Math 4441: Probability and Statistics

Tutorial 4

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1. If X_1 and X_2 are two independent binomial random variables with respective parameters $(n_1;\ p)$ and $(n_2;\ p)$, calculate the conditional PMF of X_1, given that $X_1 + X_2 = m$.

2. Consider four independent rolls of a 6-sided die. Let X be the number of 1's, and Y be the number of 2's obtained. What is the joint PMF of X and Y?

3. A source wishes to transmit data packets to a receiver over a wireless link. The packet size is _xed, the transmission of a packet requires exactly 1 millisecond. After sending the packet, the source waits 1 millisecond to receive a feedback from the destination. The receiver, in contrast, uses an error detection mechanism to determine whether all the bits of the packets are error-free or not. If the packet is received without error, the receiver sends back an ACK to the sender, which takes exactly 1 millisecond (for both the error detection and the transmission of the ACK). If the source does not receive the ACK within 1 millisecond, it retransmits the packet immediately. Assume the source continues to retransmit the packet until an ACK is received (i.e., the packet is sent successfully). Each transmission is independently corrupted (i.e., at least one bit is changed during trnasmission) with probability q.

   a) Let the random variable X counts the number of times the packet is transmitted by the source. Find the PMF of X.

   b) Let T equal the required time until the packet is successfully received. What is the relationship between T and X? What is the PMF of T?

   c) Assume that after a maximum number of retries, R, the packet is dropped. Now T represents the required time to successfully receive the packets. Find the PMF of T.

4. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second door leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.

   a) Assume that the prisoner will always selects doors 1, 2, and 3 with probabilities 0.5, 0.3, and 0.2, respectively. Find the expected number of days until the prisoner reaches freedom.

   b) Assume that the prisoner is always equally likely to choose among those doors that he has not used previously. Find the expected number of days until he reaches freedom.
   Note: If the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.

5. Roll a 4-sided fair die and let the outcome be X. Then toss a fair coin X times and let Y denote the number of heads.

   a) Find the joint probability mass function (PMF) of X and Y, $P_{XY}(x, y)$.

   b) Find the covariance of X and Y.

6. Each element in a sequence of binary data is either 1 with probability p and 0 with probability $(1 - p)$. A maximal subsequence of consecutive values having identical outcome(s) is called a run. For instance, if the outcome sequence is $1, 1, 0, 1, 1, 1, 0$, the first run is of length 2, the second is of length 1, and the third is of length 3. Find the expected length of the first run.
   [Hint: Assume a random variable that will determine the type of the first run, and then conditioning on that random variable, calculate the expected length.]