

Numerical Differentiation

Md. Mesbahose Salekeen

Introduction:

What is Numerical Differentiation? The Differential Equations we solved in our courses are solvable by analytical method $y = F(x)$, but in case practical application most of them can not be solved analytically. Thus we try to solve them numerically. There are two approaches to do such task -

1. Predictor-corrector method
2. Runge-Kutta Method

In this we will discuss about Runge-kutta Method

Runge-Kutta Method:

This method is about doing linear approximation of slopes calculated within $x_{n+1} - x_n = h_n$ such that the y_{n+1} values closely corresponds to the Taylor series approximation at that point thus overcoming the error caused by generic euler method.

The solution of

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

is to be defined by

$$y_{n+1} = y_n + \sum_{i=1}^m (w_i k_i) \quad (2)$$

where w_i are weighted values of slopes and

$$k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j) \quad (3)$$

are slope values.

We will have to find the values of α_i , β_{ij} , h_n and w_i to compute the solution. Initially $\alpha_1 = 0$, $h_n = x_{n+1} - x_n$

Expanding $F(x_{n+1})$,

$$F(x_{n+1}) = F(x_n) + F'(x_n)(x_{n+1} - x_n) + \frac{F''(x_n)(x_{n+1} - x_n)^2}{2!} + \dots$$

$$\begin{aligned} y_{n+1} - y_n &= F(x_{n+1}) - F(x_n) \\ &= (x_{n+1} - x_n)F'(x_n) + (x_{n+1} - x_n)^2 \frac{F''(x_n)}{2!} + \dots \\ &= h_n F'(x_n) + (h_n)^2 \frac{F''(x_n)}{2!} + \dots (h_n = x_{n+1} - x_n) \\ y_{n+1} - y_n &= h_n F'(x_n) + (h_n)^2 \frac{F''(x_n)}{2!} + (h_n)^3 \frac{F'''(x_n)}{3!} + \dots \end{aligned} \quad (4)$$

$$y_{n+1} - y_n = \sum_{t=1}^{\infty} (h_n)^t \frac{y_n^{(t)}}{t!} \quad (5)$$

it can be shown that,

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} \right) f \dots \text{ where} \\ &\left(\frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} \right) \text{ is a type of differential operator} \end{aligned}$$

Again,

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right) \\ &= \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)_x + f \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)_y \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right) + f \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right) = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right) f \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) + f \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial}{\partial y} \left(f \frac{\partial f}{\partial y} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} + f \left(f \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2} + f \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \\ &= \left(\frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} \right)^2 f \end{aligned}$$

and,

$$\begin{aligned}
\frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)^2 \\
&= \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)_x^2 + f \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)_y^2 \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)^2 + f \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right)^2 = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right)^2 f \\
&= \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right)^3 f
\end{aligned}$$

In addition, $y'_n = f(x_n, y_n)$, Assuming differentiability, it follows that

$$\begin{aligned}
y_n^t &= \frac{d^{t-1}}{dx^{t-1}} \{y'(x_n, y_n)\} \\
&= \frac{d^{t-1}}{dx^{t-1}} f(x, y)|_{x_n, y_n} \\
&= \left(\frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} \right)^{t-1} f(x_n, y_n)
\end{aligned} \tag{6}$$

Defining

$$D = \left(\frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} \right) \Big|_{x=x_n, y=y_n} = \left(\frac{\partial}{\partial x} + f_n \frac{\partial}{\partial y} \right) \dots f_n = f(x, y)|_{x_n, y_n} \text{ is a constant}$$

$$\begin{aligned}
D^2 &= \left(\frac{\partial}{\partial x} + f_n \frac{\partial}{\partial y} \right) \cdot \left(\frac{\partial}{\partial x} + f_n \frac{\partial}{\partial y} \right) \\
&= \left(\frac{\partial^2}{\partial x^2} + 2f_n \frac{\partial^2}{\partial x \partial y} + f_n^2 \frac{\partial^2}{\partial y^2} \right) \Big|_n
\end{aligned} \tag{7}$$

$$\begin{aligned}
\frac{d^2 f}{dx^2} \Big|_n &= \left(\frac{\partial^2 f}{\partial x^2} + 2f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \right) \Big|_n \\
&= \left(\frac{\partial^2 f}{\partial x^2} + 2f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2} \right) \Big|_n + \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} \right) \Big|_n \\
&= D^2 f \Big|_n + f_y D f \Big|_n
\end{aligned}$$

[illegible]

$$\begin{aligned}
\frac{d^3f}{dx^3} &= \frac{\partial^3 f}{\partial x^3} + 3f \frac{\partial^3 f}{\partial x^2 \partial y} + 3f^2 \frac{\partial^3 f}{\partial x \partial y^2} + f^3 \frac{\partial^3 f}{\partial y^3} + \frac{\partial f}{\partial y} \left(\frac{\partial^2 f}{\partial x^2} + 2f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2} \right) + \left(\frac{\partial f}{\partial y} \right)^2 \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \\
&+ 3 \left(\frac{\partial^2 f}{\partial x \partial y} \left\{ \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right\} + f \frac{\partial^2 f}{\partial y^2} \left\{ \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right\} \right) \\
&= \frac{\partial^3 f}{\partial x^3} + 3f \frac{\partial^3 f}{\partial x^2 \partial y} + 3f^2 \frac{\partial^3 f}{\partial x \partial y^2} + f^3 \frac{\partial^3 f}{\partial y^3} + \frac{\partial f}{\partial y} \left(\frac{\partial^2 f}{\partial x^2} + 2f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2} \right) + \left(\frac{\partial f}{\partial y} \right)^2 \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \\
&+ 3 \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right) \frac{\partial f}{\partial y}
\end{aligned}$$

$$\begin{aligned}
D^3 &= \left(\frac{\partial}{\partial x} + f_n \frac{\partial}{\partial y} \right) \left(\frac{\partial^2 f}{\partial x^2} + 2f_n \frac{\partial^2 f}{\partial x \partial y} + f_n^2 \frac{\partial^2 f}{\partial y^2} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} + 2f_n \frac{\partial^2 f}{\partial x \partial y} + f_n^2 \frac{\partial^2 f}{\partial y^2} \right) + f_n \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial x^2} + 2f_n \frac{\partial^2 f}{\partial x \partial y} + f_n^2 \frac{\partial^2 f}{\partial y^2} \right) \\
&= \frac{\partial^3 f}{\partial x^3} + 2f_n \frac{\partial^3 f}{\partial x^2 \partial y} + f_n^2 \frac{\partial^3 f}{\partial x \partial y^2} + f_n \frac{\partial^3 f}{\partial x^2 \partial y} + 2f_n^2 \frac{\partial^3 f}{\partial x \partial y^2} + f_n^3 \frac{\partial^3 f}{\partial y^3} \\
&= \frac{\partial^3 f}{\partial x^3} + 3f_n \frac{\partial^3 f}{\partial x^2 \partial y} + 3f_n^2 \frac{\partial^3 f}{\partial x \partial y^2} + f_n^3 \frac{\partial^3 f}{\partial y^3}
\end{aligned}$$

$$\left. \frac{d^3f}{dx^3} \right|_n = D^3 f + f_y D^2 f + f_y^2 Df + 3Df Df_y \Big|_n$$

putting equation 5 into 6

$$\begin{aligned}
y_{n+1} - y_n &= \sum_{t=1}^{\infty} (h_n)^t \frac{y_n^{(t)}}{t!} \\
&= \sum_{t=1}^{\infty} h_n^{t+1} \cdot \frac{\left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right)^t}{(t+1)!} \cdot f(x_n, y_n) \\
&= \left[hf + \frac{h^2}{2!} Df + \frac{h^3}{3!} \left(D^2 f + f_y Df \right) \right. \\
&\quad + \frac{h^4}{4!} \left(D^3 f + f_y D^2 f + f_y^2 Df + 3Df Df_y \right) \\
&\quad + \frac{h^5}{5!} \left(D^4 f + 6Df D^2 f_y + f_y^2 D^2 f + f_y^3 Df \right. \\
&\quad \left. \left. + 3f_{yy} (Df)^2 + f_y D^3 f + 7f_y Df Df_y \right) \right] + O(h^6)
\end{aligned} \tag{8}$$

My derivation upto this point is consistent and I can do each step with proper mathematical concepts.

$$k_i = h_n f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j)$$

$$\begin{aligned}
k_1 &= h_n f(x_n, y_n) = h_n f_n \\
k_2 &= h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21} k_1) \\
k_3 &= h_n f(x_n + \alpha_3 h_n, y_n + \beta_{31} k_1 + \beta_{32} k_2) \\
&= h_n f(x_n + \alpha_3 h_n, y_n + \beta_{31} k_1 + \beta_{32} k_1 + \beta_{32} k_2 - \beta_{32} k_1) \\
&= h_n f(x_n + \alpha_3 h_n, y_n + \sum_{j=1}^{i-1} \beta_{3j} k_j + \sum_{j=2}^{i-1} \beta_{32} (k_2 - h_n f_n)) \\
k_4 &= h_n f(x_n + \alpha_4 h_n, y_n + \beta_{41} k_1 + \beta_{42} k_2 + \beta_{43} k_3) \\
&= h_n f(x_n + \alpha_4 h_n, y_n + \beta_{41} k_1 + \beta_{42} k_1 + \beta_{43} k_1 + \beta_{42} k_2 + \beta_{43} k_3 - \beta_{42} k_1 - \beta_{43} k_1) \\
&= h_n f(x_n + \alpha_4 h_n, y_n + \sum_{j=1}^{i-1} \beta_{4j} h_n f_n + \beta_{42} (k_2 - h_n f_n) + \beta_{43} (k_3 - h_n f_n)) \\
&= h_n f(x_n + \alpha_4 h_n, y_n + \sum_{j=1}^{i-1} \beta_{4j} h_n f_n + \sum_{j=2}^{i-1} \beta_{4j} (k_j - h_n f_n))
\end{aligned}$$

Let

$$D_i = \alpha_i \frac{\partial}{\partial x} + \sum_{j=1}^{i-1} (\beta_{ij}) f_n \frac{\partial}{\partial y}$$

$$\begin{aligned}
k_i &= h_n f \left[x_n + \alpha_i h_n, y_n + \left(\sum_{j=1}^{i-1} \beta_{ij} \right) h_n f_n + \left(\sum_{j=2}^{i-1} \beta_{ij} (k_j - h_n f_n) \right) \right] \\
&= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_i + \sum_{j=2}^{i-1} \beta_{ij} (k_j - h_n f_n) \frac{\partial}{\partial y} \right)^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n}
\end{aligned}$$

Since $\alpha_1 = 0$

$$\begin{aligned}
k_1 &= h_n f(x, y) \Big|_{x=x_n, y=y_n} \\
k_2 &= h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21} k_1) = h_n f(x_n + \alpha_2 h_n, y_n + \beta_{21} h_n f_n) \\
&= h_n \sum_{t=0}^{\infty} \frac{h_n^t D_2^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n}
\end{aligned} \tag{9}$$

$$\begin{aligned}
k_3 &= h_n f(x_n + \alpha_3 h_n, y_n + \beta_{31} k_1 + \beta_{32} k_2) \\
&= h_n f \left[x_n + \alpha_3 h_n, y_n + h_n f_n (\beta_{31} + \beta_{32}) + \beta_{32} (k_2 - h_n f_n) \right] \\
&= h_n \sum_{t=0}^{\infty} \frac{\left(\alpha_i h_n \frac{\partial}{\partial x} + (\beta_{31} + \beta_{32}) h_n f_n \frac{\partial}{\partial y} + \beta_{32} (k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_3 + \beta_{32} (k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n}
\end{aligned} \tag{10}$$

$$\begin{aligned}
k_4 &= h_n f(x_n + \alpha_4 h_n, y_n + \beta_{41} k_1 + \beta_{42} k_2 + \beta_{43} k_3) \\
&= h_n f \left[x_n + \alpha_4 h_n, y_n + h_n f_n (\beta_{41} + \beta_{42} + \beta_{43}) + \beta_{41} (k_1 - h_n f_n) + \beta_{42} (k_2 - h_n f_n) \right. \\
&\quad \left. + \beta_{43} (k_3 - h_n f_n) \right] \\
&= h_n f \left[x_n + \alpha_4 h_n, y_n + \left(\sum_{j=1}^3 \beta_{4j} \right) h_n f_n + \left(\sum_{j=2}^3 \beta_{4j} \right) (k_j - h_n f_n) \right] \\
&= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right)^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n}
\end{aligned} \tag{11}$$

$$f(x_n + \alpha_i h_n, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j) = \sum_{t=0}^{\infty} \frac{h_n^t D_i^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n}$$

$$\begin{aligned}
k_1 &= h_n f(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n f_n \\
k_2 &= h_n \left(\frac{h_n^0 D_2^0}{0!} f(x, y) + \frac{h_n^1 D_2^1}{1!} f(x, y) + \frac{h_n^2 D_2^2}{2!} f(x, y) + \frac{h_n^3 D_2^3}{3!} f(x, y) + \frac{h_n^4 D_2^4}{4!} f(x, y) \right) \Big|_{x=x_n, y=y_n} \\
&\quad + O(h_n^6) \\
&= h_n f(x, y) \Big|_{x=x_n, y=y_n} + \frac{h_n^2}{1!} D_2 f(x, y) \Big|_{x=x_n, y=y_n} + \frac{h_n^3}{2!} D_2^2 f(x, y) \Big|_{x=x_n, y=y_n} + \frac{h_n^4}{3!} D_2^3 f(x, y) \Big|_{x=x_n, y=y_n} \\
&\quad + \frac{h_n^5}{4!} D_2^4 f(x, y) \Big|_{x=x_n, y=y_n} + O(h_n^6)
\end{aligned}$$

We will express $D_i^t f(x, y) \Big|_{x=x_n, y=y_n} = D_i^t f_n$. Which means the differentiation is already done than the expression is evaluated at $(x = x_n, y = y_n)$ and represents a constant value

$$(k_2 - h_n f_n) = h_n^2 D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n + \frac{h_n^4}{3!} D_2^3 f_n + \frac{h_n^5}{4!} D_2^4 f_n + O(h^6)$$

The evaluated $(k_2 - h_n f_n)$ expression is constant

$$\begin{aligned}
h_n \left[h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right]^2 f(x, y) \Big|_{x=x_n, y=y_n} &= h \left[h D_3 + \beta_{32} \left(\frac{h^2}{1!} D_2 f_n + \frac{h^3}{2!} D_2^2 f_n + \frac{h^4}{3!} D_2^3 f_n \right. \right. \\
&\quad \left. \left. + \frac{h^5}{4!} D_2^4 f_n \right) \frac{\partial}{\partial y} \right]^2 f(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n \left[h D_3 + \beta_{32} \left(\frac{h_n^2}{1!} D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n \right) \frac{\partial}{\partial y} \right] \\
&\quad \left[h_n D_3 f(x, y) + \beta_{32} \left(\frac{h_n^2}{1!} D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n \right) f_y(x, y) \right] \Big|_{x=x_n, y=y_n} \\
&= \left[h_n^2 D_3 + \beta_{32} \left(\frac{h_n^3}{1!} D_2 f_n + \frac{h_n^4}{2!} D_2^2 f_n \right) \frac{\partial}{\partial y} \right] \\
&\quad \left[h_n D_3 f(x, y) + \beta_{32} \left(\frac{h_n^2}{1!} D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n \right) f_y(x, y) \right] \Big|_{x=x_n, y=y_n} \\
&= h_n^2 D_3 [h_n D_3 f(x, y)] \Big|_{x=x_n, y=y_n} \\
&\quad + h_n^2 D_3 \left\{ \beta_{32} \left(\frac{h_n^2}{1!} D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n \right) f_y(x, y) \right\} \Big|_{x=x_n, y=y_n} \\
&\quad + \beta_{32} \left(\frac{h_n^3}{1!} D_2 f_n + \frac{h_n^4}{2!} D_2^2 f_n \right) \frac{\partial}{\partial y} [h_n D_3 f(x, y)] \Big|_{x=x_n, y=y_n} \\
&\quad + \beta_{32} \left(\frac{h_n^3}{1!} D_2 f_n + \frac{h_n^4}{2!} D_2^2 f_n \right) \frac{\partial}{\partial y} \left[\beta_{32} \left(\frac{h_n^2}{1!} D_2 f_n \right. \right. \\
&\quad \left. \left. + \frac{h_n^3}{2!} D_2^2 f_n \right) f_y(x, y) \right] \Big|_{x=x_n, y=y_n} \\
&= h_n^3 D_3^2 f_n + h_n^2 \beta_{32} D_3 \left\{ \frac{h_n^2}{1!} f_y(x, y) D_2 f_n \right\} + h_n^2 \beta_{32} D_3 \left\{ \frac{h^3}{2!} f_y(x, y) D_2^2 f_n \right\} \\
&\quad + \beta_{32} \left(\frac{h_n^3}{1!} D_2 f_n \right) \frac{\partial}{\partial y} [h_n D_3 f(x, y)] \Big|_{x=x_n, y=y_n} \\
&\quad + \beta_{32} \left(\frac{h_n^4}{2!} D_2^2 f_n \right) \frac{\partial}{\partial y} [h_n D_3 f(x, y)] \Big|_{x=x_n, y=y_n} \\
&\quad + \beta_{32} \left(\frac{h_n^3}{1!} D_2 f_n \right) \frac{\partial}{\partial y} \left[\beta_{32} \left(\frac{h_n^2}{1!} D_2 f_n \right) f_y(x, y) \right] \Big|_{x=x_n, y=y_n} \\
&= h_n^3 D_3^2 f_n + h_n^4 \beta_{32} D_3 f_{y|_n} D_2 f_n + \frac{h_n^5}{2!} \beta_{32} D_3 f_{y|_n} D_2^2 f_n \\
&\quad + h_n^4 \beta_{23} D_3 f_{y|_n} D_2 f_n + \frac{h_n^5}{2!} \beta_{23} D_3 f_{y|_n} D_2^2 f_n + h_n^5 \beta_{32}^2 (D_2 f_n)^2 f_{yy|_n} \\
&= h_n^3 D_3^2 f_n + 2h_n^4 (D_2 f_n) D_3 f_{y|_n} + h_n^5 \beta_{32} (D_2^2 f_n) D_3 f_{y|_n} + h_n^5 \beta_{32}^2 (D_2 f_n)^2 f_{yy|_n}
\end{aligned}$$

$$\begin{aligned}
h_n \left[h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right]^3 f &= \left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right) \left[h_n \left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^2 f(x, y) \right] \Big|_{x=x_n, y=y_n} \\
&\quad + O(h_n^6) \\
&= \left(h_n D_3 + \beta_{32} h_n^2 D_2 f_n \frac{\partial}{\partial y} \right) \left[h_n^3 D_3^2 f(x, y) + 2h_n^4 \beta_{32} (D_2 f_n) D_3 f_y(x, y) \right] \Big|_{x=x_n, y=y_n} \\
&\quad + O(h^6) \\
&= h_n^4 D_3^3 f(x, y) \Big|_{x=x_n, y=y_n} + 2h^5 \beta_{32} (D_2 f_n) D_3^2 f_y(x, y) \Big|_{x=x_n, y=y_n} \\
&\quad + h^5 \beta_{32} (D_2 f_n) D_3^2 f_y(x, y) \Big|_{x=x_n, y=y_n} + O(h^6)
\end{aligned}$$

$$h_n \left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^3 f(x, y) \Big|_{x=x_n, y=y_n} = h_n^4 D_3^3 f_n + 3h_n^5 \beta_{32}(D_2 f_n) D_3^2 f_{y|_n} + O(h^6)$$

$$\begin{aligned} h_n \left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^4 f(x, y) \Big|_{x=x_n, y=y_n} &= \left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right) \\ &\quad \left[h_n \left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^3 f(x, y) \right] \Big|_{x=x_n, y=y_n} \\ &\quad + O(h_n^6) \\ &= \left(h_n D_3 \right) \left[h_n^4 D_3^3 f(x, y) \right] \Big|_{x=x_n, y=y_n} + O(h_n^6) \\ &= h_n^5 D_3^4 f_n + O(h_n^6) \end{aligned}$$

$$\begin{aligned} k_3 &= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n} \\ &= h_n f_n + h_n^2 D_3 f_n + h_n \beta_{32}(k_2 - h_n f_n) f_{y|_n} + h_n \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^2}{2!} f(x, y) \Big|_{x=x_n, y=y_n} \\ &\quad + h_n \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^3}{3!} f(x, y) \Big|_{x=x_n, y=y_n} + h_n \frac{\left(h_n D_3 + \beta_{32}(k_2 - h_n f_n) \frac{\partial}{\partial y} \right)^4}{4!} f(x, y) \Big|_{x=x_n, y=y_n} + O(h^6) \\ &= h_n f_n + h_n^2 D_3 f_n + h_n \beta_{32} \left(h_n^2 D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n + \frac{h_n^4}{3!} D_2^3 f_n \right) f_{y|_n} + \frac{h_n^3}{2} D_3^2 f_n + h_n^4 \beta_{32}(D_2 f_n) D_3 f_{y|_n} \\ &\quad + \frac{h_n^5}{2} \beta_{32}(D_2^2 f_n) D_3 f_{y|_n} + \frac{h_n^5}{2} \beta_{32}^2(D_2 f_n)^2 f_{yy|_n} + \frac{h_n^4}{6} D_3^3 f_n + \frac{h_n^5}{2} \beta_{32}(D_2 f_n) D_3^2 f_{y|_n} + \frac{h_n^5}{24} D_3^4 f_n + O(h^6) \\ &= h_n f_n + h_n^2 D_3 f_n + h_n^3 \beta_{32} f_{y|_n} D_2 f_n + \frac{h_n^4}{2} \beta_{32} f_{y|_n} D_2^2 f_n + \frac{h_n^5}{6} \beta_{32} f_{y|_n} D_2^3 f_n + \frac{h_n^3}{2} D_3^2 f_n + h_n^4 \beta_{32}(D_2 f_n) D_3 f_{y|_n} \\ &\quad + \frac{h_n^5}{2} \beta_{32}(D_2^2 f_n) D_3 f_{y|_n} + \frac{h_n^5}{2} \beta_{32}^2(D_2 f_n)^2 f_{yy|_n} + \frac{h_n^4}{6} D_3^3 f_n + \frac{h_n^5}{2} \beta_{32} D_3^2 f_{y|_n} D_2 f_n + \frac{h_n^5}{24} D_3^4 f_n + O(h^6) \\ &= h_n f_n + h_n^2 D_3 f_n + h_n^3 \left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y|_n} D_2 f_n \right) + h_n^4 \left(\frac{1}{6} D_3^3 f_n + \frac{1}{2} \beta_{32} f_{y|_n} D_2^2 f_n + \beta_{32}(D_2 f_n) D_3 f_{y|_n} \right) \\ &\quad + h^5 \left(\frac{1}{24} D_3^4 f_n + \frac{1}{6} \beta_{32} f_{y|_n} D_2^3 f_n + \frac{1}{2} \beta_{32}^2(D_2 f_n)^2 f_{yy|_n} + \frac{1}{2} \beta_{32}(D_2^2 f_n) D_3 f_{y|_n} + \frac{1}{2} \beta_{32}(D_2 f_n) D_3^2 f_{y|_n} \right) + O(h^6) \\ \\ k_3 - h_n f_n &= h_n^2 D_3 f_n + h_n^3 \left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y|_n} D_2 f_n \right) + h_n^4 \left(\frac{1}{6} D_3^3 f_n + \frac{1}{2} \beta_{32} f_{y|_n} D_2^2 f_n + \beta_{32}(D_2 f_n) D_3 f_{y|_n} \right) \end{aligned} \tag{12}$$

$$\begin{aligned}
h_n \left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right] f(x, y) \Big|_{x=x_n, y=y_n} &= h_n^2 D_4 f(x, y) \Big|_{x=x_n, y=y_n} + h_n \beta_{42} (k_2 - h_n f_n) f_y(x, y) \Big|_{x=x_n, y=y_n} \\
&+ h_n \beta_{43} (k_3 - h_n f_n) f_y(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n^2 D_4 f_n + h_n \beta_{42} \left(h_n^2 D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n + \frac{h_n^4}{3!} D_2^3 f_n \right) f_y(x, y) \Big|_{x=x_n, y=y_n} \\
&+ h_n \beta_{43} \left[h_n^2 D_3 f_n + h_n^3 \left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y|n} D_2 f_n \right) \right. \\
&\left. + h_n^4 \left(\frac{1}{6} D_3^3 f_n + \frac{1}{2} \beta_{32} f_{y|n} D_2^2 f_n + \beta_{32} (D_2 f_n) D_3 f_{y|n} \right) \right] f_y(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n^2 D_4 f_n + h_n^3 \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) f_{y|n} \\
&+ h_n^4 \left(\frac{1}{2} \beta_{43} (D_3^2 f_n) + \beta_{32} \beta_{43} f_{y|n} (D_2 f_n) + \frac{1}{2} \beta_{42} (D_2^2 f_n) \right) f_{y|n} \\
&+ h_n^5 \left(\frac{1}{6} \beta_{43} D_3^3 f_n + \frac{1}{2} \beta_{32} \beta_{43} f_{y|n} D_2^2 f_n + \beta_{32} \beta_{43} (D_2 f_n) D_3 f_{y|n} + \frac{1}{6} \beta_{42} (D_2^3 f_n) \right) f_{y|n}
\end{aligned}$$

$$\begin{aligned}
h_n \left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right]^2 f(x, y) \Big|_{x=x_n, y=y_n} &= \left(h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right) \\
&\left[h_n^2 D_4 + h_n \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right] f(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n D_4 \left[h_n^2 D_4 f_n + h_n^3 \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) f_{y|n} \right] \\
&+ h_n D_4 \left[h_n^4 \left(\frac{1}{2} \beta_{43} (D_3^2 f_n) + \beta_{32} \beta_{43} f_{y|n} (D_2 f_n) + \frac{1}{2} \beta_{42} (D_2^2 f_n) \right) f_{y|n} \right] \\
&+ \beta_{42} (k_2 - h_n f_n) \frac{\partial}{\partial} \left[h_n^2 D_4 f_n + h_n^3 \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) f_{y|n} \right] \\
&+ \beta_{43} (k_3 - h_n f_n) \frac{\partial}{\partial y} \left[h_n^2 D_4 f_n + h_n^3 \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) f_{y|n} \right] + \dots \\
&= h_n^3 D_4^2 f_n + h_n^4 \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) D_4 f_{y|n} \\
&+ h_n^5 \left(\frac{1}{2} \beta_{43} (D_3^2 f_n) + \beta_{32} \beta_{43} f_{y|n} (D_2 f_n) + \frac{1}{2} \beta_{42} (D_2^2 f_n) \right) D_4 f_{y|n} \\
&+ h_n^2 \beta_{42} \left(h_n^2 D_2 f_n + \frac{h_n^3}{2!} D_2^2 f_n \right) (D_4 f_n) f_{yy|n} \\
&+ h_n^3 \beta_{42} (h_n^2 D_2 f_n) \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) f_{yy|n} \\
&+ h_n^2 \beta_{43} \left[h_n^2 D_3 f_n + h_n^3 \left(\frac{1}{2} D_3^2 f_n + \beta_{32} f_{y|n} D_2 f_n \right) \right] (D_4 f_n) f_{yy|n} \\
&+ h_n^3 \beta_{43} \left(h_n^2 D_3 f_n \right) \left(\beta_{42} (D_2 f_n) + \beta_{43} (D_3 f_n) \right) f_{yy|n}
\end{aligned}$$

$$\begin{aligned}
h_n \left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right]^3 f(x, y) \Big|_{x=x_n, y=y_n} &= \left(h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right) \\
&\left[h_n^2 D_4 + h_n \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right] f(x, y) \Big|_{x=x_n, y=y_n}
\end{aligned}$$

$$\begin{aligned}
k_4 &= h_n \sum_{t=0}^{\infty} \frac{\left(h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right)^t}{t!} f(x, y) \Big|_{x=x_n, y=y_n} \\
&= h_n f_n + h_n \left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right] f(x, y) \Big|_{x=x_n, y=y_n} + h_n \frac{\left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right]^2}{2} f(x, y) \Big|_{x=x_n, y=y_n} \\
&\quad + h_n \frac{\left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right]^3}{2} f(x, y) \Big|_{x=x_n, y=y_n} + h_n \frac{\left[h_n D_4 + \sum_{j=2}^3 \beta_{4j} (k_j - h_n f_n) \frac{\partial}{\partial y} \right]^4}{2} f(x, y) \Big|_{x=x_n, y=y_n} + \dots
\end{aligned} \tag{13}$$

$$\begin{aligned}
k_4 &= h f + h^2 D_4 f + h^3 \left(\frac{1}{2} D_4^2 f + \beta_{42} f_y D_2 f + \beta_{43} f_y D_3 f \right) + h^4 \left[\frac{1}{6} D_4^3 f + \frac{1}{2} \beta_{42} f_y D_2^2 f \right. \\
&\quad \left. + \beta_{32} \beta_{43} (f_y)^2 D_2 f + \frac{1}{2} \beta_{43} f_y D_3^2 f + \beta_{42} D_2 f D_4 f_y + \beta_{43} D_3 f D_4 f_y \right] + h^5 \left(\frac{1}{24} D_4^4 f + \frac{1}{6} \beta_{42} f_y D_2^3 f \right. \\
&\quad \left. + \frac{1}{2} \beta_{32} \beta_{43} (f_y)^2 D_2^2 f + \beta_{32} \beta_{43} f_y D_2 f D_3 f_y + \frac{1}{6} \beta_{43} f_y D_3^3 f + \frac{1}{2} \beta_{42} D_4 f_y D_2^2 f + \frac{1}{2} \beta_{43} D_4 f_y D_3^2 f \right. \\
&\quad \left. + \frac{1}{2} \beta_{42}^2 f_{yy} D_2^2 f + \beta_{42} \beta_{43} f_{yy} D_2 f D_3 f + \frac{1}{2} \beta_{43}^2 f_{yy} D_3^2 f + \frac{1}{2} \beta_{42} D_2 f D_4^2 f_y \right. \\
&\quad \left. + \frac{1}{2} \beta_{43} D_3 f D_4^2 f_y + \beta_{43} \beta_{32} f_y D_2 f D_4 f_y \right) \Big|_n + O(h^6)
\end{aligned}$$

$$\begin{aligned}
y_{n+1} - y_n &= w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4 \\
&= \left[h f + \frac{h^2}{2!} D f + \frac{h^3}{3!} \left(D^2 f + f_y D f \right) + \frac{h^4}{4!} \left(D^3 f + f_y D^2 f + f_y^2 D f + 3 D f D f_y \right) \right. \\
&\quad \left. + \frac{h^5}{5!} \left(D^4 f + 6 D f D^2 f_y + f_y^2 D^2 f + f_y^3 D f + 3 f_{yy} (D f)^2 + f_y D^3 f + 7 f_y D f D f_y \right) \right] \\
&= w_1 h f + w_2 \left\{ h f + \frac{h^2}{1!} D_2 f + \frac{h^3}{2!} D_2^2 f + \frac{h^4}{3!} D_2^3 f + \frac{h^5}{4!} D_2^4 f \right\} \\
&\quad + w_3 \left\{ h f + h^2 D_3 f + h^3 \left(\frac{1}{2} D_3^2 f + f_y \beta_{32} D_2 f \right) + h^4 \left(\frac{1}{6} D_3^3 f + \frac{1}{2} f_y \beta_{32} D_2^2 f + \beta_{32} D_3 f_y D_2 f \right) \right. \\
&\quad \left. + h^5 \left(\frac{1}{24} D_3^4 f + \frac{\beta_{32}}{6} f_y D_2^3 f + \frac{\beta_{32}^2}{2} f_{yy} (D_2 f)^2 + \frac{\beta_{32}}{2} D_3 f_y D_2^2 f + \frac{\beta_{32}}{2} D_3^2 f_y D_2 f \right) \right\} \\
&\quad + w_4 \left\{ h f + h^2 D_4 f + h^3 \left(\frac{1}{2} D_4^2 f + \beta_{42} f_y D_2 f + \beta_{43} f_y D_3 f \right) + h^4 \left(\frac{1}{6} D_4^3 f + \frac{1}{2} \beta_{42} f_y D_2^2 f \right. \right. \\
&\quad \left. \left. + \beta_{32} \beta_{43} (f_y)^2 D_2 f + \frac{1}{2} \beta_{43} f_y D_3^2 f + \beta_{42} D_2 f D_4 f_y + \beta_{43} D_3 f D_4 f_y \right) + h^5 \left(\frac{1}{24} D_4^4 f + \frac{1}{6} \beta_{42} f_y D_2^3 f \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \beta_{32} \beta_{43} (f_y)^2 D_2^2 f + \beta_{32} \beta_{43} f_y D_2 f D_3 f_y + \frac{1}{6} \beta_{43} f_y D_3^3 f + \frac{1}{2} \beta_{42} D_4 f_y D_2^2 f + \frac{1}{2} \beta_{43} D_4 f_y D_3^2 f \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \beta_{42}^2 f_{yy} D_2^2 f + \beta_{42} \beta_{43} f_{yy} D_2 f D_3 f + \frac{1}{2} \beta_{43}^2 f_{yy} D_3^2 f + \frac{1}{2} \beta_{42} D_2 f D_4^2 f_y \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \beta_{43} D_3 f D_4^2 f_y + \beta_{43} \beta_{32} f_y D_2 f D_4 f_y \right) \right\}
\end{aligned}$$

$$\begin{aligned}
w_1 + w_2 + w_3 + w_4 &= 1 \\
w_2 D_2 f + w_3 D_3 f + w_4 D_4 f &= \frac{1}{2} D f \\
\frac{1}{2} w_2 D_2^2 f + w_3 \left(\frac{1}{2} D_3^2 f + f_y \beta_{32} D_2 f \right) \\
+ w_4 \left(\frac{1}{2} D_4^2 f + \beta_{42} f_y D_2 f + \beta_{43} f_y D_3 f \right) &= \frac{1}{3!} \left(D^2 f + f_y D f \right) \\
\frac{1}{3!} w_2 D_2^3 f + w_3 \left(\frac{1}{6} D_3^3 f + \frac{1}{2} f_y \beta_{32} D_2^2 f + \beta_{32} D_3 f_y D_2 f \right) \\
+ w_4 \left(\frac{1}{6} D_4^3 f + \frac{1}{2} \beta_{42} f_y D_2^2 f + \beta_{32} \beta_{43} (f_y)^2 D_2 f \right. \\
\left. + \frac{1}{2} \beta_{43} f_y D_3^2 f + \beta_{42} D_2 f D_4 f_y + \beta_{43} D_3 f D_4 f_y \right) &= \frac{1}{4!} \left(D^3 f + f_y D^2 f + f_y^2 D f + 3 D f D f_y \right)
\end{aligned}$$

$$\begin{aligned}
D_3 &= \alpha_3 \frac{\partial}{\partial x} + \sum_{j=1}^2 \beta_{3j} f_n \frac{\partial}{\partial y} \\
D_2 f &= \alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_n \frac{\partial f}{\partial y}
\end{aligned}$$