Prove that

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

$$\begin{aligned}
L.H.S &= \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) \\
&= \prod_{k=1}^{n-1} \left(\frac{e^{\frac{k\pi}{n}i} - e^{-\frac{k\pi}{n}i}}{2i}\right) \\
&= \frac{1}{(2i)^{n-1}} \prod_{k=1}^{n-1} e^{-\frac{k\pi}{n}i}\left(e^{\frac{2k\pi}{n}i} - 1\right) \\
&= \frac{1}{(2i)^{n-1}} \prod_{k=1}^{n-1} e^{-\frac{k\pi}{n}i} \prod_{k=1}^{n-1} \left(e^{\frac{2k\pi}{n}i} - 1\right) \ldots\ldots\ldots\ldots\ldots\ldots(1)
\end{aligned}$$

Now

$$\begin{aligned}
\prod_{k=1}^{n-1} e^{-\frac{k\pi}{n}i} &= e^{-\frac{\pi i}{n}(1+2+3+\ldots\ldots\ldots\ldots+n-1)} \\
&= e^{-\frac{\pi}{n}\frac{n(n-1)i}{2}} \\
&= e^{-\frac{\pi(n-1)i}{2}} \\
&= e^{\frac{\pi}{2}i} \cdot e^{-\frac{\pi n}{2}i} \\
&= ie^{-\frac{\pi n}{2}i}
\end{aligned}$$

$$\begin{aligned}
\text{if n} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots e^{-\frac{\pi}{2}i} &= -i = (-1)^1(i)^1 \\
\text{if n} = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots e^{-\frac{2\pi}{2}i} &= -1 = (-1)^2(i)^2 \\
\text{if n} = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots e^{-\frac{3\pi}{2}i} &= i = (-1)^3(i)^3 \\
\text{if n} = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots e^{-\frac{4\pi}{2}i} &= 1 = (-1)^4(i)^4 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\text{if n} = \text{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots e^{-\frac{n\pi}{2}i} &= .. = (-1)^n(i)^n
\end{aligned}$$

So

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = (-1)^n i^{n+1}$$

from ....(1) we get

$$\begin{aligned}
\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) &= \frac{1}{(2i)^{n-1}}(-1)^n i^{n+1} \prod_{k=1}^{n-1}\left(e^{\frac{2k\pi}{n}i} - 1\right) \\
&= 2^{1-n}(-1)^{n+1} \prod_{k=1}^{n-1}\left(e^{\frac{2k\pi}{n}i} - 1\right) \\
&= (-2)^{1-n}(-1)^{n-1} \prod_{k=1}^{n-1}\left(1 - e^{\frac{2k\pi}{n}i}\right) \\
&= 2^{1-n} \prod_{k=1}^{n-1}\left(1 - e^{\frac{2k\pi}{n}i}\right) \text{....................(2)}
\end{aligned}$$

considering roots of $z^n = 1$

$$z^n = e^{2k\pi i} \text{......k = 0,1,2,3.....n-1 (a total of n roots)}$$

So, the roots are

$$z = e^{\frac{2k\pi i}{n}} \text{.........k = 0,1,2,......,(n-1)}$$

$$z^n - 1 = \prod_{k=0}^{n-1}(z - e^{\frac{2k\pi i}{n}})$$

Using binomial theorem, we get

$$(z-1)(1 + z + z^2 + ... + z^{n-1} = (z-1)\prod_{k=1}^{n-1}(z - e^{\frac{2k\pi i}{n}})$$

$$1 + z + z^2 + ... + z^{n-1} = \prod_{k=1}^{n-1}(z - e^{\frac{2k\pi i}{n}})$$

putting $z = 1$

$$n = \prod_{k=1}^{n-1}(z - e^{\frac{2k\pi i}{n}})$$

from...........(2) we get

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$