Tensor

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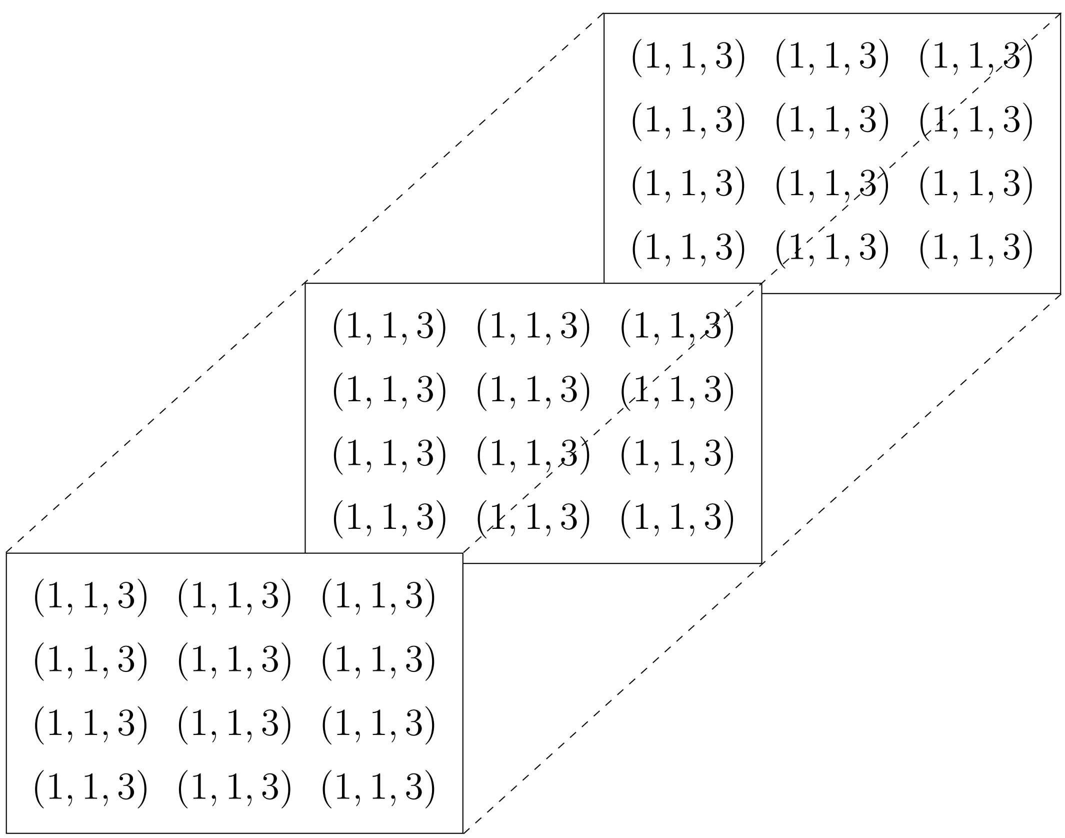
What is a tensor?

1. In basic terms tensor can be called a multidimentional array of numbers. For example,

$$[S] \qquad \begin{bmatrix} v1 \\ v2 \\ v3 \\ v4 \\ . \\ . \\ . \\ v_n \end{bmatrix} \qquad \begin{bmatrix} m_{11} & m_{12} & m_{13} & ..... & m_{1n} \\ m_{21} & m_{22} & m_{23} & ..... & m_{2n} \\ m_{31} & m_{31} & m_{32} & ..... & m_{3n} \\ m_{41} & m_{41} & m_{42} & ..... & m_{4n} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ m_{m1} & m_{m2} & m_{m3} & ..... & m_{mn} \end{bmatrix}$$

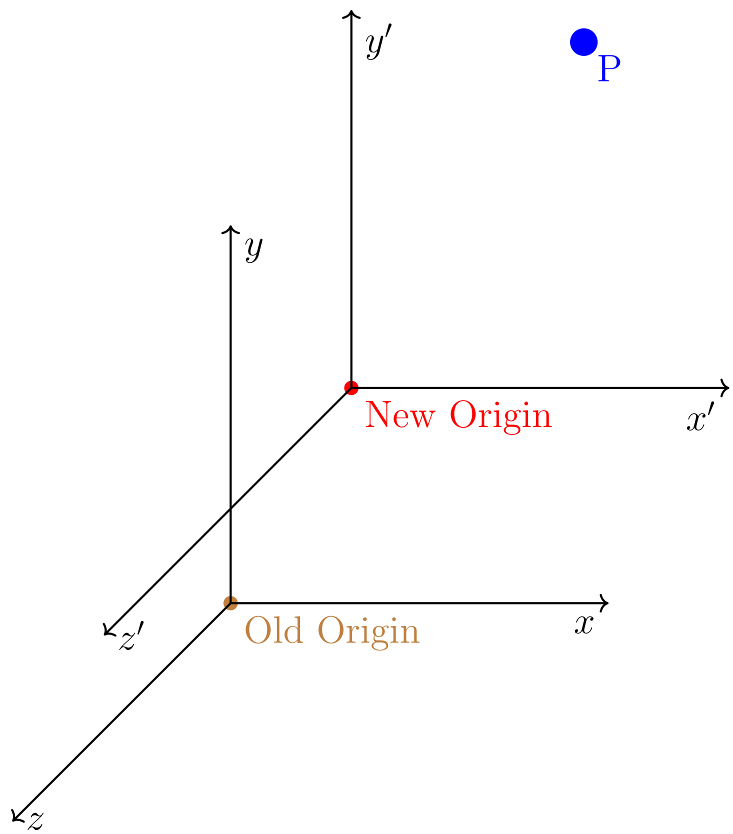


2. But this is not a clear definition of tensor. Of course we can represent tensors as arrays but that's does not give the whole picture. One better definition is:

Tensor is an object that is invariant under a change of co-ordinates system and has components that change in a special and predictable way under a change of co-ordinates system

3. A collection of vectors and covectors combined together using tensor product.(Tensor can be understood as partial derivative and gradient that transforms with jacobian matrix)

Co-Ordinate Transformation



Now let us declare three unit vector $\vec{e}_1$,$\vec{e}_2$,$\vec{e}_3$ in the old basis and three unit vector $\widetilde{\overrightarrow{e_1}}$, $\widetilde{\overrightarrow{e_2}}$, $\widetilde{\overrightarrow{e_3}}$.

Trying to express new unit vector in terms of old gives us

$$
\begin{aligned}
\widetilde{\overrightarrow{e_1}} &= \mathrm{F}_1^1\overrightarrow{e_1} + \mathrm{F}_1^2\overrightarrow{e_2} + \mathrm{F}_1^3\overrightarrow{e_3} \\
\widetilde{\overrightarrow{e_2}} &= \mathrm{F}_2^1\overrightarrow{e_1} + \mathrm{F}_2^2\overrightarrow{e_2} + \mathrm{F}_2^3\overrightarrow{e_3} \\
\widetilde{\overrightarrow{e_3}} &= \mathrm{F}_3^1\overrightarrow{e_1} + \mathrm{F}_3^2\overrightarrow{e_2} + \mathrm{F}_3^3\overrightarrow{e_3}
\end{aligned} \tag{1}
$$

We call $\overrightarrow{\mathrm{F}} = \begin{bmatrix} \mathrm{F}_1^1 & \mathrm{F}_2^1 & \mathrm{F}_3^1 \\ \mathrm{F}_1^2 & \mathrm{F}_2^2 & \mathrm{F}_3^2 \\ \mathrm{F}_1^3 & \mathrm{F}_3^3 & \mathrm{F}_3^3 \end{bmatrix}_{e_i}$ a forward matrix that transforms old basis to new basis.

and old vector in terms of new vector

$$
\begin{aligned}
\overrightarrow{e_1} &= \mathrm{B}_1^1\widetilde{\overrightarrow{e_1}} + \mathrm{B}_1^2\widetilde{\overrightarrow{e_2}} + \mathrm{B}_1^3\widetilde{\overrightarrow{e_3}} \\
\overrightarrow{e_2} &= \mathrm{B}_2^1\widetilde{\overrightarrow{e_1}} + \mathrm{B}_2^2\widetilde{\overrightarrow{e_2}} + \mathrm{B}_2^3\widetilde{\overrightarrow{e_3}} \\
\overrightarrow{e_3} &= \mathrm{B}_3^1\widetilde{\overrightarrow{e_1}} + \mathrm{B}_3^2\widetilde{\overrightarrow{e_3}} + \mathrm{B}_3^3\widetilde{\overrightarrow{e_3}}
\end{aligned} \tag{2}
$$

Similarly We call $\overrightarrow{\mathrm{B}} = \begin{bmatrix} \mathrm{B}_1^1 & \mathrm{B}_2^1 & \mathrm{B}_3^1 \\ \mathrm{B}_1^2 & \mathrm{B}_2^2 & \mathrm{B}_3^2 \\ \mathrm{B}_1^3 & \mathrm{B}_2^3 & \mathrm{B}_3^3 \end{bmatrix}$ a backward matrix that transform new basis to old basis.

To sum up

$$
\widetilde{\overrightarrow{e_j}} = \mathrm{F}_j^k\overrightarrow{e_k} \text{ and } \overrightarrow{e_k} = \mathrm{B}_k^i\widetilde{\overrightarrow{e_i}} \tag{3}
$$

from equation (3) we get

$$
\begin{aligned}
\widetilde{\overrightarrow{e_j}} &= \mathrm{F}_j^k\overrightarrow{e_k} \\
&= F_j^k B_k^i \widetilde{\overrightarrow{e_i}}
\end{aligned}
$$

To equate both sides $i = j$ and that value will need to be 1 or when $i \neq j$ the value will be 0.

Thus

$$
\mathrm{F}_j^k B_k^i = \delta_j^i
$$

$$
\delta_j^i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad \text{(this is also known as kronecker delta)} \tag{4}
$$

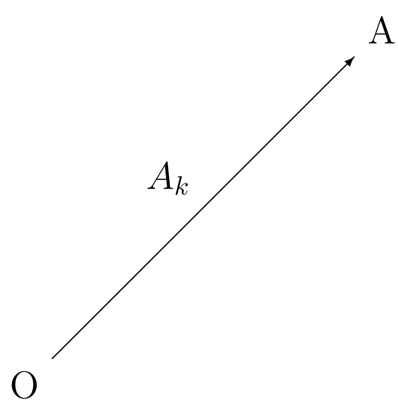
Vector

Vector is an example of Tensor.

1. Vector can be expressed as a list of numbers.

$$\vec{v} = \begin{bmatrix} v1 \\ v2 \\ v3 \\ v4 \\ . \\ . \\ . \\ v_n \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} w1 \\ w2 \\ w3 \\ w4 \\ . \\ . \\ . \\ w_n \end{bmatrix}$$

2. Vector is an arrow which has geometrical meaning; a value and a direction



But not all vectors can be represented using arrows and has physical meaning

3. A member of vector space.

***Vector Space(linear space) : a vector space (also called a linear space) is a set of objects called vectors, which may be added together and multiplied ("scaled") by numbers called scalars and the result will also be a part of that set.**

Field : In mathematics, a field is a set on which addition, subtraction, multiplication, and division are defined and behave as the corresponding operations on rational and real numbers do.

Cartesian Product : In mathematics, specifically set theory, the cartesian product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where a is in A and b is in B. In mathematical terms,

$$A \times B = (a, b) \mid a \in A \text{ and } b \in B$$

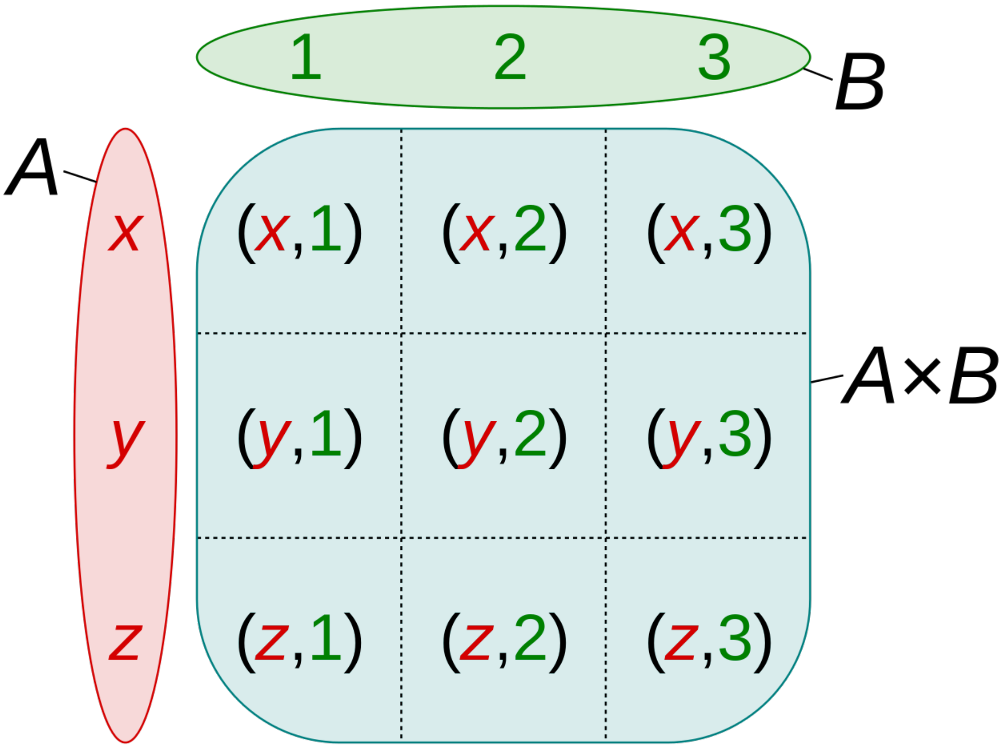


Figure 1: Cartesian Product

A vector space over a field F* is a set V together with two operations that satisfy the eight axioms listed below. In the following, V × V denotes the Cartesian product of V with itself, and → denotes a mapping from one set to another.

Table 1: Axioms

Axiom	Meaning
Associativity of vector addition	u + (v + w) = (u + v) + w
Commutativity of vector addition	u + v = v + u
Identity element of vector addition	There exists an element $0 \in V$, called the zero vector, such that v + 0 = v for all $v \in V$.
Inverse elements of vector addition	For every $v \in V$, there exists an element, $-v \in V$, called the additive inverse of v such that v + (-v) = 0
Compatibility of scalar multiplication with field multiplication	$\alpha(b\overrightarrow{v}) = (ab)\overrightarrow{v}$

Bound Vectors and Free Vectors : All displacement vectors are bound vectors. Free vectors are those representing global physical parameters. such as angular velocity of an object $\overrightarrow{\omega}$. The rotation vector can be moved along anywhere through it's rotational axis.

Vector Transformation Rules

$\overrightarrow{v} = v^1\overrightarrow{e_1} + v^2\overrightarrow{e_2} + v^3\overrightarrow{e_3} + ... + v^n\overrightarrow{e_n} = v^j\overrightarrow{e_j}$

$\overrightarrow{v} = \widetilde{v}^1\widetilde{\overrightarrow{e_1}} + \widetilde{v}^2\widetilde{\overrightarrow{e_2}} + \widetilde{v}^3\widetilde{\overrightarrow{e_3}} + ... + \widetilde{v}^n\widetilde{\overrightarrow{e_n}} = \widetilde{v}^j\widetilde{\overrightarrow{e_j}}$

vectors are invariant as a whole but not their components.

So,

$$\begin{aligned} v^j\overrightarrow{e_j} &= \widetilde{v}^j\widetilde{\overrightarrow{e_j}} \\ &= \widetilde{v}^j \mathrm{F}^k_j \overrightarrow{e_k} \end{aligned}$$

Hence, $v^k = \mathrm{F}^k_j \widetilde{v}^j$ and conversely $\widetilde{v}^k = \mathrm{B}^k_j v^j$ So, we conclude

co-ordinate transformation	vector transformation
$\boxed{\begin{aligned} \widetilde{\overrightarrow{e_j}} &= \mathrm{F}^k_j \overrightarrow{e_k} \\ \overrightarrow{e_j} &= \mathrm{B}^k_j \widetilde{\overrightarrow{e_k}} \end{aligned}}$	$\boxed{\begin{aligned} v^k &= \mathrm{F}^k_j \widetilde{v}^j \\ \widetilde{v}^k &= \mathrm{B}^k_j v^j \end{aligned}}$

Vectors are contravariant tensor because to express new vector in terms of old vector, we need backward matrix and vice versa.

Co-vector

1. basically column vector that can be written as row vector such as

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and one might think we can get this by tranforming $\begin{bmatrix} 2 & 1 \end{bmatrix}$ But it only works in orthonormal basis. There are two other basis, orthogonal basis and skew basis.

In orthonormal basis,

$$\begin{aligned} \overrightarrow{e_1} &\perp \overrightarrow{e_2} \\ \overrightarrow{e_2} &\perp \overrightarrow{e_3} \\ \overrightarrow{e_3} &\perp \overrightarrow{e_1} \end{aligned} \tag{5}$$

$$\begin{aligned} |\overrightarrow{e_1}| &= 1 \\ |\overrightarrow{e_1}| &= 1 \\ |\overrightarrow{e_1}| &= 1 \end{aligned} \tag{6}$$

In cases of orthogonal basis equation (5) holds but not equation (6) and for skew coordinates system none of them holds true.

2. Row vector that acts as a function on column vector.

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 2 \cdot 3 + 1 \cdot (-4) = 2$$

$$\alpha(\overrightarrow{v}) = \alpha_1 v^1 + \alpha_2 v^2 + ... + \alpha_n v^n = \alpha_i v^i$$
$$\therefore \alpha : \mathbf{V} \Rightarrow \mathbb{R}$$

Covectors follows two rules which conforms linearity:

$$\boxed{\begin{gathered} \alpha(\overrightarrow{v} + \overrightarrow{w}) = \alpha(\overrightarrow{v}) + \alpha(\overrightarrow{w}) \\ \alpha(n\overrightarrow{v}) = n\alpha(\overrightarrow{v}) \end{gathered}}$$

they are elements of dual vector space $\mathbf{V}^*$ which has the following property

$$\boxed{\begin{gathered} (n\alpha)(\overrightarrow{v} + \overrightarrow{w}) = n\alpha(\overrightarrow{v}) + \alpha(\overrightarrow{w}) \\ (\beta + \gamma)(\overrightarrow{v}) = \beta(\overrightarrow{v}) + \gamma(\overrightarrow{v}) \end{gathered}}$$

How can we express covector just like vectors. We will take the basis $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ for V

We will introduce special covectors called dual basis $\epsilon^1, \epsilon^2, \epsilon^3 : \mathbf{V} \Rightarrow \mathbb{R}$

- $\epsilon^1(\overrightarrow{e}_1) = 1 \qquad \epsilon^1(\overrightarrow{e}_2) = 0 \qquad \epsilon^1(\overrightarrow{e}_3) = 0$
- $\epsilon^2(\overrightarrow{e}_1) = 0 \qquad \epsilon^2(\overrightarrow{e}_2) = 1 \qquad \epsilon^2(\overrightarrow{e}_3) = 0$
- $\epsilon^3(\overrightarrow{e}_1) = 0 \qquad \epsilon^3(\overrightarrow{e}_2) = 0 \qquad \epsilon^3(\overrightarrow{e}_3) = 1$

To sum up, $\epsilon^i(\overrightarrow{e}_j) = \delta^i_j$ same as expression 4

$$\begin{aligned} \epsilon^1(\overrightarrow{v}) &= \epsilon^1(v^1\overrightarrow{e}_1) + \epsilon^1(v^2\overrightarrow{e}_2) + \epsilon^1(v^3\overrightarrow{e}_3) = v^1 \\ \epsilon^2(\overrightarrow{v}) &= \epsilon^2(v^1\overrightarrow{e}_1) + \epsilon^2(v^2\overrightarrow{e}_2) + \epsilon^2(v^3\overrightarrow{e}_3) = v^2 \\ \epsilon^3(\overrightarrow{v}) &= \epsilon^3(v^1\overrightarrow{e}_1) + \epsilon^3(v^2\overrightarrow{e}_2) + \epsilon^3(v^3\overrightarrow{e}_3) = v^3 \end{aligned}$$

The ϵ are projecting out the vector components.

$$
\begin{aligned}
\alpha(\overrightarrow{v}) &= \alpha(v^1\overrightarrow{e}_1 + v^2\overrightarrow{e}_2 + v^3\overrightarrow{e}_3) \\
&= v^1\alpha(\overrightarrow{e}_1) + v^2\alpha(\overrightarrow{e}_2) + v^3\alpha(\overrightarrow{e}_3) \\
&= \epsilon^1(\overrightarrow{v})\alpha(\overrightarrow{e_1}) + \epsilon^2(\overrightarrow{v})\alpha(\overrightarrow{e_1}) + \epsilon^3(\overrightarrow{v})\alpha(\overrightarrow{e_1})
\end{aligned}
$$

defining $\alpha(\overrightarrow{e_1}) = \alpha_1$, $\alpha(\overrightarrow{e_2}) = \alpha_2$ and $\alpha(\overrightarrow{e_3}) = \alpha_3$

we can express a covector in a linear combination of ϵ covectors

$$
\begin{aligned}
\alpha(\overrightarrow{v}) &= \alpha_1\epsilon^1(\overrightarrow{v}) + \alpha_2\epsilon^2(\overrightarrow{v}) + \alpha_3\epsilon^3(\overrightarrow{v}) \\
\alpha &= \alpha_1\epsilon^1 + \alpha_2\epsilon^2 + \alpha_3\epsilon^3
\end{aligned}
$$

Now lets take another basis $\widetilde{\overrightarrow{e_1}}, \widetilde{\overrightarrow{e_2}}, \widetilde{\overrightarrow{e_3}}$ and α can be expressed

$$
\alpha = \widetilde{\alpha_1}\widetilde{\epsilon^1} + \widetilde{\alpha_2}\widetilde{\epsilon^2} + \widetilde{\alpha_3}\widetilde{\epsilon^3}
$$

$$
\begin{aligned}
\widetilde{\alpha_1} &= \alpha(\widetilde{\overrightarrow{e_1}}) \\
\widetilde{\alpha_2} &= \alpha(\widetilde{\overrightarrow{e_2}}) \\
\widetilde{\alpha_3} &= \alpha(\widetilde{\overrightarrow{e_3}})
\end{aligned}
$$

$$
\begin{aligned}
\alpha(\overrightarrow{v}) = \alpha(v^i\overrightarrow{e_i}) &= \alpha(\widetilde{v^j}\widetilde{\overrightarrow{e_j}}) \\
= v^i\alpha(\overrightarrow{e_i}) &= \widetilde{v^j}\alpha(\widetilde{\overrightarrow{e_j}}) \\
= v^k\alpha(\overrightarrow{e_k}) &= v^k\alpha_k = \mathrm{B}^j_k v^k\widetilde{\alpha}_j
\end{aligned}
$$

Therefore,

$$
\alpha_k = \mathrm{B}^j_k\widetilde{\alpha}_j
$$

Similarly, we can find

$$
\widetilde{\alpha}_j = \mathrm{F}^i_j\alpha_i
$$

co-ordinate transformation	covector transformation
$\boxed{\begin{aligned}\widetilde{\overrightarrow{e_j}} &= \mathrm{F}^k_j\overrightarrow{e_k} \\ \overrightarrow{e_j} &= \mathrm{B}^k_j\widetilde{\overrightarrow{e_k}}\end{aligned}}$	$\boxed{\begin{aligned}\alpha_k &= \mathrm{B}^j_k\widetilde{\alpha}_j \\ \widetilde{\alpha}_k &= \mathrm{F}^j_k\alpha_j\end{aligned}}$

That's why covectors are called covariant vector.

$$
\begin{aligned}
\widetilde{\epsilon^1} &= Q_1^1\epsilon^1 + Q_2^1\epsilon^2 + Q_3^1\epsilon^3 \\
\widetilde{\epsilon^2} &= Q_1^2\epsilon^1 + Q_2^2\epsilon^2 + Q_3^2\epsilon^3 \\
\widetilde{\epsilon^3} &= Q_1^3\epsilon^1 + Q_2^3\epsilon^2 + Q_3^3\epsilon^3
\end{aligned}
$$

$$\widetilde{\epsilon^i} = Q_j^i\epsilon^j$$

$$
\begin{aligned}
\widetilde{\epsilon^i}(\widetilde{\overrightarrow{e_k}}) =& Q_j^i\epsilon^j(\widetilde{\overrightarrow{e_k}}) \\
\delta_k^i =& Q_j^i F_k^l \epsilon^j(\overrightarrow{e_l}) \\
\delta_k^i =& Q_j^i F_k^l \delta_l^j
\end{aligned}
$$

for δ_l^j has value 0 for $j \neq l$ and 1 if $j = l$.

$$\delta_k^i = Q_j^i F_k^j$$

in order to satisfy the relation $Q_j^i = B_j^i$

$$\widetilde{\epsilon^i} = B_j^i\epsilon^j \text{ and similarly } \epsilon^i = F_j^i\widetilde{\epsilon^j}$$

dual basis is of contravarient type.

Linear Maps

1. Matrices are co-ordinate version of linear maps.

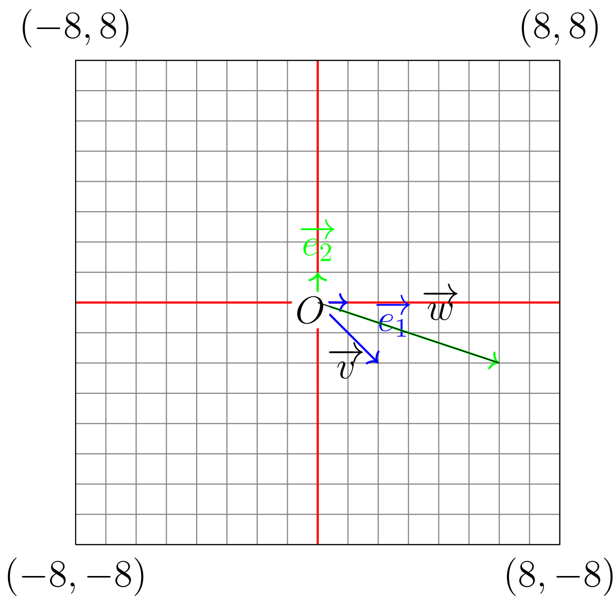
$$
\underset{\text{vector}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \quad \underset{\text{covector}}{\begin{bmatrix} 2 & 1 \end{bmatrix}} \quad \underset{\text{matrices}}{\begin{bmatrix} -3 & 4 \\ 2 & 6 \end{bmatrix}}
$$

$$
\begin{bmatrix} L_1^1 & L_2^1 & L_3^1 \\ L_1^2 & L_2^2 & L_3^2 \\ L_1^3 & L_2^3 & L_3^3 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}
$$

$$\boxed{\begin{array}{r}\text{Linear map transform input vectors}\\ \text{Linear map don't transform the basis}\end{array}}$$

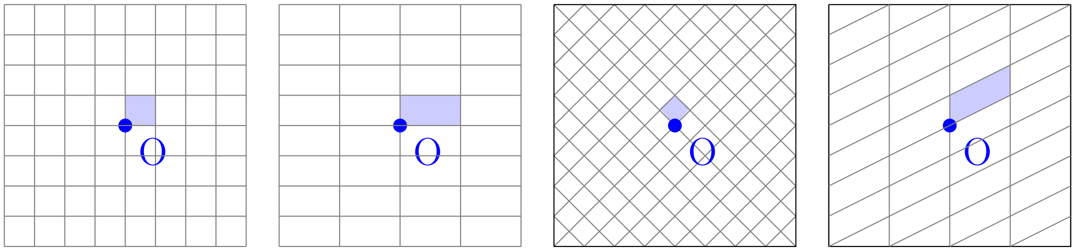
$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}_{\vec{e_i}}$$

$$\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$



2. Linear Maps(geometrically) are spatial transforms that

   1. keep gridlines parallel.
   2. keep gridlines evenly spaced
   3. keep the origin stationary



A linear map/linear transformation/vector space homomorphism/linear function is a mapping $v \to w$ between two vector spaces that preserves the operation of vector addition and scalar multiplication

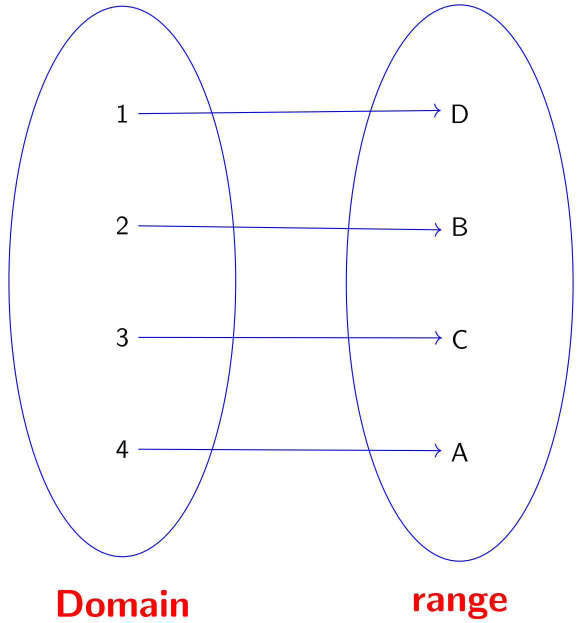


Figure 2: An injective surjective function (bijection)

Bijection

A bijection/bijective function/one-to-one correspondence or invertible function, is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set.

If a linear map is a bijection then it is called a linear isomorphism. If v = w then the linear map is called endomorphism(input vector space is identical to output vector space).

3. Linear map (abstractly)

- Maps vectors to vectors $L : V \to W$
- Add inputs or the output $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$
- scale the inputs then linear mapping or scale output of linear map of input

$$L(n\vec{v}) = nL(\vec{v})$$

In old basis, $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ with linear map L and in new basis $\widetilde{\overrightarrow{e_1}}, \widetilde{\overrightarrow{e_2}}, \widetilde{\overrightarrow{e_3}}$ with linear map $\widetilde{L}$

$$
\begin{aligned}
L(\overrightarrow{e_1}) &= \mathrm{L}_1^1\overrightarrow{e_1} + \mathrm{L}_1^2\overrightarrow{e_2} + \mathrm{L}_1^3\overrightarrow{e_3}\\
L(\overrightarrow{e_2}) &= \mathrm{L}_2^1\overrightarrow{e_1} + \mathrm{L}_2^2\overrightarrow{e_2} + \mathrm{L}_2^3\overrightarrow{e_3}\\
L(\overrightarrow{e_3}) &= \mathrm{L}_3^1\overrightarrow{e_1} + \mathrm{L}_3^2\overrightarrow{e_2} + \mathrm{L}_3^3\overrightarrow{e_3}
\end{aligned}
$$

$$
L(\overrightarrow{e_i}) = \mathrm{L}_i^k\overrightarrow{e_k}
$$

$$
L(\widetilde{\overrightarrow{e_i}}) = \widetilde{\mathrm{L}}_i^l\widetilde{\overrightarrow{e_l}}
$$

$$
\begin{aligned}
L(\overrightarrow{v}) &= L(\widetilde{v^j}\widetilde{\overrightarrow{e_j}}) = L(v^i\overrightarrow{e_i})\\
&= \widetilde{v^j}L(\widetilde{\overrightarrow{e_j}}) = L(v^i\overrightarrow{e_i})\\
&= \widetilde{v^j}\widetilde{\mathrm{L}}_j^l\widetilde{\overrightarrow{e}}_l = v^i\mathrm{L}_i^k\overrightarrow{e_k}\\
&= \widetilde{v^j}\widetilde{\mathrm{L}}_j^l\widetilde{\overrightarrow{e}}_l = \mathrm{F}_m^i\widetilde{v^m}\mathrm{L}_i^k\mathrm{B}_k^n\widetilde{\overrightarrow{e_n}} = \mathrm{F}_j^i\widetilde{v^j}\mathrm{L}_i^k\mathrm{B}_k^l\widetilde{\overrightarrow{e_l}}\\
&= \widetilde{\mathrm{L}}_j^l = \mathrm{F}_j^i\mathrm{L}_i^k\mathrm{B}_k^l
\end{aligned}
$$

similarly, $\mathrm{L}_t^s = \mathrm{F}_l^s\widetilde{\mathrm{L}}_i^l\mathrm{B}_t^i$

$$
\begin{array}{ccc}
\boxed{L(\vec{v}|_{\overrightarrow{e_i}})} & \rightarrow & \boxed{\begin{matrix} v^1\\ v^2\\ v^3\end{matrix}}\\
\uparrow & & \downarrow\\
\boxed{L(\vec{v}|_{\widetilde{\overrightarrow{e_i}}})} & \rightarrow & \boxed{\begin{matrix} \widetilde{v^1}\\ \widetilde{v^2}\\ \widetilde{v^3}\end{matrix}}
\end{array}
$$

If $\overrightarrow{\mathrm{W}} = \mathrm{F}(\vec{v}) \Rightarrow W^i = \mathrm{F}_j^{\mathrm{i}}v^j$

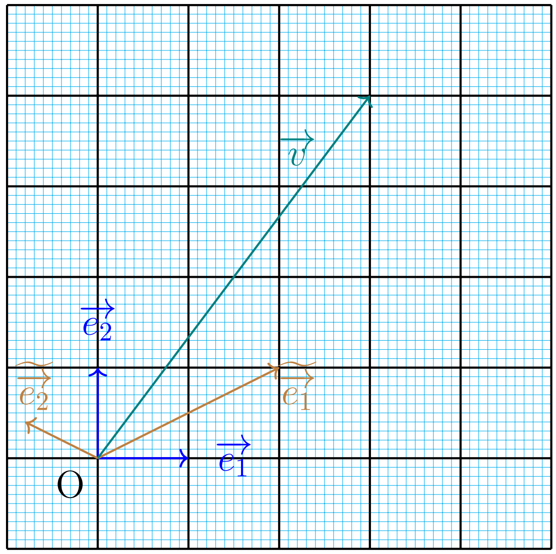
Metric Tensor

Metric tensor helps us the length and angles in space.

In order to get the length of the vector, the usual pythagorian system does not work in case of other than orthonormal basis. The actual method is to do dot product.

$$
\begin{aligned}
||v||^2 =& \vec{v} \cdot \vec{v} \\
=& (v^1\overrightarrow{e_1} + v^2\overrightarrow{e_2} + v^3\overrightarrow{e_3}) \cdot (v^1\overrightarrow{e_1} + v^2\overrightarrow{e_2} + v^3\overrightarrow{e_3}) \\
=& v^1v^1(\overrightarrow{e_1} \cdot \overrightarrow{e_1}) + v^1v^2(\overrightarrow{e_1} \cdot \overrightarrow{e_2}) + v^1v^3(\overrightarrow{e_1} \cdot \overrightarrow{e_3}) + v^2v^1(\overrightarrow{e_2} \cdot \overrightarrow{e_1}) + \\
& v^2v^2(\overrightarrow{e_2} \cdot \overrightarrow{e_2}) + v^2v^3(\overrightarrow{e_2} \cdot \overrightarrow{e_2}) + v^3v^1(\overrightarrow{e_3} \cdot \overrightarrow{e_1}) + v^3v^2(\overrightarrow{e_3} \cdot \overrightarrow{e_2}) + \\
& v^3v^3(\overrightarrow{e_3} \cdot \overrightarrow{e_3}) \\
=& (v^1)^2(\overrightarrow{e_1} \cdot \overrightarrow{e_1}) + (v^2)^2(\overrightarrow{e_2} \cdot \overrightarrow{e_2}) + (v^3)^2(\overrightarrow{e_3} \cdot \overrightarrow{e_3}) + 2v^1v^2(\overrightarrow{e_1} \cdot \overrightarrow{e_2}) + \\
& 2v^2v^3(\overrightarrow{e_2} \cdot \overrightarrow{e_3}) + 2v^1v^3(\overrightarrow{e_3} \cdot \overrightarrow{e_1}) \\
=& \begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \\
=& \widetilde{\vec{v}} \cdot \widetilde{\vec{v}} \\
=& (\widetilde{v^1})^2(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_1}}) + (\widetilde{v^2})^2(\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_2}}) + (\widetilde{v^3})^2(\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_3}}) + 2\widetilde{v^1}\widetilde{v^2}(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_2}}) + \\
& 2v^2v^3(\overrightarrow{e_2} \cdot \overrightarrow{e_3}) + 2\widetilde{v^1}\widetilde{v^3}(\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_1}}) \\
=& \begin{bmatrix} \widetilde{v^1} & \widetilde{v^2} & \widetilde{v^3} \end{bmatrix} \begin{bmatrix} \widetilde{v^1}(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_1}}) + \widetilde{v^2}(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_2}}) + \widetilde{v^3}(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_3}}) \\ \widetilde{v^1}(\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_1}}) + \widetilde{v^2}(\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_2}}) + \widetilde{v^3}(\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_3}}) \\ \widetilde{v^1}(\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_1}}) + \widetilde{v^2}(\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_2}}) + \widetilde{v^3}(\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_3}}) \end{bmatrix} \\
=& \begin{bmatrix} \widetilde{v^1} & \widetilde{v^2} & \widetilde{v^3} \end{bmatrix} \begin{bmatrix} (\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_1}}) & (\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_2}}) & (\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_3}}) \\ (\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_1}}) & (\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_2}}) & (\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_3}}) \\ (\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_1}}) & (\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_2}}) & (\widetilde{\overrightarrow{e_3}} \cdot \widetilde{\overrightarrow{e_3}}) \end{bmatrix} \begin{bmatrix} \widetilde{v^1} \\ \widetilde{v^2} \\ \widetilde{v^3} \end{bmatrix} \begin{bmatrix} \text{middle matrix is} \\ \text{called metric} \\ \text{tensor and symmetric} \\ \text{diagonally} \end{bmatrix}
\end{aligned}
$$

Example



from graph,

$$\widetilde{\overrightarrow{e_1}} = 2\overrightarrow{e_1} + \overrightarrow{e_2}$$
$$\widetilde{\overrightarrow{e_2}} = -8\overrightarrow{e_1} + 4\overrightarrow{e_2}$$

$$\boxed{\begin{aligned} &\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_1}} = 5 \\ &\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_2}} = \widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_2}} = -12 \\ &\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_2}} = 80 \end{aligned}}$$

$$\overrightarrow{e_1} = \frac{1}{4}\widetilde{\overrightarrow{e_1}} - \frac{1}{16}\widetilde{\overrightarrow{e_2}}$$
$$\overrightarrow{e_2} = \frac{1}{2}\overrightarrow{e_1} + \frac{1}{8}\overrightarrow{e_2}$$

$$(||v||)^2 = \sqrt{3^2 + 4^2} = 5$$

$$4 \cdot 3(\overrightarrow{e_2} \cdot \overrightarrow{e_1}) + 4 \cdot 4(\overrightarrow{e_2} \cdot \overrightarrow{e_2})$$

$$\boxed{\begin{aligned} &\overrightarrow{e_1} \cdot \overrightarrow{e_1} = \overrightarrow{e_2} \cdot \overrightarrow{e_2} = 1 \\ &\overrightarrow{e_1} \cdot \overrightarrow{e_2} = \overrightarrow{e_2} \cdot \overrightarrow{e_1} = 0 \end{aligned}}$$

we can write,

$$\boxed{\overrightarrow{e_i} \cdot \overrightarrow{e_j} = \delta_{ij}, \text{ in orthonormal basis}}$$

$$\begin{aligned}
\vec{v} \cdot \vec{v} =& (3\overrightarrow{e_1} + 4\overrightarrow{e_2}) \cdot (3\overrightarrow{e_1} + 4\overrightarrow{e_2}) \\
=& 3 \cdot 3(\overrightarrow{e_1} \cdot \overrightarrow{e_1}) + 3 \cdot 4(\overrightarrow{e_1} \cdot \overrightarrow{e_2}) + \\
& 4 \cdot 3(\overrightarrow{e_2} \cdot \overrightarrow{e_1}) + 4 \cdot 4(\overrightarrow{e_2} \cdot \overrightarrow{e_2}) \\
\vec{v} \cdot \vec{v} =& (||v||)^2 = 25 \therefore v = 5
\end{aligned}$$

$$\begin{aligned}
\vec{v} \cdot \vec{v} &= \left\{ 3(\frac{1}{4}\widetilde{\overrightarrow{e_1}} - \frac{1}{16}\widetilde{\overrightarrow{e_2}}) + 4(\frac{1}{2}\widetilde{\overrightarrow{e_1}} + \frac{1}{8}\widetilde{\overrightarrow{e_2}}) \right\} \cdot \left\{ 3(\frac{1}{4}\widetilde{\overrightarrow{e_1}} - \frac{1}{16}\widetilde{\overrightarrow{e_2}}) + 4(\frac{1}{2}\widetilde{\overrightarrow{e_1}} + \frac{1}{8}\widetilde{\overrightarrow{e_2}}) \right\} \\
&= (\frac{11}{4}\widetilde{\overrightarrow{e_1}} + \frac{5}{16}\widetilde{\overrightarrow{e_2}}) \cdot (\frac{11}{4}\widetilde{\overrightarrow{e_1}} + \frac{5}{16}\widetilde{\overrightarrow{e_2}}) \\
&= \frac{11}{4} \cdot \frac{11}{4}(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_1}}) + \frac{5}{16} \cdot \frac{11}{4}(\widetilde{\overrightarrow{e_1}} \cdot \widetilde{\overrightarrow{e_2}}) + \frac{11}{4} \cdot \frac{5}{16}(\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_1}}) + \frac{5}{16} \cdot \frac{5}{16}(\widetilde{\overrightarrow{e_2}} \cdot \widetilde{\overrightarrow{e_2}}) \\
&= \frac{121}{16} \cdot 5 + 2 \cdot \frac{55}{64} \cdot (-12) + \frac{25}{256} \cdot 80 \\
&= 25 = (||v||)^2 \therefore v = 5
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{v} \cdot \overrightarrow{w} =& (v^1\overrightarrow{e_1} + v^2\overrightarrow{e_2} + v^3\overrightarrow{e_3}) \cdot (w^1\overrightarrow{e_1} + w^2\overrightarrow{e_2} + w^3\overrightarrow{e_3}) \\
=& v^1w^1(\overrightarrow{e_1} \cdot \overrightarrow{e_1}) + v^1w^2(\overrightarrow{e_1} \cdot \overrightarrow{e_2}) + v^1w^3(\overrightarrow{e_1} \cdot \overrightarrow{e_3}) + v^2w^1(\overrightarrow{e_2} \cdot \overrightarrow{e_1}) + \\
& v^2w^2(\overrightarrow{e_2} \cdot \overrightarrow{e_2}) + v^2w^3(\overrightarrow{e_2} \cdot \overrightarrow{e_2}) + v^3w^1(\overrightarrow{e_3} \cdot \overrightarrow{e_1}) + v^3w^2(\overrightarrow{e_3} \cdot \overrightarrow{e_2}) + \\
& v^3w^3(\overrightarrow{e_3} \cdot \overrightarrow{e_3}) \\
=& v^iw^j(\overrightarrow{e_i} \cdot \overrightarrow{e_j}) \rightarrow g_{ij} \\
=& \widetilde{v^i}\widetilde{w^j}(\widetilde{\overrightarrow{e_i}} \cdot \widetilde{\overrightarrow{e_j}}) \rightarrow \widetilde{g_{ij}}
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\overrightarrow{e_i}} \cdot \widetilde{\overrightarrow{e_j}} = \mathrm{F}_i^k\overrightarrow{e_i} \cdot \mathrm{F}_j^l\overrightarrow{e_j} = \mathrm{F}_i^k\mathrm{F}_j^l(\overrightarrow{e_k} \cdot \overrightarrow{e_l}) \leftarrow \text{metric tensor}(g_{kl}) \\
&\overrightarrow{e_i} \cdot \overrightarrow{e_j} = \mathrm{B}_i^k\mathrm{B}_j^l(\widetilde{\overrightarrow{e_i}} \cdot \widetilde{\overrightarrow{e_j}}) \leftarrow \text{metric tensor}(\widetilde{g_{kl}})
\end{aligned}$$

$$
\begin{aligned}
||v||^2 = v^i v^j(\overrightarrow{e_i}\cdot\overrightarrow{e_j}) &= \widetilde{v^i}\widetilde{v^j}(\widetilde{\overrightarrow{e_i}}\cdot\widetilde{\overrightarrow{e_j}}) \\
&= \widetilde{v^i}\widetilde{v^j}\widetilde{g_{ij}} \\
&= \mathrm{B}^i_a\mathrm{B}^j_b v^a v^b g_{kl}\mathrm{F}^k_i\mathrm{F}^l_j \\
&= (\mathrm{B}^i_a\mathrm{F}^k_i)(\mathrm{B}^j_b\mathrm{F}^l_j)v^a v^b g_{kl} \\
&= \delta^k_a\delta^l_b v^a v^b g_{kl} \\
&= v^k v^l g_{kl}
\end{aligned}
$$

A tensor having m number of contravarient indices and n number of covariant indices is called (m,n) tensor

$$\mathrm{T}^{ijk...\text{(m no of contravarient indices)}}_{rst.....\text{(n no of covariant indices)}}$$

- tensor is a function g:V×V $\to \mathbb{R}$ (V×V denotes that it takes 2 vector as input not multiple nor cross product)
  - as a direct function $g(\vec{v},\vec{w}) \to v^i w^j g_{ij}$
  - by decomposing into a vector covector matrix

$$\begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_1}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_2}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_3}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

- Scaling inputs
  - scaling the whole is same as scaling "either" vector or covector

$$
\begin{aligned}
&a\begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_1}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_2}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_3}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \\
&= \begin{bmatrix} av^1 & av^2 & av^3 \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_1}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_2}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_3}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \\
&= \begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_1}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_2}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_3}) \\ \overrightarrow{e_3}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} av^1 \\ av^2 \\ av^3 \end{bmatrix}
\end{aligned}
$$

  - $a(v^i w^j g_{ij}) = (av^i)w^j g_{ij} = v^i(aw^j)g_{ij}$
  - a$g(\vec{v},\vec{w}) = g(a\vec{v},\vec{w}) = g(\vec{v},a\vec{w})$
- Adding inputs

- $\left(\begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} + \begin{bmatrix} u^1 + u^2 + u^3 \end{bmatrix}\right) \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$
- $(v^i + u^i)g_{ij}w^j = v^i w^j g_{ij} + u^i w^j g_{ij}$
- $g(\vec{v} + \vec{u}, \vec{w}) = g(\vec{v}, \vec{w}) + g(\vec{u}, \vec{w})$
- $\left(\begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} w^1 + t^1 \\ w^2 + t^2 \\ w^3 + t^3 \end{bmatrix}\right.$
- $v^i g_{ij}(w^j + t^j) = v^i w^j g_{ij} + v^i t^j g_{ij}$
  $g(\vec{v}, \vec{w} + \vec{t}) = g(\vec{v}, \vec{w}) + g(\vec{v}, \vec{t})$
- $g(\vec{u} + \vec{v}, \vec{w} + \vec{t}) = g(\vec{v}, \vec{w} + \vec{t}) + g(\vec{v}, \vec{w} + \vec{t}) = g(\vec{v}, \vec{w}) + g(\vec{v}, \vec{t}) + g(\vec{u}, \vec{w}) + g(\vec{u}, \vec{t})$

- $g(\vec{v}, \vec{w}) = v^i w^j g_{ij} = v^i w^j g_{ji} = g(\vec{w}, \vec{v})$
- $g(\vec{v}, \vec{v}) = ||v||^2 \geq 0$

The special and predictable way that was mentioned before.

Now we will talk about that tensor product that mentioned before to know what is tensor product.

Bilinear Forms

Metric tensor $\mathcal{B}$ is a special types of Bilinear Forms.

Bilinear Form Definition

- $\mathcal{B} : V \times V \to \mathbb{R}$
- $a\mathcal{B}(\vec{v} + \vec{w}) = \mathcal{B}(a\vec{v}, \vec{w}) = \mathcal{B}(\vec{v}, a\vec{w})$
- $a\mathcal{B}(\vec{u} + \vec{v}, \vec{w}) = \mathcal{B}(\vec{v}, \vec{w}) + \mathcal{B}(\vec{u}, a\vec{w})$
- $\mathcal{B}(\vec{v}, \vec{w} + \vec{t}) = \mathcal{B}(\vec{v}, \vec{w}) + \mathcal{B}(\vec{v}, \vec{t})$
- $\mathcal{B}(\vec{v}, \vec{w}) \to v^i w^j \mathcal{B}_{ij}$

what's the difference between metric and bilinear forms.

- metric tensor components are symmetric, that is $g_{ij} = g_{ji}$ the order of vector input does not matter
- metric tensors working on same vectors twice is a positive quantity, since outputs length

Forms

Basically a function that takes vectors as input and throws out a scaler.

$$\text{Forms} : V \times V \times V ......V \to \mathbb{R}$$

Covectors are also called a linear form as well as 1-Form. vectors are not forms they are not function. They are inputs

- $\alpha(\vec{v} + \vec{w}) = \alpha(\vec{v}) + \alpha(\vec{w})$
- $\alpha(n\vec{v}) = n\alpha(\vec{v})$

Bilinear forms are 2-Forms takes 2 vectors as inputs. Its called bilinear because if i fix one input vector and vary the other, $\mathcal{B}$ follows linearity property.

$$\begin{aligned}
\mathcal{B} =& \mathcal{B}_{kl}\epsilon^k\epsilon^l \\
=& \mathcal{B}_{kl}F_i^kF_j^l\widetilde{\epsilon}^k\widetilde{\epsilon}^j \\
\therefore \widetilde{\mathcal{B}}_{ij} =& \mathcal{B}_{kl}F_i^kF_j^l \\
& \& \\
\mathcal{B}_{kl} =& B_k^iB_l^j\widetilde{\mathcal{B}}_{ij}
\end{aligned} \tag{7}$$

Bilinear Form is not same as Metric Tensor.

Tensor Product

Let's assume two vector spaces **V** & **W** such that **V** $\to \mathbb{R}^n$ & **W**$\to \mathbb{R}^m$ are finite dimensional vector spaces. $\overrightarrow{\text{V}} \in$ **V** & $\overrightarrow{\text{W}} \in$ **W** has **n** & **m** dimentional elements respectively.

$\overrightarrow{\text{V}} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ .. \\ .. \\ .. \\ v^n \end{bmatrix}_n$	$\overrightarrow{\text{W}} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \\ .. \\ .. \\ .. \\ w^m \end{bmatrix}_m$

creating a new vector/space from these two vectors/spaces can be accomplished in two ways that is by

- stacking

$$\begin{bmatrix} v^1 \\ .. \\ v^n \\ w^1 \\ .. \\ w^m \end{bmatrix}_{m+n}$$

  direct sum $\overrightarrow{\mathrm{V}} \oplus \overrightarrow{\mathrm{W}}$ creates a vector of $\mathbf{m}+\mathbf{n}$ dimension in $\mathbf{m}+\mathbf{n}$ dimensional spaces.

- multiplying (by implementing 1×1 combination from each vector element) similar to cartesian product

$$\begin{bmatrix} v^1 w^1 \\ \dots \\ v^1 w^m \\ v^2 w^1 \\ \dots \\ v^2 w^m \\ \dots \\ v^n w^1 \\ \dots \\ v^n w^m \end{bmatrix}$$

  tensor product $\overrightarrow{\mathrm{V}} \otimes \overrightarrow{\mathrm{W}}$ creates a vector of $\mathbf{m} \cdot \mathbf{n}$ dimension in $\mathbf{m} \cdot \mathbf{n}$ dimensional spaces.

Technically, $\mathbf{V} \otimes \mathbf{W}$ is called the outer product of $\overrightarrow{\mathbf{V}}$ and $\overrightarrow{\mathbf{W}}$ and is defined by $\mathbf{V} \otimes \mathbf{W} := \mathbf{V}\mathbf{W}^T$ where $\mathbf{W}^T$ is the same as $\mathbf{W}$ but written as a row vector. (And if the entries of $\mathbf{W}$ are complex numbers, then we also replace each entry by its complex conjugate.)

Algebraic Definition

- $n(\vec{v} \otimes \alpha) = (n\vec{v}) \otimes \alpha = \vec{v} \otimes (n\alpha)$
- $\vec{v} \otimes \alpha + \vec{v} \otimes \beta = \vec{v} \otimes (\alpha + \beta)$
- $\vec{v} \otimes \alpha + \vec{u} \otimes \alpha = (\vec{v} + \vec{u}) \otimes \alpha$

Linear Map as Tensor Product

Linear Map is a vector-covector tensor product. So

$$\overrightarrow{\mathbf{V}} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ .. \\ .. \\ .. \\ v^n \end{bmatrix}_n \qquad \alpha = \begin{bmatrix} \alpha^1 & \alpha^2 & \alpha^3 & ...... & \alpha^n \end{bmatrix}$$

$$\mathbf{L}_i^j = v^j \alpha_i, \text{ pure matrices}$$

This type of linear maps are not that interesting because

$$\overrightarrow{\mathbf{V}} \otimes \alpha = \begin{bmatrix} \alpha_1 v^1 & \alpha_2 v^1 & \alpha_3 v^1 & ...... & \alpha_n v^1 \\ \alpha_1 v^2 & \alpha_2 v^2 & \alpha_3 v^2 & ..... & \alpha_n v^2 \\ \alpha_1 v^3 & \alpha_2 v^3 & \alpha_3 v^3 & ..... & \alpha_n v^3 \\ .... & .... & .... & ..... & .... \\ \alpha_1 v^n & \alpha_2 v^n & \alpha_3 v^n & ..... & \alpha_n v^n \end{bmatrix}$$

We can see each column is just some scaler of $\overrightarrow{\mathbf{V}}$. The end result will be some vectors getting stacked up on another. This linear transformation will

The Linear maps which follows $\mathbf{L}_i^j \neq v^j \alpha_i$ are more interesting. In such cases, linear map will be linear combination of vector-covector pair.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = e_1\epsilon^1 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = e_2\epsilon^1 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = e_3\epsilon^1$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = e_1\epsilon^2 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = e_2\epsilon^2 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = e_3\epsilon^2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = e_1\epsilon^3 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = e_2\epsilon^3 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = e_3\epsilon^3$$

$$L = a \cdot e_1\epsilon^1 + b \cdot e_2\epsilon^1 + c \cdot e_3\epsilon^1 + d \cdot e_1\epsilon^2 + e \cdot e_2\epsilon^2 + f \cdot e_3\epsilon^2 + g \cdot e_1\epsilon^3 + h \cdot e_2\epsilon^3 + i \cdot e_3\epsilon^3$$

If a,b,c......,h,i are components of L_i^j

$$L = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$L = L_i^j e_i \epsilon^j \text{ Linear Map}$$

so $e_i \epsilon^j$ creates the basis for all matrices that are linear maps from $\overrightarrow{V} \rightarrow \overrightarrow{V}$(vector space $\overrightarrow{V}$ to $\overrightarrow{V}$). Also $\widetilde{e_i} \widetilde{\epsilon^j}$ can be another basis for $\overrightarrow{V} \rightarrow \overrightarrow{V}$. So

$$L = \widetilde{L_i^j} \widetilde{e_i} \widetilde{\epsilon^j}$$

If L acts on a vector

$$\begin{aligned} \mathbf{W} &= \mathbf{L}(\mathbf{V}) \\ &= L_i^j e_i \epsilon^j (v^k e_k) \\ &= L_i^j e_i v^k \epsilon^j (e_k) \\ &= L_i^j e_i v^k \delta_k^j \\ &= L_i^j e_i v^j \\ &= L_i^j v^j e_i \end{aligned} \tag{8}$$

where $L_i^j v^j$ are vector components.

Bilinear Forms as Tensor Product

Linear combination of covector-covector pairs defined by

$$\mathcal{B} = \mathcal{B}_{ij} \epsilon^i \otimes \epsilon^j$$

Similar to Linear Maps. So $\mathcal{B}$ can work on two vectors and will result a scalar according to definition.

$$\begin{aligned} \mathbf{S} =& \mathcal{B}(\mathbf{V})(\mathbf{W}) \\ =& \mathcal{B}_{ij} (\epsilon^i \otimes \epsilon^j) v^k e_k w^l e_l \\ =& \mathcal{B}_{ij} v^k w^l (\epsilon^i \otimes e_k)(\epsilon^j \otimes e_l) \\ =& v^k w^l \mathcal{B}_{ij} \delta_k^i \delta_l^j \\ =& \mathcal{B}_{ij} v^i v^j \end{aligned} \tag{9}$$

In metric tensor section we wrote

$$\overrightarrow{v}\cdot\overrightarrow{w} = \begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_1}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_3}) \\ (\overrightarrow{e_2}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_3}) \\ (\overrightarrow{e_3}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$$

but we wrote the $\overrightarrow{V}$ as row vector which is unintuitive since vectors are expressed as columns. It would make more sense if we wrote

$$\overrightarrow{v}\cdot\overrightarrow{w} = \begin{bmatrix} \begin{bmatrix} (\overrightarrow{e_1}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_1}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_2}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_2}\cdot\overrightarrow{e_3}) \end{bmatrix} \begin{bmatrix} (\overrightarrow{e_3}\cdot\overrightarrow{e_1}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_2}) & (\overrightarrow{e_3}\cdot\overrightarrow{e_3}) \end{bmatrix} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$$

Kronecker Product vs Tensor Product

Tensor product deals with operation of tensors and Kronecker product with arrays. Both are represented with $\otimes$ symbol. They almost give you the same result but means the same.

Some new tensor

$$D = D^{ab}\overrightarrow{e_a}\overrightarrow{e_b} \ \& \ Q = Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k$$

How do D and Q behave under a change of basis?

$$\boxed{\begin{aligned} \widetilde{\overrightarrow{e_j}} &= F^i_j\overrightarrow{e_i} \\ \overrightarrow{e_j} &= B^i_j\widetilde{\overrightarrow{e_i}} \end{aligned}} \qquad \boxed{\begin{aligned} \widetilde{\epsilon^i} &= B^i_j\epsilon^j \\ \epsilon^i &= F^i_j\widetilde{\epsilon^j} \end{aligned}}$$

$$\begin{aligned} D &= D^{ab}\overrightarrow{e_a}\overrightarrow{e_b} \\ &= D^{ab}(B^i_a\widetilde{\overrightarrow{e_i}})(B^j_b\widetilde{\overrightarrow{e_j}}) \\ &= (B^i_aB^j_bD^{ab})\widetilde{\overrightarrow{e_i}}\widetilde{\overrightarrow{e_j}} \\ &= \widetilde{D^{ij}}\widetilde{\overrightarrow{e_i}}\widetilde{\overrightarrow{e_j}} \end{aligned}$$

$$\boxed{\begin{aligned} \widetilde{D^{ij}} &= B^i_aB^j_bD^{ab} \\ D^{ab} &= \widetilde{D^{ij}}F^a_iF^b_j \end{aligned}}$$

$$\begin{aligned} Q &= Q^a_{bc}\overrightarrow{e_a}\epsilon^b\epsilon^c \\ &= Q^a_{bc}(B^i_a\widetilde{\overrightarrow{e_i}})(F^b_j\widetilde{\epsilon^j})(F^c_k\widetilde{\epsilon^k}) \\ &= (B^i_aQ^a_{bc}F^b_jF^c_k)\widetilde{\overrightarrow{e_i}}\widetilde{\epsilon^k}\widetilde{\epsilon^j} \\ &= \widetilde{Q^i_{jk}}\widetilde{\overrightarrow{e_i}}\widetilde{\epsilon^k}\widetilde{\epsilon^j} \end{aligned}$$

$$\boxed{\begin{aligned} \widetilde{Q^i_{jk}} &= B^i_aQ^a_{bc}F^b_jF^c_k \\ Q^a_{bc} &= F^a_i\widetilde{Q^i_{jk}}B^j_bB^k_c \end{aligned}}$$

Q(D) can be evaluated for the following cases

$Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k \qquad D^{ab}\overrightarrow{e_a}\overrightarrow{e_b}$	$Q(D) = Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k(D^{ab}\overrightarrow{e_a}\overrightarrow{e_b})$
$Q^i_{jk}D^{jk}$	$\begin{aligned} Q(D) &= Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k(D^{ab}\overrightarrow{e_a}\overrightarrow{e_b}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}(\epsilon^j \otimes \overrightarrow{e_a})(\epsilon^k \otimes \overrightarrow{e_b}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}\delta^j_a\delta^k_b = Q^i_{jk}D^{jk}\overrightarrow{e_i} \end{aligned}$
$Q^i_{jk}D^{kj}$	$\begin{aligned} Q(D) &= Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k(D^{ab}\overrightarrow{e_a}\overrightarrow{e_b}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}(\epsilon^j \otimes \overrightarrow{e_b})(\epsilon^k \otimes \overrightarrow{e_a}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}\delta^j_b\delta^k_a = Q^i_{jk}D^{kj}\overrightarrow{e_i} \end{aligned}$
$Q^i_{jk}D^{kb}$	$\begin{aligned} Q(D) &= Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k(D^{ab}\overrightarrow{e_a}\overrightarrow{e_b}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}(\epsilon^j\overrightarrow{e_b})(\epsilon^k \otimes \overrightarrow{e_a}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}(\epsilon^j\overrightarrow{e_b})\delta^k_a = Q^i_{jk}D^{kb}\overrightarrow{e_i}(\epsilon^j\overrightarrow{e_b}) \end{aligned}$
$Q^i_{jk}D^{aj}$	$\begin{aligned} Q(D) &= Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k(D^{ab}\overrightarrow{e_a}\overrightarrow{e_b}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}(\epsilon^j \otimes \overrightarrow{e_b})(\epsilon^k\overrightarrow{e_a}) \\ &= Q^i_{jk}D^{ab}\overrightarrow{e_i}\delta^j_b(\epsilon^k\overrightarrow{e_a}) = Q^i_{jk}D^{aj}\overrightarrow{e_i}(\epsilon^k\overrightarrow{e_a}) \end{aligned}$

shape of $D = D^{ab}\overrightarrow{e_a}\overrightarrow{e_b}$

$$\begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \otimes \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} w^1 \\ \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} w^2 \\ \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} w^3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1w^1 \\ v^2w^1 \\ v^3w^1 \end{bmatrix} \\ \begin{bmatrix} v^1w^2 \\ v^2w^2 \\ v^3w^2 \end{bmatrix} \\ \begin{bmatrix} v^1w^3 \\ v^2w^3 \\ v^3w^3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} D^{11} \\ D^{21} \\ D^{31} \end{bmatrix} \\ \begin{bmatrix} D^{12} \\ D^{22} \\ D^{32} \end{bmatrix} \\ \begin{bmatrix} D^{13} \\ D^{23} \\ D^{33} \end{bmatrix} \end{bmatrix} \Bigg|_{\overrightarrow{e_a}\overrightarrow{e_b}}$$

shape of $Q = Q^i_{jk}\overrightarrow{e_i}\epsilon^j\epsilon^k$

$$\begin{aligned} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}\alpha_1 & \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}\alpha_2 & \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}\alpha_3 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \\ v^3\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \\ v^3\alpha_2 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3 \\ v^2\alpha_3 \\ v^3\alpha_3 \end{bmatrix} \end{bmatrix} \otimes \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \\ v^3\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \\ v^3\alpha_2 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3 \\ v^2\alpha_3 \\ v^3\alpha_3 \end{bmatrix} \end{bmatrix}\beta_1 & \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \\ v^3\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \\ v^3\alpha_2 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3 \\ v^2\alpha_3 \\ v^3\alpha_3 \end{bmatrix} \end{bmatrix}\beta_2 & \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \\ v^3\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \\ v^3\alpha_2 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3 \\ v^2\alpha_3 \\ v^3\alpha_3 \end{bmatrix} \end{bmatrix}\beta_3 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} v^1\alpha_1\beta_1 \\ v^2\alpha_1\beta_1 \\ v^3\alpha_1\beta_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2\beta_1 \\ v^2\alpha_2\beta_1 \\ v^3\alpha_2\beta_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3\beta_1 \\ v^2\alpha_3\beta_1 \\ v^3\alpha_3\beta_1 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} v^1\alpha_1\beta_2 \\ v^2\alpha_1\beta_2 \\ v^3\alpha_1\beta_2 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2\beta_2 \\ v^2\alpha_2\beta_2 \\ v^3\alpha_2\beta_2 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3\beta_2 \\ v^2\alpha_3\beta_2 \\ v^3\alpha_3\beta_2 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} v^1\alpha_1\beta_3 \\ v^2\alpha_1\beta_3 \\ v^3\alpha_1\beta_3 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2\beta_3 \\ v^2\alpha_2\beta_3 \\ v^3\alpha_2\beta_3 \end{bmatrix} & \begin{bmatrix} v^1\alpha_3\beta_3 \\ v^2\alpha_3\beta_3 \\ v^3\alpha_3\beta_3 \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$Q = \left.\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} Q^1_{11} \\ Q^2_{11} \\ Q^3_{11} \end{bmatrix} & \begin{bmatrix} Q^1_{21} \\ Q^2_{21} \\ Q^3_{21} \end{bmatrix} & \begin{bmatrix} Q^1_{31} \\ Q^2_{31} \\ Q^3_{31} \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} Q^1_{12} \\ Q^2_{12} \\ Q^3_{12} \end{bmatrix} & \begin{bmatrix} Q^1_{22} \\ Q^2_{22} \\ Q^3_{22} \end{bmatrix} & \begin{bmatrix} Q^1_{32} \\ Q^2_{32} \\ Q^3_{32} \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} Q^1_{13} \\ Q^2_{13} \\ Q^3_{13} \end{bmatrix} & \begin{bmatrix} Q^1_{23} \\ Q^2_{23} \\ Q^3_{23} \end{bmatrix} & \begin{bmatrix} Q^1_{33} \\ Q^2_{33} \\ Q^3_{33} \end{bmatrix} \end{bmatrix} \end{bmatrix}\right|_{\overrightarrow{e_i}\epsilon^j\epsilon^k}$$

Tensor Product Spaces