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# AN ALGORITHM FOR FINDING ALL VERTICES OF CONVEX POLYHEDRAL SETS*

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**Introduction.** This paper describes a method for finding all vertices of a convex polyhedral set defined by a system of linear inequalities. The simplex method of Dantzig [2, 3] provides an algorithm for finding an optimal solution to a general linear program. However, it solves the problem of finding only one optimal solution—or of showing that no optimal solution exists—and it is often desirable to find all solutions to a linear program. Closely allied is the question of finding all—not just one—optimal strategies to zero-sum two-person games with finite numbers of pure strategies (if, indeed, more than one optimal strategy exists). The algorithm described in this paper was developed to accomplish precisely these ends: finding all solutions or all optimal strategies.

Discussion of this problem has appeared in the linear programming and game theory literature. Charnes [1] mentions a method based on the simplex algorithm and Tarry's solution to the labyrinth problem of the theory of graphs. The simplex method is used to pass from one vertex, along an edge, to a neighboring vertex of the convex polyhedron (considered as a graph), and Tarry's procedure indicates the path which is to be followed. This solution results in passing through every edge exactly twice, and thus through every vertex at least $n$ times (where $n$ is the dimension of the space in which the convex polyhedral set lies). Moreover, the "Tarry data" needed to obtain this solution is prohibitive from a computational point of view, for at every vertex it is required to know about all edges adjacent, how often and in what direction they have been traversed, as well as what edge was the edge of first arrival at the vertex.

A more successful procedure is the double description method proposed by Motzkin, Raiffa, Thompson, and Thrall [5]. Essentially, it builds up the convex polyhedron by introducing the half-spaces given by the linear inequalities one at a time. Hence, it gets the vertices of a sequence of convex polyhedra, eliminating those vertices which are not in the half-space introduced and adding those new vertices which are created by the hyperplane which defines the half-space at each step, until all half-spaces are considered together and thus all vertices are found.

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We propose to give a computationally feasible procedure based on the basic iterative step of the simplex method—the "pivotal operation." Attention is focused on this approach due to the proven efficacy of the simplex method in the solution of general linear programs and zero-sum two-person games. Moreover, since the simplex method is by far the most important and most used computational scheme for solving linear programs it seems desirable to have an algorithm which can easily be attached for those cases in which all optimal solutions need to be found.

A general description of the method is given in §1. In §2 the procedure is discussed for convex polyhedra in 3-space. Section 3 extends the case $n = 3$ to any $n$-dimensional space. Section 4 contains a critical and heuristic discussion of the procedure. Finally, in the Appendix, a particular "baby" convex polyhedron is considered and the algorithm is used to find all its vertices.

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**1. General description.** Our aim is to find all vertices of the convex polyhedral set $S$ described by the system of $m$ linear inequalities in $n$ unknowns

$$(1)\quad \left.\begin{array}{ccccccccccc} a_{11}(-x_1) & + & a_{12}(-x_2) & + & \cdots & + & a_{1n}(-x_n) & + & b_1 & = & y_1 \geqq 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\ a_{m1}(-x_1) & + & a_{m2}(-x_2) & + & \cdots & + & a_{mn}(-x_n) & + & b_m & = & y_m \geqq 0 \end{array}\right\},$$

or, in abbreviated form,

$$A(-X) + b = Y \geqq 0.$$

We assume that the columns of coefficients of the matrix $A$ are linearly independent. This implies that $m \geqq n$. Notice that any such system in which is included the inequalities $x_1 \geqq 0, \cdots, x_n \geqq 0$, thus restricting the set $S$ to lie in the closed positive orthant, can have no linear dependence among the columns of $A$.

Tucker [7] uses the term *fundamental subsystem* of the auxiliary system $A(-X) + b = 0$ of equations of (1) to designate a subsystem $A^*(-X) + b^* = 0$ of equations, where $A^*$ is a nonsingular square submatrix consisting of $n$ linearly independent rows of $A$, and $b^*$ the corresponding elements of $b$. A fundamental subsystem has a unique solution $X$ which we shall call a *point*. If, moreover, the solution $X$ satisfies $A(-X) + b \geqq 0$, then $X$ is a *vertex*. Thus the vertices of $S$ are the collection of solutions $X$ to all fundamental subsystems of (1) which also satisfy the inequalities (1). We note that $y_i = 0$ corresponds to a hyperplane $H_i$ in $n$-dimensional space, and that $H_i \cap S$, if nonvacuous and not all of $S$, is a *face* of $S$.

We propose to give an iterative procedure based on the simplex method of Dantzig [2, 3], where we are, of course, without a given objective function. We shall use the simplex method to go from one point of intersection $p_i$ of $n$ hyperplanes $x_{r_1} = 0, \cdots, x_{r_k} = 0, y_{r_{k+1}} = 0, \cdots, y_{r_n} = 0$ to another such point $p_j$ where all hyperplanes associated with $p_i$ and $p_j$ are the same except for one. This step corresponds to performing one pivotal operation in the simplex tableau. In this case we say that the points $p_i$ and $p_j$ are *neighbors*. In a so called degenerate situation, when more than $n$ hyperplanes go through a single point, $p_i$ and $p_j$ may be the same point.

Geometrically stated, then, we shall find all vertices of $S$ by going from a point to one of its neighbors and follow the rules:

1. Pick a hyperplane, say $H_i$.
2. Find all vertices of $S$ which lie on $H_i$.
3. Drop the inequality or half-space requirement $y_i \geqq 0$ which corresponds to $H_i$.
4. Pick some other hyperplane $H_j$, and continue as in rule 2.

Clearly this process must come to an end in a finite number of steps since it must terminate when $m - n$ of the half-spaces are dropped, i.e., when only $n$ half-spaces remain. Note, however, that rule 3 means, in effect, that at each subsequent stage of the method we alter the convex polyhedral set $S$ by dropping out one of the half-space requirements (1) which define it; and thus we have at each stage a different polyhedral convex set, say $S_k$ at the $k$th stage. This can result in omitting no vertices of $S$ for we only drop faces of $S$ whose vertices we have already found.

The general idea of the method to be used having been introduced we raise the question: when do we know that all vertices of a certain face $H_i \cap S_k$ of $S_k$ have been found? To find all vertices of (nonvacuous) $H_i \cap S_k$ is simply to find all vertices of the convex polyhedral set $H_i \cap S_k$ (of dimension one less than $S_k$ if $S_k$ does not lie wholly within $H_i$); that is, we must find all vertices of each face of $H_i \cap S_k$. If to make sure that we have all vertices means that we must proceed as described above until $m - n$ of the $H_i$'s are dropped for each face (of each face, etc.) then the method would inspect every point of intersection of $n$ hyperplanes $H_i$; or, what is the same thing, the method would solve all fundamental subsystems of (1). But every vertex of $S$ belongs to some two-dimensional face of $S$, that is, to some $S \cap_{i=1}^{n-2} H_{r_i}$, and two-dimensional convex polyhedral sets have particularly simple properties.

Given that we are dealing with a convex polyhedral set $S$ as defined by (1) with $n > 2$, notice that rule 3 prescribes only that once $H_i$ is "exhausted", i.e., all vertices of $S$ which lie in $H_i$ have been found (if any), $y_i \geqq 0$ must be dropped. It would certainly be a pleasant thing were we able to prescribe instead that $y_i > 0$, i.e., that we never return to a point

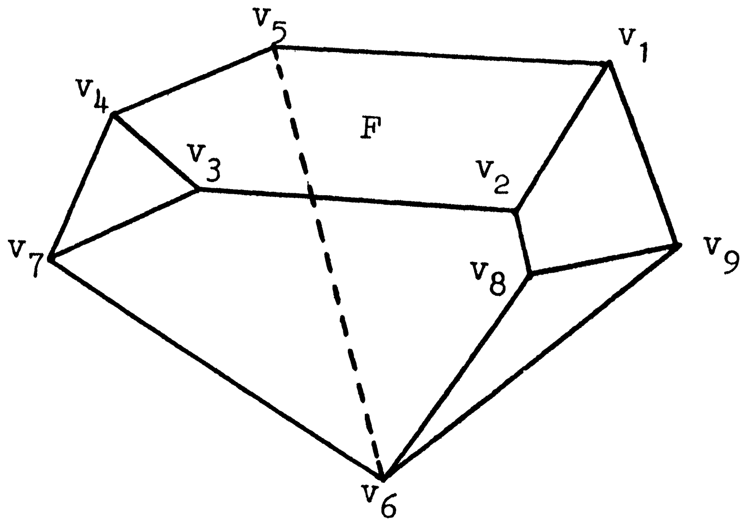


FIG. 1

of $H_i$ and we stay in the half-space $y_i > 0$. Unfortunately it is easy to construct examples which show that we cannot do this. Fig. 1 is one such example. Suppose we impose constraints $y_i > 0$, that is, once all vertices of a face $H_i$ have been found always stay in the half-space defined by its corresponding linear inequality. If we set out to find all vertices of face $F$ and start at vertex $v_1$, and go on through $v_2$, $v_3$, $v_4$, and $v_5$, then we must go to $v_6$. Continuing, we either omit $v_7$, or $v_8$ and $v_9$.

When, however, we are dealing with a convex polyhedral set $S$ as defined in (1) with $n = 2$ we can use the particularly simple properties which these sets have. Each hyperplane $H_i$ defined by (1) with $n = 2$ is a line and can have on it at most two vertices of $S$; thus, when at a point on $H_i$ we can get to a vertex on it (if any) with one step of the simplex method. If $H_i$ has no vertices of $S$ on it then we can strengthen rule 3 to read "stay in the half-space $y_i > 0$", and no vertices of $S$ can possibly be missed. $S$ may be empty. If this is the case, then we go from one point of $S$ to another, adding constraints "stay in the half-space $y_i > 0$", until no further steps are possible which do not violate the constraints, and thus we know that $S$ has no vertices. If we get to a vertex of $S$ then we need not—with one small exception—ever again go to a point of $S$ which is not a vertex; for $S$ is a closed convex polygon—with one exception—and we need merely "go all around" it, from vertex to vertex, until we are one step away from the initial vertex found. Therefore, we only need to keep a record of the first vertex found, which we do by labeling the first face of $S$ whose two vertices have been found, say $(H_i)^0$. The one small exception alluded to above is when $S$ is not bounded. In this case we "go around" it until we are stopped, then return to the initial face $(H_i)^0$ with one pivot of the simplex method, and "go around" in the other direction until we are stopped, at which point we will know that all vertices of $S$ have been found.

**2. Iterative procedure for** $n = 3$. Consider the set $S$ defined by

$$\left.\begin{array}{l} a_{11}\ (-x_1) + a_{12}\ (-x_2) + a_{13}\ (-x_3) + b_1\ = y_1 \geqq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}\ (-x_1) + a_{m2}\ (-x_2) + a_{m3}\ (-x_3) + b_m = y_m \geqq 0 \end{array}\right\}, \tag{2}$$

or, in abbreviated form,

$$A(-X) + b = Y \geqq 0,$$

where we assume that the columns of $A$ are linearly independent.

We use Tucker's [6] condensed form of the simplex tableau and thus our initial tableau has the form

(3)

| $-x_1$ | $-x_2$ | $-x_3$ | 1 | |
|---|---|---|---|---|
| $a_{11}$ | $a_{12}$ | $a_{13}$ | $b_1$ | $y_1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $a_{m1}$ | $a_{m2}$ | $a_{m3}$ | $b_m$ | $y_m$ |

We shall adopt the usual linear programming terminology, in designating, for any particular tableau, as *basic variables* those which appear in the right-hand column, and *nonbasic* those which appear in the top row. We say, for any tableau, that we have a *feasible point* if all $x_i$'s are basic and the constants in the column corresponding to 1, the *constant column*, are non-negative (except, possibly, those which correspond to the $x_i$'s). A feasible point is thus a vertex.

Our method consists in the choice of a sequence of elements on which to pivot which will assure us that all vertices of $S$ are found.

We now partition the variables of (2) into mutually disjoint sets $B_i$, $i = 0, 1, 2, 3$. Assume that we have already found all vertices which lie on hyperplanes $H_{r_1}, H_{r_2}, \cdots, H_{r_{k-1}}$, i.e., we have dropped the half-space requirements of (2) $y_{r_1} \geqq 0$, $y_{r_2} \geqq 0, \cdots, y_{r_{k-1}} \geqq 0$ and thus have left the convex polyhedral set $S_k$. We then say $y_{r_i}$ belongs to $B_3$ for $i = 1, \cdots, k - 1$. Assume that we are in the process of finding all vertices of $S_k$ which lie on the hyperplane $H_{r_k}$, i.e., all vertices of $S_k$ for which $y_{r_k} = 0$; and that we have already found all vertices on the two-dimensional face of $S_k$, $H_{r_k} \cap S_k$, for which $y_{r_k(1)} = 0, \cdots, y_{r_k(s-1)} = 0$. We then say $y_{r_k(j)}$ belongs to $B_2$ for $j = 1, \cdots, s - 1$. All $y_i$'s which belong neither to the set

$B_2$ nor the set $B_3$ belong to the set $B_1$ ; and the $x_j$'s belong to the set $B_0$ . Thus $B_0 \cup B_1 \cup B_2 \cup B_3$ is the set of all variables of the system (2). Schematically we have:

(4)

| $-y_{r_k}$ | $-y_{r_k(s)}$ | $-y_{r_k(s+1)}$ | $1$ | | |
|---|---|---|---|---|---|
| No pivots | | | | $y_{r_i}$ | $\} \in B_3$ |
| No pivots in $k$th stage | | | | $y_{r_k(j)}$ | $\} \in B_2$ |
| | | | | $y_i$ | $\} \in B_1$ |
| No pivots | | | | $x_j$ | $\} \in B_0$ |

Our immediate object is to find all vertices of $S_k$ for which $y_{r_k}$ is nonbasic. We call this the *kth stage* of the procedure.

*The kth Stage.*

i. *General rule.* Do not pivot on any element of the $y_{r_k}$ column until all vertices of $S_k$ for which $y_{r_k}$ is nonbasic have been found. Never pivot on an element which lies in a row corresponding to a $y_{r_i} \in B_3$ , for all vertices for which any of these $y_{r_i}$ are nonbasic have already been found. Likewise, never pivot on an element which lies in a row corresponding to a $y_{r_k(j)} \in B_2$ (with one exception described below), for all vertices for which any of these $y_{r_k(j)}$ are nonbasic at the same time as $y_{r_k}$ is nonbasic have been found (if any). Finally, if ever $B_1$ is null, then the $k$th stage is complete, since all lines have been investigated for vertices.

The element which is to be the next pivot depends upon the form of the current tableau. We define a tableau to be *acceptable* if all elements in the constant column which correspond to variables of $B_1$ and $B_2$ are non-negative. Similarly, the point which corresponds to this tableau is an *acceptable point.* An acceptable point is thus a vertex of the set $H_{r_k} \cap S_k$ . A vertex of $S$ is an acceptable point, but an acceptable point is not necessarily a vertex.

ii. *The current tableau is acceptable.* If the current tableau is acceptable then the pivotal element is chosen as follows:

(a) 1. Its column, say that corresponding to $y_{r_k(s)}$ , is that one of which the previous pivot was not an element (i.e., we pivot on elements in alternate columns each subsequent step).

2. Its row corresponds to a $y_i \in B_1$ and is chosen such that the new tableau is acceptable;

if such a choice is possible.

If the choice of element described is possible we perform the pivotal operation. If this was the second, or more than the second, tableau which was acceptable in the $k$th stage, then we alter, in the new tableau, the sets $B_1$ and $B_2$ by taking $y_{r_k(s)}$ out of $B_1$ and putting it into $B_2$ . Moreover, if this was the second acceptable tableau in this stage we rename $y_{r_k(s)}$ as $y^0_{r_k(s)}$ (i.e., we label $y_{r_k(s)}$ in order to be able to identify the initial acceptable point of this stage).

If such a choice in column $y_{r_k(s)}$ is impossible then we have one of three cases.

(b) There exists an element in the row corresponding to $y^0_{r_k(h)}$ and in the column corresponding to $y_{r_k(s)}$ which, if pivoted on, would lead to an acceptable point. This means we are at an acceptable point which is a neighbor of the first acceptable point of this stage, and we have found all acceptable points and thus all vertices (if any) of the two-dimensional face $H_{r_k} \cap S_k$ ; and have completed the $k$th stage.

(c) If there exists no pivot as described in (b) then we pivot on an element in the row corresponding to $y^0_{r_k(h)}$ . One of these two elements must be nonzero for, if not, $y^0_{r_k(h)}$ would never have been nonbasic in this stage. Then there must exist in the new tableau an element in the column not corresponding to $y^0_{r_k(h)}$ and in a row corresponding to a $y_i \in B_1$ , which, when pivoted on, brings us to the first acceptable point of this stage. We perform this pivotal operation. We now proceed as in (a) until no further pivots can be made. We will then have found all vertices (if any) of the $k$th stage.

(d) If there is no $y^0_{r_k(h)}$ in $B_2$ and no pivot in column $y_{r_k(s)}$ as described in (a) then either there exists a pivot in column $y_{r_k(s+1)}$ which leads to an acceptable point, or not.

1. If there is, perform the pivotal operation and alter $B_1$ and $B_2$ by taking $y_{r_k(s+1)}$ from $B_1$ and putting it into $B_2$ and continue as in (a) until no further pivots are possible (and do not go on to perform pivots as described in (b) or (c)). Then all acceptable points of this stage will have been found.
2. If there is not, then there exists only one acceptable point in this stage and hence we are done.

iii. *The current tableau is not acceptable.* If the current tableau is not acceptable then we can have had no acceptable points in this stage. We have one of three possible cases.

(a) There exists an element in the column corresponding to $y_{r_k(s)}$ or to $y_{r_k(s+1)}$ and in a row corresponding to a variable of $B_1$ on which we can pivot to reach an acceptable point. In this case we perform the pivotal operation and obtain a tableau which corresponds to an acceptable point and thus continue as in ii.

(b) There exists no element in the column corresponding to $y_{r_k(s)}$ (or to $y_{r_k(s+1)}$) as described in (a). Then there are no acceptable points for which $y_{r_k}$ and $y_{r_k(s+1)}$ are nonbasic together (nor for which $y_{r_k}$ and $y_{r_k(s)}$ are nonbasic together). Thus the pivotal element is chosen as follows:

1. Its column corresponds to $y_{r_k(s+1)}$ (or to $y_{r_k(s)}$ );
2. Its row corresponds to a $y_i \in B_1$ and is picked such that in the new tableau the constant entry which corresponds to $y_{r_k(s+1)}$ (to $y_{r_k(s)}$) is non-negative, the constant column entries corresponding to $B_2$ remain non-negative, and the least number of elements in the constant column which correspond to variables of $B_1$ are negative; if such a choice is possible.

If the choice of element described is possible we perform the pivotal operation, alter, in the new tableau, the sets $B_1$ and $B_2$ by taking $y_{r_k(s+1)}$ ($y_{r_k(s)}$) out of $B_1$ and putting it into $B_2$, and continue.

(c) If no element as described in (a) or (b) exists then the $k$th stage is completed. For we are on the two-dimensional hyperplane $H_{r_k}$ at a particular point for which $y_{r_k(s)} = 0$ and $y_{r_k(s+1)} = 0$, and any other point of $H_{r_k}$ has either $y_{r_k(s)} < 0$ or $y_{r_k(s+1)} < 0$ or $y_{r_k(j)} < 0$, for some $y_{r_k(j)} \in B_2$, and thus no point of $H_{r_k}$ can be acceptable.

iv. *End of the kth stage.*

(a) When all acceptable points of $H_{r_k}$ have been found, we pivot on an element $p$ which is in the column corresponding to $y_{r_k}$ and not in rows corresponding to variables which belong to $B_0$ or $B_3$, if possible. The row of the pivot $p$ is chosen such that the least number of elements of the column of constants corresponding to variables of $B_1$ and $B_2$ are negative. If there exists a possible pivot $p$, we perform the pivotal operation and alter the sets $B_1$, $B_2$, $B_3$, by taking $y_{r_k}$ from $B_1$ and putting it into $B_3$, and putting all variables of $B_2$ into $B_1$, and making $B_2$ null. We are now left with the set $S_{k+1}$, whose defining system of inequalities is different from that of $S_k$ only in that the constraint $y_{r_k} \geqq 0$ is dropped.

(b) If all elements of the $y_{r_k}$ column which are in rows of $B_1$ and $B_2$ are zero, then no pivot $p$ is possible, and all vertices of the convex polyhedral set $S$ have been found (if there were any). For, all fundamental subsystems of equations associated with the linear inequalities which define $S_k$ must contain the equation

$$a_{r_k,1}(-x_1) + a_{r_k,2}(-x_2) + a_{r_k,3}(-x_3) + b_{r_k} = y_{r_k} = 0,$$

and all such fundamental subsystems whose solutions satisfy the defining inequalities of $S_k$ have already been solved. Or, what is the same thing, the set $S_{k+1}$ contains no $n$ linearly independent equations $y_i = 0$, and hence no vertices. Thus all vertices of $S$ have indeed been found and the problem is solved.

The *index* of a variable $y_i$, or of its associated hyperplane $H_i$, is defined to be the number of times which $y_i$ was nonbasic at a feasible point; that is, the index of $y_i$ is increased by 1 each time it is nonbasic at a vertex of $S$. Hence, when all vertices of the set $S$ have been found, the index of $y_i$

is the number of vertices which are on face $H_i \cap S$ (unless we have degeneracy). If $H_i \cap S$ is null the index of $H_i$ is zero.

*The* $(k + 1)$*th Stage.* We now choose a new hyperplane $H_{r_{k+1}}$ and set out to find all vertices of the set $H_{r_{k+1}} \cap S_{k+1}$. $H_{r_{k+1}}$ is chosen to be that hyperplane which corresponds to a nonbasic variable $y_{r_{k+1}}$ with highest index.

*The Zeroth Stage.* There remains the question of how to begin the procedure when we are given the initial condensed tableau (3). The first aim is to make the variables $x_1$, $x_2$, $x_3$ basic. We do this by performing successive pivots on an element in each of the three columns of the first three successive tableaus, always pivoting on a row corresponding to a $y_i$ (i.e., not to an $x_j$ which has just been made basic), and so chosen that the least number of elements in the constant column are negative (ignoring those which correspond to the $x_j$). This we can always do, for if the only nonzero elements in a column corresponding to a nonbasic $x_j$ were in rows corresponding to basic $x_k$'s, then our assumption of column independence of $A$ in (2) would have been violated.

*The First Stage.* We then choose one of the nonbasic $y_i$'s, say $y_{r_1}$, and proceed to find all vertices of $S$ (if any) which lie on the hyperplane $H_{r_1}$.

*The Method Described Finds All Vertices of* $S$. Suppose that we are in the $k$th stage. To have found all vertices of $H_{r_1}, \cdots, H_{r_{k-1}}$ means that we have solved all fundamental subsystems which include any of the equations

$$a_{r_i,1}(-x_1) + a_{r_i,2}(-x_2) + a_{r_i,3}(-x_3) + b_{r_i} = 0$$

for $i = 1, \cdots, k - 1$. Thus we need only, in the sequel, consider fundamental subsystems which include none of these equations, and hence we need never pivot on an element of any row corresponding to a $y_{r_i} \in B_3$ for $i = 1, \cdots, k - 1$.

In the $k$th stage, then, we have a convex polyhedral set $S_k$ defined by the system of linear inequalities

$$a_{i1}(-x_1) + a_{i2}(-x_2) + a_{i3}(-x_3) + b_i = 0$$

for $i$ an integer, $1 \leqq i \leqq m$ and $i \neq r_1, \cdots, r_{k-1}$. An acceptable point in the $k$th stage (if there are any) is a vertex of the polyhedral convex set $S_k$ which lies in the two-dimensional hyperplane $H_{r_k} \cap S_k$ (if nonempty). Finally, owing to the simple structure of polyhedra in two dimensions we are easily able to find all its vertices, and since any vertex of $S$ which lies in the hyperplane $H_{r_k}$ must be an acceptable point, our method can omit no vertex of $S$.

**3. Iterative procedure for** $n > 3$. We now extend, by induction, the procedure described in §2. Assume that the iterative procedure is known for convex polyhedral sets $T$ which lie in a space of dimension less than or equal to $n - 1$.

Let $S$ be the convex polyhedral set defined by the system of linear inequalities

$$(5)\quad \left.\begin{array}{ccccccccc} a_{11}(-x_1) & + & a_{12}(-x_2) & + \cdots + & a_{1n}(-x_n) & + & b_1 & = & y_1 \geqq 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{mn}(-x_1) & + & a_{m2}(-x_2) & + \cdots + & a_{mn}(-x_n) & + & b_m & = & y_m \geqq 0 \end{array}\right\},$$

or, in abbreviated form,

$$A(-X) + b = Y \geqq 0,$$

where we assume that the columns of $A$ are linearly independent and thus that $m \geqq n$. We also assume, without loss of generality, that $S$ lies within no $(n - 1)$-dimensional hyperplane. As before, $H_i$ designates the $(n - 1)$-dimensional hyperplane $y_i = 0$.

The first condensed simplex tableau has the form

(6)

| $-x_1$ | $-x_2$ | $\cdots$ | $-x_n$ | $1$ | |
|---|---|---|---|---|---|
| $a_{11}$ | $a_{12}$ | | $a_{1n}$ | $b_1$ | $y_1$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $a_{m1}$ | $a_{m2}$ | | $a_{mn}$ | $b_m$ | $y_m$ |

*The Zeroth Stage.* Make all variables $x_i$, $i = 1, \cdots, n$, basic, and define $B_0$ to be the set of all $x_i$'s. The linear independence of the columns of $A$ assures us that all $x_i$'s can be made basic.

*The kth Stage.* We assume that we have found all vertices (if any) which lie on hyperplanes $H_{r_1}, \cdots, H_{r_{k-1}}$, and that the variables $y_{r_1}, \cdots, y_{r_{k-1}}$ are basic. We then say that $y_{r_1}, \cdots, y_{r_{k-1}}$ belong to the set $B_n(n \geqq 4)$. Since we have found all these vertices of $S$ we consider only the convex polyhedral set $S_k$ $(S_1 = S)$ defined by the linear inequalities

$$(7)\qquad a_{i1}(-x_1) + a_{i2}(-x_2) + \cdots + a_{in}(-x_n) + b_i = y_i \geqq 0$$

for $i$ an integer, $1 \leqq i \leqq m$ and $i \neq r_1, \cdots, r_{k-1}$. $S_k$ has as vertices all vertices of $S$ which do not lie in the hyperplanes $H_{r_1}, \cdots, H_{r_{k-1}}$ and possibly others; thus, limiting ourselves to $S_k$ can result in no loss of vertices of $S$. Hence we never pivot in a row corresponding to a variable of $B_n$.

Among the nonbasic variables we choose one with highest index (defined as in §2), say $y_{r_k}$, and the $k$th stage consists in finding all vertices of $S_k$ (if any) which lie in the hyperplane $H_{r_k}$. Schematically:

$$(8)\qquad \begin{array}{|ccccc|l} -y_{r_k} & -y^1_{r_k} & \cdots & -y^{n-1}_{r_k} & 1 & \\ \hline \multicolumn{5}{|c|}{\text{Never pivot}} & \left.\begin{matrix} y_{r_1}\\ \vdots \\ y_{r_k}\end{matrix}\right\} \in B_n \\ \hline \multicolumn{2}{|l|}{\text{No pivot in } k\text{th stage}} & \multicolumn{3}{c|}{} & y_i\text{'s} \\ \hline \multicolumn{5}{|c|}{\text{Never pivot}} & x_j\text{'s}\} \in B_0 \\ \hline \end{array}$$

Now, $H_{r_k} \cap S_k$ is a convex polyhedral set (or an empty set) which lies in a space of dimension less than or equal to $n - 1$, since it lies in $H_{r_k}$, and hence, by induction, we know how to find all its vertices.

*End of the kth Stage.* When all vertices of $H_{r_k} \cap S_k$ (if any) are found, and hence all vertices of $S$ which lie in the hyperplane $H_{r_k}$ are found, then we have found all vertices of $S$ for which $y_{r_k}$ is nonbasic. So we pivot on an element in the $y_{r_k}$ column and not in rows corresponding to variables which belong to $B_0$ or $B_n$, and alter the set $B_n$ in the new tableau by adding to it $y_{r_k}$. Also, we change sets $B_1, \cdots, B_{n-1}$ by making $B_2, \cdots, B_{n-1}$ null and letting $B_1$ contain all variables not in $B_0$ or $B_n$. We are now left with the set $S_{k+1}$, whose defining system of inequalities is different from that of $S_k$ only in that the constraint $y_{r_k} \geqq 0$ is dropped.

This process comes to an end either if all elements in the $y_{r_k}$ column and not in rows of $B_0$ and $B_n$ are zero, or if $k = m - n$. In the first case it simply means that there exist no fundamental subsystems of (5) which do not contain at least one of the equations

$$(9)\qquad a_{i1}(-x_1) + a_{i2}(-x_2) + \cdots + a_{in}(-x_n) + b_i = y_i = 0$$

for $i = r_1, \cdots, r_k$. But all fundamental subsystems which include any of the equations (9) and whose solutions satisfy (5) have already been solved. Or, what is the same thing, the set $S_{k+1}$ contains no $n$ linearly independent equations $y_i = 0$, and hence no vertices. Hence, all vertices of $S$ have been found.

In the second case, when $k = m - n$, then no further pivots are possible since $B_0 \cup B_n$ contains all basic variables. This means that there are no more fundamental subsystems of (5) to consider, for we have left a set $S_{k+1}$ which is defined by exactly $n$ inequalities of (5) and hence has at most one vertex.

**4. Some critical remarks.** The problem for which we have described a solution is the following: How are all vertices of a convex polyhedral set $S$, defined by a set of linear inequalities

$$(10)\qquad A(-X) + b = Y \geqq 0$$

to be found using the simplex method of Dantzig (without a given objective function)?

One obvious way is to use the simplex method to solve all fundamental subsystems of (10), and thus find all points of $S$. But we also wish to have an efficient method, one which requires, in general, a minimal number of steps. Likewise, we want a computationally feasible method, one that can be easily programmed. Lastly, any method—even the obvious one stated above—must include some rule which tells us when all vertices of $S$ have been found.

We have tried many methods. A number of "cutting methods"—suggested by Gomory's [4] use of such a method in his algorithm for finding optimal integer solutions to linear programs—were attempted. Generally, these methods add additional half-space requirements to the set $S$ so chosen that the vertices already found are "cut off" from $S$, and thus new convex polyhedral sets $S_k$ are formed. The difficulty is that all such methods add work, for although an $S_k$ does not have as vertex those vertices of $S$ already found, it has many vertices which are not vertices of $S$, and it is defined by a larger set of linear inequalities.

Another attempt was made by introducing a pseudo linear objective function $\phi(X)$. It serves to order the vertices of $S$ in a particular way. However, difficulties arise in trying to determine when the procedure is completed, i.e., when all vertices have been found; for the ordering implied by $\phi(X)$ does not necessarily order all vertices in a path which can be followed by successive steps of the simplex tableau.

Algebraic considerations have led us to believe that the only computationally feasible method of approaching the problem is to deal, somehow, with one variable $y_i$, or one hyperplane $H_i$, of (10) at a time. From the "bookkeeping" point of view it is an easy matter to add an extra constraint of the form "$y_i$ must remain basic," while any other conceivable sort of extra constraint which keeps one from returning to vertices already visited seems extremely difficult. Of course, a method need not never return to a vertex which it has visited. One can easily imagine roving from one vertex of $S$ to another, allowing repetitions, and never leaving the face structure of $S$. But how can one be assured of finding all vertices? And how are problems of cycling avoided? If one could follow a path through all vertices with no repetitions then these objections would be taken care of, but even the existence of such a path is in question so that the fact of its construction is even more remote.[1]

[1] In general, no such path exists, as was shown in an example by T. A. Brown in *Hamiltonian Paths on Convex Polyhedra*, RAND Corporation paper, P-209, August, 1960. See also M. L. Balinski, *On the graph structure of convex nolyhedra in n-space*, to appear in the Pacific Journal of Mathematics.

Given that the method described supplies a computationally acceptable way of keeping track of the vertices found, how efficient is it? We are able to give no proofs and hence no definite evaluation. Most important, it will certainly not find all points of $S$ for every two dimensional face contains many points of $S$ which it will not visit. Also, only after repeated steps of the method does it go to points of $S$ which are not vertices; and when many steps are carried out, and thus it gets nonfeasible more often, the process has already been accelerated, since there are fewer possible pivots. Lastly, we have a bit of empirical evidence. The method has been used on a set $S$ in four-space, defined by nine linear inequalities. The result was encouraging for 32 steps were required to find 19 vertices.[2]

The double description method [5] proceeds in an entirely different way. It adds a half-space requirement (10) at each step, while we eliminate a half-space requirement (10) at each step. It has a "point-oriented" bookkeeping system, while we have a "hyperplane-oriented" bookkeeping system; that is, iterative steps of the double description method are based on information regarding points, while iterative steps in our method are based on information regarding hyperplanes. It is thus difficult to make comparisons of computational efficiency of the methods in absence of actual trials However, it would seem that the double description method might be more efficient—in the sense that fewer nonvertex points of $S$ are found—than our method if many of the half-space requirements (10) were "superfluous," i.e., if their associated hyperplanes contained no vertices of $S$. On the other hand, the efficiency of the double description method is certainly dependent on the order in which the hyperplanes are introduced. There is no clear indication as to how dependent our method is upon the choice of starting point. If most of the half-space requirements (10) are "active" we have no definite feelings as to which method would prove to be more efficient in the sense specified above. Lastly it is worth noting again that the "format" of our method is the simplex tableau, and the basic iterative step is the pivotal operation of the simplex method. The practical advantage that derives from the use of this extensively used and highly successful format and computational routine is important: it is a simple matter to program our algorithm given a simplex method routine, and thus a simple matter

[2] The algorithm was programmed during the summer of 1960 at RAND Corporation. It has been released as a SHARE Report to be used with Philip Wolfe's RS-M1 linear programming code to find all solutions to linear programs. In an example arising from the oil industry in 65 non-negative variables, 35 equations, and having 5 nonbasic zero "reduced" costs in the "first" optimal solution, the method took 496 pivotal steps to find all 31 alternate optimal solutions. All solutions were found within about the first 200 steps.

to adjoin our algorithm to the simplex method to find all optimal solutions to a linear program.

**5. Appendix.** Consider the "baby" convex polyhedral set $S$ defined by the system of linear inequalities

$$\left.\begin{array}{rcl} -3x_1 - 2x_2 + x_3 + 6 &=& y_1 \geqq 0 \\ -3x_1 - 2x_2 - 4x_3 + 16 &=& y_2 \geqq 0 \\ -3x_1 \phantom{- 2x_2} + 4x_3 + 3 &=& y_3 \geqq 0 \\ -\tfrac{9}{4}x_1 - 4x_2 - 3x_3 + 17 &=& y_4 \geqq 0 \\ -x_1 - 2x_2 - x_3 + 10 &=& y_5 \geqq 0 \\ x_1 &=& y_6 \geqq 0 \\ x_2 &=& y_7 \geqq 0 \\ x_3 &=& y_8 \geqq 0 \end{array}\right\},$$

or, in abbreviated form,

$$(-A)X + b = Y \geqq 0.$$

The columns of $A$ are easily seen to be linearly independent. Note that the last three inequalities simply state that the half-spaces $x_1 \geqq 0$, $x_2 \geqq 0$, $x_3 \geqq 0$ contribute to the definition of the set $S$ (unless they are superfluous), and hence we shall, in the following sequence of tableaus, simply omit $y_6$, $y_7$, $y_8$ in order to shorten the length of the tableaus. We will have, then, no "zeroth stage."

In the tableaus to follow, an asterisk denotes the pivotal element. When we are looking for all vertices for which $y_i$ (or $x_i$) is nonbasic, that is, all vertices of $S$ which lie on the hyperplane $y_i = 0$ (or $x_i = 0$) not previously found (if any), we shall designate $y_i$ (or $x_i$) by an arrow. In the upper right-hand corner of each tableau will be found a triplet ($x_1$, $x_2$, $x_3$) if the tableau corresponds to a vertex of $S$. The numbers in parentheses following the nonbasic variables in the initial tableaus of each stage denote their index.

*Stage* 1.

1.

| $\downarrow$ $-x_1$ | $-x_2$ | $-x_3$ | 1 | (0, 0, 0) |
|---|---|---|---|---|
| 3 | 2* | $-1$ | 6 | $y_1$ |
| 3 | 2 | 4 | 16 | $y_2$ |
| 3 | 0 | $-4$ | 3 | $y_3$ |
| $\frac{9}{4}$ | 4 | 3 | 17 | $y_4$ |
| 1 | 2 | 1 | 10 | $y_5$ |

2.

| $\downarrow$ $-x_1$ | $-y_1$ | $-x_3$ | 1 | (0, 3, 0) |
|---|---|---|---|---|
| $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | 3 | $x_2$ |
| 0 | $-1$ | 5 | 10 | $y_2$ |
| 3 | 0 | $-4$ | 3 | $y_3$ |
| $-\frac{15}{4}$ | $-2$ | 5* | 5 | $y_4$ |
| $-2$ | $-1$ | 2 | 4 | $y_5$ |

3.

| ↓ $-x_1$ | $-y_1$ | $-y_4$ | 1 | $(0, \frac{7}{2}, 1)$ |
|---|---|---|---|---|
| $\frac{9}{8}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $\frac{7}{2}$ | $x_2$ |
| $\frac{15}{4}$ | $1^*$ | $-1$ | 5 | $y_2$ |
| 0 | $-\frac{8}{5}$ | $\frac{4}{5}$ | 7 | $y_3$ |
| $-\frac{3}{4}$ | $-\frac{2}{5}$ | $\frac{1}{5}$ | 1 | $x_3^0 \in B_2$ |
| $-\frac{1}{2}$ | $-\frac{1}{5}$ | $-\frac{2}{5}$ | 2 | $y_5$ |

4.

| ↓ $-x_1$ | $-y_2$ | $-y_4$ | 1 | $(0, 2, 3)$ |
|---|---|---|---|---|
| 0 | $-\frac{3}{10}$ | $\frac{2}{5}^*$ | 2 | $x_2$ |
| $\frac{15}{4}$ | 1 | $-1$ | 5 | $y_1 \in B_2$ |
| 6 | $\frac{8}{5}$ | $-\frac{4}{5}$ | 15 | $y_3$ |
| $\frac{3}{4}$ | $\frac{2}{5}$ | $-\frac{1}{5}$ | 3 | $x_3^0 \in B_2$ |
| $\frac{1}{4}$ | $\frac{1}{5}$ | $-\frac{3}{5}$ | 3 | $y_5$ |

5.

| ↓ $-x_1$ | $-y_2$ | $-x_2$ | 1 | $(0, 4, 0)$ |
|---|---|---|---|---|
| 0 | $-\frac{3}{4}$ | $\frac{5}{2}$ | 5 | $y_4 \in B_2$ |
| $\frac{15}{4}^*$ | $\frac{1}{4}$ | $\frac{5}{2}$ | 10 | $y_1 \in B_2$ |
| 6 | 1 | 2 | 19 | $y_3$ |
| $\frac{3}{4}$ | $\frac{1}{4}^{p}$ | $\frac{1}{2}$ | 4 | $x_3^0 \in B_2$ |
| $\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{3}{2}$ | 6 | $y_5$ |

[p] Possible pivot; hence, Stage 1 complete.

*Stage* 2.

6.

| ↓ $-y_1(3)$ | $-y_2(3)$ | $-x_2(3)$ | 1 | $(\frac{8}{3}, 0, 2)$ |
|---|---|---|---|---|
| 0 | $-\frac{3}{4}$ | $\frac{5}{2}$ | 5 | $y_4$ |
| $\frac{4}{15}$ | $\frac{1}{15}$ | $\frac{2}{3}$ | $\frac{8}{3}$ | $x_1 \in B_3$ |
| $-\frac{24}{15}$ | $\frac{3}{5}^*$ | $-2$ | 3 | $y_3$ |
| $-\frac{1}{5}$ | $\frac{1}{5}$ | 0 | 2 | $x_3$ |
| $-\frac{1}{15}$ | $-\frac{4}{15}$ | $\frac{4}{3}$ | $\frac{16}{3}$ | $y_5$ |

7.

| ↓ $-y_1$ | $-y_3$ | $-x_2$ | 1 | $(\frac{7}{3}, 0, 1)$ |
|---|---|---|---|---|
| $-2$ | $\frac{5}{4}$ | 0 | $\frac{35}{4}$ | $y_4$ |
| $\frac{4}{9}$ | $-\frac{1}{9}$ | $\frac{8}{9}$ | $\frac{7}{3}$ | $x_1 \in B_3$ |
| $-\frac{8}{3}$ | $\frac{5}{3}$ | $-\frac{10}{3}$ | 5 | $y_2$ |
| $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{2}{3}^*$ | 1 | $x_3$ |
| $-\frac{7}{9}$ | $\frac{4}{9}$ | $\frac{4}{9}$ | $\frac{20}{3}$ | $y_5$ |

8.

| ↓ $-y_1$ | $-y_3$ | $-x_3$ | 1 | $(1, \frac{3}{2}, 0)$ |
|---|---|---|---|---|
| $-2$ | $\frac{5}{4}^*$ | 0 | $\frac{35}{4}$ | $y_4$ |
| 0 | $\frac{1}{3}$ | $-\frac{4}{3}$ | 1 | $x_1 \in B_3$ |
| $-1$ | 0 | 5 | 10 | $y_2$ |
| $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $x_2^0 \in B_2$ |
| $-1$ | $\frac{2}{3}$ | $-\frac{2}{3}$ | 6 | $y_5$ |

9.

| ↓ $-y_1$ | $-y_4$ | $-x_3$ | 1 | acceptable |
|---|---|---|---|---|
| $-\frac{8}{5}$ | $\frac{4}{5}$ | 0 | 7 | $y_3 \in B_2$ |
| $\frac{8}{15}$ | $-\frac{4}{15}$ | $-\frac{4}{3}$ | $-\frac{4}{3}$ | $x_1 \in B_3$ |
| $-1$ | 0 | $5^*$ | 10 | $y_2$ |
| $-\frac{3}{10}$ | $\frac{2}{5}$ | $\frac{3}{2}$ | 5 | $x_2^0 \in B_2$ |
| $\frac{1}{15}$ | $-\frac{8}{15}$ | $-\frac{2}{3}$ | $\frac{4}{3}$ | $y_5$ |

10.

| ↓ $-y_1$ | $-y_4$ | $-y_2$ | 1 | $(\frac{4}{3}, 2, 2)$ |
|---|---|---|---|---|
| $-\frac{8}{5}^*$ | $\frac{4}{5}$ | 0 | 7 | $y_3 \in B_2$ |
| $\frac{4}{15}$ | $-\frac{4}{15}$ | $\frac{4}{15}$ | $\frac{4}{3}$ | $x_1 \in B_3$ |
| $-\frac{1}{5}$ | 0 | $\frac{1}{5}$ | 2 | $x_3 \in B_2$ |
| 0 | $\frac{2}{5}^{p}$ | $-\frac{3}{10}$ | 2 | $x_2^0 \in B_2$ |
| $-\frac{1}{15}$ | $-\frac{8}{15}$ | $\frac{2}{15}$ | $\frac{8}{3}$ | $y_5$ |

[p] Possible pivot; hence, Stage 2 complete.

*Stage* 3.

11.

| $-y_3(2)$ | $-y_4(3)$ | $-y_2(4)$ | $1$ | acceptable | |
|---|---|---|---|---|---|
| $-\frac{5}{8}$ | $-\frac{1}{2}$ | $0$ | $-\frac{35}{8}$ | $y_1$ | $\in B_3$ |
| $\frac{1}{6}$ | $-\frac{2}{15}$ | $\frac{4}{15}$ | $\frac{15}{6}$ | $x_1$ | |
| $-\frac{1}{8}$ | $-\frac{1}{10}$ | $\frac{1}{5}$ | $\frac{9}{8}$ | $x_3$ | |
| $0$ | $\frac{2}{5}*$ | $-\frac{3}{10}$ | $2$ | $x_2$ | |
| $-\frac{1}{24}$ | $-\frac{17}{30}$ | $\frac{2}{15}$ | $\frac{19}{8}$ | $y_5$ | |

No acceptable pivots in $y_3$ column.

12.

| $-y_3$ | $-x_2$ | $-y_2$ | $1$ | acceptable | |
|---|---|---|---|---|---|
| $-\frac{5}{8}$ | $\frac{5}{4}$ | $-\frac{3}{8}$ | $-\frac{15}{8}$ | $y_1$ | $\in B_3$ |
| $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{19}{6}$ | $x_1$ | |
| $-\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{8}*$ | $\frac{13}{8}$ | $x_3$ | |
| $0$ | $\frac{5}{2}^{p}$ | $-\frac{3}{4}$ | $5$ | $y_4^0$ | $\in B_2$ |
| $-\frac{1}{24}$ | $\frac{17}{12}$ | $-\frac{7}{24}$ | $\frac{125}{4}$ | $y_5$ | |

[p] Possible pivot; hence, Stage 3 complete.

*Stage* 4.

13.

| $-y_3(3)$ | $-x_2(5)$ | $-x_3(4)$ | $1$ | $(1, 0, 0)$ | |
|---|---|---|---|---|---|
| $-1$ | $2$ | $3$ | $3$ | $y_1$ | |
| $\frac{1}{3}$ | $0$ | $-\frac{4}{3}$ | $1$ | $x_1$ | $\in B_3$ |
| $-1$ | $2$ | $8$ | $13$ | $y_2$ | |
| $-\frac{3}{4}$ | $4$ | $6*$ | $\frac{59}{4}$ | $y_4$ | |
| $-\frac{1}{3}$ | $2$ | $\frac{7}{3}$ | $9$ | $y_5$ | |

No acceptable pivots in $y_3$ column.

14.

| $-y_3$ | $-x_2$ | $-y_4$ | $1$ | acceptable | |
|---|---|---|---|---|---|
| $-\frac{15}{4}$ | $0$ | $-\frac{1}{2}$ | $-\frac{25}{8}$ | $y_1$ | |
| $\frac{1}{6}$ | $\frac{8}{9}$ | $\frac{2}{9}$ | $\frac{77}{18}$ | $x_1$ | $\in B_3$ |
| $0$ | $-\frac{10}{3}$ | $-\frac{4}{3}$ | $-\frac{20}{3}$ | $y_2$ | |
| $-\frac{1}{8}$ | $\frac{2}{3}*$ | $\frac{1}{6}^{p}$ | $\frac{59}{24}$ | $x_3^0$ | $\in B_2$ |
| $-\frac{1}{24}$ | $\frac{4}{9}$ | $-\frac{7}{18}$ | $\frac{235}{72}$ | $y_5$ | |

[p] Possible pivot; hence, Stage 4 complete.

*Stage* 5.

15.

| $-y_3(3)$ | $-x_3(4)$ | $-y_4(3)$ | $1$ | acceptable | |
|---|---|---|---|---|---|
| $-\frac{15}{4}$ | $0$ | $-\frac{1}{2}$ | $-\frac{25}{8}$ | $y_1$ | |
| $\frac{1}{3}$ | $-\frac{4}{3}$ | $0$ | $1$ | $x_1$ | |
| $-\frac{5}{8}$ | $5$ | $-\frac{1}{2}$ | $\frac{45}{8}$ | $y_2$ | $\in B_3$ |
| $-\frac{3}{16}$ | $\frac{3}{2}$ | $\frac{1}{4}$ | $\frac{59}{16}$ | $x_2$ | |
| $\frac{1}{24}*$ | $-\frac{2}{3}$ | $-\frac{1}{2}$ | $\frac{13}{8}$ | $y_5$ | |

No acceptable pivot in $y_4$ column.

16.

| $-y_5$ | $-x_3$ | $-y_4$ | $1$ | acceptable | |
|---|---|---|---|---|---|
| $90$ | $-60$ | $-\frac{91}{2}$ | $142$ | $y_1$ | |
| $-8$ | $4$ | $4$ | $-12$ | $x_1$ | |
| $15$ | $-5$ | $-8$ | $30$ | $y_2$ | $\in B_3$ |
| $\frac{9}{2}$ | $-\frac{3}{2}$ | $-\frac{5}{4}$ | $11$ | $x_2$ | |
| $24^{p}$ | $-16*$ | $-12$ | $39$ | $y_3^0$ | $\in B_2$ |

[p] Possible pivot; hence, Stage 5 complete; hence, after one pivot, all vertices found.

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