Schaum - Complex Variable - Problem 1.143a

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Prove that

$$\cos(\theta) + \cos(\theta + \alpha) + \cos(\theta + 2\alpha) + ... + \cos(\alpha + n\theta) = \frac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\cos(\theta + \frac{1}{2}n\alpha)$$

Solution:

$$\text{We know}\cos(\theta) = \tfrac{1}{2}\bigg(e^{i\theta} + e^{-i\theta}\bigg)$$

$$\begin{aligned}
L.H.S &= \cos(\theta) + \cos(\theta + \alpha) + \cos(\theta + 2\alpha) + ... + \cos(\alpha + n\theta) \\
&= \frac{1}{2}\bigg(e^{i\theta} + e^{-i\theta} + e^{(\theta+\alpha)i} + e^{-(\theta+\alpha)i} + e^{(\theta+2\alpha)i} + e^{-(\theta+2\alpha)i} + ... + e^{(\theta+n\alpha)i} + e^{-(\theta+n\alpha)i}\bigg) \\
&= \frac{1}{2}e^{i\theta}\bigg(1 + e^{\alpha i} + e^{2\alpha i} + ... + e^{n\alpha i}\bigg) + \frac{1}{2}e^{-i\theta}\bigg(1 + e^{-\alpha i} + e^{-2\alpha i} + ... + e^{-n\alpha i}\bigg)
\end{aligned}$$

from geometric series $\sum_{k=0}^{n} ar^n = \frac{a(r^{n+1}-1)}{r-1}$

So

$$\begin{aligned}
L.H.S &= \frac{e^{i\theta}}{2}\bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\} + \frac{e^{-i\theta}}{2}\bigg\{\frac{1 - e^{-\alpha(n+1)i}}{1 - e^{-\alpha i}}\bigg\} \\
&= \frac{e^{i\theta}}{2}\bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\} + \frac{e^{-i\theta}}{2}\bigg\{\frac{1 - \frac{1}{e^{\alpha(n+1)i}}}{1 - \frac{1}{e^{\alpha i}}}\bigg\} \\
&= \frac{e^{i\theta}}{2}\bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\} + \frac{e^{-i\theta}}{2}\frac{e^{\alpha i}}{e^{\alpha(n+1)i}}\bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\} \\
&= \bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\}\bigg\{\frac{e^{i\theta}}{2} + \frac{1}{2e^{i\theta}}\frac{1}{e^{n\alpha}}\bigg\} \\
&= \bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\}\bigg\{\frac{e^{i\theta}}{2} + \frac{1}{2e^{(\theta+\alpha n)i}}\bigg\}..........(1)
\end{aligned}$$

Now

$$\begin{aligned}
\bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\} &= \bigg\{\frac{e^{\alpha(n+1)i} - 1}{e^{\alpha i} - 1}\bigg\}\bigg\{\frac{e^{-\alpha i} - 1}{e^{-\alpha i} - 1}\bigg\} \\
&= \frac{(e^{\alpha(n+1)i} - 1)(e^{-\alpha i} - 1)}{1 + 1 - e^{\alpha i} - e^{-\alpha i}} \\
&= \frac{e^{\alpha n i} - e^{-\alpha i} + 1 - e^{\alpha(n+1)i}}{2 - 2\cos\alpha}...........(2)
\end{aligned}$$

Also

$$\left\{\frac{e^{i\theta}}{2}+\frac{1}{2e^{(\alpha n+\theta)i}}\right\}=\frac{e^{(\alpha n+2\theta)i}+1}{2e^{(\alpha n+\theta)i}}$$
$$=\frac{(e^{(\alpha n+2\theta)i}+1)\cdot e^{-(\alpha n+\theta)i}}{2e^{(\alpha n+\theta)i}\cdot e^{-(\alpha n+\theta)i}}$$
$$=\frac{e^{i\theta}+e^{-(\alpha n+\theta)}}{2}\text{..........(3)}$$

Putting (2) and (3) into (1) we get

$$L.H.S=\frac{e^{(\alpha n+\theta)i}+e^{-(\alpha n+\theta)i}-e^{(\alpha-\theta)}-e^{-(\alpha-\theta)}+e^{i\theta}+e^{-i\theta}-e^{\{\alpha(n+1)+\theta\}i}-e^{-\{\alpha(n+1)+\theta\}i}}{2(2-2\cos\alpha)}$$
$$=\frac{\cos(\alpha n+\theta)-cos(\theta-\alpha)+cos(\theta)-cos\{\alpha(n+1)+\theta\}}{2-2\cos\alpha}$$
$$=\frac{2\sin(\frac{\alpha}{2})\sin(\alpha n+\theta+\frac{\alpha}{2})-2\sin(\theta-\frac{\alpha}{2})\sin(\frac{\alpha}{2})}{2-2cos\alpha}$$
$$=\frac{\sin(\alpha n+\theta+\frac{\alpha}{2})-\sin(\theta-\frac{\alpha}{2})}{2\sin(\frac{\alpha}{2})}$$
$$=\frac{\cos(\theta+\frac{\alpha n}{2})\sin(\frac{1}{2}(n+1)\alpha)}{\sin(\frac{\alpha}{2})}$$

[Proved]