Prof. Dr. Johannes Pfeifer — University of Cologne
Dr. Dmitry Matveev — Bank of Canada

# Derivation of the recursive pricing equation for Galí (2015), Chapter 3

## 1 Aggregation

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di \tag{1}$$

From this follows

$$Y_t = A_t \left(\frac{N_t}{\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di}\right)^{1-\alpha} \tag{2}$$

Define the term in the denominator as $S_t$ we get

$$N_{t+k} = \left(\frac{Y_{t+k}}{A_{t+k}}\right)^{\frac{1}{1-\alpha}} S_{t+k} \tag{3}$$

where

$$\begin{aligned}
S_t &= \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di & (4)\\
&= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \int_{S(i)} \left(\frac{P_{t-1}(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di & (5)\\
&= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \int_{S(i)} \left(\frac{P_{t-1}}{P_t}\frac{P_{t-1}(i)}{P_{t-1}}\right)^{-\frac{\varepsilon}{1-\alpha}} di & (6)\\
&= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \Pi_t^{\frac{\varepsilon}{1-\alpha}} \int_{S(i)} \left(\frac{P_{t-1}(i)}{P_{t-1}}\right)^{-\frac{\varepsilon}{1-\alpha}} di & (7)\\
&= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta\Pi_t^{\frac{\varepsilon}{1-\alpha}} S_{t-1} & (8)
\end{aligned}$$

From the demand function follows:

$$A_{t+k} N_{t+k|t}^{1-\alpha} = Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k} \tag{9}$$

which implies

$$N_{t+k|t} = \left(\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} \frac{Y_{t+k}}{A_{t+k}}\right)^{\frac{1}{1-\alpha}} \tag{10}$$

Therefore

$$\frac{N_{t+k|t}}{N_{t+k}} = \frac{\left(\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}\frac{Y_{t+k}}{A_{t+k}}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{Y_{t+k}}{A_{t+k}}\right)^{\frac{1}{1-\alpha}}S_t} = \frac{\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\varepsilon}{1-\alpha}}}{S_t} \tag{11}$$

Marginal costs are given by:

$$MC_{t+k|t} = \frac{W_{t+k}}{\frac{(1-\alpha)Y_{t+k|t}}{N_{t+k|t}}} = \frac{W_{t+k}}{(1-\alpha)\,A_{t+k}N_{t+k|t}^{-\alpha}} = \frac{W_{t+k}}{(1-\alpha)\,A_{t+k}N_{t+k}^{-\alpha}}\frac{N_{t+k}^{-\alpha}}{N_{t+k|t}^{-\alpha}}$$

$$= \frac{W_{t+k}}{(1-\alpha)\,A_{t+k}N_{t+k}^{-\alpha}}\left(\frac{\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\varepsilon}{1-\alpha}}}{S_{t+k}}\right)^{\alpha} = MC_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}}$$

where

$$MC_t = \frac{W_t}{(1-\alpha)\,A_tN_t^{-\alpha}S_t^{\alpha}} = \frac{W_t}{(1-\alpha)\,\frac{Y_t}{N_t}S_t}$$

# 2 Pricing FOC

The pricing FOC is given by

$$\sum_{k=0}^{\infty}\theta^k E_t\left[\Lambda_{t,t+k}Y_{t+k|t}\frac{1}{P_{t+k}}\left(P_t^* - \frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}P_{t+k}\right)\right] = 0 \tag{12}$$

where $MC_{t+k|t}$ is real marginal costs. Using the demand function, this is

$$\sum_{k=0}^{\infty}\theta^k E_t\left[\Lambda_{t,t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}Y_{t+k}\frac{1}{P_{t+k}}\left(P_t^* - \frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}P_{t+k}\right)\right] = 0 \tag{13}$$

or

$$\sum_{k=0}^{\infty}\theta^k E_t\left[\Lambda_{t,t+k}\left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon}\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon}Y_{t+k}\frac{P_t}{P_{t+k}}\left(\frac{P_t^*}{P_t} - \frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}\frac{P_{t+k}}{P_t}\right)\right] = 0 \tag{14}$$

Now plug in for the stochastic discount factor $\Lambda_{t,t+k}$:

$$\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k\frac{C_{t+k}^{-\sigma}}{C_t^{-\sigma}}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon}Y_{t+k}\left(\frac{P_t^*}{P_t} - \frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}\frac{P_{t+k}}{P_t}\right)\right] = 0 \tag{15}$$

Next, multiply by $C_t^{-\sigma}$ and collect terms:

$$\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon}Y_{t+k}\left(\frac{P_t^*}{P_t} - \frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}\frac{P_{t+k}}{P_t}\right)\right] = 0 \tag{16}$$

Using

$$\Pi_t^* = \frac{P_t^*}{P_t} \tag{17}$$

we can write this as

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}(\Pi_t^*)^{1-\varepsilon}Y_{t+k}\right] = \\ \sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon}(\Pi_t^*)^{-\varepsilon}Y_{t+k}\frac{\varepsilon}{\varepsilon-1}MC_{t+k|t}\right]\end{aligned} \tag{18}$$

Now replace idiosyncratic by average marginal costs:

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}(\Pi_t^*)^{1-\varepsilon}Y_{t+k}\right] = \\ \frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon}(\Pi_t^*)^{-\varepsilon}Y_{t+k}MC_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}}\right]\end{aligned} \tag{19}$$

Expand the fraction to get $\Pi_t^*$:

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}(\Pi_t^*)^{1-\varepsilon}Y_{t+k}\right] = \\ \frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon}(\Pi_t^*)^{-\varepsilon}Y_{t+k}MC_{t+k}\left(\frac{P_t}{P_{t+k}}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}}\left(\frac{P_t^*}{P_t}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}}\right]\end{aligned} \tag{20}$$

and bring $\Pi_t^*$ to the left:

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}(\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}}Y_{t+k}\right] = \\ \frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}Y_{t+k}MC_{t+k}\left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}}\right]\end{aligned} \tag{21}$$

Factoring out, we get

$$\begin{aligned}(\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}Y_{t+k}\right] = \\ \frac{\varepsilon}{\varepsilon-1}\sum_{k=0}^{\infty}\theta^k E_t\left[\beta^k C_{t+k}^{-\sigma}Y_{t+k}MC_{t+k}\left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}}\right]\end{aligned} \tag{22}$$

which, defining two auxiliary variables we write this as:

$$(\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}}x_{2,t} = \frac{\varepsilon}{\varepsilon-1}x_{1,t} \tag{23}$$

where

$$x_{2,t} = E_t \sum_{k=0}^{\infty} \theta^k \left[ \beta^k C_{t+k}^{-\sigma} \left( \frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] \tag{24}$$

$$= C_t^{-\sigma} Y_t + E_t \sum_{k=1}^{\infty} \theta^k \left[ \beta^k C_{t+k}^{-\sigma} \left( \frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] \tag{25}$$

$$= C_t^{-\sigma} Y_t + E_t \left( \frac{P_t}{P_{t+1}} \right)^{1-\varepsilon} \sum_{k=1}^{\infty} \theta^k \left[ \beta^k C_{t+k}^{-\sigma} \left( \frac{P_{t+1}}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] \tag{26}$$

$$= C_t^{-\sigma} Y_t + \beta\theta E_t \Pi_{t+1}^{\varepsilon-1} x_{2,t+1} \tag{27}$$

and

$$x_{1,t} = \sum_{k=0}^{\infty} \theta^k E_t \left[ \beta^k C_{t+k}^{-\sigma} Y_{t+k} MC_{t+k} \left( \frac{P_t}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \tag{28}$$

$$= C_t^{-\sigma} MC_t Y_t + E_t \sum_{k=1}^{\infty} \theta^k \left[ \beta^k C_{t+k}^{-\sigma} Y_{t+k} MC_{t+k} \left( \frac{P_t}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \tag{29}$$

$$= C_t^{-\sigma} MC_t Y_t + E_t \left( \frac{P_t}{P_{t+1}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \sum_{k=1}^{\infty} \theta^k \left[ \beta^k C_{t+k}^{-\sigma} Y_{t+k} MC_{t+k} \left( \frac{P_{t+1}}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \tag{30}$$

$$= C_t^{-\sigma} MC_t Y_t + \beta\theta E_t \Pi_{t+1}^{\varepsilon + \frac{\alpha\varepsilon}{1-\alpha}} x_{1,t+1} \tag{31}$$

$$= C_t^{-\sigma} MC_t Y_t + \beta\theta E_t \Pi_{t+1}^{\frac{\varepsilon}{1-\alpha}} x_{1,t+1} \tag{32}$$

# 3 Wage Setting

Defining $\Pi_t^{w*} = W_t^* / W_t$:

$$W_t = \left[ (1-\theta_w) \left( W_t^* \right)^{1-\varepsilon_w} + \theta_w W_{t-1}^{1-\varepsilon_w} \right]^{\frac{1}{1-\varepsilon_w}}$$

$$1 = (1-\theta_w) \left( \Pi_t^{w*} \right)^{1-\varepsilon_w} + \theta_w \left( \Pi_{t-1}^{w} \right)^{\varepsilon_w - 1}$$

$$0=\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[N_{t+k|t}\frac{C_{t+k}^{-\sigma}\left(1-\tau_{t+k}^n\right)}{\left(1+\tau_{t+k}^c\right)}\left(\frac{\varepsilon_w}{\varepsilon_w-1}\left(-\frac{\left(1+\tau_{t+k}^c\right)}{\left(1-\tau_{t+k}^n\right)}\frac{-N_{t+k|t}^{\varphi}}{C_{t+k}^{-\sigma}}\right)-\frac{W_t^*}{P_{t+k}}\right)\right]$$

$$0=\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\left(\frac{\varepsilon_w}{\varepsilon_w-1}N_{t+k|t}^{1+\varphi}-\frac{C_{t+k}^{-\sigma}\left(1-\tau_{t+k}^n\right)}{\left(1+\tau_{t+k}^c\right)}\frac{W_t^*}{P_{t+k}}N_{t+k|t}\right)\right]$$

$$N_{t+k|t}=\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d$$

$$0=\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\left(\frac{\varepsilon_w}{\varepsilon_w-1}\left(\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right)^{1+\varphi}-\frac{C_{t+k}^{-\sigma}\left(1-\tau_{t+k}^n\right)}{\left(1+\tau_{t+k}^c\right)}\frac{W_t^*}{P_{t+k}}\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right)\right]$$

$$\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\left(\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right)^{1+\varphi}\right]=\frac{\varepsilon_w-1}{\varepsilon_w}\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\frac{C_{t+k}^{-\sigma}\left(1-\tau_{t+k}^n\right)}{\left(1+\tau_{t+k}^c\right)}\frac{W_t^*}{P_{t+k}}\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right]$$

$$f_{2,t}=\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\left(\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right)^{1+\varphi}\right]$$

$$=\left(\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w}N_t^d\right)^{1+\varphi}+\sum_{k=1}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\left(\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right)^{1+\varphi}\right]$$

$$=\left(\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w}N_t^d\right)^{1+\varphi}+\beta\theta_w E_t\left(\frac{W_t^*}{W_{t+1}^*}\right)^{-\varepsilon_w(1+\varphi)}\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k\left[\left(\left(\frac{W_{t+1}^*}{W_{t+1+k}}\right)^{-\varepsilon_w}N_{t+1+k}^d\right)^{1+\varphi}\right]$$

$$=\left(\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w}N_t^d\right)^{1+\varphi}+\beta\theta_w E_t\left(\frac{W_t^*}{W_{t+1}^*}\right)^{-\varepsilon_w(1+\varphi)}f_{2,t+1}$$

$$=\left(\Pi_t^{w*}\right)^{-\varepsilon_w(1+\varphi)}\left(N_t^d\right)^{1+\varphi}+\beta\theta_w E_t\left(\frac{\Pi_t^{w*}}{\Pi_{t+1}^{w*}}\Pi_{t+1}^w\right)^{-\varepsilon_w(1+\varphi)}f_{2,t+1}$$

$$f_{1,t}=\frac{\varepsilon_w-1}{\varepsilon_w}\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\frac{C_{t+k}^{-\sigma}\left(1-\tau_{t+k}^n\right)}{\left(1+\tau_{t+k}^c\right)}\frac{W_t^*}{P_{t+k}}\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right]$$

$$=\frac{\varepsilon_w-1}{\varepsilon_w}\frac{C_t^{-\sigma}\left(1-\tau_t^n\right)}{\left(1+\tau_t^c\right)}\frac{W_t^*}{P_t}\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w}N_t^d+\frac{\varepsilon_w-1}{\varepsilon_w}\sum_{k=1}^{\infty}\left(\beta\theta_w\right)^k E_t\left[\frac{C_{t+k}^{-\sigma}\left(1-\tau_{t+k}^n\right)}{\left(1+\tau_{t+k}^c\right)}\frac{W_t^*}{P_{t+k}}\left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w}N_{t+k}^d\right]$$

$$=\frac{\varepsilon_w-1}{\varepsilon_w}\frac{C_t^{-\sigma}\left(1-\tau_t^n\right)}{\left(1+\tau_t^c\right)}\frac{W_t^*}{P_t}\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w}N_t^d+\frac{\varepsilon_w-1}{\varepsilon_w}\beta\theta_w E_t\left(\frac{W_t^*}{W_{t+1}^*}\right)^{1-\varepsilon_w}\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k\left[\frac{C_{t+1+k}^{-\sigma}\left(1-\tau_{t+1+k}^n\right)}{\left(1+\tau_{t+1+k}^c\right)}\frac{W}{P}\right.$$

$$=\frac{\varepsilon_w-1}{\varepsilon_w}\frac{C_t^{-\sigma}\left(1-\tau_t^n\right)}{\left(1+\tau_t^c\right)}\frac{W_t^*}{P_t}\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w}N_t^d+\beta\theta_w E_t\left(\frac{W_t^*}{W_{t+1}^*}\right)^{1-\varepsilon_w}f_{1,t+1}$$

$$=\frac{\varepsilon_w-1}{\varepsilon_w}\frac{C_t^{-\sigma}\left(1-\tau_t^n\right)}{\left(1+\tau_t^c\right)}\frac{W_t^*}{P_t}\left(\Pi_t^{w*}\right)^{-\varepsilon_w}N_t^d+\beta\theta_w E_t\left(\frac{\Pi_t^{w*}}{\Pi_{t+1}^{w*}}\Pi_{t+1}^w\right)^{1-\varepsilon_w}f_{1,t+1}$$

$$N_t\equiv\int_0^1 N_t^j dj=\int_0^1\left(\frac{W_t^j}{W_t}\right)^{-\varepsilon_w}N_t^d dj=N_t^d\int_0^1\left(\frac{W_t^j}{W_t}\right)^{-\varepsilon_w}dj$$

$$\Rightarrow N_t^d=\frac{N_t}{S_t^W}$$

$$S_t^W = \int\limits_0^1 \left(\frac{W_t^j}{W_t}\right)^{-\varepsilon_w} = (1-\theta)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \theta \int\limits_0^1 \left(\frac{W_{t-1}^j}{W_t}\right)^{-\varepsilon_w} dj = (1-\theta)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \theta \left(\frac{W_{t-1}}{W_t}\right)^{-\varepsilon_w} \int\limits_0^1 \left(\frac{W}{W}\right.$$

$$= (1-\theta)\left(\Pi_t^{w*}\right)^{-\varepsilon_w} + \theta \left(\frac{W_{t-1}}{W_t}\right)^{-\varepsilon_w} S_{t-1}^W$$

$$\Pi_t^w = \frac{W_t}{W_{t-1}} = \frac{\frac{W_t}{P_t}}{\frac{W_{t-1}}{P_{t-1}}} \Pi_t$$