Naïve Bayes Classifier

Conditional probability

- P(sunny | windy) = 0.8

  i.e., probability of sunny given that *windy* is all I know

- Interpreted as

  P(if Weather=sunny then Windy=windy)

- **Definition** of conditional probability:

  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$

- Alternative formulation:

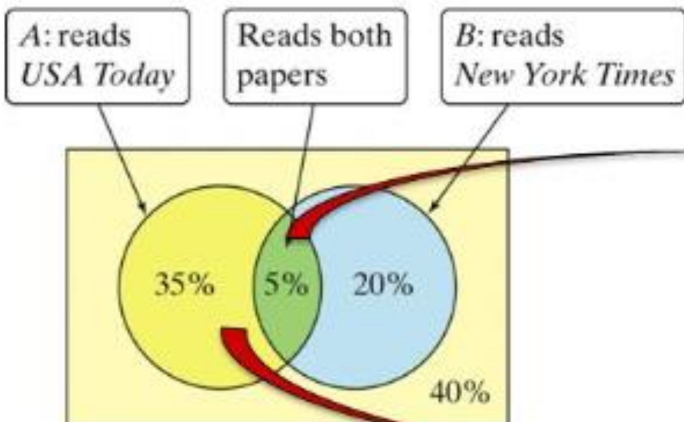
  $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$

Illustration of Conditional Probability

- **Example: Who Reads the Newspaper?**

  the residents of a large apartment complex can be classified based on the events *A*: reads *USA Today* and *B*: reads the *New York Times*. The Venn Diagram below describes the residents.

What is the probability that a randomly selected resident who reads *USA Today* also reads the *New York Times*?



$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = 0.05$$

$$P(A) = 0.40$$

$$P(B \mid A) = \frac{0.05}{0.40} = 0.125$$

There is a 12.5% chance that a randomly selected resident who reads *USA Today* also reads the *New York Times*.

Bayes' Rule

- From $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$

  we get

  Bayes' rule:

  $$P(a \mid b) = P(b \mid a)\, P(a) \,/\, P(b)$$

Breathalyzers

- A group of police officers have breathalyzers displaying false drunkenness in 5% of the cases in which the driver is sober.
- However, the breathalyzers never fail to detect a truly drunk person.
- One in a thousand drivers is driving drunk.
- Suppose the police officers then stop a driver at random, and force the driver to take a breathalyzer test. It indicates that the driver is drunk. We assume you don't know anything else about him or her. How high is the probability he or she really is drunk?

Bayes Rule in Action

P(drunk | breathalyzer_says_drunk) = P(breathalyzer_says_drunk | drunk)*P(drunk)
/ P(breathalyzer_says_drunk)

P(-drunk | breathalyzer_says_drunk) = P(breathalyzer_says_drunk | -drunk)*P(-drunk)
/ P(breathalyzer_says_drunk)

P(drunk) = 0.001
P(-drunk)=0.999
P(breathalyzer_says_drunk | drunk) = 1.00
P(breathalyzer_says_drunk | -drunk) = 0.05
Let's use alpha=1/ P(breathalyzer_says_drunk)

P(drunk | breathalyzer_says_drunk) = alpha*1.00*0.001
P(-drunk | breathalyzer_says_drunk) = alpha*0.05*0.999

P(drunk | breathalyzer_says_drunk)+ P(-drunk | breathalyzer_says_drunk)=1
alpha = 1/(1.00*0.001 + 0.05*0.999) = 19.63
P(drunk | breathalyzer_says_drunk) = 19.63*1.00*0.001 = 0.01963 or about 2%
P(-drunk | breathalyzer_says_drunk) = 19.63*0.05*0.999 = 0.9805 or about 98%

Bayes' rule -- more vars

$$P(c \mid e_1, e_2) = \frac{P(c, e_1, e_2)}{P(e_1, e_2)} = \alpha P(c, e_1, e_2)$$

$$\begin{aligned}
&= \alpha P(e_1, e_2, c) \\
&= \alpha P(e_1 \mid e_2, c) P(e_2, c) \\
&= \alpha P(e_1 \mid e_2, c) P(e_2 \mid c) P(c) \\
&= \alpha P(e_1 \mid c) P(e_2 \mid c) P(c)
\end{aligned}$$

Conditional Independence

Naive Bayes

$$P(c \mid e_1,\ldots,e_n) = \alpha P(e_1 \mid c)\ldots P(e_n \mid c)P(c)$$

- Assumption:

  Attributes are conditionally independent (given the class value)

- Although based on assumption that is almost never correct, this scheme works well in practice!

Weather Data

Outlook	Temp.	Humidity	Windy	Play
Sunny	Hot	High	False	No
Sunny	Hot	High	True	No
Overcast	Hot	High	False	Yes
Rainy	Mild	High	False	Yes
Rainy	Cool	Normal	False	Yes
Rainy	Cool	Normal	True	No
Overcast	Cool	Normal	True	Yes
Sunny	Mild	High	False	No
Sunny	Cool	Normal	False	Yes
Rainy	Mild	Normal	False	Yes
Sunny	Mild	Normal	True	Yes
Overcast	Mild	High	True	Yes
Overcast	Hot	Normal	False	Yes
Rainy	Mild	High	True	No

- A new day:

Outlook	Temp.	Humidity	Windy	Play
Sunny	Cool	High	True	?

Naïve Bayes for classification

- Classification learning: what's the probability of the class given an instance?
- **Instance (Evidence):** $E_1=e_1$, $E_2=e_1, \ldots, E_n=e_n$
- **Class** $C = \{c, \ldots\}$

- Naïve Bayes assumption: evidence can be split into independent parts (i.e. attributes of instance!)

$$P(c|E) = P(c \mid e_1, e_2, \ldots, e_n)$$
$$= P(e_1|c)\, P(e_2|c) \ldots P(e_n|c)\, P(c) \,/\, P(e_1, e_2, \ldots, e_n)$$

The weather data example

Outlook	Temp.	Humidity	Windy	Play
Sunny	Cool	High	True	?

← *Evidence E*

P(Play=yes | E) =
P(Outlook=Sunny | play=yes) *
P(Temp=Cool | play=yes) *
P(Humidity=High | play=yes) *
P(Windy=True | play=yes) *
P(play=yes) / P(E) =
= (2/9) *
(3/9) *
(3/9) *
(3/9) *
(9/14) / P(E) = 0.0053 / P(E)

Don't worry for the 1/P(E); It's alpha, the normalization constant.

Outlook	Temp.	Humidity	Windy	Play
Sunny	Hot	High	False	No
Sunny	Hot	High	True	No
Overcast	Hot	High	False	Yes
Rainy	Mild	High	False	Yes
Rainy	Cool	Normal	False	Yes
Rainy	Cool	Normal	True	No
Overcast	Cool	Normal	True	Yes
Sunny	Mild	High	False	No
Sunny	Cool	Normal	False	Yes
Rainy	Mild	Normal	False	Yes
Sunny	Mild	Normal	True	Yes
Overcast	Mild	High	True	Yes
Overcast	Hot	Normal	False	Yes
Rainy	Mild	High	True	No

The weather data example

Outlook	Temp.	Humidity	Windy	Play
Sunny	Cool	High	True	?

← *Evidence E*

P(Play=no | E) =

P(Outlook=Sunny | play=no) *

P(Temp=Cool | play=no) *

P(Humidity=High | play=no) *

P(Windy=True | play=no) *

P(play=no) / P(E) =

= (3/5) *

(1/5) *

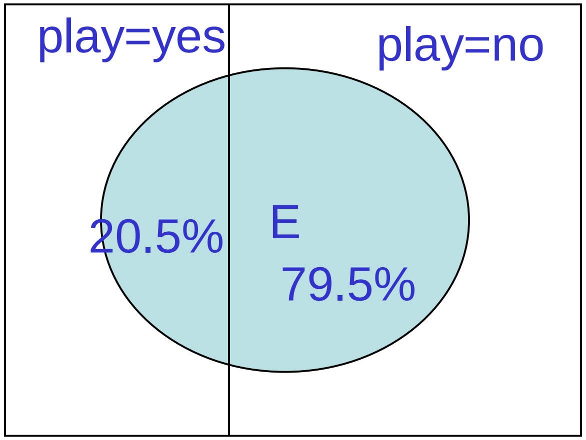
(4/5) *

(3/5) *

(5/14) / P(E) = 0.0206 / P(E)

Outlook	Temp.	Humidity	Windy	Play
Sunny	Hot	High	False	No
Sunny	Hot	High	True	No
Overcast	Hot	High	False	Yes
Rainy	Mild	High	False	Yes
Rainy	Cool	Normal	False	Yes
Rainy	Cool	Normal	True	No
Overcast	Cool	Normal	True	Yes
Sunny	Mild	High	False	No
Sunny	Cool	Normal	False	Yes
Rainy	Mild	Normal	False	Yes
Sunny	Mild	Normal	True	Yes
Overcast	Mild	High	True	Yes
Overcast	Hot	Normal	False	Yes
Rainy	Mild	High	True	No

Normalization constant



P(play=yes | E) + P(play=no | E) = 1 i.e.

0.0053 / P(E) + 0.0206 / P(E) = 1 i.e.

P(E) = 0.0053 + 0.0206

So,

P(play=yes | E) = 0.0053 / (0.0053 + 0.0206) = **20.5%**

P(play=no | E) = 0.0206 / (0.0053 + 0.0206) = **79.5%**

The “zero-frequency problem”

- What if an attribute value doesn’t occur with every class value (e.g. “Humidity = High” for class “Play=Yes”)?
  - Probability P(Humidity=High|play=yes) will be zero!
- P(Play=“Yes”|E) will also be zero!
  - No matter how likely the other values are!
- Remedy:
  - Add 1 to the count for every attribute value-class combination (Laplace estimator);
  - Add *k* (no of possible attribute values) to the denumerator. (see example on the right).

P(play=yes | E) =

P(Outlook=Sunny | play=yes) *

P(Temp=Cool | play=yes) *

P(Humidity=High | play=yes) *

P(Windy=True | play=yes) *

P(play=yes) / P(E) =

= (2/9) * (3/9) * (3/9) * (3/9) *(9/14) / P(E) = 0.0053 / P(E)

It will be instead:

Number of possible values for ‘Outlook’

Number of possible values for ‘Windy’

= ((2+1)/(9+3)) * ((3+1)/(9+3)) * ((3+1)/(9+2)) * ((3+1)/(9+2)) *((9+1)/(14+2)) / P(E) = 0.0069 / P(E)

Missing values

- **Training**: instance is not included in the frequency count for attribute value-class combination
- **Classification**: attribute will be omitted from calculation
- Example:

Outlook	Temp.	Humidity	Windy	Play
?	Cool	High	True	?

P(play=yes | E) =

P(Temp=Cool | play=yes) *

P(Humidity=High | play=yes) *

P(Windy=True | play=yes) *

P(play=yes) / P(E) =

= (4/12) * (4/11) * (4/11) *(10/16) / P(E) = 0.027 / P(E)

P(play=no | E) =

P(Temp=Cool | play=no) *

P(Humidity=High | play=no) *

P(Windy=True | play=no) *

P(play=no) / P(E) =

= (2/8) * (5/7) * (4/7) *(6/16) / P(E) = 0.0382 / P(E)

After normalization: P(play=yes | E) = **41%**, P(play=no | E) = **59%**

Dealing with numeric attributes

- Usual assumption: attributes have a normal or Gaussian probability distribution (given the class).

- Probability density function for the normal distribution is:

$$f(x \mid class) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- We approximate μ by the sample mean:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- We approximate σ^2 by the sample variance:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Weather Data with Numeric Attrib.

outlook	**temperature**	**humidity**	**windy**	**play**
sunny	85	85	FALSE	no
sunny	80	90	TRUE	no
overcast	83	86	FALSE	yes
rainy	70	96	FALSE	yes
rainy	68	80	FALSE	yes
rainy	65	70	TRUE	no
overcast	64	65	TRUE	yes
sunny	72	95	FALSE	no
sunny	69	70	FALSE	yes
rainy	75	80	FALSE	yes
sunny	75	70	TRUE	yes
overcast	72	90	TRUE	yes
overcast	81	75	FALSE	yes
rainy	71	91	TRUE	no

Outlook	Temp.	Humidity	Windy	Play
Sunny	66	90	true	?

f(Temperature=66 | yes)
=e(- ((66-m)^2 / 2*var)) / sqrt(2*3.14*var)

m = (83+70+68+64+69+75+75+72 +81)/ 9 = 73

var = ((83-73)^2 + (70-73)^2 + (68-73)^2 + (64-73)^2 + (69-73)^2 + (75-73)^2 + (75-73)^2 + (72-73)^2 + (81-73)^2)/ (9-1) = 38

f(Temperature=66 | yes)
=e(- ((66-73)^2 / (2*38))) / sqrt(2*3.14*38) = **.034**

We compute similarly:
f(Temperature=66 | no)

Weather Data

outlook	temperature	humidity	windy	play
sunny	85	85	FALSE	no
sunny	80	90	TRUE	no
overcast	83	86	FALSE	yes
rainy	70	96	FALSE	yes
rainy	68	80	FALSE	yes
rainy	65	70	TRUE	no
overcast	64	65	TRUE	yes
sunny	72	95	FALSE	no
sunny	69	70	FALSE	yes
rainy	75	80	FALSE	yes
sunny	75	70	TRUE	yes
overcast	72	90	TRUE	yes
overcast	81	75	FALSE	yes
rainy	71	91	TRUE	no

Outlook	Temp.	Humidity	Windy	Play
Sunny	66	90	true	?

f(Humidity=90 | yes)
=e(- ((90-m)^2 / 2*var)) / sqrt(2*3.14*var)

m = (86+96+80+65+70+80+70+90+75)/ 9 = 79

var = ((86-79)^2 + (96-79)^2 + (80-79)^2 + (65-79)^2 + (70-79)^2 + (80-79)^2 + (70-79)^2 + (90-79)^2 + (75-79)^2)/ (9-1) = 104

f(Humidity=90 | yes)
=e(- ((90-79)^2 / (2*104))) / sqrt(2*3.14*104) = **.022**

We compute similarly:
f(Humidity=90 | no)

Classifying a new day

- A new day E:

Outlook	Temp.	Humidity	Windy	Play
Sunny	66	90	true	?

P(play=yes | E) =
P(Outlook=sunny | play=yes) *
P(Temp=66 | play=yes) *
P(Humidity=90 | play=yes) *
P(Windy=true | play=yes) *
P(play=yes) / P(E) =
= (2/9) * (0.034) * (0.022) * (3/9) *(9/14) / P(E) = 0.000036 / P(E)

P(play=no | E) =
P(Outlook=sunny | play=no) *
P(Temp=66 | play=no) *
P(Humidity=90 | play=no) *
P(Windy=true | play=no) *
P(play=no) / P(E) =
= (3/5) * (0.0291) * (0.038) * (3/5) *(5/14) / P(E) = 0.000136 / P(E)

After normalization: P(play=yes | E) = **20.9%**, P(play=no | E) = **79.1%**

Probability densities

- Relationship between probability and density:

$$\Pr[c - \frac{\varepsilon}{2} < x < c + \frac{\varepsilon}{2}] \approx \varepsilon * f(c)$$

- But: this doesn't change calculation of a posteriori probabilities because ε cancels out

Discussion of Naïve Bayes

- Naïve Bayes works surprisingly well (even if independence assumption is clearly violated)

- Because classification doesn’t require accurate probability estimates as long as maximum probability is assigned to correct class

Tax Data – Naive Bayes

Classify: (_, No, Married, 95K, ?)

(Apply also the Laplace normalization)

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Evade**
1	Yes	Single	125K	**No**
2	No	Married	100K	**No**
3	No	Single	70K	**No**
4	Yes	Married	120K	**No**
5	No	Divorced	95K	**Yes**
6	No	Married	60K	**No**
7	Yes	Divorced	220K	**No**
8	No	Single	85K	**Yes**
9	No	Married	75K	**No**
10	No	Single	90K	**Yes**

Tax Data – Naive Bayes

Tid	Refund	Marital Status	Taxable Income	Evade
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

Classify: (_, No, Married, 95K, ?)

P(Yes) = 3/10 = 0.3

P(Refund=No|Yes) = (3+1)/(3+2) = 0.8

P(Status=Married|Yes) = (0+1)/(3+3) = 0.17

$$f(income \mid Yes) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Approximate μ with: (95+85+90)/3 =90

Approximate σ^2 with:

((95-90)^2+(85-90) ^2+(90-90) ^2)/ (3-1) = 25

f(income=95|Yes) =

e(- ((95-90)^2 / (2*25))) / sqrt(2*3.14*25) = .048

P(Yes | E) = α*.8*.17*.048*.3= α*.0019584

Tax Data

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Evade**
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

Classify: (_, No, Married, 95K, ?)

P(No) = 7/10 = .7

P(Refund=No|No) = (4+1)/(7+2) = .556

P(Status=Married|No) = (4+1)/(7+3) = .5

$$f(income \mid No) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Approximate μ with:

(125+100+70+120+60+220+75)/7 =110

Approximate σ^2 with:

((125-110)^2 + (100-110)^2 + (70-110)^2 + (120-110)^2 + (60-110)^2 + (220-110)^2 + (75-110)^2)/(7-1) = 2975

f(income=95|No) =

e(-((95-110)^2 / (2*2975))) /sqrt(2*3.14* 2975) = .00704

P(No | E) = α*.556*.5* .00704*0.7= α*.00137

Tax Data

Tid	Refund	Marital Status	Taxable Income	Evade
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

Classify: (_, No, Married, 95K, ?)

P(Yes | E) = α*.0019584

P(No | E) = α*.00137

α = 1/(.0019584 + .00137)=300.44

P(Yes|E) = 300.44 *.0019584 = 0.59

P(No|E) = 300.44 *.00137 = 0.41

We predict "Yes."