Numerical Differentiation

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Introduction:

What is Numerical Differentiation? The Differential Equations we solved in our courses are solvable by analytical method $y = F(x)$, but in case practical application most of then can not be solved analytically. Thus we try to solve them numerically. There are two approaches to do such task -

1. Predictor-corrector method
2. Runge-Kutta Method

In this we will discuss about Runge-kutta Method

Runge-Kutta Method:

This method is about doing linear approximation of slopes calculated within $x_{n+1} - x_n = h_n$ such that the y_{n+1} values closely corresponds to the Taylor series approximation at that point thus overcoming the error caused by generic euler method.

The solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1}$$

is to be defined by

$$y_{n+1} = y_n + \sum_{i=1}^{m}(w_i k_i) \tag{2}$$

where w_i are weighted values of slopes and

$$k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j) \tag{3}$$

are slope values.

We will have to find the values of α_i, β_{ij}, h_n and w_i to compute the solution. Initially $\alpha_1 = 0, h_n = x_{n+1} - x_n$

Expanding $F(x_{n+1})$,

$$F(x_{n+1}) = F(x_n) + F'(x_n)(x_{n+1} - x_n) + \frac{F''(x_n)(x_{n+1} - x_n)^2}{2!}..........$$

$$\begin{aligned} y_{n+1} - y_n &= F(x_{n+1}) - F(x_n) \\ &= (x_{n+1} - x_n)F'(x_n) + (x_{n+1} - x_n)^2 \frac{F''(x_n)}{2!} + ... \\ &= h_n F'(x_n) + (h_n)^2 \frac{F''(x_n)}{2!} + ...(h_n = x_{n+1} - x_n) \\ y_{n+1} - y_n &= h_n F'(x_n) + (h_n)^2 \frac{F''(x_n)}{2!} + (h_n)^3 \frac{F'''(x_n)}{3!} + ... \end{aligned} \tag{4}$$

$$y_{n+1} - y_n = \sum_{t=1}^{\infty} (h_n)^t \frac{y_n^{(t)}}{t!} \tag{5}$$

it can be shown that,

$$\frac{\mathrm{df}}{\mathrm{dx}} = \frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y} = (\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y})f ......\text{ where}$$
$$(\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y}) \text{ is a type of differential operator}$$

Again,

$$\begin{aligned} \frac{\mathrm{d^2f}}{\mathrm{dx^2}} &= \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right) \\ &= \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)_x + f\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)_y \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right) + f\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right) = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)f \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) + f\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} + f\frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} + f\left(f\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2} + f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y} \\ &= (\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y})^2 f \end{aligned}$$

and,

$$
\begin{aligned}
\frac{\mathrm{d^3f}}{\mathrm{dx^3}} &= \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)^2 \\
&= \left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)^2_x + f\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)^2_y \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)^2 + f\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^2 f \\
&= \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^3 f
\end{aligned}
$$

In addition, $y'_n = f(x_n, y_n)$, Assuming differentiability, it follows that

$$
\begin{aligned}
y^t_n &= \frac{\mathrm{d}^{t-1}}{\mathrm{dx}^{t-1}}\left\{y'(x_n, y_n)\right\} \\
&= \frac{\mathrm{d}^{t-1}}{\mathrm{dx}^{t-1}} f(x,y)_{|x_n,y_n} \\
&= \left(\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y}\right)^{t-1} f(x_n, y_n)
\end{aligned}
\tag{6}
$$

Defining

$$
D = \left(\frac{\partial}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial}{\partial y}\right)\Big|_{x=x_n, y=y_n} = \left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right)....f_n = f(x,y)|_{x_n,y_n} \text{ is a constant}
$$

$$
\begin{aligned}
D^2 &= \left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right) \\
&= \Big(\frac{\partial^2}{\partial x^2} + 2f_n\frac{\partial^2}{\partial x\partial y} + f_n^2\frac{\partial^2}{\partial y^2}\Big)\Big|_n
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\frac{\mathrm{d^2f}}{\mathrm{dx^2}}\Big|_n &= \Big(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\Big)\Big|_n \\
&= \Big(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + f^2\frac{\partial^2 f}{\partial y^2}\Big)\Big|_n + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{dy}}{\mathrm{dx}}\frac{\partial f}{\partial y}\right)\Big|_n \\
&= D^2 f\Big|_n + f_y D f\Big|_n
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d^3f}}{\mathrm{dx^3}} =& \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial f}{\partial x}+f\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x}+f\frac{\partial f}{\partial y}\right)^3 \\
=&\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2}+2f\frac{\partial^2 f}{\partial x\partial y}+f^2\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}+f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right) \\
=&\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2}\right)+\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)\left(2f\frac{\partial^2 f}{\partial x\partial y}\right)+\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)\left(f^2\frac{\partial^2 f}{\partial y^2}\right) \\
&+\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right)+\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)\left(f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right) \\
=&\underline{\tfrac{\partial^3 f}{\partial x^3}+f\tfrac{\partial^3 f}{\partial x^2\partial y}}+2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+\underline{2f\tfrac{\partial^3 f}{\partial x^2\partial y}}+\underline{2f\tfrac{\partial^2 f}{\partial x\partial y}\tfrac{\partial f}{\partial y}}+\underline{2f^2\tfrac{\partial^3 f}{\partial x\partial y^2}+f^2\tfrac{\partial^3 f}{\partial x\partial y^2}} \\
&+2f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+\underline{f^2\tfrac{\partial f}{\partial y}\tfrac{\partial^2 f}{\partial y^2}}+f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}+\underline{f^3\tfrac{\partial^3 f}{\partial y^3}}+\underline{\tfrac{\partial^2 f}{\partial x^2}\tfrac{\partial f}{\partial y}}+\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} \\
&+\underline{\tfrac{\partial f}{\partial x}\left(\tfrac{\partial f}{\partial y}\right)^2}+2f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}+\underline{f\left(\tfrac{\partial f}{\partial y}\right)^3}+2f^2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} \\
=&\underline{\tfrac{\partial^3 f}{\partial x^3}+3f\tfrac{\partial^3 f}{\partial x^2\partial y}+3f^2\tfrac{\partial^3 f}{\partial x\partial y^2}+f^3\tfrac{\partial^3 f}{\partial y^3}}+2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+\underline{2f\tfrac{\partial^2 f}{\partial x\partial y}\tfrac{\partial f}{\partial y}}+2f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} \\
&+\underline{f^2\tfrac{\partial f}{\partial y}\tfrac{\partial^2 f}{\partial y^2}}+f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}+\underline{\tfrac{\partial^2 f}{\partial x^2}\tfrac{\partial f}{\partial y}}+\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}+\underline{\tfrac{\partial f}{\partial x}\left(\tfrac{\partial f}{\partial y}\right)^2} \\
&+2f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}+\underline{f\left(\tfrac{\partial f}{\partial y}\right)^3}+2f^2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} \\
=&\underline{\tfrac{\partial^3 f}{\partial x^3}+3f\tfrac{\partial^3 f}{\partial x^2\partial y}+3f^2\tfrac{\partial^3 f}{\partial x\partial y^2}+f^3\tfrac{\partial^3 f}{\partial y^3}}+\underline{\tfrac{\partial f}{\partial y}\left(\tfrac{\partial^2 f}{\partial x^2}+2f\tfrac{\partial^2 f}{\partial x\partial y}+f^2\tfrac{\partial^2 f}{\partial y^2}\right)}+\underline{\left(\tfrac{\partial f}{\partial y}\right)^2\left(\tfrac{\partial f}{\partial x}+f\tfrac{\partial f}{\partial y}\right)} \\
&+2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+2f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} \\
&+2f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}+2f^2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} \\
=&\underline{\tfrac{\partial^3 f}{\partial x^3}+3f\tfrac{\partial^3 f}{\partial x^2\partial y}+3f^2\tfrac{\partial^3 f}{\partial x\partial y^2}+f^3\tfrac{\partial^3 f}{\partial y^3}}+\underline{\tfrac{\partial f}{\partial y}\left(\tfrac{\partial^2 f}{\partial x^2}+2f\tfrac{\partial^2 f}{\partial x\partial y}+f^2\tfrac{\partial^2 f}{\partial y^2}\right)}+\underline{\left(\tfrac{\partial f}{\partial y}\right)^2\left(\tfrac{\partial f}{\partial x}+f\tfrac{\partial f}{\partial y}\right)} \\
&+3\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+3f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+3f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}+3f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} \\
=&\underline{\tfrac{\partial^3 f}{\partial x^3}+3f\tfrac{\partial^3 f}{\partial x^2\partial y}+3f^2\tfrac{\partial^3 f}{\partial x\partial y^2}+f^3\tfrac{\partial^3 f}{\partial y^3}}+\underline{\tfrac{\partial f}{\partial y}\left(\tfrac{\partial^2 f}{\partial x^2}+2f\tfrac{\partial^2 f}{\partial x\partial y}+f^2\tfrac{\partial^2 f}{\partial y^2}\right)}+\underline{\left(\tfrac{\partial f}{\partial y}\right)^2\left(\tfrac{\partial f}{\partial x}+f\tfrac{\partial f}{\partial y}\right)} \\
&+3\left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+f\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}+f^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}+f\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d^3f}}{\mathrm{dx^3}} =& \underline{\frac{\partial^3 f}{\partial x^3} + 3f\frac{\partial^3 f}{\partial x^2 \partial y} + 3f^2\frac{\partial^3 f}{\partial x \partial y^2} + f^3\frac{\partial^3 f}{\partial y^3}} + \underline{\frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right)} + \underline{\left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)} \\
&+3\Big(\frac{\partial^2 f}{\partial x \partial y}\Big\{\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\Big\} + f\frac{\partial^2 f}{\partial y^2}\Big\{\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\Big\}\Big) \\
=& \underline{\frac{\partial^3 f}{\partial x^3} + 3f\frac{\partial^3 f}{\partial x^2 \partial y} + 3f^2\frac{\partial^3 f}{\partial x \partial y^2} + f^3\frac{\partial^3 f}{\partial y^3}} + \underline{\frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right)} + \underline{\left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)} \\
&+3\Big(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\Big)\Big(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\Big)\frac{\partial f}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
D^3 =& \left(\frac{\partial}{\partial x} + f_n\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2} + 2f_n\frac{\partial^2 f}{\partial x \partial y} + f_n^2\frac{\partial^2 f}{\partial x^2}\right) \\
=& \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2} + 2f_n\frac{\partial^2 f}{\partial x \partial y} + f_n^2\frac{\partial^2 f}{\partial y^2}\right) + f_n\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + 2f_n\frac{\partial^2 f}{\partial x \partial y} + f_n^2\frac{\partial^2 f}{\partial y^2}\right) \\
=& \frac{\partial^3 f}{\partial x^3} + 2f_n\frac{\partial^3 f}{\partial x^2 \partial y} + f_n^2\frac{\partial^3 f}{\partial x \partial y^2} + f_n\frac{\partial^3 f}{\partial x^2 \partial y} + 2f_n^2\frac{\partial^3 f}{\partial x \partial y^2} + f_n^3\frac{\partial^3 f}{\partial y^3} \\
=& \frac{\partial^3 f}{\partial x^3} + 3f_n\frac{\partial^3 f}{\partial x^2 \partial y} + 3f_n^2\frac{\partial^3 f}{\partial x \partial y^2} + f_n^3\frac{\partial^3 f}{\partial y^3}
\end{aligned}
$$

$$
\left.\frac{\mathrm{d^3f}}{\mathrm{dx^3}}\right|_n = D^3 f + f_y D^2 f + f_y^2 Df + 3DfDf_y\Big|_n
$$

putting equation 5 into 6

$$
\begin{aligned}
y_{n+1} - y_n =& \sum_{t=1}^{\infty}(h_n)^t\frac{y_n^{(t)}}{t!} \\
=& \sum_{t=1}^{\infty} h_n^{t+1}\cdot\frac{(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y})^t}{(t+1)!}\cdot f(x_n, y_n) \\
=& \Bigg[hf + \frac{h^2}{2!}Df + \frac{h^3}{3!}\Big(D^2 f + f_y Df\Big) \\
&+ \frac{h^4}{4!}\Big(D^3 f + f_y D^2 f + f_y^2 Df + 3DfDf_y\Big) \\
&+ \frac{h^5}{5!}\Big(D^4 f + 6DfD^2 f_y + f_y^2 D^2 f + f_y^3 Df \\
&+ 3f_{yy}(Df)^2 + f_y D^3 f + 7f_y DfDf_y\Big)\Bigg] + O(h^6)
\end{aligned}
\tag{8}
$$

My derivation upto this point is consistent and I can do each step with proper mathematical concepts.

$$
k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1}\beta_{ij}k_j)
$$

$$
\begin{aligned}
k_1 &= h_n f(x_n, y_n) = h_n f_n \\
k_2 &= h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21} k_1) \\
k_3 &= h_n f(x_n + \alpha_3 h_n, y_n + \beta_{31} k_1 + \beta_{32} k_2) \\
&= h_n f(x_n + \alpha_3 h_n, y_n + \beta_{31} k_1 + \beta_{32} k_1 + \beta_{32} k_2 - \beta_{32} k_1) \\
&= h_n f(x_n + \alpha_3 h_n, y_n + \sum_{j=1}^{i-1} \beta_{3j} k_j + \sum_{j=2}^{i-1} \beta_{32}(k_2 - h_n f_n)) \\
k_4 &= h_n f(x_n + \alpha_4 h_n, y_n + \beta_{41} k_1 + \beta_{42} k_2 + \beta_{43} k_3) \\
&= h_n f(x_n + \alpha_4 h_n, y_n + \beta_{41} k_1 + \beta_{42} k_1 + \beta_{43} k_1 + \beta_{42} k_2 + \beta_{43} k_3 - \beta_{42} k_1 - \beta_{43} k_1) \\
&= h_n f(x_n + \alpha_4 h_n, y_n + \sum_{j=1}^{i-1} \beta_{4j} h_n f_n + \beta_{42}(k_2 - h_n f_n) + \beta_{43}(k_3 - h_n f_n)) \\
&= h_n f(x_n + \alpha_4 h_n, y_n + \sum_{j=1}^{i-1} \beta_{4j} h_n f_n + \sum_{j=2}^{i-1} \beta_{4j}(k_j - h_n f_n))
\end{aligned}
$$

Let

$$
D_i = \alpha_i \frac{\partial}{\partial x} + \sum_{j=1}^{i-1} (\beta_{ij}) f_n \frac{\partial}{\partial y}
$$

$$
\begin{aligned}
k_i &= h_n f\left[x_n + \alpha_i h_n, y_n + (\sum_{j=1}^{i-1} \beta_{ij}) h_n f_n + \left(\sum_{j=2}^{i-1} \beta_{ij}(k_j - h_n f_n)\right)\right] \\
&= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_i + \sum_{j=2}^{i-1} \beta_{ij}(k_j - h_n f_n) \frac{\partial}{\partial y}\right)^t}{t!} f(x,y)\Big|_{x=x_n, y=y_n}
\end{aligned}
$$

Since $\alpha_1 = 0$

$$
\begin{aligned}
k_1 =& h_n f(x,y)\Big|_{x=x_n, y=y_n} \\
k_2 =& h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21} k_1) = h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21} h_n f_n) \\
=& h_n \sum_{t=0}^{\infty} \frac{h_n^t D_2^t}{t!} f(x,y)\Big|_{x=x_n, y=y_n}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
k_3 =& h_n f(x_n + \alpha_3 h_n, y_n + \beta_{31} k_1 + \beta_{32} k_2) \\
=& h_n f\left[x_n + \alpha_3 h_n, y_n + h_n f_n(\beta_{31} + \beta_{32}) + \beta_{32}(k_2 - h_n f_n)\right] \\
=& h_n \sum_{t=0}^{\infty} \frac{\left(\alpha_i h_n \frac{\partial}{\partial x} + (\beta_{31} + \beta_{32}) h_n f_n \frac{\partial}{\partial y} + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y}\right)^t}{t!} f(x,y)\Big|_{x=x_n, y=y_n} \\
=& h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y}\right)^t}{t!} f(x,y)\Big|_{x=x_n, y=y_n}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
k_4 =&h_n f(x_n + \alpha_4 h_n, y_n + \beta_{41}k_1 + \beta_{42}k_2 + \beta_{43}k_3) \\
=&h_n f\Big[x_n + \alpha_2 h_n, y_n + h_n f_n(\beta_{41} + \beta_{42} + \beta_{43}) + \beta_{41}(k_1 - h_n f_n) + \beta_{42}(k_2 - h_n f_n) \\
&+ \beta_{43}(k_3 - h_n f_n)\Big] \\
=&h_n f\Big[x_n + \alpha_4 h_n, y_n + (\sum_{j=1}^{3} \beta_{4j}) h_n f_n + (\sum_{j=2}^{3} \beta_{4j})(k_j - h_n f_n)\Big] \\
=&h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_4 + \sum_{j=2}^{3} \beta_{4j}(k_j - h_n f_n)\frac{\partial}{\partial y}\right)^t}{t!} f(x,y)\big|_{x=x_n, y=y_n}
\end{aligned} \tag{11}
$$

$$
f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij}k_j) = \sum_{t=0}^{\infty} \frac{h_n^t D_i^t}{t!} f(x,y)\big|_{x=x_n, y=y_n}
$$

$$
\begin{aligned}
k_1 =&h_n f(x,y)\big|_{x=x_n, y=y_n} \\
=&h_n f_n \\
k_2 =&h_n\left(\frac{h_n^0 D_2^0}{0!} f(x,y) + \frac{h_n^1 D_2^1}{1!} f(x,y) + \frac{h_n^2 D_2^2}{2!} f(x,y) + \frac{h_n^3 D_2^3}{3!} f(x,y) + \frac{h_n^4 D_2^4}{4!} f(x,y)\right)\Bigg|_{x=x_n, y=y_n} \\
&+O(h_n^6) \\
=&h_n f(x,y)\big|_{x=x_n, y=y_n} + \frac{h_n^2}{1!} D_2 f(x,y)\big|_{x=x_n, y=y_n} + \frac{h_n^3}{2!} D_2^2 f(x,y)\big|_{x=x_n, y=y_n} + \frac{h_n^4}{3!} D_2^3 f(x,y)\big|_{x=x_n, y=y_n} \\
&+\frac{h_n^5}{4!} D_2^4 f(x,y)\big|_{x=x_n, y=y_n} + O(h_n^6)
\end{aligned}
$$

We will express $D_i^t f(x,y)\big|_{x=x_n, y=y_n} = D_i^t f_n$. Which means the differentiation is already done than the expression is evaluated at $(x = x_n, y = y_n)$ and represents a constant value

$$
(k_2 - h_n f_n) \ = h_n^2 D_2 f_n + \frac{h^3}{2!} D_2^2 f_n + \frac{h_n^4}{3!} D_2^3 f_n + \frac{h_n^5}{4!} D_2^4 f_n + O(h^6)
$$

The evaluated $(k_2 - h_n f_n)$ expression is constant

$$
\begin{aligned}
&h_n\left[h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right]^2 f(x,y)\Big|_{x=x_n,y=y_n}\\
&=h\left[hD_3+\beta_{32}\left(\frac{h^2}{1!}D_2f_n+\frac{h^3}{2!}D_2^2f_n+\frac{h^4}{3!}D_2^3f_n+\frac{h^5}{4!}D_2^4f_n\right)\frac{\partial}{\partial y}\right]^2 f(x,y)\Big|_{x=x_n,y=y_n}\\
&=h_n\left[hD_3+\beta_{32}\left(\frac{h_n^2}{1!}D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)\frac{\partial}{\partial y}\right]\left[h_nD_3f(x,y)+\beta_{32}\left(\frac{h_n^2}{1!}D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)f_y(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&=\left[h_n^2D_3+\beta_{32}\left(\frac{h_n^3}{1!}D_2f_n+\frac{h_n^4}{2!}D_2^2f_n\right)\frac{\partial}{\partial y}\right]\left[h_nD_3f(x,y)+\beta_{32}\left(\frac{h_n^2}{1!}D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)f_y(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&=h_n^2D_3\left[h_nD_3f(x,y)\right]\Big|_{x=x_n,y=y_n}+h_n^2D_3\left\{\beta_{32}\left(\frac{h_n^2}{1!}D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)f_y(x,y)\right\}\Big|_{x=x_n,y=y_n}\\
&+\beta_{32}\left(\frac{h_n^3}{1!}D_2f_n+\frac{h_n^4}{2!}D_2^2f_n\right)\frac{\partial}{\partial y}\left[h_nD_3f(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&+\beta_{32}\left(\frac{h_n^3}{1!}D_2f_n+\frac{h_n^4}{2!}D_2^2f_n\right)\frac{\partial}{\partial y}\left[\beta_{32}\left(\frac{h_n^2}{1!}D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)f_y(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&=h_n^3D_3^2f_n+h_n^2\beta_{32}D_3\left\{\frac{h_n^2}{1!}f_y(x,y)D_2f_n\right\}+h_n^2\beta_{32}D_3\left\{\frac{h^3}{2!}f_y(x,y)D_2^2f_n\right\}+\beta_{32}\left(\frac{h_n^3}{1!}D_2f_n\right)\frac{\partial}{\partial y}\left[h_nD_3f(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&+\beta_{32}\left(\frac{h_n^4}{2!}D_2^2f_n\right)\frac{\partial}{\partial y}\left[h_nD_3f(x,y)\right]\Big|_{x=x_n,y=y_n}+\beta_{32}\left(\frac{h_n^3}{1!}D_2f_n\right)\frac{\partial}{\partial y}\left[\beta_{32}\left(\frac{h_n^2}{1!}D_2f_n\right)f_y(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&=h_n^3D_3^2f_n+h_n^4\beta_{32}D_3f_{y_{|n}}D_2f_n+\frac{h_n^5}{2!}\beta_{32}D_3f_{y_{|n}}D_2^2f_n+h_n^4\beta_{23}D_3f_{y_{|n}}D_2f_n+\frac{h_n^5}{2!}\beta_{23}D_3f_{y_{|n}}D_2^2f_n+h_n^5\beta_{32}^2(D_2f_n)^2f_{yy_{|n}}\\
&=h_n^3D_3^2f_n+2h_n^4(D_2f_n)D_3f_{y_{|n}}+h_n^5\beta_{32}(D_2^2f_n)D_3f_{y_{|n}}+h_n^5\beta_{32}^2(D_2f_n)^2f_{yy_{|n}}
\end{aligned}
$$

$$
\begin{aligned}
&h_n\left[h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right]^3 f\\
&=\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)\left[h_n\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)^2\right]f(x,y)\Big|_{x=x_n,y=y_n}+O(h_n^6)\\
&=\left(h_nD_3+\beta_{32}h_n^2D_2f_n\frac{\partial}{\partial y}\right)\left[h_n^3D_3^2f(x,y)+2h_n^4\beta_{32}(D_2f_n)D_3f_y(x,y)\right]\Big|_{x=x_n,y=y_n}+O(h^6)\\
&=h_n^4D_3^3f(x,y)\Big|_{x=x_n,y=y_n}+2h^5\beta_{32}(D_2f_n)D_3^2f_y(x,y)\Big|_{x=x_n,y=y_n}+h^5\beta_{32}(D_2f_n)D_3^2f_y(x,y)\Big|_{x=x_n,y=y_n}+O(h^6)
\end{aligned}
$$

$$
h_n\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)^3 f(x,y)\Big|_{x=x_n,y=y_n}=h_n^4D_3^3f_n+3h_n^5\beta_{32}(D_2f_n)D_3^2f_{y_{|n}}+O(h^6)
$$

$$
\begin{aligned}
h_n\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)^4 f(x,y)\Big|_{x=x_n,y=y_n}&=\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)\\
&\quad\left[h_n\left(h_nD_3+\beta_{32}(k_2-h_nf_n)\frac{\partial}{\partial y}\right)^3 f(x,y)\right]\Big|_{x=x_n,y=y_n}\\
&\quad+O(h_n^6)\\
&=\left(h_nD_3\right)\left[h_n^4D_3^3f(x,y)\right]\Big|_{x=x_n,y=y_n}+O(h_n^6)\\
&=h_n^5D_3^4f_n+O(h_n^6)
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n)\frac{\partial}{\partial y}\right)^t}{t!} f(x,y)\Big|_{x=x_n, y=y_n} \\
&= h_n f_n + h_n^2 D_3 f_n + h_n \beta_{32}(k_2 - h_n f_n) f_{y_{|n}} + h_n \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n)\frac{\partial}{\partial y}\right)^2}{2!} f(x,y)\Big|_{x=x_n, y=y_n} \\
&+ h_n \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n)\frac{\partial}{\partial y}\right)^3}{3!} f(x,y)\Big|_{x=x_n, y=y_n} + h_n \frac{\left(h D_3 + \beta_{32}(k_2 - h f)\frac{\partial}{\partial y}\right)^4}{4!} f(x,y)\Big|_{x=x_n, y=y_n} + O(h^6) \\
&= h_n f_n + h_n^2 D_3 f_n + h_n \beta_{32}\left(h_n^2 D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n + \frac{h^4}{3!} D_2^3 f_n\right) f_{y_{|n}} + \frac{h_n^3}{2} D_3^2 f_n + h_n^4 \beta_{32}(D_2 f_n) D_3 f_{y_{|n}} \\
&+ \frac{h_n^5}{2}\beta_{32}(D_2^2 f_n) D_3 f_{y_{|n}} + \frac{h_n^5}{2}\beta_{32}^2 (D_2 f_n)^2 f_{yy_{|n}} + \frac{h_n^4}{6} D_3^3 f_n + \frac{h^5}{2}\beta_{32}(D_2 f_n) D_3^2 f_{y_{|n}} + \frac{h^5}{24} D_3^4 f_n + O(h^6) \\
&= h_n f_n + h_n^2 D_3 f_n + h_n^3 \beta_{32} f_{y_{|n}} D_2 f_n + \frac{h_n^4}{2}\beta_{32} f_{y_{|n}} D_2^2 f_n + \frac{h_n^5}{6}\beta_{32} f_{y_{|n}} D_2^3 f_n + \frac{h_n^3}{2} D_3^2 f_n + h_n^4 \beta_{32}(D_2 f_n) D_3 f_{y_{|n}} \\
&+ \frac{h_n^5}{2}\beta_{32}(D_2^2 f_n) D_3 f_{y_{|n}} + \frac{h_n^5}{2}\beta_{32}^2 (D_2 f_n)^2 f_{yy_{|n}} + \frac{h_n^4}{6} D_3^3 f_n + \frac{h_n^5}{2}\beta_{32} D_3^2 f_{y_{|n}} D_2 f_n + \frac{h_n^5}{24} D_3^4 f_n + O(h^6) \\
&= h_n f_n + h_n^2 D_3 f_n + h_n^3\left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y_{|n}} D_2 f_n\right) + h_n^4\left(\frac{1}{6} D_3^3 f_n + \frac{1}{2}\beta_{32} f_{y_{|n}} D_2^2 f_n + \beta_{32}(D_2 f_n) D_3 f_{y_{|n}}\right) \\
&+ h^5\left(\frac{1}{24} D_3^4 f_n + \frac{1}{6}\beta_{32} f_{y_{|n}} D_2^3 f_n + \frac{1}{2}\beta_{32}^2 (D_2 f_n)^2 f_{yy_{|n}} + \frac{1}{2}\beta_{32}(D_2^2 f_n) D_3 f_{y_{|n}} + \frac{1}{2}\beta_{32}(D_2 f_n) D_3^2 f_{y_{|n}}\right) + O(h_n^6)
\end{aligned}
\tag{12}
$$

$$
k_3 - h_n f_n = h_n^2 D_3 f_n + h_n^3\left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y_{|n}} D_2 f_n\right) + h_n^4\left(\frac{1}{6} D_3^3 f_n + \frac{1}{2}\beta_{32} f_{y_{|n}} D_2^2 f_n + \beta_{32}(D_2 f_n) D_3 f_{y_{|n}}\right)
$$

$$
\begin{aligned}
& h_n\left[h_n D_4 + \sum_{j=2}^{3} \beta_{4j}(k_j - h_n f_n)\frac{\partial}{\partial y}\right] f(x,y)\big|_{x=x_n, y=y_n} \\
&= h_n^2 D_4 f(x,y)\big|_{x=x_n, y=y_n} + h_n \beta_{42}(k_2 - h_n f_n) f_y(x,y)\big|_{x=x_n, y=y_n} + h_n \beta_{43}(k_3 - h_n f_n) f_y(x,y)\big|_{x=x_n, y=y_n} \\
&= h_n^2 D_4 f_n + h_n \beta_{42}\left(h_n^2 D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n + \frac{h_n^4}{3!} D_2^3 f_n\right) f_y(x,y)\big|_{x=x_n, y=y_n} + h_n \beta_{43}\Bigg[h_n^2 D_3 f_n + h_n^3\left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y_{|n}} D_2 f_n\right) \\
&+ h_n^4\left(\frac{1}{6} D_3^3 f_n + \frac{1}{2}\beta_{32} f_{y_{|n}} D_2^2 f_n + \beta_{32}(D_2 f_n) D_3 f_{y_{|n}}\right)\Bigg] f_y(x,y)\big|_{x=x_n, y=y_n} \\
&= h_n^2 D_4 f(x,y)\Big|_{x=x_n, y=y_n} + h_n^3\left(\beta_{42}(D_2 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n} + \beta_{43}(D_3 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n}\right) \\
&+ h_n^4\left(\frac{1}{2}\beta_{42}(D_2^2 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n} + \frac{1}{2}\beta_{43}(D_3^2 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n} + \beta_{32}\beta_{43} f_{y_{|n}}(D_2 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n}\right) \\
&+ h_n^5\left(\frac{1}{6}\beta_{42}(D_2^3 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n} + \frac{1}{6}\beta_{43}(D_3^3 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n} + \frac{1}{2}\beta_{32}\beta_{43} f_{y_{|n}}(D_2^2 f_n) f_y(x,y)\Big|_{x=x_n, y=y_n}\right. \\
&\left. + \beta_{32}\beta_{43}(D_2 f_n) D_3 f_{y_{|n}} f_y(x,y)\Big|_{x=x_n, y=y_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&h_n\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^2 f(x,y)\big|_{x=x_n,y=y_n}\\
&=\left(h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right)\left[h_n^2D_4+h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]f(x,y)\big|_{x=x_n,y=y_n}\\
&=h_nD_4\left[h_n^2D_4f(x,y)\big|_{x=x_n,y=y_n}+h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)f_y(x,y)\big|_{x=x_n,y=y_n}\right]\\
&+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\left[h_n^2D_4f(x,y)\big|_{x=x_n,y=y_n}+h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)f_y(x,y)\big|_{x=x_n,y=y_n}\right]\\
&=h_nD_4\left[h_n^2D_4f(x,y)\big|_{x=x_n,y=y_n}\right]+h_nD_4\left[h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)f_y(x,y)\big|_{x=x_n,y=y_n}\right]\\
&+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\left[h_n^2D_4f(x,y)\big|_{x=x_n,y=y_n}\right]+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\left[h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)f_y(x,y)\big|_{x=x_n,y=y_n}\right]\\
&=h_n^3D_4^2f(x,y)\big|_{x=x_n,y=y_n}+h_n^2\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+h_n^2\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\left[D_4f(x,y)\big|_{x=x_n,y=y_n}\right]\\
&+h_n\left(\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\right)^2 f_{yy}(x,y)\big|_{x=x_n,y=y_n}\\
&=h_n^3D_4^2f(x,y)\big|_{x=x_n,y=y_n}+h_n^2\beta_{42}(k_2-h_nf_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+h_n^2\beta_{43}(k_3-h_nf_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}\\
&+h_n^2\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\left[D_4f(x,y)\big|_{x=x_n,y=y_n}\right]+h_n\left[\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\right]^2 f_{yy}(x,y)\big|_{x=x_n,y=y_n}\\
&=h_n^3D_4^2f(x,y)\big|_{x=x_n,y=y_n}+h_n^2\beta_{42}\left(h_n^2D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)D_4f_y(x,y)\big|_{x=x_n,y=y_n}\\
&+h_n^2\beta_{43}\left[h_n^2D_3f_n+h_n^3\left(\frac{1}{2}D_3^2f_n+\beta_{32}f_{y_{|n}}D_2f_n\right)\right]D_4f_y(x,y)\big|_{x=x_n,y=y_n}+h_n^2\left[\beta_{42}(k_2-h_nf_n)+\beta_{43}(k_3-h_nf_n)\right]D_4f_y(x,y)\big|_{x=x_n,y=y_n}\\
&+h_n\left[\beta_{42}(k_2-h_nf_n)+\beta_{43}(k_3-h_nf_n)\right]^2 f_{yy}(x,y)\big|_{x=x_n,y=y_n}\\
&=h_n^3D_4^2f(x,y)\big|_{x=x_n,y=y_n}+h_n^2\beta_{42}\left(h_n^2D_2f_n+\frac{h_n^3}{2!}D_2^2f_n\right)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+h_n^4\beta_{43}D_3f_nD_4f_y(x,y)\Big|_{x=x_n,y=y_n}\\
&+h_n^5\beta_{43}\left(\frac{1}{2}D_3^2f_n+\beta_{32}f_{y_{|n}}D_2f_n\right)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+h_n^2\beta_{42}\left[h_n^2D_2f_n+\frac{h_n^3}{2}D_2^2f_n\right]D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\\
&+h_n^2\beta_{43}\left[h_n^2D_3f_n+h_n^3\left(\frac{1}{2}D_3^2f_n+\beta_{32}f_{y_{|n}}D_2f_n\right)\right]D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+h_n\left[\beta_{42}h_n^2D_2f_n+\beta_{43}h_n^2D_3f_n\right]^2 f_{yy}(x,y)\Big|_{x=x_n,y=y_n}\\
&=h_n^3D_4^2f(x,y)\big|_{x=x_n,y=y_n}+h_n^4\left[2\beta_{42}(D_2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+2\beta_{43}(D_3f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\right]\\
&+h_n^5\left[\frac{1}{2}\beta_{42}(D_2^2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{43}(D_3^2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+\beta_{32}\beta_{43}f_{y_{|n}}(D_2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}\right.\\
&+\frac{1}{2}\beta_{42}(D_2^2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{43}(D_3^2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+\beta_{32}\beta_{43}f_{y_{|n}}(D_2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}\\
&\left.+\beta_{42}^2(D_2f_n)^2f_{yy}(x,y)\big|_{x=x_n,y=y_n}+\beta_{43}^2(D_3f_n)^2f_{yy}(x,y)\big|_{x=x_n,y=y_n}+2\beta_{42}\beta_{43}(D_2f_n)(D_3f_n)f_{yy}(x,y)\big|_{x=x_n,y=y_n}\right]\\
&=h_n^3D_4^2f(x,y)\big|_{x=x_n,y=y_n}+h_n^4\left(2\beta_{42}(D_2f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}+2\beta_{43}(D_3f_n)D_4f_y(x,y)\big|_{x=x_n,y=y_n}\right)\\
&+h_n^5\left(\beta_{42}(D_2^2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+\beta_{43}(D_3^2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+2\beta_{32}\beta_{43}f_{y_{|n}}(D_2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\right.\\
&\left.+\beta_{42}^2(D_2f_n)^2f_{yy}(x,y)\Big|_{x=x_n,y=y_n}+\beta_{43}^2(D_3f_n)^2f_{yy}(x,y)\Big|_{x=x_n,y=y_n}+2\beta_{42}\beta_{43}(D_3f_n)(D_2f_n)f_{yy}(x,y)\Big|_{x=x_n,y=y_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&h_n\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^3 f(x,y)\Big|_{x=x_n,y=y_n}\\
&=\left(h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right)\left[h_n^2D_4+h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^2 f(x,y)\Big|_{x=x_n,y=y_n}\\
&=\left(h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right)\left[h_n^3D_4^2f(x,y)\Big|_{x=x_n,y=y_n}+h_n^4\left(2\beta_{42}(D_2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+2\beta_{43}(D_3f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\right)\right]\\
&=h_nD_4\left[h_n^3D_4^2f(x,y)\Big|_{x=x_n,y=y_n}+h_n^4\left(2\beta_{42}(D_2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+2\beta_{43}(D_3f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\right)\right]+\\
&+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\left[h_n^3D_4^2f(x,y)\Big|_{x=x_n,y=y_n}\right]\\
&=h_n^4D_4^3f(x,y)\Big|_{x=x_n,y=y_n}+h_n^5\left(2\beta_{42}(D_2f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}+2\beta_{43}(D_3f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}\right.\\
&\left.+\beta_{42}(D_2f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}+\beta_{43}(D_3f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}\right)\\
&=h_n^4D_4^3f(x,y)\Big|_{x=x_n,y=y_n}+h_n^5\left(3\beta_{42}(D_2f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}+3\beta_{43}(D_3f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&h_n\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^4 f(x,y)\Big|_{x=x_n,y=y_n}\\
&=\left(h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right)\left[h_n^2D_4+h_n\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^3 f(x,y)\Big|_{x=x_n,y=y_n}\\
&=h_nD_4\left[h_n^4D_4^3f(x,y)\Big|_{x=x_n,y=y_n}\right]+O(h_n^6)\\
&=h_n^5D_4^4f(x,y)\Big|_{x=x_n,y=y_n}+O(h_n^6)
\end{aligned}
$$

$$
\begin{aligned}
k_4&=h_n\sum_{t=0}^{\infty}\frac{\left(h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right)^t}{t!}f(x,y)\Big|_{x=x_n,y=y_n}\\
&=h_nf_n+h_n\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]f(x,y)\Big|_{x=x_n,y=y_n}+h_n\frac{\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^2}{2!}f(x,y)\Big|_{x=x_n,y=y_n}\\
&+h_n\frac{\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^3}{3!}f(x,y)\Big|_{x=x_n,y=y_n}+h_n\frac{\left[h_nD_4+\sum_{j=2}^{3}\beta_{4j}(k_j-h_nf_n)\frac{\partial}{\partial y}\right]^4}{4!}f(x,y)\Big|_{x=x_n,y=y_n}+\ldots\\
&=h_nf(x,y)\Big|_{x=x_n,y=y_n}+h_n^2D_4f(x,y)\Big|_{x=x_n,y=y_n}+h_n^3\left[\frac{1}{2}D_4^2f(x,y)\Big|_{x=x_n,y=y_n}+\beta_{42}(D_2f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}\right.\\
&\left.+\beta_{43}(D_3f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}\right]+h_n^4\left[\frac{1}{6}D_4^3f(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{42}(D_2^2f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{43}(D_3^2f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}\right.\\
&\left.+\beta_{32}\beta_{43}f_{y|_n}(D_2f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}+\beta_{42}(D_2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+\beta_{43}(D_3f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\right]\\
&+h_n^5\left[\frac{1}{24}D_4^4f(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{6}\beta_{42}(D_2^3f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{6}\beta_{43}(D_3^3f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{32}\beta_{43}f_{y|_n}(D_2^2f_n)f_y(x,y)\Big|_{x=x_n,y=y_n}\right.\\
&+\beta_{32}\beta_{43}(D_2f_n)D_3f_{y|_n}f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{42}(D_2^2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{43}(D_3^2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}\\
&+\beta_{32}\beta_{43}f_{y|_n}(D_2f_n)D_4f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{42}^2(D_2f_n)^2f_{yy}(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{43}^2(D_3f_n)^2f_{yy}(x,y)\Big|_{x=x_n,y=y_n}\\
&\left.+\beta_{42}\beta_{43}(D_3f_n)(D_2f_n)f_{yy}(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{42}(D_2f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}+\frac{1}{2}\beta_{43}(D_3f_n)D_4^2f_y(x,y)\Big|_{x=x_n,y=y_n}\right]
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
k_4 =& hf + h^2 D_4 f + h^3\left(\frac{1}{2}D_4^2 f + \beta_{42} f_y D_2 f + \beta_{43} f_y D_3 f\right) + h^4\left[\frac{1}{6}D_4^3 f + \frac{1}{2}\beta_{42} f_y D_2^2 f\right.\\
&\left. + \beta_{32}\beta_{43}(f_y)^2 D_2 f + \frac{1}{2}\beta_{43} f_y D_3^2 f + \beta_{42} D_2 f D_4 f_y + \beta_{43} D_3 f D_4 f_y\right] + h^5\left(\frac{1}{24}D_4^4 f + \frac{1}{6}\beta_{42} f_y D_2^3 f\right.\\
& + \frac{1}{2}\beta_{32}\beta_{43}(f_y)^2 D_2^2 f + \beta_{32}\beta_{43} f_y D_2 f D_3 f_y + \frac{1}{6}\beta_{43} f_y D_3^3 f + \frac{1}{2}\beta_{42} D_4 f_y D_2^2 f + \frac{1}{2}\beta_{43} D_4 f_y D_3^2 f\\
& + \frac{1}{2}\beta_{42}^2 f_{yy} D_2^2 f + \beta_{42}\beta_{43} f_{yy} D_2 f D_3 f + \frac{1}{2}\beta_{43}^2 f_{yy} D_3^2 f + \frac{1}{2}\beta_{42} D_2 f D_4^2 f_y\\
&\left.\left. + \frac{1}{2}\beta_{43} D_3 f D_4^2 f_y + \beta_{43}\beta_{32} f_y D_2 f D_4 f_y\right)\right|_n + O(h^6)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} - y_n &= w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4\\
&= \left[hf + \frac{h^2}{2!}Df + \frac{h^3}{3!}\left(D^2 f + f_y Df\right) + \frac{h^4}{4!}\left(D^3 f + f_y D^2 f + f_y^2 Df + 3DfDf_y\right)\right.\\
&\left. + \frac{h^5}{5!}\left(D^4 f + 6DfD^2 f_y + f_y^2 D^2 f + f_y^3 Df + 3f_{yy}(Df)^2 + f_y D^3 f + 7f_y DfDf_y\right)\right]\\
&= w_1 hf + w_2\left\{hf + \frac{h^2}{1!}D_2 f + \frac{h^3}{2!}D_2^2 f + \frac{h^4}{3!}D_2^3 f + \frac{h^5}{4!}D_2^4 f\right\}\\
&+ w_3\left\{hf + h^2 D_3 f + h^3\left(\frac{1}{2}D_3^2 f + f_y\beta_{32} D_2 f\right) + h^4\left(\frac{1}{6}D_3^3 f + \frac{1}{2}f_y\beta_{32}D_2^2 f + \beta_{32} D_3 f_y D_2 f\right)\right.\\
&\left. + h^5\left(\frac{1}{24}D_3^4 f + \frac{\beta_{32}}{6}f_y D_2^3 f + \frac{\beta_{32}^2}{2}f_{yy}(D_2 f)^2 + \frac{\beta_{32}}{2}D_3 f_y D_2^2 f + \frac{\beta_{32}}{2}D_3^2 f_y D_2 f\right)\right\}\\
&+ w_4\left\{hf + h^2 D_4 f + h^3\left(\frac{1}{2}D_4^2 f + \beta_{42} f_y D_2 f + \beta_{43} f_y D_3 f\right) + h^4\left(\frac{1}{6}D_4^3 f + \frac{1}{2}\beta_{42} f_y D_2^2 f\right.\right.\\
&\left. + \beta_{32}\beta_{43}(f_y)^2 D_2 f + \frac{1}{2}\beta_{43} f_y D_3^2 f + \beta_{42} D_2 f D_4 f_y + \beta_{43} D_3 f D_4 f_y\right) + h^5\left(\frac{1}{24}D_4^4 f + \frac{1}{6}\beta_{42} f_y D_2^3 f\right.\\
& + \frac{1}{2}\beta_{32}\beta_{43}(f_y)^2 D_2^2 f + \beta_{32}\beta_{43} f_y D_2 f D_3 f_y + \frac{1}{6}\beta_{43} f_y D_3^3 f + \frac{1}{2}\beta_{42} D_4 f_y D_2^2 f + \frac{1}{2}\beta_{43} D_4 f_y D_3^2 f\\
& + \frac{1}{2}\beta_{42}^2 f_{yy} D_2^2 f + \beta_{42}\beta_{43} f_{yy} D_2 f D_3 f + \frac{1}{2}\beta_{43}^2 f_{yy} D_3^2 f + \frac{1}{2}\beta_{42} D_2 f D_4^2 f_y\\
&\left.\left. + \frac{1}{2}\beta_{43} D_3 f D_4^2 f_y + \beta_{43}\beta_{32} f_y D_2 f D_4 f_y\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
w_1 + w_2 + w_3 + w_4 &= 1\\
w_2 D_2 f + w_3 D_3 f + w_4 D_4 f &= \frac{1}{2}Df\\
\frac{1}{2}w_2 D_2^2 f + w_3\left(\frac{1}{2}D_3^2 f + f_y\beta_{32} D_2 f\right)&\\
+ w_4\left(\frac{1}{2}D_4^2 f + \beta_{42} f_y D_2 f + \beta_{43} f_y D_3 f\right) &= \frac{1}{3!}\left(D^2 f + f_y Df\right)\\
\frac{1}{3!}w_2 D_2^3 f + w_3\left(\frac{1}{6}D_3^3 f + \frac{1}{2}f_y\beta_{32} D_2^2 f + \beta_{32} D_3 f_y D_2 f\right)&\\
+ w_4\left(\frac{1}{6}D_4^3 f + \frac{1}{2}\beta_{42} f_y D_2^2 f + \beta_{32}\beta_{43}(f_y)^2 D_2 f\right.&\\
\left. + \frac{1}{2}\beta_{43} f_y D_3^2 f + \beta_{42} D_2 f D_4 f_y + \beta_{43} D_3 f D_4 f_y\right) &= \frac{1}{4!}\left(D^3 f + f_y D^2 f + f_y^2 Df + 3DfDf_y\right)
\end{aligned}
$$

$$
\begin{aligned}
\omega_1+\omega_2+\omega_3+\omega_4 &= 1\\
\omega_2 D_2 f+\omega_3 D_3 f+\omega_4 D_4 f &= \frac{1}{2}Df\\
\therefore \frac{1}{2}\omega_2 D_2^2 f+\frac{1}{2}\omega_3 D_3^2 f+\frac{1}{2}\omega_4 D_2^4 &= \frac{1}{6}D^2 f\\
\omega_3\beta_{32}D_2 f+\omega_4\Big(\beta_{42}D_2 f+\beta_{43}D_3 f\Big) &= \frac{1}{6}Df\\
\frac{1}{6}\omega_2 D_2^3 f+\frac{1}{6}\omega_3 D_3^3 f+\frac{1}{6}\omega_4 D_4^3 f &= \frac{1}{24}D^3 f\\
\frac{1}{2}\omega_3\beta_{32}D_2^2 f+\frac{1}{2}\omega_4\beta_{42}D_2^2 f+\frac{1}{2}\omega_4\beta_{43}D_3^2 f &= \frac{1}{24}D^2 f\\
\omega_4\beta_{32}\beta_{43}D_2 f &= \frac{1}{24}Df\\
\omega_3\beta_{32}D_2 f D_3 f_y+\omega_4\beta_{42}D_2 f D_4 f_y+\omega_4\beta_{43}D_3 f D_4 f_y &= \frac{1}{8}Df Df_y
\end{aligned}
$$

$$
\begin{aligned}
\omega_1+\omega_2+\omega_3+\omega_4 &= 1\\
\omega_2\left(\frac{D_2 f}{Df}\right)+\omega_3\left(\frac{D_3 f}{Df}\right)+\omega_4\left(\frac{D_4 f}{Df}\right) &= \frac{1}{2}\\
\omega_2\left(\frac{D_2^2 f}{D^2 f}\right)+\omega_3\left(\frac{D_3^2 f}{D^2 f}\right)+\omega_4\left(\frac{D_4^2 f}{D^2 f}\right) &= \frac{1}{3}\\
\omega_3\beta_{32}\left(\frac{D_2 f}{Df}\right)+\omega_4\left\{\beta_{42}\left(\frac{D_2 f}{Df}\right)+\beta_{43}\left(\frac{D_3 f}{Df}\right)\right\} &= \frac{1}{6}\\
\omega_2\left(\frac{D_2^3 f}{D^3 f}\right)+\omega_3\left(\frac{D_3^3 f}{D^3 f}\right)+\omega_4\left(\frac{D_4^3 f}{D^3 f}\right) &= \frac{1}{4}\\
\omega_3\beta_{32}\left(\frac{D_2^2 f}{D^2 f}\right)+\omega_4\left\{\beta_{42}\left(\frac{D_2^2 f}{D^2 f}\right)+\beta_{43}\left(\frac{D_3^2 f}{D^2 f}\right)\right\} &= \frac{1}{12}\\
\omega_4\beta_{32}\beta_{43}\left(\frac{D_2 f}{Df}\right) &= \frac{1}{24}\\
\omega_3\beta_{32}\left(\frac{D_2 f}{Df}\right)\left(\frac{D_3 f_y}{Df_y}\right)+\omega_4\left\{\beta_{42}\left(\frac{D_2 f}{Df}\right)\left(\frac{D_4 f_y}{Df_y}\right)+\beta_{43}\left(\frac{D_3 f}{Df}\right)\left(\frac{D_4 f_y}{Df_y}\right)\right\} &= \frac{1}{8}
\end{aligned}
$$

In order to make a general theory that works for all differential equation, the value of $\omega_i, \alpha_i, \beta_{ij}$ has to be independent of $f(x, y)$.

$$
\begin{aligned}
D &= \frac{\partial}{\partial x}+f_n\frac{\partial}{\partial y}\\
D_i &= \alpha_i\frac{\partial}{\partial x}+\sum_{j=1}^{i-1}\beta_{ij}f_n\frac{\partial}{\partial y}
\end{aligned}
$$

$\frac{D_i f}{Df}=\frac{\alpha_i\frac{\partial f}{\partial x}+\sum_{j=1}^{i-1}\beta_{ij}f_n\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}+f_n\frac{\partial f}{\partial y}}$

$\frac{D_i f_y}{Df_y}=\frac{\alpha_i\frac{\partial f_y}{\partial x}+\sum_{j=1}^{i-1}\beta_{ij}f_n\frac{\partial f_y}{\partial y}}{\frac{\partial f_y}{\partial x}+f_n\frac{\partial f_y}{\partial y}}$

The only way to make these expressions free of $f(x, y)$ is to let the numerator and denominator cancels each other.

That will be only possible when,

$$\frac{\alpha_i \frac{\partial f}{\partial x} + \sum_{j=1}^{i-1} \beta_{ij} f_n \frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x} + f_n \frac{\partial f}{\partial y}} \Rightarrow \frac{\alpha_i \frac{\partial f}{\partial x} + \sum_{j=1}^{i-1} \beta_{ij} f_n \frac{\partial f}{\partial y}}{\alpha_i \frac{\partial f}{\partial x} + \alpha_i f_n \frac{\partial f}{\partial y}} \cdot \alpha_i$$

$$\frac{\alpha_i \frac{\partial f_y}{\partial x} + \sum_{j=1}^{i-1} \beta_{ij} f_n \frac{\partial f_y}{\partial y}}{\frac{\partial f_y}{\partial x} + f_n \frac{\partial f_y}{\partial y}} \Rightarrow \frac{\alpha_i \frac{\partial f_y}{\partial x} + \sum_{j=1}^{i-1} \beta_{ij} f_n \frac{\partial f_y}{\partial y}}{\alpha_i \frac{\partial f_y}{\partial x} + \alpha_i f_n \frac{\partial f_y}{\partial y}} \cdot \alpha_i$$

$$\therefore \alpha_i = \sum_{j=1}^{i-1} \beta_{ij} \text{ and that infers to } D_i = \alpha_i D$$

$$\begin{aligned}
\omega_1 + \omega_2 + \omega_3 + \omega_4 &= 1 \\
\omega_2\alpha_2 + \omega_3\alpha_3 + \omega_4\alpha_4 &= \frac{1}{2} \\
\omega_2\alpha_2^2 + \omega_3\alpha_3^2 + \omega_4\alpha_4^2 &= \frac{1}{3} \\
\omega_3\beta_{32}\alpha_2 + \omega_4\Bigg(\beta_{42}\alpha_2 + \beta_{43}\alpha_3\Bigg) &= \frac{1}{6} \\
\omega_2\alpha_2^3 + \omega_3\alpha_3^3 + \omega_4\alpha_4^3 &= \frac{1}{4} \\
\omega_3\beta_{32}\alpha_2^2 + \omega_4\Bigg(\beta_{42}\alpha_2^2 + \beta_{43}\alpha_3^2\Bigg) &= \frac{1}{12} \\
\omega_4\beta_{32}\beta_{43}\alpha_2 &= \frac{1}{24} \\
\omega_3\beta_{32}\alpha_2\alpha_3 + \omega_4\Bigg(\beta_{42}\alpha_2\alpha_4 + \beta_{43}\alpha_3\alpha_4\Bigg) &= \frac{1}{8}
\end{aligned}$$