QF1100 Midterms Cheatsheet

Accumulation function and Interest

Accumulation function

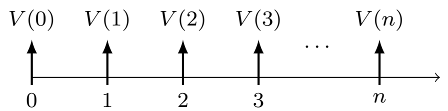
If V(t) is the value of an investment at time t, the ratio

$$a(t) = \frac{V(t)}{V(0)}$$

is the accumulation function. a(t) is a measure of how good the investment is. a(1) = 1

Cashflow Diagram

A cash flow diagram visualises a(t). The lower half of a cashflow diagram denotes time, while the upper half denotes payments.



Simple and Compound Interest

Simple interest: $a(t) = 1 + tr\%, for\ t \geq 0$
Compound interest: $a(t) = (1 + r\%)^t, for\ t \geq 0$

Frequency of Compounding

A *nominal* interest rate of r% is compounded p times annually (or convertible p-thly) if the year is divided into p equal periods and interest is paid over each period is $\frac{r\%}{p}$. p is the *frequency of compounding.*
If the nominal interest rate of r% is compounded p times annually, the effective interest rate is

$$r_{e\%} = (1 + \frac{r\%}{p})^p - 1$$

The accumulation function is

$$a(t) = (1 + r_{e\%})^t = (1 + \frac{r\%}{p})^{pt}$$

2 nominal interest rates are said to be **equivalent** if and only if they yield the same interest rate, i.e.

$$(1 + \frac{r^{(p)}}{p})^p = (1 + \frac{r^{(q)}}{q})^q$$

$r^{(p)}$ is the nominal interest rate compounded p-thly annually

Continuous Compounding

For a fixed nominal interest rate r, the more we increase the frequency of compounding, the larger the effective interest rate. When the frequency of compounding tends to infinity, the interest is *compounded continuously*

$$1 + r_e = \lim_{p \to \infty} (1 + \frac{r}{p})^p = e^r$$

Correspondingly, the accumulation function is

$$a(t) = (1 + r_e)^t = e^{rt}$$

Force of Interest

The **force of interest** at time t, of an investment product with activation function a(t), is

$$\delta(t) = \frac{a'(t)}{a(t)} = [ln(a(t))]'$$

Suppose an investment product accumulates like continuously compounded interests, i.e. $a(t) = e^{\delta t}$, then $\delta(t) = \frac{\delta e^{\delta t}}{e^{\delta t}} = \delta$.
One is indifferent between an investment product with $\delta(t)$ as force of interest at time t, and a deposit with nominal interest rate of $\delta(t)$ compounded continuously
From definition of force of interest,

$$a(t) = \exp \int_0^t \delta(\mathrm{u})\mathrm{du}$$

If $0 < s < t$, then

$$a(s,t) = \frac{a(t)}{a(s)} = exp \int_s^t \delta(\mathrm{u})\mathrm{du}$$

is the value of investment at time s when $1 is invested at time s. Hence, the *principle of consistency*:
For $0 < t_0 < t_1 < t_2 < ... < t_n$,

$$a(t_0, t_n) = a(t_0, t_1)a(t_1, t_2)...a(t_{n-1}, t_n)$$

Present Value and Equivalence

Present Value and Time Value

Let a(t) be the accumulation function of a bank deposit. Let c be an amount guaranteed to receive T time periods later. Then, the *present value* of c is

$$\frac{c}{a(T)}$$

For a cash flow consisting of a series of payments, with c_i received at time t_i,

$$\vec{C} = (c_1, t_1), (c_2, t_2), ..., (c_n, t_n)$$

the *present value*,$PV(\vec{C})$, is defined by

$$PV(\vec{C}) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}$$

Time value of $\vec{C}$ at time t, $TV_t(\vec{C})$ is given by

$$TV_t(\vec{C}) = PV(\vec{C}) \times a(t)$$

Suppose effective annual interest rates are constant at r, and t is measured in years, then $a(t) = (1 + r)^t$. Then,

$$PV(\vec{C}) = \sum_{i=1}^{n} \frac{c_i}{(1 + r)^{t_i}} \ and\ TV_t(\vec{C}) = \sum_{i=1}^{n} \frac{c_i}{(1 + r)^{t_i - t}}$$

If $t_i = i - 1$, then

$$\vec{C} = (c_1, t_1), (c_2, t_2), ..., (c_n, t_n) = (c_1, 0), (c_2, 1), ..., (c_n, n - 1)$$
$$= (c_1, c_2, ..., cn)$$

Principle of Equivalence

Two cash flows are **equivalent** if and only if they have the same present value.

Internal Rate of Return(IRR)

Given a cash flow $\vec{C} = (c_1, t_1), (c_2, t_2), ..., (c_n, t_n)$, the equation

$$PV(\vec{C}) = \sum_{i=1}^{n} \frac{c_i}{(1 + r)^{t_i}} = 0$$

is known as the **equation of value**.
Any non negative solution, r, for the equation of value is known as the yield, or IRR. IRR is the prevailing interest rate such that one is indifferent between $\vec{C}$ and $0

Annuities

An **annuity** is a series of payments made at regular intervals.
A **perpetuity** is an annuity with infinite payments.

Loans

Loans are repaid by a series of installment payments made at periodic intervals. If L is the amount of loan taken at time $t = 0$ and $\vec{C} = (c_1, t_1), (c_2, t_2), ..., (c_n, t_n)$ is the series of repayments, then

$$L = PV(\vec{C})$$

Loan Balance

Loan Balance immediately after the m-th installment is paid is the Time Value at t=m of the remaining *(n-m)* installment payments.
Sometimes, we need to determine n based on the Loan itself.

$$\vec{C} = \overbrace{(A, A, ..., \underbrace{A + B}_{n\text{-th payment}}\ )}^{n \text{ payments}}$$

To find n, find the largest n such that *$PV(\vec{C})$* of n payments of $A \leq L$ and *$PV(\vec{C})$* of $n + 1$ payments of $A > L$

Bonds

Bonds are issued by governments/corporations that want to borrow money. It is a written contract between issuer (borrower) and investors (lenders/bond holders), and can be freely traded before the maturity date.

Bond Risks

Interest Rate Risk: Risk due to prevailing fluctuating interest rates (may be more profitable to put money in bank deposit vs buy the bond)
Default Risk: Credit-worthiness of bond issuer Risk that the bond issuer will default on coupon payments
Liquidity Risk: Risk due to liquidity/ease of buying and selling bond **We assume these risk do not exist i.e. interest rates are constant, no default/liquidity risk**

Basic Bond Terminology

1. **F**: Face value/par value of bond - amount based on which periodic interest payments are computed
2. **R**: Redemption value/maturity value of bond - amount to be repaid at the end of the loan. Typically same as F
3. **c%**: Coupon rate - bond's interest payments, represented as percentage of par value, to be paid regularly to investors during the term of the loan
4. **Maturity Date**: Or redemption date - date on which the loan will be fully repaid. Additionally, **m** denotes the no. of coupon payments per year, and **n** denotes the total no. of coupon payments

Important: When the bond is issued, the information is specified and FIXED throughout the duration of the loan
Cash flow of the bond is given by

$$\vec{C} = \left((-P, 0), (\frac{c\%F}{m}, \frac{1}{m}), (\frac{c\%F}{m}, \frac{2}{m}), \dots, (\frac{c\%F}{m} + R, \frac{n}{m})\right)$$

Subsequently, we assume $R = F$

Bond yields

For any time t in the lifetime of the bond, the nominal yield of the bond is the nominal internal rate of return compounded m times per annum of holding the bond from time t to maturity. If $P(t)$ is the price of bond at time t, then nominal yield, $\lambda(t)\%$, satisfies

$$P(t) = \overbrace{\frac{R}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}}^{\text{Redemption value } R \text{ discounted}} + \overbrace{\sum_{i=1}^{n-tm} \frac{c\%F/m}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{i}}}^{\text{final } n\text{-}tm \text{ coupon payments}}$$

If $R = F$, then

$$P(t) = F\left[\frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \frac{c}{\lambda(t)}\left(1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right)\right]$$

$$= F + F\left(\frac{c - \lambda(t)}{\lambda(t)}\right)\left[1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right]$$

Bond is priced at time t

1. at a *premium* if $P(t) > F$ and iff $c > \lambda(t)$
2. at *par* if $P(t) = F$ and iff $c = \lambda(t)$
3. at a *discount* if $P(t) < F$ and iff $c < \lambda(t)$

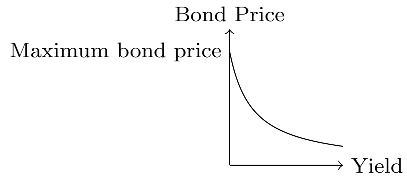
Effective yield, $\lambda_e(t)\%$, satisfies

$$P(t) = \frac{R}{(1 + \lambda_e(t)\%)^{\frac{n}{m} - t}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{(1 + \lambda_e(t)\%)^{\frac{i}{m}}}$$

Note: $P(\frac{n}{m}) = R$

Price-yield Relationship

Price $P(t)$ of the bond at time t is a decreasing function of the nominal yield $\lambda(t)$. Furthermore, this graph is convex.



Common types of bonds

Zero coupon bond: a bond that pays no coupons. At any time t, the price $P(t)$ and effective yield $\lambda_e(t)\%$ of a N-year zero-coupon bond with maturity value R satisfies

$$P(t) = \overbrace{\frac{R}{(1 + \lambda_e(t)\%^{N-t}}}^{\text{Discount } R \text{ by } N-t \text{ years}}$$

Perpetual bond or *consol*: a bond that never matures. At any time $t > 0$, the price $P(t)$ and nominal yield $\lambda(t)\%$ of a perpetual bond with coupon c paid m times annually satisfies

$$P(t) = \frac{cF}{\lambda(t)}$$

Pricing a bond

To price a bond, we make the following simplifying assumptions

- Interest rates constant over lifetime of bond $\iff$ yield is constant (reasonable if economic conditions are stable)
- Yield at any point of time = interest rates (reasonable if no significant default/liquidity risks). Then, price of bond = PV

Given this, we can price the bond at every point of time by replacing $\lambda(t)$ **by current interest rates**

Sensitivity of bond prices to interest rates

Macaulay Duration and Average Holding Times

Macaulay duration of any cash flow $\vec{C} = ((c_1, t_1), (c_2, t_2), \dots, (c_n, t_n))$ is the quantity

$$D = \frac{\sum_{i=1}^{n} t_i \cdot PV(C_i)}{PV(\vec{C}}) = \sum_{i=1}^{n} w_i t_i$$

where $PV(C_i)$ is the present value of C_i and $w_i = \frac{PV(C_i)}{PV(\vec{C})}$.
The Macaulay duration is the average time each dollar in $PV(\vec{C}$ needs to be held before it can be redeemed by investor.
For infinite cash flow $\vec{C} = ((c_1, t_1), (c_2, t_2), \dots)$:

$$D = \frac{\sum_{i=1}^{\infty} t_i \cdot PV(C_i)}{PV(\vec{C})}$$

For zero-coupon bond:

$$D = t_n$$

For bond *redeemable at par* $(R = F)$ and pays n coupons at a frequency of m payments a year. Suppose constant nominal bond yield $\lambda\%$ and coupon rate $c\%$. Cash flow is given by

$$\vec{C} = \left(\left(\frac{c\%F}{m}, t_1\right), \dots, \left(\frac{c\%F}{m}, t_{n-1}\right), \left(\frac{c\%F}{m} + F, t_n\right)\right)$$

where $t_i = \frac{i}{m}$, such that

$$D = \frac{1}{P}\left[\sum_{i=1}^{n} \overbrace{\frac{\frac{c\%F}{m}}{\left(1 + \frac{\lambda\%}{m}\right)^{i}}}^{PV(C_i)} \cdot \overbrace{\frac{i}{m}}^{t_i} + \overbrace{\frac{F}{\left(1 + \frac{\lambda\%}{m}\right)^{n}} \cdot \frac{n}{m}}^{PV(R) \times t_n}\right]$$

where

$$P = \sum_{i=1}^{n} \frac{\frac{c\%F}{m}}{\left(1 + \frac{\lambda\%}{m}\right)^{i}} + \frac{F}{\left(1 + \frac{\lambda\%}{m}\right)^{n}}$$

Then,

$$D = \frac{1 + \frac{\lambda\%}{m}}{\lambda\%} - \frac{1 + \frac{\lambda\%}{m} + n\left(\frac{c\%}{m} - \frac{\lambda\%}{m}\right)}{c\%\left[\left(1 + \frac{\lambda\%}{m}\right)^{n} - 1\right] + \lambda\%}$$

For a perpetual bond $(n \to \infty)$,

$$D = \frac{1 + \frac{\lambda\%}{m}}{\lambda\%}$$

If bond priced at par $(\lambda = c)$,

$$D = \frac{1 + \frac{c\%}{m}}{c\%}\left(1 - \frac{1}{\left(1 + \frac{c\%}{m}\right)^{n}}\right)$$

Modified Duration and Sensitivity

Modified duration measures the sensitivity per unit dollar of the present value P of a cash flow to interest rates λ

$$D_M = \frac{\left(\frac{dP}{d\lambda}\right)}{P}$$

By differentiating P wrt λ,

$$\frac{dP}{d\lambda} = \frac{1}{1 + \frac{\lambda}{m}} DP$$

Then,

$$D_M = \frac{1}{1 + \frac{\lambda}{m}} D$$

By linear approximation,

$$P(\lambda + \Delta\lambda) \approx P(\lambda) - (D_M P) \cdot \Delta\lambda$$

The corresponding change in P, ΔP is

$$\Delta P \approx -(D_M P) \cdot \Delta\lambda$$