Shaum series - Complex Variable - Problem 1.130

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prove that

$$\cos^n(\phi) = \frac{1}{2^{n-1}}\{\cos n\phi + \cos(n-2)\phi + \frac{n(n-1)}{2!}\cos(n-4)\phi + ... + R_n\}$$

where

$$R_n = \begin{cases} \frac{n!}{[\frac{(n-1)}{2}]![\frac{(n+1)}{2}]!}\cos(\phi) & \text{if n is odd} \\ \frac{n!}{2[(\frac{n}{2})!]^2} & \text{if n is even} \end{cases}$$

Proof
We know,

$$\cos(\phi) = \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$$

so

$$\cos^n(\phi) = \frac{1}{2^n}\left(e^{i\phi} + e^{-i\phi}\right)^n$$

$$\begin{aligned}
\cos^n(\phi) &= \frac{1}{2^n}\left(e^{i\phi} + e^{-i\phi}\right)^n \\
&= \frac{1}{2^n}\{{}^nC_0 e^{ni\phi} +^n C_1 e^{(n-1)i\phi}e^{-i\phi} +^n C_2 e^{(n-2)i\phi}e^{-2i\phi} + ... \\
&+^n C_{n-1}e^{\{n-(n-1)\}i\phi}e^{-(n-1)i\phi} +^n C_n e^{i\phi}e^{-ni\phi}\} \\
&= \frac{1}{2^n}\{e^{ni\phi} + ne^{(n-2)i\phi} + \frac{n(n-1)}{2!}e^{(n-4)i\phi} + ... + ne^{-(n-2)i\phi} + e^{-in\phi}\} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{(n-2k)i\phi}
\end{aligned}$$

so

$$\cos^n(\phi) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{(n-2k)i\phi}....................(1)$$

from law of combination we know,

$${}^nC_r =^n C_{n-r}$$

so

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{(n-2k)i\phi} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{n-k}e^{(2k-n)i\phi}$$

which gives

$$
\begin{aligned}
\cos^n(\phi) &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{n-k}e^{(2k-n)i\phi} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{n-k}e^{-(n-2k)i\phi} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{-(n-2k)i\phi}
\end{aligned}
$$

$\frac{(1)+(2)}{2}$ gives

$$
\begin{aligned}
\cos^n(\phi) &= \frac{1}{2^n}\{\sum_{k=0}^{n}\frac{1}{2}\left(\binom{n}{k}e^{(n-2k)i\phi}+\sum_{k=0}^{n}\binom{n}{k}e^{-(n-2k)i\phi}\}\right) \\
&= \frac{1}{2^n}\{\sum_{k=0}^{n}\binom{n}{k}\cos(n-2k)\phi\} \\
&= \frac{1}{2^n}\{\cos(n\phi)+n\cos(n-2)\phi+\frac{n(n-1)}{2!}\cos(n-4)\phi+........+n\cos(n-2)\phi+\cos(n\phi)\} \\
&\qquad ....(3)
\end{aligned}
$$

if n is odd there will be n+1 even terms

Table 1: odd relationship

0	1	2	.....	$\frac{n-1}{2}$
n	n-1	n-2	...	$\frac{n+1}{2}$

from..(3) we get

$$
\begin{aligned}
\cos^n(\phi) = &\frac{1}{2^n}\{\cos(n\phi)+n\cos(n-2)\phi+\frac{n(n-1)}{2!}\cos(n-4)\phi+....+ \\
&\binom{n}{\frac{n-1}{2}}\cos\{n-2\frac{(n-1)}{2}\}+\binom{n}{\frac{n+1}{2}}\cos\{n-2\frac{(n+1)}{2}\}+ \\
&.......+n\cos(n-2)\phi+\cos(n\phi)\}.........................(4)
\end{aligned}
$$

Now

$$
\binom{n}{(\frac{n-1}{2})} = \binom{n}{(\frac{n+1}{2})} = \frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!}
$$

from ...(4) we get

$$
\cos^n(\phi) = \frac{1}{2^{n-1}}\{\cos(n\phi)+n\cos(n-2)\phi+\frac{n(n-1)}{2}\cos(n-4)\phi+...............+\frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!}\cos(\phi)\}
$$

if n is even there will be n+1 odd terms

Table 2: even relationship

0	1	2	.....	$\frac{n}{2}-1$	$\frac{n}{2}$
n	n-1	n-2	...	$\frac{n}{2}+1$	

from ....(3) we get

$$\cos^n(\phi) = \frac{1}{2^n}\{\cos(n\phi) + n\cos(n-2)\phi + \frac{n(n-1)}{2!}\cos(n-4)\phi + ...+$$
$$\frac{n!}{[(\frac{n}{2})!]^2} + ... + n\cos(n-2)\phi + \cos(n\phi)\}$$
$$= \frac{1}{2^{n-1}}\{\cos(n\phi) + n\cos(n-2)\phi + \frac{n(n-1)}{2!}\cos(n-4)\phi + ...$$
$$.. + \frac{n!}{[(\frac{n}{2})!]^2}\}$$

[Proved]