Prove that for any integer $m \leq 1$

$$(z+a)^{2m} - (z-a)^{2m} = 4maz \prod_{k=1}^{m-1} \{z^2 + a^2 cot(\frac{k\pi}{2m})\}$$

Solution

From roots of $(z+a)^{2m} = (z-a)^{2m}$

$$\left(\frac{z+a}{z-a}\right)^{2m} = 1$$

Next

$$(\frac{z+a}{z-a}) = e^{(\frac{2\pi k}{2m})i}$$
$$\Rightarrow (\frac{z}{a}) = \frac{e^{\frac{\pi k}{m}}+1}{e^{\frac{\pi k}{m}}-1}$$
$$\Rightarrow z = a\frac{e^{\frac{\pi k}{m}}+1}{e^{\frac{\pi k}{m}}-1}$$

Here

$$\begin{aligned} z &= a \cdot \frac{(e^{\frac{\pi k}{m}}+1)(e^{-\frac{\pi k}{m}}-1)}{(e^{\frac{\pi k}{m}}-1)(e^{-\frac{\pi k}{m}}-1)} \\ &= a \cdot \frac{-2i\sin(\frac{\pi k}{m})}{2-2\cos(\frac{\pi k}{m})} \\ &= ia\cot(\frac{\pi k}{2m}) ......k = \pm 1, \pm 2, \pm 3, ..., \pm(m-1) \end{aligned}$$

So

$$\begin{aligned} (z+a)^{2m} - (z-a)^{2m} &= Cz \prod_{k=1}^{m-1} \{z - ia\cot(\frac{\pi k}{2m})\}\{z + ia\cot(\frac{\pi k}{m})\} \\ &= Cz \prod_{k=1}^{m-1} \{z^2 + a^2\cot^2(\frac{\pi k}{2m})\}..................(1) \end{aligned}$$

From binomial expansion

$$\begin{aligned} (z+a)^{2m} - (z-a)^{2m} &= ({}^{2m}C_0 z^{2m} +^{2m} C_1 z^{2m-1}a +^{2m} C_2 z^{2m-2}a^2 + ... +^{2m} C_{2m-1} z a^{2m-1} \\ &+^{2m} C_{2m} a^{2m}) - ({}^{2m}C_0 z^{2m} -^{2m} C_1 z^{2m-1}a +^{2m} C_2 z^{2m-2}a^2 \\ &..... -^{2m} C_{2m-1} z a^{2m-1} +^{2m} C_{2m} a^{2m}) \\ &= 2\left({}^{2m} C_1 z^{2m-1}a +^{2m} C_3 z^{2m-3}a^3 + ... +^{2m} C_{2m-1} z a^{2m-1}\right) \\ &= 2 \cdot 2m \cdot a \cdot z \left(z^{2m-2} + \frac{(2m-1)(2m-2)}{3!} z^{2m-4}a^2 + ... + a^{2m-2}\right).........(2) \end{aligned}$$

From (1) and (2) equating the right side, we get

$$C = 4ma$$

putting C in (1) we get

$$(z+a)^{2m} - (z-a)^{2m} = 4maz\prod_{k=1}^{m-1}\{z^2 + a^2 cot(\frac{k\pi}{2m})\}$$

[Proved]