Numerical Differentiation

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1 Introduction:

What is Numerical Differentiation? The Differential Equations we solved in our courses are solvable by analytical method $y = F(x)$, but in case practical application most of then can not be solved analytically. Thus we try to solve them numerically. There are two approaches to do such task -

1. predictor-corrector method
2. Runge-Kutta Method

In this we will discuss about Runge-kutta Method

2 Runge-Kutta Method:

This method is about doing linear approximation of slopes calculated within h_n than do eular method thus overcoming the error caused by generic euler method.

The solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1}$$

is to be defined by

$$y_{n+1} = y_n + \sum_{i=1}^{m}(w_i k_i) \tag{2}$$

where w_i are weighted values of slopes and

$$k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j) \tag{3}$$

are slope values.

We will have to find the values of α_i, β_{ij}, h_n and w_i to compute the solution. Initially $\alpha_1 = 0, h_n = x_{n+1} - x_n$

Expanding $F(x_{n+1})$,

$$F(x_{n+1}) = F(x_n) + F'(x_n)(x_{n+1} - x_n) + \frac{F''(x_n)(x_{n+1} - x_n)^2}{2!}..........$$

$$
\begin{aligned}
y_{n+1} - y_n &= F(x_{n+1}) - F(x_n) \\
&= (x_{n+1} - x_n)F'(x_n) + (x_{n+1} - x_n)^2 \frac{F''(x_n)}{2!} + ... \\
&= h_n F'(x_n) + (h_n)^2 \frac{F''(x_n)}{2!} + ...(h_n = x_{n+1} - x_n) \\
y_{n+1} - y_n &= h_n F'(x_n) + (h_n)^2 \frac{F''(x_n)}{2!} + (h_n)^3 \frac{F'''(x_n)}{3!} + ...
\end{aligned}
$$

$$
y_{n+1} - y_n = \sum_{t=1}^{\infty} (h_n)^t \frac{y_n^{(t)}}{t!} \tag{4}
$$

In addition, $y_n' = f(x_n, y_n)$, Assuming differentiability, it follows that

$$
y_n^t = \frac{\mathrm{d}^{t-1}}{\mathrm{dx}^{t-1}} = (\frac{\mathrm{d}}{\mathrm{dx}} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\mathrm{d}}{\mathrm{dy}})^{t-1} f(x_n, y_n) \tag{5}
$$

it can be shown that,

$$
\begin{aligned}
\frac{\mathrm{df}}{\mathrm{dx}} = \frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial f}{\partial y} &= (\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial}{\partial y}) f ...... \text{ where} \\
&(\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial}{\partial y}) \text{ is a differential operator}
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{\mathrm{d^2f}}{\mathrm{dx^2}} &= \frac{\mathrm{d}}{\mathrm{dx}} \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial f}{\partial y} \right) \\
&= \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial f}{\partial y} \right)_x + f \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial f}{\partial y} \right)_y \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial f}{\partial y} \right) + f \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial f}{\partial y} \right) = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right) f \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) + f \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \\
&= \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial}{\partial y} \left(f \frac{\partial f}{\partial y} \right) \\
&= \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} + f \left(f \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \right) \\
&= \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2} + f \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \\
&= (\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}} \frac{\partial}{\partial y})^2 f
\end{aligned}
$$

Defining

$$D = \left(\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y}\right)\Big|_n = \left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right)$$

$$\begin{aligned} D^2 &= \left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right) \\ &= \left(\frac{\partial^2}{\partial x^2} + 2f_n\frac{\partial^2}{\partial x\partial y} + f_n^2\frac{\partial^2}{\partial y^2}\right)\Big|_n \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\mathrm{d^2f}}{\mathrm{dx^2}}\Big|_n &= \left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right)\Big|_n \\ &= \left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right)\Big|_n + \frac{\partial f}{\partial y}\left(\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y}\right)\Big|_n \\ &= D^2 f + f_y Df\Big|_n \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d^3f}}{\mathrm{dx^3}} =& \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)^3 \\ =& \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right) \\ =& \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2}\right) + \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(2f\frac{\partial^2 f}{\partial x\partial y}\right) + \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(f^2\frac{\partial^2 f}{\partial y^2}\right) \\ &+ \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right) + \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right) \\ =& \underline{\tfrac{\partial^3 f}{\partial x^3} + f\tfrac{\partial^3 f}{\partial x^2\partial y}} + 2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + \underline{2f\tfrac{\partial^3 f}{\partial x^2\partial y}} + \underline{2f\tfrac{\partial^2 f}{\partial x\partial y}\tfrac{\partial f}{\partial y}} + \underline{2f^2\tfrac{\partial^3 f}{\partial x\partial y^2} + f^2\tfrac{\partial^3 f}{\partial x\partial y^2}} \\ &+ 2f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + \underline{f^2\tfrac{\partial f}{\partial y}\tfrac{\partial^2 f}{\partial y^2}} + f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + \underline{f^3\tfrac{\partial^3 f}{\partial y^3}} + \underline{\tfrac{\partial^2 f}{\partial x^2}\tfrac{\partial f}{\partial y}} + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} \\ &+ \underline{\tfrac{\partial f}{\partial x}\left(\tfrac{\partial f}{\partial y}\right)^2} + 2f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} + \underline{f\left(\tfrac{\partial f}{\partial y}\right)^3} + 2f^2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} \\ =& \underline{\tfrac{\partial^3 f}{\partial x^3} + 3f\tfrac{\partial^3 f}{\partial x^2\partial y} + 3f^2\tfrac{\partial^3 f}{\partial x\partial y^2} + f^3\tfrac{\partial^3 f}{\partial y^3}} + 2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + \underline{2f\tfrac{\partial^2 f}{\partial x\partial y}\tfrac{\partial f}{\partial y}} + 2f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} \\ &+ \underline{f^2\tfrac{\partial f}{\partial y}\tfrac{\partial^2 f}{\partial y^2}} + f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + \underline{\tfrac{\partial^2 f}{\partial x^2}\tfrac{\partial f}{\partial y}} + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} + \underline{\tfrac{\partial f}{\partial x}\left(\tfrac{\partial f}{\partial y}\right)^2} \\ &+ 2f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} + \underline{f\left(\tfrac{\partial f}{\partial y}\right)^3} + 2f^2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} \\ =& \underline{\tfrac{\partial^3 f}{\partial x^3} + 3f\tfrac{\partial^3 f}{\partial x^2\partial y} + 3f^2\tfrac{\partial^3 f}{\partial x\partial y^2} + f^3\tfrac{\partial^3 f}{\partial y^3}} + \underline{\tfrac{\partial f}{\partial y}\left(\tfrac{\partial^2 f}{\partial x^2} + 2f\tfrac{\partial^2 f}{\partial x\partial y} + f^2\tfrac{\partial^2 f}{\partial y^2}\right)} + \underline{\left(\tfrac{\partial f}{\partial y}\right)^2\left(\tfrac{\partial f}{\partial x} + f\tfrac{\partial f}{\partial y}\right)} \\ &+ 2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + 2f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} \\ &+ 2f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} + 2f^2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d^3f}}{\mathrm{dx^3}} =& \frac{\partial^3 f}{\partial x^3} + 3f\frac{\partial^3 f}{\partial x^2\partial y} + 3f^2\frac{\partial^3 f}{\partial x\partial y^2} + f^3\frac{\partial^3 f}{\partial y^3} + \frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\
&+3\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + 3f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + 3f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + 3f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} \\
=& \frac{\partial^3 f}{\partial x^3} + 3f\frac{\partial^3 f}{\partial x^2\partial y} + 3f^2\frac{\partial^3 f}{\partial x\partial y^2} + f^3\frac{\partial^3 f}{\partial y^3} + \frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\
&+3\Big(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} + f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}\Big) \\
=& \frac{\partial^3 f}{\partial x^3} + 3f\frac{\partial^3 f}{\partial x^2\partial y} + 3f^2\frac{\partial^3 f}{\partial x\partial y^2} + f^3\frac{\partial^3 f}{\partial y^3} + \frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\
&+3\Big(\frac{\partial^2 f}{\partial x\partial y}\Big\{\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\Big\} + f\frac{\partial^2 f}{\partial y^2}\Big\{\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\Big\}\Big) \\
=& \frac{\partial^3 f}{\partial x^3} + 3f\frac{\partial^3 f}{\partial x^2\partial y} + 3f^2\frac{\partial^3 f}{\partial x\partial y^2} + f^3\frac{\partial^3 f}{\partial y^3} + \frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\
&+3\Big(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\Big)\Big(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\Big)\frac{\partial f}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
D^3 =& \left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2} + 2f_n\frac{\partial^2 f}{\partial x\partial y} + f_n^2\frac{\partial^2 f}{\partial x^2}\right) \\
=& \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2} + 2f_n\frac{\partial^2 f}{\partial x\partial y} + f_n^2\frac{\partial^2 f}{\partial y^2}\right) + f_n\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f_n\frac{\partial^2 f}{\partial x\partial y} + f_n^2\frac{\partial^2 f}{\partial y^2}\right) \\
=& \frac{\partial^3 f}{\partial x^3} + 2f_n\frac{\partial^3 f}{\partial x^2\partial y} + f_n^2\frac{\partial^3 f}{\partial x\partial y^2} + f_n\frac{\partial^3 f}{\partial x^2\partial y} + 2f_n^2\frac{\partial^3 f}{\partial x\partial y^2} + f_n^3\frac{\partial^3 f}{\partial y^3} \\
=& \frac{\partial^3 f}{\partial x^3} + 3f_n\frac{\partial^3 f}{\partial x^2\partial y} + 3f_n^2\frac{\partial^3 f}{\partial x\partial y^2} + f_n^3\frac{\partial^3 f}{\partial y^3}
\end{aligned}
$$

$$
\frac{\mathrm{d^3f}}{\mathrm{dx^3}}\Big|_n = D^3 f + f_y D^2 f + f_y^2 Df + 3Df\,Df_y\Big|_n
$$

putting equation 4 into 5

$$
\begin{aligned}
y_{n+1} - y_n &= \sum_{t=1}^{\infty}(h_n)^t\frac{y_n^{(t)}}{t!} \\
&= \sum_{t=1}^{\infty} h_n^{t+1}\cdot\frac{(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y})^t}{(t+1)!}\cdot f(x_n, y_n) \\
&= \Bigg[hf + \frac{h^2}{2!}Df + \frac{h^3}{3!}\Big(D^2 f + f_y Df\Big) \\
&+ \frac{h^4}{4!}\Big(D^3 f + f_y D^2 f + f_y^2 Df + 3Df\,Df_y\Big) \\
&+ \frac{h^5}{5!}\Big(D^4 f + 6Df\,D^2 f_y + f_y^2 D^2 f + f_y^3 Df \\
&+ 3f_{yy}(Df)^2 + f_y D^3 f + 7f_y Df\,Df_y\Big)\Bigg] + O(h^6)
\end{aligned} \tag{7}
$$

The expansion of equation 2 can be done using taylor series expansion

$$
\begin{aligned}
f(x_n+\alpha_i h_n, y_n+\sum_{j=1}^{i-1}\beta_{ij}k_j) =& f(x_n,y_n)+\left[\alpha_i h_n\frac{\partial}{\partial y}+\sum(\beta_{ij})h_n f_n\frac{\partial}{\partial y}\right]f(x_n,y_n)\\
&+\frac{1}{2!}\left[\alpha_i h_n\frac{\partial}{\partial y}+\sum(\beta_{ij})h_n f_n\frac{\partial}{\partial y}\right]^2 f(x_n,y_n)\\
&+\frac{1}{3!}\left[\alpha_i h_n\frac{\partial}{\partial y}+\sum(\beta_{ij})h_n f_n\frac{\partial}{\partial y}\right]^3 f(x_n,y_n)\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\frac{1}{n!}\left[\alpha_i h_n\frac{\partial}{\partial y}+\sum(\beta_{ij})h_n f_n\frac{\partial}{\partial y}\right]^n f(x_n,y_n)
\end{aligned}
$$

$$
f(x_n+\alpha_i h_n, y_n+\sum_{j=1}^{i-1}\beta_{ij}k_j)=\sum_{t=0}^{\infty}\frac{\left(\alpha_i h_n\frac{\partial}{\partial x}+\sum(\beta_{ij})h_n f_n\frac{\partial}{\partial y}\right)^t}{t!}f(x_n,y_n) \tag{8}
$$

Let

$$
D_i=\alpha_i\frac{\partial}{\partial x}+\sum(\beta_{ij})f_n\frac{\partial}{\partial y}
$$

$$
f(x_n+\alpha_i h_n, y_n+\sum_{j=1}^{i-1}\beta_{ij}k_j)=\sum_{t=0}^{\infty}\frac{h_n^t D_i^t}{t!}f(x_n,y_n)
$$

Since $\alpha_1=0$

$$
\begin{aligned}
k_1 =& h_n f(x_n,y_n)\\
k_2 =& h_n f(x_n+\alpha_2 h_n, y_n+\beta_{21}k_1)=h_n f(x_n+\alpha_2 h_n, y_n+\beta_{21}h_n f_n)\\
=& h_n\sum_{t=0}^{\infty}\frac{h_n^t D_2^t}{t!}f(x_n,y_n)\\
k_3 =& h_n f(x_n+\alpha_3 h_n, y_n+\beta_{31}k_1+\beta_{32}k_2)\\
=& h_n f\left[x_n+\alpha_2 h_n, y_n+h_n f_n(\beta_{31}+\beta_{32})+\beta_{32}(k_2-h_n f_n)\right]\\
=& \sum_{t=0}^{\infty}\frac{\left(\alpha_i h_n\frac{\partial}{\partial x}+(\beta_{31}+\beta_{32})h_n f_n\frac{\partial}{\partial y}+\beta_{32}(k_2-h_n f_n)\frac{\partial}{\partial y}\right)^t}{t!}f(x_n,y_n)\\
=& h_n\sum_{t=0}^{\infty}\frac{\left(h_n D_3+\beta_{32}(k_2-h_n f_n)\frac{\partial}{\partial y}\right)^t}{t!}f(x_n,y_n) \qquad (9)\\
k_4 =& h_n f(x_n+\alpha_4 h_n, y_n+\beta_{41}k_1+\beta_{42}k_2+\beta_{43}k_3)\\
=& h_n f\Big[x_n+\alpha_2 h_n, y_n+h_n f_n(\beta_{41}+\beta_{42}+\beta_{43})+\beta_{41}(k_1-h_n f_n)+\beta_{42}(k_2-h_n f_n)\\
&+\beta_{43}(k_3-h_n f_n)\Big]\\
=& h_n f\left[x_n+\alpha_4 h_n, y_n+(\sum_{j=1}^{3}\beta_{4j})h_n f_n+(\sum_{j=2}^{3}\beta_{4j})(k_j-h_n f_n)\right]\\
=& h_n\sum_{t=0}^{\infty}\frac{\left(h_n D_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_n f_n)\frac{\partial}{\partial y}\right)^t}{t!}f(x_n,y_n)
\end{aligned}
$$

$$
\begin{aligned}
k_i &= h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j) \\
&= h_n f\Bigg[x_n + \alpha_4 h_n, y_n + (\sum_{j=1}^{i-1} \beta_{ij}) h_n f_n + \Bigg(\sum_{j=2}^{i-1} \beta_{ij}(k_j - h_n f_n)\Bigg)\Bigg] \\
&= h_n \sum_{t=0}^{\infty} \frac{\Bigg(h_n D_i + \sum_{j=2}^{i-1} \beta_{ij}(k_j - h_n f_n)\frac{\partial}{\partial y}\Bigg)^t}{t!} f(x_n, y_n)
\end{aligned}
$$

$$
\begin{aligned}
k_1 =& hf(x,y)\Big|_n \\
k_2 =& h\Bigg(\frac{h^0 D_2^0}{0!} f + \frac{h^1 D_2^1}{1!} f + \frac{h^2 D_2^2}{2!} f + \frac{h^3 D_2^3}{3!} f + \frac{h^4 D_2^4}{4!} f\Bigg)\Bigg|_n + O(h^6) \\
=& hf + \frac{h^2}{1!} D_2 f + \frac{h^3}{2!} D_2^2 f + \frac{h^4}{3!} D_2^3 + \frac{h^5}{4!} D_2^4 f\Bigg|_n + O(h^6)
\end{aligned}
$$

$$
(k_2 - hf) \; = h^2 D_2 f + \frac{h^3}{2!} D_2^2 f + \frac{h^4}{3!} D_2^3 f + \frac{h^5}{4!} D_2^4 f + O(h^6)
$$

$$
\begin{aligned}
h\left(hD_3+\beta_{32}(k_2-hf)\frac{\partial}{\partial y}\right)^2 f =& h\left[hD_3+\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f+\frac{h^4}{3!}D_2^3f+\frac{h^5}{4!}D_2^4f\right)\frac{\partial}{\partial y}\right]^2 f\\
=& h\left[hD_3+\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f+\frac{h^4}{3!}D_2^3f+\frac{h^5}{4!}D_2^4f\right)\frac{\partial}{\partial y}\right]\\
&\left[hD_3f+\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f+\frac{h^4}{3!}D_2^3f+\frac{h^5}{4!}D_2^4f\right)f_y\right]\\
=& h\left(hD_3\left[hD_3f+\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f+\frac{h^4}{3!}D_2^3f+\frac{h^5}{4!}D_2^4f\right)f_y\right]\right)+\\
&h\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)\frac{\partial}{\partial y}\left[hD_3f+f_y\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)\right]\\
=& h^3D_3^2f+h^4\beta_{32}D_3(D_2f\cdot f_y)+\frac{h^5}{2}\beta_{32}D_3(D_2^2f\cdot f_y)\\
&+h\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)\frac{\partial}{\partial y}\left[hD_3f+f_y\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)\right]\\
=& h^3D_3^2f+h^4\beta_{32}D_3(D_2f\cdot f_y)+\frac{h^5}{2}\beta_{32}D_3(D_2^2f\cdot f_y)\\
&+h\beta_{32}\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)hD_3f_y\\
&+h\beta_{32}^2\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)\frac{\partial}{\partial y}\left\{\left(\frac{h^2}{1!}D_2f+\frac{h^3}{2!}D_2^2f\right)f_y\right\}\\
=& h^3D_3^2f+h^4\beta_{32}D_3(f_yD_2f)+\frac{h^5}{2}\beta_{32}D_3(f_yD_2^2f)+h^4\beta_{32}D_3f_yD_2f+\frac{h^5}{2}\beta_{32}D_3f_yD_2^2f\\
&+h^5\beta_{32}^2D_2f\frac{\partial}{\partial y}\left(f_yD_2f\right)\\
=& h^3D_3^2f+h^4\beta_{32}D_3(f_yD_2f)+h^4\beta_{32}D_3f_yD_2f+\frac{h^5}{2}\beta_{32}D_3(f_yD_2^2f)+\frac{h^5}{2}\beta_{32}D_3f_yD_2^2f\\
&+h^5\beta_{32}^2f_{yy}(D_2f)^2\\
=& h^3D_3^2f+2h^4\beta_{32}D_3(f_yD_2f)+h^5\beta_{32}D_3(f_yD_2^2f)+h^5\beta_{32}^2f_{yy}(D_2f)^2
\end{aligned}
$$

$$
\begin{aligned}
h\left(hD_3+\beta_{32}(k_2-hf)\frac{\partial}{\partial y}\right)^3 f =& \left(hD_3+\beta_{32}(k_2-hf)\frac{\partial}{\partial y}\right)\left(h^3D_3^2f+2h^4\beta_{32}D_3(f_yD_2f)\right)+O(h^6)\\
=& \left(hD_3+\beta_{32}h^2D_2f\frac{\partial}{\partial y}\right)\left(h^3D_3^2f+2h^4\beta_{32}D_3(f_yD_2f)\right)+O(h^6)\\
=& h^4D_3^3f+2h^5\beta_{32}D_3^2f_yD_2f+h^5\beta_{32}D_3^2f_yD_2f+O(h^6)\\
=& h^4D_3^3f+3h^5\beta_{32}D_3^2f_yD_2f+O(h^6)
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= h\sum_{t=0}^{\infty}\frac{\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)^t}{t!}f(x,y)\\
&= h\Bigg(f+hD_3f+\beta_{32}(k_2-hf)f_y+\frac{\left(hD_3+\beta_{32}(k_2-hf)\frac{\partial}{\partial y}\right)^2}{2!}f+\frac{\left(hD_3+\beta_{32}(k_2-hf)\frac{\partial}{\partial y}\right)^3}{3!}f\\
&+\frac{\left(hD_3+\beta_{32}(k_2-hf)\frac{\partial}{\partial y}\right)^4}{4!}f\Bigg)+O(h^6)\\
&= hf+h^2D_3f+h\beta_{32}\left(h^2D_2f+\frac{h^3}{2!}D_2^2f+\frac{h^4}{3!}D_2^3f\right)f_y+\frac{h^3}{2}D_3^2f+h^4\beta_{32}D_3f_yD_2f+\frac{h^5}{2}\beta_{32}D_3f_yD_2^2f\\
&+\frac{h^5}{2}\beta_{32}^2f_{yy}(D_2f)^2+\frac{h^4}{6}D_3^3f+\frac{h^5}{2}\beta_{32}D_3^2f_yD_2f+\frac{h^5}{24}D_3^4f\\
&= hf+h^2D_3f+h^3f_y\beta_{32}D_2f+\frac{h^4}{2}f_y\beta_{32}D_2^2f+\frac{h^5}{6}f_y\beta_{32}D_2^3f+\frac{h^3}{2}D_3^2f+h^4\beta_{32}D_3(f_yD_2f)+\frac{h^5}{2}\beta_{32}D_3f_yD_2^2f\\
&+\frac{h^5}{2}\beta_{32}^2f_{yy}(D_2f)^2+\frac{h^4}{6}D_3^3f+\frac{h^5}{2}\beta_{32}D_3^2f_yD_2f+\frac{h^5}{24}D_3^4f\\
&= hf+h^2D_3f+h^3\left(\frac{1}{2}D_3^2f+f_y\beta_{32}D_2f\right)+h^4\left(\frac{1}{6}D_3^3f+\frac{1}{2}f_y\beta_{32}D_2^2f+\beta_{32}D_3f_yD_2f\right)\\
&+h^5\left(\frac{1}{24}D_3^4+\frac{\beta_{32}}{6}f_yD_2^3f+\frac{\beta_{32}^2}{2}f_{yy}(D_2f)^2+\frac{\beta_{32}}{2}D_3f_yD_2^2f+\frac{\beta_{32}}{2}D_3^2f_yD_2^2f\right)+O(h^6)
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
k_4 =&hf+h^2D_4f+h^3\left(\frac{1}{2}D_4^2f+\beta_{42}f_yD_2f+\beta_{43}f_yD_3f\right)+h^4\Bigg[\frac{1}{6}D_4^3f+\frac{1}{2}\beta_{42}f_yD_2^2f\\
&+\beta_{32}\beta_{43}(f_y)^2D_2f+\frac{1}{2}\beta_{43}f_yD_3^2f+\beta_{42}D_2fD_4f_y+\beta_{43}D_3fD_4f_y\Bigg]+h^5\Bigg(\frac{1}{24}D_4^4f+\frac{1}{6}\beta_{42}f_yD_2^3f\\
&+\frac{1}{2}\beta_{32}\beta_{43}(f_y)^2D_2^2f+\beta_{32}\beta_{43}f_yD_2fD_3f_y+\frac{1}{6}\beta_{43}f_yD_3^3f+\frac{1}{2}\beta_{42}D_4f_yD_2^2f+\frac{1}{2}\beta_{43}D_4f_yD_3^2f\\
&+\frac{1}{2}\beta_{42}^2f_{yy}D_2^2f+\beta_{42}\beta_{43}f_{yy}D_2fD_3f+\frac{1}{2}\beta_{43}^2f_{yy}D_3^2f+\frac{1}{2}\beta_{42}D_2fD_4^2f_y\\
&+\frac{1}{2}\beta_{43}D_3fD_4^2f_y+\beta_{43}\beta_{32}f_yD_2fD_4f_y\Bigg)\Bigg|_n+O(h^6)
\end{aligned}
$$