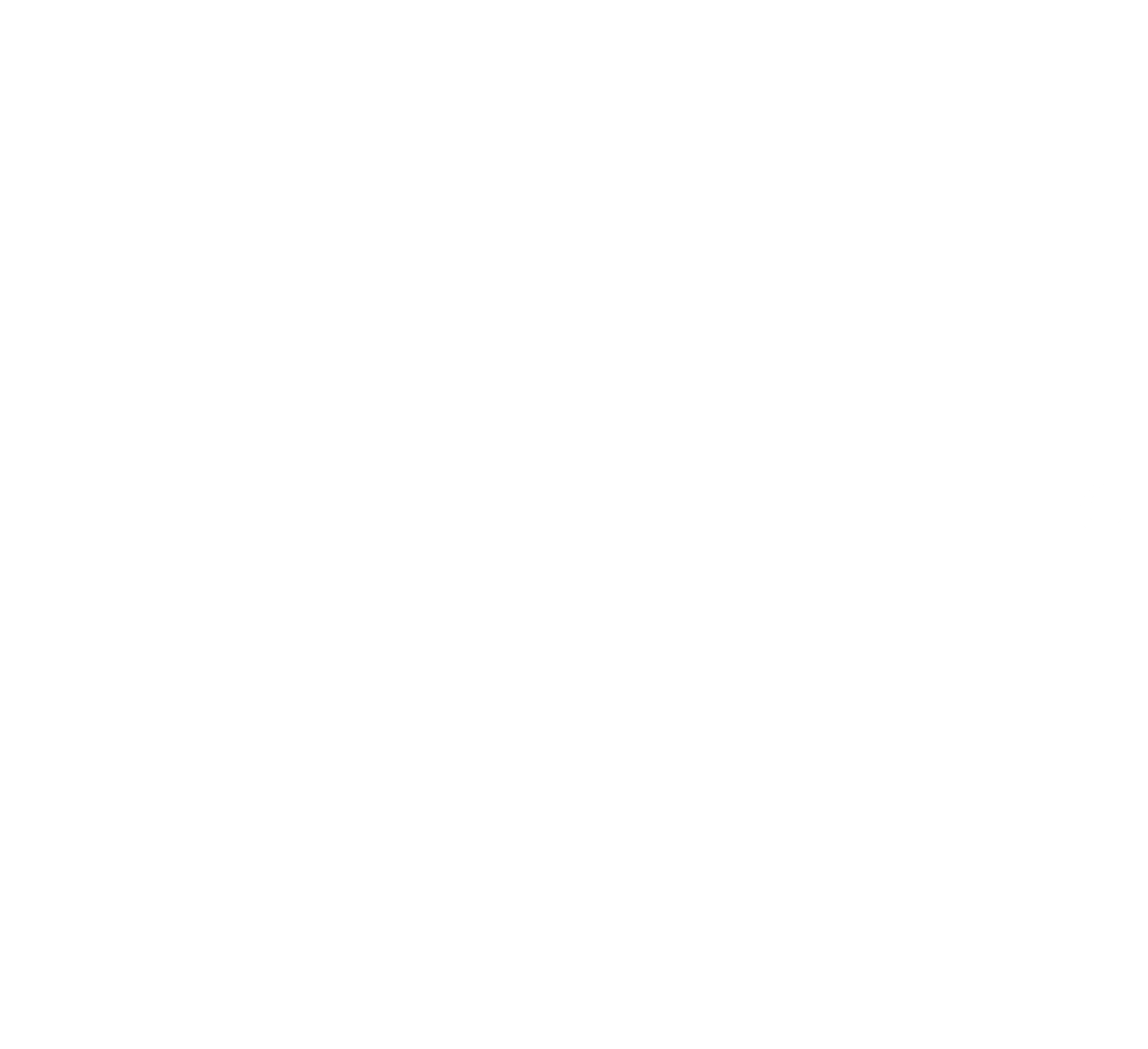
– circuit

A path or circuit is said to be simple if it does not contain the same edge more than once.

### Directed Graph



For a pair of vertices, if there is a path from to and a path from to , the vertices are said to be strongly connected. In a strongly connected graph, all vertices are strongly connected.



Here, the vertices are weakly connected since there is a path from to , but there is no path from to .

## 10.5 Euler – Hamilton Paths and Circuits

A Euler Path is a path containing all the edges of a graph exactly once.

A Hamilton Path is a path that contains all vertices exactly once.

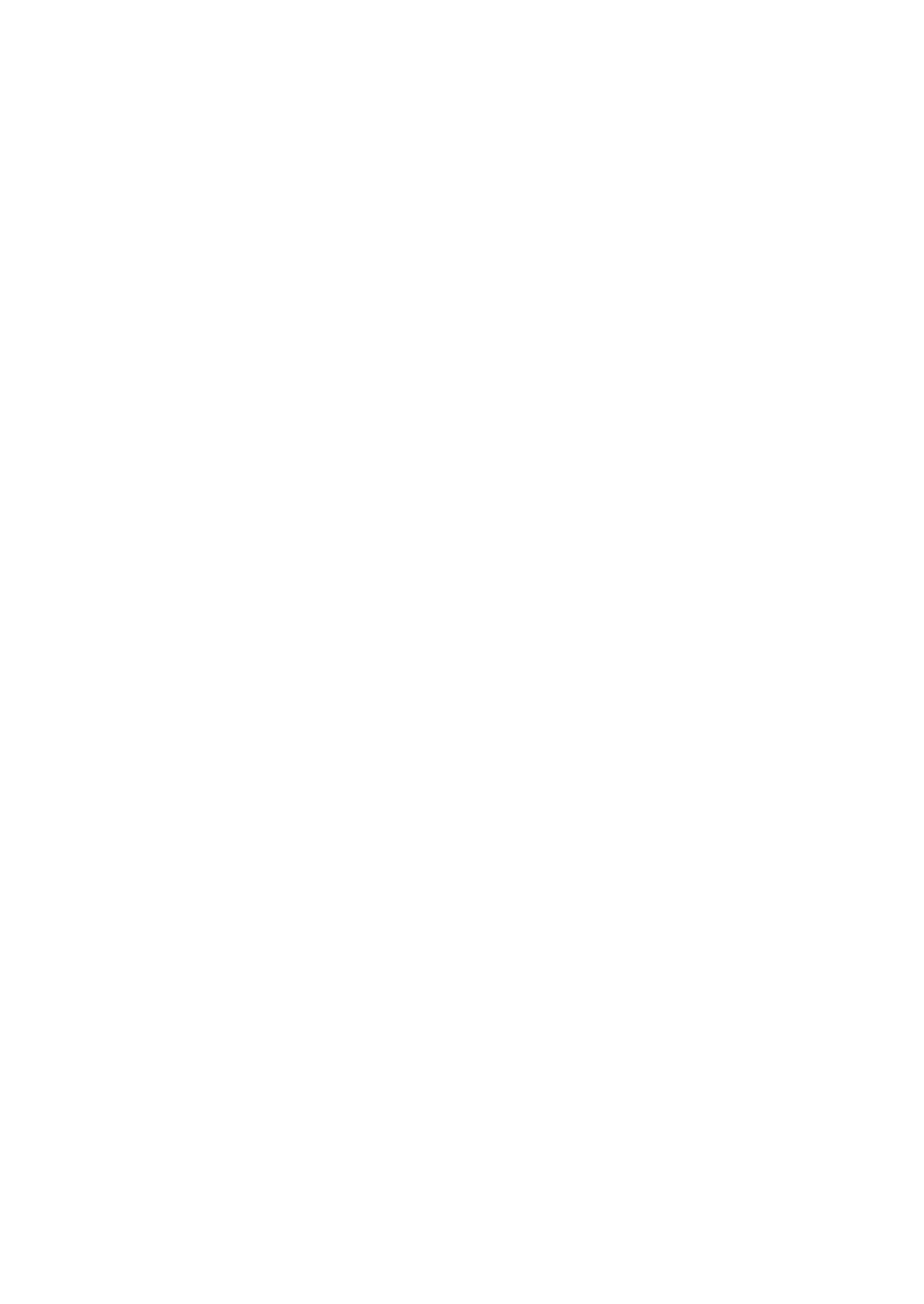
Both of these types of paths cover all the vertices.

Theorem 1: A connected multigraph with at least two vertices has a Euler Circuit if and only if each vertex has an even degree.

Theorem 2: A connected multigraph has a Euler Path and no Euler Circuit if and only if there are exactly two vertices with odd degree.

Dirac’s Theorem: If is a simple graph with vertices, where , such that the degree of each vertex is at least , then has a Hamilton Circuit.

Ore’s Theorem: If is a simple graph with vertices where , such that for each pair of non-adjacent vertices , the following rule is true, then has a Hamilton Circuit.



For the above circuit, so the condition is met. Next, we look at the degrees of each of the vertices.

Here, . Thus, every vertex has a degree of at least , and this is a Hamilton Circuit according to Dirac’s Theorem.

Following Ore’s Theorem,

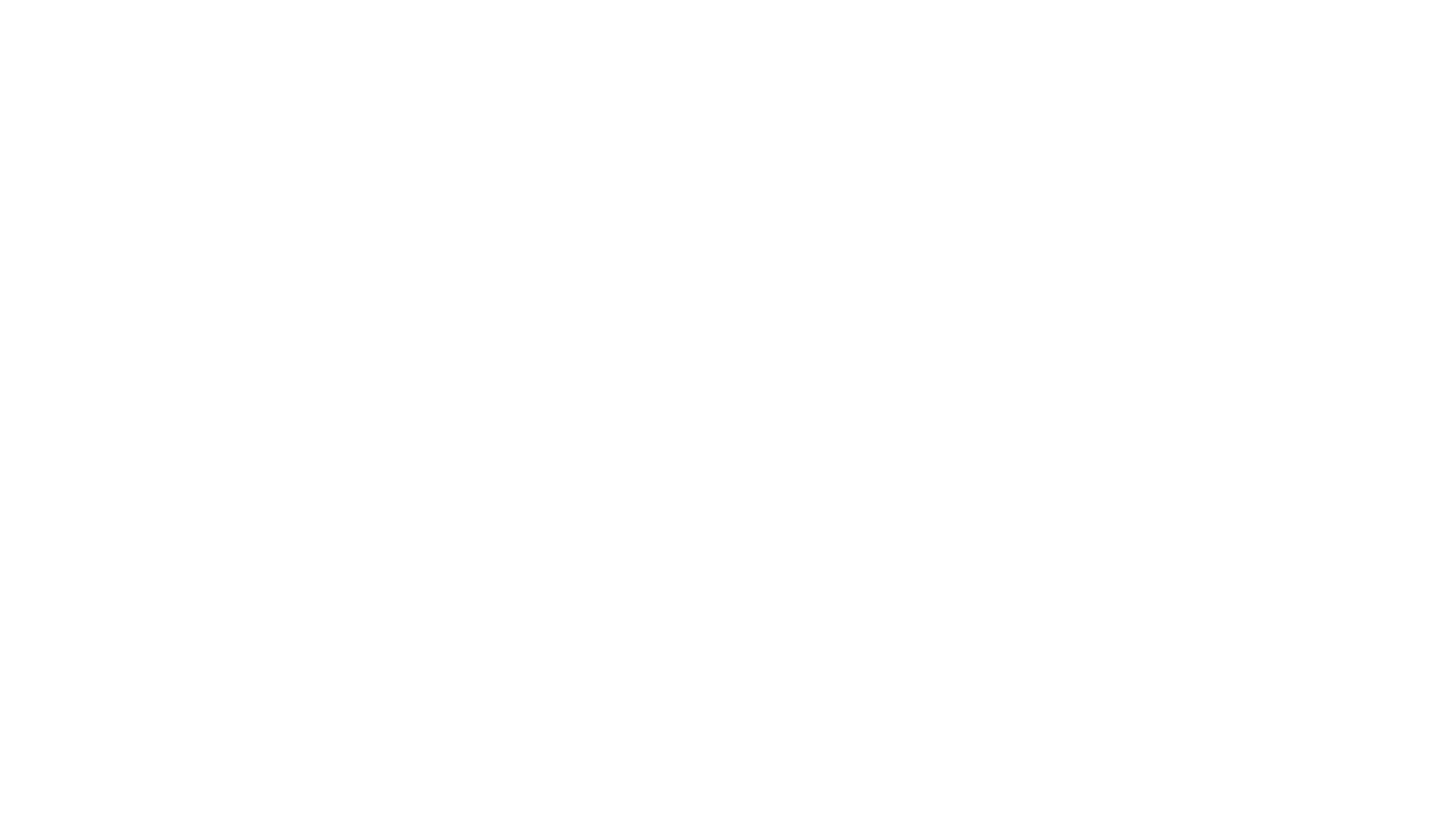
For ,

For ,

These are the only possible non-adjacent pair of vertices, and for both, . Thus, this is a Hamilton Circuit according to Ore’s Theorem.

## 10.8 Graph Coloring

The minimum number of colors required to color the vertices of a graph , such that no two adjacent vertices have the same color, is known as the chromatic number of the graph.



For the above diagram, , and must be given different colors since they are all adjacent to each other. can be given the same color as , can be the same color as and can be the same color as . Finally, can be the same color as . Thus, the graph has a chromatic number of .