**Directional Derivatives and Gradients**

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We already know how to find the **derivative** of a function, i.e. the **slope** of the function. We also technically know how to find the derivative of a function along a **specific direction**, specifically the -axis and -axis, using **partial derivatives**.

We can actually find the derivative along **any vector**, , not just the and axes. This is called the **directional derivative** along that vector, denoted as .

Thus, if the vector is , then .

The above vector can also be written in **polar form** as , where is the angle with the positive -axis, which will give us as .

If we consider that defines a **surface**, then there is a **plane** that passes through the point and is **parallel** to . This plane **intersects the surface** to create a **curve**, .



The slope of the surface at the point **directly above** , in the direction of is the **slope of the curve** at that point.

The plane that intersects the surface to form also **intersects the plane** along the **line** . This line can be represented by the following **parametric equations**:

These two equations can be used to reach **any point**, , that lies on the line . A corresponding point exists on the curve that is **directly above** , .

The **distance** between and is given by:

From this, we can write that the **slope** of the **secant line** through and is:

Allowing to **approach** , we can say that the **directional derivative** of in the direction of , denoted here as , is given by:

Finding the directional derivative using this approach is like finding the derivative of a single-variable function using the **limit approach**.

A simpler approach is to use **partial derivatives**:

This is what we saw at the beginning.

We can prove that this is true, since setting to be in the direction of the -axis will result in the directional derivative being the same as and setting it to be in the direction of the -axis will result in the directional derivative being the same as .

All these equations can of course be **extended** for more dimensions.

Example

Find the directional derivative of at the point in the direction of .

## Gradient of a Function of Two Variables

The **gradient** of a function of two variables is a **vector-valued function** of two variables.

If is a function such that and exist, the gradient of is denoted as:

Thus, the **gradient** is just the **slope** of a function in **vector** form.

Example

For , the gradient at the point is given by:

Because the gradient of is a **vector**, the **directional derivative** of in the direction can be written as

Thus, the **directional derivative** is the **dot product** of the **gradient** and the **direction vector**.

Example

Find the directional derivative of at in the direction from to .

## Applications of the Gradient

### Properties of the Gradient

1. If , then for all .
2. The direction of maximum increase of is given by . The maximum value of is .
3. The direction of minimum increase of is given by . The minimum value of is .
4. If is differentiable at and , then is normal to the level curve through .