**Chapter 08: Generating Random Variates**

Table of Contents

[Distributions 3](#_Toc87178157)

[Inverse Transform Method 4](#_Toc87178158)

[Algorithm 6](#_Toc87178159)

[Exponential Distribution 7](#_Toc87178160)

[Uniform Distribution 8](#_Toc87178161)

[Weibull Distribution 8](#_Toc87178162)

[Discrete Distributions 8](#_Toc87178163)

[Convolution Method 11](#_Toc87178164)

[-Erlang Distribution 11](#_Toc87178165)

[Acceptance-Rejection Method 12](#_Toc87178166)

[Composition Method 16](#_Toc87178167)

When we generate **random numbers**, we obtain samples values of a random variable with a **uniform continuous distribution** within the **range** .

When we generate **random variates**, we obtain samples values of a random variable with **any distribution** and with **any range** other than the distribution and range that defines random numbers.

## Distributions

The actual values we generate will be **sample values** based on the **distribution**. If we draw a histogram for the values, the shape will be similar to the distribution.

For the case of random numbers, if we generate values and divide the range into sections of length, then each section will have roughly values on average. In reality, one section may have values while another has .

For random variates, things start to get trickier, since the distributions are not as simple. For example, if the distribution is exponential, there will be more values towards the beginning of the range and fewer towards the end. The density will be higher towards the beginning. If the distribution is normal, there will be more values towards the centre of the range.

For random variates, the exact values we will get depend on the parameters of the distribution. For example, in the case of the normal distribution, of the values are within the range of from , are within the range from . When generating random variates, we are simply **controlling the frequency** with which values are generated. For each section of the PDF curve, the values along the -axis will be generated at a frequency that corresponds to the PDF curve.

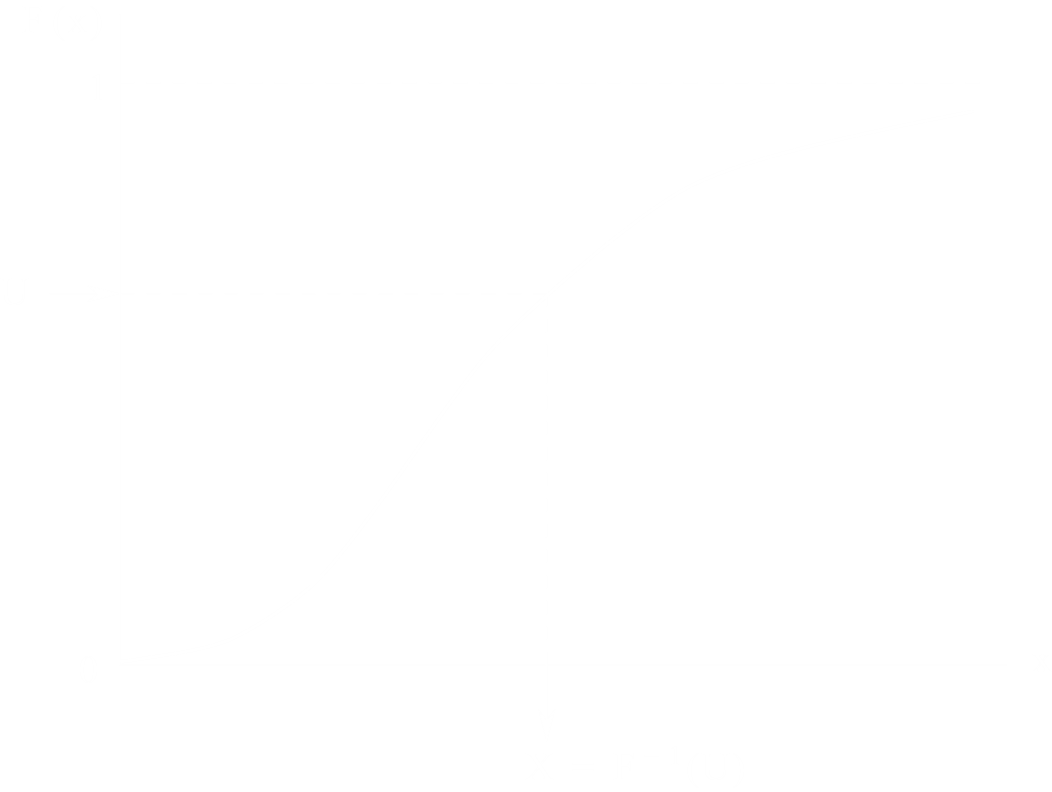
Consider that we have a normal distribution with and . If we generate values, roughly or values will be in the range . or will be in the range .

## Inverse Transform Method

To be able to generate random values from a given distribution, we need to know the mathematics behind the process. In earlier chapters, we saw formulae for generating random values. Now we will learn how those formulae came to be.

Whenever we are generating a **random variate**, we will first need a **random number**, denoted by or . We will be converting the random number into the required random variate.

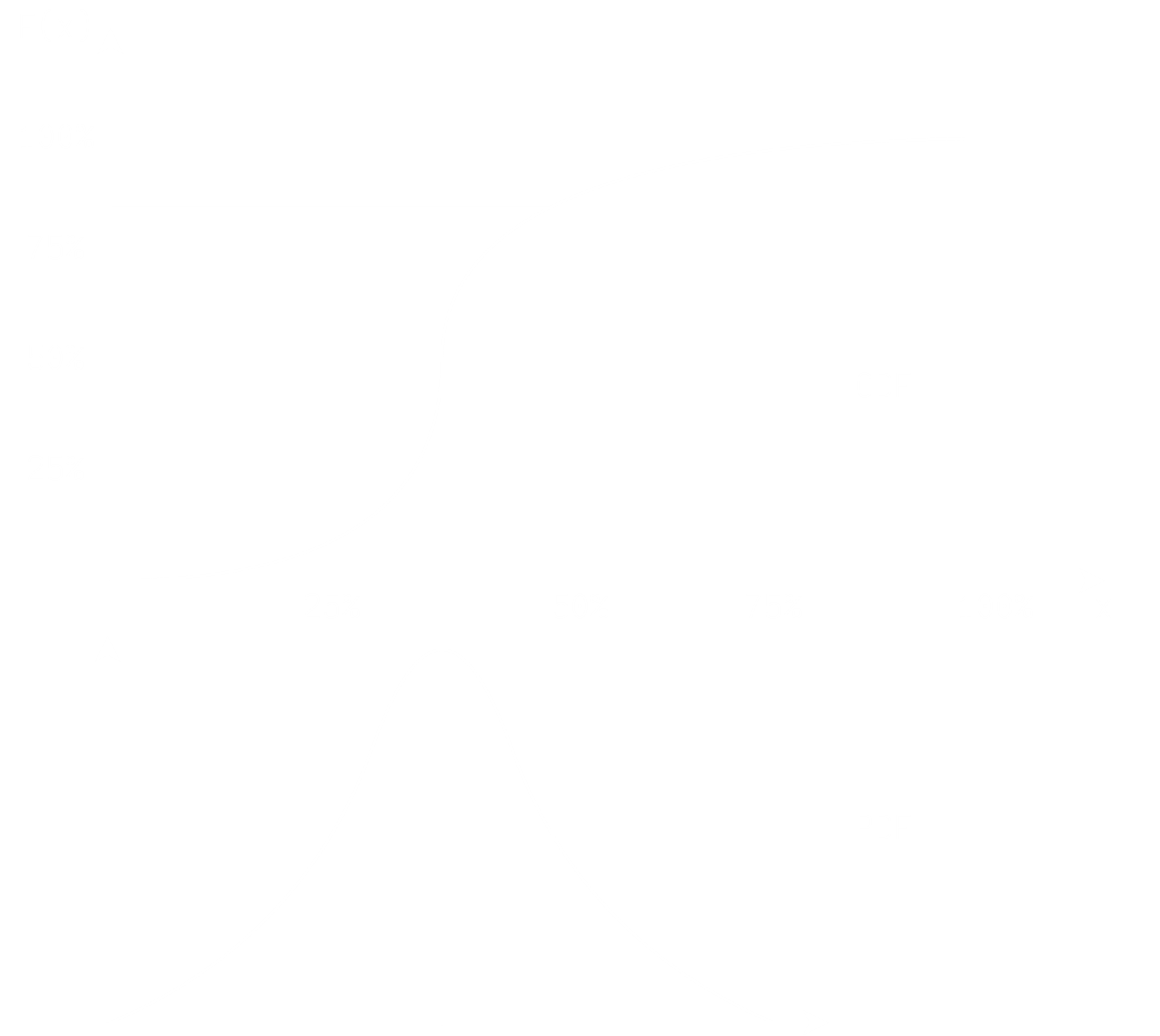
The first approach to generating random variates is called the **Inverse Transform Method**. Under this approach, for the **CDF** of a given distribution, , the set of random values, , can be found using the **inverse** of the distribution with as the parameter, .



We first need to discuss why this works. For a given **function**, for each value of , we get some **result**. For the case of probability distributions, the result is the **CDF value**. The inverse function on the other hand, we are given a **result** and asked to find the  **value** which causes this CDF value.

In the graph above, we actually have the CDF curve for a **normal distribution**. Measuring the **rate of change** of a **PDF curve** will give us the corresponding **CDF curve**. In the above graph, the **slope** towards the **centre** is much **steeper** than the slope towards the beginning and the end. Thus, the PDF curve for this will look like a normal distribution.

The values of **,** by nature, are **uniform**. If we have a value of on the -axis, this just means of the values of are below this point and are above it. Because of this behaviour, the values on the **-axis**, combined with the actual **CDF curve**, will be able to perfectly give us the correct values of .



Thus, if we generate random numbers and use those numbers as -axis values, the corresponding values we will get on the -axis will follow the required distribution. For this example, of the corresponding vales will be in the central, steep section, while the rest will be in the less steep sections.

In summary:

1. The **PDF** of a random variable, , tells the **probability** of observing a specified value of . The graph for the PDF distributed these probabilities across a range. Thus, if is high in a particular region, we will observe more values in that region.
2. The PDF is the **derivative** of the **CDF**, . It tells us about the rate of change of the CDF. If is steep, it means is high at that point. If is flat, it means is low at that point.
3. is **uniformly distributed** on . **More values** of hit the **steep part** of , meaning more values of will be observed from the steep part.

### Algorithm

Step 1:

1. Generate , where .
2. Set .
3. Return .

Say .

Thus, we generate a random number, , calculate and return it.

### Exponential Distribution

Consider another example, where the **PDF** is given.

,

To be able to generate random variates, we need to first find the **CDF**. We do this by **integrating** the PDF. Keep in mind that the integration must be from to .

From here, we can find .

Now all we need to do is find generate a random number, and use the above formula to calculate from there. Note that it is acceptable to replace with , since the distribution will be unaffected and a small amount of time can be saved.

### Uniform Distribution

,

### Weibull Distribution

,

### Discrete Distributions

Consider that we have a **PMF** instead of a PDF.

Thus, the CDF is

In this case, we first generate a **random number**, . Next, we find the **smallest integer** such that

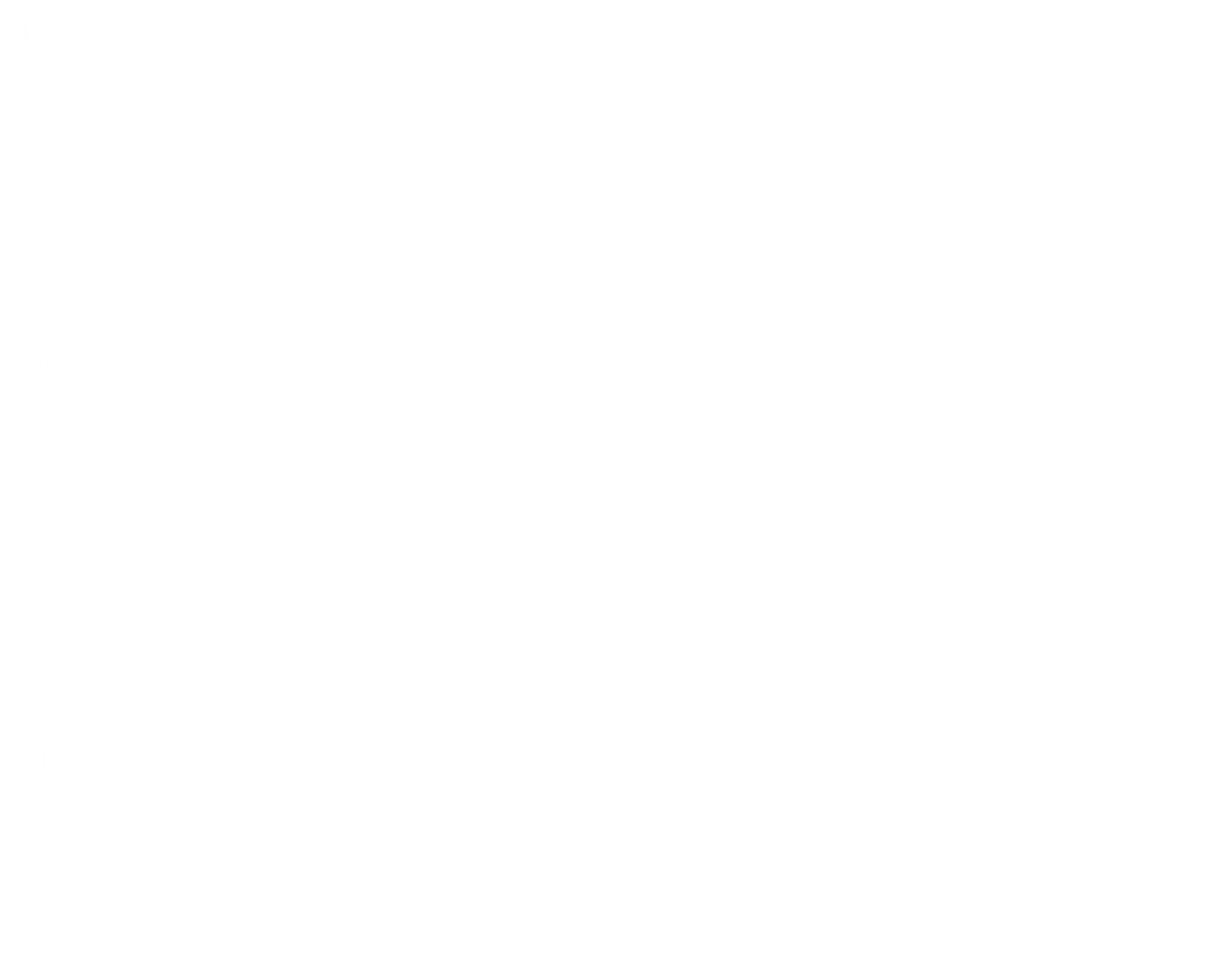
From here, we return .

This should be easier to understand with a diagram.



Example

Say we have a CDF similar to that for a six-sided die.



Every vertical increase is .

If , then must be , since and .

If , then , since and .

To be able to implement this programmatically, we would need to use a series of **if-else statements**.

## Convolution Method

Say we have a **series of random variables**, through , each with their own PDFs and another random variable , which is the **sum** of the previous random variables.

In this case, find a closed form solution for might be very difficult or even impossible. Instead, we can generate values for **each of the**  random variables using the **inverse transform method** and then **sum them** to find .

### -Erlang Distribution

The **-Erlang Distribution** is the sum of IID exponential random variables with **common mean** .

Using the **inverse transform method**

## Acceptance-Rejection Method

In the **Acceptance-Rejection Method**, we generate random variates based on a **different distribution**, , from the one we are provided with, . We choose a **simple distribution** to make things easier for ourselves, for example a uniform distribution.

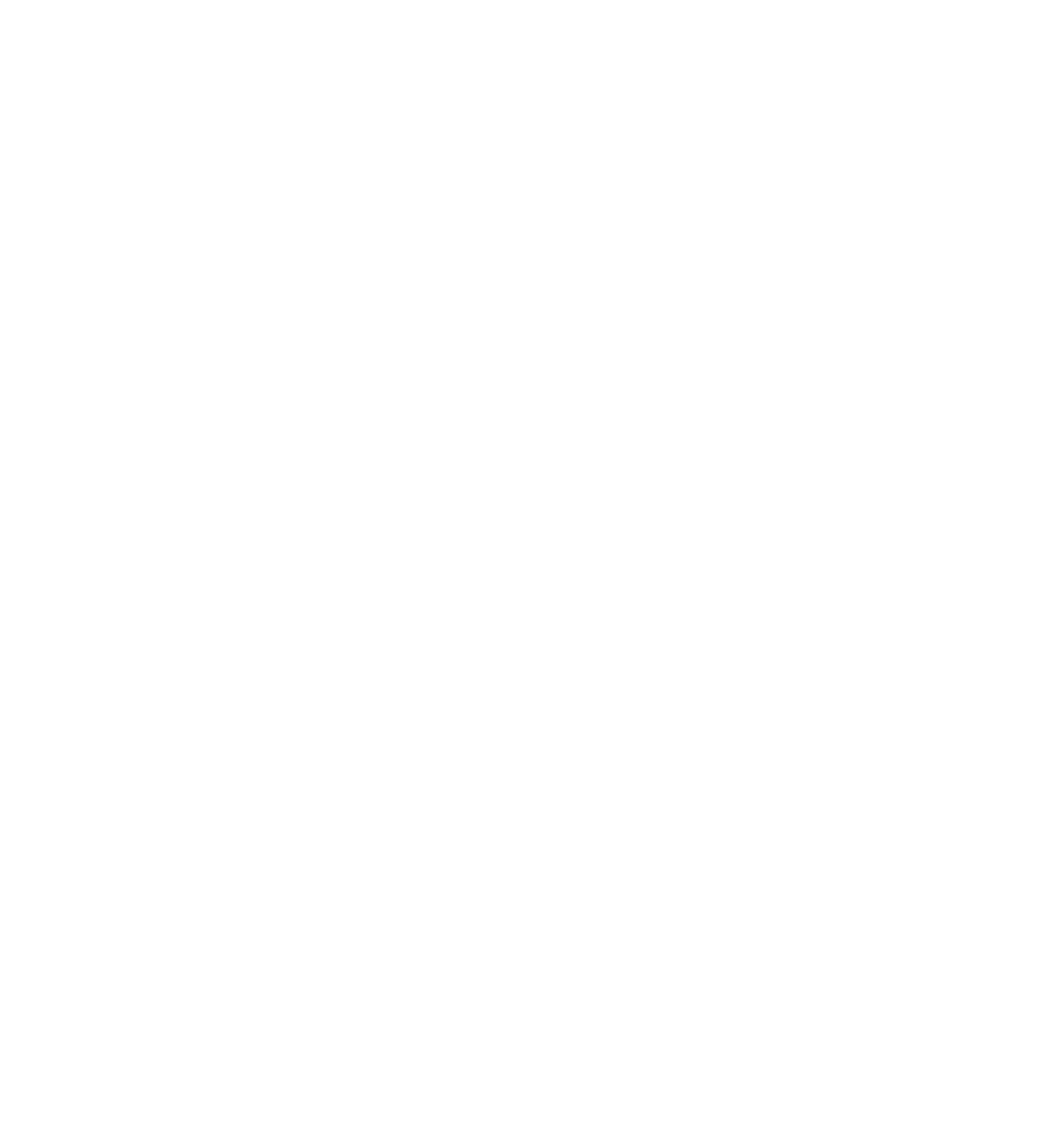
For each value we generate, we simply check if the value matches the provided distribution. If it does, we **accept** it. Otherwise, we **reject** it. Thus, we are using our own distribution and from that distribution, taking values that agree with the given distribution.

For example, say we are given an exponential distribution and we choose to use a uniform distribution instead. Thus, we will accept values towards the start of the range more often than we will accept vales towards the end of the range. In this way, the actual values we get will mimic the exponential distribution.

Note that we need to preserve the **range** of the provided distribution in whatever distribution we choose to use.

In the Acceptance-Rejection Method, we deal with **PDFs**. Thus, if we are given a CDF, we need to convert it to a PDF.

Next, we define a function by ourselves, such that . This is called a **majoring function**, which is a function that has a higher value for all values of . To make out lives easy, we choose the majoring function to be a **uniform distribution**.



**cannot be a PDF**, since the area under and for all values of . The actual area underneath is

We need to find a function, , which **is a PDF**.

We are **dividing by** , since the area under is when it needs to be . Thus, it is times greater.

Next, we can generate a **random variate** with the PDF . We accept if it matches certain conditions and reject it otherwise. This condition is

Here, is another **random number**.

Consider why this works. For the example above, at the very centre of the graph, the values of and will be almost exactly the same. Thus, the ratio will be , meaning whatever we get, we will accept . On the other hand, towards either end, the values of will be far greater than , which means the ratio will be far less. This will make it far less likely that we get a number which is even less than that. Thus, mostly, we will reject those numbers.

Example

For this PDF, the maximum value is . Thus, we can choose the following majoring function:

,

Since ,

Thus, the acceptance conditions are:

if

if

Example

,

To find the majoring function in this case, we can differentiate the function and find the maximum value.

## Composition Method

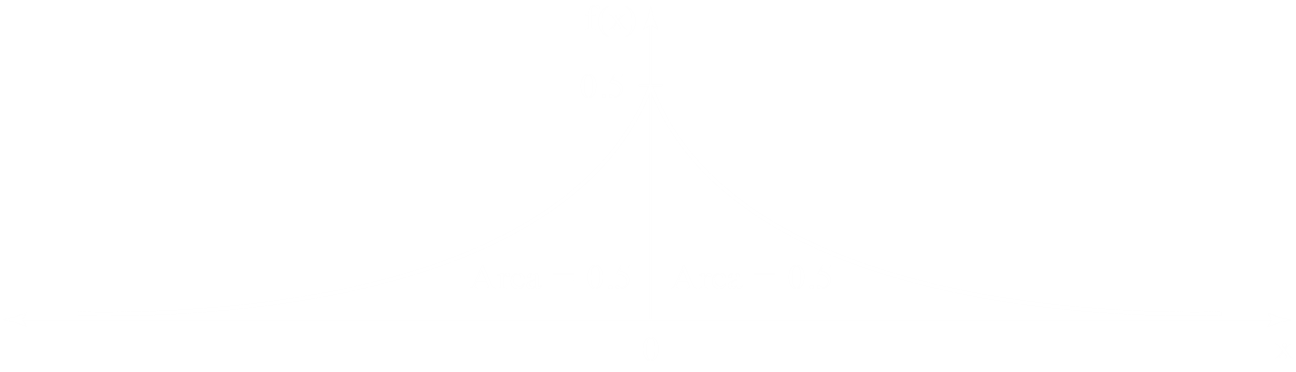
The **Composition Method** is used to represent a complex distribution as a sum of several easier distributions. However, this sum is required to be a **convex combination** (as opposed to a linear combination).

A **linear combination** is the sum of several unknown terms, each multiplied by a scalar value. For example, is a linear combination, where , and are scalar values and , and are the unknowns. The important thing is that two unknowns cannot be multiplied, i.e. or is not valid.

A **convex combination** is like a linear combination, except that the scalar values being used **sum up to** .

In the composition method, we represent the CDF in a way such that it creates a convex combination, i.e. , where and each is a simpler distribution than . We could also use the PDF, as in .

Consider that we have a **double exponential distribution**, also called a **Laplace distribution**.



Here, .

Here, and can be consider as indicator functions, where the range is . Thus,

Notice that this means that the double exponential distribution is a convex combination, with and and and . From here, we can generate a random variate.

1. **Generate** and , **two random numbers**.
2. If , , and thus, .

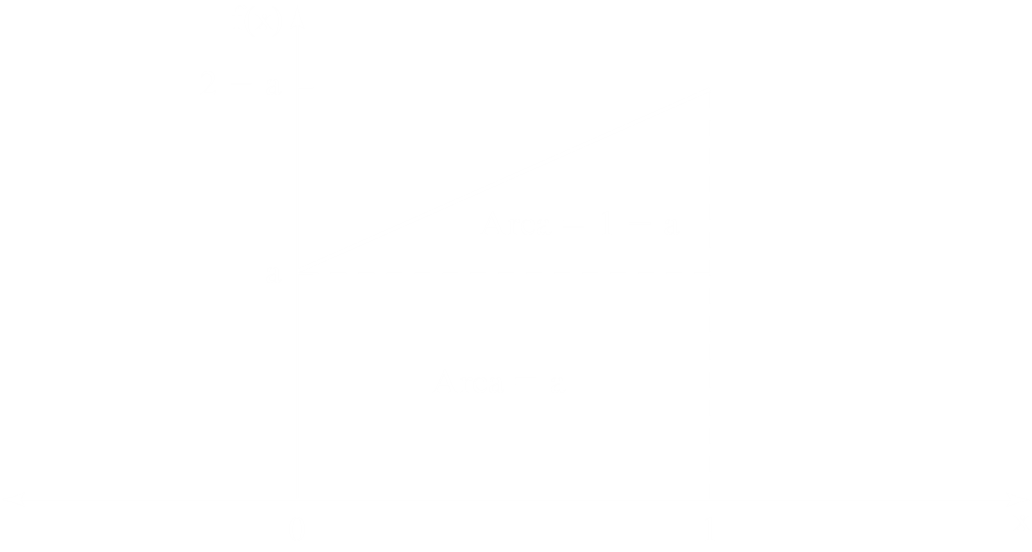
If , , and thus, .

The two possibilities are **equally divided** because and are equally divided. If a different ratio were used, the two equations above would change accordingly.

1. **Return**  for the value of .

The fact that helps us here. If this were not the case, we would have some random values for which the original function would be **invalid** (since would be more than ), which adds complexity.

Now consider that for .



In this scenario, we have a weird shape and dividing it along the  **– axis** does not help us, since we just end up with two smaller weird shapes. Instead, we divide it along the  **– axis**.

Thus,

where and . We can see this by dividing the area along the , with the top half being since the bottom half is . Additionally, and .

Based on the values of and ,

1. Generate and .
2. If , .

If , , or .

1. Return for the value of .