Math 4441: Probability and Stochastic Processes

Tutorial Class 3: Continuous Random Variables and Joint Random Variables

Following problems will be discussed in the tutorial class.

1. A point is chosen at random on a line segment of length L. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

2. A professor pays 25 cents for each blackboard error made in the lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid by the professor can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$. What is the probability that Y_{20} exceeds 1000?

3. A circle of radius 1 is inscribed in a square with sides of length 2. A point is selected at random from the square. Find the probability that the point is inside the circle. Note that by a point being selected at random from the square we mean that the point is selected in a way that all the subsets of equal areas of the square are equally likely to contain the point.

4. Let R be the bounded region between $y = x$ and $y = x^2$. A random point is selected from R. If the x-coordinate and the y-coordinate of the point are represented by random variables X and Y, respectively.
   a) Find the joint probability density function (PDF) of X and Y, $P_{XY}(x, y)$.
   b) Find the marginal probability density function of X.
   c) Find $E[X]$.

5. When a certain car breaks down, the time it takes to fix it (in hours) is a random variable with the probability density function
$$f_X(x) = \begin{cases} ce^{-3x} & \text{if } 0 \leq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$
Calculate the value of c. Also, find the probability that when the car breaks down, it takes at most 30 minutes to fix it.

6. A point (X, Y) is selected randomly from the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.
   a) Find the joint probability density function of X and Y.
   b) Find $F_{XY}(x, y)$.

7. Let the joint probability mass function of random variables X and Y be given by
$$P_{XY}(x, y) = \begin{cases} \frac{1}{70} x(x + y), & x = 1, 2, 3, \quad y = 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$
Find $\text{Cov}[X, Y]$.

8. Two fair dice are rolled. The maximum and minimum of the outcomes are denoted by X and Y, respectively
   a) Calculate the joint probability mass function of X and Y.

b) Find the marginal probability mass functions of X and Y.
c) Find $E[X]$ and $E[Y]$.

9. Suppose that, on average, the number of β-particles emitted from a radioactive substance is four every second. What is the probability that it takes at least 2 seconds before the next two β-particles are emitted?

10. A beam of length l, rigidly supported at both ends, is hit suddenly at a random point. This leads to a break in the beam at a position X units from the right end. If $\frac{X}{l}$ is beta with parameters $\alpha = \beta = 3$, find $E[X]$, $\mathrm{Var}[X]$, and $P[\frac{l}{5} < X < \frac{l}{4}]$.